The Beilinson-Bernstein Localization Theorem for the affine Grassmannian

by

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Abstract

This thesis mainly deals with the study of the category $\widehat{\mathfrak{g}}_{crit}$-mod of admissible modules for the affine Kac-Moody algebra at the critical level $\widehat{\mathfrak{g}}_{crit}$ and the study of its center $Z(\widehat{\mathfrak{g}}_{crit}\text{-mod})$. The language used in this work is the one of *chiral algebras*, viewed as either D-modules over a smooth curve X or as a collection of sheaves over powers $X^{(n)}$ of the curve. In particular, we study the chiral algebra $\mathcal{A}_{crit}$ corresponding to the Kac-Moody algebra $\widehat{\mathfrak{g}}_{crit}$ and its center $\mathfrak{Z}_{crit}$. By considering the categories of $\mathcal{A}_{crit}$-modules and $\mathfrak{Z}_{crit}$-modules supported at some point x in X, we recover the categories $\widehat{\mathfrak{g}}_{crit}$-mod and $Z(\widehat{\mathfrak{g}}_{crit}\text{-mod})$-mod respectively. In this thesis we also study the chiral algebra $\mathcal{D}_{crit}$ of *critically twisted differential operators on the loop group* $G((t))$ and its relation to the category of D-modules over the affine Grassmannian $\mathrm{Gr}_{G,x} = G((t))/G[[t]]$.

In the first part of the thesis, we consider the chiral algebra $\mathcal{A}_{crit}$ and its center $\mathfrak{Z}_{crit}$. The commutative chiral algebra $\mathfrak{Z}_{crit}$ admits a canonical deformation into a non-commutative chiral algebra $\mathcal{W}_{\hbar}$. We will express the resulting first order deformation via the chiral algebra $\mathcal{D}_{crit}$ of chiral differential operators on $G((t))$ at the critical level.

In the second part of the thesis, we consider the Beilinson-Drinfeld Grassmannian Gr_G and the factorization category of D-modules on it. We try to describe this category in algebraic terms. For this, we first express this category as the factorization category $\mathcal{D}_{crit}\text{-mod}^{JG}$ of chiral $\mathcal{D}_{crit}$-modules which are equivariant with respect to the action of a certain factorization group JG. Then we express the factorization category of chiral $\mathfrak{Z}_{crit}$-modules as the category of modules over the factorization space $\mathrm{Op}^{\circ}_{\check{\mathfrak{g}}}$ of *opers on the punctured disc.* Using the *Drinfeld-Sokolov reduction* Ψ, we construct a chiral algebra $\mathcal{B}$ and a functor Γ_{Ψ} from the category of D-modules on Gr_G to the category of chiral $\mathcal{B}$-modules that are supported on a certain sub-scheme of $\mathrm{Op}^{\circ}_{\check{\mathfrak{g}}}$. We conjecture that this functor establishes an equivalence between these two categories.

Thesis Supervisor: Dennis Gaitsgory
Title: Professor of Mathematics

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Contents

Chapter 1

Introduction

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over an algebraically closed field k of characteristic 0, and let G be the corresponding adjoint algebraic group. Fix a smooth curve X over k, and a point $x \in X$. Let t be a coordinate near x. For any invariant bilinear form κ, denote by $\hat{\mathfrak{g}}_\kappa$ the Kac-Moody algebra given as the corresponding central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}((t))$ by $\mathbb{C}$. Denote by κ_{Kill} the Killing form on $\mathfrak{g}$ and let κ_{crit} be the *critical level* $\kappa_{crit} = -1/2\kappa_{Kill}$. For the critical level, denote by U'_{crit} the appropriately completed twisted enveloping algebra of $\widehat{\mathfrak{g}}_{crit}$. Consider the category $\widehat{\mathfrak{g}}_{crit}$-mod of continuous modules over U'_{crit}. These are the same as discrete admissible $\widehat{\mathfrak{g}}_{crit}$-modules on which $1 \in \mathbb{C}$ acts as the identity. At the critical level, unlike any other κ, the center of the category $\widehat{\mathfrak{g}}_{crit}$-mod is non-trivial. For instance, in the case of $\mathfrak{sl}_2$, it is generated by the *Sugawara operators.* We denote by $\hat{\mathfrak{z}}_{crit}$ the center

$$\hat{\mathfrak{z}}_{crit} := Z(\widehat{\mathfrak{g}}_{crit}\text{-mod})$$

This peculiarity makes the theory of $\widehat{\mathfrak{g}}_{crit}$-modules more interesting and more complicated.

Denote by $\check{\mathfrak{g}}$ the Langlands dual Lie algebra to $\mathfrak{g}$, and let X be a smooth curve over the complex numbers. Denote by $\mathrm{Op}_{\check{\mathfrak{g}},X}$ the space of $\check{\mathfrak{g}}$-opers on X, introduced in [BD2]. Roughly speaking, an oper is a triple $(\mathcal{F}_{\check{G}}, \mathcal{F}_{\check{B}}, \nabla)$, where $\mathcal{F}_{\check{G}}$ is a $\check{G}$-bundle on X, $\mathcal{F}_{\check{B}}$ is a reduction to a fixed Borel $\check{B} \subset \check{G}$ and ∇ is a connection on $\mathcal{F}_{\check{G}}$ satisfying certain properties. For every point $x \in X$, we denote by $\mathrm{Op}_{\check{\mathfrak{g}}}(D_x^\circ)$ the ind-scheme of opers on the punctured disc $D_x^\circ = \mathrm{Spec}(\mathbb{C}((t)))$, and by $\mathrm{Op}_{\check{\mathfrak{g}}}(D_x)$ the scheme of regular opers i.e., opers on the disc $D = \mathrm{Spec}(\mathbb{C}[[t]])$.

The interplay between the representation theory of $\widehat{\mathfrak{g}}_{crit}$ and the space $\mathrm{Op}_{\check{\mathfrak{g}},X}$ is given by a theorem of Feigin and Frenkel. In [FF], they prove the existence of an isomorphism of commutative topological algebras $\hat{\mathfrak{z}}_{crit} \simeq \mathrm{Fun}(\mathrm{Op}_{\check{\mathfrak{g}}}(D_x^\circ))$. In the work [FG2] of D. Gaitsgory and E. Frenkel, they define a closed sub-ind-scheme of $\mathrm{Op}_{\check{\mathfrak{g}}}(D_x^\circ)$, denoted by $\mathrm{Op}_{\check{\mathfrak{g}},x}^{unr}$, corresponding to *unramified opers*. This ind-scheme consists of those opers that are unramified as local systems. We denote by $\widehat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$ the subcategory of $\widehat{\mathfrak{g}}_{crit}$-mod on which the action of $Fun(\mathrm{Op}_{\check{\mathfrak{g}}}(D_x^\circ))$ factors through $Fun(\mathrm{Op}_{\check{\mathfrak{g}}}(D_x))$. Similarly we denote by $\widehat{\mathfrak{g}}_{crit}\text{-mod}_{unr}$ the subcategory consisting of modules on which the action of $Fun(\mathrm{Op}_{\check{\mathfrak{g}}}(D_x^\circ))$ factors through the quotient $Fun(\mathrm{Op}_{\check{\mathfrak{g}},x}^{unr})$.

Our basic tool in this paper is the theory of *chiral algebras* as introduced in [BD]. In fact, we will see in 2.1.1 how we can attach to the $\mathcal{D}_X$-algebra $\mathrm{Op}_{\check{\mathfrak{g}},X}$ a *commutative chiral algebra*, and more generally, how an affine $\mathcal{D}_X$-scheme corresponds to a commutative chiral algebra. This suggests that the theory of chiral algebras is a more suitable tool for the study of the above categories. In particular we will use the chiral algebra $\mathcal{A}_\kappa$ attached to $\hat{\mathfrak{g}}_\kappa$ as defined in [AG]. A chiral algebra on X, is a $\mathcal{D}_X$-module $\mathcal{A}$ equipped with a map $\{\,,\,\} : j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \to \Delta_!(\mathcal{A})$, called the *chiral product*, satisfying certain properties, where $\Delta : X \to X^2$ denotes the diagonal embedding and j the inclusion of the complement. We will denote by $[,]_{\mathcal{A}}$ the restriction of the chiral product to $\mathcal{A} \boxtimes \mathcal{A} \hookrightarrow j_*j^*(\mathcal{A} \boxtimes \mathcal{A})$. The rising interest in the theory of chiral algebras has a twofold motivation. The first is that it has numerous applications in the study of conformal field theory in two dimensions. The second is that, as explained in [BD], chiral algebras are the same as *factorization algebras* on $Ran(X)$, i.e., a sequence of quasi-coherent sheaves $\mathcal{A}^{(n)}$ for every power of the curve X^n, satisfying certain factorization properties. For instance, a co-unital affine factorization space on $Ran(X)$ is the same as a commutative chiral algebra on X. This description makes the understanding of factorization algebras, and of modules over them, easier.

1.1 W-algebras and chiral differential operators on the loop group

Consider the chiral algebra $\mathcal{A}_\kappa$ attached to $\hat{\mathfrak{g}}_\kappa$. For $\kappa = \kappa_{crit} = -\frac{1}{2}\kappa_{kill}$ denote by $\mathfrak{z}_{crit}$ the center of $\mathcal{A}_{crit} := \mathcal{A}_{\kappa_{crit}}$. This is a commutative chiral algebra with the property that the fiber $(\mathfrak{z}_{crit})_x$ over any point $x \in X$ is equal to the commutative algebra $\mathrm{End}_{\widehat{\mathfrak{g}}_{crit}}(\mathbb{V}^0_{\mathfrak{g},crit})$, where

$$\mathbb{V}^0_{\mathfrak{g},crit} := \mathrm{Ind}^{\widehat{\mathfrak{g}}_{crit}}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}\mathbb{C}.$$

The chiral algebra $\mathfrak{z}_{crit}$ is closely related to the center $\hat{\mathfrak{z}}_{crit}$ of the category $\widehat{\mathfrak{g}}_{crit}$-mod. In fact, for any chiral algebra $\mathcal{A}$ and any point $x \in X$, we can form an associative topological algebra $\hat{\mathcal{A}}_x$ with the property that its discrete continuous modules are the same as $\mathcal{A}$-modules supported at x (see [BD] 3.6.6). In this case the topological associative algebra corresponding to $\mathfrak{z}_{crit}$ is isomorphic to $\hat{\mathfrak{z}}_{crit}$.

As we mentioned before, the importance of choosing the level κ to be κ_{crit} relies on the fact that the center $\hat{\mathfrak{z}}_{crit}$ happens to be very big. Another crucial feature of the critical level is that $\hat{\mathfrak{z}}_{crit}$ carries a natural Poisson structure, obtained by considering the one parameter deformation of κ_{crit} given by $\kappa_{crit} + \hbar\kappa_{kill}$, as explained below in the language of chiral algebras. Moreover, according to [FF, F1, F2], the center $\hat{\mathfrak{z}}_{crit}$ is isomorphic, as Poisson algebra, to the *quantum Drinfeld-Sokolov reduction* of $U'(\widehat{\mathfrak{g}}_{crit})$ introduced in [FF] and [FKW]. In particular the above reduction provides a quantization of the Poisson algebra $\hat{\mathfrak{z}}_{crit}$ that will be central in this work.

Since the language we have chosen is the one of chiral algebras, we will now reformulate these properties for the algebra $\mathfrak{z}_{crit}$.

The commutative chiral algebra $\mathfrak{z}_{crit}$ can be equipped with a Poisson structure which can be described in either of the following two equivalent ways:

- For any $\hbar \neq 0$ let κ be any non critical level $\kappa = \kappa_{crit} + \hbar\kappa_{kill}$ and denote by $\mathcal{A}_\hbar$ the chiral algebra $\mathcal{A}_\kappa$. Let z and w be elements of $\mathfrak{z}_{crit}$. Let z_κ and w_κ be any two families of elements in $\mathcal{A}_\hbar$ such that $z = z_\kappa$ and $w = w_\kappa$ when $\hbar = 0$. Define the Poisson bracket of z and w to be

$$\{z, w\} = \frac{[z_\kappa, w_\kappa]_{\mathcal{A}_\hbar}}{\hbar} \pmod{\hbar}.$$

- The functor Ψ_X of semi-infinite cohomology introduced in [FF] (which is the analogous of the quantum Drinfeld-Sokolov reduction mentioned before and whose main properties will be recalled later), produces a 1-parameter family of chiral algebras $\{\mathcal{W}_\hbar\} := \{\Psi_X(\mathcal{A}_\hbar)\}$ such that $\mathcal{W}_0 \simeq \mathfrak{z}_{crit}$. Define the Poisson structure on $\mathfrak{z}_{crit}$ as

$$\{z, w\} = \frac{[\widetilde{z}_\hbar, \widetilde{w}_\hbar]_{\mathcal{W}_\hbar}}{\hbar} \pmod{\hbar}$$

  where $z = \widetilde{z}_\hbar|_{\hbar=0}$ and $w = \widetilde{w}_\hbar|_{\hbar=0}$.

Although the above two expressions look the same, we'd like to stress the fact that, unlike the second construction, in the first we are not given any deformation of $\mathfrak{z}_{crit}$. In other words the elements z_κ and w_κ do not belong to the center of $\mathcal{A}_\kappa$ (that in

fact is trivial). It is worth noticing that the associative topological algebras $\hat{\mathcal{W}}_h$ associated to them (usually denoted by $W_\hbar$) are the well known *W-algebras.*
As in the case of usual algebras, the Poisson structure on $\mathfrak{z}_{crit}$ gives the sheaf of Kähler differentials $\Omega^1(\mathfrak{z}_{crit})$ a structure of Lie* algebroid. A remarkable feature, when dealing with chiral algebras, is that the existence of a quantization $\{\mathcal{W}_\hbar\}$ of $\mathfrak{z}_{crit}$ allows us to construct what is called a *chiral extension* $\Omega^c(\mathfrak{z}_{crit})$ of the Lie* algebroid $\Omega^1(\mathfrak{z}_{crit})$, and moreover, as it is explained in [BD] 3.9.11, this establishes an equivalence of categories between 1-st order quantizations of $\mathfrak{z}_{crit}$ and chiral extensions of $\Omega^1(\mathfrak{z}_{crit})$. This equivalence is the point of departure for this work.

1.1.1 Main Theorem

In [BD], the highly non-trivial notion of *chiral extension of Lie*-algebroid* is introduced. Chiral extensions form a gerbe over a certain Picard category; in particular, such extensions may not even exist. Given a chiral extension $\mathfrak{L}^c$ of a Lie*-algebroid $\mathfrak{L}$, we can form its chiral envelope $U(\mathfrak{L}^c)^{ch}$. For example, for a $\mathcal{D}_X$-space Y, and a chiral extension of the Lie*-algebroid Θ_Y of vector fields on Y, its chiral envelope is a chiral algebra of twisted chiral differential operators on Y.

This project consists of comparing two, a priori different, chiral extensions of the Lie*-algebroid $\Omega^1(\mathfrak{z}_{crit})$. The first extension is defined using quantum W-algebras, as explained before. Namely we consider the 1-paramenter family of chiral algebras $\{\mathcal{W}_\hbar\} := \{\Psi_X(\mathcal{A}_\hbar)\}$. As is shown in [FF], the cohomology of $\Psi_X(\mathcal{A}_{crit}) = \Psi_X(\mathcal{A}_0)$ is concentrated in degree zero, and its 0-th cohomology is isomorphic to the center $\mathfrak{z}_{crit}$. Therefore the chiral algebras $\{\mathcal{W}_\hbar\}$ provide a 1-parameter family deformation of $\mathfrak{z}_{crit}$, giving rise to the same Poisson structure as introduced above. According to [BD], such a quantization gives rise to a chiral extension $\Omega^c(\mathfrak{z}_{crit})$ of $\Omega^1(\mathfrak{z}_{crit})$. The second extension is given via its chiral envelope.

We start with the chiral algebra $\mathcal{D}_{crit}$ of critically-twisted differential operators on the loop group $G((t))$, introduced in [AG]. The algebra $\mathcal{D}_{crit}$ admits two embeddings of $\mathcal{A}_{crit}$, corresponding to right and left invariant vector fields on $G((t))$. The two embedding l and r of $\mathcal{A}_{crit}$ into $\mathcal{D}_{crit}$ endow the fiber $(\mathcal{D}_{crit})_x$ with a structure of $\widehat{\mathfrak{g}}_{crit}$-bimodule. The fiber can therefore be decomposed according to these actions as explained below.

Consider the topological commutative algebra $\hat{\mathfrak{Z}}_{crit}$. For a dominant weight λ, let V^λ be the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ and let $\mathbb{V}^\lambda_{\mathfrak{g},crit}$ be the $\widehat{\mathfrak{g}}_{crit}$-module given by

$$\mathbb{V}^\lambda_{\mathfrak{g},crit} := U(\widehat{\mathfrak{g}}_{crit}) \underset{U(\mathfrak{g}[[t]]\oplus\mathbb{C})}{\otimes} V^\lambda.$$

The action of the center $\hat{\mathfrak{Z}}_{crit}$ on $\mathbb{V}^\lambda_{\mathfrak{g},crit}$ factors as follows

$$\hat{\mathfrak{Z}}_{crit} \twoheadrightarrow \mathfrak{z}^\lambda_{crit} := \mathrm{End}(\mathbb{V}^\lambda_{\mathfrak{g},crit}).$$

Denote by I^λ the kernel of the above map, and consider the formal neighborhood of $\mathrm{Spec}(\mathfrak{z}^\lambda_{crit})$ inside $\mathrm{Spec}(\hat{\mathfrak{Z}}_{crit})$. Let $\widehat{\mathfrak{g}}_{crit}\text{-mod}^{G[[t]]}$ be the full subcategory of $\widehat{\mathfrak{g}}_{crit}$-modules such that the action of $\mathfrak{g}[[t]]$ can be integrated to an action of $G[[t]]$. We have the following Lemma.

Lemma 1.1.1. *Any module M in $\widehat{\mathfrak{g}}_{crit}$-mod$^{G[[t]]}$ can be decomposed into a direct sum of submodules M_λ such that each M_λ admits a filtration whose subquotients are annihilated by I^λ.*

As a bimodule over $\widehat{\mathfrak{g}}_{crit}$ the fiber at any point $x \in X$ of $\mathcal{D}_{crit}$ is $G[[t]]$ integrable with respect to both actions, hence we have two direct sum decompositions of $(\mathcal{D}_{crit})_x$ corresponding to the left and right action of $\widehat{\mathfrak{g}}_{crit}$, as explained in [FG2]:

$$(\mathcal{D}_{crit})_x = \bigoplus_{\lambda\, dominant} (\mathcal{D}_{crit})^\lambda_x,$$

where $(\mathcal{D}_{crit})_x^\lambda$ is the direct summand supported on the formal completion of Spec$(\mathfrak{z}_{crit}^\lambda)$. Denote by $\mathcal{D}_{crit}^0$ the $\mathcal{D}_X$-module corresponding to $(\mathcal{D}_{crit})_x^0$. It is easy to see that $\mathcal{D}_{crit}^0$ is in fact a chiral algebra.

Since the fiber of $\mathcal{A}_{crit}$ at x is isomorphic to $\mathbb{V}_{\mathfrak{g},crit}^0$, the embeddings l and r must land in the chiral algebra $\mathcal{D}_{crit}^0$. Hence we have

$$\mathcal{A}_{crit} \xrightarrow{l,r} \mathcal{D}_{crit}^0 \hookrightarrow \mathcal{D}_{crit}.$$

The above two embeddings give $\mathcal{D}_{crit}^0$ a structure of $\mathcal{A}_{crit}$-bimodule, hence it makes sense to apply the functor of semi-infinite cohomology Ψ_X to it with respect to both actions. Let us denote by $\mathcal{C}_{crit}^0$ the resulting chiral algebra

$$\mathcal{C}_{crit}^0 := (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_{crit}^0).$$

The main result of this work is the following.

Theorem 1.1. *The chiral envelope $U(\Omega^c(\mathfrak{Z}_{crit}))$ of the extension*

$$0 \to \mathfrak{Z}_{crit} \to \Omega^c(\mathfrak{Z}_{crit}) \to \Omega(\mathfrak{Z}_{crit}) \to 0,$$

given by the quantization $\{\mathcal{W}_\hbar := \Psi_X(\mathcal{A}_\hbar)\}$ of the center $\mathfrak{Z}_{crit}$, is isomorphic to the chiral algebra $\mathcal{B}^0$.

1.2 D-modules over the affine Grassmannian

1.2.1 The Beilinson-Bernestein Localization Theorem

Recall the theorem of A. Beilinson and J. Bernstein, that realizes $\mathcal{D}$-modules on the flag variety G/B as modules over the associative algebra given as the quotient of $U(\mathfrak{g})$ by the maximal ideal of the center defined by its action on the trivial $\mathfrak{g}$-

module. The main body of this second project tries to develop an analogue of the above theorem in the affine case. More precisely, denote by $\mathrm{Gr}_{G,x}$ the affine Grassmannian

$$\mathrm{Gr}_{G,x} := G((t))/G[[t]]$$

This is an ind-scheme of finite type classifying G-bundles on X with a given trivialization on $X - x$. On the affine Grassmannian we can define the category D-mod$(\mathrm{Gr}_{G,x})$ of D-modules, and, as in the finite dimensional case, we are interested in describing it in different terms. In particular, we would like to have an *algebraic* description of it, where, by algebraic, we mean a description of it as modules over some associative algebra. However, it turns out that the category that can be realized as such is a *critically*-twisted version of D-mod$(\mathrm{Gr}_{G,x})$. The reason being that this new category is related to the category $\widehat{\mathfrak{g}}_{crit}$-mod and therefore to the topological algebra $Fun(\mathrm{Op}_{\check{\mathfrak{g}}}(D_x^\circ))$. This category, denoted by D_{crit}-mod$(\mathrm{Gr}_{G,x})$ is constructed in the following way. As it is explained in [BD2], there exist a canonical line bundle

$$\mathcal{L}_{crit,x} \to \mathrm{Gr}_{G,x}$$

on $\mathrm{Gr}_{G,x}$. Critically twisted D-modules on $\mathrm{Gr}_{G,x}$ are just $\mathcal{O}$-modules on $\mathrm{Gr}_{G,x}$ with an action of a particular sheaf $\mathcal{D}_{\mathcal{L}_{crit,x}}$ attached to $\mathcal{L}_{crit,x}$. The functor $\mathcal{M} \to \mathcal{M} \otimes \mathcal{L}_{crit,x}$ defines an equivalence of categories

$$D\text{-mod}(\mathrm{Gr}_{G,x}) \xrightarrow{\sim} D_{crit}\text{-mod}(\mathrm{Gr}_{G,x}), \tag{1.1}$$

therefore describing the RHS as modules over some associative algebra, would also describe the category of D-modules on $\mathrm{Gr}_{G,x}$ as such. We can start by considering the functor Γ of global sections on $\mathrm{Gr}_{G,x}$ as a functor from the category D_{crit}-mod$(\mathrm{Gr}_{G,x})$ to the category of vector spaces $Vect$. It can be shown that the action of $\mathcal{D}_{\mathcal{L}_{crit,x}}$ on a module $\mathcal{M}$, gives a $\widehat{\mathfrak{g}}_{crit}$-module structure on the vector space

$\Gamma(\mathrm{Gr}_{G,x}, \mathcal{M})$. We therefore have a functor

$$\Gamma : D_{crit}\text{-mod}(\mathrm{Gr}_{G,x}) \to \widehat{\mathfrak{g}}_{crit}\text{-mod}.$$

However, unlike the finite dimensional case, this functor does not establish an equivalence of categories. It is not hard to see that Γ factors through the sub-category $\widehat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$. However, the resulting functor $\Gamma : D_{crit}\text{-mod}(\mathrm{Gr}_{G,x}) \to \widehat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$ is not an equivalence. Instead, the following conjecture was proposed in [FG3]:

- The action of the groupoid $\mathrm{Isom}_{\mathrm{Op}_{\mathfrak{g}}(D_x)} = \mathrm{Op}_{\check{\mathfrak{g}}}(D_x) \underset{\cdot/G}{\times} \mathrm{Op}_{\check{\mathfrak{g}}}(D_x)$ on $\mathfrak{z}_{crit}$ lifts to an action on $\widehat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$ compatible with the action of $G((t))$.

- The functor Γ establishes an equivalence between $D_{crit}\text{-mod}(\mathrm{Gr}_{G,x})$ and the category $(\widehat{\mathfrak{g}}_{crit}\text{-mod}_{reg})^{\mathrm{Isom}_{\mathrm{Op}_{\mathfrak{g}}(D_x)}}$ of $\mathrm{Isom}_{\mathrm{Op}_{\mathfrak{g}}(D_x)}$-equivariant objects in $\widehat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$.

The above conjecture shows that Γ, viewed as a forgetful functor, does not realize $D_{crit}\text{-mod}(\mathrm{Gr}_{G,x})$ as $\mathcal{B}$-mod, for some associative algebra $\mathcal{B}$.
In understanding how to describe the category $D_{crit}\text{-mod}(\mathrm{Gr}_{G,x})$, the questions that arise are the following.

- Is there a different way of describing D-modules on $\mathrm{Gr}_{G,x}$?
- What do we mean by algebraic description?

An answer to the above questions is given by the notion of *modules over a chiral algebra*. In fact critically-twisted D-modules on the affine Grassmannian can be described as chiral modules for the chiral algebra $\mathcal{D}_{crit}$ satisfying certain properties. In the second project, we will define a chiral algebra $\mathcal{B}$ and a functor

$$\Gamma_{\Psi,x} : D_{crit}\text{-mod}(\mathrm{Gr}_{G,x}) \to \mathcal{B}\text{-mod}_x,$$

which *will* realize the LHS as modules supported at $x \in X$ for the chiral algebra $\mathcal{B}$. We believe that the functor $\Gamma_{\Psi,x}$ defines an equivalence between the RHS and a certain subcategory of $\mathcal{B}\text{-mod}_x$ defined using the action of $\hat{\mathfrak{z}}_{crit}$ on $\widehat{\mathfrak{g}}_{crit}\text{-mod}$. However, we were only able to show the promised equivalence assuming a conjecture concerning the functor Ψ_X.

1.2.2 Construction of the functor

Consider the chiral algebra $\mathcal{A}_{crit}$. For this chiral algebra we have an equivalence between the category $\mathcal{A}_{crit}\text{-mod}_x$ of $\mathcal{A}_{crit}$-modules supported at x, and the category $\widehat{\mathfrak{g}}_{crit}\text{-mod}$ introduced earlier. Let Ψ_X denote the functor of semi-infinite cohomology, which from now on will be simply called the *Quantum Drinfeld-Sokolov* reduction.

$$\Psi_X : \{\text{ Chiral } \mathcal{A}_{crit}\text{-modules}\} \to \{\text{Chiral } \mathfrak{z}_{crit}\text{-modules}\}.$$

By the theorem of Feigen and Frenkel we have an equivalence

$$\mathfrak{z}_{crit}\text{-mod}_x \simeq \mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}}}(D_x^\circ)) := \left\{\text{discrete continuous } Fun(\mathrm{Op}_{\check{\mathfrak{g}}}(D_x^\circ))\text{-modules}\right\}.$$

If we restrict the functor Ψ_X to the category $\mathcal{A}_{crit}\text{-mod}_x$ of $\mathcal{A}_{crit}$-modules supported at x, we therefore have a functor

$$\Psi_x : \mathcal{A}_{crit}\text{-mod}_x \to \mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}}}(D_x^\circ)).$$

Consider now the chiral algebra $\mathcal{D}_{crit}$ of chiral differential operators on the loop group $G((t))$. Recall the two embeddings

$$\mathcal{A}_{crit} \xrightarrow{l} \mathcal{D}_{crit} \xleftarrow{r} \mathcal{A}_{crit},$$

corresponding to left and right invariant vector fields on $G((t))$. Chiral $\mathcal{D}_{crit}$-modules supported at x should be thought as D-modules on $G((t))$. In particular, if we denote by π the projection

$$\pi : G((t)) \twoheadrightarrow \mathrm{Gr}_{G,x},$$

given a module $\mathcal{M} \in D_{crit}\text{-mod}(\mathrm{Gr}_{G,x})$, we can regard $\Gamma(G((t)), \pi^*(\mathcal{M}))$ as an object in $\mathcal{D}_{crit}\text{-mod}_x$. We define $\Gamma_{\Psi,x}$ to be

$$\begin{array}{rccc} \Gamma_{\Psi,x} : & D_{crit}\text{-mod}(\mathrm{Gr}_{G,x}) & \longrightarrow & \widehat{\mathfrak{g}}_{crit}\text{-mod} \\ & \mathcal{M} & \mapsto & (Id \boxtimes \Psi_X)(\Gamma(G((t)), \pi^*(\mathcal{M}))). \end{array}$$

Denote by $\mathcal{B}$ the chiral algebra

$$\mathcal{B} := (id \boxtimes \Psi_X)(\mathcal{D}_{crit}).$$

By construction, the action of $\widehat{\mathfrak{g}}_{crit}$ on $\Gamma_{\Psi,x}(\mathcal{M})$ can be lifted to an action of $\mathcal{B}$. Recall now the ind sub-scheme $\mathrm{Op}^{unr}_{\check{\mathfrak{g}},x}$ of $\mathrm{Op}_{\check{\mathfrak{g}}}(D^\circ_x)$, and denote by $\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},x})$ the category of continuous discrete $Fun(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},x})$-modules. We have the following.

Conjecture 1.2.1. *The functor $\Gamma_{\Psi,x}$ establishes an equivalence of categories*

$$D\text{-}mod(Gr_{G,x}) \underset{\cdot\otimes\mathcal{L}_{crit,x}}{\simeq} D_{crit}\text{-}mod(Gr_{G,x}) \underset{\Gamma_{\Psi,x}}{\simeq} \mathcal{B}\text{-}mod_{unr,x}$$

where $\mathcal{B}$-mod_{unr} denotes the category of $\mathcal{B}$-modules supported at $x \in X$, which are supported on $Op^{unr}_{\mathfrak{g},x}$ when regarded as objects in $QCoh^!(Op_{\check{\mathfrak{g}}}(D^\circ_x))$.

1.2.3 The factorization picture

In trying to prove conjecture 1.2.1 we immediately realized that we needed an understanding the categories involved as the point x moves. More generally, for n distinct points $x_1, \ldots, x_n$ on X, we need to understand the categories $D_{crit}\text{-mod}(\mathrm{Gr}_{G,x_1}) \otimes$

$\cdots \otimes D_{crit}$-mod(Gr_{G,x_n}) and $\mathcal{B}$-mod$_{unr,x_1} \otimes \cdots \otimes \mathcal{B}$-mod$_{unr,x_n}$, as the n points move, hence, in particular, as they collide.

In the second project we will state a conjectural equivalence of categories over any power X^n of the curve. In particular it will imply the equivalence of conjecture 1.2.1, by "taking the fiber at $x \in X$". We will then explain how the conjecture 1.2.1 would follow from a conjecture concerning the functor Ψ_X in its factorization version, as explained below. The formulation of this conjecture uses the description of chiral algebras in term of factorization algebras together with the notion of factorization spaces.

An important example of factorization space is given by the *Beilinson-Drinfeld Grassmannian* Gr_G on $Ran(X)$. This is given by the assignment

$$I \to \mathrm{Gr}_{G,I},$$

where I is a finite set, and $\mathrm{Gr}_{G,I}$ is the space over X^I given in the following way. For an affine scheme S, an S point of $\mathrm{Gr}_{G,I}$ consists of a map $S \xrightarrow{\phi} X^I$, a G-bundle P_G on $X_S := S \times X$ and a trivialization of P_G on $X_S - \cup_{i \in I} \Gamma_{\phi_i}$, where Γ_{ϕ_i} denotes the graph of the i-th component of ϕ in $S \times X$. In particular, for $I = \{1, \ldots, n\}$, the fiber of Gr_{G,X^I} at any $(x_1, \ldots, x_n)$, with $x_i \neq x_j$, is the product of the corresponding affine Grassmannians over each x_i. This property of the Beilinson-Drinfeld Grassmannian is, indeed, one of the data in the definition of a factorization space.
On each $\mathrm{Gr}_{G,I}$ there is a well defined notion of D-modules on it, and a well defined notion of critically-twisted D-modules. We denoted this category by D_{crit}-mod$(\mathrm{Gr}_{G,I})$. The latter is given using a line bundle $\mathcal{L}_{crit}$ on Gr_G, i.e. a collection

$$I \to \mathcal{L}_{crit,I}$$

of line bundles over $\mathrm{Gr}_{G,I}$. We will be interested in the *factorization category* $D_{crit}\text{-mod}(\mathrm{Gr}_G)$ given by the assignment

$$I \to D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}).$$

As it is explained in [NR], given a chiral algebra $\mathcal{A}$, we can define the category $\mathcal{A}\text{-mod}_I$ of chiral $\mathcal{A}$-modules over X^I. The assignment $I \to \mathcal{A}\text{-mod}_I$ defines a factorization category, simply denoted by $\mathcal{A}$-mod. When $\mathcal{A}$ is commutative, in 3.1.1, we will explain how to describe the category $\mathcal{A}\text{-mod}_I$ as the category of modules over a space $M_I\mathfrak{Y}_{\mathcal{A}}$ over X^I,

$$M_I\mathfrak{Y}_{\mathcal{A}} \to X^I$$

canonically attached to $\mathcal{A}$.

Consider the commutative chiral algebra $\mathcal{A} = \mathfrak{z}_{crit}$. We denote by $\mathrm{Op}^{\circ}_{\check{\mathfrak{g}}}$ the factorization space $\mathrm{Op}^{\circ}_{\check{\mathfrak{g}}} = \{I \to \mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I} := M_I\mathfrak{Y}_{\mathfrak{z}_{crit}}\}$ that should be thought as the factorization version of opers on the punctured disc. For each I, the algebra of function on $\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I}$ has a structure of topological algebra over X^I and we have an equivalence

$$\mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I}) := \{\text{discrete continuous } Fun(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I})\text{-modules }\} \simeq \mathfrak{z}_{crit}\text{-mod}_I.$$

We denote by $\mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}}})$ the factorization category given by the assignment

$$I \to \mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I}).$$

For every I, we can define a certain sub-functor $\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I} \subset \mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I}$ corresponding to the space of unramified opers from the previous section. It can be shown that this sub-functor is represented by an affine-ind-scheme. This gives rise to a topology on the algebra of functions $Fun(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})$. As before, we will denote by $\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})$ the category of discrete continuous $Fun(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})$-modules. By adapting the construction of Ψ_X to the factorization picture, we define functors

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I \to \mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I}),$$

and use them to construct functors $\Gamma_{\Psi,I} : D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}) \to \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I}))$. Using the above functors, we arrive at the formulation of the $Ran(X)$-version of conjecture 1.2.1.

Conjecture 1.2.2. *The collection $\{I \to \Gamma_{\Psi,I}\}$ together with the equivalence 1.1 give rise to an equivalence of factorization categories*

$$D\text{-}mod(Gr_G) \underset{\Gamma_{\Psi}\circ(\cdot\otimes\mathcal{L}_{crit})}{\xrightarrow{\sim}} \mathcal{B}\text{-}mod(QCoh^!(Op^{unr}_{\check{\mathfrak{g}}})), \tag{1.2}$$

where $\mathcal{B}$-mod$(QCoh^!(Op^{unr}_{\check{\mathfrak{g}}}))$ denotes the factorization category $\{I \to \mathcal{B}\text{-}mod(QCoh^!(Op^{unr}_{\check{\mathfrak{g}},I}))\}$ of $\mathcal{B}$-modules on X^I which are supported on $Op^{unr}_{\mathfrak{g},I}$ when regarded as modules over $Op^{\circ}_{\check{\mathfrak{g}},I}$.

1.2.4 The main conjecture

As we mentioned before, the above conjecture, formally follows from a conjecture concerning the functors Ψ_I. More precisely, consider the group $\mathcal{D}_X$-scheme $J_X(G)$ as defined in 3.1.7. It acts on the category $\mathcal{A}_{crit}$-mod$_X$ of chiral $\mathcal{A}_{crit}$-modules on X. We can therefore consider the sub-category $\mathcal{A}_{crit}\text{-mod}_X^{J_X(G)}$ of strongly $J_X(G)$-equivariant objects in $\mathcal{A}_{crit}$-mod$_X$. For instance, if we consider $\mathcal{A}_{crit}$-modules supported at x, then the category $\mathcal{A}_{crit}\text{-mod}_x^{J_X(G)}$ is the category $\widehat{\mathfrak{g}}_{crit}\text{-mod}^{G[[t]]}$ consisting of $\widehat{\mathfrak{g}}_{crit}$-modules on which the action of $\mathfrak{g}[[t]]$ can be integrated to an action of $G[[t]]$. Consider the functor Ψ_X,

$$\Psi_X : \mathcal{A}_{crit}\text{-mod}_X \to \mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},X}).$$

This functor has being studied by D. Gaitsgory and E. Frenkel. In particular, in [FG2] they show that, when restricted to the sub-category $\mathcal{A}_{crit}\text{-mod}_X^{J_X(G)}$ it defines an equivalence

$$\mathcal{A}_{crit}\text{-mod}_X^{J_X(G)} \underset{\Psi_X}{\xrightarrow{\sim}} \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},X}). \tag{1.3}$$

Similarly to the above, as it is explained in 3.1.7 and 3.3.3, there exists a factorization group $JG = \{I \to JG_I\}$ acting on the factorization category $\mathcal{A}_{crit}$-mod. We can consider the sub-category $\mathcal{A}_{crit}\text{-mod}_I^{JG}$ of $\mathcal{A}_{crit}$-modules on X^I, which are strongly JG-equivariant, and consider the restriction of Ψ_I to this category,

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I^{JG} \to \mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I}).$$

The main conjecture is the following.

Conjecture 1.2.3. *The collection $\Psi := \{I \to \Psi_I\}$ defines an equivalence of factorization categories*

$$\mathcal{A}_{crit}\text{-}mod^{JG} \underset{\Psi}{\xrightarrow{\sim}} QCoh^!(Op^{unr}_{\check{\mathfrak{g}}}).$$

1.2.5 How conjecture 1.2.3 implies conjecture 1.2.2

We will briefly explain how the equivalence in 1.2.3 would imply the equivalence

$$D\text{-mod}(\mathrm{Gr}_G) \underset{\Gamma_\Psi \circ (\cdot \otimes \mathcal{L}_{crit})}{\xrightarrow{\sim}} \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}}})).$$

As it is explained in 3.1.3, given a factorization category $\mathcal{C}$ we can define (chiral)-algebra objects in $\mathcal{C}$. Moreover, as it is explained in 3.1.8, for an algebra object $\mathcal{A}$ in a factorization category $\mathcal{C}$, we can define a factorization category $\mathcal{A}$-mod($\mathcal{C}$) of $\mathcal{A}$-modules in $\mathcal{C}$. Consider now the factorization algebra

$$\mathcal{A}_{crit}\text{-mod}^{JG} = \left\{I \to \mathcal{A}_{crit}\text{-mod}_I^{JG}\right\}.$$

Using the right embedding $\mathcal{A}_{crit} \xrightarrow{r} \mathcal{D}_{crit}$, we consider $\mathcal{D}_{crit}$ as an algebra object in $\mathcal{A}_{crit}\text{-mod}^{JG}$. We can therefore consider the factorization category $\mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}^{JG})$. Similarly, we can consider the chiral algebra $\mathcal{B} = \Psi_X(\mathcal{D}_{crit})$ as an algebra object in the category $\mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}}}^{unr})$. Conjecture 1.2.3 implies that we have an equivalence of factorization categories

$$\mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}^{JG}) \simeq \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}}}^{unr})). \tag{1.4}$$

In proposition 3.3.3 and in theorem 3.1 we will shown the following two facts.

- We have an equivalence $\mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}^{JG}) \simeq \mathcal{D}_{crit}\text{-mod}^{JG}$.

- There exist an equivalence of factorization categories

$$D_{crit}\text{-mod}(\mathrm{Gr}_G) \xrightarrow{\sim} \mathcal{D}_{crit}\text{-mod}^{JG}.$$

Therefore the equivalence (1.4) can be written as

$$D_{crit}\text{-mod}(\mathrm{Gr}_G) \xrightarrow{\sim} \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}}}^{unr})),$$

which immediately implies the equivalence stated in 1.2.2 after tensoring with the factorization line bundle $\mathcal{L}_{crit}$ as explained before.

1.3 Organization of the thesis

- In Chapter 2 we start by recalling the definition of the basic objects that will be used in this thesis. In particular, we will recall the *classical* definition of chiral algebras. We will focus on commutative chiral algebras and on Lie*-algebroids acting on them. For a commutative chiral algebra $\mathcal{R}$ we will be interested in studying *Poisson* structures on it. In particular, given a Poisson structure on $\mathcal{R}$, in 2.2 we will define the notion of *quantizations modulo* $\hbar^2$ of it. This will be used to state the equivalence between such quantizations of $\mathcal{R}$ and *chiral extensions* of the Lie*-algebroid $\Omega^1(\mathcal{R})$ presented in [BD]. More precisely, If we denote by $\mathcal{Q}^{ch}(\mathcal{R})$ the groupoid of $\mathbb{C}[\hbar]/\hbar^2$-deformations of a chiral-Poisson algebra $\mathcal{R}$, and by $\mathcal{P}^{ch}(\Omega^1(\mathcal{R}))$ the groupoid of chiral extensions of $\Omega^1(\mathcal{R})$, there is a functor

$$\mathcal{P}^{ch}(\Omega^1(\mathcal{R})) \to \mathcal{Q}^{ch}(\mathcal{R}). \tag{1.5}$$

In [BD] 3.9.10. they show that the above functor is an equivalence. In section 2.3 we will consider the chiral algebra $\mathcal{A}_{crit}$ and its center $\mathfrak{Z}_{crit}$. We will define a Poisson structure on it, and a quantization $\{\mathcal{W}_\hbar\}$ of this Poisson structure. The quantization will be constructed using the Drinfeld-Sokolov reduction Ψ_X, that will be introduced at the end of 2.3.1, as a special case of the BRST reduction, that will be studied in 2.3.1. We will then consider the chiral extension $\Omega^c(\mathfrak{Z}_{crit})$ of $\Omega^1(\mathfrak{Z}_{crit})$ given by the above equivalence. The main theorem of this chapter will describe $\Omega^c(\mathfrak{Z}_{crit})$ in terms of the chiral algebra $\mathcal{D}_{crit}$ of critically twisted differential operators on the loop group, whose definition will be recalled in 2.4.1. More precisely, we will show that the chiral envelope of $\Omega^c(\mathfrak{Z}_{crit})$ coincides with the chiral algebra $\mathcal{C}^0_{crit} = (\Psi_X \boxtimes \Psi_X)(\mathcal{D}^0_{crit})$. The proof of this theorem will rely on the explicit construction of the inverse to the functor in (1.5) that will be given in 2.2.2. In fact, since the proof of the equivalence 1.5 presented in [BD] does not provide such inverse, a large part of this chapter will be taken by this construction. The last section will be devoted to the proof of theorem 2.3. In section 2.4.2 we will give an alternative formulation of the Theorem that consists in finding a map F from $\Omega^c(\mathfrak{Z}_{crit})$ to $\mathcal{C}^0_{crit}$ with some particular properties. In section 2.4.3 we will finally define the map F and conclude the proof of the Theorem.

- Chapter 3 is divided into 7 sections. The main conjecture 1.2.3 will only appear in section 3.6, the reason being that its formulation needs some foundational preliminary notions that will be given in the previous sections.

  In section 3.1 we introduce the right categorical setting in which we will be working. In particular, since conjecture 1.2.2 states an equivalence between two factorization categories, we will define the notion of *abelian category over Ran(X)* and the notion of factorization category. We will then define the notion of *factorization algebra* in a factorization category $\mathcal{C}$ and relate this notion to the notion of chiral algebra. In 3.1.5 we will relate the notion of commutative chiral algebras to the notion of *factorization spaces.* We will then address our attention to the factorization category of modules over a chiral algebra $\mathcal{A}$. In 3.1.8 we will see how this notion plays out in the case of a commutative chiral algebra.

  In 3.2 we recall the definition of action of a group G on a category, and, in 3.3 we define the notion of *action of a $\mathcal{D}_X$-group-scheme $\mathcal{G}$ on a factorization category.* In particular, in 3.3.1, we will study the action of the group $\mathcal{D}_X$-scheme $\mathcal{G}$ of on the factorization category $\mathcal{A}$-mod of chiral $\mathcal{A}$-modules, as defined in 3.2. We will be interested in the category $\mathcal{A}\text{-mod}^{\mathcal{G}}$ of *srtongly $\mathcal{G}$-equivariant* objects in $\mathcal{A}$-mod. In 3.3.3, we will apply this to the group $\mathcal{D}_X$-scheme $J_X(G)$ of *jets into* G, defined in 3.1.7, acting on the factorization category $\mathcal{D}_{crit}$-mod of $\mathcal{D}_{crit}$-modules.

  In section 3.4 we recall the definition of the *Beilinson-Drinfeld Grassmannian* Gr_G. This will be defined as a factorization space, i.e. we will have a space $\mathrm{Gr}_{G,I}$ over X^I for every finite set I. We will then explain, in 3.23, how to define the category D-mod$(\mathrm{Gr}_{G,I})$ of D-modules on each $\mathrm{Gr}_{G,I}$. In 3.24, we will then construct a line bundle $\mathcal{L}_{crit,I}$ over $\mathrm{Gr}_{G,I}$ and define the category D_{crit}-mod$(\mathrm{Gr}_{G,I})$ of *critically-twisted-D-modules* on $\mathrm{Gr}_{G,I}$. In 3.4.4 we will show how we can describe the category D_{crit}-mod$(\mathrm{Gr}_{G,I})$ in terms of the category $\mathcal{D}_{crit}$-mod$_I$ of $\mathcal{D}_{crit}$-modules on X^I. In fact, we will show that the former category is equivalent to the category of strongly $J_X(G)$-equivariant objects in $\mathcal{D}_{crit}$-mod$_I$. We will then move to the definition of the factorization space $\mathrm{Op}_{\check{\mathfrak{g}}}$ corresponding to the chiral algebra $\mathfrak{Z}_{crit}$. More precisely, in 3.5 we will recall the definition of opers on the punctured disc $\mathrm{Op}_{\mathfrak{g}}(D_x^\circ)$ as given in [BD2] and construct the factorization space $\mathrm{Op}_{\mathfrak{g}}^\circ$ corresponding to it. We will then define the co-unital factorization space $\mathrm{Op}_{\mathfrak{g}}$ of *regular oper*, and the factorization space $\mathrm{Op}_{\mathfrak{g}}^{unr}$, corresponding to *opers on the disc*, and to *unramified opers* as defined in [FG2].

In 3.6 we will finally state the main conjecture from which we will derive conjecture 1.2.2. We will explain in details how to construct the factorization functor $\{I \to \Psi_I\}$, where

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I \to \mathfrak{Z}_{crit}\text{-mod}_I = \mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}},I}^\circ),$$

denotes the *Drinfeld-Sokolov* reduction for modules over X^I that will be explained in 3.6.1. We will finally recall the equivalence (1.3) and state the conjecture 1.2.3.

The last section combines together all the results from the previous ones to finally come to the proof of conjecture 1.2.2. We will in fact use the Drinfeld-Sokolov reduction on X to define the chiral algebra $\mathcal{B} = \Psi_X(\mathcal{D}_{crit})$, then we will use results from section 3.6 and 3.4 to first define a functor from $\mathcal{D}_{crit}$-mod to the category $\mathcal{B}$-mod$(\mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}}}^\circ))$ of $\mathcal{B}$-modules in $\mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}}}^\circ)$, as defined in 3.1.8. The equivalence showed in 3.1 between strongly-equivariant $J_X(G)$-

objects in $\mathcal{D}_{crit}$-mod and the category D_{crit}-mod$(\mathrm{Gr}_{G,x})$ will yield the factorization functor $\{I \to \Gamma_{\Psi,I}\}$, where $\Gamma_{\Psi,I}$ is

$$\Gamma_{\Psi,I} : D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}) \to \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I})).$$

Assuming conjecture 1.2.3, we will finally show that the above functors induce equivalences of categories

$$D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}) \underset{\Gamma_{\Psi,I}}{\simeq} \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})),$$

and this will conclude the proof of conjecture 1.2.2, and therefore of conjecture 1.2.1.

- The Appendix is devoted to an explanation of how we think conjecture 1.2.3 can be proven. We will present two different approaches. The first, presented in A.1, consists in constructing a functor

$$\Phi_I : \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I}) \to \mathcal{A}_{crit}\text{-mod}_I^{JG},$$

and show that Φ_I and Ψ_I are mutually inverse equivalences of categories. The second approach, presented in A.2, consists in deducing the equivalence $\mathcal{A}_{crit}\text{-mod}_I^{JG} \xrightarrow{\sim} \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})$ over X^I from the equivalence over X given in (1.3). More generally, given a factorization functor $G : \mathcal{C} \to \mathcal{D}$ between two abelian factorization categories inducing an equivalence $G_X : \mathcal{C}_X \xrightarrow{\sim} \mathcal{D}_X$, we will explain what conditions on it would guarantee equivalences $G_I : \mathcal{C}_{X^I} \xrightarrow{\sim} \mathcal{D}_{X^I}$ over X^I.

Chapter 2

W-algebras and chiral differential operators at the critical level

2.1 Chiral algebras

We start this chapter by introducing the notion of *chiral algebra* as presented in [BD]. We will see later, in section 3.1.3, how chiral algebras can be described as factorization algebras, i.e. a sequence of quasi-coherent sheaves on X^n satisfying some properties. Since we will only use the latter description in the second chapter, we prefer giving the classical definition here. Throughout this chapter $\Delta : X \hookrightarrow X \times X$ will denote the diagonal embedding and $j : U \to X \times X$ its complement, where $U = (X \times X) - \Delta(X)$.

For any two sheaves $\mathcal{M}$ and $\mathcal{N}$ denote by $\mathcal{M} \boxtimes \mathcal{N}$ the external tensor product $\pi_1^* \mathcal{M} \underset{\mathcal{O}_{X \times X}}{\otimes} \pi_2^* \mathcal{N}$, where π_1 and π_2 are the two projections from $X \times X$ to X. For a right $\mathcal{D}_X$-module $\mathcal{M}$ define the extension $\Delta_!(\mathcal{M})$ as

$$\Delta_!(\mathcal{M}) := j_* j^* (\Omega_X \boxtimes \mathcal{M}) / \Omega_X \boxtimes \mathcal{M}.$$

Sections of $\Delta_!(\mathcal{M})$ can be thought as distributions on $X \times X$ with support on the diagonal and with values on $\mathcal{M}$. If $\mathcal{M}$ and $\mathcal{N}$ are two right $\mathcal{D}_X$-modules, we will denote by $\mathcal{M} \overset{!}{\otimes} \mathcal{N}$ the right $\mathcal{D}_X$-module $\mathcal{M} \otimes \mathcal{N} \otimes \Omega_X^*$.

We will now recall the definition of unital *chiral algebras* as presented in [BD].

Definition 2.1.1. A unital chiral algebra $\mathcal{A}$ is as a right $\mathcal{D}_X$-module $\mathcal{A}^{cl}$ on X equipped with a $\mathcal{D}_X$-module homomorphism

$$\mu : j_*j^*(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl}) \to \Delta_!(\mathcal{A}^{cl})$$

where $j : X \times X - \Delta(X) \to X \times X \leftarrow X : \Delta$, and an embedding

$$i : \Omega_X \hookrightarrow \mathcal{A}^{cl}$$

satisfying the following conditions:

- (skew-symmetry) $\mu = -\sigma_{12} \circ \mu \circ \sigma_{12}$.

- (Jacobi identity) $\mu_{1\{23\}} = \mu_{\{12\}3} + \mu_{2\{13\}}$.

- (unit) The following diagram commutes:

$$\begin{tikzcd} j_*j^*(\Omega_X \boxtimes \mathcal{A}^{cl}) \arrow[r] \arrow[d] & j_*j^*(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl}) \arrow[d] \\ \Delta_!(\mathcal{A}^{cl}) \arrow[r, "id"] & \Delta_!(\mathcal{A}^{cl}) \end{tikzcd}$$

  where the vertical map on the left comes from the sequence

$$\Omega_X \boxtimes \mathcal{A}^{cl} \to j_*j^*(\Omega_X \boxtimes \mathcal{A}^{cl}) \to \Delta_!\Delta^!(\Omega_X \boxtimes \mathcal{A}^{cl})[1] \simeq \Delta_!(\mathcal{A}^{cl})$$

  and σ_{12} is the induced action on $\mathcal{A}$ by permuting the variables of X^2.

The Jacobi identity above means the following: if we denote by j_{123} the inclusion of the subset of X^3 where all the $x_i's$ are different and by Δ_{ij} the inclusion of the diagonal $x_i = x_j$, then $\mu_{1\{23\}} : j_*j^*(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl}) \to \Delta_!(\mathcal{A}^{cl})$ is defined as the composition

$$\begin{aligned} j_*j_{123}^*(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl}) \xrightarrow{\mu} \; & j_*j^*_{x_1\neq x_2, x_1\neq x_3}(\mathcal{A}^{cl} \boxtimes \Delta_{(23)!}\mathcal{A}^{cl}) \simeq \\ & \simeq \Delta_{(23)!}j_*j^*_{x_1\neq x_3}(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl}) \xrightarrow{\mu} \Delta_{(123)!}(\mathcal{A}^{cl}), \end{aligned}$$

the map $\mu_{\{12\}3}$ is the composition

$$\begin{aligned} j_*j_{123}^*(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl}) \xrightarrow{\mu} \; & j_*j^*_{x_1\neq x_3, x_2\neq x_3}(\Delta_{(12)!}\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl}) \simeq \\ & \simeq \Delta_{(12)!}(j_*j^*_{x_2\neq x_3}(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl})) \xrightarrow{\mu} \Delta_{(123)!}(\mathcal{A}^{cl}), \end{aligned}$$

and the map $\mu_{2\{13\}}$ is gives as

$$\begin{aligned} j_*j_{123}^*(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl}) \xrightarrow{\mu} \; & j_*j^*_{x_2\neq x_3, x_1\neq x_2}(\Delta_{(13)!}\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl}) \simeq \\ & \simeq \Delta_{(13)!}j_*j^*_{x_2\neq x_3}(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl}) \xrightarrow{\mu} \Delta_{(123)!}(\mathcal{A}^{cl}). \end{aligned}$$

The Jacobi identity means that, as a map taking place on X^3, the alternating sum of the above maps is zero.

2.1.1 Commutative chiral algebras

As in the world of classical algebras, there is a well defined notion of *commutative chiral algebra.* We will see how these are the same as affine $\mathcal{D}_X$-schemes. Moreover, in 3.1.6, we will relate the factorization description of commutative chiral algebras to the notion of *co-unital factorization spaces.*

Definition 2.1.2. Let $(\mathcal{R}, \mu)$ be a unital chiral algebra. $\mathcal{R}$ is called *commutative* if the composition

$$\mathcal{R} \boxtimes \mathcal{R} \longrightarrow j_* j^* (\mathcal{R} \boxtimes \mathcal{R}) \xrightarrow{\mu} \Delta_! (\mathcal{R}) \tag{2.1}$$

vanishes.

Denote by $\mathcal{R}^l$ the left $\mathcal{D}_X$-module given as $\mathcal{R}^l := \mathcal{R} \otimes \Omega_X^*$. The diagram in (2.1) implies, and is in fact equivalent, that μ factors through a map

$$\begin{tikzcd} j_* j^* (\mathcal{R} \boxtimes \mathcal{R}) \arrow[r, "\mu"] \arrow[dr] & \Delta_! (\mathcal{R}) \\ & \Delta_! (\mathcal{R} \overset{!}{\otimes} \mathcal{R}) \arrow[u] \end{tikzcd}$$

Therefore we obtain a map

$$\mathcal{R} \overset{!}{\otimes} \mathcal{R} \to \mathcal{R}.$$

This map yields a commutative product (because of the skew-symmetry) $\mathcal{R}^l \otimes \mathcal{R}^l \xrightarrow{m} \mathcal{R}^l$, making $\mathcal{R}^l$ a $\mathcal{D}_X$-algebra. On the other hand, if we are given a $\mathcal{D}_X$-algebra $\mathcal{R}^l$, i.e. a left $\mathcal{D}_X$-module with a map of $\mathcal{D}_X$-modules $\mathcal{R}^l \otimes \mathcal{R}^l \to \mathcal{R}^l$, we can consider $\mathcal{R} := (\mathcal{R}^l)^r$ and the composition

$$j_* j^* (\mathcal{R} \boxtimes \mathcal{R}) = j_* j^* (\mathcal{R}^l \boxtimes \mathcal{R}^l) \otimes j_* j^* (\Omega_X \boxtimes \Omega_X) \to \Delta_! \Delta^! (\mathcal{R}^l \boxtimes \mathcal{R}^l) \otimes \Delta_! (\Omega_X) =$$

$$= \Delta_! (\mathcal{R}^l \otimes \mathcal{R}^l) \otimes \Delta_! (\Omega_X) \xrightarrow{m \otimes id} \Delta_! (\mathcal{R}^l) \otimes \Delta_! (\Omega_X) = \Delta_! (\mathcal{R})$$

where m is the product map of $\mathcal{R}^l$. This is a chiral operation on $\mathcal{R}$. The above establishes an equivalence

$$\{\mathcal{D}_X\text{-algebras } \mathcal{R}^l\} \rightleftarrows \{\text{Commutative chiral algebras } \mathcal{R}\}. \tag{2.2}$$

For instance, in the case $\mathcal{R} = \Omega_X$, with chiral product defined as $\mu(f(x,y)dx \boxtimes dy) = f(x,y) dx \wedge dy \pmod{\Omega^2_{X \times X}}$, you simply recover the commutative product on the sheaf of functions on X, which is in fact the left $\mathcal{D}_X$-module corresponding to Ω_X.

2.1.2 Chiral envelope of Lie*-algebras

The chiral algebras that we are mostly interested in are those that can be constructed from Lie* algebras by taking their *chiral envelope.* Let L be a Lie*-algebra, as introduced in [BD]. A Lie*-algebra L is, in particular, a right $\mathcal{D}_X$-modules with a map

$$[\, ,\,]_L : L \boxtimes L \to \Delta_*(L),$$

satisfying certain properties. The natural embedding

$$\mathcal{M} \boxtimes \mathcal{M} \hookrightarrow j_*j^*(\mathcal{M} \boxtimes \mathcal{M})$$

defines an obvious forgetful functor

$$\{\text{ chiral algebras }\} \to \{\text{Lie}^*\text{-algebras}\}\,.$$

We will denote by $\mathcal{A}^{Lie}$ the Lie*-algebra corresponding to the chiral algebra $\mathcal{A}$. The above functor admits a left adjoint U,

$$U : \{\text{Lie}^*\text{-algebras}\} \to \{\text{chiral algebras }\}\,.$$

Given a Lie*-algebra L, we define its chiral envelope to be the chiral algebra $U(L)$. In particular, by definition, we have

$$\mathrm{Hom}^{ch}(\mathcal{A}, U(L)) \simeq \mathrm{Hom}^{Lie}(\mathcal{A}^{Lie}, L).$$

The chiral algebra $U(L)$ is generated by the image of L which is a Lie*-subalgebra of $U(L)$. The corresponding filtration on $U(L)$ is called the Poincare'-Birkhoff-Witt filtration. We have a canonical surjection

$$Sym^{\cdot} L \twoheadrightarrow \mathrm{gr}_{\cdot} U(L).$$

As it is explained in [BD], when L is $\mathcal{O}_X$-flat, the above surjection is an isomorphism.

2.1.3 The chiral algebra $\mathcal{A}_\kappa$

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra. Recall the Lie*-algebra $L_\mathfrak{g} = \mathfrak{g} \otimes \mathcal{D}_X$ as defined in [AG]. For every symmetric invariant bilinear form

$$\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C},$$

consider the pairing $\widetilde{\kappa}$,

$$\begin{array}{rccc} \widetilde{\kappa}: & (\mathfrak{g} \otimes \mathcal{O}_X) \times (\mathfrak{g} \otimes \mathcal{O}_X) & \to & \Omega_X \\ & (a,b) & \mapsto & \kappa(da,b), \end{array}$$

and extend this pairing to a map $\widetilde{\kappa}_{\mathcal{D}_X} : L_\mathfrak{g} \otimes L_\mathfrak{g} \to \Delta_*(\Omega_X \otimes \mathcal{D}_X)$. The composition $\kappa_{\mathcal{D}_X}$ of $\widetilde{\kappa}_{\mathcal{D}_X}$ with the map $\Omega_X \otimes \mathcal{D}_X \to \Omega_X$ defines a 2-cocycle on the Lie*-algebra $L_\mathfrak{g}$. We define$L_\mathfrak{g}^\kappa$ to be the Lie*-algebra extension corresponding to this cocycle. We will denote by $\mathcal{A}_\kappa$ the *twisted* chiral envelope of $L_\mathfrak{g}^\kappa$,

$$\mathcal{A}_\kappa := U(L_\mathfrak{g}^\kappa)/\mathbf{1} - 1,$$

where $\mathbf{1}$ denotes the embedding of Ω_X given by the identity in $U(L_\mathfrak{g}^\kappa)$ and 1 denotes the embedding of Ω_X given by the construction of $L_\mathfrak{g}^\kappa$.

Remark 2.1.1. Given a bilinear form κ, we can consider the Lie-algebra extension $\widehat{\mathfrak{g}}_\kappa$ given as

$$0 \to \mathbb{C} \cdot \mathbb{1} \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0,$$

with bracket given by

$$[af(t), bg(t)] = [a,b]f(t)g(t) + \kappa(a,b)\mathrm{Res}(fdg) \cdot \mathbb{1},$$

where a and b are elements in $\mathfrak{g}$, and $\mathbb{1}$ is the central element.

Recall that we denoted by U'_κ the appropriately completed twisted enveloping algebra of $\widehat{\mathfrak{g}}_\kappa$, and by $\widehat{\mathfrak{g}}_\kappa$-mod the category consisting of $\widehat{\mathfrak{g}}_\kappa$-modules M on which the central element $\mathbb{1}$ acts as the identity and such that, for every $m \in M$, the action of at^N on it is zero for $N >> 0$. Consider the chiral algebra $\mathcal{A}_\kappa$, and the category $\mathcal{A}_\kappa$-mod$_x$ of $\mathcal{A}_\kappa$-modules supported at $x \in X$. Recall the associative topological algebra $\hat{\mathcal{A}}_x$ attached to $\mathcal{A}$, with the property that its discrete continuous modules are the same as $\mathcal{A}$-modules supported at x. When we take $\mathcal{A}$ to be $\mathcal{A}_\kappa$, the topological associative algebra $\hat{\mathcal{A}}_{\kappa x}$ is isomorphic to U'_κ. In particular we have an equivalence of categories

$$\widehat{\mathfrak{g}}_\kappa\text{-mod} \simeq \mathcal{A}_\kappa\text{-mod}x.$$

We will be interested in the critical level $\kappa = \kappa_{crit} := -1/2\kappa_{Kill}$. Denote by $\mathcal{A}_{crit}$ the chiral algebra $\mathcal{A}_{\kappa_{crit}}$ and by $\mathfrak{Z}_{crit}$ its center. The importance of choosing the level κ to be κ_{crit} relies on the fact that the center $Z(\widehat{\mathfrak{g}}_{crit}\text{-mod})$ of the category $\widehat{\mathfrak{g}}_{crit}$-mod $:= U'_{crit}$-mod, happens to be very big, unlike any other level $\kappa \neq \kappa_{crit}$ where the center is in fact just $\mathbb{C}$, as shown in [FF]. The chiral algebra $\mathfrak{Z}_{crit}$ is closely related to the center $Z(\widehat{\mathfrak{g}}_{crit}\text{-mod})$, in fact we have an isomorphism

$$\hat{\mathfrak{Z}}_{crit,x} \simeq Z(\widehat{\mathfrak{g}}_{crit}\text{-mod}).,\tag{2.3}$$

in particular $Z(\widehat{\mathfrak{g}}_{crit}\text{-mod})$-mod is equivalent to the category $\mathfrak{Z}_{crit}$-mod$_x$ of $\mathfrak{Z}_{crit}$-modules supported at x.

2.1.4 Lie*-algebroids and $\mathcal{R}$-extensions

Let $\mathcal{R}$ a commutative chiral algebra. In this section we will recall the definitions of Lie*-$\mathcal{R}$ algebroids and *chiral* $\mathcal{R}$-extension of such. These definitions will be used in 2.1.6 to define the notion of *chiral envelope* of a chiral $\mathcal{R}$-extension of an algebroid

$\mathfrak{L}$.

Let $(\mathcal{R},\, m : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R})$ be a commutative chiral algebra. Given a Lie* algebra L, we say that L acts on $\mathcal{R}$ by derivations if we are given a Lie* L-action on $\mathcal{R}$

$$\tau : \mathcal{R} \boxtimes \mathcal{R} \to \Delta_*(\mathcal{R}),$$

which is a derivation of the product m.

Definition 2.1.3. Let L be a Lie* algebra acting by derivations on $\mathcal{R}$ via a map τ. An *$\mathcal{R}$-extension* of L is a $\mathcal{D}_X$-module L^c fitting in the short exact sequence

$$0 \to \mathcal{R} \to L^c \xrightarrow{\pi} L \to 0$$

together with a Lie* algebra structure on L^c such that π is a morphism of Lie* algebras and the adjoint action of L^c on $\mathcal{R} \subset L^c$ coincides with $\tau \circ \pi$.

Definition 2.1.4. A Lie* $\mathcal{R}$-algebroid $\mathcal{L}$ is a Lie* algebra with a central action of $\mathcal{R}$ (a map $\mathcal{R} \otimes \mathcal{L} \to \mathcal{L}$) and a Lie* action $\tau_{\mathcal{L}}$ of $\mathcal{L}$ on $\mathcal{R}$ by derivations such that

- $\tau_{\mathcal{L}}$ is $\mathcal{R}$-linear with respect to the $\mathcal{L}$-variable.

- The adjoint action of $\mathcal{L}$ is a $\tau_{\mathcal{L}}$-action of $\mathcal{L}$ (as a Lie* algebra) on $\mathcal{L}$ (as an $\mathcal{R}$-module).

In the next definitions we consider objects equipped with a chiral action of $\mathcal{R}$ instead of just a central one.

Definition 2.1.5. Let $\mathcal{R}$ be a commutative chiral algebra, and $\mathcal{L}$ be a Lie* $\mathcal{R}$-algebroid. A *chiral $\mathcal{R}$-extension* of $\mathcal{L}$ is a $\mathcal{D}_X$-module $\mathcal{L}^c$ such that

$$0 \to \mathcal{R} \xrightarrow{i} \mathcal{L}^c \to \mathcal{L} \to 0, \tag{2.4}$$

together with a Lie* bracket and a chiral $\mathcal{R}$-module structure $\mu_{\mathcal{R},\mathcal{L}^c}$ on $\mathcal{L}^c$ satisfying the following properties:

- The arrows in (2.4) are compatible with the Lie* algebra and chiral $\mathcal{R}$-module structures.

- The chiral operations $\mu_{\mathcal{R}}$ and $\mu_{\mathcal{R},\mathcal{L}^c}$ are compatible with the Lie* actions of $\mathcal{L}^c$.

- The $*$ operation that corresponds to $\mu_{\mathcal{R},\mathcal{L}^c}$ (i.e. the restriction of $\mu_{\mathcal{R},\mathcal{L}^c}$ to $\mathcal{R}\boxtimes\mathcal{L}^c$) is equal to $-i\circ\sigma\circ\tau_{\mathcal{L}^c,\mathcal{R}}\circ\sigma$, where $\tau_{\mathcal{L}^c,\mathcal{R}}$ is the $\mathcal{L}^c$-action on $\mathcal{R}$ given by the projection $\mathcal{L}^c\to\mathcal{L}$ and the $\mathcal{L}$ action $\tau_{\mathcal{L},\mathcal{R}}$ on $\mathcal{R}$ and σ is the transposition of variables. In other words the following diagram commutes

$$\begin{tikzcd}
\mathcal{R}\boxtimes\mathcal{L}^c \arrow[r] \arrow[d] & j_*j^*(\mathcal{R}\boxtimes\mathcal{L}^c) \arrow[r, "\mu_{\mathcal{R},\mathcal{L}^c}"] & \Delta_!(\mathcal{L}^c). \\
\mathcal{R}\boxtimes\mathcal{L} \arrow[r, "\sigma\circ\tau_{\mathcal{L},\mathcal{R}}\circ\sigma"'] & \Delta_!(\mathcal{R}) \arrow[ur, "\Delta_!(i)"'] &
\end{tikzcd}$$

Remark 2.1.2. The triples $(\mathcal{R},\mathcal{L},\mathcal{L}^c)$ form a category in the obvious manner. For fixed $\mathcal{R}$ and $\mathcal{L}$, the chiral $\mathcal{R}$-extensions of $\mathcal{L}$ form a groupoid, denoted by $\mathcal{P}^{ch}(\mathcal{L})$. It is important to notice that $\mathcal{P}^{ch}(\mathcal{L})$ is *not* a Picard groupoid. The notion of trivial chiral $\mathcal{R}$-extension of $\mathcal{L}$ makes no sense. However, if we denote by $\mathcal{P}^{cl}(\mathcal{L})$ the Picard groupoid of *classical* $\mathcal{L}$-extensions, i.e. extensions in the category of Lie*-algebroids, then we have that the Bear difference of two chiral extensions is a classical one, therefore we have the following.

Proposition 2.1.1. *If $\mathcal{P}^{ch}(\mathcal{L})$ is non-empty, then it is a $\mathcal{P}^{cl}(\mathcal{L})$-torsor.*

Definition 2.1.5 can be extended by replacing $\mathcal{R}$ with any chiral algebra $\mathcal{C}$ endowed with a central action of $\mathcal{R}$. More precisely a *chiral $\mathcal{C}$-extension* of $\mathcal{L}$ is a

$\mathcal{D}_X$-module $\mathcal{L}^c$ such that

$$0 \to \mathcal{C} \to \mathcal{L}^c \to \mathcal{L} \to 0, \tag{2.5}$$

together with a Lie* bracket and a chiral $\mathcal{R}$-module structure $\mu_{\mathcal{R},\mathcal{L}^c}$ on $\mathcal{L}^c$ such that:

- The arrows in (2.5) are compatible with the Lie* algebra and chiral $\mathcal{R}$-module structures.

- The chiral operations $\mu_{\mathcal{C}}$ and $\mu_{\mathcal{R},\mathcal{L}^c}$ are compatible with the Lie* actions of $\mathcal{L}^c$.

- The structure morphism $\mathcal{R} \to \mathcal{C}$ is compatible with the Lie* actions of $\mathcal{L}^c$.

- The $*$ operation that corresponds to $\mu_{\mathcal{R},\mathcal{L}^c}$ (i.e. $\mu_{\mathcal{R},\mathcal{L}^c}$ restricted to $\mathcal{R} \boxtimes \mathcal{L}^c$) is equal to $-i \circ \sigma \circ \tau_{\mathcal{L}^c,\mathcal{R}} \circ \sigma$, where $\tau_{\mathcal{L}^c,\mathcal{R}}$ is the $\mathcal{L}^c$-action on $\mathcal{R}$, σ is the transposition of variables and i is the composition of the structure morphism $\mathcal{R} \to \mathcal{C}$ and the embedding $\mathcal{C} \subset \mathcal{L}^c$.

Definition 2.1.6. The *chiral envelope* of the chiral extension $(\mathcal{R}, \mathcal{C}, \mathcal{L}^c, \mathcal{L})$ is a pair $(U(\mathcal{C}, \mathcal{L}^c), \phi^c)$, where $U(\mathcal{C}, \mathcal{L}^c)$ is a chiral algbera and ϕ^c is a homomorphism of $\mathcal{L}^c$ into $U(\mathcal{C}, \mathcal{L}^c)$, satisfying the following universal property. For every chiral algebra $\mathcal{A}$ and any morphism $f : \mathcal{L}^c \to \mathcal{A}$ such that:

- f is a morphism of Lie* algebras.

- f restricts to a morphism of chiral algebras on $\mathcal{C} \subset \mathcal{L}^c$.

- f is a morphism of chiral-$\mathcal{R}$-modules (where the $\mathcal{R}$-action on $\mathcal{A}$ is the one given by the above point), there exist a unique map $\overline{f} : U(\mathcal{C}, \mathcal{L}^c) \to \mathcal{A}$ that makes the following diagram commutative

$$\begin{tikzcd} \mathcal{L}^c \arrow[r, "f"] \arrow[d, "\phi^c"'] & \mathcal{A} . \\ U(\mathcal{C}, \mathcal{L}^c) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

It is shown in [BD] that such object exists. When $\mathcal{C} = \mathcal{R}$ we will simply write $U(\mathcal{L}^c)$ instead of $U(\mathcal{R}, \mathcal{L}^c)$.

2.2 Quantization-deformation of commutative chiral algebras

Definition 2.2.1. Let $\mathcal{R}$ be a commutative chiral algebra. $\mathcal{R}$ is called a *chiral-Poisson algebra* if it is endowed with a Lie*-bracket, called the *chiral-Poisson bracket* $\{\, , \,\} : \mathcal{R} \boxtimes \mathcal{R} \to \Delta_!(\mathcal{R})$ that is a derivation of $\mathcal{R}$ in the sense of 2.1.4.

Example 2.2.1. Let $\mathcal{A}_t$ be a one-parameter flat family of chiral algebras; i.e., $\mathcal{A}_t$ is a chiral $k[t]$-algebra which is flat as a $k[t]-$module. Assume that $\mathcal{A} := \mathcal{A}_{t=0} := \mathcal{A}_t / t\mathcal{A}_t$ is a commutative chiral algebra. This means that the Lie*-bracket $[\, , \,]_t$ of $\mathcal{A}_t$ is divisible by t. Thus $\{\, , \,\}_t := t^{-1}[\, , \,]_t$ is a Lie*-bracket on $\mathcal{A}_t$. Reducing this picture modulo t, we see that $\mathcal{A}$ is a chiral-Poisson algebra, with bracket

$$\{\, , \,\} := \{\, , \,\}_{t=0}$$

One calls $\mathcal{A}_t$ the quantization of the coisson algebra $(\mathcal{A}, \{\, , \,\})$ with respect to the parameter t.

2.2.1 Quantizations of chiral-Poisson algebras

As in the usual Poisson setting, one can consider quantizations mod t^{n+1}, $n \geq 0$, of a given chiral-Poisson algebra $(\mathcal{A}, \{\, , \})$. Namely, these are triples $(\mathcal{A}^{(n)}, \{\, , \}_{(n)})$, where $\mathcal{A}^{(n)}$ is a flat chiral $k[t]/t^{n+1}$-algebra, $\{\, , \}_{(n)}$ a $k[t]/t^{n+1}$-bilinear Lie*-bracket on $\mathcal{A}_{(n)}$ such that $t\{\, , \}$ equals the Lie*-bracket for the chiral algebra structure, and $\alpha : \mathcal{A}_{(n)}/t\mathcal{A}_{(n)} \to \mathcal{A}$ an isomorphism of chiral algebras that sends $\{\, , \}_{(n)}$ (mod t) to $\{\, , \}$. Quantizations modulo t^{n+1} form a groupoid.

Now, let $\mathcal{R}$ be a commutative chiral algebra. As it is explained in [BD] 1.4.18, a Poisson structure on $\mathcal{R}$ gives the module $\Omega^1(\mathcal{R})$ a structure of a Lie* algebroid. In fact, the bracket $\{\, , \}$ yields a Lie*-$\mathcal{R}$ algebroid structure on $\mathcal{R} \otimes \mathcal{R}$. One checks easily that the kernel of the projection $\mathcal{R} \otimes \mathcal{R} \to \Omega^1(\mathcal{R})$, $a \otimes b \to adb$, is an ideal in $\mathcal{R} \otimes \mathcal{R}$, therefore $\Omega^1(\mathcal{R})$ inherits the Lie*-$\mathcal{R}$ algebroid structure.
Now consider the following: given a chiral extension

$$0 \to \mathcal{R} \to \Omega^c(\mathcal{R}) \to \Omega^1(\mathcal{R}) \to 0,$$

consider the pull-back of the above sequence via the differential $d : \mathcal{R} \to \Omega^1(\mathcal{R})$. The resulting short exact sequence is a $\mathbb{C}[\hbar]/\hbar^2$-deformation of the chiral-Poisson algebra $\mathcal{R}$[1]. If we denote by $\mathcal{Q}^{ch}(\mathcal{R})$ the groupoid of $\mathbb{C}[\hbar]/\hbar^2$-deformations of the chiral-Poisson algebra $\mathcal{R}$, and by $\mathcal{P}^{ch}(\Omega^1(\mathcal{R}))$ the groupoid of chiral $\mathcal{R}$-extensions of $\Omega^1(\mathcal{R})$ as defined in 2.1.2, the above map defines a functor

$$\mathcal{P}^{ch}(\Omega^1(\mathcal{R})) \to \mathcal{Q}^{ch}(\mathcal{R}).$$

In [BD] 3.9.10. the following is shown.

Theorem 2.1. *The above functor defines an equivalence between* $\mathcal{P}^{ch}(\Omega^1(\mathcal{R}))$ *and*

[1] If $\{\, , \}$ denotes the Poisson bracket on $\mathcal{R}$, this is indeed a quantization of $(\mathcal{R}, 2\{\, , \})$

$\mathcal{Q}^{ch}(\mathcal{R})$.

The above equivalence is the point of departure of this work.

Quantization of the center $\mathfrak{z}_{crit}$

Recall the commutative chiral algebra $\mathfrak{z}_{crit}$ defined in 2.1.3. We will see later, that the Drinfeld-Sokolov reduction Ψ_X, introduced in 2.3.1, produces a 1-parameter family of chiral algebras $\{\mathcal{W}_\hbar\} := \{\Psi_X(\mathcal{A}_\hbar)\}$ such that $\mathcal{W}_0 \simeq \mathfrak{z}_{crit}$. According to 2.2.1, we therefore have a Poisson structure on $\mathfrak{z}_{crit}$ defined by

$$\{z, w\} = \frac{[\widetilde{z}_\hbar, \widetilde{w}_\hbar]_{\mathcal{W}_\hbar}}{\hbar} \pmod{\hbar}$$

where $z = \widetilde{z}_\hbar|_{\hbar=0}$ and $w = \widetilde{w}_\hbar|_{\hbar=0}$. It follows from the definition of the functor Ψ_X that this Poisson structure coincides with the one from 1.1. Consider now the following diagram:

$$\begin{tikzcd}
\mathcal{P}^{ch}(\Omega^1(\mathcal{R})) \arrow[r] \arrow[d, "\simeq"] & \{\text{Lie}^* \text{ algebroid structures on } \Omega^1(\mathcal{R})\} \arrow[d, "\simeq"] \\
\mathcal{Q}^{ch}(\mathcal{R}) \arrow[r] & \{\text{Chiral-Poisson structures on } \mathcal{R}\}.
\end{tikzcd}$$

A natural question to ask is the following: if we consider the quantization of $\mathfrak{z}_{crit}$ introduced before, how does the corresponding chiral extension of $\Omega^1(\mathfrak{z}_{crit})$ look like?

The answer to the above question is the main body of this chapter.

2.2.2 Construction of the inverse

In this section, we will give an explicit construction of $\Omega^c(\mathcal{R})$ for an arbitrary chiral-Poisson algebra $\mathcal{R}$. In the case where $\mathcal{R} = \mathfrak{z}_{crit}$ we will see how this chiral extension relates to the *chiral algebra of differential operators on the loop group* $G((t))$ at the critical level introduced in [AG], where G is the algebraic group of adjoint type corresponding to $\mathfrak{g}$.

Let $\mathcal{R}$ be a commutative-Poisson chiral algebra and $\{\mathcal{R}_\hbar\}$ a quantization of the Poisson structure. The equivalence of categories from Theorem 2.1 states the existence of a chiral extension

$$0 \to \mathcal{R} \to \Omega^c(\mathcal{R}) \to \Omega^1(\mathcal{R}) \to 0$$

However the proof of this theorem doesn't provide a construction of it. This section will be devoted to the construction of the above extension.
Starting from the Lie* algebra extension

$$0 \to \mathcal{R} \to \mathcal{R}^c \to \mathcal{R} \to 0,$$

where $\mathcal{R}^c := \mathcal{R}_\hbar/\hbar^2\mathcal{R}_\hbar$ acts on $\mathcal{R}$ via the projection $\mathcal{R}^c \to \mathcal{R}$ and the Poisson bracket on $\mathcal{R}$, we will first construct a chiral extension (see Definition 2.1.5) $\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$ fitting into

$$0 \to \mathcal{R} \to \mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c) \to \mathcal{R} \otimes \mathcal{R} \to 0,$$

where $\mathcal{R} \otimes \mathcal{R}$ is viewed as a Lie* algebroid using the Poisson structure on $\mathcal{R}$. The chiral extension $\Omega^c(\mathcal{R})$ will be then defined as a quotient $\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$.
More generally, in 2.2.2-2.2.2 we will explain how to construct a chiral extension $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$ fitting into

$$0 \to \mathcal{R} \to \mathrm{Ind}^{ch}_{\mathcal{R}}(L^c) \to \mathcal{R} \otimes L \to 0 \tag{2.6}$$

for every Lie* algebra L acting on $\mathcal{R}$ by derivations and every extension

$$0 \to \mathcal{R} \to L^c \to L \to 0.$$

The case where $L = \mathcal{R}$ and $L^c = \mathcal{R}^c$, will be presented in 2.2.2 as a particular case of the above general construction.

Definition of $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$

Let $(\mathcal{R}, \mu)$ be a commutative chiral algebra and let L be a Lie* algebra acting on $\mathcal{R}$ by derivations via the map τ. The induced $\mathcal{R}$-module $\mathcal{R} \otimes L$ has a unique structure of Lie* $\mathcal{R}$-algebroid such that the morphism $1_{\mathcal{R}} \otimes id_L : L \to \mathcal{R} \otimes L$ is a morphism of Lie* algebras compatible with their actions on $\mathcal{R}$. Note that we have an obvious map

$$i : L \to \mathcal{R} \otimes L.$$

The Lie* algebroid $\mathcal{R} \otimes L$ is called *rigidified.* More generally we have the following definition.

Definition 2.2.2. A Lie* algebroid $\mathcal{L}$ is called *rigidified* if we are given a Lie* algebra L acting on $\mathcal{R}$ via the map τ, and an inclusion $i : L \to \mathcal{L}$, such that $\mathcal{R} \otimes L \xrightarrow{\sim} \mathcal{L}$.

Let $\mathcal{L}$ be a rigidified Lie* algebroid. Consider the map that sends a chiral extension of $\mathcal{L}$

$$0 \to \mathcal{R} \to \mathcal{L}^c \to \mathcal{L} \to 0$$

to the $\mathcal{R}$ extension of L given by considering the pull-back of the map $i : L \to \mathcal{L}$. Denote by $\mathcal{P}^{cl}(\mathcal{L})$ (resp. $\mathcal{P}^{ch}(\mathcal{L})$) the groupoid of classical (resp. chiral) extensions of $\mathcal{L}$ (where by classical we mean extensions in the category of Lie* algebroids), and by $\mathcal{P}(L, \tau)$ the Picard groupoid of $\mathcal{R}$-extensions of L. Clearly the map mentioned above (that can be equally defined for classical extensions as well), defines two functors

$$\mathcal{P}^{cl}(\mathcal{L}) \to \mathcal{P}(L, \tau), \quad \mathcal{P}^{ch}(\mathcal{L}) \to \mathcal{P}(L, \tau).$$

As it is explained in [BD] 3.9.9. the following is true.

Proposition 2.2.1. *If L is $\mathcal{O}_X$ flat, then these maps define an equivalence of groupoids*

$$\mathcal{P}^{cl}(\mathcal{L}) \xrightarrow{\sim} \mathcal{P}(L,\tau), \quad \mathcal{P}^{ch}(\mathcal{L}) \xrightarrow{\sim} \mathcal{P}(L,\tau), \tag{2.7}$$

Given a Lie* algebra extension $0 \to \mathcal{R} \to L^c \to L \to 0$, define $\mathrm{Ind}^{cl}_{\mathcal{R}}(L^c)$ (resp. $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$) to be the classical (resp. chiral) extension corresponding to the above sequence under the equivalences stated in the above proposition.

In 2.2.2 we will briefly recall the construction of the inverse functors to (2.7) in the classical and chiral setting respectively (as presented in [BD]). However in 2.2.2 we will give a different construction of the inverse functor in the chiral setting, i.e. a different construction of the chiral extension $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$ associated to any $\mathcal{R}$-extension of L. The latter construction will be used to define the chiral extension $\Omega^c(\mathcal{R})$.

The classical setting

For the "classical" map $\mathcal{P}^{cl}(\mathcal{L}) \to \mathcal{P}(L,\tau)$, to an extension

$$0 \to \mathcal{R} \to L^c \to L \to 0, \tag{2.8}$$

the inverse functor associates the classical extension $\mathrm{Ind}^{cl}_{\mathcal{R}}(L^c)$ of the Lie* algebroid $\mathcal{R} \otimes L = \mathcal{L}$ given by the push-out of the extension

$$0 \to \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \otimes L^c \to \mathcal{R} \otimes L \to 0$$

via the map $m : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$.

The construction of the inverse functor in the "chiral" setting given in [BD] (i.e. the construction of $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$), uses the following two facts:

- $\mathcal{P}^{ch}(\mathcal{L})$ has a structure of $\mathcal{P}^{cl}(\mathcal{L})$-torsor under Baer sum.

- $\mathcal{P}^{ch}(\mathcal{L})$ is non empty.

The first fact follows from condition 3) in the definition of chiral $\mathcal{R}$-extension, which guarantees that the Baer difference of two chiral extensions is a classical one. In other words the action of $\mathcal{R}$ on the sum of two chiral extensions is automatically central.

The non emptiness of $\mathcal{P}^{ch}(\mathcal{L})$ follows from the existence of a distinguished chiral $\mathcal{R}$-extension $\mathrm{Ind}_{\mathcal{R}}(L)$ attached to every Lie* algebra L acting on $\mathcal{R}$. Such object is defined by the following:

Definition-Proposition 2.2.1. *Suppose that we are given a Lie* algebra L acting by derivations on $\mathcal{R}$ via the map τ, and let $\mathcal{L}$ be a rigidified Lie* algebroid (see Definition 2.2.2), so we have a morphism of Lie* algebras $i : L \to \mathcal{L}$ such that $\mathcal{R} \otimes L \xrightarrow{\sim} \mathcal{L}$. Then there exist a chiral extension $Ind_{\mathcal{R}}(L)$ equipped with a lifting $\bar{i} : L \to Ind_{\mathcal{R}}(L)$ such that $\bar{i}$ is a morphism of Lie* algebras and the adjoint action of L on $\mathcal{R}$ via $\bar{i}$ equals τ. The pair $(Ind_{\mathcal{R}}(L), \bar{i})$ is unique.*

The proof of this proposition can be found in [BD] 3.9.8. However in 2.2.2 we will recall the construction of $\mathrm{Ind}_{\mathcal{R}}(L)$ and of the map $\bar{i} : L \to \mathrm{Ind}_{\mathcal{R}}(L)$.

To finish the construction of $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$ (or in other words, the construction of the inverse to the functor $\mathcal{P}^{ch}(\mathcal{R}) \to \mathcal{P}(L, \tau)$), we use the classical extension $\mathrm{Ind}^{cl}_{\mathcal{R}}(L^c)$ given in 2.2.2 together with the $\mathcal{P}^{cl}(\mathcal{L})$-action on $\mathcal{P}^{ch}(\mathcal{L})$. To the extension $0 \to \mathcal{R} \to L^c \to L \to 0$ we associate the chiral $\mathcal{R}$-extension

$$\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c) := \mathrm{Ind}^{cl}_{\mathcal{R}}(L^c) \underset{Baer}{+} \mathrm{Ind}_{\mathcal{R}}(L)$$

of $\mathcal{R} \otimes L$ by $\mathcal{R}$, where $\mathrm{Ind}_{\mathcal{R}}(L)$ is the distinguished classical extension defined in 2.2.1. Note that, after pulling back the extension

$$0 \to \mathcal{R} \to \mathrm{Ind}^{ch}_{\mathcal{R}}(L^c) \to \mathcal{L} \simeq \mathcal{R} \otimes L \to 0$$

via the map $L \to \mathcal{L} \simeq \mathcal{R} \otimes L$, we obtain the Baer sum of the trivial extension (corresponding to $\mathrm{Ind}_{\mathcal{R}}(L)$) with L^c, i.e. we recover the initial Lie* extension $0 \to \mathcal{R} \to L^c \to L \to 0$ as we should.

Construction of $\mathrm{Ind}_{\mathcal{R}}(L)$.

In this subsection we want to recall the construction and the main properties of the distinguished chiral extension $\mathrm{Ind}_{\mathcal{R}}(L)$ given by Definition-Proposition 2.2.1.

Given a Lie* algebra L acting on $\mathcal{R}$ by derivations, we can consider the action map $\mathcal{R} \boxtimes L \to \Delta_!(\mathcal{R})$ and consider the following push out:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R} \boxtimes L \arrow[r] \arrow[d] & j_*j^*(\mathcal{R} \boxtimes L) \arrow[r] \arrow[d] & \Delta_!(\mathcal{R} \otimes L) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Delta_!(\mathcal{R}) \arrow[r] & \Delta_!(\mathcal{R}) \oplus j_*j^*(\mathcal{R} \boxtimes L)/\mathcal{R} \boxtimes L \arrow[r] & \Delta_!(\mathcal{R} \otimes L) \arrow[r] & 0.
\end{tikzcd}$$

The term in the middle is a $\mathcal{D}_X$-module supported on the diagonal, hence by Kashiwara's Theorem (see [?] Theorem 4.30) it corresponds to a $\mathcal{D}_X$-module on X. This $\mathcal{D}_X$-module has a structure of chiral extension and will be our desired $\mathrm{Ind}_{\mathcal{R}}(L)$ (i.e. we have $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(L)) \simeq \Delta_!(\mathcal{R}) \oplus j_*j^*(\mathcal{R} \boxtimes L)/\mathcal{R} \boxtimes L$).

Remark 2.2.1. By construction we have inclusions $\mathcal{R} \to \mathrm{Ind}_{\mathcal{R}}(L)$ and a lifting

$\bar{i} : L \to \mathrm{Ind}_{\mathcal{R}}(L)$ of $i : L \to \mathcal{R} \otimes L$. In fact we can consider the following diagram

$$
\begin{tikzcd}
0 \arrow[r] & \Omega_X \boxtimes L \arrow[r] \arrow[d] & j_*j^*(\Omega_X \boxtimes L) \arrow[r] \arrow[d] & \Delta_!(\Omega_X \overset{!}{\otimes} L) \simeq \Delta_!(L) \arrow[r] \arrow[d, "\Delta_!(i)=\Delta_!(unit \otimes id)"] \arrow[ddl, "\Delta_!(\bar{i})"'] & 0 \\
0 \arrow[r] & \mathcal{R} \boxtimes L \arrow[r] \arrow[d] & j_*j^*(\mathcal{R} \boxtimes L) \arrow[r] \arrow[d, "\pi"] & \Delta_!(\mathcal{R} \overset{!}{\otimes} L) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Delta_!(\mathcal{R}) \arrow[r] & \Delta_!(\mathcal{R}) \oplus j_*j^*(\mathcal{R} \boxtimes L) /_{\mathcal{R} \boxtimes L} \arrow[r] & \Delta_!(\mathcal{R} \overset{!}{\otimes} L) \arrow[r] & 0.
\end{tikzcd}
$$

By looking at the composition of the two vertical arrows in the middle, it is not hard to see that this composition factors through $\Delta_!(L)$. In fact the most left vertical arrow from $\Omega_X \boxtimes L$ to $\Delta_!(\mathcal{R})$ is zero. We define $\bar{i}$ to be the map corresponding (under the Kashiwara's equivalence) to $\Delta_!(\bar{i})$.

As it is shown in [BD] 3.3.6. the inclusions $\mathcal{R} \to \mathrm{Ind}_{\mathcal{R}}(L)$, $\bar{i} : L \to \mathrm{Ind}_{\mathcal{R}}(L)$ and the chiral operation $j_*j^*(\mathcal{R} \boxtimes L) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}(L))$, uniquely determine a chiral action of $\mathcal{R}$ on $\mathrm{Ind}_{\mathcal{R}}(L)$ and a Lie* bracket on it. In other words they give $\mathrm{Ind}_{\mathcal{R}}(L)$ a structure of chiral $\mathcal{R}$-extension.

Note that this chiral $\mathcal{R}$-extension corresponds, under the equivalence given by Theorem 2.2.1 (i.e. after we pull-back the extension via the map $\Psi_X : L \to \mathcal{R} \otimes \mathcal{L}$), to the trivial extension of L by $\mathcal{R}$ in $\mathcal{P}(L, \tau)$. To summarize we have seen that:

- If a Lie* algebra L acts on $\mathcal{R}$ we can construct the distinguished chiral extension $\mathrm{Ind}_{\mathcal{R}}(L)$ of $\mathcal{L}$ with a lifting $\bar{i} : L \to \mathrm{Ind}_{\mathcal{R}}(L)$ of the canonical map $i : L \to \mathcal{L}$.

- From an extension $0 \to \mathcal{R} \to L^c \to L \to 0$ we can construct a chiral extension $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$ with a map $L^c \to \mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$ given by the pull-back of $L \to \mathcal{L}$.

Remark 2.2.2. Clearly, if we have the extension $0 \to \mathcal{R} \to L^c \to L \to 0$, we can also consider L^c as a Lie* algebra acting on $\mathcal{R}$ via the projection $L^c \to \mathcal{R}$. In other words we forget about the extension and we only remember the Lie* algebra L^c. From point one of the above summary we can construct the distinguished chiral extension $\mathrm{Ind}_{\mathcal{R}}(L^c)$ corresponding to this L^c action on $\mathcal{R}$, together with a map $\bar{i} : L^c \to \mathrm{Ind}_{\mathcal{R}}(L^c)$.

Different construction of $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$.

We will now explain a different construction of the chiral extension

$$0 \to \mathcal{R} \to \mathrm{Ind}^{ch}_{\mathcal{R}}(L^c) \to \mathcal{L} \simeq \mathcal{R} \otimes L \to 0$$

that will be used later to construct $\Omega^c(\mathcal{R})$.

As it is explained in the Remark 2.2.2, given an $\mathcal{R}-$extension

$$0 \to \mathcal{R} \xrightarrow{k} L^c \to L \to 0,$$

we can consider the action of L^c on $\mathcal{R}$ given by the projection $L^c \to L$ and construct the distinguished chiral extension $\mathrm{Ind}_{\mathcal{R}}(L^c)$. This is a chiral $\mathcal{R}$-extension fitting into

$$0 \to \Delta_!(\mathcal{R}) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c)) \to \Delta_!(\mathcal{R} \otimes L^c) \to 0,$$

where $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c)) \simeq \Delta_!(\mathcal{R}) \oplus j_*j^*(\mathcal{R} \boxtimes L^c) /_{\mathcal{R} \boxtimes L^c}$. Since we ultimately want an extension of $\mathcal{R}$ by $\mathcal{R} \otimes L$, we have to quotient the above sequence by some additional relations. We will in fact obtain $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$ by taking the quotient of $\mathrm{Ind}_{\mathcal{R}}(L^c)$ by the image of the difference of two maps from $\mathcal{R} \otimes \mathcal{R} \to \mathrm{Ind}_{\mathcal{R}}(L^c)$.

The above maps are given (under the Kashiwars's equivalence) by the following two maps from $\Delta_!(\mathcal{R}\otimes\mathcal{R})$ to $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c))$.

1) The first map is given by the composition

$$\Delta_!(\mathcal{R}\otimes\mathcal{R}) \xrightarrow{m} \Delta_!(\mathcal{R}) \hookrightarrow \Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c)).$$

2) For the second map, consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}\boxtimes\mathcal{R} \arrow[r] \arrow[d, "id\boxtimes k"] & j_*j^*(\mathcal{R}\boxtimes\mathcal{R}) \arrow[r] \arrow[d, "k\boxtimes id"] & \Delta_!(\mathcal{R}\otimes\mathcal{R}) \arrow[r] \arrow[d, "id\otimes k"] & 0 \\
0 \arrow[r] & \mathcal{R}\boxtimes L^c \arrow[r] \arrow[d] & j_*j^*(\mathcal{R}\boxtimes L^c) \arrow[r] \arrow[d, "\pi"] & \Delta_!(\mathcal{R}\otimes L^c) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Delta_!(\mathcal{R}) \arrow[r] & \underset{\Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c))}{\underset{\cong}{\Delta_!(\mathcal{R})\oplus j_*j^*(\mathcal{R}\boxtimes L^c)/\mathcal{R}\boxtimes L^c}} \arrow[r] & \Delta_!(\mathcal{R}\otimes L^c) \arrow[r] & 0.
\end{tikzcd}$$

We claim that the composition of the two vertical arrows in the middle (i.e. $\pi\circ(k\boxtimes id)$) factors through $\Delta_!(\mathcal{R}\otimes\mathcal{R})$. In fact since the action of L^c on $\mathcal{R}$ is given by the projection $L^c\to\mathcal{R}$, the copy of $\mathcal{R}$ inside L^c via k acts by zero. Hence the composition of the left most vertical arrows is zero, which shows that there is a well defined map

$$\overline{k}:\Delta_!(\mathcal{R}\otimes\mathcal{R})\to\Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c)).$$

The quotient of $\mathrm{Ind}_{\mathcal{R}}(L^c)$ by the image of the difference of the above maps is exactly $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$.

Remark 2.2.3. Note that the inclusion $L^c\to\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$ mentioned in the summary in 2.2.2 corresponds to the composition

$$\Delta_!(L^c)\xrightarrow{\Delta_!(\overline{i})}\Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c))\twoheadrightarrow\Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)). \tag{2.9}$$

A special case: deformations of $\mathcal{R}$.

Let $(\mathcal{R}, m : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R})$ be a commutative chiral algebra given as $\mathcal{R} := \mathcal{R}_\hbar / \hbar \mathcal{R}_\hbar$, where $\{\mathcal{R}_\hbar\}$ is a family of chiral algebras. Denote by $\{\,,\}$ the Poisson bracket on $\mathcal{R}$ defined as

$$\{z, w\} = \frac{1}{\hbar}[z_\hbar, w_\hbar]_\hbar \pmod{\hbar},$$

where $z_\hbar = z \pmod{\hbar}$, $w_\hbar = w \pmod{\hbar}$ and $[\,,]_\hbar$ denotes the Lie* bracket on $\mathcal{R}_\hbar$ induced by the chiral product $\mu_\hbar$ restricted to $\mathcal{R}_\hbar \boxtimes \mathcal{R}_\hbar$.
Consider the quotient $\mathcal{R}^c = \mathcal{R}_\hbar / \hbar^2 \mathcal{R}_\hbar$. This is a Lie* algebra with bracket $[\,,]_c$ defined by

$$[\overline{z_\hbar}, \overline{w_\hbar}]_c = \overline{\frac{1}{\hbar}[z_\hbar, w_\hbar]_h}.$$

Consider the short exact sequence

$$0 \to \mathcal{R} \xrightarrow{\cdot\hbar} \mathcal{R}^c \to \mathcal{R} \to 0, \tag{2.10}$$

and let us regard $\mathcal{R}^c$ as a Lie* algebra acting on $\mathcal{R}$ via the projection $\mathcal{R}^c \to \mathcal{R}$ followed by the Poisson bracket multiplied by[2] $1/2$. This sequence is an $\mathcal{R}$-extension of $\mathcal{R}$ in the sense we introduced in Definition 2.1.3, therefore, from what we have seen in 2.2.2, we can construct a chiral $\mathcal{R}$-extension of $\mathcal{R} \otimes \mathcal{R}$ by $\mathcal{R}$ (here $L = \mathcal{R}$ and $L^c = \mathcal{R}^c$)

$$0 \to \mathcal{R} \to \mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c) \to \mathcal{R} \otimes \mathcal{R} \to 0. \tag{2.11}$$

Below we will use the above chiral extension to define the chiral algebroid $\Omega^c(\mathcal{R})$.

The construction of $\Omega^c(\mathcal{R})$.

We can now proceed to the construction of $\Omega^c(\mathcal{R})$. Recall that, because of the Poisson bracket on $\mathcal{R}$, the sheaf $\Omega^1(\mathcal{R})$ acquires a structure of a Lie* algebroid.

[2]This correction is due to the fact that, as we saw in 2.2, the equivalence stated in Theorem 2.1 gives a quantization of $1/2\{\,,\}$.

Recall that we denoted by $\mathcal{Q}^{ch}(\mathcal{R})$ the groupoid of $\mathbb{C}[\hbar]/\hbar^2$-deformations of our chiral-Poisson algebra $\mathcal{R}$, and that we want to understand how to construct the inverse to the functor

$$\mathcal{P}^{ch}(\Omega^1(\mathcal{R})) \to \mathcal{Q}^{ch}(\mathcal{R}),$$

that assigns to a chiral extension $0 \to \mathcal{R} \to \Omega^c(\mathcal{R}) \to \Omega^1(\mathcal{R}) \to 0$, its pull-back via the differential $d : \mathcal{R} \to \Omega^1(\mathcal{R})$.

The inverse functor will be constructed as follows: for any object in $\mathcal{Q}^{ch}(\mathcal{R})$, i.e. to for any extension $0 \to \mathcal{R} \xrightarrow{\cdot\hbar} \mathcal{R}^c \to \mathcal{R} \to 0$, we will consider the chiral extension

$$0 \to \mathcal{R} \to \mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c) \to \mathcal{R} \otimes \mathcal{R} \to 0$$

described in the previous subsection. We will quotient $\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$ by some additional relations in order to impose the Leibniz rule on $\mathcal{R} \otimes \mathcal{R}$. These relations will be given, under Kashiwara's equivalence, as the image of a map from $\Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c)$ to $\Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c))$. More precisely, we will construct a map from $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ to $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(\mathcal{R}^c))$ such that the composition with the projection $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(\mathcal{R}^c)) \to \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c))$ vanishes when restricted to $\mathcal{R}^c \boxtimes \mathcal{R}^c$. Hence it will induce a map $\Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \to \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c))$. Form the sequence (2.11) we will therefore obtain a chiral $\mathcal{R}$-extension $\widetilde{\Omega^c(\mathcal{R})}$ of the Lie* algebroid $\Omega^1(\mathcal{R})$

$$0 \to \mathcal{R}' \to \widetilde{\Omega^c(\mathcal{R})} \to \Omega^1(\mathcal{R}) \to 0.$$

We will then check that $\mathcal{R}'$, which a priori is a quotient of $\mathcal{R}$, is in fact $\mathcal{R}$ itself, and that the pull-back via the differential $d : \mathcal{R} \to \Omega^1(\mathcal{R})$ is the original sequence (2.10), with induced Poisson bracket given by $\{,\}$. This will imply that $\widetilde{\Omega^c(\mathcal{R})}$ is in fact the chiral extension $\Omega^c(\mathcal{R})$ given by Theorem 2.1.

The map from $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ to $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(\mathcal{R}^c))$ is defined as the sum of the following three maps:

1. The first map α_1 is given by the composition

$$j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \to j_*j^*(\mathcal{R} \boxtimes \mathcal{R}^c) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}(\mathcal{R}^c)),$$

   where the first map comes from the projection $\mathcal{R}^c \to \mathcal{R}$.

2. The second map α_2 is obtained from the first one by interchanging the roles of the factors in $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$.

3. For the third map α_3, note that the chiral bracket $\mu_\hbar$ on $\mathcal{R}_\hbar$ gives rise to a map

$$\cdot\mu_c : j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \to \Delta_!(\mathcal{R}^c)$$

   and we compose it with the canonical map $\Delta_!(\mathcal{R}^c) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}(\mathcal{R}^c))$.

Now consider the linear combination $\alpha_1 - \alpha_2 - \alpha_3$ as a map from $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ to $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(\mathcal{R}^c))$. If we compose this map with the inclusion $\mathcal{R}^c \boxtimes \mathcal{R}^c \to j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ and the projection onto $\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$, it is easy to see that the map vanishes. More precisely we have the following:

Lemma 2.2.1. *The composition*

$$\mathcal{R}^c \boxtimes \mathcal{R}^c \hookrightarrow j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \xrightarrow{\alpha_1-\alpha_2-\alpha_3} \Delta_!(Ind_{\mathcal{R}}(\mathcal{R}^c)) \twoheadrightarrow \Delta_!(Ind^{ch}_{\mathcal{R}}(\mathcal{R}^c))$$

vanishes. Thus it defines a map $Leib : \Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \to \Delta_!(Ind^{ch}_{\mathcal{R}}(\mathcal{R}^c))$.

Proof. Since the action of $\mathcal{R}^c$ on $\mathcal{R}$ is given by the projection $\mathcal{R}^c \to \mathcal{R}$ and the Poisson bracket on $\mathcal{R}$ multiplied by $1/2$, and because of the relation $\sigma \circ \{\,,\,\} \circ \sigma = -\{\,,\,\}$, the maps α_1 and α_2 factor as

$$\begin{tikzcd}
\mathcal{R}^c \boxtimes \mathcal{R}^c \arrow[r] \arrow[dr, "\alpha_1=\frac{1}{2}\{,\}=-\alpha_2"'] & \Delta_!(\mathrm{Ind}_{\mathcal{R}}(\mathcal{R}^c)) \arrow[r] & \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)) . \\
& \Delta_!(\mathcal{R}) \arrow[ur] &
\end{tikzcd}$$

Note that the above wouldn't have been true if we hadn't used the relation in $\mathrm{Ind}_{\mathcal{R}}(\mathcal{R}^c)$ as well. Moreover the third map, when composed with the projection to $\Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c))$ is exactly

$$\mathcal{R}^c \boxtimes \mathcal{R}^c \to \mathcal{R} \boxtimes \mathcal{R} \xrightarrow{\{,\}} \Delta_!(\mathcal{R}) \to \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)),$$

hence the combination $\alpha_1 - \alpha_2 - \alpha_3$ is indeed zero. From the above we therefore get a map $\Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \to \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c))$.

$\square$

We define $\widetilde{\Omega^c(\mathcal{R})}$ to be the quotient of $\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$ by the image of the corresponding map from $\mathcal{R}^c \otimes \mathcal{R}^c$ to $\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$ under the Kashiwara's equivalence.

Remark 2.2.4. Note that the map $\mathcal{R}^c \otimes \mathcal{R}^c \to \mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$ indeed factors through $\mathcal{R}^c \otimes \mathcal{R}^c \twoheadrightarrow \mathcal{R} \otimes \mathcal{R}$. To show this it is enough to show that the map $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \to \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c))$ factors through $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \to j_*j^*(\mathcal{R} \boxtimes \mathcal{R})$. If so, then the diagram below would imply that the composition $\mathcal{R} \boxtimes \mathcal{R} \to \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c))$ is zero, and we are done:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}^c \boxtimes \mathcal{R}^c \arrow[r] \arrow[d] & j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \arrow[r] \arrow[d] & \Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \arrow[r] \arrow[d] & 0 . \\
0 \arrow[r] & \mathcal{R} \boxtimes \mathcal{R} \arrow[r] & j_*j^*(\mathcal{R} \boxtimes \mathcal{R}) \arrow[r] \arrow[d] & \Delta_!(\mathcal{R} \otimes \mathcal{R}) \arrow[r] & 0 \\
& & \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)) & &
\end{tikzcd}$$

To show that the map factors as

$$\begin{tikzcd} j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \arrow[r] \arrow[d] & \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)) \\ j_*j^*(\mathcal{R} \boxtimes \mathcal{R}) \arrow[ur] & \end{tikzcd}$$

we need to show that the composition of the map $\alpha_1 - \alpha_2 - \alpha_3$ with the two embeddings $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}) \hookrightarrow j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ and $j_*j^*(\mathcal{R} \boxtimes \mathcal{R}^c) \hookrightarrow j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ is zero. We'll do only one of them (the second one can be done similarly). For the first embedding the map α_2 is zero (since we are projecting the second $\mathcal{R}^c$ onto $\mathcal{R}$) whereas the first map (because of the relations in $\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$) is equal to minus the composition

$$j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}) \to j_*j^*(\mathcal{R} \boxtimes \mathcal{R}) \xrightarrow{\mu} \Delta_!(\mathcal{R}) \to \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c))$$

which is exactly the third map when restricted to $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R})$.

Recall that we defined $\widetilde{\Omega^c(\mathcal{R})}$ to be the quotient of $\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$ by the image of the map *Leib* from 2.2.1 obtained using the combination α_1-α_2-α_3. By construction we have a short exact sequence

$$0 \to \mathcal{R}' \to \widetilde{\Omega^c(\mathcal{R})} \to \Omega^1(\mathcal{R}) \to 0, \tag{2.12}$$

where $\mathcal{R}'$ is a certain quotient of $\mathcal{R}$. In the rest of this section we will show that the above extension is in fact isomorphic to the extension of $\Omega^1(\mathcal{R})$ given in Theorem 2.1. This is equivalent to the following:

Proposition 2.2.2. *Consider the extension of $\Omega^1(\mathcal{R})$ given by (2.12). Then we have*

1. *$\mathcal{R}' = \mathcal{R}$.*

2. *The pull-back of (2.12) via the differential $d : \mathcal{R} \to \Omega^1(\mathcal{R})$ is the original sequence (2.10).*

Proof. To show that $\mathcal{R}' = \mathcal{R}$, consider the chiral extension given by the equivalence of Theorem 2.1. This is an extension of $\Omega^1(\mathcal{R})$ such that the pull back via the differential $\mathcal{R} \to \Omega^1(\mathcal{R})$ is the sequence (2.10). that is, we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{R} \arrow[r, "i"] & \Omega^c(\mathcal{R}) \arrow[r] & \Omega^1(\mathcal{R}) \arrow[r] & 0, \\ 0 \arrow[r] & \mathcal{R} \arrow[r] \arrow[u] & \mathcal{R}^c \arrow[r, "\pi"] \arrow[u, "d^c"] & \mathcal{R} \arrow[r] \arrow[u, "d"] & 0 \end{tikzcd} \tag{2.13}$$

with d^c a derivation, i.e. as maps from $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ to $\Delta_!(\Omega^c(\mathcal{R}))$, we have $d^c(\mu_c) = \mu_{\mathcal{R},\Omega^c(\mathcal{R})}(\pi, d^c) - \sigma \circ \mu_{\mathcal{R},\Omega^c(\mathcal{R})} \circ \sigma(d^c, \pi)$, where μ_c is the chiral product on $\mathcal{R}^c$ and $\mu_{\mathcal{R},\Omega^c(\mathcal{R})}$ is the chiral action of $\mathcal{R}$ on $\Omega^c(\mathcal{R})$. We claim that there is a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \Delta_!(\mathcal{R}) \arrow[r] \arrow[d, "id"] & \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)) \arrow[r] \arrow[d] & \Delta_!(\mathcal{R} \otimes \mathcal{R}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Delta_!(\mathcal{R}) \arrow[r] & \Delta_!(\Omega^c(\mathcal{R})) \arrow[r] & \Delta_!(\Omega^1(\mathcal{R})) \arrow[r] & 0 \end{tikzcd}$$

that factors through $0 \to \Delta_!(\mathcal{R}') \to \Delta_!(\widetilde{\Omega^c(\mathcal{R})}) \to \Delta_!(\Omega^1(\mathcal{R})) \to 0$, and moreover induces an isomorphism from $\Omega^1(\mathcal{R})$ to $\Omega^1(\mathcal{R})$. This would imply that $\mathcal{R}'$, which a priori is a quotient of $\mathcal{R}$, is in fact $\mathcal{R}$ itself. Furthermore, the fact that it is an isomorphism on $\Omega^1(\mathcal{R})$, would also imply that $\widetilde{\Omega^c(\mathcal{R})} \simeq \Omega^c(\mathcal{R})$, hence the pull-back via $d : \mathcal{R} \to \Omega^1(\mathcal{R})$ would indeed be the original sequence $0 \to \mathcal{R} \to \mathcal{R}^c \to \mathcal{R} \to 0$.

To prove the claim, consider the map $d^c : \mathcal{R}^c \to \Omega^c(\mathcal{R})$ given by (2.13). Using the chiral $\mathcal{R}$-module structure $\mu_{\mathcal{R},\Omega^c(\mathcal{R})}$ on $\Omega^c(\mathcal{R})$, we can consider the composition

$$j_*j^*(\mathcal{R} \boxtimes \mathcal{R}^c) \xrightarrow{i \boxtimes d^c} j_*j^*(\mathcal{R} \boxtimes \Omega^c(\mathcal{R})) \xrightarrow{\mu_{\mathcal{R},\Omega^c(\mathcal{R})}} \Delta_!(\Omega^c(\mathcal{R})).$$

The above composition can be extended to a map from $\Delta_!(\mathcal{R}) \oplus j_*j^*(\mathcal{R} \boxtimes \mathcal{R}^c) \to \Delta_!(\Omega^c(\mathcal{R}))$, by setting the map to be $\Delta_!(i)$ on $\Delta_!(\mathcal{R})$. It is straightforward to check that this map factors through a map D^c

$$D^c : \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)) \to \Delta_!(\Omega^c(\mathcal{R})).$$

Note that, by construction, the resulting map $\overline{D}^c : \mathcal{R} \otimes \mathcal{R} \to \Omega^1(\mathcal{R})$ is the one given by $z \otimes w \mapsto zdw$, for z and w in $\mathcal{R}$, and that the kernel of this map is just the ideal defining the Leibniz rule.

To show that D^c factors through $0 \to \Delta_!(\mathcal{R}') \to \Delta_!(\widetilde{\Omega^c(\mathcal{R})}) \to \Delta_!(\Omega^1(\mathcal{R})) \to 0$, we need to show that the composition of D^c with the map

$$Leib : \Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c))$$

given in 2.2.2. vanishes. Hence we are left with checking that the composition

$$\Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \xrightarrow{Leib} \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)) \xrightarrow{\Delta_!(D^c)} \Delta_!(\Omega^c(\mathcal{R}))$$

is zero. For this, recall that the map *Leib* was constructed using the linear combination $\alpha_1 - \alpha_2 - \alpha_3$ of three maps α_1, α_2 and α_3 from $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$. By looking at the map

$$j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \xrightarrow{\alpha_1 - \alpha_2 - \alpha_3} \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)) \xrightarrow{\Delta_!(D^c)} \Delta_!(\Omega^c(\mathcal{R})),$$

we see that the condition on d^c being a derivation, implies that the above composition vanishes. Indeed $\Delta_!(D^c) \circ \alpha_1$ is given by

$$j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \xrightarrow{\pi \boxtimes id} j_*j^*(\mathcal{R} \boxtimes \mathcal{R}^c) \xrightarrow{id \boxtimes d^c} j_*j^*(\mathcal{R} \boxtimes \Omega^c(\mathcal{R})) \xrightarrow{\mu_{\mathcal{R},\Omega^c(\mathcal{R})}} \Delta_!(\Omega^c(\mathcal{R})).$$

The map $\Delta_!(D^c) \circ \alpha_2$ is given by the above by applying the transposition of variables

σ, whereas the third map

$$j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \xrightarrow{\mu_c} \Delta_!(\mathcal{R}^c) \to \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)) \xrightarrow{D^c} \Delta_!(\Omega^c(\mathcal{R}))$$

is equal to $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \xrightarrow{\mu_c} \Delta_!(\mathcal{R}^c) \xrightarrow{\Delta_!(d^c)} \Delta_!(\Omega^c(\mathcal{R}))$. Therefore the above maps coincide with the terms in the relation $d^c(\mu_c) = \mu_{\mathcal{R},\Omega^c(\mathcal{R})}(\pi, d^c) - \sigma \circ \mu_{\mathcal{R},\Omega^c(\mathcal{R})} \circ \sigma(d^c, \pi)$, and hence $\Delta_!(D^c) \circ \mathit{Leib}$ in zero. Note that the resulting map

$$\Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)) \overset{\pi}{\twoheadrightarrow} \Delta_!(\mathcal{R} \otimes \mathcal{R}) \xrightarrow{\Delta_!(\overline{D^c})} \Delta_!(\Omega^1(\mathcal{R}))$$

induces an isomorphism

$$\Delta_!(\Omega^1(\mathcal{R})) \simeq \Delta_!(\mathcal{R} \otimes \mathcal{R})/\mathrm{Im}(\pi \circ \mathit{Leib}) \xrightarrow{\sim} \Delta_!(\Omega^1(\mathcal{R})).$$

This conclude the proof of the proposition. $\square$

2.3 Quantization of the center $\mathfrak{z}_{crit}$

Recall the commutative chiral algebra $\mathfrak{z}_{crit}$ defined as the center of $\mathcal{A}_{crit}$. As we have mentioned in the introduction, $\mathfrak{z}_{crit}$ can be equipped with a chiral-Poisson structure in the following two equivalent way:

- For any $\hbar \neq 0$ let κ be any non critical level $\kappa = \kappa_{crit} + \hbar\kappa_{kill}$ and denote by $\mathcal{A}_\hbar$ the chiral algebra $\mathcal{A}_\kappa$. Let z and w be elements of $\mathfrak{z}_{crit}$. Let z_κ and w_κ be any two families of elements in $\mathcal{A}_\hbar$ such that $z = z_\kappa$ and $w = w_\kappa$ when $\hbar = 0$. Define the Poisson bracket of z and w to be

$$\{z, w\} = \frac{[z_\kappa, w_\kappa]_{\mathcal{A}_\hbar}}{\hbar} \pmod{\hbar}.$$

We will now introduce the Drinfeld-Sokolov reduction Ψ_X and explain how this functor provides a 1-parameter family of chiral algebras $\{\mathcal{W}_\hbar\} := \{\Psi_X(\mathcal{A}_\hbar)\}$ such that $\mathcal{W}_0 \simeq \mathfrak{z}_{crit}$, that are a quantization of $\mathfrak{z}_{crit}$.

2.3.1 The Drinfeld-Sokolov reduction

We start by recalling the BRST complex for a chiral algebra $\mathcal{A}$ as presented in [BD]. We will also recall the BRST reduction for $\mathcal{A}$-modules over X. Finally, in 2.3.1 we will define the Drifeld-Sokolov reduction Ψ_X as a special case of the above,

$$\Psi_X : \mathcal{A}_\hbar\text{-mod} \to \mathcal{W}_\hbar\text{-mod},$$

and use it to define the quantization $\{\mathcal{W}_\hbar\}$ of $\mathfrak{z}_{crit}$.

The BRST reduction

For a finite dimensional Lie algebra L, consider the Lie*-algebra $\mathfrak{L} := L \otimes \mathcal{D}_X$ over X. As it is explained in [BD], we can construct the *Clifford Chiral algebra* $\mathcal{C}l^\cdot(\mathfrak{L})$ given by

$$\mathcal{C}l^\cdot(\mathfrak{L}) := U(\mathfrak{L}[1] \oplus \mathfrak{L}^*[-1] \oplus \Omega_X)'.$$

We can also consider the PBW-filtration on $\mathcal{C}l^\cdot(\mathfrak{L})$ (e.g. $\mathcal{C}l^\cdot_1(\mathfrak{L}) = \mathfrak{L}[1] \oplus \mathfrak{L}^*[-1] \oplus \Omega_X$) and the adjoint action ad of $\mathfrak{L}$ on itself. We define $\mathfrak{L}^{Tate}$ to be the bull-back of the following short exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_X \arrow[r] & \mathcal{C}l^0_{\leq 2}(\mathfrak{L}) \arrow[r] & \mathcal{C}l^0_{\leq 2}(\mathfrak{L})/\Omega_X \simeq \mathfrak{gl}(\mathfrak{L}) \arrow[r] & 0\,. \\
0 \arrow[r] & \Omega_X \arrow[r] \arrow[u] & \mathfrak{L}^{Tate} \arrow[r] \arrow[u, "a"] & \mathfrak{L} \arrow[r] \arrow[u, "ad"] & 0
\end{tikzcd}$$

We have also an embedding $i : \mathfrak{L}[1] \to \mathcal{C}l^{-1}_1(\mathfrak{L})$. Consider now the natural map $l : \mathfrak{L}^{-Tate} \to U(\mathfrak{L}^{-Tate})'$, where $U(\mathfrak{L}^{-Tate})'$ is the chiral twisted enveloping algebra of $\mathfrak{L}^{-Tate}$. If we denote by $\mathcal{A}^{\cdot}$ the chiral algebra $U(\mathfrak{L}^{-Tate})' \otimes \mathcal{C}l^{\cdot}(\mathfrak{L})$ we get a map

$$Lie := a + l : \mathfrak{L} \to \mathcal{A}^0$$

given by $g \mapsto g \otimes 1 + 1 \otimes a(g)$. If we denote by $\mathfrak{L}_{\dagger}$ the DG Lie*-algebra $Cone(\mathfrak{L} \xrightarrow{id} \mathfrak{L})$, we can view i and Lie as the components of a map of graded Lie*-algebras

$$\phi : \mathfrak{L}_{\dagger} \to \mathcal{A}^{\cdot}.$$

We now define a differential on $\mathcal{A}$ in the following way. As it is explained in [BD] 1.4.10, there is an action of $\mathfrak{L}_{\dagger}$ on the DG-algebra $Sym(\mathfrak{L}^*[-1])$ compatible with the differential δ of $Sym(\mathfrak{L}^*[-1])$ (see [BD] 3.8.9) and one has the following:

Lemma 2.3.1. *The operations*

$$[Lie, id_{\mathfrak{L}^*[-1]}],\ [i, \delta|_{\mathfrak{L}^*[-1]}] : \mathfrak{L} \boxtimes \mathfrak{L}^*[-1] \to \Delta(\mathcal{A}^1)$$

coincide.

The above lemma allows us to define the map χ. In fact it tells us that the map

$$\mu(Lie, id_{\mathfrak{L}^*[-1]}) - \mu(i, \delta|_{\mathfrak{L}^*[-1]}) : j_*j^*(\mathfrak{L} \boxtimes \mathfrak{L}^*[-1]) \to \Delta_*(\mathcal{A}^1[1])$$

vanishes on $\mathfrak{L} \boxtimes \mathfrak{L}^*$, where μ is the chiral operation on $\mathcal{A}^{\cdot}$. Hence we get a map

$$\chi : \mathfrak{L} \otimes \mathfrak{L}^* \to \mathcal{A}^1[1]. \tag{2.14}$$

We have $h(\mathfrak{L} \otimes \mathfrak{L}^*) \simeq \mathrm{End}(\mathfrak{L})$. We define Q to be the image of the identity endomorphism of $\mathfrak{L}$, projected onto $h(\mathcal{A}^1[1])$,

$$Q = \chi(Id_{\mathfrak{L}}) \in h(\mathcal{A}^1[1]).$$

We have the following proposition:

Proposition 2.3.1. *Q is the unique element in $h(\mathcal{A}^1[1])$ such that $[Q, i(g)] = Lie(g)$ for every $g \in \mathfrak{L}$, moreover $[Q,Q] = 0$. In particular if we denote by d the Lie action of Q on $\mathcal{A}^{\cdot}$, then d is a derivation of $\mathcal{A}^{\cdot}$ of degree 1 and square 0 and the map*

$$\phi : \mathfrak{L}_{\dagger} \to (\mathcal{A}^{\cdot}, d)$$

is a map of DG Lie-algebras.*

The complex $BRST(\mathcal{A}) = (\mathcal{A}^{\cdot}, d)$ is called the BRST-reduction of $U(\mathfrak{L}^{-Tate})'$. For a map of chiral algebras $f : U(\mathfrak{L}^{-Tate})' \to \mathcal{R}$, the complex $BRST(\mathcal{A}^{\cdot}_{\mathcal{R}} := \mathcal{R} \otimes \mathcal{C}l^{\cdot}(\mathfrak{L}), d_{\mathcal{R}} := [f(Q), \,])$ is called the BRST reduction of $\mathcal{R}$. The cohomology of $BRST(\mathcal{A}_{\mathcal{R}})$ is called the *semiinfinite cohomology* of $\mathcal{R}$.

BRST reduction for modules

For an $\mathcal{A}^{\cdot}$-chiral module $\mathcal{M}$ on X, the Lie-action $Q_{\mathcal{M}}$ of Q on it is a derivation of square 0 and degree 1. If we denote such derivation by $d_{\mathcal{M}}$, then $(\mathcal{M}, d_M)$ is a $BRST(\mathcal{A}^{\cdot})$-module. In particular we can take an $U(\mathfrak{L}^{-Tate})'$-module $\mathcal{M}$ and consider the $\mathcal{A}^{\cdot}$-module $\mathcal{M} \otimes \mathcal{C}l^{\cdot}(\mathfrak{L})$. Moreover, if we are given a map of chiral algebras $f : U(\mathfrak{L}^{-Tate})' \to \mathcal{R}$ we can do the same construction for any $\mathcal{R}$-module $\mathcal{M}$ and get a $BRST(\mathcal{A}^{\cdot}_{\mathcal{R}})$-module. Hence we obtain a functor:

$$\{\mathcal{R}\text{-modules on } X\} \longrightarrow \{BRST(\mathcal{A}^{\cdot}_{\mathcal{R}})\text{-modules on } X\}. \tag{2.15}$$

Drinfeld-Sokolov reduction

In the above framework, we can take L to be the nilpotent sub-algebra $\mathfrak{n}$ of $\mathfrak{g}$. We denote by $\mathfrak{L}_{\mathfrak{n}}$ the corresponding Lie*-algebra. Given any invariant bilinear form

$\kappa = \kappa_{crit} + \hbar\kappa_{Kill}$ on $\mathfrak{g}$, we see that the natural map

$$\mathfrak{L}_{\mathfrak{n}} \to \mathcal{A}_{\hbar}$$

lifts to a map $\mathfrak{L}_{\mathfrak{n}}^{-tate} \to \mathcal{A}_{\hbar}$, and therefore to a map

$$U(\mathfrak{L}_{\mathfrak{n}}^{-tate})' \xrightarrow{f} \mathcal{A}_{\hbar}.$$

We can consider the complex $BRST(\mathcal{A}_{\hbar} \otimes \mathcal{C}l(\mathfrak{L}_{\mathfrak{n}}))$. This complex, comes equipped with a differential $d = f(Q)$, as explained above. However, to define the *Drinfeld-Sokolov reduction* we will consider a new differential $d_n = d + d_0$, where d_0 is defined in the following way.

Let $\{e_\alpha, \alpha \in \Pi\}$ be a basis of $\mathfrak{n}$, and let χ_0 be the non-degenerate character of $\mathfrak{n}$ given by

$$\chi_0(e_\alpha) = \begin{cases} 1, & \text{if } \alpha \text{ is simple} \\ 0, & \text{otherwise.} \end{cases}$$

As it is explained in [BD] 2.6.8. this defines a map $\overline{\chi}_0 : \mathfrak{L}_{\mathfrak{n}} \to \Omega_X$ that we can regard as an element in $\mathfrak{L}_{\mathfrak{n}}^* \subset \mathcal{C}l(\mathfrak{L}_{\mathfrak{n}}) \to \mathcal{A}_{\hbar} \otimes \mathcal{C}l(\mathfrak{L}_{\mathfrak{n}})$. We define d_0 to be $d_0 := [\overline{\chi}_0, \,]$. Clearly $d_0^2 = 0$, moreover d_0 commutes with the differential d.

Definition 2.3.1. We define the Drinfeld-Sokolov reduction of $\mathcal{A}_{\hbar}$ to be the DG-chiral algebra

$$BRST^{\chi}(\mathcal{A}_{\hbar}) := (\mathcal{A}_{\hbar} \otimes \mathcal{C}l(\mathfrak{L}_{\mathfrak{n}}), d_n = d + d_0).$$

We have the following remarkable theorem, proved in [FB].

Theorem 2.2.
- *For any $\kappa = \kappa_{crit} + \hbar\kappa_{Kill}$, the cohomology of the complex $(\mathcal{A}_{\hbar} \otimes \mathcal{C}l(\mathfrak{L}_{\mathfrak{n}}), d_n = d + d_0)$ is concentrated only in degree zero. Thus, this DG-chiral algebra reduces to a plain chiral algebra $\mathcal{W}_{\hbar} := H^0((\mathcal{A}_{\hbar} \otimes \mathcal{C}l(\mathfrak{L}_{\mathfrak{n}}), d_n = d + d_0))$ called the W-algebra.*

- *The chiral algebra $\mathcal{W}_0$ is isomorphic to the center $\mathfrak{Z}_{crit}$ of $\mathcal{A}_{crit}$. Moreover this gives a quantization of $\mathfrak{Z}_{crit}$ viewed as a chiral-Poisson algebra.*

From now on, we will denote by Ψ_X the functor $BRST^{\chi}$. As we have seen in 2.15, this defines a functor

$$\Psi_X : \mathcal{A}_{\hbar}\text{-mod}_X := \{\mathcal{A}_{\hbar}\text{-mod on } X\} \to \mathcal{W}\text{-mod}_X.$$

We will be interested in $\kappa = \kappa_{crit}$. By the above theorem, for the critical level we can re-write the above functor as

$$\Psi_X := BRTS^{\chi} : \mathcal{A}_{crit}\text{-mod}_X \to \mathfrak{Z}_{crit}\text{-mod}_X. \tag{2.16}$$

2.4 Main theorem

2.4.1 The chiral algebra of twisted differential operators on the loop group

We will now recall the chiral algebra $\mathcal{D}_{\hbar}$ of κ *twisted differential operators on the loop group* as presented in [AG], for $\kappa = \kappa_{crit} + \hbar\kappa_{Kill}$.

Let G be an algebraic group and consider the group scheme $G \times X$ on X. Let $J_X(G)$ be the corresponding $\mathcal{D}_X$-scheme, where, by J_X we denoted the functor

$$J_X : \{\mathcal{O}_X\text{-schemes}\} \to \{\mathcal{D}_X\text{-schemes}\}, \tag{2.17}$$

right adjoint to the forgetful functor

$$For : \{\mathcal{D}_X\text{-schemes}\} \to \{\mathcal{O}_X\text{-schemes}\}.$$

Consider the Lie*-algebra $L_{\mathfrak{g}}$, and the invariant bilinear form $\kappa = \kappa_{crit} + \hbar\kappa_{Kill}$ on $\mathfrak{g}$. The algebra $\mathcal{D}_\hbar$ is constructed using the Lie*-algebra $\mathcal{O}_{J_X(G)} \oplus L_{\mathfrak{g}}$. More precisely, it is defined as

$$\mathcal{D}_{crit} := U(\mathcal{O}_{J_X(G)} \oplus L_{\mathfrak{g}})/I,$$

where I is the ideal in $U(\mathcal{O}_{J_X(G)} \oplus L_{\mathfrak{g}})$ generated by the kernel of the map

$$U(\mathcal{O}_{J_X(G)}) \to J_X(G).$$

As it is explained in [AG], the fiber $(\mathcal{D}_\hbar)_x$ of $\mathcal{D}_\hbar$ at $x \in X$ is isomorphic to

$$(\mathcal{D}_\hbar)_x \simeq U(\hat{\mathfrak{g}}_\kappa) \underset{U(\mathfrak{g}[[t]] \oplus \mathbb{C})}{\otimes} \mathcal{O}_{G[[t]]}.$$

Moreover $\mathcal{D}_\hbar$ comes equipped with two embeddings

$$\mathcal{A}_\hbar \xrightarrow{l_\hbar} \mathcal{D}_\hbar \xleftarrow{r_\hbar} \mathcal{A}_{-\hbar} \tag{2.18}$$

corresponding to left and right invariant vector fields on the loop group $G((t))$. In particular, for $\hbar = 0$, we have $\mathcal{A}_\hbar = \mathcal{A}_{-\hbar} = \mathcal{A}_{crit}$ and $\mathcal{D}_\hbar = \mathcal{D}_{crit}$. Therefore we obtain two different embeddings, $l := l_0$ and $r := r_0$ of $\mathcal{A}_{crit}$ into $\mathcal{D}_{crit}$

$$\mathcal{A}_{crit} \xrightarrow{l} \mathcal{D}_{crit} \xleftarrow{r} \mathcal{A}_{crit}.$$

If we restrict these two embeddings to $\mathfrak{Z}_{crit}$, as it is explained in [FG] Theorem 5.4, we have

$$l(\mathfrak{Z}_{crit}) = l(\mathcal{A}_{crit}) \cap r(\mathcal{A}_{crit}) = r(\mathfrak{Z}_{crit}).$$

Moreover the two compositions

$$\mathfrak{Z}_{crit} \hookrightarrow \mathcal{A}_{crit} \xrightarrow{l} \mathcal{D}_{crit} \xleftarrow{r} \mathcal{A}_{crit} \hookleftarrow \mathfrak{Z}_{crit}$$

are intertwined by the automorphism $\eta : \mathfrak{Z}_{crit} \to \mathfrak{Z}_{crit}$ given by the involution of the Dynkin diagram that sends a weight λ to $-w_0(\lambda)$ (i.e. when restricted to $\mathfrak{Z}_{crit}$ we have $l = r \circ \eta$).

The two embedding l and r of $\mathcal{A}_{crit}$ into $\mathcal{D}_{crit}$ endow the fiber $(\mathcal{D}_{crit})_x$ with a structure of $\widehat{\mathfrak{g}}_{crit}$-bimodule. The fiber can therefore be decomposed according to these actions as explained in the introduction.

These decompositions coincide up to the involution η and we have

$$(\mathcal{D}_{crit})_x = \bigoplus_{\lambda \, dominant} (\mathcal{D}_{crit})_x^\lambda,$$

where $(\mathcal{D}_{crit})_x^\lambda$ is the direct summand supported on the formal completion of $\mathrm{Spec}(\mathfrak{z}_{crit}^\lambda)$. Recall that we denote by $\mathcal{D}^0_{crit}$ the chiral algebra corresponding to $(\mathcal{D}_{crit})_x^0$. The embeddings l and r give $\mathcal{D}^0_{crit}$ a structure of $\mathcal{A}_{crit}$-bimodule, and we denote by $\mathcal{C}^0_{crit}$ the resulting chiral algebra

$$\mathcal{C}^0_{crit} := (\Psi_X \boxtimes \Psi_X)(\mathcal{D}^0_{crit}).$$

The following theorem relates the chiral algebra $\mathcal{D}^0_{crit}$ to the extension of $\Omega^1(\mathfrak{Z}_{crit})$ arising from the quantization of $\mathfrak{Z}_{crit}$ given by the 1-parameter deformation $\{\mathcal{W}_\hbar\} = \{\Psi_X(\mathcal{A}_\hbar)\}$ defined by 2.2, under the equivalence of theorem 2.1. This is the main result of this chapter.

Theorem 2.3. *The chiral envelope $U(\Omega^c(\mathfrak{Z}_{crit}))$ of the extension*

$$0 \to \mathfrak{Z}_{crit} \to \Omega^c(\mathfrak{Z}_{crit}) \to \Omega(\mathfrak{Z}_{crit}) \to 0,$$

given by the quantization $\{\mathcal{W}_\hbar := \Psi_X(\mathcal{A}_\hbar)\}$ of the center $\mathfrak{Z}_{crit}$, is isomorphic to the chiral algebra $\mathcal{C}^0_{crit}$.

2.4.2 Proof of the theorem

The proof of Theorem 2.3 will be organized as follows: we will give an alternative formulation of the Theorem that consists in finding a map F from $\Omega^c(\mathfrak{Z}_{crit})$ to $\mathcal{C}^0_{crit}$ with some particular properties. The definition of the above map will rely on the explicit construction of the chiral extension $\Omega^c(\mathfrak{Z}_{crit})$ that was given in Section 2.2.2. In Section 2.4.3 we will finally define the map F and conclude the proof of the Theorem.

Reformulation of the Theorem

We will show how to prove Theorem 2.3 assuming the existence of a map F from $\Omega^c(\mathfrak{Z}_{crit})$ to $\mathcal{C}^0_{crit}$. In order to do so, we will use the fact that both $U(\Omega^c(\mathfrak{Z}_{crit}))$ and $\mathcal{C}^0_{crit}$ can be equipped with filtrations as explained below.
The chiral algebra $U(\Omega^c(\mathfrak{Z}_{crit}))$, being the chiral envelope of the extension

$$0 \to \mathfrak{Z}_{crit} \to \Omega^c(\mathfrak{Z}_{crit}) \to \Omega^1(\mathfrak{Z}_{crit}) \to 0,$$

has its standard *Poincaré-Birkhoff-Witt filtration.* In fact, more generally, given a chiral-extension $(\mathcal{R}, \mathcal{C}, \mathcal{L}^c, \mathcal{L})$, using the notations from Definition 2.1.6, we can define a PBW filtration on $U(\mathcal{C}, \mathcal{L}^c)$ as the filtration generated by $U(\mathcal{C}, \mathcal{L}^c)_0 := \phi^c(\mathcal{C})$ and

$$U(\mathcal{C}, \mathcal{L}^c)_1 := \mathrm{Im}(j_* j^*(\mathcal{L}^c \boxtimes \mathcal{C}) \xrightarrow{\phi^c \boxtimes \phi^c|_{\mathcal{C}}}$$
$$\to j_* j^*(U(\mathcal{C}, \mathcal{L}^c) \boxtimes U(\mathcal{C}, \mathcal{L}^c)) \to \Delta_!(U(\mathcal{C}, \mathcal{L}^c))).$$

Moreover, according to [BD] 3.9.11. we have the folloing.

Theorem 2.4. *If $\mathcal{R}$ and $\mathcal{C}$ are $\mathcal{O}_X$ flat and $\mathcal{L}$ is a flat $\mathcal{R}$-module then we have an* *isomorphism*

$$\mathcal{C} \underset{\mathcal{R}}{\otimes} Sym^{\cdot}_{\mathcal{R}}\mathcal{L} \xrightarrow{\simeq} gr_{\cdot}U(\mathcal{C}, \mathcal{L}^c).$$

By applying the above to the case where $\mathcal{C} = \mathcal{R} = \mathfrak{Z}_{crit}$ and the extension of $\mathcal{L} = \Omega^1(\mathfrak{Z}_{crit})$ given by $\mathcal{L}^c = \Omega^c(\mathfrak{Z}_{crit})$ we get

$$\mathrm{gr}_{\cdot}U(\Omega^c(\mathfrak{Z}_{crit})) \simeq \mathrm{Sym}^{\cdot}_{\mathfrak{Z}_{crit}}\Omega^1(\mathfrak{Z}_{crit}).$$

The filtration on $\mathcal{C}^0_{crit}$ is defined using the functor Ψ_X from 2.3.1.

Recall that, for any central charge $\kappa = \hbar\kappa_{kill} + \kappa_{crit}$, the functor Ψ_X assigns to a chiral $\mathcal{A}_\hbar$-module a $\Psi_X(\mathcal{A}_\hbar) = \mathcal{W}_\hbar$-module. In particular, for every chiral algebra $\mathcal{B}$, and every morphism of chiral algebras $\phi : \mathcal{A}_\hbar \to \mathcal{B}$ we have

$$\Psi_X : \left\{ \begin{array}{c} \text{chiral algebra morphism} \\ \phi : \mathcal{A}_\hbar \to \mathcal{B} \end{array} \right\} \to \left\{ \begin{array}{c} \text{chiral algebra morphism} \\ \Psi_X(\phi) : \mathcal{W}_\hbar \to \Psi_X(\mathcal{B}) \end{array} \right\}.$$

Moreover recall that for $\hbar = 0$ we have $\Psi_X(\mathcal{A}_{crit}) \simeq \mathfrak{Z}_{crit}$.

As it is explained in [FG], the chiral algebra $\mathcal{C}^0_{crit}$ can be described as

$$(\Psi_X \boxtimes \Psi_X)(U(\mathcal{C}, \mathcal{L}^c)) \xrightarrow{\sim} \mathcal{C}^0_{crit} = (\Psi_X \boxtimes \Psi_X)(\mathcal{D}^0_{crit}),$$

for some particular chiral algebra $\mathcal{C}$ and chiral extension $\mathcal{L}^c$. Hence it carries a canonical filtration induced by the PBW-filtration on $U(\mathcal{C}, \mathcal{L}^c)$. We will recall below the definitions of the chiral algebra $\mathcal{C}$ and the chiral extension $\mathcal{L}^c$.

THE RENORMALIZED CHIRAL ALGEBROID. Recall that [FG] Proposition 4.5. shows the existence of a chiral extension $\mathcal{A}^{ren,\tau}$ that fits into the following exact sequence

$$0 \to (\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}) \to \mathcal{A}^{ren,\tau} \to \Omega^1(\mathfrak{Z}_{crit}) \to 0,$$

which is a chiral extension of $(\mathcal{A}_{crit} \underset{\mathfrak{z}_{crit}}{\otimes} \mathcal{A}_{crit})$ in the sense we introduced in Definition 2.1.5. In particular, if we consider the chiral envelope $U((\mathcal{A}_{crit} \underset{\mathfrak{z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau})$, by Theorem 2.4 we have

$$\begin{aligned} &\mathrm{gr}_{.}(U((\mathcal{A}_{crit} \underset{\mathfrak{z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau})) \simeq \\ &\simeq (\mathcal{A}_{crit} \underset{\mathfrak{z}_{crit}}{\otimes} \mathcal{A}_{crit}) \underset{\mathfrak{z}_{crit}}{\otimes} \mathrm{Sym}^{\cdot}_{\mathfrak{z}_{crit}}(\Omega^1(\mathfrak{z}_{crit})). \end{aligned}$$

The chiral envelope $U((\mathcal{A}_{crit} \underset{\mathfrak{z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau})$ is closely related to the chiral algebra $\mathcal{D}^0_{crit}$, in fact in [FG] the following is proved:

Theorem 2.5. *We have an embedding G of the chiral extension $\mathcal{A}^{ren,\tau}$ into $\mathcal{D}_{crit}$ such that the maps l and r are the compositions of this embedding with the canonical maps*

$$\mathcal{A}_{crit} \rightrightarrows (\mathcal{A}_{crit} \underset{\mathfrak{z}_{crit}}{\otimes} \mathcal{A}_{crit}) \xrightarrow{G} \mathcal{A}^{ren,\tau}.$$

The embedding extends to a homomorphism of chiral algebras

$$U((\mathcal{A}_{crit} \underset{\mathfrak{z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau}) \to \mathcal{D}_{crit}$$

and the latter is an isomorphism into $\mathcal{D}^0_{crit}$.

Therefore we see that $\mathcal{C}^0_{crit}$ is given by applying the functor $\Psi_X \boxtimes \Psi_X$ to the chiral envelope $U(\mathcal{C}, \mathcal{L}^c)$, for

$$\mathcal{C} = (\mathcal{A}_{crit} \underset{\mathfrak{z}_{crit}}{\otimes} \mathcal{A}_{crit}), \text{ and } \mathcal{L}^c = \mathcal{A}^{ren,\tau}.$$

In particular, since the functor Ψ_X is exact, we obtain a filtration on $\mathcal{C}^0_{crit}$ induced from the PBW-filtration on $U(\mathcal{A}_{crit} \otimes \mathcal{A}_{crit}, \mathcal{A}^{ren,\tau})$ such that

$$\mathrm{Sym}^{\cdot}_{\mathfrak{z}_{crit}}\Omega^1(\mathfrak{z}_{crit}) \xrightarrow{\sim} \mathrm{gr}_{.}\mathcal{B}^0,$$

where we used the fact that $\Psi_X(\mathcal{A}_{crit}) \simeq \mathfrak{Z}_{crit}$.

Remark 2.4.1. Note that if we apply the functor Ψ_X to the two embeddings in (2.18), we obtain two embeddings

$$\mathcal{W}_\hbar \xrightarrow{l_\hbar} (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar) \xleftarrow{r_\hbar} \mathcal{W}_{-\hbar}$$

such that $l := l_0 = r_0 \circ \eta =: r \circ \eta$, where we are denoting simply by $l_\hbar$ and $r_\hbar$ the maps $\Psi_X(l_\hbar)$ and $\Psi_X(r_\hbar)$ respectively. In particular, for $\hbar = 0$, we obtain two embeddings l and r of $\mathfrak{Z}_{crit}$ into $(\Psi_X \boxtimes \Psi_X)(\mathcal{D}_{crit})$ that differs by η. Moreover the image of the two maps lands in $\mathcal{B}^0$, therefore we obtain two embeddings

$$\mathfrak{Z}_{crit} \xrightarrow{l} \mathcal{B}^0 \xleftarrow{r} \mathfrak{Z}_{crit}.$$

From the above construction it is clear that $\mathfrak{Z}_{crit}$ corresponds to the 0-th part of the filtration defined on $\mathcal{B}^0$. Moreover, by the definition of the map G from Theorem 2.5 (see [FG]), the embedding $\mathfrak{Z}_{crit} \hookrightarrow \mathcal{B}^0$ induced by the inclusion $(\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}) \hookrightarrow U((\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau})$ under $\Psi_X \boxtimes \Psi_X$, coincides with l.

Suppose now that we are given a map $F : \Omega^c(\mathfrak{Z}_{crit}) \to \mathcal{C}^0_{crit}$ satisfying the conditions stated in Definition 2.1.6. By the universal property of the chiral envelope, we automatically get a map

$$U(\Omega^c(\mathfrak{Z}_{crit})) \to \mathcal{C}^0_{crit}.$$

Clearly not every such map will induce an isomorphism between the two chiral algebras. Theorem 2.3 can be reformulated as saying that there exists a map as above, that gives rise to an isomorphism $U(\Omega^c(\mathfrak{Z}_{crit})) \xrightarrow{\sim} \mathcal{C}^0_{crit}$. More precisely we have the following:

Theorem 2.6. *There exists a map $F : \Omega^c(\mathfrak{z}_{crit}) \to (\mathcal{C}^0_{crit})_1 \hookrightarrow \mathcal{C}^0_{crit}$ compatible with the $\mathfrak{z}_{crit}$ structure on both sides that restricts to the embedding l of chiral algebras on $\mathfrak{z}_{crit}$ such that the following diagram commutes*

$$\begin{tikzcd} \Omega^c(\mathfrak{z}_{crit})/\mathfrak{z}_{crit} \arrow[rr, "F"] & & (\mathcal{C}^0_{crit})_1/\mathfrak{z}_{crit}. \\ & \Omega^1(\mathfrak{z}_{crit}) \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

We will now show how Theorem 2.3 follows from Theorem 2.6. The proof Theorem 2.6 will occupy the rest of the article.

Proof of (Theorem 2.6 $\Rightarrow$ Theorem 2.3). To prove Theorem 2.3 we need to show that the above F induces an isomorphism $U(\Omega^c(\mathfrak{z}_{crit})) \xrightarrow{\sim} \mathcal{C}^0_{crit}$. This amounts to showing that the following diagram commutes for every i:

$$\begin{tikzcd} \mathrm{gr}_{i+1}U(\Omega^c(\mathfrak{z}_{crit})) \arrow[rr, "F"] & & \mathrm{gr}_{i+1}\mathcal{C}^0_{crit}. \\ & \mathrm{Sym}^{i+1}_{\mathfrak{z}_{crit}}\Omega^1(\mathfrak{z}_{crit}) \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] & \end{tikzcd} \tag{2.19}$$

But this follows from the fact that the above filtrations are generated by their first two terms. In fact, more generally, for any chiral envelope $U(\mathcal{L}^c)$, we have

$$\Delta_!(\mathrm{gr}_{i+1}U(\mathcal{L}^c)) :=$$

$$= \mathrm{Im}\begin{pmatrix} j_*j^*(U(\mathcal{L}^c)_1 \boxtimes U(\mathcal{L}^c)_i) \to \\ \Delta_!(U(\mathcal{L}^c)) \end{pmatrix} \Big/ \mathrm{Im}\begin{pmatrix} j_*j^*(U(\mathcal{L}^c)_1 \boxtimes U(\mathcal{L}^c)_{i-1}) \to \\ \Delta_!(U(\mathcal{L}^c)) \end{pmatrix}.$$

It is not hard to see that the isomorphism $\mathrm{Sym}^{i+1}_{\mathfrak{z}_{crit}}\Omega^1(\mathfrak{z}_{crit}) \xrightarrow{\sim} \mathrm{gr}_{i+1}U(\Omega^c(\mathfrak{z}_{crit}))$

(and similarly for $\mathcal{C}^0_{crit}$) is the one induced by the map

$$j_*j^*(\Omega^1(\mathfrak{z}_{crit}) \boxtimes \mathrm{Sym}^i_{\mathfrak{z}_{crit}}) \to \Delta_!(\mathrm{gr}_{i+1} U(\Omega^c(\mathfrak{z}_{crit}))),$$

that in fact vanishes when restricted to $\Omega^1(\mathfrak{z}_{crit}) \boxtimes \mathrm{Sym}^i_{\mathfrak{z}_{crit}}$, and factors through the action of $\mathfrak{z}_{crit}$. Therefore the diagram (2.19) commutes by induction on i. $\square$

2.4.3 Construction of the map F

Recall that Theorem 2.6 amounts to the construction of a map of Lie* algebras $F : \Omega^c(\mathfrak{z}_{crit}) \to \mathcal{C}^0_{crit}$ compatible with the $\mathfrak{z}_{crit}$-structure on both sides and such that:

1. F restricts to the embedding l (given in Remark 2.4.1) on $\mathfrak{z}_{crit}$.

2. The following diagram commutes:

$$\begin{tikzcd}
\Omega^c(\mathfrak{z}_{crit})/\mathfrak{z}_{crit} \arrow[rr, "F"] & & (\mathcal{C}^0_{crit})_1/\mathfrak{z}_{crit}. \\
& \Omega^1(\mathfrak{z}_{crit}) \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

Remark 2.4.2. Since $\Delta_!(\Omega^c(\mathfrak{z}_{crit}))$ was constructed as a quotient of $\Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit}))$ and since, by definition,

$$\Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) = \Delta_!(\mathfrak{z}_{crit}) \oplus j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit})/\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit},$$

to construct any map F from $\Omega^c(\mathfrak{z}_{crit})$ to $\mathcal{C}^0_{crit}$ we can proceed as follows:

- first we construct a map $f : \mathfrak{z}^c_{crit} \to \mathcal{C}^0_{crit}$.

- Using the chiral bracket μ' on $\mathcal{C}^0_{crit}$ we consider the composition

$$j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit}) \xrightarrow{l \boxtimes f} j_*j^*(\mathcal{C}^0_{crit} \boxtimes \mathcal{C}^0_{crit}) \xrightarrow{\mu'} \Delta_!(\mathcal{C}^0_{crit}).$$

  This composition yields a map

$$\hat{F} : \Delta_!(\mathfrak{z}_{crit}) \oplus j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit}) \to \Delta_!(\mathcal{C}^0_{crit}),$$

  by sending $\Delta_!(\mathfrak{z}_{crit})$ to $\mathcal{C}^0_{crit}$ via $\Delta_!(l)$.

- We check that the above map factors through a map

$$\widetilde{F} : \Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) \to \Delta_!(\mathcal{C}^0_{crit}).$$

- We check that in fact if factors through $\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) \to \Delta_!(\mathcal{C}^0_{crit})$.

- We verify that the relations defining $\Delta_!(\Omega^c(\mathfrak{z}_{crit}))$ as a quotient of $\Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit}))$ are satisfied, i.e. that $\overline{F}$ gives the desired map F from $\Omega(\mathfrak{z}_{crit})^c$ to $\mathcal{C}^0_{crit}$ under the Kashiwara equivalence.

Remark 2.4.3. Note that any map F constructed as before, automatically satisfies the first condition in 2.4.3, hence to prove Theorem 2.6, once the map f is defined, we only have to verify that condition (2) in 2.4.3 is satisfied, i.e. that the diagram above commutes.

Definition of the map f.

We will now define the map $f : \mathfrak{z}^c_{crit} \to \mathcal{C}^0_{crit}$ and hence, according to the first two points in 2.4.2, the map $\hat{F} : \Delta_!(\mathfrak{z}_{crit}) \oplus j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit}) \to \Delta_!(\mathcal{C}^0_{crit})$. Assuming that it factors through a map $F : \Omega^c(\mathfrak{z}_{crit}) \to \mathcal{C}^0_{crit}$, we will then show that it satisfies the second condition in 2.4.3. This will conclude the proof of Theorem

2.6. The poof that it factors through $\Omega^c(\mathfrak{Z}_{crit})$ (which amounts to the proof of the remaining last three points in 2.4.2) will be postponed until 2.4.4. To define the map $f : \mathfrak{Z}^c_{crit} \to \mathfrak{C}^0_{crit}$ we will use the following three facts:

1. There exist two embeddings

$$\mathfrak{Z}_{crit} \xrightarrow{l} \mathfrak{C}^0_{crit} \xleftarrow{r} \mathfrak{Z}_{crit}$$

constructed by applying the functor Ψ_X to the two embeddings in (2.18). In fact, by doing it, we obtain two maps

$$\mathcal{W}_\hbar \xrightarrow{l_\hbar} (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar) \xleftarrow{r_\hbar} \mathcal{W}_{-\hbar}$$

such that $l := l_0 = r_0 \circ \eta =: r \circ \eta$, where we are denoting by $l_\hbar$ and $r_\hbar$ the maps $\Psi_X(l_\hbar)$ and $\Psi_X(r_\hbar)$ respectively. The two embedding of $\mathfrak{Z}_{crit}$ correspond to the above maps when $\hbar = 0$.

2. There is a well defined map

$$e : \mathcal{W}_\hbar \to \mathcal{W}_{-\hbar}.$$

In fact, since $\mathcal{W}_\hbar = \Psi_X(\mathcal{A}_\hbar)$, and since $\mathcal{A}_{-\hbar}$ is isomorphic to $\mathcal{A}_\hbar$ as vector space with the action of $\mathbb{C}[\hbar]$ modified to $\hbar \cdot a = -\hbar a$, $a \in \mathcal{A}_{-\hbar}$, we can consider the map $\mathcal{W}_\hbar \to \mathcal{W}_{-\hbar}$ that simply sends $\hbar$ to $-\hbar$.

3. The involution $\eta : \mathfrak{Z}_{crit} \to \mathfrak{Z}_{crit}$ can be extended to a map $\eta : \mathcal{W}_\hbar \to \mathcal{W}_\hbar$ by setting $\eta(h) = h$.

We define f in the following way: for every $z_\hbar \in \mathfrak{Z}^c_{crit} = \mathcal{W}_\hbar/\hbar^2\mathcal{W}_\hbar$ we set

$$f(z_\hbar) = \frac{1}{2}\frac{l_\hbar(z_\hbar) - r_\hbar(\eta(e(z_\hbar)))}{\hbar} \pmod{\hbar}.$$

This is a well defined element in $\mathcal{C}^0_{crit}$ because $l_0 = r_0 \circ \tau$, i.e. the numerator vanishes mod $\hbar$.

Assuming the proposition below, we will now show that the resulting F satisfies condition (2) of 2.4.3, which, according to Remark 2.4.3, concludes the proof of Theorem 2.6. Proposition 2.4.1 will be proved later in 2.4.4.

Proposition 2.4.1. *The map*

$$\hat{F} : \Delta_!(\mathfrak{Z}_{crit}) \oplus j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \to \Delta_!(\mathcal{C}^0_{crit}),$$

obtained by using $f : \mathfrak{Z}^c_{crit} \to \mathcal{C}^0_{crit}$ from above, factors through a map $F : \Delta_!(\Omega^c(\mathfrak{Z}_{crit})) \to \Delta_!(\mathcal{C}^0_{crit})$.

End of the proof of Theorem 2.6

Proof. We are now ready to finish the proof of Theorem 2.6, which, according to Remark 2.4.3, amounts to check that

$$\begin{tikzcd}
\Omega^c(\mathfrak{Z}_{crit})/\mathfrak{Z}_{crit} \arrow[rr, "F"] & & (\mathcal{C}^0_{crit})_1/\mathfrak{Z}_{crit} \\
& \Omega^1(\mathfrak{Z}_{crit}) \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

commutes. In order to do so, we will show that it commutes when composed with the map $d : \mathfrak{Z}_{crit} \to \Omega^1(\mathfrak{Z}_{crit})$. By looking at the composition

$$\mathfrak{Z}_{crit} \xrightarrow{d} \Omega^1(\mathfrak{Z}_{crit}) \to \Omega^c(\mathfrak{Z}_{crit})/\mathfrak{Z}_{crit} \xrightarrow{F} (\mathcal{C}^0_{crit})_1/\mathfrak{Z}_{crit},$$

we see that, for $z \in \mathfrak{Z}_{crit}$, the resulting map is

$$z \mapsto \frac{1}{2} \frac{l_\hbar(z_\hbar) - r_\hbar(\eta(e(z_\hbar)))}{\hbar} \pmod{\hbar}, \tag{2.20}$$

where $z_\hbar$ is any lifting of z to $\mathfrak{Z}^c_{crit}$. Note that this map is well defined only after taking the quotient of $\mathcal{C}^0_{crit}$ by $\mathfrak{Z}_{crit}$.

For the other composition, we first need to recall how the isomorphism $\Omega^1(\mathfrak{Z}_{crit}) \xrightarrow{\sim} (\mathcal{C}^0_{crit})_1/\mathfrak{Z}_{crit}$ was constructed. Recall from 2.4.2 that the filtration on $\mathcal{C}^0_{crit}$ is the one induced (under $\Psi_X \boxtimes \Psi_X$) from the isomorphism G given in Theorem 2.5. Therefore the isomorphism above is the one corresponding to the composition

$$\begin{aligned}(\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}) \underset{\mathfrak{Z}_{crit}}{\otimes} \Omega^1(\mathfrak{Z}_{crit}) \quad &\xrightarrow{\sim} U(\mathcal{A}^{ren,\tau})_1/(\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}) \xrightarrow{G} \\ &\to \mathcal{D}^0_{crit}/l(\mathcal{A}_{crit}) + r(\mathcal{A}_{crit})\end{aligned}$$

under $(\Psi_X \boxtimes \Psi_X)$ (here, for simplicity, we are denoting the chiral envelope $U((\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau})$ by $U(\mathcal{A}^{ren,\tau})$). If we consider the inclusion of $\Omega^1(\mathfrak{Z}_{crit})$ followed by the first arrow from above, it is clear that the image in $U(\mathcal{A}^{ren,\tau})/(\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit})$ is $G[[t]] \times G[[t]]$ invariant. In particular it means that the image of $\Omega^1(\mathfrak{Z}_{crit})$ maps to $(\Psi_X \boxtimes \Psi_X)(U(\mathcal{A}^{ren,\tau}))/\mathfrak{Z}_{crit}$. Now, by looking at the definition of the map G (see [FG] 5.5.), we see that the the map

$$\mathfrak{Z}_{crit} \xrightarrow{d} \Omega^1(\mathfrak{Z}_{crit}) \to (\Psi_X \boxtimes \Psi_X)(U(\mathcal{A}^{ren,\tau}))_1/\mathfrak{Z}_{crit} \xrightarrow{(\Psi_X \boxtimes \Psi_X)(G)} (\mathcal{C}^0_{crit})_1/\mathfrak{Z}_{crit},$$

is indeed given by (2.20). This completes the proof of Theorem 2.3. $\square$

We will now give the proof of Proposition 2.4.1, which will occupy the rest of the chapter.

2.4.4 Proof of Proposition 2.4.1

Recall that the proof of Proposition 2.4.1 consists in showing the following:

1. the map $\hat{F} : \Delta_!(\mathfrak{z}_{crit}) \oplus j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit}) \to \Delta_!(\mathcal{C}^0_{crit})$ factors through a map

$$\widetilde{F} : \Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) \to \Delta_!(\mathcal{C}^0_{crit}).$$

2. the map $\widetilde{F}$ factors through $\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) \to \Delta_!(\mathcal{C}^0_{crit})$.

3. The relations defining $\Delta_!(\Omega^c(\mathfrak{z}_{crit}))$ as a quotient of $\Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit}))$ are satisfied, i.e. $\overline{F}$ gives the desired map F from $\Omega(\mathfrak{z}_{crit})^c$ to $\mathcal{C}^0_{crit}$ under the Kashiwara equivalence.

For this we will need the following Lemma.

Lemma 2.4.1. *The composition*

$$\mathcal{W}_\hbar \boxtimes \mathcal{W}_{-\hbar} \xrightarrow{l_\hbar \boxtimes r_\hbar} (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar) \boxtimes (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar) \xrightarrow{\mu'} \Delta_!((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar))$$

is zero.

Proof. In [FG] Lemma 5.2 it is shown that the composition

$$\mathcal{A}_\hbar \boxtimes \mathcal{A}_{-\hbar} \xrightarrow{l_\hbar \boxtimes r_\hbar} \mathcal{D}_\hbar \boxtimes \mathcal{D}_\hbar \xrightarrow{\mu'} \Delta_!(\mathcal{D}_\hbar)$$

is zero. In other words the two embeddings centralize each other. The Lemma then, immediately follows by applying the functor $(\Psi_X \boxtimes \Psi_X)$.

$\square$

Proof of (1).To prove that $\hat{F}$ factors through

$$\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) \to \Delta_!(\mathcal{C}^0_{crit})$$

we use Lemma 2.4.1. Recall that we defined f from $\mathfrak{z}^c_{crit}$ to $\mathcal{C}^0_{crit}$ to be

$$f(z_\hbar) = \frac{1}{2}\frac{l_\hbar(z_\hbar) - r_\hbar(\eta(e(z_\hbar)))}{\hbar} \pmod{\hbar}.$$

Because of the above Lemma, it is clear that, when we consider the inclusion $\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit} \hookrightarrow j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit})$ and the composition with the map to $\Delta_!(\mathcal{C}^0_{crit})$, the resulting map factors as:

$$\begin{tikzcd}
\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit} \arrow[r] \arrow[d] & j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit}) \arrow[r] & \Delta_!(\mathcal{C}^0_{crit}) , \\
\mathfrak{z}_{crit} \boxtimes \mathfrak{z}_{crit} \arrow[r, "\frac{1}{2}\{\,,\}"] & \Delta_!(\mathfrak{z}_{crit}) \arrow[ur, "\Delta_!(l)"'] &
\end{tikzcd}$$

which implies that the map factors through a map $\widetilde{F} : \Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) \to \Delta_!(\mathcal{C}^0_{crit})$.

Remark 2.4.4. Note that when we restrict the map $f : \mathfrak{z}^c_{crit} \to \mathcal{C}^0_{crit}$ to $\mathfrak{z}_{crit} \xrightarrow{\cdot\hbar} \mathfrak{z}^c_{crit}$, because of the flip from $\hbar$ to $-\hbar$ in the definition of e, we simply obtain the inclusion $\mathfrak{z}_{crit} \xrightarrow{l} \mathcal{C}^0_{crit}$.

Proof of (2). Now we want to check that the relations defining $\Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit}))$ as a quotient of $\Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit}))$ are satisfied, i.e. that $\widetilde{F}$ factors through a map $\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) \to \Delta_!(\mathcal{C}^0_{crit})$.

First of all, recall that to pass from $\Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit}))$ to $\Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit}))$ we took the quotient by the image of the difference of two maps from $\Delta_!(\mathfrak{z}_{crit} \otimes \mathfrak{z}_{crit})$ to $\Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit}))$. The first map was given by

$$\Delta_!(\mathfrak{z}_{crit} \otimes \mathfrak{z}_{crit}) \xrightarrow{m} \Delta_!(\mathfrak{z}_{crit}) \to \Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})), \tag{2.21}$$

while the second map was induced by the composition

$$j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}_{crit}) \xrightarrow{id\boxtimes\cdot\hbar} j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit}) \twoheadrightarrow \Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})),$$

which vanishes on $\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit} \hookrightarrow j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit})$. When we compose the map (2.21) with $\widetilde{F}$, we get $l \circ m = \mu' \circ (l \boxtimes l)$. However when we compose the second map with

$$j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \xrightarrow{l\boxtimes f} j_*j^*(\mathcal{C}^0_{crit} \boxtimes \mathcal{C}^0_{crit}) \xrightarrow{\mu'} \Delta_!(\mathcal{C}^0_{crit}),$$

because of Remark 2.4.4, we see that this map corresponds to $\mu' \circ (l \boxtimes l)$ hence the difference of the images goes to zero under $\widetilde{F}$.

Proof of (3). Now we are left with checking that $\widetilde{F}$ factors through

$$\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \to \Delta_!(\Omega^c(\mathfrak{Z}_{crit})).$$

This will occupy the rest of the article. Recall that $\Delta_!(\Omega^c(\mathcal{R}))$ was given as a quotient of $\Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit}))$ by the map *Leib*. The Leibniz relation was given as the image of a map

$$\Delta_!(\mathfrak{Z}^c_{crit} \otimes \mathfrak{Z}^c_{crit}) \to \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit}))$$

and this map was the sum of three maps, α_1, α_2 and α_3, from $j_*j^*(\mathfrak{Z}^c_{crit} \boxtimes \mathfrak{Z}^c_{crit})$ which vanished on $\mathfrak{Z}^c_{crit} \boxtimes \mathfrak{Z}^c_{crit}$. Hence we want to check that the composition

$$\Delta_!(\mathfrak{Z}^c_{crit} \otimes \mathfrak{Z}^c_{crit}) \xrightarrow{Leib} \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \xrightarrow{\overline{F}} \Delta_!(\mathcal{C}^0_{crit})$$

vanishes. Instead of considering the map from $\Delta_!(\mathfrak{Z}^c_{crit} \otimes \mathfrak{Z}^c_{crit})$ we can consider the three maps

$$j_*j^*(\mathfrak{Z}^c_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \overset{\alpha_1}{\underset{\alpha_3}{\overset{\alpha_2}{\rightrightarrows}}} \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \xrightarrow{\overline{F}} \Delta_!(\mathcal{C}^0_{crit}), \tag{2.22}$$

and show that the composition $\overline{F} \circ (\alpha_1 - \alpha_2 - \alpha_3)$ is zero. Recall that the first map, α_1, was given by projecting onto $j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit})$, i.e.

$$j_*j^*(\mathfrak{z}^c_{crit} \boxtimes \mathfrak{z}^c_{crit}) \longrightarrow j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit}) \xrightarrow[\beta]{} \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) \;,$$

where β denotes the second component of the projection

$$\Delta_!(\mathfrak{z}_{crit}) \oplus j_*j^*(\mathfrak{z}_{crit} \boxtimes \mathfrak{z}^c_{crit}) \twoheadrightarrow \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})).$$

The second map, α_2, was given by $\sigma \circ \alpha_1 \circ \sigma$, and the third map α_3 was given by the composition

$$\begin{aligned} j_*j^*(\mathfrak{z}^c_{crit} \boxtimes \mathfrak{z}^c_{crit}) \xrightarrow{\mu_c} \Delta_!(\mathfrak{z}^c_{crit}) \xrightarrow{\bar{i}} \Delta_!(\mathrm{Ind}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) \twoheadrightarrow \\ \twoheadrightarrow \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})). \end{aligned}$$

When we compose α_3 with the map $\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{z}_{crit}}(\mathfrak{z}^c_{crit})) \to \Delta_!(\mathcal{B}^0)$, it is easy to see that the unit axiom implies that the composition is equal to

$$j_*j^*(\mathfrak{z}^c_{crit} \boxtimes \mathfrak{z}^c_{crit}) \xrightarrow{\mu_c} \Delta_!(\mathfrak{z}^c_{crit}) \xrightarrow{\Delta_!(f)} \Delta_!(\mathcal{C}^0_{crit}). \tag{2.23}$$

Now consider the chiral algebra $(\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)$ and denote by $\mu'_\hbar$ its chiral operation. Consider the map

$$f_\hbar : \mathcal{W}_\hbar \to (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)$$
$$f_\hbar(z_\hbar) = \frac{1}{2}\frac{l_\hbar(z_\hbar) - r_\hbar(\eta(e(z_\hbar)))}{\hbar} \in (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_{g,\hbar}),$$

(i.e. we are not taking this element $\pmod{\hbar}$). It is clear that the three maps α''_1, α''_2 and α''_3 given by

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{l_\hbar \boxtimes f_\hbar} j_*j^*((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar) \boxtimes (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)) \xrightarrow{\mu'_\hbar}$$

$$\xrightarrow{\mu'_\hbar} \Delta_!((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)) \to \Delta_!(\mathcal{C}^0_{crit}), \qquad (\alpha''_1)$$

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{(r_\hbar \circ \eta \circ e \boxtimes f_\hbar)\circ\sigma} j_*j^*((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar) \boxtimes (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar) \xrightarrow{\sigma\circ\mu'_\hbar}$$

$$\xrightarrow{\sigma\circ\mu'_\hbar} \Delta_!((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)) \to \Delta_!(\mathcal{C}^0_{crit}), \qquad (\alpha''_2)$$

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{\mu_\hbar} \Delta_!(\mathcal{W}_\hbar) \xrightarrow{\Delta_!(f_\hbar)} \Delta_!((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)) \to \Delta_!(\mathcal{C}^0_{crit}), \qquad (\alpha''_3)$$

respectively, vanish on $j_*j^*(\hbar^2(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar)) \hookrightarrow j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar)$, in particular they define well defined maps from $j_*j^*(\mathfrak{Z}^c_{crit} \boxtimes \mathfrak{Z}^c_{crit})$ to $\Delta_!(\mathcal{C}^0_{crit})$. Moreover the resulting maps coincide with α_1, α_2 and α_3 composed with $\overline{F}$. In fact, the first and the last coincide by definition. For the second one, simply note that, modulo $\hbar$, the map $r_\hbar \circ \eta \circ e$ equals l.

By the above, to show that the combination of the three maps given in (2.22) is zero, it is enough to check that the combination $\alpha''_1 - \alpha''_2 - \alpha''_3$ of the above three maps vanishes.

Let us denote by α'_1, α'_2 and α'_3 the maps from $j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar)$ to $\Delta_!((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar))$ corresponding to α''_1, α''_2 and α''_3 respectively (i.e. before taking the maps $\pmod{\hbar}$). We will show that the combination $\alpha'_1 - \alpha'_2 - \alpha'_3$ is already zero.

Because $(\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)$ is h-torsion free, it is enough to show that the three maps agree after multiplication by h. But now note that each of the maps

$$h\alpha'_1,\ h\alpha'_2,\ h\alpha'_3 \in \mathrm{Hom}(j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar), \Delta_!((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar))),$$

is the sum of two terms, and the sum of the resulting six maps is zero. Indeed $h\alpha_1'$ equals the sum of the following two maps:

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{l_\hbar \boxtimes l_\hbar} j_*j^*((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)) \xrightarrow{\mu_\hbar'} \Delta_!(\mathcal{D}_\hbar), \tag{2.24}$$

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{l_\hbar \boxtimes r_\hbar \circ \eta \circ e} j_*j^*((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)) \xrightarrow{\mu_\hbar'} \Delta_!(\mathcal{D}_\hbar). \tag{2.25}$$

On the other hand, the map $h\alpha_3'$ is given by the sum of the following

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{\mu_\hbar} \Delta_!(\mathcal{W}_\hbar) \xrightarrow{\Delta_!(l_\hbar)} \Delta_!(\mathcal{D}_\hbar), \tag{2.26}$$

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{\eta \circ \mu_\hbar \circ (e \boxtimes e)} \Delta_!(\mathcal{W}_{-\hbar}) \xrightarrow{\Delta_!(r_\hbar)} \Delta_!(\mathcal{D}_\hbar). \tag{2.27}$$

It is clear that the map (2.24) equals minus the map (2.26). Similarly, the relation $\mu_\hbar' = -\sigma \circ \mu_\hbar' \circ \sigma$ guarantees that the two maps summing up to $h\alpha_2'$, given by

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{(r_\hbar \circ \eta \circ e \boxtimes l_\hbar) \circ \sigma} j_*j^*((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)) \xrightarrow{\sigma \circ \mu_\hbar'} \Delta_!(\mathcal{D}_\hbar),$$

and

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{(r_\hbar \circ \eta \circ e \boxtimes r_\hbar \circ \eta \circ e) \circ \sigma} j_*j^*((\Psi_X \boxtimes \Psi_X)(\mathcal{D}_\hbar)) \xrightarrow{\sigma \circ \mu_\hbar'} \Delta_!(\mathcal{D}_\hbar),$$

cancel with the remaining maps (2.25) and (2.27) respectively.

Hence the composition $\alpha_1 - \alpha_2 - \alpha_3$ as a map from $j_*j^*(\mathfrak{Z}_{crit}^c \boxtimes \mathfrak{Z}_{crit}^c)$ to $\Delta_!(\mathcal{C}_{crit}^0)$ is zero, i.e. the map $\overline{F} : \Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}^{ch}(\mathfrak{Z}_{crit}^c)) \to \Delta_!(\mathcal{C}_{crit}^0)$ factors as

$$\begin{tikzcd}
\Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}^{ch}(\mathfrak{Z}_{crit}^c)) \arrow[r, "\overline{F}"] \arrow[d] & \Delta_!(\mathcal{C}_{crit}^0) . \\
\Delta_!(\Omega(\mathfrak{Z}_{crit})^c) \arrow[ur, "F"'] &
\end{tikzcd}$$

By Kashiwara we obtain the desired map $F : \Omega(\mathfrak{z}_{crit})^c \to \mathcal{C}^0_{crit}$, and this concludes the proof of Theorem 2.6.

Chapter 3

Localization theorem for the affine Grassmannian

3.1 Factorization categories and factorization spaces

In this section we start by defining the appropriate categorical setting that will be needed to formulate Conjecture 1.2.2. In 3.1.1-3.1.5 we recall the definition of the *Ran space* and define the notion of factorization category, algebras in a factorization category, factorization spaces and factorization groups. We will then recall the factorization description of chiral algebras.

In 3.2 we will define and study the factorization category $\mathcal{A}$-mod $= \{I \to \mathcal{A}\text{-mod}_I\}$ of chiral $\mathcal{A}$-modules. In particular, in 3.1.8, given a commutative chiral algebra $\mathcal{B}$, we will describe the category $\mathcal{B}$-mod$_I$ of chiral-$\mathcal{B}$-modules on X^I as the category of modules over a sheaf of topological associative algebras on X^I canonically attached to $\mathcal{B}$.

In 3.2 we will recall what it means for a group G to act on a category $\mathcal{C}$, as explained in [FG3]. In Section 3.3 we will generalize the above to a factorization group $\mathcal{G}$ acting on a factorization category. The factorization category we will focus on will be the factorization category $\mathcal{A}$-mod of chiral $\mathcal{A}$-modules. In particular, we will study the factorization sub-category $\mathcal{A}\text{-mod}^{\mathcal{G}}$ consisting of strongly $\mathcal{G}$-equivariant objects in $\mathcal{A}$-mod, as defined in 3.3.1. In 3.3.3, we will apply this to the chiral algebra $\mathcal{D}_{crit}$ and the group $\mathcal{D}_X$-scheme JG of *Jets into* G as defined in 3.1.7.

3.1.1 The Ran space

Let k be an algebraically closed field of characteristic zero, and denote by ${}^f\mathrm{Sch}_{aff/k}$ the category of affine schemes of finite type over k. For a category $\mathcal{D}$, denote by $Pshv(\mathcal{D})$ the category of pre-sheaves on it, i.e.

$$Pshv(\mathcal{D}) := \{\text{ functors } \mathcal{D}^{op} \to \mathbf{Gpd}_\infty\},$$

where $\mathbf{Gpd}_\infty$ is the category of ∞-groupoids. Fix a smooth curve X over k.

Definition 3.1.1. The *Ran space* on X is the functor of points

$$\begin{array}{ccc} {}^f\mathrm{Sch}^{op}_{aff/k} & \xrightarrow{Ran(X)} & \mathbf{Set} \\ S & \mapsto & \{F \subset \mathrm{Hom}(S,X) \,:\, F \text{ is finite}\}. \end{array}$$

The Ran space is more commonly defined as the colimit of the diagrams

$$\mathbf{Fin}^{op}_{sur} \xrightarrow{I \to X^I} Pshv({}^f\mathrm{Sch}_{aff/k}),$$

where $\mathbf{Fin}^{op}_{sur}$ denotes the category of finite sets with surjections as morphisms, and a surjection of finite sets $\phi : I \twoheadrightarrow J$ maps to the corresponding diagonal embedding $X^J \hookrightarrow X^I$.

Note that a k-point $Spec(k) \to Ran(X)$ of $Ran(X)$ consists of a finite collection F of points $F \subset X(k)$. In this perspective, we can think of $Ran(X)$ as the moduli space for finite subsets of X. More generally, for an affine scheme S, an S-point of $Ran(X)$ consists of a finite set of maps $F = \{f_1, \dots, f_n\} \subset \mathrm{Hom}(S,X)$. We associate to F the closed subscheme in $X_S := X \times S$ given by the union of the graphs

$\Gamma_{f_i} \subset X_S$.

We will be interested in collections of categories over finite powers of the curve X satisfying certain properties. We will use the notion of *tensor product of abelian categories* as introduced in [FG7] 17.1.

As it is explained in *loc. cit.* the tensor product of two abelian categories is an abelian category satisfying a certain universal property. In particular it may not exist. We will always work in the following framework, in which the tensor product does exist.

Let $A \xrightarrow{f} A'$ be a homomorphism of commutative algebras. Consider the abelian categories A-mod and A'-mod. The map f gives rise to a monoidal action of A-mod on A'-mod, moreover the monoidal action $A\text{-mod} \times A'\text{-mod} \to A'\text{-mod}$ is right-exact and commutes with direct sums. Assume now that we have an abelian category $\mathcal{C}$ endowed with a monoidal action of A-mod such that the action map is right-exact and commutes with direct sums. Under the above assumptions, as it is shown in [FG7], the tensor product

$$A'\text{-mod} \underset{A\text{-mod}}{\otimes} \mathcal{C},$$

exists. For instance, we can consider an abelian category $\mathcal{C}$, with a map from A to the center $Z(\mathcal{C})$ of it. This map endows $\mathcal{C}$ with a monoidal action of A-mod, satisfying the properties above. In this case the tensor product $A'\text{-mod} \underset{A\text{-mod}}{\otimes} \mathcal{C}$ can be described as follows. This is an abelian category whose objects are objects C in $\mathcal{C}$, endowed with an additional action of A', such that the two actions of A coincide, where one action is the one coming from $f : A \to A'$, and the second is the one coming from the map $A \to Z(\mathcal{C})$. Morphisms are morphisms $C_1 \to C_2$ in $\mathcal{C}$ that intertwine the A'-action.

The categories of interest can be described as categories over the space $Ran(X)$.

More precisely, we have the following definition.

Definition 3.1.2. A category $\mathcal{C}$ over the Ran is the following data:

- For a finite set I we have an abelian category $\mathcal{C}_{X^I}$ over X^I, in other words, we are given an action of the monoidal category $\mathrm{QCoh}(X^I)$ on $\mathcal{C}_{X^I}$. We require the monoidal action to be right-exact and to commute with direct sums.

- We have pairs of mutually adjoint functors

$$\Delta_\phi^* : \mathcal{C}_{X^I} \rightleftarrows \mathcal{C}_{X^J} : \Delta_{\phi,*}$$

  for every surjection $\phi : I \twoheadrightarrow J$.

- We require that the induced functor

$$\mathrm{QCoh}(X^J) \underset{\mathrm{QCoh}(X^I)}{\otimes} \mathcal{C}_{X^I} \xrightarrow{\Delta_{\phi,*}\otimes Id_{\mathcal{C}_{X^I}}} \mathcal{C}_{X^J}$$

  be an equivalence (as categories acted on by $\mathrm{QCoh}(X^J)$).

We define the category $\mathcal{C}$ to be

$$\varprojlim_{\Delta^*_{\phi:I\to J}} \mathcal{C}_{X^I}.$$

Given a map $f^I : Y^I \to X^I$ and a category $\mathcal{C}$ over $\mathrm{Ran}(X)$, we denote by $f^*(\mathcal{C})_{Y^I}$ the category over Y^I given as

$$I \mapsto f^*(\mathcal{C})_{Y^I} := \mathrm{QCoh}(Y^I) \underset{\mathrm{QCoh}(X^I)}{\otimes} \mathcal{C}_{X^I},$$

where $\mathrm{QCoh}(X^I)$ acts on $\mathrm{QCoh}(Y^I)$ via the map $f^{I,*}$.

Definition 3.1.3. A functor F between two categories $\mathcal{C}$ and $\mathcal{D}$ over $\mathrm{Ran}(X)$, consists of a family of functors

$$F_I : \mathcal{C}_{X^I} \to \mathcal{D}_{X^I},$$

such that, for $\phi : I \twoheadrightarrow J$, we are given isomorphisms of functors $\Delta_\phi^* \circ F_I \simeq F_J \circ \Delta_\phi^*$, compatible with higher compositions.

3.1.2 Factorization categories

For every partition $\pi : I \twoheadrightarrow J$ of I, i.e. $I = \sqcup_{j \in J} I_j$, where $I_j = \pi^{-1}(j)$, consider the open subset

$$j^{(I/J)} : U^{(I/J)} \to X^I,$$

where $U^{(I/J)} = \left\{ (x_i) \in X^I \,|x_i \neq x_j \text{ if } \pi(i) \neq \pi(j) \right\}$.

Definition 3.1.4. A *factorization category* is a category $\mathcal{C}$ over $\mathrm{Ran}(X)$ such that, for every partition $\pi : I \twoheadrightarrow J$ of I, we are given equivalences

$$(j^{(I/J)})^*(\mathcal{C}_{X^I}) \simeq \mathcal{C}_{X^{I_1}} \otimes \cdots \otimes \mathcal{C}_{X^{I_n}}|_{U^{(I/J)}}\,, \tag{3.1}$$

compatible with refinements of partitions.

Example 3.1.1. The most obvious example of factorization category is given by the category $\mathrm{QCoh}(Ran(X))$ of quasi-coherent sheaves on $\mathrm{Ran}(X)$, given by

$$I \mapsto \mathrm{QCoh}(Ran(X))_{X^I} := \mathrm{QCoh}(X^I).$$

Definition 3.1.5. Given two factorization categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is called a *factorization* functor if it is compatible with the equivalences in (3.1), in the sense that

$$(F_I)|_{U^{(I/J)}} \simeq (F_{I^1} \otimes \cdots \otimes F_{I^n})|_{U^{(I/J)}}.$$

3.1.3 factorization algebras

Let $\mathcal{C}$ be a factorization category over $\mathrm{Ran}(X)$. We are interested in *chiral algebra objects* in $\mathcal{C}$. In particular, we are interested is *unital chiral algebras* in the factorization category $\mathrm{QCoh}(Ran(X))$. These correspond to *chiral algebras* as defined in 2.1.

Definition 3.1.6. A (chiral) algebra $\mathcal{A}$ in $\mathcal{C}$ is the assignment $I \to \mathcal{A}_I \in \mathcal{C}_{X^I}$, such that

- For every surjection $\phi : I \twoheadrightarrow J$, we have an isomorphism $\Delta_\phi^*(\mathcal{A}_I) \simeq \mathcal{A}_J$.

- For every partition $\pi : I \twoheadrightarrow J$ of I, we are given an isomorphism

$$(j^{(I/J)})^*(\mathcal{A}_I) \simeq \mathcal{A}_{I_1} \boxtimes \cdots \boxtimes \mathcal{A}_{I_n}|_{U^{(I/J)}}\,.$$

Definition 3.1.7. A *unital chiral algebra* over a curve X is an algebra in the factorization algebra $\mathrm{QCoh}(Ran(X))$ that satisfies the following.

- (unit) There exist a global section $1 \in \mathcal{A}_X$, called *the unit*, with the property that for every local section $f \in \mathcal{A}(U)$, $U \subset X$, the section $1 \boxtimes f \in \mathcal{A}_{X^2}|_{U^2 - \Delta(X)}$, (defined by the factorization isomorphism), extends across the diagonal, and restricts to $f \in \mathcal{A} \simeq \mathcal{A}_{X^2}|_{\Delta(X)}$.

Remark 3.1.1. As it is explained in [BD], chiral algebras as defined in 2.1 are the same as unital factorization algebras in $\mathrm{QCoh}(Ran(X))$. In fact we have the following proposition.

Proposition 3.1.1. *There is an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{Unital factorization algebras} \\ \textit{in } QCoh(Ranx(X)) \end{array} \right\} \xrightarrow{\sim} \{\textit{Unital chiral algebras}\}\,,$$

given by the assignment $\mathcal{A} \mapsto \mathcal{A}^{cl} := \mathcal{A}_X \otimes \Omega_X$.

Given a factorization algebra $\mathcal{A}$ in a factorization category $\mathcal{C}$, there is a notion of *modules over* $\mathcal{A}$. The category of $\mathcal{A}$-modules will in fact be a factorization category, defined as follows.

Definition 3.1.8. Given a factorization algebra $\mathcal{A}$ in $\mathcal{C}$, the factorization category $\mathcal{A}$-mod$(\mathcal{C})$ of $\mathcal{A}$-modules in $\mathcal{C}$ is the factorization category defined by the assignment $I \to \mathcal{A}\text{-mod}(\mathcal{C}_{X^I})$, where $\mathcal{A}\text{-mod}(\mathcal{C}_{X^I})$ is the category whose objects are collections $\mathcal{M}_{I\sqcup K} \in \mathcal{C}_{X^{I\sqcup K}}$, for every finite set K satisfying the following:

- For every surjection $\phi : K \twoheadrightarrow K'$, we have an isomorphism $\Delta_\phi^*(\mathcal{M}_{I\sqcup K}) \simeq \mathcal{M}_{I\sqcup K'}$.

- For every partition $\pi : I \sqcup K \twoheadrightarrow K'$ of $I \sqcup K$, such that $I \subset \pi^{-1}(k') =: \overline{I}$ for some $k' \in K'$, we are given an isomorphism

$$(j^{(I\sqcup K/K')})^*(\mathcal{M}_{I\sqcup K}) \simeq \mathcal{A}_{I_1} \boxtimes \cdots \boxtimes \mathcal{A}_{I_{n-1}} \boxtimes \mathcal{M}_{\overline{I}}\big|_{U^{(I\sqcup K/K')}},$$

  where $I \sqcup K = I_1 \sqcup \cdots \sqcup I_{n-1} \sqcup \overline{I}$.

- The above isomorphisms are compatible with sequence of surjections $I \sqcup K \xrightarrow{\pi} K' \xrightarrow{\pi'} K''$ in the obvious sense, where

$$I \sqcup K = \overline{I} \sqcup_{\substack{k\in K' \\ k\neq k'}} I_{\pi^{-1}(k)} \text{ and } \overline{I} = \widetilde{I} \sqcup_{\substack{k''\in K'' \\ k''\neq \overline{k}}} \overline{I}_{k''}, \text{ with } I \subset \widetilde{I}.$$

For a chiral algebra $\mathcal{A}$, we can consider the factorization category

$$\mathcal{A}\text{-mod} := \{\ \textit{unital}\ \mathcal{A}\text{-modules in QCoh}(Ran(X))\} \tag{3.2}$$

These are $\mathcal{A}$-modules $\mathcal{M}$ in QCoh$(Ran(X))$ such that

- (unit) If $1 \in \mathcal{A}_X$ denotes the unit section in $\mathcal{A}$, then for every local section $f \in \mathcal{M}(U)$, $U \subset X^I$, the section $1 \boxtimes f \in \mathcal{M}_{X^{\{*\}\sqcup I}}|_{U(\{*\}\sqcup I/I)-\Delta(X^I)}$, (defined by the factorization isomorphism), extends across the diagonal, and restricts to $f \in \mathcal{M} \simeq \mathcal{M}_{X^{\{*\}\sqcup I}}|_{\Delta(X^I)}$.

3.1.4 Chiral modules over X^I

From proposition 3.1.1, it is natural to expect the factorization category $\mathcal{A}$-mod from (3.2) to have a different description in terms of the right $\mathcal{D}_X$-module $\mathcal{A}^{cl}$. Given a unital chiral algebra $\mathcal{A}^{cl}$, and a finite set I, we can in fact define the notion of chiral $\mathcal{A}^{cl}$-modules on X^I in the following way.

For any finite set K and embedding $\phi : I \to K$, consider the $|I|$-dimensional closed sub-scheme $H_\phi \subset X^K$ given by the union of the diagonal sub-schemes

$$H_\phi := \bigcup_{\pi : K_1 \to I} \{x_i = y_{\pi(i)}, \text{ for } i \in K_1\} \subset X^{K_1} \times X^I,$$

where $K = K_1 \sqcup I$, and x_i and y_j are coordinates on X^{K_1} and X^I respectively. Consider the following diagram

$$\begin{tikzcd} H_\phi \arrow[r, "i"] & X^{K_1} \times X^I \arrow[dl, "p_1"] \arrow[dr, "p_2"] & U_\phi, \arrow[l, "j"'] \\ X^{K_1} & & X^I \end{tikzcd}$$

where U_ϕ denotes the complement of H_ϕ inside $X^{K_1} \times X^I$. For a quasi-coherent sheaf $\mathcal{M}_I$ on X^I, denote by $\Gamma_\phi(\mathcal{M}_I)$ the module over $X^{K_1} \times X^I$ given by

$$\Gamma_\phi(\mathcal{M}_I) := i_* i^*(p_2^*(\mathcal{M}_I)).$$

Definition 3.1.9. A chiral $\mathcal{A}^{cl}$-module $\mathcal{M}^I$ on X^I is a quasi-coherent sheaf on X^I

along with a map

$$\mu^I : j_*j^*(\mathcal{A}^{cl} \boxtimes \mathcal{M}^I) \to \Gamma_{I \subset I \sqcup \{*\}}(\mathcal{M}^I)$$

such that the following is satisfied.

- (unit) The following diagram commutes:

$$\begin{tikzcd}
j_*j^*(\Omega_X \boxtimes \mathcal{A}^{cl}) \arrow[r] \arrow[d] & j_*j^*(\mathcal{A}^{cl} \boxtimes \mathcal{M}^I) \arrow[d] \\
\Gamma_{I \subset I \sqcup \{*\}}(\mathcal{M}^I) \arrow[r, "id"] & \Gamma_{I \subset I \sqcup \{*\}}(\mathcal{M}^I)
\end{tikzcd}$$

- (Lie action) $\mu^I_{\{1,2\},3} = \mu^I_{2,\{1,3\}} - \mu^I_{1,\{2,3\}}$ where

$\mu^I_{1,\{2,3\}} : j_*j^*(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl} \boxtimes \mathcal{M}^I) \to j_*j^*(\mathcal{A}^{cl} \boxtimes \Gamma_{I \subset I \sqcup \{*\}}(\mathcal{M}^I)) \to \Gamma_{I \subset I \sqcup \{*,*\}}(\mathcal{M}^I)$,
$\mu^I_{2,\{1,3\}} = \mu_{1,\{2,3\}} \circ \sigma^*_{12}$, and

$$\begin{aligned}
\mu^I_{\{1,2\},3} : j_*j^*(\mathcal{A}^{cl} \boxtimes \mathcal{A}^{cl} \boxtimes \mathcal{M}^I) &\to \Delta_{12*} j_*j^*(\mathcal{A}^{cl} \boxtimes \mathcal{M}^I) \to \\
&\to \Delta_{12*}(\Gamma_{I \subset I \sqcup \{*\}}(\mathcal{M}^I)) \hookrightarrow \quad \Gamma_{I \subset I \sqcup \{*,*\}}(\mathcal{M}^I).
\end{aligned}$$

As it is explained in [NR], we have an equivalence of categories between the category $\mathcal{A}$-mod$_I$ introduced in (3.2), and the category of chiral $\mathcal{A}^{cl}$-modules on X^I, where $\mathcal{A}^{cl} = \mathcal{A}_X \otimes \Omega_X$.

Proposition 3.1.2. *For every I finite set, there is an equivalence of categories*

$$\mathcal{A}\text{-}mod_I \xrightarrow{\sim} \left\{ chiral\ \mathcal{A}^{cl}\text{-}modules\ on\ X^I \right\},$$

given by the assignment $\mathcal{M}_I \to \mathcal{M}^I := \mathcal{M}_I \otimes \Omega^{|I|}_{X^I}$.

Given a $\mathcal{A}^{cl}$-module $\mathcal{M}^I$ over X^I, the corresponding factorization module is constructed inductively. For instance, for the finite set $I \sqcup \{*\}$, the module $\mathcal{M}_{I \sqcup \{*\}}$ is defined as the kernel of the chiral action

$$\mu^I : j_* j^*(\mathcal{A}^{cl} \boxtimes \mathcal{M}^I) \to \Gamma_{I \subset I \sqcup \{*\}}(\mathcal{M}^I),$$

which is surjective thanks to the unit axiom. The inductive step, constructing the modules $\mathcal{M}_{I \sqcup K}$ for any finite set K, can be found in [NR] 3.5.

Now, let $\mathcal{A}^{cl}$ and $\mathcal{B}^{cl}$ be two chiral algebras and

$$\phi : \mathcal{A}^{cl} \to \mathcal{B}^{cl}$$

a morphism of chiral algebras. Let $\mathcal{A}$ and $\mathcal{B}$ be the corresponding factorization algebras. Consider the factorization category $\mathcal{C} = \mathcal{A}$-mod. Since the map ϕ endows $\mathcal{B}^{cl}$ with a chiral $\mathcal{A}^{cl}$-module structure, and therefore makes it an $\mathcal{A}^{cl}$-module over X (and $\mathcal{B}_I$ and $\mathcal{A}^{cl}$-module over X^I), we see that, by proposition 3.1.2, $\mathcal{B}$ becomes a factorization-algebra object in the category $\mathcal{A}$-mod. In fact, for every I we take $(\mathcal{B})_I \in \mathcal{A}\text{-mod}_{X^I}$ to be $\mathcal{B}_I$ itself, and for every K we take $(\mathcal{B})_{I \sqcup K}$ to be the object in $\mathcal{A}\text{-mod}_{X^{I \sqcup K}}$ given by the inverse to the functor in 3.1.2. We can therefore consider the factorization category $\mathcal{B}$-mod($\mathcal{A}$-mod) as defined in definition 3.1.8. We have the following proposition.

Proposition 3.1.3. *We have an equivalence of factorization categories*

$$\mathcal{B}\text{-}mod(\mathcal{A}\text{-}mod) \simeq \mathcal{B}\text{-}mod.$$

Proof. For every finite set I, there is a tautologically defined functor

$$\mathcal{B}\text{-mod}(\mathcal{A}\text{-mod}_I) \to \mathcal{B}\text{-mod}_I,$$

and proposition 3.1.2 says that this functor is an equivalence. In fact, let us consider an object $\mathcal{M}_I \in \mathcal{B}\text{-mod}(\mathcal{A}\text{-mod}_I)$. By definition, this corresponds to a collection of objects $\mathcal{M}_{I\sqcup K} \in \mathcal{A}\text{-mod}_{I\sqcup K}$, satisfying the factorization property

$$(j^{(I/J)})^*(\mathcal{M}_{I\sqcup K}) \simeq \mathcal{B}_{I_1} \boxtimes \cdots \boxtimes \mathcal{B}_{I_{n-1}} \boxtimes \mathcal{M}_{\overline{I}}\big|_{U^{(I/J)}},$$

for every partition $\pi : I \sqcup K \twoheadrightarrow K'$ of $I \sqcup K$ such that $I \subset \pi^{-1}(k') =: \overline{I}$ for some $k \in K'$. However, each $\mathcal{M}_{I\sqcup K}$, being an object of $\mathcal{A}\text{-mod}_{I\sqcup K}$, corresponds to a collection of objects $\mathcal{M}_{I\sqcup K\sqcup J} \in \mathcal{A}\text{-mod}_{I\sqcup K\sqcup J}$ satisfies the factorization property

$$(j^{(I/J)})^*(\mathcal{M}_{I\sqcup K\sqcup J}) \simeq \mathcal{A}_{I'_1} \boxtimes \cdots \boxtimes \mathcal{A}_{I'_{n-1}} \boxtimes \mathcal{M}_{\overline{I'}}\Big|_{U^{(I/J)}},$$

for every partition $I'_1 \sqcup \cdots \sqcup I'_{n-1} \sqcup \overline{I'}$ of $I \sqcup K \sqcup J$, such that $I \sqcup K \subset \overline{I'}$. We will therefore think of the object $\mathcal{M}_I$ as a collection of objects $\mathcal{M}_{I\sqcup K\sqcup J}$ satisfying the factorization property for $\mathcal{B}$ with respect to the K finite set, and the factorization property for $\mathcal{A}$ with respect to the finite set J.

Clearly, given an object $\mathcal{M}_I \in \mathcal{B}\text{-mod}(\mathcal{A}\text{-mod}_I)$, the collection $\mathcal{M}_{I\sqcup K}$ defines a $\mathcal{B}$-module structure on $\mathcal{M}_I$. Hence we have the functor mentioned above. Conversely, given an object $\mathcal{N}_I \in \mathcal{B}\text{-mod}_I$, and hence a collection of objects $\mathcal{N}_{I\sqcup K} \in \mathcal{B}\text{-mod}_{I\sqcup K}$, we can construct the modules $\mathcal{N}_{I\sqcup K\sqcup J}$ in $\mathcal{A}\text{-mod}_{I\sqcup K\sqcup J}$ using proposition 3.1.2 in the following way. Using the object $\mathcal{N}_{I\sqcup K}$ we can construct a chiral-$\mathcal{B}^{cl}$-module structure on it. More precisely, consider the map $\phi : I \sqcup K \to I \sqcup K \sqcup \{*\}$, and the corresponding stratification $H_\phi \xrightarrow{i} X \times X^{I\sqcup K} \xleftarrow{j} U_\phi$. Denote by $\mathcal{N}_{\{*\}}$ the object $\mathcal{N}_{I\sqcup K\sqcup\{*\}}$, and consider the Cousin complex for $\mathcal{N}_{\{*\}}$ given by the above stratification,

$$0 \to \mathcal{N}_{\{*\}} \to j_*j^*(\mathcal{B} \boxtimes \mathcal{N}_{I\sqcup K}) \to i_*i^*(\mathcal{N}_{\{*\}})[1] \to 0.$$

It is not hard to see that the unit axiom for $\mathcal{B}$ implies a natural isomorphism $i_*i^*(\mathcal{N}_{\{*\}})[1] \simeq \Gamma_\phi(\mathcal{N}_{I\sqcup K})$, and that the second map in the above sequence gives rise to a $\mathcal{B}^{cl}$ action on the right-D-module $\mathcal{N}^r_{I\sqcup K} := \mathcal{N}_{I\sqcup K} \otimes \Omega^{|I\sqcup K|}_{X^{I\sqcup K}}$. Therefore, given the object $\mathcal{N}_{I\sqcup K}$, we obtain a $\mathcal{B}^{cl}$-module $\mathcal{N}^r_{I\sqcup K}$ on $X^{I\sqcup K}$. Using the map of chiral algebras $\psi : \mathcal{A}^{cl} \to \mathcal{B}^{cl}$, we can regard it as a $\mathcal{A}^{cl}$-module on $X^{I \sqcup K}$. Now, using proposition 3.1.2, we can construct the corresponding factorization $\mathcal{A}$-module, i.e. the collection $\mathcal{N}_{I \sqcup K \sqcup J}$. $\square$

3.1.5 Factorization spaces

We will now introduce the notion of *factorization space.* In particular we will be interested in the factorization space attached to a $\mathcal{D}_X$-scheme $\mathcal{Y}$. For $\mathcal{D}_X$-schemes of the form $\mathcal{Y} = J_X(Z)$, where Z is an affine $\mathcal{O}_X$-scheme, and J_X is the functor defined in 2.17. The corresponding factorization space will be studied in more details in 3.1.3. In 3.1.7 we will focus on the case of $Z = G$, for an affine group-scheme G over X.

Definition 3.1.10. A *factorization space* $\mathcal{Y}$ is the assignment $I \to \mathcal{Y}_I$, for I finite set, and $\mathcal{Y}_I$ a space over X^I, such that

- For every surjection $\phi : I \twoheadrightarrow J$, we are given isomorphisms $\mathcal{Y}_I|_{X^J} \simeq \mathcal{Y}_J$
- For every partition $\pi : I \twoheadrightarrow J$ of I, we are given isomorphisms

$$(j^{(I/J)})^*(\mathcal{Y}_I) \simeq \mathcal{Y}_{I_1} \times \cdots \times \mathcal{Y}_{I_n}|_{U^{(I/J)}}\,.$$

A factorization space $\mathcal{Y}$ is called *co-unital* if it comes with a collection of maps

$$\mathcal{Y}_I \to X^{I_1} \times \mathcal{Y}_{I_2}$$

for each partition $I = I_1 \sqcup I_2$ which extends the corresponding map over the complement of the diagonal. We demand that these be isomorphisms over the formal neighborhood of the diagonal.

Example 3.1.2. We will see later that the space of opers on X , as defined in [BD2] can be organized into a co-unital factorization space $\mathrm{Op}_{\mathfrak{g}}$. Moreover, we will have a sequence of inclusions

$$\mathrm{Op}_{\mathfrak{g}} \subset \mathrm{Op}_{\mathfrak{g}}^{unr} \subset \mathrm{Op}_{\mathfrak{g}}^{\circ},$$

where the last two (non-counital) factorization spaces $\mathrm{Op}_{\mathfrak{g}}^{unr}$ and $\mathrm{Op}_{\mathfrak{g}}^{\circ}$ generalize the notion of *unramified opers* and *opers on the punctured disc* as introduced in [FG3].

Remark 3.1.2. Given a factorization space $\mathcal{Y}$, we can define the factorization category $\mathrm{QCoh}(\mathcal{Y})$ of quasi-coherent sheaves on it, by the assignment

$$I \to \mathrm{QCoh}(\mathcal{Y}_I).$$

In the special case in which we take $\mathcal{Y}_I$ to be X_I, we recover the factorization category $\mathrm{QCoh}(Ran(X))$ from example 3.1.1.

3.1.6 From $\mathcal{D}_X$-schemes to factorization spaces

Given a $\mathcal{D}_X$-scheme $\mathcal{Y}$, we will now define and study factorization spaces $J\mathcal{Y} = \{I \to J_I\mathcal{Y}\}$ and $M\mathcal{Y} = \{I \to M_I\mathcal{Y}\}$ canonically attached to $\mathcal{Y}$. In studying these spaces, we will use the notions of ind-scheme and formal schemes. In particular, given a map $\phi : S \xrightarrow{(\phi_1,\ldots,\phi_n)} X^I$ we will consider the union $\cup_{i\in I}\Gamma_{\phi_i}$ of the graphs Γ_i's inside $X \times S$, and the completion $\widehat{\cup_{i\in I}\Gamma_{\phi_i}}$ of $X \times S$ along $\cup_{i\in I}\Gamma_{\phi_i}$. As it is explained below, we can regard $\widehat{\cup_{i\in I}\Gamma_{\phi_i}}$ as a scheme or as a formal scheme. When regarded as a scheme, we will denote it by $D_{\underline{\phi}}$ and by $D_{\underline{\phi}}^{\circ}$ the complement of Γ_ϕ inside D_ϕ. We start by recalling the definition of ind-scheme.

Definition 3.1.11. An ind-scheme X is a presheaf on the category of affine schemes, that can be represented by a filtered family of schemes,

$$X = \varinjlim_{\alpha} X_\alpha,$$

where the transition maps $i_{\beta,\alpha} : X_\alpha \to X_\beta$ are closed embeddings, and the limit is taken inside the category $Pshv({}^f\mathrm{Sch}_{aff/k})$ as defined in 3.1.1.
We say that an ind-scheme X is of ind-finite type if each scheme X_α is.

By a *formal scheme* we mean an ind-scheme whose reduced part is a scheme. Let now S be a space over X^I, we will now explain the difference between the formal scheme $\widehat{\cup_{i\in I}\Gamma_{\phi_i}}$ and the scheme $D_{\underline{\phi}}$. Assume for a moment that X is the affine line, S is just a point, and I is the one element set. Let $x \in X$ be the point corresponding to $pt \to X$. Denote by $\hat{D}_x$ the formal scheme

$$\hat{D}_x := \varinjlim_n Spec(\mathbb{C}[x]/x^n).$$

We could have also considered the disc as a non-formal scheme, in other words we could have considered the scheme D_x

$$D_x := Spec(\varprojlim_n \mathbb{C}[x]/x^n).$$

By definition of direct/inverse limit, for an affine scheme S, we have

$$\mathrm{Hom}(\hat{D}_x, S) \simeq \mathrm{Hom}(D_x, S).$$

However, the formal disc $\hat{D}_x$ is "too small", in the sense that it makes no sense to talk about *the formal punctured disc* $\hat{D}_x - x$. The same is not true if we consider the disc D_x. In fact, the latter, as a scheme, contains the closed point x, and we denote by D_x° the complement of x inside D_x.
To pass to the general situation, if S is an affine scheme mapping to X^I, we can consider the formal-scheme $\widehat{\cup_{i\in I}\Gamma_{\phi_i}}$. We write it as

$$\widehat{\cup_{i\in I}\Gamma_{\phi_i}} = \varinjlim_n Spec(R/I_n),$$

where R is a topological algebra, whose topology is given by I_n's. As before, this formal-scheme is not good enough for us. In particular it doesn't contain $\cup_{i\in I}\Gamma_{\phi_i}$ as a closed sub-scheme. Therefore it makes more sense to consider the scheme $D_{\underline{\phi}}$

$$D_{\underline{\phi}} := Spec(\varprojlim_n R/I_n). \tag{3.3}$$

The scheme $D_{\underline{\phi}}$ has a closed sub-scheme $\cup_{i\in I}\Gamma_{\phi_i}$, and we can consider the complement

$$D^{\circ}_{\underline{\phi}} := D_{\underline{\phi}} - \cup_{i\in I}\Gamma_{\phi_i}. \tag{3.4}$$

The factorization space attached to a $\mathcal{D}_X$-scheme

Let $\mathcal{Y}$ be a $\mathcal{D}_X$-scheme over X. In the case that $\mathcal{Y}$ is affine, this is the same as a $\mathcal{D}_X$-algebra, hence, as we saw in 2.2, the same as a commutative chiral algebra. In the general case, given a $\mathcal{D}_X$-space $\mathcal{Y}$, we can construct a factorization space $J\mathcal{Y}$ as follows. For every finite set I, and test scheme S, we define $J_I\mathcal{Y}(S)$ to be

$$J_I\mathcal{Y}(S) = \left\{ \begin{array}{c} S \xrightarrow{(\phi_1,\ldots,\phi_n)} X^I \\ \alpha \text{ horizontal section } D_{\underline{\phi}} \to \mathcal{Y} \end{array} \right\},$$

where by horizontal we mean a map of $\mathcal{D}_X$-schemes. The above construction defines a functor

$$\{\mathcal{D}_X\text{-spaces}\} \to \{\text{ Factorization spaces }\}.$$

The factorization space $J\mathcal{Y}$ naturally sits inside the factorization space $M\mathcal{Y}$ defined as follows. For every finite set I and test scheme S, we define $M_I\mathcal{Y}(S)$ to be

$$M_I\mathcal{Y} := \left\{ \begin{array}{c} S \xrightarrow{(\phi_1,\ldots,\phi_n)} X^I \\ \alpha \text{ horizontal section } D^{\circ}_{\underline{\phi}} \to \mathcal{Y} \end{array} \right\}, \tag{3.5}$$

where $D^\circ_{\underline{\phi}}$ is the scheme defined in 3.4. Clearly $J_I\mathcal{Y}(S)$ corresponds to those sections that are regular along $\cup_{i\in I}\Gamma_{\phi_i}$.

Remark 3.1.3. Given a commutative chiral algebra $\mathcal{B}^{cl}$, we can consider the corresponding $\mathcal{D}_X$-algebra $\mathcal{B}^l := \mathcal{B}^{cl}\otimes\Omega^*_X$. Since $\mathcal{B}^l$ corresponds to an affine $\mathcal{D}_X$-scheme, we can construct the corresponding factorization space. It is easy to see that the unit axiom on $\mathcal{B}^{cl}$ translate into the co-unital property of the corresponding factorization space. Therefore we have a functor

$$\{\text{Commutative chiral algebras}\} \to \{\text{ affine counital factorization sapces }\}.$$

3.1.7 Factorization groups

Let $\mathcal{G}_X$ be an affine $\mathcal{D}_X$-group scheme. From the above construction, we obtain a factorization group $\mathcal{D}_X$-scheme, i.e. for each power of the curve X^I, we have an affine group scheme $\mathcal{G}_I := J_I\mathcal{G}_X$ over X^I.

Consider now a group scheme G on X, and let $J_X(G)$ be the corresponding $\mathcal{D}_X$-scheme, where J_X is the functor defined in (2.17). Denote simply by JG the factorization space $JJ_X(G)$. In this case, by the adjunction property of $J_X : \{\mathcal{O}\text{-schemes}\} \to \{\mathcal{D}_X\text{-schemes }\}$, we have

$$JG_I(S) = \left\{ \begin{array}{c} S \xrightarrow{(\phi_1,\ldots,\phi_n)} X^I \\ \alpha \text{ section } D_{\underline{\phi}} \to G \end{array} \right\}.$$

We denote by MG_I the ind-scheme of *meromorphic jets* $MG_I := M_IJ_X(G)$,

$$MG_I(S) = \left\{ \begin{array}{c} S \xrightarrow{(\phi_1,\ldots,\phi_n)} X^I \\ \alpha \text{ section } D^\circ_{\underline{\phi}} \to G \end{array} \right\}. \tag{3.6}$$

We will mostly focus on the quotient MG_I/JG_I. We will see later, that its closed points can be described as the set of G-bundles on X with a given trivialization outside a finite set of points.

3.1.8 Chiral modules over a commutative chiral algebra

Let now $\mathcal{B}^{cl}$ be a commutative chiral algebra. In this section, we will see how we can describe the category of $\mathcal{B}$-modules on X^I as quasi-coherent sheaves over the space $M_I \mathcal{Y}_{\mathcal{B}}$ defined in 3.5, for the $\mathcal{D}_X$-scheme $\mathcal{Y}_{\mathcal{B}}$ corresponding to $\mathcal{B}$. More generally, in 3.1.1for a chiral algebra $\mathcal{B}$, not necessarily commutative, we will construct a sheaf of topological algebras $\mathcal{B}^{(I)}$ over X^I, such that modules over it will be equivalent to the category of chiral-$\mathcal{B}$-modules on X^I. For this we will first need to recall some constructions regarding the chiral envelope $U(L)$ of a Lie*-algebra L.

For a Lie*-algebra L, we will start by recalling the definitions of *Lie*-modules* and *chiral L-modules* over X^I.

Definition 3.1.12.
- A Lie*-L-module $\mathcal{M}^I$ on X^I is a quasi-coherent sheaf on X^I along with a map

$$\mu^I : L \boxtimes \mathcal{M}^I \to \Gamma_{I \subset I \sqcup \{*\}}(M)$$

  such that the following is satisfied.

  - (Lie action) $\mu^I_{\{1,2\},3} = \mu^I_{2,\{1,3\}} - \mu^I_{1,\{2,3\}}$ where

$$\begin{aligned}
&\mu^I_{1,\{2,3\}} = \mu^I \circ \mu^I : L \boxtimes L \boxtimes M \to L \boxtimes \Gamma_{I \subset I \sqcup \{*\}}(M) \to \Gamma_{I \subset I \sqcup \{*,*\}}(M),\\
&\mu^I_{2,\{1,3\}} = \mu_{1,\{2,3\}} \circ \sigma^*_{12}, \text{ and}\\
&\mu^I_{\{1,2\},3} = \mu^I \circ \mu_L : L \boxtimes L \boxtimes M \to \Delta_*(L) \boxtimes M \to \Delta_*(\Gamma_{I \subset I \sqcup \{*\}}(M)) \hookrightarrow \Gamma_{I \subset I \sqcup \{*,*\}}(M).
\end{aligned}$$

- A chiral L-module $\mathcal{M}_I$ on X^I is quasi-coherent sheaf on X^I along with a map

$$\mu^I : j_* j^*(L \boxtimes \mathcal{M}^I) \to \Gamma_{I \subset I \sqcup \{*\}}(M)$$

such that the following is satisfied.

– (Lie action) Let $U' \subset X^2 \times X^I$ be the complement of the diagonals $\{y_1 = x_i\}$ and $\{y_2 = x_i\}$. Denote by j' the inclusion $j' : U' \to X^2 \times X^I$. We require that $\mu^I_{\{1,2\},3} = \mu^I_{2,\{1,3\}} - \mu^I_{1,\{2,3\}}$ where

$$\mu^I_{1,\{2,3\}} : j'_* j'^*(L \boxtimes L \boxtimes M) \to j_* j^*(L \boxtimes L \boxtimes M) \to \Gamma_{I \subset I \sqcup \{*,*\}}(M),$$
$$\mu^I_{2,\{1,3\}} = \mu_{1,\{2,3\}} \circ \sigma^*_{12}, \text{ and}$$

$$\begin{aligned}\mu^I_{\{1,2\},3} : j'_* j'^*(L \boxtimes L \boxtimes M) \to \quad & \Delta_{12*} j_* j^*(L \boxtimes M) \to \\ & \to \Delta_{12*}(\Gamma_{I \subset I \sqcup \{*\}}(M)) \hookrightarrow \quad \Gamma_{I \subset I \sqcup \{*,*\}}(M).\end{aligned}$$

Unlike in the world of usual Lie algebras, for a Lie*-algebra L, the category of Lie*-modules on X^I is not equivalent to the category of chiral $U(L)$-modules on X^I. Chiral $U(L)$-modules are in fact equivalent to chiral-L-modules, in the sense that there exist an induction functor Ind^I establishing an equivalence of categories

$$Ind^I : \{\text{Chiral } L\text{-modules on } X^I\} \xrightarrow{\sim} \{\text{ chiral } U(L)\text{-modules on } X^I\}. \tag{3.7}$$

We can also describe chiral and Lie*-L-modules over X^I as modules for some particular sheaves of topological Lie algebras on X^I. For this, consider the following diagram

$$\begin{tikzcd} H_{(I \subset I \sqcup \{*\})} \arrow[r, "i"] & X \times X^I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & U. \arrow[l, "j"'] \\ X & & X^I \end{tikzcd}$$

Given a Lie*-algebra L, let

$$\mathfrak{L}^{(I)}_0 := h_\Gamma(p_1^!(L)[-I]) \quad \text{and} \quad \mathfrak{L}^{(I)} := h_\Gamma(j_* j^*(p_1^!(L)[-I])), \tag{3.8}$$

where, for a sheaf $\mathcal{M}$ on $X \times X^I$, we define $h_\Gamma(\mathcal{M})$ to be

$$h_\Gamma(\mathcal{M}) := \varprojlim_{\xi} p_{2,*}(\mathcal{M}/\mathcal{M}_\xi),\ \mathcal{M}_\xi \text{ such that } \mathcal{M}/\mathcal{M}_\xi \text{ is supported on } H.$$

The objects in (3.8) are sheaves of topological $\mathcal{O}_{X^I}$-modules, moreover they have a structure of Lie algebra, coming from the Lie*-algebra structure on L.
As it is explained in [NR], we have the following proposition.

Proposition 3.1.4. *Let L be a Lie*-algebra. Then the category of Lie*-modules (resp. chiral L-modules) on X^I is equivalent to the category of $\mathfrak{L}_0^{(I)}$-modules (resp. $\mathfrak{L}^{(I)}$-modules).*

Chiral algebras and topological algebras attached to them

Let $\mathcal{B}^{cl}$ be a chiral algebra. We will use the previous subsection to describe chiral $\mathcal{B}^{cl}$-modules over X^I as modules over a sheaf of topological associative algebras.
The ideas is the following: consider a classical associative algebra B, and denote by B^{Lie} the corresponding Lie algebra. We have an obvious forgetful functor from the category of B-modules to the category of chiral-B^{Lie}-modules. Since chiral-B^{Lie}-modules are the same as $U(B^{Lie})$-modules we therefore have a functor

$$B\text{-mod} \to U(B^{Lie})\text{-mod}.$$

Let now K be the kernel of the natural map $U(B^{Lie}) \to B$. It is clear that the functor above defines an equivalence

$$B\text{-mod} \simeq U(B^{Lie})/K\text{-mod}.$$

We can apply the same idea to the world of chiral algebras. When the chiral algebra $\mathcal{B}^{cl}$ is commutative, we can furthermore describe the (commutative) algebra corresponding to $U(B^{Lie})/K$ as some scheme over X.

Consider the (commutative)-Lie*-algebra $\mathcal{B}^{Lie}$ corresponding to $\mathcal{B}^{cl}$. Denote by K the ideal in $U(\mathcal{B}^{Lie})$ generated by the kernel of the natural map

$$U(\mathcal{B}^{Lie}) \to \mathcal{B}^{cl}.$$

From the definition it is clear that we have the following lemma.

Lemma 3.1.1. *The functor Ind^I from (3.7) induces an equivalence of categories*

$$Ind^I : \{ \text{ Chiral } \mathcal{B}^{cl}\text{-modules on } X^I \ \} \simeq \{ \text{ chiral-}U(\mathcal{B}^{Lie})/K\text{-modules on } X^I \ \}. \tag{3.9}$$

Let's now consider the sheaf of topological (commutative) Lie-algebras defined in (3.8) for $L = \mathcal{B}^{Lie}$. Denote them simply by $\mathcal{B}_0^{(I)}$ and $\mathcal{B}^{(I)}$ respectively. Because of the chiral algebra structure on $\mathcal{B}^{cl}$, we have maps

$$U(\mathcal{B}^{(I)}) \to \mathcal{B}^{(I)}.$$

Denote by $K^{(I)}$ the ideal generated by the kernel of the above maps. Consider the equivalence of proposition 3.1.4

$$\{ \text{ chiral-}U(\mathcal{B}^{Lie})\text{-modules on } X^I \ \} \simeq \{U(\mathcal{B}^{(I)})\text{-modules on } X^I\}.$$

We have the following proposition.

Proposition 3.1.5. *The composition of the above functor with (3.9) induces an equivalence*

$$\{ \text{ Chiral } \mathcal{B}^{cl}\text{-modules on } X^I\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{modules for the topological} \\ \textit{associative algebra } U(\mathcal{B}^{(I)})/K^{(I)} \end{array} \right\}.$$

Proof. Clearly the functor $\{\ \mathcal{B}^{cl}\text{-modules on } X^I\} \to \{\ U(\mathcal{B}^{(I)})\text{-modules }\}$ given by proposition 3.1.4 factors through

$\{$ modules for the topological associative algebra $U(\mathcal{B}^{(I)})/K^{(I)}\} \to \{\ U(\mathcal{B}^{(I)})\text{-modules }\}$.

On the other hand, given a module $\mathcal{M}_I$ for the associative algebra $U(\mathcal{B}^{(I)})/K^{(I)}$, we can consider it as a Lie-$\mathcal{B}^{(I)}$-module. By the equivalence (3.7), together with proposition 3.1.4, we can consider the module $Ind^I(\mathcal{M}_I)$ as a chiral $U(\mathcal{B}^{Lie})$-module over X^I, However, the fact that $\mathcal{M}_I$ was in fact a module for $\mathcal{B}^{(I)}$ when regarded as an algebra, implies that the action of $U(\mathcal{B}^{Lie})$ on $Ind^I(\mathcal{M}_I)$ factors through the chiral algebra $U(\mathcal{B}^{Lie})/K$. Now, by Lemma 3.1.1 we therefore have that $\mathcal{M}_I$ itself is in fact a $\mathcal{B}^{cl}$-module on X^I.

□

Example 3.1.3. The commutative case: For a commutative chiral algebra $\mathcal{B}^{cl}$, denote by $\mathcal{Y}_\mathcal{B}$ the corresponding co-unital affine factorization $\mathcal{D}_X$-space. Recall from 3.1.6 and 3.5 that we have constructed spaces $J_I\mathcal{Y}_\mathcal{B}$ and $M_I\mathcal{Y}_\mathcal{B}$ over X^I. Denote by p^I the natural map

$$p^I : M_I\mathcal{Y}_\mathcal{B} \to X^I,$$

and by $\mathcal{O}^{rel}_{M_I\mathcal{Y}_\mathcal{B}}$ the object $\mathcal{O}^{rel}_{M_I\mathcal{Y}_\mathcal{B}} := p^I_*(\mathcal{O}_{M_I\mathcal{Y}_\mathcal{B}})$. Note that we have isomorphisms $\mathrm{Sym}(\mathcal{B}^{(I)})/K^{(I)} \simeq \mathcal{B}^{(I)}$ (as sheaves) over X^I. For $I = \{*\}$, the fiber of $\mathcal{B}^{(\{*\})}$ at any point $x \in X$ coincides, by construction, with the topological associative algebra $\hat{\mathcal{B}}^{ass}_x = \varprojlim_j \hat{\mathcal{B}}^{ass}_{xj}$ introduced in [BD] 3.6. As it is explained in *loc. cit* 2.4.7, for this topological algebra, the ind-scheme $Spf(\hat{\mathcal{B}}^{ass}_x) := \varinjlim_j Spec(\hat{\mathcal{B}}^{ass}_{xj})$ is the space of horizontal sections of $Spec(\mathcal{B})$ over the formal punctured disc D°_x. In other words, we have

$$\mathcal{B}^{(\{*\})}_x \simeq (\mathcal{O}^{rel}_{M_{\{*\}}\mathcal{Y}_\mathcal{B}})_x.$$

Let now I be a finite set with n elements, and $\overline{x} = (x_1, \ldots, x_n)$ be a point in X^I, with $x_i \neq x_j$ for all i and j. Since both $\mathcal{B}^{(I)}$ and $M_I\mathcal{Y}_{\mathcal{B}}$ factorize, we have

$$(\mathcal{B}^{(I)})_{\overline{x}} \simeq \hat{\mathcal{B}}^{ass}_{x_1} \otimes \cdots \otimes \hat{\mathcal{B}}^{ass}_{x_n} \simeq (\mathcal{O}^{rel}_{M_{\{*\}}\mathcal{Y}_{\mathcal{B}}})_{x_1} \otimes \cdots \otimes (\mathcal{O}^{rel}_{M_{\{*\}}\mathcal{Y}_{\mathcal{B}}})_{x_n} \simeq (\mathcal{O}_{M_I\mathcal{Y}_{\mathcal{B}}})_{\overline{x}}$$

and therefore we have

$$\mathcal{B}^{(I)} \simeq \mathcal{O}^{rel}_{M_I\mathcal{Y}_{\mathcal{B}}}.$$

By the above, proposition 3.1.5 can be re-formulated in this particular case in the following way.

Corollary 3.1.1. *For a chiral algebra $\mathcal{B}^{cl}$ we have the following equivalence*

$$\left\{ \textit{ Chiral } \mathcal{B}^{cl}\textit{-modules on } X^I \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{continuous modules for the sheaf of topological} \\ \textit{associative algebras } \mathcal{O}^{rel}_{M_I\mathcal{Y}_{\mathcal{B}}} \end{array} \right\}.$$

In the special case in which the commutative chiral algebra $\mathcal{B}^{cl}$ comes as the $\mathcal{D}_X$-algebra corresponding to a $\mathcal{O}_X$-algebra under the functor J_X from (2.17), we will simply write JZ_I and MZ_I for the spaces $J_I\mathcal{Y}_{J_X(Z)}$ and $M_I\mathcal{Y}_{J_X(Z)}$ respectively. Note that, by the adjunction property of J_X we have

$$MZ_I(S) := \left\{ \begin{array}{c} S \xrightarrow{(\phi_1,\ldots,\phi_n)} X^I \\ \alpha \text{ section } D^\circ_{\underline{\phi}} \to M \end{array} \right\},$$

and JZ_I is the subfunctor consisting of those sections that are well defined on $\cup_{i\in I}\Gamma_{\phi_i}$. Therefore in this case we take $\mathcal{B}^{cl}$ to be $\mathcal{O}_{J_X(Z)}$ and we have an equivalence

$$\left\{ \text{ Chiral } \mathcal{O}_{J_X(Z)}\text{-modules on } X^I \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{continuous modules for the topological} \\ \text{associative algebra } \mathcal{O}^{rel}_{MZ_I} \end{array} \right\}.$$

3.2 Action of a group scheme on a category

Let $\mathcal{C}$ be an abelian category, and let G be a group scheme. A *weak action* of G on $\mathcal{C}$ consists of functors:

$$act_S^* : \mathrm{QCoh}(S) \otimes \mathcal{C} \to \mathrm{QCoh}(G \times S) \otimes \mathcal{C},$$

functorial in $S \to G$, and two functorial isomorphisms related to these functors.

- (unit) The first isomorphism is between the identity functor in $\mathcal{C}$ and the composition
$$\mathcal{C} \xrightarrow{act^*} \mathrm{QCoh}(G) \otimes \mathcal{C} \to \mathcal{C},$$
where the second arrow corresponds to the restriction to $1 \in G$.

- (associativity constrain) The second isomorphism is between the two functors $\mathcal{C} \to \mathrm{QCoh}(G \times G) \otimes \mathcal{C}$ given by the following diagram
$$\begin{tikzcd} \mathcal{C} \arrow[r, "act^*"] \arrow[d, "act^*"] & \mathrm{QCoh}(G) \otimes \mathcal{C} \arrow[d, "act_G^*"] \\ \mathrm{QCoh}(G) \otimes G \arrow[r, "mult^*"] & \mathrm{QCoh}(G \times G) \otimes \mathcal{C}. \end{tikzcd}$$

Example 3.2.1. The tautological map

$$triv^* : \mathcal{C} \to \mathrm{QCoh}(G) \otimes \mathcal{C}, \quad C \mapsto \mathcal{O}_G \otimes C,$$

defines an action of G on $\mathcal{C}$. We will refer to this action as the *trivial* G-action on $\mathcal{C}$.

Definition 3.2.1. We say that an element $C \in \mathcal{C}$ is weakly equivariant, if it comes equipped with an isomorphism

$$act^*(C) \simeq triv^*(C) \tag{3.10}$$

which is compatible with the associativity constraint of the G-action on $\mathcal{C}$.

We denote by $\mathcal{C}^{w,G}$ the category consisting of weakly G-equivariant objects.

Example 3.2.2. If we take $\mathcal{C} = Vect$, with the trivial G-action, then we have $Vec^{w,G} = Rep(G)$.

3.2.1 Strong action on $\mathcal{C}$

For a group scheme (or ind-scheme) G, we let $G_{(1)} = Spf(\mathbb{C} \oplus \epsilon \cdot \mathfrak{g}^*)$ denote the first infinitesimal neighborhood of the unit $1 \in G$, and we let $\widehat{G}_1$ be the formal completion of G at the unit. A weak action of G on $\mathcal{C}$ is called *strong* if either of the following equivalent conditions are satised:

- We are given have functorial isomorphisms between the functors act_S^* for any pair of innitesimally close points $\phi, \phi' : S \to G$, satisfying certain compatibility conditions.

- We are given functorial trivializations of act_S^* for any $S \to \widehat{G}_1$, respecting the unit, the multiplication, and the adjoint action of G on $\widehat{G}_1$.

Remark 3.2.1. The second condition above is actually equivalent to a weaker version. It is enough to be given a trivialization not on $\widehat{G}_1$, but on $G_{(1)}$:

$$act^*|_{G_{(1)}} \simeq triv^*|_{G_{(1)}} \tag{3.11}$$

which is compatible with the unit and the Lie algebra structure.

Definition 3.2.2. Given a category $\mathcal{C}$ with a strong action of G, an object in $C \in \mathcal{C}^{w,G}$ is called *strongly equivariant*, if the two isomorphisms

$$act^*(C) \simeq triv^*(C)$$

coming from (3.10) and (3.11) coincide.

We denote by $\mathcal{C}^G$ the category consisting of strongly equivariant objects.

3.2.2 The case $\mathcal{C} = A$-mod.

Let us consider a scheme (or ind-scheme as defined in chapter ??) X and a group scheme G acting on it. Consider the map

$$act : G \times X \to X.$$

This defines a functor $act^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(G) \otimes \mathrm{QCoh}(X)$, and it defines an action of G on $\mathrm{QCoh}(X)$ in the above sense. We have also the projection $triv : G \times X \to X$, and we can consider the diagram

$$G \times X \underset{triv}{\overset{act}{\rightrightarrows}} X .$$

In this case, the objects of $\mathrm{QCoh}(X)^{w,G}$ are exactly those modules over X whose pull-back on $G \times X$ along the above two maps are isomorphic. Obviously we have a functor

$$\mathrm{QCoh}(X)^{w,G} \to \mathrm{QCoh}(X/G).$$

The example 3.2.2 corresponds to the case $X = Spec(k)$.

Let now X be affine, $X = Spec(A)$, and suppose that A is acted on by a group G. Then we have an action of G on the category of A-modules, that corresponds to the above action on $\mathrm{QCoh}(X)$.

More generally, let A be an associative algebra acted on by G. This defines a weak action of G on the category of A-modules. In fact, we have a map

$$G \times A\text{-mod} \to A\text{-mod}$$

given by $(g, M) \to gM$, where $gM \simeq M$ as vector spaces, but the action of A on it is defined by $a.(gm) = ((g.a)m)$. This defines the required map

$$A\text{-mod} \to \mathrm{QCoh}_G \otimes A\text{-mod}, \quad M \mapsto \widetilde{M},$$

where $\widetilde{M}(g) = gM \in A$-mod. The objects of $(A\text{-mod})^{w,G}$ are those A-modules M that are endowed with an action of G.

As we have seen earlier, given a scheme (or an ind-scheme X) acted on by G, we have a weak-action of G on the category $\mathrm{QCoh}(X)$. However, by considering the category of D-modules on it, we see how this category carries a strong action of G. The weakly equivariant objects in $\mathcal{D}_X$-mod are exactly the weakly equivariant $\mathcal{D}_X$-modules, and the strongly equivariant objects are the same as D-modules on the quotient X/G.
As before, if we take an affine scheme $X = Spec(A)$, we can translate what it means for the G-action to be strong in terms of the G-action on A. In this case, a strong action on A-modules, translates into the existence of a map

$$\mathfrak{g} \to A$$

that coincides with the derived action of $\mathfrak{g}$ on A coming from the G-action on it.

More generally, let A be any associative algebra, acted on by G via a map $G \to Aut(A)$. The action of G on the category of A-modules is strong if the derivative of the above map $\mathfrak{g} \to Der(A)$ factors through the algebra if inner derivations via a

G-equivariant map ϕ,

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d, "\psi"] & Der(A). \\ A \arrow[ur] & \end{tikzcd}$$

The objects of $(A\text{-mod})^G$ are those A-modules M for which the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d(act_G)"] \arrow[d] & End(M), \\ A \arrow[ur, "act"] & \end{tikzcd}$$

where $act_G : G \to Aut(M)$ is the G-action of G on M coming from the forgetful functor

$$(A\text{-mod})^{w,G} \to Vect^{w,G} \simeq Rep(G).$$

3.3 Action of a factorization group on a factorization category

We now want to explain what it means for a factorization category to be acted on by a factorization group $\mathcal{G}$.

Let X be a smooth curve over k, and let $\mathcal{C}$ be a factorization category and $\mathcal{G} = \{I \to \mathcal{G}_I\}$ a factorization group (see 3.1.7).

Definition 3.3.1. By a *weak action* of $\mathcal{G}$ on $\mathcal{C}$ we mean a collection of functors

$$act_I^* : \mathcal{C}_{X^I} \to \mathrm{QCoh}(\mathcal{G}_I) \underset{\mathrm{QCoh}(X^I)}{\otimes} \mathcal{C}_{X^I},$$

for every finite set I, compatible with the factorization isomorphisms. These functors should satisfy the following:

- (unit) The first isomorphism is between the identity functor in $\mathcal{C}_{X^I}$ and the composition

$$\mathcal{C}_{X^I} \xrightarrow{act_I^*} \mathrm{QCoh}(\mathcal{G}_I) \underset{\mathrm{QCoh}(X^I)}{\otimes} \mathcal{C}_{X^I} \to \mathcal{C}_{X^I},$$

where the second arrow corresponds to the restriction to the composition with the pull-back along the identity section $X^I \to \mathcal{G}_I$.

- (associativity costraint) The second isomorphism is between the two functors $\mathcal{C}_{X^I} \to \mathrm{QCoh}(\mathcal{G}_I \times \mathcal{G}_I) \otimes \mathcal{C}_{X^I}$ given by the following diagram

$$\begin{tikzcd}
\mathcal{C}_{X^I} \arrow[r, "act_I^*"] \arrow[d, "act_I^*"] & \mathrm{QCoh}(\mathcal{G}_I) \underset{\mathrm{QCoh}(X^I)}{\otimes} \mathcal{C}_{X^I} \arrow[d, "act_{\mathcal{G},I}^*"] \\
\mathrm{QCoh}(\mathcal{G}_I) \underset{\mathrm{QCoh}(X^I)}{\otimes} \mathcal{C}_{X^I} \arrow[r, "mult_I^*"] & \mathrm{QCoh}(\mathcal{G}_I \times \mathcal{G}_I) \otimes \mathcal{C}_{X^I}.
\end{tikzcd}$$

Example 3.3.1. For every I, the tautological map

$$triv_I^* : \mathcal{C}_{X^I} \to \mathrm{QCoh}(\mathcal{G}_I) \underset{\mathrm{QCoh}(X^I)}{\otimes} \mathcal{C}_{X^I}, \quad C \mapsto \mathcal{O}_{\mathcal{G}_I} \otimes C,$$

defines an action of $\mathcal{G}$ on $\mathcal{C}$. We will refer to this action as the *trivial* $\mathcal{G}$-action on $\mathcal{C}$.

•

Definition 3.3.2. Let $\mathcal{C}$ be a factorization category acted on by a factorization group $\mathcal{G}$. We say that an object $\mathcal{M} \in \mathcal{C}$ is weakly equivariant, if for every I we have

$$act_I^*(\mathcal{M}) \simeq triv_I^*(\mathcal{M}).$$

We will be interested in the case of a group $\mathcal{D}_X$-scheme acting on a chiral algebra $\mathcal{A}$. Although we can define what it means for an action of a factorization group to be strong, we will spell out the definition only in the case $\mathcal{C} = \mathcal{A}$-mod.

3.3.1 Action on the category $\mathcal{A}$-mod

Let $\mathcal{A}$ be a chiral algebra, and let $\mathcal{G}_X$ be a group $\mathcal{D}_X$-scheme. Denote by $\mathcal{G}$ the corresponding factorization group. We want to apply the discussion above to the case of $\mathcal{C} = \mathcal{A}$-mod. For this, we first need to define what it means for the $\mathcal{D}_X$-scheme $\mathcal{G}_X$ to act on the chiral algebra $\mathcal{A}$.

Given a group $\mathcal{D}_X$-scheme $\mathcal{G}_X$, we consider its coordinate ring $\mathcal{O}_{\mathcal{G}_X}$ as a commutative chiral algebra endowed with a map

$$\delta : \mathcal{O}_{\mathcal{G}_X} \to \mathcal{O}_{\mathcal{G}_X} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{\mathcal{G}_X}$$

of chiral algebras, i.e such that the following diagram commutes

$$\begin{tikzcd}
j_*j^*(\mathcal{O}_{\mathcal{G}_X} \otimes \mathcal{O}_{\mathcal{G}_X}) \arrow[r] \arrow[d] & \Delta_*(\mathcal{O}_{\mathcal{G}_X}) \arrow[d] \\
j_*j^*(\mathcal{O}_{\mathcal{G}_X} \otimes \mathcal{O}_{\mathcal{G}_X} \boxtimes \mathcal{O}_{\mathcal{G}_X} \otimes \mathcal{O}_{\mathcal{G}_X}) \arrow[r] & \Delta_*(\mathcal{O}_{\mathcal{G}_X} \otimes \mathcal{O}_{\mathcal{G}_X}).
\end{tikzcd}$$

Definition 3.3.3. An action of a group $\mathcal{D}_X$-scheme $\mathcal{G}_X$ on a chiral algebra $\mathcal{A}$, is a $\mathcal{D}_X$-map of chiral algebras

$$\mathcal{A} \xrightarrow{act} \mathcal{A} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{\mathcal{G}_X},$$

such that $(act \otimes id) \circ act = (id \otimes \delta) \circ act$.

The condition on *act* to be a map of chiral algebras, translates into the commutativity of the following diagram:

$$\begin{tikzcd}
j_*j^*(\mathcal{A} \otimes \mathcal{A}) \arrow[r] \arrow[d] & \Delta_*(\mathcal{A}) \arrow[d] \\
j_*j^*(\mathcal{A} \otimes \mathcal{O}_{\mathcal{G}_X} \boxtimes \mathcal{A} \otimes \mathcal{O}_{\mathcal{G}_X}) \arrow[r] & \Delta_*(\mathcal{A} \otimes \mathcal{O}_{\mathcal{G}_X}).
\end{tikzcd}$$

In this case the category $\mathcal{A}$-mod$_{X^I}$ is acted on by the factorization group $\mathcal{G}$ attached to $\mathcal{G}_X$, in other words, we have functors

$$\mathcal{A}\text{-mod}_{X^I} \to \mathcal{A}\text{-mod}_{X^I} \underset{\mathrm{QCoh}(X^I)}{\otimes} \mathrm{QCoh}(\mathcal{G}_I).$$

To see this, note that the action of $\mathcal{G}_X$ on $\mathcal{A}$ defines an automorphism ψ^{act} of the chiral algebra $\mathcal{A} \otimes \mathcal{O}_{\mathcal{G}_X}$. This automorphism is defines as the composition

$$\psi^{act} : \mathcal{A} \otimes \mathcal{O}_{\mathcal{G}_X} \xrightarrow{act \otimes id} \mathcal{A} \otimes \mathcal{O}_{\mathcal{G}_X} \otimes \mathcal{O}_{\mathcal{G}_X} \xrightarrow{id_{\mathcal{A}} \otimes m} \mathcal{A} \otimes \mathcal{O}_{\mathcal{G}_X},$$

where m denotes the commutative product on $\mathcal{O}_{\mathcal{G}_X}$ when regarded as a commutative chiral algebra on X. We denote by ψ^{act}_I the map $\psi^{act}_I : \mathcal{A}_I \otimes \mathcal{O}_{\mathcal{G}_I} \to \mathcal{A}_I \otimes \mathcal{O}_{\mathcal{G}_I}$ corresponding to ψ^{act}. Moreover, we have an obvious forgetful functor

$$\mathcal{O}_{\mathcal{G}_X}\text{-mod}_{X^I} \to \mathrm{QCoh}(\mathcal{G}_I).$$

We define the functor $\mathcal{A}\text{-mod}_{X^I} \to \mathcal{A}\text{-mod}_{X^I} \underset{\mathrm{QCoh}(X^I)}{\otimes} \mathrm{QCoh}(\mathcal{G}_I)$ in the following way. To a module $\mathcal{M}_I$ over X^I we assign the image under the above forgetful functor of the object $\widetilde{\mathcal{M}}_I \in \mathcal{A}\text{-mod}_{X^I} \underset{\mathrm{QCoh}(X^I)}{\otimes} \mathcal{O}_{\mathcal{G}_X}\text{-mod}_{X^I}$ defined in the following way. For every finite set K, we take $(\widetilde{\mathcal{M}}_I)_K$ to be $\mathcal{M}_{I \sqcup K} \otimes \mathcal{O}_{\mathcal{G}_{I \sqcup K}}$ and for every partition $\pi : I \sqcup K \twoheadrightarrow K'$ of $I \sqcup K$, such that $I \subset \pi^{-1}(k') =: \overline{I}$ for some $k \in K'$, we define factorization isomorphisms as the composition

$$\begin{aligned}
(j^{(I/J)})^*(\widetilde{\mathcal{M}}_{I \sqcup K}) = (j^{(I/J)})^*(\mathcal{M}_{I \sqcup K} \otimes \mathcal{O}_{\mathcal{G}_{I \sqcup K}}) &\xrightarrow{\sim} \\
\xrightarrow{\sim} \mathcal{A}_{I_1} \boxtimes \cdots \boxtimes \mathcal{A}_{I_{n-1}} \boxtimes \mathcal{M}_{\overline{I}} \otimes \mathcal{O}_{\mathcal{G}_{I_1}} \boxtimes \cdots \boxtimes \mathcal{O}_{\mathcal{G}_{I_{n-1}}} \boxtimes \mathcal{O}_{\mathcal{G}_{\overline{I}}}\Big|_{U^{(I/J)}} &\simeq \\
\simeq \mathcal{A}_{I_1} \otimes \mathcal{O}_{\mathcal{G}_{I_1}} \boxtimes \cdots \boxtimes \mathcal{A}_{I_{n-1}} \otimes \mathcal{O}_{\mathcal{G}_{I_{n-1}}} \boxtimes \mathcal{M}_{\overline{I}} \otimes \mathcal{O}_{\mathcal{G}_{\overline{I}}}\Big|_{U^{(I/J)}} &\to \\
\xrightarrow{\psi^{act}_{I_1} \boxtimes \cdots \boxtimes \psi^{act}_{I_{n-1}} \boxtimes id} \mathcal{A}_{I_1} \otimes \mathcal{O}_{\mathcal{G}_{I_1}} \boxtimes \cdots \boxtimes \mathcal{A}_{I_{n-1}} \otimes \mathcal{O}_{\mathcal{G}_{I_{n-1}}} \boxtimes \mathcal{M}_{\overline{I}} \otimes \mathcal{O}_{\mathcal{G}_{\overline{I}}}\Big|_{U^{(I/J)}} &\simeq \\
\simeq \mathcal{A}_{I_1} \boxtimes \cdots \boxtimes \mathcal{A}_{I_{n-1}} \boxtimes \mathcal{M}_{\overline{I}} \otimes \mathcal{O}_{\mathcal{G}_{I_1}} \boxtimes \cdots \boxtimes \mathcal{O}_{\mathcal{G}_{I_{n-1}}} \boxtimes \mathcal{O}_{\mathcal{G}_{\overline{I}}}\Big|_{U^{(I/J)}} &
\end{aligned}$$

,

where $I \sqcup K = I_1 \sqcup \cdots \sqcup I_{n-1} \sqcup \overline{I}$, and where the first isomorphisms is the one coming from the structure of $\mathcal{A}$-module on X^I on $\mathcal{M}_I$.

The objects in $(\mathcal{A}\text{-mod}_{X^I})^{w,\mathcal{G}_I}$ are those chiral modules $\mathcal{M}_I$ endowed with an action of the group $\mathcal{D}_X$-scheme $\mathcal{G}_I$.

3.3.2 Strong action on the category $\mathcal{A}$-mod

We now want to understand what it means to have a strong action of $\mathcal{G}$ on $\mathcal{A}$-mod. For this, we will need some preliminaries concerning the notion of Lie$^!$-coalgebras.

Definition 3.3.4. A Lie$^!$-coalgebra on X is a $\mathcal{D}_X$-module $\check{\mathfrak{L}}$ on X endowed with a map $\delta : \check{\mathfrak{L}} \to \check{\mathfrak{L}} \overset{!}{\otimes} \check{\mathfrak{L}}$ satisfying

a) $(Id + \tau) \circ \delta = 0$

b) $(Id + \nu + \nu^2) \circ (id + \delta) \circ \delta = 0$

where $\tau(v \otimes w) = w \otimes v$ and $\nu(v \otimes w \otimes u) = w \otimes u \otimes v$.

Remark 3.3.1. Consider now $\mathfrak{L} = Hom_{\mathcal{D}_X}(\check{\mathfrak{L}}, \mathcal{D}_X \otimes \Omega_X)$. This $\mathcal{D}_X$-module has a structure of a Lie*-algebra. In fact, more generally, for any $\mathcal{D}_X$-modules M, N, V, having a map

$$M \to N \overset{!}{\otimes} V$$

is the same as having a map

$$M \boxtimes \check{N} \to \Delta_*(V)$$

and iterating the process again gives a map $M \boxtimes \check{N} \boxtimes \check{V} \to \Delta_*(\Omega_X)$. This game allows as to construct a map $\mathfrak{L} \boxtimes \mathfrak{L} \to \Delta_*(L)$ from the original one. The Jacobi identity follows easily from conditions a) and b). Hence we obtain that $\mathfrak{L}$ is a Lie* algebra if and only if $\check{\mathfrak{L}}$ is a Lie$^!$-coalgebra. Note also that in the same way we can show that for any $\check{\mathfrak{L}}$-module, the map

$$M \to M \otimes \check{\mathfrak{L}} \tag{3.12}$$

gives us a map $M \boxtimes \mathfrak{L} \to \Delta_*(M)$ which defines an action of $\mathfrak{L}$ on M.

Consider now a group $\mathcal{D}_X$-scheme $\mathcal{G}_X$ and consider the unit section $X \xrightarrow{u} \mathcal{G}_X$. As it is explained in [NR], $\check{\mathfrak{L}}_{\mathcal{G}_X} := u^*(\Omega_{\mathcal{G}_X/X})$ has a natural structure of a Lie$^!$-coalgebra. In fact this is very similar to the fact that for an algebraic group G, Ω_G has a structure of a Lie coalgebra. As we have said before, we can now consider the Lie*-algebra

$$\mathfrak{L}_{\mathcal{G}_X} = \mathrm{Hom}_{\mathcal{G}_X}(\check{\mathfrak{L}}_{\mathcal{G}_X}, \mathcal{D}_X \otimes \Omega_X).$$

More generally, we can consider the group schemes $\mathcal{G}_I$ over X^I, and take the pull back along the identity section $X^I \xrightarrow{u_I} \mathcal{G}_I$ of the sheaf of differentials of $\mathcal{G}_I$. We denote such pull back by $\check{\mathfrak{L}}_{\mathcal{G}_I}$

$$\check{\mathfrak{L}}_{\mathcal{G}_I} := u_I^*(\Omega_{\mathcal{G}_I/X^I}).$$

In this case the $\mathcal{D}_{X^I}$-module $\check{\mathfrak{L}}_{\mathcal{G}_I}$ acquires a structure of coalgebra over X^I. Similarly to what we have seen before, a module $\mathcal{M}_I$ for $\mathfrak{L}_{\mathcal{G}_I}$, naturally becomes a comodule for the coalgebra $\check{\mathfrak{L}}_{\mathcal{G}_I}$,

$$\{\ \mathfrak{L}_{\mathcal{G}_I}\text{-modules}\ \} \leftrightarrow \{\ \check{\mathfrak{L}}_{\mathcal{G}_I}\text{-comodules}\ \}. \tag{3.13}$$

Let now $\mathfrak{L}_{\mathcal{G}_X}$ be the Lie*-algebra defined earlier, and let $\mathfrak{L}^{(I)}_{\mathcal{G}_X,0}$ be the sheaf of topological Lie algebras defined in 3.8. Following [NR] , we have the following proposition.

Proposition 3.3.1. *Let $\mathcal{G}_X$ be an affine $\mathcal{D}_X$-group scheme with Lie*-algebra* $\mathfrak{L}_{\mathcal{G}_X} =$

$\mathrm{Hom}_{\mathcal{G}_X}(\check{\mathfrak{L}}_{\mathcal{G}_X}, \mathcal{D}_X \otimes \Omega_X)$. *Then the Lie coalgebra* $\check{\mathfrak{L}}_{\mathcal{G}_I}$ *is isomorphic to* $\check{\mathfrak{L}}^{(I)}_{\mathcal{G}_X,0}$.

Now note that, given a factorization group $\mathcal{G}$, a module $\mathcal{M}_I$ for $\mathcal{G}_I$ over X^I naturally becomes a comodule for $\check{\mathfrak{L}}_{\mathcal{G}_I}$. In fact the coaction map is obtained in the same way as in the case of $X = pt$ (in which case $\check{\mathfrak{L}}_{\mathcal{G}_X} = \mathfrak{g}^*$), i.e. you consider the composition

$$M \to M \otimes \mathcal{O}_G \to M \otimes \mathcal{O}_G/(e)^2 \to M \otimes \mathfrak{g}^*$$

where (e) is the maximal ideal in $\mathcal{O}_G$ corresponding to the identity element. In particular, by proposition 3.1.4, and the equivalence (3.13), the we have the following corollary.

Corollary 3.3.1. *For a* $\mathcal{D}_X$*-group scheme* $\mathcal{G}_X$*, there is a functor*

$$\{\mathcal{G}_I\text{-}modules\} \to \left\{Lie^*\text{-}\mathfrak{L}_{\mathcal{G}_X}\text{-}modules\ on\ X^I\right\}$$

Consider now a chiral algebra $\mathcal{A}$ with an action of $\mathcal{G}_X$

$$\mathcal{A} \to \mathcal{A} \otimes \mathcal{O}_{\mathcal{G}_X}.$$

As for usual algebras, corollary 3.3.1 allows us to derive, from such map, a Lie*-action

$$\mathfrak{L}_{\mathcal{G}_X} \boxtimes \mathcal{A} \to \Delta_*(\mathcal{A}) \tag{3.14}$$

of the Lie*-algebra $\mathfrak{L}_{\mathcal{G}_X}$ on it. We can now define what it means for the $\mathcal{G}$-action on $\mathcal{A}$-mod to be strong.

Definition 3.3.5. Let $\mathcal{A}$ be a chiral algebra acted upon by a group $\mathcal{G}_X$-scheme $\mathcal{G}_X$. This action is called *strong* if there exists a $\mathcal{G}_X$-equivariant map

$$\mathfrak{L}_{\mathcal{G}_X} \longrightarrow \mathcal{A}$$

such that the composition

$$\mathfrak{L}_{\mathcal{G}_X} \boxtimes \mathcal{A} \to \mathcal{A} \boxtimes \mathcal{A} \to \Delta_*(\mathcal{A}) \tag{3.15}$$

coincides with (3.14).

Consider now the action of $\mathcal{G}$ on the category $\mathcal{A}\text{-mod}_{X^I}$ of $\mathcal{A}$-modules on X^I. Let $\mathcal{M}_I$ be an object in $\mathcal{A}\text{-mod}_{X^I}^{w,\mathcal{G}}$. In particular it is a $\mathcal{G}_I$-module.

Definition 3.3.6. A module $\mathcal{M}_I$ in $\mathcal{A}\text{-mod}_{X^I}^{w,\mathcal{G}}$ is called strongly $\mathcal{G}$-equivariant if the two actions of the topological Lie algebra $\mathfrak{L}_{\mathcal{G}_X,0}^{(I)}$ on it coincide. Where the first action comes from the map

$$\mathfrak{L}_{\mathcal{G}_X} \to \mathcal{A}, \tag{3.16}$$

and the second action comes from the action of $\mathcal{G}_I$ and by corollary 3.3.1.

Equivalently, we see that $\mathcal{M}_I$ is strongly equivariant, if the action of $\mathfrak{L}_{\mathcal{G}_X,0}^{(I)}$ coming from 3.16, can be integrated to an action of the group $\mathcal{G}_I$. We will denote by $\mathcal{A}\text{-mod}_I^{\mathcal{G}}$ the category of strongly $\mathcal{G}$-equivariant objects in $\mathcal{A}\text{-mod}_I := \mathcal{A}\text{-mod}_{X^I}$, and by $\mathcal{A}\text{-mod}^{\mathcal{G}}$ the factorization category given by the assignment $I \to \mathcal{A}\text{-mod}_I^{\mathcal{G}}$.

Consider now the following general set-up. Recall from 3.1.3 that a chiral algebra morphism $\phi : \mathcal{A}^{cl} \to \mathcal{B}^{cl}$ defines an equivalence of factorization categories

$$\mathcal{B}\text{-mod}(\mathcal{A}\text{-mod}) \to \mathcal{B}\text{-mod}.$$

Suppose now that the chiral algebras $\mathcal{A}^{cl}$ and $\mathcal{B}^{cl}$ are acted on by a group $\mathcal{D}_X$-scheme $\mathcal{G}_X$ in a compatible way, in other words, suppose that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}^{cl} \arrow[r, "\delta_1"] \arrow[d, "\psi"'] & \mathcal{A}^{cl} \otimes \mathcal{O}_{\mathcal{G}_X} \arrow[d, "\psi \otimes Id"] \\ \mathcal{B}^{cl} \arrow[r, "\delta_2"] & \mathcal{B}^{cl} \otimes \mathcal{O}_{\mathcal{G}_X} \end{tikzcd} \quad (3.17)$$

Suppose that this action is strong. Denote by k_1 and k_2 the two $\mathcal{G}_X$-equivariant maps

$$\mathcal{L}_{\mathcal{G}_X} \xrightarrow{k_1} \mathcal{A}^{cl}, \quad \text{and} \quad \mathcal{L}_{\mathcal{G}_X} \xrightarrow{k_2} \mathcal{B}^{cl}.$$

The commutativity of the above diagram implies that the following diagram also commutes

$$\begin{tikzcd} \mathcal{L}_{\mathcal{G}_X} \boxtimes \mathcal{A}^{cl} \arrow[r] \arrow[d, "id \boxtimes \psi"'] & \Delta_*(\mathcal{A}^{cl}). \arrow[d, "\Delta_*(\psi)"] \\ \mathcal{L}_{\mathcal{G}_X} \boxtimes \mathcal{B}^{cl} \arrow[r] & \Delta_*(\mathcal{B}^{cl}) \end{tikzcd}$$

We have the following proposition:

Proposition 3.3.2. *In the conditions above, if the chiral algebra $\mathcal{B}^{cl}$ is in $\mathcal{A}$-mod$^{\mathcal{G}_X}$, then we have an equivalence*

$$\mathcal{B}\text{-}mod(\mathcal{A}\text{-}mod^{\mathcal{G}}) \simeq \mathcal{B}\text{-}mod^{\mathcal{G}}.$$

Proof. First of all, note that $\mathcal{B}^{cl}$ being in $\mathcal{A}$-mod$^{\mathcal{G}}$ is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} \mathcal{A}^{cl} \arrow[rr, "\psi"] & & \mathcal{B}^{cl}. \\ & \mathcal{L}_{\mathcal{G}_X} \arrow[ul, "k_2"] \arrow[ur, "k_1"'] & \end{tikzcd} \quad (3.18)$$

Now, let $\mathcal{M}_I$ be a strongly equivariant object in $\mathcal{B}$-mod($\mathcal{A}$-mod$_I$). This is the same as a collection of objects $\mathcal{M}_{I \sqcup K \sqcup J}$ satisfying the factorization property for $\mathcal{B}$ with respect to the finite set K, and the factorization property for $\mathcal{A}$ with respect to the finite set J. However, as explained earlier, being strongly $\mathcal{G}_X$-equivariant as a

$\mathcal{A}$-module on X^I is the same as requiring that the Lie algebra action of $\mathfrak{L}^{(I)}_{\mathcal{G}_X,0}$ on the module $\mathcal{M}_I$ can be integrated to an action of $\mathcal{G}_I$. However, when we regard $\mathcal{M}_I$ as an object in $\mathcal{B}\text{-mod}_I$, and hence we look at the $\mathfrak{L}^{(I)}_{\mathcal{G}_X,0}$-action on it coming from the map k_2, the commutativity of (3.18) implies that this action is also integrable. Moreover, the $\mathcal{G}_I$ action coming from it, corresponds to the $\mathcal{G}_I$-action on $\mathcal{M}_I$ coming from the weakly equivariance, in virtue of 3.17. This implies that the module $\mathcal{M}_I$, is naturally an object of $\mathcal{B}\text{-mod}_I^{\mathcal{G}_X}$.

□

3.3.3 Strong action on the category $\mathcal{D}_{crit}$-mod

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and with an invariant bilinear form κ. Recall the chiral algebras $\mathcal{A}_{crit}$ and $\mathcal{D}_{crit}$ defined in 2.1.3 and 2.4.1. In this section we will define a strong action of the group $\mathcal{D}_X$-scheme $J_X(G)$ on $\mathcal{A}_{crit}$ and $\mathcal{D}_{crit}$. We will therefore use the notion of "action of a factorization group on a factorization category" developed in 3.3.

Action of $J_X(G)$ on the chiral algebra $\mathcal{A}_\kappa$

Recall the construction of $\mathcal{A}_\kappa$ given in 2.1.3. It is constructed as the twisted-chiral envelope of the Lie*-algebra $L^\kappa_\mathfrak{g} = \mathfrak{g} \otimes \mathcal{D}_X \oplus \Omega(X)$.
We have a natural action of $L_\mathfrak{g}$ on $L^\kappa_\mathfrak{g}$ defined by

$$g \cdot (h + \omega) = [g, h]_{L_\mathfrak{g}} + \kappa_{\mathcal{D}_X}(g, h) + \omega,$$

for $g \in L_\mathfrak{g}$ and $(h + \omega) \in L^\kappa_\mathfrak{g}$.
Now consider the group-scheme $J_X(G)$. Since the bilinear form κ is $\mathrm{Ad}(G)$-invariant, we have a well defined action of $J_X(G)$ on $L^\kappa_\mathfrak{g}$ given as follows. Let k' be a horizontal section of $J_X(G)$ over X. This section corresponds to a section k of G over X. Set

$k' \cdot (h+\omega)$ to be

$$k' \cdot (h+\omega) := Ad_k(h) + \kappa_{\mathcal{D}_X}(k^{-1}dk, h) + \omega.$$

Strong action of $J_X(G)$ on $\mathcal{D}_{crit}$-mod

Recall from 2.4.1 the chiral algebra $\mathcal{D}_{crit}$. A better way of describing it is by using the factorization picture. For this, we can define it in the following way. For every finite set I, we define $\mathcal{D}_{crit,I}$ to be

$$\mathcal{D}_{crit,I} := U(\mathfrak{L}^{(I)}_{crit}) \underset{U(\mathfrak{L}^{(I)}_{0,crit})}{\otimes} \mathcal{O}_{JG_I},$$

where $\mathfrak{L}^{(I)}_{crit}$ and $\mathfrak{L}^{(I)}_{0,crit}$ are the topological Lie algebras over X^I attached to $L^{crit}_{\mathfrak{g}}$ as defined in (3.8). As it is proven in [AG], $\mathcal{D}_{crit}$ comes equipped with two embeddings

$$\mathcal{A}_{crit} \xrightarrow{l} \mathcal{D}_{crit} \xleftarrow{r} \mathcal{A}_{crit}. \tag{3.19}$$

These two embedding endow $\mathcal{D}_{crit}$ with a structure of chiral $\mathcal{A}_{crit}$-bimodule. In particular, by considering the right $\mathcal{A}_{crit}$-action, we have the following lemma.

Lemma 3.3.1. *The chiral algebra $\mathcal{D}_{crit}$ is an algebra in $\mathcal{A}_{crit}$-mod.*

From the factorization description of $\mathcal{D}_{crit}$, we see that the group $\mathcal{D}_X$-scheme $J_X(G)$ acts on it via right-multiplication. We also have a natural map

$$L_{\mathfrak{g}} \to \mathcal{D}_{crit}$$

given by the composition

$$L_{\mathfrak{g}} \to \mathcal{A}_{crit} \xrightarrow{r} \mathcal{D}_{crit}.$$

It is not hard to see that, from the construction of the right embedding $r : \mathcal{A}_{crit} \to \mathcal{D}_{crit}$ given in [AG], the Lie*-action of $L_{\mathfrak{g}}$ on $\mathcal{D}_{crit}$ coming from the above map, coincides with the Lie*-action coming from the $J_X(G)$-action on it. Therefore we have a strong action of $J_X(G)$ on $\mathcal{D}_{crit}$, and hence fore, a strong action of $J_X(G)$ on the category $\mathcal{D}_{crit}$-mod.

Recall that we have defined an action of $J_X(G)$ on $\mathcal{A}_{crit}$ whose induced Lie*-$L_{\mathfrak{g}}$-action is given by the natural map

$$L_{\mathfrak{g}} \to \mathcal{A}_{crit}$$

followed by the Lie*-bracket on $\mathcal{A}_{crit}$. Consider now the chiral algebra map

$$\mathcal{A}_{crit} \xrightarrow{r} \mathcal{D}_{crit}.$$

Both $\mathcal{A}_{crit}$ and $\mathcal{D}_{crit}$ are equipped with a strong action of $J_X(G)$, and moreover, the maps from $L_{\mathfrak{g}}$ to $\mathcal{A}_{crit}$ and $\mathcal{D}_{crit}$ for these actions fit into the commutative diagram

$$\begin{tikzcd} \mathcal{A}_{crit} \arrow[r, "r"] & \mathcal{D}_{crit}. \\ L_{\mathfrak{g}} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Consider now the factorization categories $\mathcal{A}_{crit}$-mod^{JG} and $\mathcal{D}_{crit}$-mod^{JG}. By lemma 3.3.1 it makes sense to consider the factorization category $\mathcal{D}_{crit}$-mod($\mathcal{A}_{crit}$-mod^{JG}). By proposition 3.3.2, we have the following.

Proposition 3.3.3. *We have an equivalence of factorization categories*

$$\mathcal{D}_{crit}\text{-}mod(\mathcal{A}_{crit}\text{-}mod^{JG}) \simeq \mathcal{D}_{crit}\text{-}mod^{JG}.$$

In other words, $\mathcal{D}_{crit}$ is a strongly $J_X(G)$-equivariant objects in $\mathcal{A}_{crit}$-mod.

3.4 The Beilinson-Drinfeld Grassmannian and critically twisted D-modules on it

In this section we are going to introduce the main players of this work: the *Beilinson-Drinfeld Grassmannian* and the category of *critically-twisted D-modules on it.* We will use the language introduced in Section 3.1. In particular, the Beilinson-Drinfeld Grassmannian Gr_G will be a factorization space, and the category of D-modules on it a factorization category.

For a smooth curve X over k, the *Beilinson-Drinfeld Grassmannian*, denoted by Gr_G, generalizes the well-known affine Grassmannian $\mathrm{Gr}_{G,x}$ classifying G-bundles on X with a given trivialization outside a point $x \in X$. For every finite set I, we define a space $\mathrm{Gr}_{G,I}$ over X^I. The factorization space Gr_G is given by the assignment

$$I \to \mathrm{Gr}_{G,I}.$$

We start by defining, in 3.4.2, the *local* Beilinson-Drinfeld Grassmannian. We will then show how the local Beilinson-Drinfeld Grassmannian is equivalent to Gr_G. In proposition 3.4.1 we will show how this equivalence allows us to present the spaces $\mathrm{Gr}_{G,I}$ as quotients of two group schemes over X^I. In 3.4.2 we will then define the category $D_{crit}\text{-mod}(\mathrm{Gr}_G)$ of critically-twisted D-modules on Gr_G, using the existence of a canonical line bundle $\mathcal{L}_{crit,I}$ over $\mathrm{Gr}_{G,I}$, as explained in 3.4.3.

3.4.1 The Beilinson-Drinfeld Grassmannian

Let G be a semi-simple algebraic group of adjoint type, and let $S = Spec(A)$ be an affine scheme. Before going into the definition of Gr_G, we will recall some notions regarding families of bundles/G-bundles over X. These will be used in 3.21 to obtain a more manageable description of Gr_G.

For every set I, with $|I| = n$, and a map $\underline{\phi} = (\phi_1, \ldots, \phi_n)$

$$S \overset{\phi}{\rightarrow} X^I,$$

recall the schemes $D_{\underline{\phi}}$ and $D^{\circ}_{\underline{\phi}}$ defined in 3.3 and 3.4.

Definition 3.4.1. For X and S as above, we define the category of *gluing data* to be the category of triples $(\mathcal{M}_{X^{\circ}_S}, \mathcal{M}_{D_{\underline{\phi}}}, \gamma)$, where $\mathcal{M}_{X^{\circ}_S}$ is a bundle on X°_S, $\mathcal{M}_{D_{\underline{\phi}}}$ is a bundle over $D_{\underline{\phi}}$ and γ is an isomorphism

$$\gamma : \mathcal{M}_{X^{\circ}_S}|_{D^{\circ}_{\underline{\phi}}} \simeq \mathcal{M}_{D_{\underline{\phi}}}|_{D^{\circ}_{\underline{\phi}}}.$$

Morphisisms in this category are defined as morphisisms of vector bundles compatible with the isomorphisisms γ's.

Consider now a vector bundle $\mathcal{M}$ on X_S. The assignment

$$\mathcal{M} \rightarrow (\mathcal{M}|_{X^{\circ}_S}, \mathcal{M}|_{D_{\underline{\phi}}}, id)$$

defines a functor from the category of vector bundles on X_S to the category of gluing data. Moreover, by Beauville-Laszlo theorem, this functor is an equivalence. In the case of a Noetherian ring A, this is also a consequence of faithfully flat descent by looking at the diagram

$$\begin{array}{ccc} D^{\circ}_{\underline{\phi}} & \longrightarrow & X^{\circ}_S \\ \downarrow & & \downarrow \\ D_{\underline{\phi}} & \longrightarrow & X_S \end{array}$$

If instead of considering vector bundles on X_S, we consider G-bundles, the above statement remains true, and the same functor defines an equivalence

$$\{G\text{-bundles on } X_S\} \xrightarrow{Res} \left\{ \begin{array}{c} (P_{G,X^{\circ}_S}, P_{G,D_{\underline{\phi}}}, \gamma), \\ \text{where } \gamma : P_{G,X^{\circ}_S}|_{D^{\circ}_{\underline{\phi}}} \simeq P_{G,D_{\underline{\phi}}}|_{D^{\circ}_{\underline{\phi}}} \end{array} \right\} \tag{3.20}$$

where P_{G,X_S° and $P_{G,D_{\underline{\phi}}}$ are G-bundles on X_S° and $D_{\underline{\phi}}$ respectively.
We will now define the *local* and *global* Beilinson-Drinfeld Grassmannian and show how the two notions coincide.

Definition 3.4.2. For every finite set I, and test scheme S, we define $\mathrm{Gr}_{G,I}^{loc}$ to be

$$\mathrm{Gr}_{G,I}^{loc}(S) = \left\{ \begin{array}{c} S \xrightarrow{\underline{\phi}} X, \ (P_{G,D_{\underline{\phi}}}, \gamma) \\ \text{where } \gamma : P_{G,D_{\underline{\phi}}}|_{D_{\underline{\phi}}^\circ} \simeq P_{G,D_{\underline{\phi}}}^0|_{D_{\underline{\phi}}^\circ} \end{array} \right\},$$

where $P_{G,D_{\underline{\phi}}}^0$ denotes the trivial bundle on $D_{\underline{\phi}}$.

The assignment

$$I \to \mathrm{Gr}_{G,I}^{loc}$$

defines a factorization space, called *local Beilinson-Drinfeld Grassmannian.* The global version of the above is what is called the *Beilinson-Derinfeld Grassmannian* Gr_G. It is defined in the following way.

Definition 3.4.3. For every finite set I, and test scheme S, we define the space $\mathrm{Gr}_{G,I}$ over X^I to be

$$\mathrm{Gr}_{G,I}(S) = \left\{ \begin{array}{c} S \xrightarrow{\underline{\phi}} X, \ (P_{G,X_S}, \gamma) \\ \text{where } \gamma : P_{G,X_S}|_{X_S^\circ} \simeq P_{G,X_S}^0|_{X_S^\circ} \end{array} \right\}.$$

Clearly, the restriction functor, defines a map

$$\mathrm{Gr}_{G,I} \to \mathrm{Gr}_{G,I}^{loc}.$$

Moreover, if we have a pair $(P_{G,D_{\underline{\phi}}}, \gamma)$ in $\mathrm{Gr}_{G,I}^{loc}$, we can consider the object $(P_{G,X_S^\circ}^0, P_{G,D_{\underline{\phi}}}, \gamma)$ in the category of gluing data. Under the equivalence (3.20), this object corresponds to a G-bundle on X_S, with a trivialization on X_S°, i.e. it corresponds to an object in $\mathrm{Gr}_{G,I}$. In other words, for every I, we have an equivalence

$$\mathrm{Gr}_{G,I} \simeq \mathrm{Gr}^{loc}_{G,I} \tag{3.21}$$

The above equivalence allows us to describe the space $\mathrm{Gr}_{G,I}$ as a quotient of a group ind-scheme by a group scheme. Recall from 3.1.7 the group-scheme JG_I and MG_I over X^I. Then we have the following:

Proposition 3.4.1. *For every I, the space $Gr_{G,I}$ can be described as the quotient*

$$Gr_{G,I} \simeq MG_I/JG_I.$$

Proof. For every affine scheme $S = Spec(A)$, and $S \to X^I$, we can regard $MG_I(S)$ as

$$MG_I(S) = \mathrm{Hom}(Spec(A((t_1, \dots, t_n))), G) \simeq \left\{ \text{automorphisms of the trivial } G\text{-bundle } P^0_{G,D^\circ_{\underline{\phi}}} \right\},$$

where $t_i = (t - \phi_i^*(t))$, for t a local coordinate on X. Therefore, given an element $g \in MG_I(S)$, we can define an element in $\mathrm{Gr}^{loc}_{G,I}(S)$ simply by taking $P_{G,D_{\underline{\phi}}}$ to be the trivial G-bundle $P^0_{G,D_{\underline{\phi}}}$ on $D_{\underline{\phi}}$, and γ to be given by g. This assignment defines a map

$$MG_I(S) \twoheadrightarrow \mathrm{Gr}^{loc}_{G,I}(S),$$

and the fibers are acted simply transitively by the group of automorphisms of the trivial G-bundle on $D_{\underline{\phi}}$, which is isomorphic to $JG_I(S)$. Therefore we have $MG_I(S)/JG_I(S) \simeq \mathrm{Gr}^{loc}_{G,I}(S)$, and, by the equivalence (3.21), we have

$$MG_I/JG_I \xrightarrow{\sim} \mathrm{Gr}_{G,I}.$$

$\square$

In the next section we will be interested in the category of D-modules on $\mathrm{Gr}_{G,I}$. However, the presentation of $\mathrm{Gr}_{G,I}$ given in proposition 3.4.1 does not make clear how to define such category. However, as it is shown in [BD2], a remarkable feature of $\mathrm{Gr}_{G,I}$ is that it can be represented as the inductive limit of schemes of finite type. In fact we have the following.

Proposition 3.4.2. *The functor $Gr_{G,I}$ is represented by an ind-scheme of ind-finite type (see remark ?? for the definition). Moreover, if G is reductive, then $Gr_{G,I}$ is ind-projective.*

3.4.2 Critically twisted D-modules on Gr_G

We will now define the category of D-modules over the Beilinson-Drinfeld Grassmannian. Since for each I the space $\mathrm{Gr}_{G,I}$ can be represented by an ind-scheme of ind-finite type, we start by developing the notion of D-modules on every such scheme. Recall that if X is a classical scheme, we have a well defined forgetful functor $\mathcal{D}_X\text{-mod} \to \mathrm{QCoh}(X)$. When X is an ind-scheme, we will explain below the correct replacement for the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X that will be used to construct the *ind-version* of the above forgetful functor.

We start with the definition of the category $\mathrm{QCoh}^!(X)$ replacing the usual notion of quasi-coherent sheaves on X. Let X be an ind-scheme $X = \varinjlim_\alpha X_\alpha$, and denote by $i_{\beta,\alpha}$ the closed ambeddings $X_\alpha \xrightarrow{i_{\beta,\alpha}} X_\beta$. We have a pair of adjoint functors

$$i_{\beta,\alpha,*} : \mathrm{QCoh}(X_\alpha) \rightleftarrows \mathrm{QCoh}(X_\beta) \; : i^!_{\beta,\alpha} \,.$$

Consider the category $\mathcal{C} := \varprojlim_{i^!_{\beta,\alpha}} \mathrm{QCoh}(X_\alpha)$. By definition, we have a map maps

$$\mathcal{C} \xrightarrow{i^!_\alpha} \mathrm{QCoh}(X_\alpha).$$

Moreover, following [JB], we have the following.

Proposition 3.4.3. *The functors $i^!_\alpha : \mathcal{C} \to QCoh(X_\alpha)$ admit left adjoints $i_{\alpha,*}$,*

$$i_{\alpha,*} : QCoh(X_\alpha) \to \mathcal{C}.$$

We will denote by $\mathrm{QCoh}^!(X)$ the category defined as

$$\mathrm{QCoh}^!(X) := \varinjlim_{i_{\beta,\alpha,*}} \mathrm{QCoh}(X_\alpha). \tag{3.22}$$

For two objects $\mathcal{F}$ and $\mathcal{F}'$ in $\mathrm{QCoh}^!(X)$, a morphism $\phi : \mathcal{F} \to \mathcal{F}'$ is a collection of maps $\phi_\alpha : \mathcal{F}_\alpha \to \mathcal{F}'_\alpha$ compatible with $i_{\beta,\alpha,*}$. As it is shown in [JB], we have the following.

Proposition 3.4.4. *The category $QCoh^!(X)$ is equivalent to $\mathcal{C}$, i.e.*

$$\varinjlim_{i_{\beta,\alpha,*}} QCoh(X_\alpha) \simeq \varprojlim_{i^!_{\beta,\alpha}} QCoh(X_\alpha).$$

Remark 3.4.1. The importance of the above proposition is the following. The example of ind-scheme we should have in mind, it that of an affine ind-scheme, i.e the scheme corresponding to an abelian, complete, separated topological ring whose toplogy is generated by a ltered system of open ideals $\{I_i\}$, s.t. $I_i + I_j$ is finitely generated over $I_i \cap I_j$. The ind-scheme X is given as

$$X = \varinjlim_i X_i := \varinjlim_i Spec(A/I_i).$$

We are interested in the category of continuous discrete A-modules. This category is given by the following.

Denote by $i_{i,j}$ the embeddings $i_{i,j} : X_j \to X_i$, and consider the functors $i^!_{i,j}$ as functors

$$i^!_{i,j} : (A/I_i)\text{-mod} \to (A/I_j)\text{-mod}.$$

The category $A^{c,dis}$-mod of continuous discrete A-modules, is by definition

$$A^{c,dis}\text{-mod} := \varprojlim_{i^!_{i,j}} (A/I_i)\text{-mod}.$$

Since we can re-write the above as $\varprojlim_{i^!_{i,j}} \mathrm{QCoh}(X_i)$, it seems natural to think that this is the right category to consider. However, the presentation of it as a limit, has the disadvantage that it is not clear how to compute maps out of it. However, proposition 3.4.4 says that the category $A^{c,dis}$-mod can also be described as the colimit under the maps

$$i_{i,j,*} : (A/I_j)\text{-mod} \to (A/I_i)\text{-mod}.$$

We will now define the category of D-modules on an ind-scheme $X = \varinjlim X_\alpha$, with $X'_\alpha s$ schemes of finite type. Recall that for a closed embedding $i_{\beta,\alpha} : X_\alpha \to X_\beta$, we have a natural exact functor

$$i_{\beta,\alpha,!} : D\mathrm{mod}(X_\alpha) \to D\text{-mod}(X_\beta),$$

satisfying $F_\beta \circ i_{\beta,\alpha,!} = i_{\beta,\alpha,*} \circ F_\alpha$, where F_γ denotes the forgetful functor F_γ : D-mod$(X_\gamma) \to \mathrm{QCoh}(X_\gamma)$. We define the category D-mod(X) to be

$$D\text{-mod}(X) := \varinjlim_{i_{\beta,\alpha,!}} D\text{-mod}(X_\alpha).$$

Note that we clearly have a forgetful functor

$$F : D\text{-mod}(X) \to \mathrm{QCoh}^!(X).$$

Consider the Beilinson-Drinfeld Grassmannian Gr_G given by the assignment

$$I \to \mathrm{Gr}_{G,I}.$$

By proposition 3.4.1 and 3.4.2, we have:

- The functor $\mathrm{Gr}_{G,I}$ can be represented by the quotient MG_I/JG_I.

- The functor $\mathrm{Gr}_{G,I}$ is represented by an ind-scheme of ind-finite type.

We will write $\mathrm{Gr}_{G,I}$ as

$$\mathrm{Gr}_{G,I} = \varinjlim_i Y_i^I.$$

In virtue of the above, it makes sense to consider the category $\mathrm{QCoh}^!(\mathrm{Gr}_G)$ given by

$$I \to \mathrm{QCoh}^!(\mathrm{Gr}_{G,I}),$$

and the category of D-modules on Gr_G defined as

$$D\text{-mod}(\mathrm{Gr}_{G,I}) := \varinjlim_{k^I_{i,j,!}} D\text{-mod}(Y_i^I). \tag{3.23}$$

The assignment $I \to D\text{-mod}(\mathrm{Gr}_{G,I})$ defines a factorization category, denoted by $D\text{-mod}(\mathrm{Gr}_G)$.

Twisted D-modules on Gr_G

We will be interested in the category of twisted-D-modules on Gr_G. These are defined as modules on Gr_G endowed with an action of a sheaf of twisted-differential operators on Gr_G. In particular, we will be interested in the category $D_{crit}\text{-mod}(\mathrm{Gr}_G)$ of critically-twisted differential operators on Gr_G. We will now recall the definition of these objects.

Recall that, as it is explained in [BB], given a *Picard algebroid* on X, i.e. a sheaf of Lie algebras $\mathcal{P}$ on X, such that

$$0 \to \mathcal{O}_X \to \mathcal{P} \xrightarrow{\pi} T_X \to 0,$$

and such that, for any η and η' in $\mathcal{P}$ and $f \in \mathcal{O}_X$ we have $[\eta, f\eta'] = f[\eta, \eta'] + (\pi(\eta)f)\eta'$, we can consider the algebra $D_{\mathcal{P}}$. This is the universal algebra equipped with morphisms $i : \mathcal{O}_X \hookrightarrow D_{\mathcal{P}}$ and $i_{\mathcal{P}} : \mathcal{P} \hookrightarrow D_{\mathcal{P}}$ such that

- i is a morphism of algebras.
- $i_{\mathcal{P}}$ is a morphism of Lie algebras.
- for $f \in \mathcal{O}_X$, $\eta \in \mathcal{P}$ one has $i_{\mathcal{P}}(f\eta) = i(f)i_{\mathcal{P}}(\eta)$ and $[i_{\mathcal{P}}(\eta), i(f)] = i(\pi(\eta)f)$.

We call $D_{\mathcal{P}}$ a sheaf of twisted differential operators on X.
Consider now a line bundle $\mathcal{L}$ on X, and the algebroid $\mathcal{P}_{\mathcal{L}}$ defined as the algebroid of $\mathbb{G}_m$-invariant vector fields on the principal $\mathbb{G}_m$-bundle associated to $\mathcal{L}$. We have maps

$$0 \to \mathcal{O}_X \to \mathcal{P}_{\mathcal{L}} \to T_X \to 0,$$

making $\mathcal{P}_{\mathcal{L}}$ a Picard algebroid over X.

Definition 3.4.4. We define the category $D_{\mathcal{L}}$-mod(X) of $\mathcal{L}$-twisted $\mathcal{D}_X$-modules to be the category of $\mathcal{O}_X$-modules endowed with an action of the sheaf $D_{\mathcal{P}_{\mathcal{L}}}$.

Note that we have an equivalence of categories

$$\mathcal{D}_X\text{-mod} \simeq D_{\mathcal{L}}\text{-mod}(X)$$

given by $\mathcal{M} \to \mathcal{M} \otimes \mathcal{L}$.

Critically twisted differential operators on $\mathrm{Gr}_{G,I}$

Let's now go back to the space $\mathrm{Gr}_{G,I} = MG_I/JG_I$ over X^I and the category D-mod$(\mathrm{Gr}_{G,I})$ as defined in (3.23). For every I, we will construct a line bundle $\mathcal{L}_{crit,I}$

$$\mathcal{L}_{crit,I} \to \mathrm{Gr}_{G,I},$$

and consider the category D_{crit}-mod$(\mathrm{Gr}_{G,I})$ defined as

$$D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}) := D_{\mathcal{L}_{crit,I}}\text{-mod}(\mathrm{Gr}_{G,I}). \tag{3.24}$$

As we pointed out before, we have the following proposition

Proposition 3.4.5. *For every I, there exists an equivalence of categories*

$$D\text{-}mod(Gr_{G,I}) \simeq D_{crit}\text{-}mod(Gr_{G,I}),$$

given by $\mathcal{M}_I \to \mathcal{M}_I \otimes \mathcal{L}_{crit,I}$.

3.4.3 Construction of the line bunde $\mathcal{L}_{crit}$ over the Beilinson-Drinfeld Grassmannian

We will first recall the definition of the line bundle $\mathcal{L}_{crit,x}$ over the affine Grassmannian $\mathrm{Gr}_{G,x}$ presented in [BD2]. We will then generalize this construction for the spaces $\mathrm{Gr}_{G,I}$.

We will start by recalling some definitions from Tate linear algebra.

Definition 3.4.5. A Tate vector space V is a complete topological vector space having a base of neighborhoods of 0 consisting of commensurable vector subspaces.

- A subspace $P \subset V$ is *bounded* if for every open subspace $U \subset V$ there exist a finite set $\{v_1, \dots, v_n\} \in V$ such that $P \subset U + kv_1 + \dots + kv_n$.

- A *c-lattice* in V is an open bounded subspace.

- A *d-lattice* in V is a discrete subspace $\Gamma \subset V$, such that $V = \Gamma + P$, for some c-lattice $P \subset V$.

Let x be a point in X and t a coordinate around it. Consider the formal disc D_x, and the formal punctured disc D_x°. Denote by $\widehat{\Omega^1(\mathcal{R})}$ the ring $\mathbb{C}[[t]]$ and by $\widehat{\mathcal{K}}$ the field $\mathbb{C}((t))$.

Example 3.4.1. Given a vector bundle $\mathcal{Q}$ on X equipped with a non-degenerate symmetric form

$$\mathcal{Q} \otimes \mathcal{Q} \to \Omega_X,$$

and a point $x \in X$, we can consider $\mathcal{Q} \otimes \widehat{\Omega^1(\mathcal{R})} \subset \mathcal{Q} \otimes \widehat{\mathcal{K}}$. The vector space $V := \mathcal{Q} \otimes \widehat{\mathcal{K}}$ is a Tate vector space, moreover it is equipped with a symmetric nondegenerate form given by the residue. The subspace $L := \mathcal{Q} \otimes \widehat{\Omega^1(\mathcal{R})}$ is a c-lattice in it. Moreover it is a Lagrangian subspace.
More generally, for every non-empty finite set of points $S \subset X$, we have the Tate vector spaces and corresponding Lagrangians,

$$L := \oplus_{x \in S} \mathcal{Q} \otimes \widehat{\Omega^1(\mathcal{R})}_x \subset V := \oplus_{x \in S} \mathcal{Q} \otimes \widehat{\mathcal{K}}_x.$$

As a special case of the above example, given a square root $\mathcal{L}$ of the line bundle Ω_{D_x}, and a vector space W with a non-degenrerae symmetric form

$$W \times W \to \mathbb{C},$$

we can consider the vector bundle $\mathcal{Q} := \mathcal{L} \otimes W$. Let now W be equal to the Lie algebra $\mathfrak{g}$, and consider the killing form κ_{Kill} on $\mathfrak{g}$. The Tate vector space $V_{\mathcal{L}}$,

$$V_{\mathcal{L}} := \mathcal{L} \otimes_{\widehat{\Omega^1(\mathcal{K})}} (\mathfrak{g} \otimes \widehat{\mathcal{K}}),$$

carries a non-degenerate bilinear form given by

$$V_{\mathcal{L}} \otimes V_{\mathcal{L}} \xrightarrow{\kappa_{kill}} \mathcal{L} \otimes \mathcal{L} \otimes \widehat{\mathcal{K}} \simeq \Omega_{D_x^\circ} \xrightarrow{Res} \mathbb{C}.$$

Consider the Lagrangian subspace $L_{\mathcal{L}}$, equals to

$$L_{\mathcal{L}} := \mathcal{L} \otimes \mathfrak{g}.$$

Denote by $\mathcal{C}l(V)$ the Clifford algebra associated to V and by M the irreducible $\mathcal{C}l(V)$-module

$$M := \mathcal{C}l(V)/\mathcal{C}l(V)L.$$

Denote by $Lagr(V)$ the ind-scheme of Lagrangian c-lattices in $V = V_{\mathcal{L}}$ as defined in [BD2] 4.3.2.

There exist a canonical line bundle $\mathcal{P}_M$ on $Lagr(V)$ defined as follows.

Definition 3.4.6. We define $\mathcal{P}_M$ to be the line bundle over $Lagr(V)$ whose fiber over $L' \in Lagr(V)$ is

$$\mathcal{P}_{M,L'} = M^{L'} = \{m \in M |\, L' \cdot m = 0\}.$$

Consider now the map $\phi : G((t))/G[[t]] \to Lagr(V)$ given by

$$g \to gLg^{-1}.$$

Following [BD2] 4.6.11 we have the following definition.

Definition 3.4.7. We define the line bundle $\mathcal{L}_{crit,x}$ on $\mathrm{Gr}_{G,x}$ to be the pull-back, along ϕ of the line bundle $\mathcal{P}_M$;

$$\mathcal{L}_{crit,x} := \phi^*\mathcal{P}_M \to \mathrm{Gr}_{G,x}.$$

We will now try to generalize the above to powers of X. In particular, we will construct a sheaf of Tate vector spaces over X^I.

Let us fix a square root $\mathcal{L}$ of the canonical bundle Ω_X. Consider the $\mathcal{D}_X$-module $\mathcal{V}_X$ given as

$$\mathcal{V}_X = \mathcal{L} \otimes_{\mathcal{O}_X} (\mathfrak{g} \otimes \mathcal{D}_X).$$

As before, the killing form on X together with the fact that $\mathcal{L} \otimes \mathcal{L} \simeq \Omega_X$, defines a symmetric bracket

$$\mathcal{V}_X \otimes \mathcal{V}_X \to \Delta_*(\Omega_X). \tag{3.25}$$

In particular, we have a skew-symmetric pairing on $\mathcal{V}_X[1]$, and therefore a Lie*-algebra structure on the direct sum $\mathcal{V}[1] \oplus \Omega_X$. Define $\mathcal{C}l(\mathcal{V}_X)$ to be the twisted enveloping chiral algebra of $\mathcal{V}[1] \oplus \Omega_X$,

$$\mathcal{C}l(\mathcal{V}_X) := U'(\mathcal{V}[1] \oplus \Omega_X),$$

where we regard $\mathcal{V}[1]$ as a commutative Lie*-algebra. For every finite set I, consider the sheaves of topological vectors spaces

$$L^{(I)} = h_\Gamma(p_1^!(\mathcal{V}_X)[-I]) \quad \text{and} \quad V^{(I)} := h_\Gamma(j_*j^*(p_1^!(\mathcal{V}_X)[-I])),$$

where the functor h_Γ is the one defined in 3.8. The map (3.25) gives us a map

$$(\,,\,)^{(I)} : V^{(I)} \otimes V^{(I)} \to h_\Gamma(j_*j^*(p_1^!(\Omega_X)[-I])). \tag{3.26}$$

Note that, if we take the fiber of $V^{(I)}$ and $L^{(I)}$ at $(x_1, \ldots, x_n) \in X^I$, we recover the

Tate vector space and the c-lattice from example 3.4.1. Moreover the map (3.26) becomes

$$(\oplus_{i=1}^{n}\Omega\otimes\widehat{\mathcal{K}}_{x_i})\times(\oplus_{i=1}^{n}\Omega\otimes\widehat{\mathcal{K}}_{x_i})\xrightarrow{(Res_1,\ldots,Res_n)}\mathbb{C}\times\cdots\times\mathbb{C}. \tag{3.27}$$

The above topological vector spaces will be our family of Tate vector spaces. More generally, we have the following definition.

Definition 3.4.8. Let R be a commutative ring. A Tate R-module is a topological R-module isomorphic to $P\oplus Q^*$, where P and Q are infinite direct sums of finitely generated projective R-modules.

- A *c-lattice* in a Tate R-module V is an open bounded submodule $P\subset V$ such that V/P is projective.

- A *d-lattice* in V is a submodule $\Gamma\subset V$, such that for some c-lattice P, one has $\Gamma\cap P=0$ and $V/(\Gamma+P)$ is a projective module of finite type.

Let's go back to the topological vector space $V^{(I)}$ over $X^{(I)}$.

Definition 3.4.9. We say that a c-lattice $L'\subset V^{(I)}$ is Lagrangian, if for any geometric point $(x_1,\ldots,x_n)\in X^I$, we have that the maps in (3.27) define n non-degenerate forms on the quotient $V^{(I)}/L'$.

Denote by $Lagr(V^{(I)})$ the ind-scheme of Lagrangian sub-spaces in $V^{(I)}$. In particular, note that $L^{(I)}\in Lagr(V^{(I)})$. Consider now the chiral $\mathcal{C}l(\mathcal{V}_X)$-module over X^I equal to

$$\mathcal{M}_I=\mathcal{C}l(\mathcal{V}_X)_I.$$

From the definition of $\mathcal{C}l(\mathcal{V}_X)$, proposition 3.1.4, and the fact that we have maps

$$V^{(I)}\to h_\Gamma(j_*j^*(p_1^!(\mathcal{V}_X\oplus\Omega_X)[-I])),$$

we see that $V^{(I)}$ acts on any chiral-$\mathcal{C}l(\mathcal{V}_X)$-module on X^I. In particular it acts on $\mathcal{M}_I$. Similarly to the above, we have the following definition.

Definition 3.4.10. We define the line bundle $\mathcal{P}_{\mathcal{M}_I}$ on $Lagr(V^{(I)})$ to be the line bundle whose fiber over $L' \in Lagr(V^{(I)})$ is

$$\mathcal{P}_{\mathcal{M}_I,L'} = \mathcal{M}_I^{L'} = \{m \in \mathcal{M}_I |\, L' \cdot m = 0\}.$$

The adjoint action of G on $\mathfrak{g}$, defines an action of MG_I on $V^{(I)}$, that we will still denote by Ad. Consider now the map $\phi_I : MG_I/JG_I \to Lagr(V^{(I)})$ given by

$$g \to Ad_g(L^{(I)}).$$

Definition 3.4.11. Define the line bundle $\mathcal{L}_{crit,I}$ over $\mathrm{Gr}_{G,I} \simeq MG_I/JG_I$ to be thepull-back along ϕ_I of $\mathcal{P}_{\mathcal{M}_I}$;

$$\mathcal{L}_{crit,I} := \phi_I^* \mathcal{P}_{\mathcal{M}_I} \to \mathrm{Gr}_{G,I}.$$

We therefore arrive to the following definition.

Definition 3.4.12. Let Gr_G be the Beilinson-Drinfeld Grassmannian. We define the factorization category D_{crit}-mod(Gr_G) of critically twisted D-modules on Gr_G to be the category given by the assignment

$$I \to D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}) = D_{\mathcal{L}_{crit,I}}\text{-mod}(\mathrm{Gr}_{G,I}),$$

where $D_{\mathcal{L}_{crit,I}}$-mod$(\mathrm{Gr}_{G,I})$ is defined as in definition 3.4.4.

The reason why they are called critically twisted is given by the following proposition (see [BD2]). Let $\widehat{\mathfrak{g}}_\kappa$ be the Lie algebra given in 2.1.1.

Proposition 3.4.6. *Denote by π the projection*

$$\pi : G((t)) \to Gr_{G,x}.$$

Consider the pull back along π of the line bundle $\mathcal{L}_{crit,x}$. Denote by $\widetilde{G((t))}$ the corresponding $\mathbb{G}_m$-bundle on $G((t))$. Then, the Lie algebra corresponding to the extension

$$1 \to \mathbb{G}_m \to \widetilde{G((t))} \to G((t)) \to 1,$$

is equal to

$$0 \to \mathbb{C} \to \widehat{\mathfrak{g}}_{crit} \to \mathfrak{g}((t)) \to 0,$$

where $\widehat{\mathfrak{g}}_{crit}$ denotes the Kac-Moody algebra at the critical level $\kappa_{crit} = -1/2\kappa_{kill}$.

3.4.4 D-modules on the Beilinson-Drinfeld Grassmannian as chiral $\mathcal{D}_{crit}$-modules

Consider the factorization category $\mathcal{D}_{crit}\text{-mod}^{JG} = \{I \to \mathcal{D}_{crit}\text{-mod}_I^{JG}\}$ of strongly JG-equivariant $\mathcal{D}_{crit}$-modules. We want to relate this category to the factorization category $D_{crit}\text{-mod}(\mathrm{Gr}_G) = \{I \to D_{crit}\text{-mod}(\mathrm{Gr}_{G,I})\}$ of critically-twisted D-modules on the Beilinson-Drinfeld Grassmannian Gr_G. We start by considering $\mathcal{D}_{crit}$-modules supported at some point $x \in X$, where this relation is completely understood (see [AG]). We will then pass to the categories $\mathcal{D}_{crit}\text{-mod}_I^{JG}$ and $D_{crit}\text{-mod}(\mathrm{Gr}_{G,I})$ over X^I.

The equivalence over the point

Recall from [AG], that to specify a structure of a chiral $\mathcal{D}_{crit}$-module supported at x on a vector space M is the same as to endow it with continuous (w.r. to the discrete topology on M) actions of $\Omega^1(\mathcal{R})_{G((t))}$ and $\widehat{\mathfrak{g}}_{crit}$ compatible in the sense that for $\eta \in \widehat{\mathfrak{g}}_{crit}$, $f \in \mathcal{O}_{G((t))}$ and $m \in M$,

$$\eta.(f.m) = f.(\eta.m) + Lie_{\eta^l}(f).m,$$

where η^l is the corresponding left-invariant vector field on $G((t))$. This follows from the construction of $\mathcal{D}_{crit}$ and from the fact that M, when viewed as a chiral module for $J_X(G)$ supported at x, becomes a module for $\widehat{J_X(G)}^{ass,x}$, and that

$$\widehat{J_X(G)}^{ass,x} \simeq \mathcal{O}_{G((t))}. \tag{3.28}$$

The right embedding of $\mathcal{A}_{crit}$ into $\mathcal{D}_{crit}$ given by (2.18), endows M with a structure of right $\widehat{\mathfrak{g}}_{crit}$-module. This action is compatible with the $\mathcal{O}_{G((t))}$-action, in the sense that for $\xi \in \widehat{\mathfrak{g}}_{crit}$, $f \in \mathcal{O}_{G((t))}$ and $m \in M$,

$$\xi.(f.m) = f.(\xi.m) + Lie_{\xi^r}(f).m,$$

where ξ^r is the corresponding right-invariant vector eld on $G((t))$.

Consider now the category $\mathcal{D}_{crit}\text{-mod}_x$ defined as

$$\mathcal{D}_{crit}\text{-mod}_x^{G[[t]]} := (\mathcal{D}_{crit}\text{-mod}_X^{JG})_x.$$

In other words, we are looking at those $\mathcal{D}_{crit}$-modules at x on which the right action of $\mathfrak{g}[[t]] \subset \widehat{\mathfrak{g}}_{crit}$ can be integrated to an action of $J_X(G)_x = G[[t]]$. In the above, we regard a module $M \in \mathcal{D}_{crit}\text{-mod}_x$ as a $\mathfrak{g}[[t]]$-module by means of the right action of $\widehat{\mathfrak{g}}_{crit}$ on it and the fact that the sequence

$$0 \to \mathbb{C} \to \widehat{\mathfrak{g}}_{crit} \to \mathfrak{g}((t)) \to 0,$$

splits over $\mathfrak{g}[[t]]$.

Consider now the affine Grassmannian $\mathrm{Gr}_{G,x} = G((t))/G[[t]]$, and the category

D_{crit}-mod$(\mathrm{Gr}_{G,x})$ of critically twisted D-modules on it. Recall that, by subsection 3.4.2, this category is isomorphic to the colimit

$$D_{crit}\text{-mod}(\mathrm{Gr}_{G,x}) := \varinjlim_{k_{i,j,!}} D\text{-mod}(Y_i),$$

where $\mathrm{Gr}_{G,x} = \varinjlim Y_i$. In particular, recall that we have a forgetful functor D_{crit}-mod$(\mathrm{Gr}_{G,x}) \to$ QCoh$^!(\mathrm{Gr}_{G,x})$. We can describe the category $\mathcal{D}_{crit}\text{-mod}_x^{G[[t]]}$ as D-modules on $\mathrm{Gr}_{G,x}$.
In fact, we have the following proposition.

Proposition 3.4.7. *There exist an equivalence of categories*

$$D_{crit}\text{-}mod(Gr_{G,x}) \xrightarrow{\sim} \mathcal{D}_{crit}\text{-}mod_x^{G[[t]]}$$

Proof. The proof of the above proposition can be found in [AG]. However it is useful to recall how the functor is constructed. Let $\mathcal{M}$ be an object in D_{crit}-mod$(\mathrm{Gr}_{G,x})$, and denote by π the projection $\pi : G((t)) \to \mathrm{Gr}_{G,x}$. Consider the pull back $\pi^*(\mathcal{M})$. We can define on the vector space $\Gamma(G((t)), \pi^*(\mathcal{M}))$ a structure of chiral $\mathcal{D}_{crit}$-module at x in the following way. The module $\Gamma(G((t)), \pi^*(\mathcal{M}))$ is naturally a discrete $\mathcal{O}_{G((t))}$-module, and therefore, by (3.28) a $J_X(G)$-module supported at x. Moreover, the projection π is right-G-invariant, the right D-module structure on $\mathcal{M}$, gives rise to the action of $\mathfrak{g}((t))$ on $\pi^*(\mathcal{M})$, therefore, $\Gamma(G((t)), \pi^*(\mathcal{M}))$ is indeed a chiral $\mathcal{D}_{crit}$-module supported at x. The fact that it belongs to $\mathcal{D}_{crit}\text{-mod}_x^{G[[t]]}$ follows from noticing that the right action of $\mathfrak{g}[[t]]$ on it coincides with the $G[[t]]$-action coming from the $G[[t]]$-equivariant structure on $\pi^*(\mathcal{M})$. □

The equivalence over X^I

Let's now consider the category $\mathcal{D}_{crit}$-mod$_I$ of $\mathcal{D}_{crit}$-modules on X^I. Consider the $J_X(G)$-action on $\mathcal{D}_{crit}$ as defined earlier. Recall that the Lie*-$L_{\mathfrak{g}}$-action coming from the $J_X(G)$-action coincides with the $L_{\mathfrak{g}}$-action coming from the composition

$$L_{\mathfrak{g}} \to \mathcal{A}_{crit} \xrightarrow{r} \mathcal{D}_{crit}.$$

We are interested in the category $\mathcal{D}_{crit}\text{-mod}_I^{J_X(G)}$ of strongly $J_X(G)$-equivariant objects in $\mathcal{D}_{crit}\text{-mod}_I$. Objects in this category can be described as modules $\mathcal{M}_I \in \mathcal{D}_{crit}\text{-mod}_I$ on which the Lie action of $\mathfrak{L}_{G,0}^{(I)}$ can be integrated to an action of the group scheme JG_I over X^I. Note that, by considering the case of $I = \emptyset$, we recover the discussion before, where $\mathfrak{L}_{G,0}^{\emptyset}$ is exactly $\mathfrak{g}[[t]]$.

We will start by describing the category $\mathcal{D}_{crit}\text{-mod}_I$ in a more suitable way. Recall the group ind-scheme MG_I of meromorphic jets defined in (3.6). Let p^I be the map

$$p^I : MG_I \to X^I,$$

and denote by $\Gamma_{rel}^{MG_I}$ the functor

$$\begin{array}{ccc} \mathrm{QCoh}^!(MG_I) & \xrightarrow{\Gamma_{rel}^{MG_I}} & \{ \text{ discrete } \mathcal{O}_{MG_I}\text{-modules}\} \\ \mathcal{F}_I & \to & p_*^I(\mathcal{F}_I). \end{array}$$

Note that this functor corresponds to the functor $\Gamma(G((t)),\ )$ if we take the fiber at $x \in X$ for $I = \{*\}$. Denote by $\mathcal{O}_{MG_I}^{rel}$ the sheaf of topological algebras over X^I given as

$$\Gamma_{rel}^{MG_I}(\mathcal{O}_{MG_I}) \simeq \mathcal{O}_{MG_I}^{rel}.$$

We have the following proposition.

Proposition 3.4.8. *To specify a structure of a chiral $\mathcal{D}_{crit}$-module on X^I on a quasi-coherent sheaf $\mathcal{M}_I$ is the same as to endow it with continuous (w.r. to the discrete topology on M) actions of $\mathcal{O}_{MG_I}^{rel}$ and $\mathfrak{L}_{crit}^{(I)}$ compatible in the sense that for*

$\eta \in \mathfrak{L}_{\mathfrak{g}}^{(I)}$, $f \in \mathcal{O}_{JG_I}$ and $m \in \mathcal{M}_I$,

$$\eta.(f.m) = f.(\eta.m) + Lie_{\eta^l}(f).m,$$

where η^l is the corresponding left-invariant vector field on JG_I. Where $\mathfrak{L}_{crit}^{(I)}$ denotes the sheaf of topological Lie algebras defined in (3.8) for $L = L_{\mathfrak{g}}^{crit}$.

Proof. From the construction of $\mathcal{D}_{crit}$, we see that a chiral $\mathcal{D}_{crit}$-module $\mathcal{M}_I$ on X^I, is in particular a module for $U(\mathcal{O}_{J_X(G)})/K$, where K is the kernel of the map $U(\mathcal{O}_{J_X(G)}) \to \mathcal{O}_{J_X(G)}$. Since, for a Lie*-algebra L, chiral modules for $U(L)$ on X^I are in bijection with Lie modules for $\mathfrak{L}^I$ defined in (3.8), we see that $\mathcal{M}_I$ is naturally a Lie*-module for the commutative topological sheaf of Lie algebras over X^I

$$h_\Gamma(j_*j^*(p_1^!(\mathcal{O}_{J_X(G)})[-|I|])) \tag{3.29}$$

However the sheaf of topological algebras in (3.29) coincides with $\mathcal{O}_{MG_I}^{rel}$, therefore $\mathcal{M}_I$ becomes a continuous $\mathcal{O}_{MG_I}^{rel}$-module, i.e. a module over the group ind-scheme MG_I. Moreover this is an equivalence between chiral modules for $U(\mathcal{O}_{J_X(G)})/K$ over X^I and discrete $\mathcal{O}_{MG_I}^{rel}$-modules, i.e. objects in $\mathrm{QCoh}^!(MG_I)$, as explained in corollary 3.1.1. Now, from the definition of $\mathcal{D}_{crit}$ in factorization terms, it is also clear that $\mathcal{M}_I$ comes equipped with a $\mathfrak{L}_{crit}^{(I)}$-action and that the latter needs to be compatible with the former $\mathcal{O}_{MG_I}$-action. □

Consider now the Beilinson-Drinfeld Grassmannian Gr_G. As it is explained by proposition 3.4.1, for every finite set I, we can describe the space $\mathrm{Gr}_{G,I}$ as

$$\mathrm{Gr}_{G,I} \simeq MG_I/JG_I.$$

The above quotient is an ind-scheme,

$$\mathrm{Gr}_{G,I} = \varinjlim_i Y_i^I,$$

and, as before, each Y_i^I is of finite type.

Consider now the category $D_{crit}\text{-mod}(\mathrm{Gr}_{G,I})$ of critically-twisted D-modules on it. Denote by π_I the projection

$$\pi_i : MG_I \to \mathrm{Gr}_{G,I}.$$

For a D-module $\mathcal{F}_I$ in $D_{crit}\text{-mod}(\mathrm{Gr}_{G,I})$, consider the pull-back $\pi_I^*(\mathcal{F}_I)$. This is, by definition, an object in $\mathrm{QCoh}^!(MG_I)$, and, therefore, the object

$$\mathcal{M}_I := \Gamma_{rel}^{MG_I}(\pi_I^*(\mathcal{F}_I))$$

is a discrete module for $\mathcal{O}_{MG_I}^{rel}$ (see example 3.1.3 for the definition of $\mathcal{O}_{MG_I}^{rel}$). We claim that there is a natural $\mathcal{D}_{crit}$-module structure on $\mathcal{M}_I$. In fact, we have the following theorem.

Theorem 3.1. *There exist an equivalence of factorization categories*

$$D_{crit}\text{-}mod(Gr_G) \xrightarrow{\sim} \mathcal{D}_{crit}\text{-}mod^{JG},$$

given by $\mathcal{F}_I \to \Gamma_{rel}^{MG_I}(\pi_I^*(\mathcal{F}_I))$.

Proof. As we explained before, the object $\mathcal{M}_I = \Gamma_{rel}^{MG_I}(\pi_I^*(\mathcal{F}_I))$ is a discrete $\mathcal{O}_{MG_I}^{rel}$-module, and therefore, a chiral $U(J_X(G))/K$-module over X^I. Now, the (negative) of the action of $\mathfrak{L}_{crit}^0$ on $\mathcal{F}_I$ gives rise to an action of the same Lie algebra on $\mathcal{M}_I$, compatible with the $\mathcal{O}_{MG_I}^{rel}$-action. Therefore, by proposition 3.4.8, the objects $\mathcal{M}_I$ is a $\mathcal{D}_{crit}$-module on X^I. Now we claim that the right action of $\mathfrak{L}_{crit}$ on $\mathcal{M}_I$ coming from the right embedding of $\mathcal{A}_{crit}$, is obtained by derivating the JG_I-action on $\pi_I^*(\mathcal{F}_I)$

coming from the equivariant map π_I. This would imply that $\mathcal{M}_I$ is indeed strongly $J_X(G)$-equivariant. This fact is proved by repeating the argument presented in [AG] Proposition 6.7.

$\square$

3.5 The space of Opers

In this section we will recall the definition of *Opers* as given in [BD2]. In particular, given a curve X and a point $x \in X$, we will recall the definition of opers on the disc D_x (resp. punctured disc D_x°) and its presentation as a scheme (resp. ind-scheme). We will also generalize the above definitions in order to obtain factorization spaces and study the factorization categories of modules over them. In 3.5.1 we introduce the factorization space $\mathrm{Op}_{\mathfrak{g}}$ corresponding to the $\mathcal{D}_X$-scheme of opers. In 3.5.2 we construct the factorization space $\mathrm{Op}_{\mathfrak{g}}^\circ$ corresponding to opers on the puncured disc, and show how this can be represented as an ind-scheme. We will then introduce the factorization space $\mathrm{Op}_{\mathfrak{g}}^{unr}$ of *unramified* opers.

3.5.1 The space of opers

Let X be any smooth curve, G a simple algebraic group of adjoint type, and $B \subset G$ a Borel subgroup. For a B-bundle P_B on X, denote by P_G the induced G-torsor $P_G = G \underset{B}{\times} P_B$. We have the corresponding twisted Lie-algebras $\mathfrak{b}_P := \mathfrak{b}_{P_B} = \mathfrak{b} \underset{B}{\times} P_B$ and $\mathfrak{g}_G := \mathfrak{g}_{P_G} = \mathfrak{g} \underset{G}{\times} P_G \simeq \mathfrak{g} \underset{B}{\times} P_B$. The Lie algebra $\mathfrak{g}_G$ is equipped with a standard filtration, induced from the filtration on $\mathfrak{g}$ given by the choice of $\mathfrak{b}$. We have $\mathfrak{g}^{-r} = \mathfrak{g}$, and $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{n}]$; in particular we have $\mathfrak{g}^0 = \mathfrak{b}$ and $\mathfrak{g}^1 = \mathfrak{b}$. Let now ∇ be a connection on P_G. For any connection ∇' preserving P_B, we can think of $\nabla - \nabla'$ as an element in $\mathfrak{g}_G \otimes \Omega_X = \mathfrak{g}_B \otimes \Omega_X$. We denote by $c(\nabla)$ the projection onto $(\mathfrak{g}/\mathfrak{b})_B \otimes \Omega_X$

$$c(\nabla) := (\nabla - \nabla') \mod \mathfrak{b}_B.$$

Definition 3.5.1. An *oper* on X, is a pair (P_B, ∇), where P_B is a B-bundle on X, ∇ is a connection on the induced G-bundle P_G satisfying the following:

- $c(\nabla) \in (\mathfrak{g}^{-1}/\mathfrak{b})_B \otimes \Omega_X \subset (\mathfrak{g}/\mathfrak{b})_B \otimes \Omega_X.$

- For any simple negative root α, we have that the α-component $c(\nabla)^{\alpha} \in \mathfrak{g}_G^{-\alpha} \otimes \Omega_X$ does not vanish on X.

Equivalently, we can think of opers in the following way. Lets choose a trivialization of P_B, and let ∇_0 be the tautological connection on it. Denote by Π the set of simple roots of $\mathfrak{g}$. Then, as it is explained in [FG3], an oper is given by an equivalence $B(X)$-class of connections ∇ of the form

$$\nabla = \nabla_0 + \sum_{\alpha \in \Pi} \phi_\alpha \cdot \mathbf{f}_\alpha + \mathbf{q},$$

where each ϕ_α is a nowhere vanishing one-form on X, and $\mathbf{q}$ is a $\mathfrak{b}$-valued one form. Changing the trivialization of P_B by $g : X \to B$, the connection ∇ get transformed into $\nabla' = g^{-1}\nabla g - g^{-1}dg$.

The above makes sense in families. Indeed, if S is a $\mathcal{D}_X$-scheme $S \xrightarrow{\phi} X$, then we have a well defined notion of G-bundle with a connection ∇ along X. It is a G-bundle $P_G \xrightarrow{\pi} S$ on S, such that P_G is a $\mathcal{D}_X$-scheme and the map π is horizontal, i.e. a map of $\mathcal{D}_X$-schemes. We define opers over S to be the set consisting of pairs (P_B, ∇), where P_B is a B-bundle on S, and ∇ is a connection along X on the induced G-bundle P_G such that the conditions above are satisfied, with Ω_X, replaced by $\phi^*(\Omega_X)$. It can be shown that the above functor is represented by an affine $\mathcal{D}_X$-scheme, denoted by $\mathrm{Op}_{\mathfrak{g},X}$. According to 3.1.6, we therefore obtain a factorization space $\{I \to \mathrm{Op}_{\mathfrak{g},I} := J_I \mathrm{Op}_{\mathfrak{g},X}\}$ that we will simply denoted by $\mathrm{Op}_{\mathfrak{g}}$. Therefore we have spaces $\mathrm{Op}_{\mathfrak{g},I}$ over X^I, where for any test scheme S,

$$\mathrm{Op}_{\mathfrak{g},I}(S) = \left\{ \begin{array}{c} S \xrightarrow{(\phi_1,\ldots,\phi_n)} X^I, \, (P_G, P_B, \nabla), \\ \text{where } P_G \text{ is a } G\text{-bundle on } D_{\underline{\phi}},\ P_B \text{ is a reduction to } B, \\ \text{and } \nabla \text{ is a connection on } P_G, \text{ satisfying the oper condition} \end{array} \right\}.$$

Note that, if we take $S = Spec(k)$, and I to be the set with one element, then $\mathrm{Op}_{\mathfrak{g},X}(S) =: \mathrm{Op}_{\mathfrak{g}}(D_x)$ is the space of *regular opers* introduced in [FG3]. By construction we have the following.

Proposition 3.5.1. *The assignment $I \to Op_{\mathfrak{g},I}$ defines a co-unital factorization space $Op_{\mathfrak{g}}$. Moreover, $Op_{\mathfrak{g},X}$ is affine, in particular it correspond to a commutative chiral algebra on X.*

3.5.2 Opers on the punctured disc

Recall now example 3.1.3. In particular recall that, for an affine $\mathcal{D}_X$-scheme $\mathcal{Y}$, we have defined spaces $M_I\mathcal{Y}$ over X^I, that contain $J_I\mathcal{Y}$, where

$$M_I\mathcal{Y}(S) := \left\{ \begin{array}{c} S \xrightarrow{(\phi_1,\dots,\phi_n)} X^I \\ \alpha \text{ horizontal section } D^\circ_{\underline{\phi}} \to Z \end{array} \right\},$$

where $D^\circ_{\underline{\phi}}$ is the scheme defined in 3.4. This construction, in the special case of $\mathcal{Y} = \mathrm{Op}_{\mathfrak{g},X}$, generalizes the notion of opers on the punctured disc $\mathrm{Op}_{\mathfrak{g}}(D^\circ_x)$ introduced in [BD2]. It is defined in the following way.

Definition 3.5.2. For every I finite set, and test scheme S, we define $\mathrm{Op}^\circ_{\mathfrak{g},I}$ to be the space over X^I given by

$$\mathrm{Op}^\circ_{\mathfrak{g},I}(S) = \left\{ \begin{array}{c} S \xrightarrow{(\phi_1,\dots,\phi_n)} X^I,\ (P_G, P_B, \nabla), \\ \text{where } P_G \text{ is a } G\text{-bundle on } D^\circ_{\underline{\phi}},\ P_B \text{ is a reduction to } B \\ \text{and } \nabla \text{ is a connection on } P_G, \text{ satisfying the oper condition} \end{array} \right\}.$$

Note that, if we take $S = Spec(k)$, and I to be the set with one element, then $\mathrm{Op}^\circ_{\mathfrak{g},X}(S) = \mathrm{Op}_{\mathfrak{g}}(D^\circ_x)$. In particular, from example 3.1.3 we have the following.

Proposition 3.5.2. *The assignment $I \to Op^\circ_{\mathfrak{g},I}$ defines a factorization space $Op^\circ_{\mathfrak{g}}$.*

Remark 3.5.1. Note that, as expected, the above factorization space, in contrast to $\mathrm{Op}_{\mathfrak{g}}$, is not co-unital. However, as it is explained in corollary 3.1.1, if we consider the chiral algebra $\mathcal{O}_{\mathrm{Op}_{\mathfrak{g},X}}$, then we have an equivalence

$$\left\{ \text{ Chiral } \mathcal{O}_{\mathrm{Op}_{\mathfrak{g},X}}\text{-modules on } X^I \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{continuous modules for the topological} \\ \text{associative } \mathcal{O}_{X^I}\text{-algebra } \mathcal{O}^{rel}_{\mathrm{Op}^\circ_{\mathfrak{g},I}} \end{array} \right\},$$

where $\mathcal{O}^{rel}_{M_I Z}$ was defined in example 3.1.3.

Relation with the center $\mathfrak{Z}_{crit}$

Recall the commutative chiral algebra $\mathfrak{Z}_{crit}$ defined as the center of $\mathcal{A}_{crit}$. Recall that this chiral algebra is related to the space of $\check{G}$-opers, where $\check{G}$ denotes the Langlands dual group of G. In fact in [FF] they prove the following theorem.

Theorem 3.2. *For the critical level κ_{crit}, we have an isomorphism of chiral algebras $\mathfrak{Z}_{crit} \simeq \mathcal{O}_{(Op_{\check{\mathfrak{g}},X})}$.*

The above theorem allows us to describe the category of chiral $\mathfrak{Z}_{crit}$-modules over X^I in terms of modules over the topological algebra $\mathcal{O}^{rel}_{\mathrm{Op}^\circ_{\mathfrak{g},I}}$. From the previous remark we have the following equivalence

$$\mathfrak{Z}_{crit}\text{-mod}_I := \left\{ \text{ Chiral } \mathfrak{Z}_{crit}\text{-modules on } X^I \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{continuous modules for the topological} \\ \text{associative } \mathcal{O}_{X^I}\text{-algebra } \mathcal{O}^{rel}_{\mathrm{Op}^\circ_{\check{\mathfrak{g}},I}} \end{array} \right\}. \tag{3.30}$$

We will denote by $\mathrm{QCoh}^!(\mathrm{Op}^\circ_{\check{\mathfrak{g}},I})$ the category on the right hand side of the above equivalence. We will denote by $\mathrm{QCoh}^!(\mathrm{Op}^\circ_{\check{\mathfrak{g}}})$ the factorization category given by the assignment

$$I \to \mathrm{QCoh}^!(\mathrm{Op}^\circ_{\check{\mathfrak{g}},I}).$$

Equivalence (3.30) can be regarded as an equivalence of factorization categories

$$\mathfrak{Z}_{crit}\text{-mod} \simeq \mathrm{QCoh}^!(\mathrm{Op}^\circ_{\check{\mathfrak{g}}}).$$

3.5.3 Unramified opers

Recall now the sub-functor $\mathrm{Op}^{unr}_{\mathfrak{g},x} \subset \mathrm{Op}_{\mathfrak{g}}(D^{\circ}_x)$ of *unramified opers on D_x* introduced in [FG5] and [FG2]. It consists of opers on D°_x, that are unramified when regarded as G-local systems. In other words, are those pairs (P_B, ∇) such that ∇ is $G((t))$-Gauge equivalent to the trivial connection $\nabla_0 = dt$. As it is explained in [FG4] the space $\mathrm{Op}^{unr}_{\mathfrak{g},x}$ can be described as a closed sub-scheme of $\mathrm{Op}_{\mathfrak{g}}(D^{\circ}_x)$, in particular, the algebra of functions $\mathcal{O}_{\mathrm{Op}^{unr}_{\mathfrak{g},x}}$ on $\mathrm{Op}^{unr}_{\mathfrak{g},x}$ has a structure of a topological algebra. We denote by $\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\mathfrak{g},x})$ the category of continuous discrete modules over this algebra. We will now define the factorization space corresponding to $\mathrm{Op}^{unr}_{\mathfrak{g},x}$.

Definition 3.5.3. For every I finite set, and test scheme S, such that $S \xrightarrow{(\phi_1,\ldots,\phi_n)} X^I$, we define the space $\mathrm{Op}^{unr}_{\mathfrak{g},I}$ over X^I by

$$\mathrm{Op}^{unr}_{\mathfrak{g},I}(S) = \left\{ \begin{array}{c} (P_B, \nabla), \text{ where } (P_B, \nabla) \text{ is an oper on } D^{\circ}_{\underline{\phi}}, \\ \text{and the pair } (P_G, \nabla) \text{ can be extended to the entire } D_{\underline{\phi}} \end{array} \right\}.$$

Note that, if we take $S = Spec(k)$, and I to be the set with one element, then $\mathrm{Op}^{unr}_{\mathfrak{g},X}(S) = \mathrm{Op}^{unr}_{\mathfrak{g},x}$. From the definition, we have the following lemma.

Lemma 3.5.1. *The assignment $I \to Op^{unr}_{\mathfrak{g},I}$ defines a factorization space $Op^{unr}_{\mathfrak{g}}$.*

Remark 3.5.2. It can be showed that the algebra $\mathcal{O}^{rel}_{\mathrm{Op}^{unr}_{\mathfrak{g},I}}$ has a structure of a topological algebra over X^I. As before, we denote by $\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\mathfrak{g},I})$ the category

$$\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\mathfrak{g},I}) := \left\{ \begin{array}{c} \text{continuous modules for the topological} \\ \text{associative } \mathcal{O}_{X^I}\text{-algebra } \mathcal{O}^{rel}_{\mathrm{Op}^{unr}_{\mathfrak{g},I}} \end{array} \right\}.$$

3.6 The Conjecture

Recall the Drinfeld-Sokolov reduction Ψ_X as defined in 2.3.1. Consider the category $\mathcal{A}_{crit}\text{-mod}^{JG}_X$ consisting of strongly JG-equivariant chiral modules on X as defined in 3.3.6. Consider the restriction of Ψ_X to this category

$$\Psi_X : \mathcal{A}_{crit}\text{-mod}_X^{JG} \to \mathfrak{Z}_{crit}\text{-mod}_X.$$

The above functor has been studied by D. Gaitsgoy and E. Frenke. In [FG2] they show the following.

Theorem 3.3. *Let Ψ_X be the Drinfeld-Sokolov reduction at the critical level.*

1. *The functor Ψ_X, restricted to $\mathcal{A}_{crit}$-mod$_X^{JG}$ is exact.*

2. *It defines an equivalence of categories*

$$\Psi_X : (\mathcal{A}_{crit}\text{-}mod_X)^{JG} \xrightarrow{\sim} QCoh^!(Op^{unr}_{\check{\mathfrak{g}},X}),$$

where we regard $QCoh^!(Op^{unr}_{\check{\mathfrak{g}},X})$ as a sub-category of $\mathfrak{Z}_{crit}$-mod$_X$ via the equivalence 3.30 and the inclusion $Op^{unr}_{\check{\mathfrak{g}},X} \subset Op^{\circ}_{\check{\mathfrak{g}},X}$.

3.6.1 The conjecture over X^I

Consider the factorization category $\mathcal{A}_{crit}\text{-mod} = \{I \to \mathcal{A}_{crit}\text{-mod}_I\}$. Recall that in 3.3.3 we have defined a strong action of the group $\mathcal{D}_X$-scheme $J_X(G)$ on the category $\mathcal{A}_{crit}$-mod. For every finite set I, we are interested in the category $\mathcal{A}_{crit}$-mod$_I^{JG}$ of strongly $J_X(G)$-equivariant objects in $\mathcal{A}_{crit}$-mod$_I$. Conjecture 1.2.3 states that we have a description of this category similar to the one provided by theorem 3.3.
In order to state the conjecture, in the next sub-section we define the Drifeld-Sokolov reduction Ψ_I over X^I as a functor

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I \to \mathfrak{Z}_{crit}\text{-mod}_I.$$

Using 3.30, the functor Ψ_I can be seen as a functor

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I \to D(\mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I})).$$

The assignment $I \to \Psi_I$ defines a factorization functor Ψ

$$\Psi : \mathcal{A}_{crit}\text{-mod} \to D(\mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}}})).$$

We consider the restriction of Ψ_I to $\mathcal{A}_{crit}\text{-mod}_I^{JG}$,

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I^{JG} \to D(\mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I})).$$

The main conjecture is that the same equivalence as the one in 3.3 holds for modules over X^I.

Conjecture 3.6.1. *Consider the functors Ψ_I,*

$$\Psi_I : \mathcal{A}_{crit}\text{-}mod_I^{JG} \to D(QCoh^!(Op^{\circ}_{\check{\mathfrak{g}},I})).$$

1. *The above functor is exact.*

2. *The image of Ψ_i is contained in $QCoh^!(Op^{unr}_{\check{\mathfrak{g}},I})$.*

3. *The collection of functors $\Psi = \{I \to \Psi_I\}$ establishes an equivalence of factorization categories*

$$\mathcal{A}_{crit}\text{-}mod^{JG} \xrightarrow{\Psi} QCoh^!(Op^{unr}_{\check{\mathfrak{g}}}).$$

Drinfeld-Sokolov reduction for modules over X^I

Recall from section 2.3.1 the BRST-reduction. Recall that, for any Lie*-algebra $\mathfrak{L}$ and any map of chiral algebras $f : U(\mathfrak{L}^{-Tate}) \to \mathcal{R}$, we defined a functor BRST,

$$BRST : \mathcal{R}\text{-mod}_X \to BRST(\mathcal{R} \otimes \mathcal{C}l(\mathfrak{L}))\text{-mod}_X.$$

We will start by generalizing the BRST reduction to modules over X^I. In fact, the construction of Ψ_I follows from the possibility of extending the BRST-reduction for $\mathcal{A} = U(\mathfrak{L}^{-Tate})' \otimes \mathcal{C}l^{\cdot}(\mathfrak{L})$-modules over X to $\mathcal{A}$-modules over X^I. All we have to do, is to be able to define a differential on any $\mathcal{A}$-module $\mathcal{M}_I$ over X^I

BRST-reduction for modules on X^I

Lets $\mathcal{M}_I$ be a chiral $\mathcal{A}$-module on X^I and let us regard it as a Lie*-$\mathcal{A}$-module. By proposition 3.1.4, $\mathcal{M}_I$ is therefore a module for the topological Lie algebra $\mathcal{A}_0^{(I)} = h_\Gamma(p_1^!(\mathcal{A})[-I])$ (see Definition 3.8). We define the differential $d_{\mathcal{M}_I}$ on $\mathcal{M}_I$ to be the action on it of the element $Q^{(I)}$, where

$$Q^{(I)} \in \mathcal{A}_0^{(I)}$$

is the section on $\mathcal{A}_0^{(I)}$ corresponding to the image of the identity endomorphism under the map $\chi : \mathfrak{L} \otimes \mathfrak{L}^* \to \mathcal{A}^1[1]$, where χ is the map defined in 2.14. The pair $(\mathcal{M}_I, d_{\mathcal{M}_I})$ is naturally a BRST($\mathcal{A}$,)-module on X^I. If we are given a map of chiral algebras $f : U(\mathfrak{L}^{-Tate})' \to \mathcal{R}$ and a $\mathcal{A}_{\mathcal{R}}$-module $\mathcal{M}_I$ on X^I, then its BRST-reduction will be a $BRST(\mathcal{A}^{\cdot}_{\mathcal{R}})$-module on X^I. Therefore, for every I, we have functors

$$\left\{\mathcal{A}_{\mathcal{R}}\text{-modules on } X^I\right\} \xrightarrow{BRST_I} \left\{BRST(\mathcal{A}^{\cdot}_{\mathcal{R}})\text{-modules on } X^I\right\}.$$

We have the following proposition.

Proposition 3.6.1. *Given a map of chiral algebras $f : U(\mathfrak{L}^{-Tate})' \to \mathcal{R}$, the assignment $I \to BRST_I$ defines a factorization functor*

$$BRST : \mathcal{A}_{\mathcal{R}}\text{-}mod \to BRST(\mathcal{A}_{\mathcal{R}})\text{-}mod.$$

Proof. For every surjection $\phi : I \twoheadrightarrow J$, and for every partition $\pi : I \twoheadrightarrow J$ of I, we need to show that

$$\Delta_{\phi} \circ BRST_I \simeq BRST_J \circ \Delta_{\phi}^*,$$

$$BRST_I|_{U(I/J)} \simeq (BRST_{I_1} \otimes \cdots \otimes BRST_{I_n})|_{U(I/J)}, \tag{3.31}$$

where $I = \sqcup_i^n I_i$. These both follow from the fact that the topological Lie algebra $\mathcal{A}_0^I$ factorizes, and we have $\Delta_{\phi}^*(Q^{(I)}) = Q^{(J)}$, and $Q^{(I)}|_{U(I/J)}$ corresponds to the product of the corresponding $Q^{(I_j)}$'s in $\mathcal{A}_0^{I_j}$. $\square$

If we consider the natural map $U(\mathfrak{L}_{\mathfrak{n}})' \to \mathcal{A}_{crit}$, and the chiral $\mathcal{C}l(\mathfrak{L}_{\mathfrak{n}})$ $\mathcal{C}l(\mathfrak{L}_{\mathfrak{n}})_I$ over X^I, then, given an $\mathcal{A}_{crit}$-module $\mathcal{M}_I$, we can consider the corresponding $\mathcal{A}_{crit} \otimes \mathcal{C}l(\mathfrak{L}_{\mathfrak{n}})-$module $\mathcal{M}_I \otimes \mathcal{C}l(\mathfrak{L}_{\mathfrak{n}})_I$. Therefore we have functors

$$BRST_I : \mathcal{A}_{crit}\text{-mod}_I \to BRST(\mathcal{A}_{crit} \otimes \mathcal{C}l(\mathfrak{L}_{\mathfrak{n}}))\text{-mod}_I.$$

As we have explained in 2.3.1, we can furthermore modify the differential using the character χ to obtain the Drinfeld-Sokolov reduction Ψ_I,

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I \to BRST^{\chi}(\mathcal{A}_{crit} \otimes \mathcal{C}l(\mathfrak{L}_{\mathfrak{n}}))\text{-mod}_I \simeq \mathfrak{Z}_{crit}\text{-mod}_I,$$

where the last isomorphism follows from theorem 2.2. Now, according to (3.30), we can rewrite the above as

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I \to D(\mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I})).$$

3.7 The localization Conjecture for the Beilinson-Drinfeld Grassmannian

In this section we define a factorization algebra $\mathcal{B}$ and a factorization functor Γ_Ψ : $\mathcal{D}_{crit}$-mod $\to$ $\mathcal{B}$-mod(QCoh$^!$(Op$^\circ_{\check{\mathfrak{g}}}$)). Assuming conjecture 3.6.1, we show that Γ_Ψ induces an equivalence of categories between D-modules on the Beilinson-Drinfeld Grassmannian, and the factorization category of $\mathcal{B}$-modules in QCoh$^!$(Op$^{unr}_{\check{\mathfrak{g}}}$), as stated in conjecture 1.2.2.

Recall the factorization category $\mathcal{A}_{crit}$-mod, the factorization space Op$^\circ_{\check{\mathfrak{g}}}$ and its factorization sub-space Op$^{unr}_{\check{\mathfrak{g}}}$. Consider the factorization categories QCoh$^!$(Op$^\circ_{\check{\mathfrak{g}}}$) and QCoh$^!$(Op$^{unr}_{\check{\mathfrak{g}}}$) as defined in 3.5.2 and 3.5.3. Recall the functor Ψ_I,

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I \to D(\mathrm{QCoh}^!(\mathrm{Op}^\circ_{\check{\mathfrak{g}},I})).$$

By conjecture 3.6.1, the above functor, restricted to the category $\mathcal{A}_{crit}\text{-mod}_I^{JG}$ is exact and induces an equivalence of categories

$$\mathcal{A}_{crit}\text{-mod}_I^{JG} \underset{\Psi_I}{\xrightarrow{\sim}} \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I}).$$

3.7.1 Definition of the functor Γ_Ψ

Recall the chiral algebra $\mathcal{D}_{crit}$ from 3.3.3. By Lemma 3.3.1, the corresponding factorization algebra is a factorization algebra in $\mathcal{A}_{crit}$-mod, and moreover it has a natural action of the factorization group JG, making it an algebra in $\mathcal{A}_{crit}\text{-mod}^{JG}$. We can therefore consider the factorization category

$$\mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}^{JG}),$$

where, for each finite set I, we take $\mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}^{JG})_I$ to be $\mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}_I^{JG})$ as defined in definition 3.1.8.
Since $\mathcal{D}_{crit,I}$ is an object in $\mathcal{A}_{crit}\text{-mod}^{JG}$, it makes sense to consider the object $\mathcal{B}_I$, where

$$\mathcal{B}_I := \Psi_I(\mathcal{D}_{crit,I}).$$

By definition, the assignment $I \to \mathcal{B}_I$ defines a factorization algebra in $\mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}}})$. However, by the second point of conjecture 3.6.1 it is in fact a factorization algebra in $\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}}})$. We will consider the factorization category

$$\mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}}})),$$

as defined in 3.1.8. For every I, the composition

$$\mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}_I^{JG}) \xrightarrow{For} \mathcal{A}_{crit}\text{-mod}_I^{JG} \xrightarrow{\Psi_I} \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I}),$$

lifts to a functor, that we will still denote by Ψ_I,

$$\Psi_I : \mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}_I^{JG}) \to \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})).$$

We denote simply by Ψ the collection of functors $\{I \to \Psi_I\}$;

$$\Psi : \mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}^{JG}) \to \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})). \tag{3.32}$$

Recall now the Beilinson-Drinfeld Grassmaniann Gr_G, and the factorization category $D_{crit}\text{-mod}(\mathrm{Gr}_G)$ of critically twisted D-modules on Gr_G, given by the assignment

$$I \to D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}).$$

In proposition 3.3.3 and in theorem 3.1 we have shown the following two facts.

- We have an equivalence $\mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}^{JG}) \simeq \mathcal{D}_{crit}\text{-mod}^{JG}$.

- There exist an equivalence of factorization categories

$$D_{crit}\text{-mod}(\mathrm{Gr}_G) \xrightarrow{\sim} \mathcal{D}_{crit}\text{-mod}^{JG}.$$

Recall that the second equivalence is constructed as follows. For every I, denote by π_I the projection

$$\pi_i : MG_I \to \mathrm{Gr}_{G,I}.$$

For a D-module $\mathcal{F}_I$ in $D_{crit}\text{-mod}(\mathrm{Gr}_{G,I})$, consider the pull-back $\pi_I^*(\mathcal{F}_I)$. This is, by definition, an object in $\mathrm{QCoh}^!(MG_I)$, and, therefore, the object

$$\mathcal{M}_I = \Gamma_{rel}^{MG_I}(\pi_I^*(\mathcal{F}_I)) := p_*^I(\pi_I^*(\mathcal{F}_I))$$

is a discrete module for $\mathcal{O}_{MG_I}^{rel} = p_*^I(\mathcal{O}_{MG_I})$, where p^I is the map $p^I : MG_I \to X^I$. The equivalence above is given by the functor Γ_I

$$\begin{array}{rccc} \Gamma_I : & D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}) & \to & \mathcal{D}_{crit}\text{-mod}_I^{JG} \\ & \mathcal{F}_I & \mapsto & \Gamma_{rel}^{MG_I}(\pi_I^*(\mathcal{F}_I)). \end{array}$$

Let's now consider the composition

$$\begin{array}{l} D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}) \underset{\Gamma_I}{\xrightarrow{\sim}} \mathcal{D}_{crit}\text{-mod}_I^{JG} \simeq \ \mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}_I^{JG}) \xrightarrow{\Psi_I} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \to \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}},I}^{unr})). \end{array}$$

Denote by $\Gamma_{\Psi,I}$ the resulting factorization functor

$$\Gamma_{\Psi,I} : D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}) \to \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}},I}^{unr})).$$

We can finally state the main corollary of conjecture 3.6.1, from which conjecture 1.2.2 will follows using the equivalence

$$D\text{-mod}(\mathrm{Gr}_{G,I}) \simeq D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}),$$

of proposition 3.4.5.

Conjecture 3.7.1. *The collection $\{I \to \Gamma_{\Psi,I}\}$ gives rise to an equivalence of factorization categories*

$$D_{crit}\text{-}mod(Gr_G) \underset{\Gamma_\Psi}{\xrightarrow{\sim}} \mathcal{B}\text{-}mod(QCoh^!(Op^{unr}_{\check{\mathfrak{g}}})). \tag{3.33}$$

We will prove theorem 3.7.1 assuming the conjecture 3.6.1

proof of conjecture 3.7.1. By Theorem 3.6.1 2, the chiral algebra $\mathcal{B}$ is an algebra in the factorization category $\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}}})$. In particular, for every I, it makes sense to consider the category $\mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I}))$. Recall now that the functor $\Gamma_{\Psi,I}$ is constructed as the composition

$$D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}) \underset{\Gamma_I}{\xrightarrow{\sim}} \mathcal{D}_{crit}\text{-mod}_I^{JG} \simeq \mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}_I^{JG}) \xrightarrow{\Psi_I} \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})).$$

However, conjecture 3.6.1 3. implies that the last functor gives rise to an equivalence

$$\mathcal{D}_{crit}\text{-mod}(\mathcal{A}_{crit}\text{-mod}_I^{JG}) \xrightarrow{\Psi_I} \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})),$$

therefore, we indeed get an equivalence

$$D_{crit}\text{-mod}(\mathrm{Gr}_{G,I}) \underset{\Gamma_{\Psi,I}}{\xrightarrow{\sim}} \mathcal{B}\text{-mod}(\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})),$$

over X^I, for every I, i.e. we have that Γ_Ψ establishes an equivalence of factorization categories.

$\square$

As we mentioned before, using the equivalence between D_{crit}-mod(Gr_G) and D-mod(Gr_G) we arrive at the algebraic description of the category of D-modules on the Grassmannian Gr_G and of the category of D-modules on the affine Grassmannian $\mathrm{Gr}_{G,x}$.

Theorem 3.4. *The composition*

$$D\text{-}mod(Gr_G) \xrightarrow{\otimes \mathcal{L}_{crit}} D_{crit}\text{-}mod(Gr_G) \xrightarrow{\Gamma_\Psi} \mathcal{B}\text{-}mod(QCoh^!(Op_{\check{\mathfrak{g}}}^{unr}))$$

is an equivalence of factorization categories.

Corollary 3.7.1. *The functor* $\mathcal{M} \mapsto \Gamma_{\Psi,x}(\mathcal{M} \otimes \mathcal{L}_{crit,x})$ *establishes an equivalence of categories*

$$D\text{-}mod(Gr_{G,x}) \simeq \mathcal{B}\text{-}mod_{unr,x}$$

where $\mathcal{B}\text{-}mod_{unr}$ *denotes the category of* $\mathcal{B}$*-modules supported at* $x \in X$*, which are supported on* $Op_{\mathfrak{g},x}^{unr}$ *when regarded as objects in* $QCoh^!(Op_{\check{\mathfrak{g}}}(D_x^\circ))$.

Appendices

Appendix A

How to prove conjecture 3.6.1

In this appendix we will explain how we think conjecture 3.6.1 could be proven. We will present two different approaches. More precisely, in A.1, we will try to construct an inverse to the functor

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I^{JG} \to \mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}},I}^{unr}).$$

While, in A.2 we will try to deduce conjecture 3.6.1 from the equivalence $\Psi_X : \mathcal{A}_{crit}\text{-mod}_X^{JG} \xrightarrow{\sim} \mathrm{QCoh}^!(\mathrm{Op}_{\check{\mathfrak{g}},X}^{unr})$ of theorem 3.3.

A.1 First approach

Recall the BRST-reduction introduced in 2.3.1. Given a Lie* algebra $\mathfrak{L}$, and a finite set I, it defines functors

$$BRST_I : \left\{U(\mathfrak{L}^{-Tate})\text{-modules on } X^I\right\} \to \left\{\mathrm{BRST}(\mathcal{A}^{\cdot})\text{-modules on } X^I\right\},$$

where $\mathcal{A}^{\cdot}$ id the dg-chiral algebra $\mathcal{A}^{\cdot} = U(\mathfrak{L})^{-Tate} \otimes \mathcal{C}l(\mathfrak{L})$. Consider now the Lie*-algebra $L_{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{D}_X$. As it is explained in [AG], for this Lie*-algebra, the extension $L_{\mathfrak{g}}^{-Tate}$ corresponds to the extension $L_{\mathfrak{g}}^{-\kappa_{Kill}}$ given by the Killing form $-\kappa_{Kill}$ on $\mathfrak{g}$.

We therefore have a collection of functors

$$BRST_I : \left\{U(L_{\mathfrak{g}}^{-Kill})\text{-modules on } X^I\right\} \to \left\{\mathrm{BRST}(\mathcal{A}^{\cdot})\text{-modules on } X^I\right\}.$$

Since the critical level κ_{crit} is equal to $-1/2\kappa_{Kill}$, given any two $\mathcal{A}_{crit}$-modules $\mathcal{M}$ and $\mathcal{N}$, we can regard the tensor product $\mathcal{M} \otimes \mathcal{N}$ as a $U(L_{\mathfrak{g}}^{-\kappa_{Kill}}) = \mathcal{A}_{-\kappa_{Kill}}$-module. This gives rise to a pairing

$$\begin{array}{rccc} BRST_I : & \mathcal{A}_{crit}\text{-mod}_I \otimes \mathcal{A}_{crit}\text{-mod}_I & \to & Vect \\ & \mathcal{M} \otimes \mathcal{N} & \mapsto & BRST_I(\mathcal{M} \otimes \mathcal{N}). \end{array}$$

Recall now the chiral algebra $\mathcal{D}_{crit}$. A remarkable feature of this chiral algebra, that was shown in [AG] and [FG7], is the following.

Proposition A.1.1. *Let $\mathcal{M}$ be a chiral $\mathcal{A}_{crit}$-module on X^I. Consider the $\mathcal{A}_{crit}$-module $(\mathcal{D}_{crit})_I$ corresponding to $\mathcal{D}_{crit}$. Then we have*

$$BRST_I(\mathcal{M} \otimes (\mathcal{D}_{crit})_I) \simeq \mathcal{M}.$$

A.1.1 Construction of the inverse to Ψ_I

Consider now the functor

$$\Psi_I : \mathcal{A}_{crit}\text{-mod}_I^{JG} \to \mathrm{QCoh}^!(\mathrm{Op}^{\circ}_{\check{\mathfrak{g}},I}).$$

A generalization of the argument presented in [FG7] can be used to show the second point of the above conjecture.

lemma A.1.1. *The image of the functor Ψ_I restricted to the category $\mathcal{A}_{crit}$-mod$_I^{JG}$ is contained in $QCoh^!(Op_{\check{\mathfrak{g}},I}^{unr})$.*

We can therefore consider Ψ_I as a functor from $\mathcal{A}_{crit}$-mod$_I^{JG}$ to the category

$\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})$. We now want to construct an inverse Φ_I to Ψ_I,

$$\Phi_I : \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I}) \to \mathcal{A}_{crit}\text{-mod}_I^{JG}.$$

Consider the chiral algebra $\mathcal{D}_{crit}$, and recall that it admits two embedding l and r of $\mathcal{A}_{crit}$. Denote by $\mathcal{C}_{crit}$ the chiral algebra

$$\mathcal{C}_{crit} := (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_{crit}) \in \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I}) \otimes \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I}).$$

Th chiral algebra $\mathcal{C}_{crit}$ naturally defines a functor

$$< . >_I^*: Vect \to \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I}) \otimes \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I}),$$

simply by sending the unit object $\mathbb{C}$ to $< \mathbb{C} >^* := \mathcal{C}_{crit}$.
The existence of the functor Φ_I would follow from the following proposition.

Proposition A.1.2. *For every finite set I there exists a pairing*

$$< . \otimes . >_I: QCoh^!(Op^{unr}_{\check{\mathfrak{g}},I}) \otimes QCoh^!(Op^{unr}_{\check{\mathfrak{g}},I}) \to Vect,$$

such that, the following two properties are satisfied:

- *the composition*

$$QCoh^!(Op^{unr}_{\check{\mathfrak{g}},I}) \xrightarrow{<,>_I^*\otimes Id} QCoh^!(Op^{unr}_{\check{\mathfrak{g}},I}) \otimes QCoh^!(Op^{unr}_{\check{\mathfrak{g}},I}) \otimes QCoh^!(Op^{unr}_{\check{\mathfrak{g}},I}) \xrightarrow{Id\otimes<.\otimes.>_I}$$
$$\xrightarrow{Id\otimes<.\otimes.>_I} QCoh^!(Op^{unr}_{\check{\mathfrak{g}},I})$$

  is the identity functor.

- *For $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{A}_{crit}$-mod$_I^{JG}$, we have*

$$< \Psi_I(\mathcal{M}) \otimes \Psi_I(\mathcal{N}) >_I := < (\Psi_I \boxtimes \Psi_I)(\mathcal{M} \otimes \mathcal{N}) >_I \simeq BRST_I(\mathcal{M} \otimes \mathcal{N}). \quad \text{(A.1)}$$

Let's now show how proposition A.1.2 would allow us to construct the inverse functor Φ_I. Recall that we denoted by $\mathcal{B}$ the chiral algebra

$$\mathcal{B} = (Id \boxtimes \Psi_X)(\mathcal{D}_{crit}) \in (\mathcal{A}_{crit}\text{-mod}_X)^{J_X(G)} \otimes \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},X}).$$

Given an object $\mathcal{F}$ in $\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})$, we define $\Phi_I(\mathcal{F})$ to be:

$$\Phi_I(\mathcal{F}) :=< \mathcal{B}_I \otimes \mathcal{F} >_I \in \mathcal{A}_{crit}\text{-mod}_I^{JG},$$

where we regard $\mathcal{B}_I$ as an object in $\mathcal{A}_{crit}\text{-mod}_I^{JG} \otimes \mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})$. Using proposition A.1.1, we can immediately check that the composition $\Phi_I \circ \Psi_I$ is the identity on $\mathcal{A}_{crit}\text{-mod}_I^{JG}$. In fact, for $\mathcal{M}$ a $\mathcal{A}_{crit}$-module on X^I, we have

$$\Phi_I(\Psi_I(\mathcal{M})) =< (Id \boxtimes \Psi_I)(\mathcal{D}_{crit}) \otimes \Psi_I(\mathcal{M}) >_I \simeq BRST^{-Kill}(\mathcal{D}_{crit} \otimes \mathcal{M}) \simeq \mathcal{M}.$$

Similarly, we can show that the composition $\Psi_I \circ \Phi_I$ is the identity on $\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})$. In fact, by the first property in A.1.2, for $\mathcal{N}$ in $\mathrm{QCoh}^!(\mathrm{Op}^{unr}_{\check{\mathfrak{g}},I})$, we have

$$\begin{aligned}\Psi_I(\Phi_I(\mathcal{N})) = \quad & \Psi_I(< (Id \boxtimes \Psi_X)(\mathcal{D}_{crit})_I \otimes \mathcal{N} >) \simeq \quad < (\Psi_X \boxtimes \Psi_X)(\mathcal{D}_{crit})_I \otimes \mathcal{N} >= \\ =< \mathcal{C}_{crit} \otimes \mathcal{N} >= \quad & (<,>_I^* \otimes Id) \circ (Id \otimes < . \otimes . >_I)(\mathcal{N}) \quad \simeq \mathcal{N}.\end{aligned}$$

A.2 Second approach

Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian factorization categories. Let $G : \mathcal{C} \to \mathcal{D}$ be a factorization functor, $G = \{G_I : \mathcal{C}_I \to \mathcal{D}_I\}$. The idea of the second approach is to try to understand what it takes for G to establish an equivalence of categories over X^I, if we assume that

- G induces an equivalence $\mathcal{C}_X \to \mathcal{D}_X$ (over one copy of the curve).

We start by the following general proposition.

Proposition A.2.1. *Let $G : \mathcal{C} \to \mathcal{D}$ be a factorization functor between two factorization categories $\mathcal{C}$ and $\mathcal{D}$. If for every finite set I, G_I induces an equivalence on* Hom*'s and* Ext^1*'s, then G_I is an equivalence.*

Proof. This follows from the following lemma.

Lemma A.2.1. *Let $G : \mathcal{C}_1 \to \mathcal{C}_2$ be an exact functor between abelian categories. Assume that for $X, Y \in \mathcal{C}_1$ the maps*

$$\mathrm{Hom}_{\mathcal{C}_1}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_2}(G(X), G(Y)),$$

$$Ext^1_{\mathcal{C}_1}(X, Y) \to Ext^1_{\mathcal{C}_2}(G(X), G(Y))$$

are isomorphisms. If G admits a right adjoint functor F which is conservative, then G is an equivalence.

In fact, under the above assumptions, the functor G_I admits a right adjoint F_I which is conservative. To show this, we have to show that, for every $\mathcal{N} \in \mathcal{D}_{X^I}$, the functor

$$\mathcal{M} \mapsto \mathrm{Hom}_{\mathcal{D}_{X^I}}(G(\mathcal{M}), \mathcal{N})$$

is representable, where $\mathcal{M} \in \mathcal{C}_{X^I}$. Denote by $\mathcal{C}^0_{X^I}$ the subcategory of compact objects. Consider the category of pairs (X, f), where $X \in \mathcal{C}^0_{X^I}$ and $f \in G_I(X)$. Morphisms between (X, f) and (X', f') are maps $\phi : X \to X'$, such that $\phi_*(f') = f$. It is easy to see that the ob ject

$$\underset{(X,f)}{colim} X$$

represents the functor G_I.

To show that F_I is conservative, it is enough to show that for every $\mathcal{N} \in \mathcal{D}_{X^I}$, there exists $\mathcal{M} \in \mathcal{C}_{X^I}$ such that $\mathrm{Hom}_{\mathcal{D}_I}(G_I(\mathcal{M}), \mathcal{N})$ is non zero. For this, consider the exact triangle

$$i_*i^*(\mathcal{N}) \to \mathcal{N} \to j_*j^*(\mathcal{N}).$$

If $i_*i^*(\mathcal{N})$ is non zero, then it is in the image of G_I, by induction on n, since G is an equivalence over X. Hence we are done. If $i_*i^*(\mathcal{N})$ is zero, then $\mathcal{N}$ is quasi-isomorphic to $j_*j^*(\mathcal{N})$, and we are done for the same reasons. $\square$

Proof of A.2.1. The fully faithfulness assumption on G implies that the adjunction map induces an isomorphism between the composition $F \circ G$ and the identity functor on $\mathcal{C}_1$. We have to show that the second adjunction map is also an isomorphism. For $X' \in \mathcal{C}_2$ let Y' and Z' be the kernel and cokernel, respectively, of the adjunction map $G \circ F(X') \to X'$. Being a right adjoint functor, F is left-exact, hence we obtain an exact sequence

$$0 \to F(Y') \to F \circ G \circ F(X') \to F(X').$$

But since $F(X') \to F \circ G(F(X'))$ is an isomorphism, we obtain that $F(Y') = 0$. Since F is conservative, this implies that $Y' = 0$. Suppose that $Z' \neq 0$. Since $F(Z') = 0$, there exists an ob ject $Z \in \mathcal{C}_1$ with a non-zero map $G(Z) \to Z'$. Consider the induced extension

$$0 \to G \circ F(X') \to W' \to G(Z) \to 0.$$

Since G induces a bijection on Ext_1 , this extension can be obtained from an extension

$$0 \to F(X') \to W \to Z \to 0$$

in $\mathcal{C}_1$. In other words, we obtain a map $G(W) \to X'$, which does not factor through $G \circ F(X') \subset X'$, which contradicts the (G, F) adjunction. $\square$

A.2.1 The case $G = \Psi$

Now, consider the factorization functor Ψ. By lemma A.1.1, we can regard it as a factorization functor $\Psi : \mathcal{A}_{crit}\text{-mod}^{JG} \to D(\mathrm{QCoh}^{!}(\mathrm{Op}_{\mathfrak{g}}^{unr}))$. By proposition A.2.1, we have that conjecture 3.6.1 is equivalent to the following.

- For every I, the functor Ψ_I is exact.
- For every I, the functor Ψ_I induces an isomorphism on Hom's and Ext^1's.

In trying to show these two points, we will use the assumption on Ψ_X being an equivalence. For the first point, we have the following:

Proposition A.2.2. *If the functor Ψ_I, restricted to the category of strongly JG-equivariant objects*

$$\Psi_I : \mathcal{A}_{crit}\text{-}mod_I^{JG_I} \to D(QCoh^{!}(Op_{\mathfrak{g},I}^{\circ}))$$

is right exact, then it is exact.

Proof. Consider the following general setting. Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian factorization categories. Let $F : D(\mathcal{C}) \to D(\mathcal{D})$ be a factorization functor. Assume that

- $F_X : \mathcal{C}_X \to D(\mathcal{D}_X)$ is exact.
- For every I, $F_I(D^{\leq 0}(\mathcal{C}_I)) \subset D^{\leq 0}(\mathcal{D}_I)$.

Then the functors $\mathcal{F}_I$ are also exact. This simply follows from the fact that under these hypothesis, we also have

$$F_I(D^{\geq 0}(\mathcal{C}_I)) \subset D^{\geq 0}(\mathcal{D}_I).$$

In fact, by induction on I, for $\mathcal{M}_I \in D^{\geq 0}(\mathcal{C}_I)$, we can consider the triangle

$$\mathcal{M}_I \to j_*(\mathcal{M}'_{I-\{*\}}) \to cone(\mathcal{M}_I \to j_*(\mathcal{M}'_{I-\{*\}})).$$

Note that the second term of the above sequence is in $D^{\geq 0}(\mathcal{C}_I)$, and moreover, since $F_{I-\{*\}}$ is exact, we have that

$$F_I(j_*(\mathcal{M}'_{I-\{*\}})) \in D^{\geq 0}(\mathcal{D}_I).$$

The same reasoning applies to the last term of the triangle, since the object $cone(\mathcal{M}_I \to j_*(\mathcal{M}'_{I-\{*\}}))$ can be written as the colimit of push-forward of modules in $D^{\geq 0}(\mathcal{C}_{I-\{*\}})$, and F commutes with both colimits and Δ_*. Therefore we see that, by applying F_I, we get

$$F_I(\mathcal{M}_I) \to \underbrace{F_I(j_*(\mathcal{M}'_{I-\{*\}})) \to F_I(cone(\mathcal{M}_I \to j_*(\mathcal{M}'_{I-\{*\}})))}_{\in \mathcal{D}_I^{\geq 0}},$$

which implies $F_I(\mathcal{M}_I) \in D^{\geq 0}(\mathcal{D}_I)$. □

For the second point, we proceed as follows. If $\mathcal{C}$ in an abelian category, we say that an object $C \in \mathcal{C}$ is *quasi-perfect* if for any directed system of objects, the natural map

$$\mathrm{Ext}^i_{\mathcal{C}}(C, \varinjlim C_i) \to \varinjlim_i \mathrm{Ext}^i_{\mathcal{C}}(C, C_i)$$

is an isomorphism for every $i \geq 0$.

Assume that $G : \mathcal{C} \to \mathcal{D}$ is a continuous factorization functor between factorization categories as before. Suppose that:

- each $\mathcal{C}_{X^I}$ is generated by quasi-perfect objects.
- G induces an equivalence $\mathcal{C}_X \to \mathcal{D}_X$ (over one copy of the curve).

We have the following proposition.

Proposition A.2.3. *Suppose that, in the conditions above, $G_I : \mathcal{C}_{X^I} \to \mathcal{D}_{X^I}$ preserves quasi-perfect objects. Then G is an equivalence.*

Proof. By proposition A.2.1, it is enough to show that G_I induces equivalences on Hom's and Ext^1's We will deal with the case n=2, the general case can be treated similarly. We need to show that for every $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\mathcal{C}_{X^2}$, $\mathrm{Hom}(\mathcal{M}_1, \mathcal{M}_2) \to \mathrm{Hom}(G_2(\mathcal{M}_1), G_2(\mathcal{M}_2))$ and $\mathrm{Ext}^1(\mathcal{M}_1, \mathcal{M}_2) \to \mathrm{Ext}^1(G_2(\mathcal{M}_1), G_2(\mathcal{M}_2))$ are equivalences. We can assume that $\mathcal{M}_1$ is quasi-perfect. Since Δ_* and j_* have both left adjoints, and F commutes with them by definition, the statement is true for $\mathcal{M}_2$ of the form $j_*(\mathcal{M})$ and $\Delta_*(\mathcal{M})$, for $\mathcal{M} \in \mathcal{C}_X$, where we are taking the $\mathcal{O}$-module direct image. For arbitrary $\mathcal{M}_2$, we can consider the exact triangle

$$\mathcal{M}_2 \to j_*(\mathcal{M}_2') \to \mathcal{M}_2'',$$

where $\mathcal{M}_2''$ is supported set-theoretically on the diagonal. Now, such $\mathcal{M}_2''$ can be written as a colimit $\mathcal{M}_2'' = \underset{i \in I}{colim} \Delta_*(\mathcal{M}_i)$. By applying $\mathrm{Hom}(\mathcal{M}_1, -)$ we get

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(\mathcal{M}_1, \mathcal{M}_2) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathcal{M}_1, j_*(\mathcal{M}_2')) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(\mathcal{M}_1, \underset{i \in I}{colim} \Delta_*(\mathcal{M}_i)), \arrow[d, "\simeq"] \\
\cdots \arrow[r] & \mathrm{Hom}(G_2(\mathcal{M}_1), G_2(\mathcal{M}_2)) \arrow[r] & \mathrm{Hom}(G_2(\mathcal{M}_1), G_2(j_*(\mathcal{M}_2'))) \arrow[r] & \mathrm{Hom}(G_2(\mathcal{M}_1), G_2(\underset{i \in I}{colim} \Delta_*(\mathcal{M}_i
\end{tikzcd}$$

where the last isomorphism follows from the fact that $\mathcal{M}_1$ is compact, therefore $G_2(\mathcal{M}_1)$ is, and Hom out of them commutes with colimits. Note that the same argument applies for Ext^1's. $\square$

By proposition A.2.3 and proposition A.2.2 we see that conjecture 3.6.1 is equivalent to the following:

1. The functor Ψ_I is right exact.
2. The category $\mathcal{A}_{crit}\text{-mod}_I^{JG}$ is generated by quasi-perfect objects.
3. The functor Ψ_I preserves quasi-perfectness.

Currently, we don't know how the first point can be shown. For the second point, for any n-duple of dominant weights $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$, denotes by $\mathcal{A}_{crit}^{\underline{\lambda}}$ the

$\mathcal{A}_{crit}$-module on X^I given as

$$\mathcal{A}^{\underline{\lambda}}_{crit} := \mathrm{Ind}^{\mathfrak{L}^{(I)}_{crit}}_{\mathfrak{L}^{(I)}_{G,0} \oplus \Omega^n_{X^I}} (V^{\lambda_1} \otimes \cdots \otimes V^{\lambda_n} \otimes \mathcal{O}_{X^I}), \tag{A.2}$$

where $\mathfrak{L}^{(I)}_{G,0}$ and $\mathfrak{L}^{(I)}_{crit}$ are the topological sheaves of Lie algebras introduced in 3.8. In particular, for $\underline{\lambda} = (0, \ldots, 0)$, we recover the factorization algebra $(\mathcal{A}_{crit})_I$ attached to $\mathcal{A}_{crit}$.

We have the following proposition.

Proposition A.2.4. *The category $\mathcal{A}_{crit}$-mod^{JG} is generated by the objects $\mathcal{A}^{\underline{\lambda}}_{crit}$ and these objects are quasi-perfect.*

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