EXPONENTIATION IN BANACH STAR ALGEBRAS

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In their recently published book Bonsall and Duncan ask the following question ((1), p. 65): if A is a Banach star algebra with an identity, is it true that

$$\exp(a^*) = (\exp a)^* \quad (a \in A)? \tag{1}$$

Of course, if the involution $*$ is continuous, this result does hold. However, in this note, we record a counter-example. In fact, the example is a slight modification of a result of Loy (3), in turn based on a construction in (2), which demonstrates the possible non-uniqueness of the exponential of an element in a commutative Banach algebra with two inequivalent complete norms.

Proposition 1. *There exists a commutative Banach star algebra B with identity and an element $b \in B$ for which* $\exp(b^*) \neq (\exp b)^*$.

Proof. Let A denote the disc algebra, let X denote the Banach space $C[0, 1]$ (with the usual notation), and let $T \in B(X)$ be the Volterra operator defined by

$$(Tx)(t) = \int_0^t x(s)ds \quad (x \in X).$$

Then, with respect to the operation $(f, x) \to f.x = f(T)x$, $A \times X \to X$, X is a commutative Banach A-module (cf. (2)). Now let $\mathfrak{A} = A \oplus X$, let

$$(a_1, x_1)(a_2, x_2) = (a_1 a_2, a_1.x_2 + a_2.x_1),$$

and let $\|(a, x)\|_1 = \|a\| + \|x\|$. Then $\mathfrak{A}_1 = (\mathfrak{A}, \|.\|_1)$ is a commutative Banach algebra with identity.

Let $*$ be the standard involution on A defined by $f^*(z) = \overline{f(\bar{z})}$ $(f \in A)$, where $^-$ denotes complex-conjugation, and define $*: \mathfrak{A} \to \mathfrak{A}$ by

$$(f, x)^* = (f^*, \bar{x}) \quad ((f, x) \in \mathfrak{A}).$$

To check that $*$ is indeed an involution on $\mathfrak{A}$, it suffices to check that

$$\overline{(f.x)} = f^*.\bar{x} \quad (f \in A, x \in X).$$

But, if $f(z) = \sum_0^\infty \lambda_n z^n$, so that $f^*(z) = \sum_0^\infty \bar{\lambda}_n z^n$,

$$f^*.\bar{x} = \sum_0^\infty \bar{\lambda}_n T^n \bar{x} = \sum_0^\infty \bar{\lambda}_n \overline{(T^n x)} = \overline{(f.x)},$$

as required. Thus $(\mathfrak{A}_1, *)$ is a Banach star algebra.

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Let $D: A \to X$ be a discontinuous derivation with $D(p) = 0$ for p a polynomial, but with $D(E) = x_0 \neq 0$, where $E(z) = e^z$. Such a derivation is constructed in (2). Define

$$\|(f, x)\|_2 = \|f\| + \|Df - x\| \quad ((f, x) \in \mathfrak{A}).$$

(This norm is introduced in (3).) It is clear that $\| \cdot \|_2$ is an algebra norm on $\mathfrak{A}$. We check directly that it is complete. Let $((f_n, x_n))$ be a $\| \cdot \|_2$-Cauchy sequence, so that (f_n) is Cauchy in A and $(Df_n - x_n)$ is Cauchy in X. Let $f_n \to f \in A$ and let $Df_n - x_n \to y \in X$. Then it is clear that

$$(f_n, x_n) \to (f, Df - y) \tag{2}$$

with respect to $\| \cdot \|_2$. Thus, $\mathfrak{A}_2 = (\mathfrak{A}, \| \cdot \|_2)$ is a commutative Banach algebra.

Let B denote the direct sum $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ with the norm

$$\|(a, b)\| = \max\{\|a\|_1, \|b\|_2\},$$

with the product

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2),$$

and the involution

$$(a, b)^* = (b^*, a^*).$$

Then B is a commutative Banach star algebra with a discontinuous involution ((1), Example 36.11).

Let $a = (z, 0) \in \mathfrak{A}$, where z denotes the coordinate functional in A, and let $\exp_i a$ denote the exponential of a in $\mathfrak{A}_i$ $(i = 1, 2)$. Precisely,

$$\exp_i a = \lim_{\|\cdot\|_i} \sum_{k=0}^{n} \frac{a^k}{k!} \quad \text{as } n \to \infty.$$

Then $\exp_1 a = (E, 0)$, but, by (2), $\exp_2 a = (E, x_0) \neq \exp_1 a$. Finally, let $b = (a, 0) \in B$. Then, noting that a is self-adjoint in $\mathfrak{A}$,

$$\exp(b^*) = \exp(0, a^*) = \exp(0, a) = (0, \exp_2 a),$$

whereas

$$(\exp b)^* = (\exp(a, 0))^* = (\exp_1 a, 0)^* = (0, \exp_1 a),$$

so that $\exp(b^*) \neq (\exp b)^*$, and we have constructed the required example.

Remark. By taking $h = (a, a) \in B$ we obtain an Hermitian element h in B for which $\exp h \neq (\exp h)^*$, so answering a question raised by B. E. Johnson in the review of (3), *Math. Reviews* **50** #2918.

We do have a positive result in a special case. We say that a commutative Banach algebra A *has a unique functional calculus* if, for each $a \in A$, there is a unique unital algebra homomorphism $\Theta_a : \mathcal{O}_{\sigma(a)} \to A$ such that $\Theta_a(z) = a$. Here $\sigma(a)$ denotes the spectrum of a in $\mathbb{C}$, and $\mathcal{O}_{\sigma(a)}$ denotes the algebra of analytic functions on $\sigma(a)$ (cf. (2)).

Proposition 2. *Let* $(A, *)$ *be a commutative Banach algebra with a unique functional calculus. Then* $\exp(a^*) = (\exp a)^*$ $(a \in A)$.

Proof. Let $a \in A$. First note that $\sigma(a^*) = \overline{\sigma(a)}$, so that, if $F \in \mathcal{O}_{\sigma(a)}$, then $F^* \in \mathcal{O}_{\sigma(a^*)}$, where $F^*(z) = \overline{F(\bar{z})}$. Define $\Theta_a^* : \mathcal{O}_{\sigma(a)} \to A$ by

$$\Theta_a^*(F) = [\Theta_{a^*}(F^*)]^* \quad (F \in \mathcal{O}_{\sigma(a)}).$$

Then Θ_a^* is clearly a unital algebra homomorphism, and

$$\Theta_a^*(z) = [\Theta_{a^*}(z)]^* = (a^*)^* = a,$$

so that, by the uniqueness of the functional calculus $\Theta_a^* = \Theta_a$. In particular, $(\exp(a^*))^* = \exp a$, as required.

Note that a Banach algebra with a finite-dimensional radical has a unique functional calculus (2). It is easy to give examples of such algebras which have a discontinuous involution.

REFERENCES

(1) F. F. Bonsall and J. Duncan, *Complete normed algebras* (Springer-Verlag, 1973).

(2) H. G. Dales, The uniqueness of the functional calculus, *Proc. London Math. Soc.* **27** (1973), 638-648.

(3) R. J. Loy, Commutative Banach algebras with non-unique complete norm topology, *Bull. Austral. Math. Soc.* **10** (1974), 409-420.

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