Robust Distributed Routing in Dynamical Networks with Cascading Failures

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***Abstract*— We consider a dynamical formulation of network flows, whereby the network is modeled as a switched system of ordinary differential equations derived from mass conservation laws on directed graphs with a single origin-destination pair and a constant inflow at the origin. The rate of change of the density on each link of the network equals the difference between the inflow and the outflow on that link. The inflow to a link is determined by the total flow arriving to the tail node of that link and the routing policy at that tail node. The outflow from a link is modeled to depend on the current density on that link through a flow function. Every link is assumed to have finite capacity for density and the flow function is modeled to be strictly increasing up to the maximum density. A link becomes inactive when the density on it reaches the capacity. A node fails if all its outgoing links become inactive, and such node failures can propagate through the network due to rerouting of flow. We prove some properties of these dynamical networks and study the resilience of such networks under distributed routing policies with respect to perturbations that reduce link-wise flow functions. In particular, we propose an algorithm to compute upper bounds on the maximum resilience over all distributed routing policies, and discuss examples that highlight the role of cascading failures on the resilience of the network.**

I. Introduction

Network flows provide a fruitful modeling framework for transport phenomena, with many applications of interest, e.g., road traffic, data, and production networks. They entail a fluid-like description of the macroscopic motion of particles, which are routed from their origins to their destinations via intermediate nodes: we refer to standard textbooks, such as [1], for a thorough treatment.

In this paper, we consider a dynamical framework for studying network flows, as proposed in our earlier work [2], [3]. In particular, we study *dynamical networks*, modeled as systems of ordinary differential equations derived from mass conservation laws on directed graphs with a single origin-destination pair and a constant total inflow at the origin. The rate of change of the density on each link of the network equals the difference between the *inflow* and the *outflow* of that link. The latter is modeled to depend on the current density on that link through a *flow function*. On the other hand, the way the total outflow at a non-destination node gets split among its outgoing links depends on the *routing policy* at that node. We focus on *distributed* routing policy, characterized by the property that the proportion of total outflow routed to the outgoing links of a node is allowed to depend only on *local information*, consisting of the current densities on the outgoing links of the same node.

The novel modeling element in the present contribution is that every link is assumed to have finite capacity for density. The flow function is modeled to be strictly increasing as density increases from zero up to the maximum density. A link becomes inactive when the density on it reaches the capacity. This, in particular, is a discontinuous version of the fundamental traffic diagram from traffic engineering [4], where the flow functions are modeled to be continuously increasing up to a critical density and then continuously decreasing to zero up to the maximum density. Such a feature allows for the possibility of spill-backs and cascaded failures in our model, which was absent from the model considered in [2], [3].

Our main objective is studying the resilience of such dynamical networks with respect to perturbations that reduce the flow capacity of their links. We measure the magnitude of a perturbation as the sum of the link-wise flow capacity reductions and define the margin of resilience as the minimum magnitude of a perturbation which makes a dynamical network previously in equilibrium converge to a state in which no flow reaches the destination node. In fact, once the density reaches its maximum capacity on a link, the corresponding outflow is zero, and the link becomes irreversibly inactive. If all the outgoing links of a node become inactive, the node fails irreversibly, and in turn all its incoming links become inactive. As a consequence some other links may experience an overload, possibly reaching their density capacity, thus becoming inactive ever since. Through this mechanism, link and node failures propagate through the network.

Models for cascades in general complex networks are given in [5], [6], [7], while domain-specific models are provided in [8] (power networks), [9] (financial networks),and [10] (supply networks). There has also been work on understanding the role of human decisions on such cascading phenomena, especially in the context of financial networks, e.g., see [11], [12]. However, most of these models rely on a stochastic model for initiation of failure and its propagation. In this paper, however, we propose a deterministic dynamical framework for cascading failures that are particularly relevant for transportation networks.

The main contribution of this paper are as follows: we introduce a novel dynamical model of network flow that allows for spill-backs and cascaded failures; we study

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some properties of such dynamical networks, including a dichotomy stating that either the asymptotic outflow equals the inflow, or it is null; we provide an upper bound on the margin of resilience for a tree-like dynamical network in terms of an easily computable static function of the link flow capacities and the initial equilibrium flow; we discuss an insightful example showing how the spill back effect can be useful in improving the resilience of the dynamical network.

Before proceeding, let us gather here some preliminary notation to be used throughout the paper. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ be the set of nonnegative reals. When $\mathcal{A}$ and $\mathcal{B}$ are finite sets, $|\mathcal{A}|$ will denote the cardinality of $\mathcal{A}$, $\mathbb{R}^{\mathcal{A}}$ (respectively, $\mathbb{R}_+^{\mathcal{A}}$) will stay for the space of real-valued (nonnegative-real-valued) vectors whose components are indexed by elements of $\mathcal{A}$, and $\mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ for the space of matrices whose real entries are indexed by pairs of elements in $\mathcal{A}\times\mathcal{B}$. $\mathbf{1}$ will stand for the all-one vector, whose size will be clear from the context. Let $\mathrm{cl}(\mathcal{X})$ be the closure of a set $\mathcal{X} \subseteq \mathbb{R}^{\mathcal{A}}$. For $x \in \mathbb{R}$, let $[x]^+ := \max\{0, x\}$. A directed multigraph is the pair $(\mathcal{V}, \mathcal{E})$ of a finite set $\mathcal{V}$ of nodes, and of a multiset $\mathcal{E}$ of links consisting of ordered pairs of nodes (i.e., we allow for parallel links). If $e = (v, w) \in \mathcal{E}$ is a link, we shall write $\sigma(e) = v$ and $\tau(e) = w$ for its tail and head node, respectively. The sets of outgoing and incoming links of a node $v \in \mathcal{V}$ will be denoted by $\mathcal{E}_v^+ := \{e \in \mathcal{E} : \sigma(e) = v\}$ and $\mathcal{E}_v^- := \{e \in \mathcal{E} : \tau(e) = v\}$, respectively.

II. Problem statement

We briefly summarize the dynamical flow framework introduced in [2], [3], highlighting the key differences.

A. Dynamical networks with cascading failures

Definition 1 (Flow network): A *flow network* $\mathcal{N} = (\mathcal{T}, \mu)$ is the pair of a *topology*, described by a finite directed multigraph $\mathcal{T} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{0, 1, \ldots, n\}$ is the node set and $\mathcal{E}$ is the link multiset, and a family of *flow functions* $\mu := \{\mu_e : [0, \rho_e^{\max}] \to \mathbb{R}_+\}_{e\in\mathcal{E}}$ describing the functional dependence $f_e = \mu_e(\rho_e)$ of the flow on the density on every link $e \in \mathcal{E}$. The quantity $\rho_e^{\max}$ is referred to as the *density capacity* of a link $e \in \mathcal{E}$, while its *flow capacity* is defined as $f_e^{\max} := \sup\{\mu_e(\rho_e) : \rho_e \in [0, \rho_e^{\max}]\}$.

Throughout, we shall assume that the topology $\mathcal{T}$ contains a unique origin (i.e., a node $v \in \mathcal{V}$ such that $\mathcal{E}_v^-$ is empty), and a unique destination (i.e., a node $v \in \mathcal{V}$ such that $\mathcal{E}_v^+$ is empty). In addition, we shall assume that there exists a path in $\mathcal{T}$ to the destination node from every other node in $\mathcal{V}$. With no loss of generality, we shall assume the origin and the destination nodes to be labeled by 0, and n, respectively. Note that, unlike the model in [2], [3], we do not impose acyclicity assumption at this stage. However, we shall formulate more stringent assumptions on the network topology in Section III. Moreover, we shall focus on flow functions satisfying the following:

Assumption 1 (Flow function): For every link $e \in \mathcal{E}$, the density capacity $\rho_e^{\max}$ is finite. Moreover, the flow function $\mu_e : [0, \rho_e^{\max}] \to \mathbb{R}_+$ is continuously differentiable and increasing on $[0, \rho_e^{\max})$, and such that $\mu_e(0) = \mu_e(\rho_e^{\max}) = 0$.

Example 1 (Flow function): For $f_{\mathrm{e}}^{\max}, \rho_e^{\max}, \alpha > 0$, let

$$\mu_e(\rho_e) = \begin{cases} f_{\mathrm{e}}^{\max}\,(\rho_e/\rho_e^{\max})^\alpha & \text{if} \quad \rho_e \in [0, \rho_e^{\max}) \\ 0 & \text{if} \quad \rho_e = \rho_e^{\max}. \end{cases}$$

Let $\mathcal{R} := \times_{e\in\mathcal{E}}[0, \rho_e^{\max}]$ and $\mathcal{F} := \times_{e\in\mathcal{E}}[0, f_{\mathrm{e}}^{\max})$ the sets of admissible density vectors, and flow vectors, respectively. Write $f := \{f_e : e \in \mathcal{E}\} \in \mathcal{F}$, and $\rho := \{\rho_e : e \in \mathcal{E}\} \in \mathcal{R}$, for the vectors of flows and of densities, respectively, on the different links. We shall compactly denote by $f = \mu(\rho)$ the functional relationship between density and flow vectors, interpreting μ as a (surjective) map from $\mathcal{R}$ to $\mathcal{F}$. For an inflow $\lambda_0 \geq 0$, we shall consider the set of equilibrium flows

$$\mathcal{F}^*(\lambda_0) := \Big\{ f^* \in \mathcal{F} : \sum\nolimits_{e\in\mathcal{E}_0^+} f_e^* = \lambda_0\,, \quad \sum\nolimits_{e\in\mathcal{E}_v^+} f_e^* = \sum\nolimits_{e\in\mathcal{E}_v^-} f_e^*\,, \quad \forall\, 0 < v < n \Big\}\,.$$

An origin-destination cut is some $\mathcal{U} \subseteq \mathcal{V}$ such that $0 \in \mathcal{U}$ and $n \notin \mathcal{U}$. For $\mathcal{U} \subseteq \mathcal{V}$, let $\mathcal{E}_{\mathcal{U}}^+ := \{e \in \mathcal{E} : \sigma(e) \in \mathcal{U}, \tau(e) \notin \mathcal{U}\}$ be the set of links with tail node in $\mathcal{U}$ and head node in $\mathcal{V} \setminus \mathcal{U}$, and put

$$C(\mathcal{N}, \mathcal{U}) := \sum\nolimits_{e\in\mathcal{E}_{\mathcal{U}}^+} f_{\mathrm{e}}^{\max}\,.$$

The *min-cut capacity* of the flow network $\mathcal{N}$ is

$$C(\mathcal{N}) := \min_{\mathcal{U}} C(\mathcal{N}, \mathcal{U})\,,$$

where the minimization runs over all the origin-destination cuts. The min-cut max-flow theorem implies that $\mathcal{F}^*(\lambda_0) \neq \emptyset$ if and only if $C(\mathcal{N}) \geq \lambda_0$, a condition that we shall assume to hold throughout the paper, in order to avoid trivialities.

For every non-destination node $0 \leq v < n$, the simplex of probability vectors over $\mathcal{E}_v^+$ will be denoted by $\mathcal{S}_v := \{p \in \mathbb{R}_+^{\mathcal{E}_v^+} : \sum_{e\in\mathcal{E}_v^+} p_e = 1\}$, while $\mathcal{R}_v := \times_{e\in\mathcal{E}_v^+}[0, \rho_e^{\max}]$ and $\mathcal{F}_v := \times_{e\in\mathcal{E}_v^+}[0, f_e^{\max}]$ will stand for the sets of all admissible local density vectors, and flow vectors, respectively. $\mathcal{R}_v^* := \mathcal{R}_v \setminus \{\rho_v^*\}$, where $\rho_v^* := \{\rho_e^{\max} : e \in \mathcal{E}_v^+\}$, will denote the set of all but the fully congested local density vectors. The notation $f^v := \{f_e : e \in \mathcal{E}_v^+\} \in \mathcal{F}_v$, and $\rho^v := \{\rho_e : e \in \mathcal{E}_v^+\} \in \mathcal{R}_v$ will stand for the vectors of flows and densities, respectively, on the outgoing links of a node v.

We now introduce the notion of distributed routing policy.

Definition 2 (Distributed routing policy): A *distributed routing policy* for a network $\mathcal{N}$ is a family of functions $\mathcal{G} := \{G^v : \mathcal{R}_v^* \to \mathcal{S}_v\}_{0\leq v<n}$ describing the ratio in which the outflow from each non-destination node v gets split among its outgoing link set $\mathcal{E}_v^+$, as a function of the observed current density ρ^v on the outgoing links themselves. For all $0 \leq v < n$, the function G^v is assumed to be continuously differentiable and such that

(i) if for some $e \in \mathcal{E}_v^+$, $\rho_e = \rho_e^{\max}$, then $G_e^v(\rho^v) = 0$;

(ii) $\dfrac{\partial}{\partial\rho_e} G_j^v(\rho^v) \geq 0$ for all $\rho^v \in \mathcal{R}_v^*$, $e \neq j \in \mathcal{E}_v^+$.

The two salient features of Definition 2 are the *local information constraint* which allows the routing policy $G^v(\rho^v)$ to depend only on the density ρ^v on the set $\mathcal{E}_v^+$ of outgoing links of the non-destination node v, and the constraint (i), which models the fact that no flow can be routed to a

fully congested link. Observe that every $\rho^v \in \mathcal{R}_v^*$ has at least one non-zero component, so that the aforementioned constraint does not prevent one from meeting the constraint $\sum_e G_e^v(\rho^v) = 1$. The additional condition (ii) is a rather natural in that it states that the fraction of flow routed towards any link does not decrease when the density in some other link is increased. It is reminiscent of the notion of cooperative dynamical system [13], and in fact implies certain useful monotonicity properties of the solution of the dynamical network. In fact, routing policies with this property were proven to be optimal in terms of robustness in the our earlier work [2], [3] on dynamical networks with infinite density capacity on the links.

We are now ready to introduce the dynamical network. Let $\mathcal{N}$ be a flow network satisfying Assumption 1, $\mathcal{G}$ a distributed routing policy as per Definition 2, and $\lambda_0 \geq 0$ a constant inflow at the origin. Consider the dynamical system whose state is the density vector $\rho(t) \in \mathcal{R}$ evolving in time according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_e(t) = \chi_{\sigma(e)}(t)\lambda_{\sigma(e)}(t)G_e^{\sigma(e)}\left(\rho^{\sigma(e)}(t)\right) - \chi_{\tau(e)}(t)f_e(t)\,, \tag{1}$$

for all $e \in \mathcal{E}$, where $f(t) = \mu(\rho(t))$ and

$$\lambda_v(t) := \begin{cases} \lambda_0 & \text{if} \quad v = 0 \\ \sum_{e\in\mathcal{E}_v^-} f_e(t) & \text{if} \quad v > 0\,, \end{cases} \tag{2}$$

is the incoming flow at node $v \in \mathcal{V}$, while

$$\chi_v(t) := \begin{cases} 1 - \prod_{e\in\mathcal{E}_v^+}(1-\xi_e(t)) & \text{if} \quad v < n \\ & \text{if} \quad v = n\,, \end{cases} \tag{3}$$

$$\xi_e(t) := \mathbf{1}_{[0,\rho_e^{\max})}(\rho_e(t))\,, \tag{4}$$

are the activation status indicators of a node $v \in \mathcal{V}$, and of a link $e \in \mathcal{E}$. We shell refer to the dynamics (1)-(4) as to the dynamical network associated to the triple $(\mathcal{N}, \mathcal{G}, \lambda_0)$.

Equation (1) states that the rate of change of the density on a link e outgoing from some non-destination node v is given by the difference between $\lambda_v(t)G_e^v(\rho^v(t))$, i.e., the portion of the total outflow at node v which is routed to link e, and $f_e(t)$, i.e., the flow on link e. These equations model conservation of mass both at every non-destination node and on the links of the flow network. In particular, when $\chi_v(t) = 0$, no flow can be absorbed by any of the outgoing links of node v, and (1) implies that no flow comes out of any of the incoming links of node v. Observe that the distributed routing policy $G^v(\rho^v)$ induces a local feedback which couples the dynamics of the flow on the different links. In fact, the dynamical network (1)-(3) should be interpreted as an $|\mathcal{E}|$-dimensional switched system. Existence and uniqueness of a solution for every initial density $\rho(0) \in \mathcal{R}$ then follow from the differentiability assumptions on the flow function μ and the routing policy $\mathcal{G}$ by standard arguments.

The most novel feature of the dynamics (1)-(4) resides in the role of the link and node activation status indicators $\xi_e(t)$, and $\chi_v(t)$. Indeed, observe that, if $\xi_e(t^*) = 0$ for some t^*, then $\xi_e(t) = 0$ for all $t \geq t^*$. This is a direct consequence of the fact that $\lambda_{\sigma(e)}(t)G_e^v(\rho^v) - \mu_e(\rho_e) = 0$ whenever $\rho_e = \rho_e^{\max}$. Once the density reaches its maximum capacity on a link, the corresponding outflow is zero, and the link becomes irreversibly inactive. On the other hand, (3) implies that a node becomes inactive, or fails, when all the outgoing links do so, and thus it remains inactive ever since. In turn, this drops the outflow of all its incoming links to zero so that they are bound to become inactive. As a consequence some other links may experience an overload, possibly reaching their density capacity, thus becoming inactive ever since. Through this mechanism, link and node failures can propagate through the network.

The monotonicity property of the status indicators are stated in the following Proposition together with another fundamental property of the dynamical network.

Proposition 1: Let $\mathcal{N}$ be a flow network satisfying Assumption 1, $\mathcal{G}$ be a distributed routing policy as per Definition 2, and $\lambda_0 \geq 0$ a constant inflow at the origin node. Consider the dynamical network (1)-(4) associated to $(\mathcal{N}, \mathcal{G}, \lambda_0)$. Then, for any initial density vector $\rho(0) \in \mathcal{R}$, the activation status indicators $\xi_e(e)$ and $\chi_v(t)$, of every link $e \in \mathcal{E}$ and every node $v \in \mathcal{V}$ are non-increasing in t. Moreover, either of the alternatives

$$\lim_{t\to\infty} \lambda_n(t) = \lambda_0\,, \tag{5}$$

$$\lim_{t\to\infty} \lambda_n(t) = 0 \tag{6}$$

holds.

Proof The first part of the claim was already proven. In order to prove the second part, fix some $\tau > 0$, and define

$$\zeta(t) := ||\rho(t+\tau) - \rho(t)||_1\,, \quad \beta(t) := \sum_{e\in\mathcal{E}_n^-} |f_e(t+\tau) - f_e(t)|\,.$$

Let $t^* > 0$ be the time of the last link failure. Arguing in the a way analogous to [2, Lemmas 1 and 2] one finds that

$$\zeta(t) \leq \zeta(t^*) + \int_{t^*}^{t} \beta(s)\mathrm{d}s\,, \qquad \forall t \geq t^*\,,$$

so that $\beta(t)$ is integrable. Then a standard argument (exploiting the boundedness of its time-derivative, see again the proof of [3, Lemma 2] shows that the vector $\{f_e(t) : e \in \mathcal{E}_n^-\}$ is converging. Hence, *a fortiori*, $\lambda_n(t) = \sum_{e\in\mathcal{E}_v^+} f_e(t)$ is convergent. Finally, it is not hard to check that, if $\chi_0(t) = 1$ for all $t \geq 0$, then $\lim_{t\to\infty}\lambda_n(t) = \lambda_0$, while $\lim_{t\to\infty}\lambda_n(t) = 0$ if $\chi_0(t) = 0$ for $t \geq t^*$. ■

The second part of Proposition 1 states a fundamental dichotomy in the behavior of the dynamical network we are considering: either all the asymptotic outflow equals the constant inflow, or it is zero. Such dichotomy is a direct consequence of the boundedness of the density capacities and can in fact be contrasted with the behavior of dynamical networks with infinite density capacity studied in [2], [3], were the notion of α transferring network is meaningful for all $\alpha \in (0, 1]$. This motivates the following:

Definition 3: Let $\mathcal{N}$ be a flow network satisfying Assumption 1, $\mathcal{G}$ be a distributed routing policy as per Definition 2, and $\lambda_0 \geq 0$ a constant inflow at the origin node. The dynamical network (1)-(3) associated to $(\mathcal{N}, \mathcal{G}, \lambda_0)$ is said to be *transferring* with respect to some initial density vector $\rho(0) \in \mathcal{R}$ if (5) holds.

B. Perturbations and resilience

We shall consider persistent perturbations of the dynamical transport network (1) that reduce the flow functions on the links, as per the following:

Definition 4 (Admissible perturbation): An *admissible perturbation* of a network $\mathcal{N} = (\mathcal{T}, \mu)$, satisfying Assumption 1, is a network $\tilde{\mathcal{N}} = (\mathcal{T}, \tilde{\mu})$, with the same topology $\mathcal{T}$, and a family of perturbed flow functions $\tilde{\mu} := \{\tilde{\mu}_e : [0, \rho_e^{\max}] \to \mathbb{R}_+\}_{e\in\mathcal{E}}$, such that, for every $e \in \mathcal{E}$, $\tilde{\mu}_e$ satisfies Assumption 1, as well as $\tilde{\mu}_e(\rho_e) \leq \mu_e(\rho_e)$, for all $\rho_e \in [0, \rho_e^{\max}]$. We accordingly let $\tilde{f}_{\mathsf{e}}^{\max} := \sup\{\tilde{\mu}_e(\rho_e) : \rho_e \in [0, \rho_e^{\max}]\}$. The *magnitude* of an admissible perturbation is defined as

$$||\delta||_1 = \sum_e \delta_e \,,$$

where

$$\delta \in \mathbb{R}_+^{\mathcal{E}} \,, \qquad \delta_e := \sup_{\rho_e\in[0,\rho_e^{\max}]} \{\mu_e(\rho_e) - \tilde{\mu}_e(\rho_e)\} \;,\; e \in \mathcal{E} \,. \tag{7}$$

Remark 1: Note that under the above definition of an admissible perturbation, we let $\rho_e^{\max}$ of the perturbed flow function to be the same as that of the original flow function.

Given a dynamical transport network associated to a flow $\mathcal{N}$, a distributed routing policy $\mathcal{G}$, a constant inflow λ_0, and an admissible perturbation $\tilde{\mathcal{N}}$, we shall refer to the dynamical network associated to the triple $(\tilde{\mathcal{N}}, \mathcal{G}, \lambda_0)$ as the *perturbed dynamical network*.

We can now define the notion of *margin of resilience*.

Definition 5: (*Margin of resilience*) Let $\mathcal{N}$ be a flow network satisfying Assumption 1, $\mathcal{G}$ be a distributed routing policy as per Definition 2, and $\lambda_0 \geq 0$ a constant inflow at the origin node. Consider the dynamical network (1)-(4) associated to $(\mathcal{N}, \mathcal{G}, \lambda_0)$. For any $\rho^\circ \in \mathcal{R}$, the margin of resilience $\gamma(\mathcal{N}, \rho^\circ)$ is defined as the infimum magnitude of all the admissible perturbations $\tilde{\mathcal{N}}$ for which the perturbed dynamical network $(\tilde{\mathcal{N}}, \mathcal{G}, \lambda_0)$ is not transferring with respect to the initial density vector $\tilde{\rho}(0) = \rho^\circ$.

In the rest of this paper we shall focus on estimating the margin of resilience of dynamical networks. A first such estimate is provided in terms of the min-cut capacity of the network. In fact, it is not hard to show that $C(\mathcal{N}) - \lambda_0$ is an upper bound on the margin of resilience of the dynamical network associated to $(\mathcal{N}, \mathcal{G}, \lambda_0)$. Observe that in [2] the network capacity, $C(\mathcal{N})$, was found to be the maximal margin of weak resilience of a dynamical network without either finite density capacity, or local information constraints on the routing policy. In the following section, we shall derive a tighter bound on the margin of resilience in the presence of finite density capacity and local information constraints.

III. Upper bound on the resilience

In this section, we present the main result providing an upper bound on the margin of resilience of a dynamical network. Throughout, we shall assume that ρ° is an equilibrium for the unperturbed dynamical flow network $(\mathcal{N}, \mathcal{G}, \lambda_0)$, with corresponding flow $f^\circ = \mu(\rho^\circ)$. We shall also assume that the topology $\mathcal{T}$ is tree-like, i.e., the only node reachable from the origin by two distinct paths is the destination one. This in particular implies that the nodes $v \in \mathcal{V} = \{0, \dots, n\}$ have been labeled in such a way that $\sigma(e) < \tau(e)$ for every $e \in \mathcal{E}$.

Before proceeding we introduce some preliminary notation. For a flow network satisfying Assumption 1, and an equilibrium flow $f^\circ \in \mathcal{F}^*(\lambda_0)$, let

$$R_v(\mathcal{N}, f^\circ) := \sum_{e\in\mathcal{E}_v^+} f_{\mathsf{e}}^{\max} - f_e^\circ \,,$$

be the residual capacity of a non-destination node $0 \leq v < n$, and let

$$R(\mathcal{N}, f^\circ) := \min\{R_v(\mathcal{N}, f^\circ) : 0 \leq v < n\}$$

be the minimal node residual capacity of the network.

For every non-destination node $0 \leq v < n$, and $\lambda \geq 0$, define

$$\mathcal{X}_v(\lambda) := \left\{x \in \times_e [0, f_{\mathsf{e}}^{\max}] : \sum\nolimits_e (f_{\mathsf{e}}^{\max} - x_e) \leq \lambda\right\} . \tag{8}$$

where the product/sum index e is intended to run over $\mathcal{E}_v^+$. Further, let $d_n = +\infty$. For $v = n-1, \dots, 1, 0$, iteratively define

$$d_v := \min\left\{c_v(x) : x \in \mathcal{X}_v(\lambda_v^\circ)\right\} \,, \qquad \lambda_v^\circ := \sum\nolimits_{e\in\mathcal{E}_v^+} f_e^\circ \,, \tag{9}$$

where

$$c_v(x) := \sum\nolimits_{e\in\mathcal{E}_v^+} \min\{x_e, d_{\tau(e)}\} \,. \tag{10}$$

The intuition behind this definition is the following: $c_v(x)$ is the cost that an hypothetical malicious adversary has to face in order to reduce the sum of the maximal flow capacities of the outgoing links of a node v below the inflow λ_v°, thus causing the eventual link's failure. In order to compute such cost, for every outgoing link e, the minimum between the flow capacity reduction x_e and the previously computed cost to induce a failure of the head node $\tau(e)$ is considered.

Observe that $\mathcal{X}_v = \mathcal{X}_v(\lambda_v^\circ)$ is a non-empty convex polytope, and the cost function c is concave over $\mathcal{X}_v$. Hence, the minimization (9) can be restricted to the finite set of extremal points of $\mathcal{X}_v$, i.e., points which cannot be written as the convex combination of other points in $\mathcal{X}_v$. Let $\mathcal{X}_v^*$ be the set of those extremal points x^* of $\mathcal{X}_v$ in which $c_v(x)$ achieves its minimum. For all $x^* \in \mathcal{X}_v^*$, define $\Delta_v(x^*) \subseteq \mathbb{R}_+^{\mathcal{E}}$ as follows. Let $\mathcal{J} := \{e \in \mathcal{E}_v^+ : x_e^* < d_{\tau(e)}\}$, and define $\delta^* \in \mathbb{R}^{\mathcal{E}}$ by $\delta_j^* = x_j^*$ for all $j \in \mathcal{J}$, and $\delta_e^* = 0$ for any $e \in \mathcal{E} \setminus J$. Then, let $\mathcal{K} := \{\tau(e) : e \in \mathcal{E}_v^+ \setminus \mathcal{J}\}$, and define

$$\Delta_v(x^*) := \left\{ \delta^* + \sum_{k\in\mathcal{K}} \delta_{\tau(k)} : \delta_k \in \Delta_k \,, \forall k \in \mathcal{K} \right\} .$$

Finally, put

$$\Delta_v := \bigcup_{x^*\in\mathcal{X}_v^*} \Delta_v(x^*) \,.$$

In order to get some intuition on the above definition, it is convenient to think of Δ_v as the set of extremal minimum cost perturbation magnitudes that cause the eventual failure of node v. This motivates the following

Definition 6: Let $\mathcal{N}$ be an tree-like flow network satisfying Assumption 1, $\lambda_0 \geq 0$ a constant inflow, and $f^\circ \in \mathcal{F}^*(\lambda_0)$ an equilibrium flow. Let

$$\Gamma(\mathcal{N}, f^\circ) := d_0\,, \qquad \Delta(\mathcal{N}, f^\circ) := \Delta_0\,,$$

where d_0 and Δ_0 are the outcomes of the foregoing iterative definition.

Observe that the computation of $\Gamma(\mathcal{N}, f^\circ) := d_0$ can be considered computationally feasible as it involves iteratively solving n minimizations of concave functions (in fact, it is possible to cast it as a linear program) on polytopes in $\mathcal{R}_v$ defined by $|\mathcal{E}_v^+| + 1$ inequalities.

Two simple properties of the above definition are gathered in the following proposition, whose proof is omitted because of space limitations.

Proposition 2: Let $\mathcal{N}$ be a flow network satisfying Assumption 1, $\lambda_0 > 0$ a constant inflow, and $f^\circ \in \mathcal{F}^*(\lambda_0)$ an equilibrium flow. Then,

$$R(\mathcal{N}, f^\circ) \leq \Gamma(\mathcal{N}, f^\circ) \leq C(\mathcal{N}) - \lambda_0\,.$$

The main result of this section is stated below.

Theorem 1 (Upper bound on the margin of resilience): Let $\mathcal{N}$ be a tree-like flow network satisfying Assumption 1, $\lambda_0 > 0$ a constant inflow, and $\mathcal{G}$ a distributed routing policy. Assume that the dynamical network associated to $(\mathcal{N}, \mathcal{G}, \lambda_0)$ admits an equilibrium density vector $\rho^\circ \in \mathcal{R}$. Then, the margin of the resilience is upper bounded as:

$$\gamma(\rho^\circ, \mathcal{N}) \leq \Gamma(f^\circ, \mathcal{N})\,, \tag{11}$$

where $f^\circ := \mu(\rho^\circ)$.

Proof We provide a brief sketch of the proof, in order to convey the main ideas, and refer the reader to a forthcoming longer version of the manuscript for the details. For simplicity, we shall assume that $\lambda_v^\circ > 0$ for all $0 \leq v < n$.

Let us start by choosing a perturbation magnitude vector $\delta \in \Delta(\mathcal{N}, f^\circ)$. To each such vector, one can associate a subset of nodes $\mathcal{U} \subseteq \mathcal{V}$ which includes 0, as well as all those nodes v such that $\delta_e > 0$ for some $e \in \mathcal{E}_v^+$. For the ease of the rest of the proof it is convenient to explicit the natural ordering of $\mathcal{U}$, by writing $\mathcal{U} = \{u_k : 0 \leq k \leq l\}$, where $0 = u_0 < v_1 < \ldots < v_l$. The proof proceeds by defining an admissible perturbation $\tilde{\mathcal{N}} = (\mathcal{T}, \tilde{\mu})$ such that $||\mu(\rho_e) - \tilde{\mu}_e(\rho_e)||_\infty$ is arbitrarily close to δ_e on every link $e \in \mathcal{E}$, and the perturbed dynamical network associated to $(\mathcal{N}, \mathcal{G}, \lambda_0)$ is not transferring. This is performed inductively on $k = l, \ldots, 1, 0$. First, let us consider node $v = u_l$, and observe that, because of the way δ has been defined, necessarily $\delta_e = x_e^*$ for all $e \in \mathcal{E}_v^+$, where $x^* \in \mathcal{X}_v^*$ is an extremal minimizer of c_v over $\mathcal{X}_v(\lambda_v^\circ)$. Since $\lambda_v^\circ > 0$, we can approximate x^* arbitrarily well by some $\tilde{x}^* \in \times_{e \in \mathcal{E}_v^+}(0, f_{\mathrm{e}}^{\max})$ such that $\sum_{e \in \mathcal{E}_v^+} f_{\mathrm{e}}^{\max} - \tilde{x}_e^* < \lambda_v^o$. Then, we can define an admissible perturbation $\tilde{\mathcal{N}}^l = (\mathcal{T}, \tilde{\mu}^l)$ such that $\tilde{\mu}_e^l \equiv (1 - \tilde{x}_e^*/f_{\mathrm{e}}^{\max})\mu_e$ for all $e \in \mathcal{E}_v^+$, and $\tilde{\mu}_e^l \equiv \mu_e$ for every $e \in \mathcal{E} \setminus \mathcal{E}_v^+$. The first key property of this perturbation is that, since none of the links $e \notin \mathcal{E}_v^+$ is perturbed, the tree-likeness of the network topology implies that the dynamics of the edges $e \in \mathcal{E}$ with $\sigma(e) < v$ are unaffected by the perturbation, until the first node failure. This implies that, in the perturbed dynamic network $(\tilde{\mathcal{N}}^k, \mathcal{G}, \lambda_0)$, one has

$$\lambda_v(t) = \lambda_v^\circ > \sum_{e \in \mathcal{E}_v^+} f_{\mathrm{e}}^{\max} - \tilde{x}_e^* = \sum_{e \in \mathcal{E}_v^+} \tilde{f}_e^{\max}\,,$$

until the first node failure, which in turn implies that in finite time $\rho_e(t)$ converges to $\rho_e^{\max}$, and hence $\xi_e(t) = 0$, for all $e \in \mathcal{E}_v^+$, so that node v necessarily fails in finite time.

Now, one can proceed by considering a perturbation $\tilde{\mathcal{N}}^{l-1}$ which analogously reduces the flow capacity on both $\mathcal{E}_{u_l}^+$, and $\mathcal{E}_{u_{l-1}}^+$. By exploiting property (ii) on Definition 2 it is possible to prove that the dynamical network is monotone in the sense of [13], i.e., preserving the natural partial order of $\mathcal{R}$ ($\rho \geq \rho'$ if $\rho_e \geq \rho'_e$ for all e), both with respect to the initial condition, and the flow function μ. This implies that, with the same initial condition $\rho(0)$, the solution of the perturbed dynamical network $(\tilde{\mathcal{N}}^{l-1}, \mathcal{G}, \lambda_0)$ is dominated by the one of $(\tilde{\mathcal{N}}^l, \mathcal{G}, \lambda_0)$. Hence, since node u_l fails in finite time in $(\tilde{\mathcal{N}}^l, \mathcal{G}, \lambda_0)$, it does so also in $(\tilde{\mathcal{N}}^{l-1}, \mathcal{G}, \lambda_0)$, and arguing in a similar way as before, one is able to show that also node u_{l-1} fails in finite time in $(\tilde{\mathcal{N}}^{l-1}, \mathcal{G}, \lambda_0)$. Then, the argument can be iterated until proving that node $u_0 = 0$ fails in finite time in $(\tilde{\mathcal{N}}^0, \mathcal{G}, \lambda_0)$, which implies the claim. ■

Let us conclude this section with the following remark. Theorem 1 states that $\Gamma(f^\circ, \mathcal{N})$ is an upper bound on the margin of resilience $\gamma(\rho^\circ, \mathcal{N})$ of a tree-like dynamical network with respect to some initial equilibrium. The reason why this upper bound may fail to be tight can be intuitively grasped by observing that $\Gamma(f^\circ, \mathcal{N})$ takes into account just backward propagations of node failures. However, it could occur that, e.g., in a tree-like network, perturbations supported far from the root node cause the eventual failure first of one of the branches stemming from the root node. However, because of the lack of information about the non-disruptive but still potentially resilience-reducing effect of the perturbation on the surviving branches, the inflow at the root node gets split among the surviving branches in a potentially suboptimal way, leading to the eventual failure of such branches as well, and therefore making the dynamical network non transferring. Such kind of patterns are not taken into account by the analysis presented in this section.

IV. Examples

In this section, we present an example to illustrate the effect of cascades on the margin of resilience, and compare it with corresponding results from our prior work [2], [3] where we computed margins of resilience for dynamical networks with no cascading failures.

In [2], we computed margin of *weak* resilience of a dynamical network when there is no bound on the maximum density on the links, and the flow functions are monotonically increasing. The margin of weak resilience is defined to be the infimum magnitude of all admissible perturbations for which the outflow from the destination node of the perturbed dynamical network is not asymptotically positive. In particular, we showed that the margin of weak resilience in that setting is equal to the maximum flow capacity of the network. It is easy to construct examples to demonstrate that

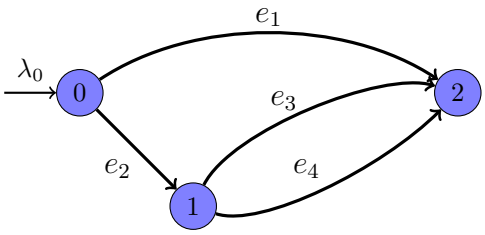

Fig. 1. A sample network topology.

the margin of (weak) resilience can decrease due to presence of cascading failures.

In [3], we computed margin of *strong* resilience of a dynamical network when there is no bound on the maximum density on the links, and the flow functions are monotonically increasing. The margin of strong resilience is defined to be the infimum magnitude of all admissible perturbations for which (analogous to definition 5) the outflow from the destination node of the perturbed dynamical network is not asymptotically equal to λ_0. In particular, we showed that the margin of strong resilience in that setting is equal to the minimum node residual capacity of the network. We demonstrate that, when the links have finite capacity for densities as in this paper, the margin of strong resilience could be possibly *greater* than the minimum node residual capacity. We illustrate this point through the following example.

Consider the topology shown in Figure 1. Let the flow functions μ be such that: $f_{e_1}^{\max} = 3$, $f_{e_2}^{\max} = 1.5$, $f_{e_3}^{\max} = f_{e_4}^{\max} = 0.75$. Let the equilibrium flows be: $f_{e_1}^{\circ} = f_{e_2}^{\circ} = 1$ and $f_{e_3}^{\circ} = f_{e_4}^{\circ} = 0.5$. The min node residual capacity with these parameters is 0.5. We now show that, if nodes 0 and 1 are implementing distributed routing policy as in Definition 2, then even with a disturbance of magnitude $0.6 > 0.5$, then network is still fully transferring. First, consider a specific disturbance of magnitude 0.6 under which the perturbed flow functions are: $\tilde{\mu}_{e_1} = \mu_{e_1}$, $\tilde{\mu}_{e_2} = \mu_{e_2}$, $\tilde{\mu}_{e_3} = 3\mu_{e_3}/5$ and $\tilde{\mu}_{e_4} = 3\mu_{e_4}/5$. For the perturbed network, $f_{e_3}^{\max} + f_{e_4}^{\max} = 0.9 < 1 = f_{e_3}^{\circ} + f_{e_4}^{\circ}$. Therefore, after a finite time, both $\tilde{\rho}_{e_3}(t)$ and $\tilde{\rho}_{e_4}(t)$ hit the respective maximum capacity on densities. at which point χ_1 becomes zero. As a consequence the outflow term for link e_2 becomes zero after this time. If the routing policy G at node 0 has property that $G_{e_2}^0(\tilde{\rho}^v) = 0$ if $\tilde{\rho}_{e_2} = \hat{\rho}_{e_2}$, the inflow of 2 at node 0 is routed to link e_1 and hence the network maintains its fully transferring property. In general, for any other disturbance of magnitude 0.6, in the worst-case, the inflow to node 1 would be such that it could exceed the sum of perturbed capacities of links e_3 and e_4 and hence making $\chi_1 = 1$, after which one can repeat the argument to show that all the inflow of 2 at node 0 is transferred to link e_1 and the network maintains its transferring property.

This example shows that spill-backs act as backward propagators of information to upstream routing policies (in this example, routing policy at node 0 gets information about links e_3 and e_4 through spill-backs). Such additional information about downstream links by routing policies increases resilience.

V. Conclusion

In this paper, we studied dynamical network modeled as switched systems of ordinary differential equations derived from mass conservation laws on directed graphs with single origin-destination pair and constant inflow at the origin. The main features of this framework are finiteness of the maximum density capacity on the links, and the local information constraint on the routing policies which govern the way the outflow of a node gets split among its outgoing links as a function of the current density on those links only. Because of the finiteness assumption on the link density capacities, the model allows for cascaded link and node failures. Our analysis has focused on resilience properties of such networks against persistent perturbations that reduce the flow capacities of their links. Beyond the introduction of the model and the derivation of some of its fundamental properties, our main contribution is an upper bound on the margin of resilience, defined as the minimum magnitude of a perturbation causing the eventual failure of the origin node, thus preventing any flow to reach destination. Such upper bound may fail to be tight because it only takes into account indirect backward propagation patterns of failures.

In future, we plan to complement the analysis presented here with lower bounds for specific routing policies and extend the analysis to networks with cycles.

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