Strategic Dynamic Vehicle Routing with Spatio-Temporal Dependent Demands

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***Abstract*— We study a dynamic vehicle routing problem where demands are strategically placed in the region by an adversarial agent with unitary capacity operating from a depot. In particular, we focus on the following problem: a system planner seeks to design dynamic vehicle routing policies for a vehicle that minimize the average waiting time of a typical demand, defined as the time difference between the moment the demand is placed in the region until its location is visited by the vehicle; while the agent aims at the opposite, strategically choosing the spatial distribution to place demands. We model the problem as a complete information zero-sum game and characterize an equilibrium in the limiting case where the vehicle travels arbitrarily slower than the agent. We show that such an equilibrium is constituted by a routing policy based on performing successive traveling salesperson tours through outstanding demands and a unique power-law spatial density centered at the depot location.**

I. Introduction

In the recent past, considerable efforts have been devoted to dynamic vehicle routing problems, where the objective is to cooperatively assign and schedule demands among a team of vehicles for service requests that are realized in a dynamic fashion over a region of interest [3], [4], [7]. These problems provide a rich framework for a variety of applications such as surveillance missions, environmental monitoring, automated material handling and transportation networks. Throughout the existing literature, demands are assumed to be generated over time by an exogenous process that is unaffected by the routing policies, and in particular is non-adversarial [7]. A recurrent theme is that demands are either customers that need to be picked up, raw material or merchandise to be delivered, failures that must be serviced by a mobile repair person, etc. However, there are many scenarios, including surveillance missions, where there is an inherent conflict of interest between the process generating demands and the system planner designing routing policies. Moreover, even in non-adversarial scenarios the system planner may not have perfect information about the underlying process generating demands and a study of strategic dynamic vehicle routing can add insight into policies that are robust with respect to such uncertainty. To the best of our knowledge, settings with these characteristics have not yet been studied.

In this paper, we consider the following problem: a system planner seeks to design dynamic vehicle routing policies for a vehicle that minimize the average waiting time of a typical demand, defined as the time difference between the moment the demand is placed in the region until its location is visited by a vehicle; while an adversarial agent with unitary capacity operating from a depot, aims at the opposite, strategically choosing the spatio-temporal stochastic process of demands. A novel feature of this setup is that, since demand generation is tied to the motion of the agent, there is a dependence between the spatial and temporal aspect of the demand generation process: the point process is thus completely specified by the spatial distribution. This is in stark contrast with the conventional setup for dynamic vehicle routing problems, where the spatial and temporal components of the demand generation process are typically assumed to be independent.

We model the problem and its inherent pure conflict of interests as a complete information zero-sum game with two players: the system planner and the adversarial agent, with the average system time being the utility function. In the limiting regime when the vehicle travels arbitrarily slower than the adversarial agent, we show that the game has a finite value and we characterize an equilibrium (or saddle point) of the game. This saddle point is shown to consist of a routing policy performing successive traveling salesperson (TSP) tours through outstanding demands and a unique power-law spatial density centered at the depot location. The saddle point routing policy is the one proposed in [4], where it is shown to be optimal for the setup where the demands are generated by an arbitrary spatio-temporal renewal process with a very high arrival rate. In order to rigorously determine the saddle point spatial distribution for the adversary we rely on Fenchel (conjugate) duality [15] and results from [5], [6] concerning the maximization of concave integral functionals subject to linear equality constraints. Such mathematical tools could also be brought to bear on the study of dynamic vehicle routing with partial information about demand generation, where the goal is to design routing policies with performance guarantees under worst case scenarios.

The rest of the paper is organized as follows: Section II contains basic notions on classic convex theory, as well as duality results for partially finite optimization (optimization of a functional subject to a finite number of constraints); Section III describes the problem and the zero-sum game theoretic formulation; in Section IV we review the recursive TSP-based routing policy and its optimality for the conventional dynamic vehicle routing setup, and we show that the value of the game is obtained by maximizing a

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functional over the space of integrable functions, where the maximizer constitutes the saddle point spatial density of demands; simulations are presented in Section VI, while conclusions and future work are discussed in Section VII.

II. Mathematical Preliminaries

A. Basic Concepts and Notation

A point $\mathbf{x}$ in the n-dimensional Euclidean space $\mathbb{R}^n$ will be conceived as a column vector, where x_i denotes its i-th component. The inner product of two vectors in $\mathbb{R}^n$ will be written as $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \sum_i x_i y_i$. The non-negative orthant is the set $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0\}$, where $\geq$ is to be understood component-wise.

We now introduce some basic concepts from convex analysis, as found in [2], [15]. The *epigraph* of an extended real-valued function $f : X \subseteq \mathbb{R}^n \to [-\infty, +\infty]$ is the set

$$\text{epi } f = \{(\mathbf{x}, w) \in \mathbb{R}^{n+1} : \mathbf{x} \in X, w \in \mathbb{R}, f(\mathbf{x}) \leq w\};$$

its *effective domain* is defined as $\text{dom } f = \{\mathbf{x} \in X : f(\mathbf{x}) < \infty\}$, which is the projection of $\text{epi } f$ on $\mathbb{R}^n$. We say that the function f is *proper* if $f(\mathbf{x}) < +\infty$ for at least one $\mathbf{x} \in X$ and $f(\mathbf{x}) > -\infty$ for all $\mathbf{x} \in X$; and f is said to be *closed*, if its epigraph is a closed set. A proper convex function $f : \mathbb{R} \to (-\infty, +\infty]$ is said to be: *essentially strictly convex* if f is strictly convex on strictly convex on $\text{dom } f$; and *essentially smooth* if f is differentiable on the interior of $\text{dom } f$ and $\|f'(\mathbf{x}_k)\| \to \infty$ for any sequence $\{\mathbf{x}_k\}$ in the interior of $\text{dom } f$ such that $\mathbf{x}_k \to \mathbf{x}$ with $\mathbf{x}$ in the boundary of $\text{dom } f$. Given a set $X \subseteq \mathbb{R}^n$, we define its *indicator function* $\delta : \mathbb{R}^n \to (-\infty, +\infty]$ as

$$\delta(\mathbf{x}|X) = \begin{cases} 0 & \text{if } \mathbf{x} \in X, \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

The *affine hull* of a subset X of $\mathbb{R}^n$ is the smallest affine set containing X. The *relative interior* of X, denoted $\text{ri } X$, is defined as the interior which results when X is considered as a subset of its affine hull. The key property of relative interiors is that if X is a nonempty convex set, then $\text{ri } X$ is nonempty and convex as well (in contrast to the interior of X, which is certainly convex but might be empty). Often, finding the relative interior of a set based on its definition might be cumbersome. The next lemma, as stated in [2], provides an equivalent characterization for convex sets.

Lemma 1: Let X be a nonempty convex set. Then, $\mathbf{x} \in \text{ri } X$ if and only if, for every $\mathbf{y} \in X$ there exists a scalar $\alpha > 0$ such that $\mathbf{x} + \alpha(\mathbf{x} - \mathbf{y}) \in X$.

We end this section with a generalized version of the classical Weierstrass theorem concerning the existence of minima of an extended real-valued function.

Theorem 1: Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a closed proper extended real-valued function. If there exists $\eta \in \mathbb{R}$ such that the level set $\{\mathbf{x} \in \text{dom } f : f(\mathbf{x}) \leq \eta\}$ is nonempty and bounded, then the set of minima of f over $\mathbb{R}^n$ is nonempty and compact.

B. Conjugate Functions and Fenchel Duality

Let V and $V^\star$ be vector spaces, equipped with a bilinear product $\langle \cdot, \cdot \rangle$ on the product space $V \times V^\star$, and consider a convex function $f : V \to [-\infty, +\infty]$. The *(Fenchel) conjugate function* of f with respect to $\langle \cdot, \cdot \rangle$, is a function $f^\star : V^\star \to [-\infty, +\infty]$ defined as

$$f^\star(\mathbf{x}^\star) := \sup\{\langle \mathbf{x}, \mathbf{x}^\star \rangle - f(\mathbf{x}) : \mathbf{x} \in V\}. \tag{2}$$

Fenchel's duality theory is concerned with the problem of minimizing the difference of two proper functions, $f - g$, convex and concave respectively. The following duality theorem resides in the connection between minimizing $f - g$ (convex) and maximizing $g^\star - f^\star$ (concave).

Theorem 2: Let V and $V^\star$ be vector spaces paired by a bilinear product $\langle \cdot, \cdot \rangle$ defined on $V \times V^\star$. Let $\mathbf{A} : V \to \mathbb{R}^n$ be a linear map with adjoint $\mathbf{A}^T$, and let $f : V \to (-\infty, +\infty]$ and $g : \mathbb{R}^n \to [-\infty, +\infty)$ be proper functions, convex and concave respectively. If the constraint qualification

$$\text{ri } (\mathbf{A}\text{dom } f) \cap \text{ri } (\text{dom } g) \neq \emptyset$$

is satisfied, then

$$\inf_{\mathbf{x} \in V} \{f(\mathbf{x}) - g(\mathbf{A}\mathbf{x})\} = \sup_{\boldsymbol{\xi} \in \mathbb{R}^n} \{g^\star(\boldsymbol{\xi}) - f^\star(\mathbf{A}^T \boldsymbol{\xi})\},$$

with the supremum on the right being attained when finite.

The reader is referred to [15] for the proof in the case where V has finite dimension, and to [6] when V is infinite-dimensional. The latter case is often called *partially finite* because the linear operator $\mathbf{A}$ maps V into $\mathbb{R}^n$.

C. Partially Finite Convex Programming in $\mathcal{L}_1$

Let $\mathcal{S} \in \mathbb{R}^n$ be a finite Lebesgue measure set, and let $h : \mathbb{R} \to \mathbb{R}$ be a closed proper convex function. Consider the (convex) functional $\mathcal{I} : \mathcal{L}_1(\mathcal{S}) \to [-\infty, +\infty]$ defined as in [16] by

$$\mathcal{I}(\varphi) = \int_{\mathcal{S}} h(\varphi(\mathbf{x})) d\mathbf{x}. \tag{3}$$

Now, consider the following optimization problem:

$$\inf \ \mathcal{I}(\varphi) \quad \text{s.t.} \quad \mathbf{A}\varphi = \mathbf{b}, \ \varphi \in \mathcal{L}_1(\mathcal{S}), \tag{4}$$

where $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{A} : \mathcal{L}_1(\mathcal{S}) \to \mathbb{R}^n$ is a continuous linear map with components $A_i \in \mathcal{L}_\infty(\mathcal{S})$ defined by

$$(\mathbf{A}\varphi)_i = \int_{\mathcal{S}} A_i(\mathbf{x}) \varphi(\mathbf{x}) d\mathbf{x}, \quad \text{for} \quad i = 1, \ldots, n. \tag{5}$$

The most widely encountered instance of (4) is the problem of entropy optimization (see [10] and references therein), where the goal is to describe the statistical properties of an underlying stochastic process from a finite set of measurements of its moments.

Problem (4) is amenable to solve through Fenchel duality. Let $V = \mathcal{L}_1(\mathcal{S})$ and $V^\star = \mathcal{L}_\infty(\mathcal{S})$, then it is possible to define a bilinear product on $V \times V^\star$ by,

$$(\varphi, \varphi^\star) \longmapsto \langle \varphi, \varphi^\star \rangle := \int_{\mathcal{S}} \varphi(\mathbf{x}) \varphi^\star(\mathbf{x}) d\mathbf{x}. \tag{6}$$

As the following result [14] shows, to compute the convex conjugate of the integral functional (3) with the bilinear product defined in (6), we may just conjugate the integrand.

Proposition 1: Let $\mathcal{S}$ be a finite measure set in $\mathbb{R}^n$, and let V and $V^\star$ be as above with bilinear product given by (6). Then, for any $\varphi^\star \in V^\star$, we have

$$\mathcal{I}^\star(\varphi^\star) = \int_{\mathcal{S}} h^\star(\varphi^\star(\mathbf{x}))d\mathbf{x}. \quad (7)$$

A direct application of Theorem 2 and Proposition 1, with $f := \mathcal{I}$ and $g(\mathbf{A}\varphi) := -\delta(\mathbf{A}\varphi - \mathbf{b}|\mathbf{0})$, yields the next result.

Corollary 1: Consider the problem defined by (4) and (5), and assume that the constraint qualification

$$\mathbf{b} \in \mathrm{ri}\ (\mathbf{A}\mathrm{dom}\ \mathcal{I}), \quad (8)$$

holds. Then, (4) is equal to

$$\sup\left\{\langle \boldsymbol{\xi}, \mathbf{b}\rangle - \mathcal{I}^\star(\mathbf{A}^T\boldsymbol{\xi}) : \boldsymbol{\xi} \in \mathbb{R}^n\right\}, \quad (9)$$

where $\mathbf{A}^T : \mathbb{R}^n \to \mathcal{L}_\infty(\mathcal{S})$ is the adjoint map, given by $\mathbf{A}^T\boldsymbol{\xi} := \sum_{i=1}^n \xi_i A_i$. Moreover, the supremum on the right-hand side of (9) is attained by some $\boldsymbol{\xi}^*$ whenever finite.

Problem (4) will be referred to as the *primal problem*, and $\varphi \in \mathcal{L}_1(\mathcal{S})$ the *primal variable*; (9) is the *dual problem*, and the vector $\boldsymbol{\xi} \in \mathbb{R}^n$ the *dual variable* or simply *multiplier*. The dual is always a convex problem, regardless of the structure of the primal. The following proposition gives sufficient conditions for the uniqueness of $\boldsymbol{\xi}^*$.

Proposition 2: If the set of constraint functions $\{A_i\}_{i=1}^n$ is linearly independent and $h^\star$ is essentially strictly convex, then any optimal dual solution is unique.

We now have all the ingredients required to state the chief result in [5], which yields the existence, uniqueness and characterization of the primal optimal solution $\varphi^*(\mathbf{x}) \in \mathcal{L}_1(\mathcal{S})$ in terms of $(h^\star)'$, the optimal dual solution $\boldsymbol{\xi}^*$ and the linear operator of constraints $\mathbf{A}$.

Theorem 3: Consider the primal-dual pair (4)–(9) of Corollary 1. Assume that h is an essentially strictly convex and essentially smooth function, and suppose the following condition is satisfied:

$$\Delta := \lim_{x\to\infty} \frac{h(x)}{x} > \operatorname*{ess\,sup}_{\mathbf{x}\in\mathcal{S}} \mathbf{A}^T\boldsymbol{\xi}^*(\mathbf{x}). \quad (10)$$

Then, the primal optimal solution is given by,

$$\varphi^*(\mathbf{x}) := (h^\star)'(\mathbf{A}^T\boldsymbol{\xi}^*(\mathbf{x})) = (h^\star)'\left(\sum_{i=1}^n \xi_i^* A_i(\mathbf{x})\right), \quad (11)$$

where $\boldsymbol{\xi}^* \in \mathbb{R}^n$ is the dual optimal solution.

The proof of Theorem 3 builds on results derived in [14] regarding the subgradients of convex integral functionals, and is mainly based on differentiating the dual objective function at the optimum.

III. Problem Description

Consider a bounded set $\mathcal{S} \subseteq \mathbb{R}^2$ with $\mu(\mathcal{S}) > 0$, where $\mu(\cdot)$ is the Lebesgue measure. Let $\overline{\mathcal{S}}$ be closure of $\mathcal{S}$ and assume that for every $\mathbf{x} \in \overline{\mathcal{S}}$ there exists a ball $\mathcal{B}$ centered at $\mathbf{x}$, such that $\mu(\mathcal{B} \cap \mathcal{S}) > 0$. An infinite number of targets/demands are stored in a depot located, without any loss of generality, at the origin $\mathbf{0}$. These targets are picked up from the depot, carried and dropped in $\mathcal{S}$ by an adversarial agent traveling in straight lines at unit speed. The placement locations are sampled independently from a spatial density $\varphi : \mathcal{S} \to \mathbb{R}_+$. We assume that the agent has unitary target carrying capacity, i.e., he returns to the depot in between placing successive targets, and when he returns to the depot he spends an average time $\tau > 0$. The rate at which demands are placed in $\mathcal{S}$ by the agent adopting spatial distribution φ is thus given by

$$\lambda_\varphi = \frac{1}{2\mathbb{E}_\varphi[\|\mathbf{X}\|] + \tau}, \quad (12)$$

where $\mathbf{X}$ is the location of an arbitrary demand. From (12) we observe that the demand generation stochastic process is completely specified by the spatial density φ. In particular, in order to sustain a higher rate, the density has to be more concentrated around $\mathbf{0}$; alternately, as demands are distributed further away from the depot, the smaller is the arrival rate λ_φ. This dependence between the temporal rate and spatial density of demands is a novel feature in our formulation as compared to conventional setup for dynamical vehicle routing problems, where the temporal and spatial components of the demand point process are typically assumed to be independent.

Demands have to be serviced by a vehicle[1], traveling at speed v. In order to service a target, the vehicle has to physically travel to the location of the target, and we assume without loss of generality, that the on-site service time is zero. A routing policy is said to be stable if the expected number of outstanding demands is bounded almost surely at all times. In this paper, we are further interested in *spatially unbiased* policies. A policy is said to be spatially unbiased if for every pair of sets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$,

$$\mathbb{E}[T|\mathbf{X} \subseteq \mathcal{S}_1] = \mathbb{E}[T|\mathbf{X} \subseteq \mathcal{S}_2],$$

where T represents its waiting time. The results for spatially biased policies follow along similar lines.

Let Π denote the class of spatially unbiased stable policies and let $\mathcal{F} = \left\{\varphi : \mathcal{S} \to \mathbb{R}_+ \text{ s.t. } \int_{\mathcal{S}} \varphi(\mathbf{x})d\mathbf{x} = 1\right\}$ be the set of spatial probability distributions with support $\mathcal{S}$. Let $T_i(\pi, \varphi)$ represent the time elapsed from the moment the agent places i-th demand at its location until the vehicle reaches its location, while the agent is placing targets according to distribution $\varphi \in \mathcal{F}$ and the vehicle is implementing routing policy $\pi \in \Pi$. Define the *system time* $\overline{T} : \Pi \times \mathcal{F}$ by

$$\overline{T}(\pi, \varphi) := \limsup_{i\to\infty} \mathbb{E}_\varphi[T_i(\pi, \varphi)]. \quad (13)$$

Expressions for the average system time $\overline{T}$ in dynamic vehicle routing problems are available only in the extremes of system load λ/v, light or heavy. Here we focus on the heavy load regime, $\lambda/v \to \infty$. However, in this setup the

[1] We assume a single vehicle for the sake of simplicity. The analysis presented here does not change qualitatively when there are multiple vehicles.

target generation rate, as given by (12), is intertwined with the spatial density φ; therefore, we perform our analysis for the limiting case $v \to 0^+$ so that $\lambda_\varphi / v \to +\infty$ for any $\varphi \in \mathcal{F}$.

In the context described above, we consider a two-player complete information zero-sum game between the system planner seeking to design routing policy for the vehicle and the adversarial agent placing demands, with the system time defined in (13) as the utility function. In other words, in this strictly competitive setting, the agent will seek to maximize the system time, while the goal of the system planner is exactly the opposite. A solution, or equilibrium, of the game will be a pair $(\pi^*, \varphi^*) \in \Pi \times \mathcal{F}$ for which

$$\sup_{\varphi \in \mathcal{F}} \inf_{\pi \in \Pi} \overline{T}(\pi, \varphi) = \overline{T}(\pi^*, \varphi^*) = \inf_{\pi \in \Pi} \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi, \varphi). \quad (14)$$

A pair (π^*, φ^*) satisfying condition (14) is called a *saddle point* for the function $\overline{T}$. In this paper, we are interested in characterizing one such saddle point.

IV. An Optimal Routing Policy

Consider the following policy proposed in [4], which we will refer to as π^*:

Unbiased TSP-based Routing Policy: Let r be a large enough positive integer. From a central point in $\mathcal{S}$ partition $\mathcal{S}$ into r sets $\mathcal{S}_1, \ldots, \mathcal{S}_r$, such that $\int_{\mathcal{S}_k} \varphi(\mathbf{x}) d\mathbf{x} = 1/r$. Within each set of the partition, form sets of demands with size n/r, and as these sets are constructed, deposit them in a queue and service them in a "first come, first served" fashion. The service of each set is achieved by constructing a TSP tour and following it in an arbitrary direction. Finally, optimize over n.

It was shown in [4] that in the limit as $\lambda / v \to +\infty$,

$$\overline{T}(\pi^*, \varphi) \geq \frac{\beta^2}{2v^2} \lambda_\varphi \left(\int_{\mathcal{S}} \sqrt{\varphi(\mathbf{x})} d\mathbf{x} \right)^2 \quad (15)$$

for any $\varphi \in \mathcal{F}$, where $\beta \simeq 0.7120$ is a constant that appears in the asymptotic result for the length of the shortest path in the Traveling Salesman Problem (TSP) over the Euclidean plane [1]. Moreover, it was proved in [18] that $\inf_{\pi \in \Pi} \overline{T}(\pi, \varphi) = \overline{T}(\pi^*, \varphi)$ for all $\varphi \in \mathcal{F}$. Hence,

$$\sup_{\varphi \in \mathcal{F}} \overline{T}(\pi^*, \varphi) = \sup_{\varphi \in \mathcal{F}} \inf_{\pi \in \Pi} \overline{T}(\pi, \varphi) \leq \inf_{\pi \in \Pi} \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi, \varphi),$$

where the last inequality follows from the min-max inequality, which holds true on any product space (see e.g. [2]). Since $\inf_{\pi \in \Pi} \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi, \varphi) \leq \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi^*, \varphi)$ (by definition of infimum), we arrive at

$$\sup_{\varphi \in \mathcal{F}} \inf_{\pi \in \Pi} \overline{T}(\pi, \varphi) = \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi^*, \varphi) = \inf_{\pi \in \Pi} \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi, \varphi).$$

Therefore, if there exists a $\varphi^* \in \mathcal{F}$ such that $\overline{T}(\pi^*, \varphi^*) = \sup_{\varphi \in \mathcal{F}} \overline{T}(\pi^*, \varphi)$, then it would constitute, together with π^*, a saddle point of $\overline{T}$ as $v \to 0^+$. The next section is devoted to finding such a φ^*.

Remark: Note that the complete information assumption implies that the system planner has perfect knowledge of the spatial density φ, needed to implement π^*. This could, for example, be achieved if the system planner has an estimator to learn φ. We leave the investigation of such learning/adaptive routing policies for future work.

V. The Saddle Point Spatial Density

The optimal spatial density that will maximize the system time as $v \to 0$ when the routing policy is π^* will emerge as the solution to the following optimization problem:

$$\sup_{\varphi} \; \lambda_\varphi \left(\int_{\mathcal{S}} \sqrt{\varphi(\mathbf{x})} d\mathbf{x} \right)^2 \quad \text{s.t.} \quad \varphi \in \mathcal{F}. \quad (16)$$

In its original form, problem (16) is the product between a convex and a concave function. Hence, it is not convex thus hard to tackle. However, applying a logarithmic transformation to the objective function and introducing a new variable $\gamma := \log \lambda_\varphi$ yields the following equivalent formulation

$$\sup_{\gamma, \varphi} \; \gamma + 2 \log \int_{\mathcal{S}} \sqrt{\varphi(\mathbf{x})} d\mathbf{x} \quad \text{s.t.} \quad \gamma \in \Gamma, \, \varphi \in \mathcal{F}. \quad (17)$$

Since $0 \leq \mathbb{E}_\varphi[\|\mathbf{X}\|] \leq \max_{\mathbf{x} \in \overline{\mathcal{S}}} \|\mathbf{x}\|$, from (12) it can be easily seen that

$$\Gamma = - \left[\log \left(2 \max_{\mathbf{x} \in \overline{\mathcal{S}}} \|\mathbf{x}\| + \tau \right), \log \tau \right] \subset \mathbb{R}. \quad (18)$$

Expressing the dependence of φ on the real variable γ allows us to rewrite (17) as

$$\sup_{\gamma \in \Gamma} \left\{ \gamma + 2 \sup_{\varphi \in \mathcal{F}_\gamma} \log \int_{\mathcal{S}} \sqrt{\varphi(\mathbf{x})} d\mathbf{x} \right\}, \quad (19)$$

where

$$\mathcal{F}_\gamma = \left\{ \varphi : \varphi \in \mathcal{F}, \, \mathbb{E}_\varphi[\|\mathbf{X}\|] = \frac{e^{-\gamma} - \tau}{2} \right\}. \quad (20)$$

Problem (19) decouples the spatio-temporal dependence of the stochastic process of demand locations, splitting (16) into two connected sub-problems: one for the spatial component, and another one for the temporal component. The former entails a maximization over an infinite-dimensional space to determine an optimal parametric family of spatial probability densities, parametrized by γ; the latter is a scalar maximization which yields the optimal rate, therefore completely identifying the optimal density from the previously found parametric family, rendering the solution to (16) and the saddle point density for the game.

A. The Optimal Parametric Family

Given $\gamma \in \Gamma$, we wish to solve

$$\sup \left\{ \log \int_{\mathcal{S}} \sqrt{\varphi(\mathbf{x})} d\mathbf{x} : \varphi \in \mathcal{F}_\gamma \right\},$$

or equivalently,

$$\inf \; \mathcal{I}(\varphi) := \int_{\mathcal{S}} -\sqrt{\varphi(\mathbf{x})} d\mathbf{x} \quad \text{s.t.} \quad \varphi \in \mathcal{F}_\gamma. \quad (21)$$

First let us note that, as stated by the following lemma, problem (21) is feasible for every $\gamma \in \Gamma$ and has a value of zero when γ lays at the boundary of Γ in (18).

Lemma 2: For every $\gamma \in \text{int } \Gamma$, there exists a density $\varphi \in \mathcal{F}_\gamma$. Moreover, when γ belongs to the boundary of Γ the value of (21) is zero.

Proof: Let $\mathbf{y} \in \mathcal{A} = \{\mathbf{x} \in \overline{\mathcal{S}} : \mathbf{x} \in \arg\max_{\mathbf{x}} \|\mathbf{x}\|\}$, and note that $\mu(\mathcal{A}) = 0$. Because of the smoothness assumption imposed on the set $\mathcal{S}$, there exist balls $\mathcal{B}$ and $\mathcal{B}'$ centered at $\mathbf{0}$ and $\mathbf{y}$ with radius r and r' respectively, such that $\mu(\mathcal{B} \cap \mathcal{S}) > 0$ and $\mu(\mathcal{B}' \cap \mathcal{S}) > 0$. Clearly, we can reduce the radii and still maintain the same property. Let φ_r and $\varphi_{r'}$ denote the densities associated with uniform distributions defined over $\mathcal{B}$ and $\mathcal{B}'$, respectively. Then, $\lim_{r \to 0} \mathbb{E}_{\varphi_r}[\|\mathbf{X}\|] = 0$ and $\lim_{r' \to 0} \mathbb{E}_{\varphi_{r'}}[\|\mathbf{X}\|] = \max_{\mathbf{x} \in \overline{\mathcal{S}}} \|\mathbf{x}\|$. Furthermore, these limiting values will only be achieved by singular distributions with supports over $\{\mathbf{0}\}$ and $\mathcal{A}$, respectively; therefore, the integral in (21) is zero for γ on the boundary of Γ defined in (18). Now consider a density φ defined as a linear combination of φ_r and $\varphi_{r'}$, with support over $\mathcal{B} \cup \mathcal{B}'$. Then, by the linearity of the expectation we get for every $\alpha \in [0, 1]$,

$$\mathbb{E}_\varphi[\|\mathbf{X}\|] = \alpha \mathbb{E}_{\varphi_r}[\|\mathbf{X}\|] + (1 - \alpha)\mathbb{E}_{\varphi_{r'}}[\|\mathbf{X}\|].$$

Thus, given $\gamma \in \Gamma$ with appropriate choices of r, r' and α we can always construct a density $\varphi \in \mathcal{F}_\gamma$. ∎

Define the continuous linear map $\mathbf{A} : \mathcal{L}_1(\mathcal{S}) \to \mathbb{R}^2$ with linearly independent components given by,

$$\mathbf{A}\varphi = \begin{pmatrix} \int_{\mathcal{S}} \varphi(\mathbf{x}) d\mathbf{x} \\ \int_{\mathcal{S}} \|\mathbf{x}\| \varphi(\mathbf{x}) d\mathbf{x} \end{pmatrix}, \tag{22}$$

and let $\mathbf{b}_\gamma \in \mathbb{R}^2$ be the column vector $\left(1, \frac{e^{-\gamma} - \tau}{2}\right)$. Then, expressing the constraints that define $\mathcal{F}_\gamma$ in (20) in terms of $\mathbf{A}$ and $\mathbf{b}_\gamma$, we can rewrite problem (21) as

$$\inf\ \mathcal{I}(\varphi) \quad \text{s.t.} \quad \mathbf{A}\varphi = \mathbf{b}_\gamma,\ \varphi \geq 0,\ \varphi \in \mathcal{L}_1(\mathcal{S}). \tag{23}$$

Since the objective function is convex in φ and the equality constraints defining $\mathcal{F}_\gamma$ are linear, problem (21) is convex thus amenable to solve through Lagrange duality. The Lagrangian for this problem is the function $L : \mathcal{L}_1(\mathcal{S}) \times \mathbb{R}^2 \to \mathbb{R}$ defined as $L(\varphi, \boldsymbol{\xi}) := \mathcal{I}(\varphi) + \langle \boldsymbol{\xi}, \mathbf{b}_\gamma - \mathbf{A}\varphi \rangle$. If some constraint qualifications are met, assuring that strong duality holds, then the solution to (21) can be obtained by solving

$$\sup_{\boldsymbol{\xi} \in \mathbb{R}^2} \inf_{\varphi \geq 0} L(\varphi, \boldsymbol{\xi}).$$

For the minimization over φ one might be tempted to differentiate the Lagrangian (in the Fréchet sense [12]); however, the Lagrangian is nowhere differentiable since the positive cone $\{\varphi \in \mathcal{L}_1(\mathcal{S}) : \varphi \geq 0\}$ has empty interior and its complement is dense in $\mathcal{L}_1(\mathcal{S})$. As a result, this approach cannot be rigorously justified (see [5] for further discussions).

To bypass this technical difficulty we will cast problem (21) under the conjugate duality framework presented in Section II-C. Defining the function

$$h(x) := -\sqrt{x} + \delta(x|\mathbb{R}_+), \quad \text{for all} \quad x \in \mathbb{R}, \tag{24}$$

we can further rewrite (23) as

$$\inf\ \mathcal{I}_+(\varphi) := \int_{\mathcal{S}} h(\varphi(\mathbf{x})) d\mathbf{x} \quad \text{s.t.} \quad \mathbf{A}\varphi = \mathbf{b}_\gamma,\ \varphi \in \mathcal{L}_1(\mathcal{S}). \tag{25}$$

Note that the integrand h is proper and convex. Moreover, we claim that it is also closed. Indeed, consider a sequence $\{x_k, w_k\} \subseteq \text{epi } h$ such that $(x_k, w_k) \to (x, w)$ as $k \to \infty$. We can assume that $\{x_k\} \subseteq \text{dom } h$ given that restricting a function to its effective domain does not change its epigraph. Then, since $\text{dom } h = [0, \infty)$ is a closed set, it follows that $x \in \text{dom } h$, thus $w \geq \lim_{k \to \infty} -\sqrt{x_k} = -\sqrt{x}$. This implies that $(x, w) \in \text{epi } h$, which shows the closedness of h. Consequently, formulation (25) exhibits the same structure as (4), and the results in Section II-C are applicable.

The next conjugate duality theorem will be of great significance in the subsequent analysis. It determines the dual of (25) and states that the duality gap is zero. Before we formally state and prove the theorem, we need the following key lemma.

Lemma 3: $\mathbf{b}_\gamma \in \text{ri }(\mathbf{A}\text{dom } \mathcal{I}_+)$, for every $\gamma \in \text{int } \Gamma$.

Proof: By definition, $\text{ri }(\mathbf{A}\text{dom } \mathcal{I}_+)$ is equal to

$$\text{ri }\{\mathbf{d} \in \mathbb{R}^2 : \exists\, \varphi \in \mathcal{L}_1(\mathcal{S}) \cap \text{dom } \mathcal{I}_+ \text{ s.t. } \mathbf{A}\varphi = \mathbf{d}\},$$

and because $\{\varphi \in \mathcal{L}_1(\mathcal{S}) : \varphi \geq 0\} \subset \text{dom } \mathcal{I}_+$, it follows from Lemma 1 that $\text{ri }(\mathbf{A}\text{dom } \mathcal{I}_+) = \{\mathbf{d} \in \mathbb{R}^2 : \mathbf{d} > 0\}$. The fact that $\mathbf{b}_\gamma > 0$ yields the claimed result. ∎

Theorem 4: Let Γ be defined as in (18), and let $\text{int } \Gamma$ denote its interior. Then, for every $\gamma \in \text{int } \Gamma$ the dual of problem (25) is given by

$$D(\gamma) := \sup_{\boldsymbol{\xi} \in \mathbb{R}^2} \left\{ \langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle + \int_{\mathcal{S}} \frac{d\mathbf{x}}{4\mathbf{A}^T \boldsymbol{\xi}(\mathbf{x})} : \mathbf{A}^T \boldsymbol{\xi} < 0 \right\}, \tag{26}$$

where $\mathbf{A}^T : \mathcal{S} \to \mathcal{L}_\infty(\mathcal{S})$ is the adjoint map, given by $\mathbf{A}^T \boldsymbol{\xi}(\mathbf{x}) = \xi_1 + \xi_2 \|\mathbf{x}\|$. Furthermore, (26) admits a unique solution $\boldsymbol{\xi}^*(\gamma)$, and the optimal value achieved is finite and equal to the infimum in (25).

Proof: The dual of problem (25) is given by,

$$\sup_{\boldsymbol{\xi} \in \mathbb{R}^2} \left\{ \langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle - \mathcal{I}_+^\star(\mathbf{A}^T \boldsymbol{\xi}) \right\},$$

and since the conjugate of the integrand function h defined in (24) is,

$$h^\star(y) = \sup_{x \geq 0} \{xy + \sqrt{x}\} = \begin{cases} -\frac{1}{4y} & y < 0, \\ \infty & \text{otherwise,} \end{cases} \tag{27}$$

by Proposition 1 it must be equal to

$$\sup_{\boldsymbol{\xi} \in \mathbb{R}^2} \left\{ \langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle + \int_{\mathcal{S}} \frac{d\mathbf{x}}{4\mathbf{A}^T \boldsymbol{\xi}(\mathbf{x})} : \mathbf{A}^T \boldsymbol{\xi} < 0 \right\}.$$

From Lemma 3 it follows that for every $\gamma \in \text{int } \Gamma$ the constraint qualification (8) is satisfied, and Corollary 1 implies that (26) is equal to (25) (thus equal to (21)).

Over the set

$$\mathcal{M} = \{\boldsymbol{\xi} \in \mathbb{R}^2 : \mathbf{A}^T \boldsymbol{\xi}(\mathbf{x}) < 0, \text{ for all } \mathbf{x} \in \mathcal{S}\}, \tag{28}$$

that describes the maximization space in the dual problem, we must have $\langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle < 0$. To see this, consider an arbitrary $\gamma \in \text{int } \Gamma$ and let φ be a density with support over $\mathcal{S}$ such that $\mathbf{A}\varphi = \mathbf{b}_\gamma$, whose existence is guaranteed by Lemma

2. Then, for every $\mathbf{x} \in \mathcal{S}$ we have $\varphi(\mathbf{x})(\xi_1 + \xi_2\|\mathbf{x}\|) < 0$; hence,

$$\langle \boldsymbol{\xi}, \mathbf{b}_\gamma \rangle = \xi_1 \int_{\mathcal{S}} \varphi(\mathbf{x}) d\mathbf{x} + \xi_2 \int_{\mathcal{S}} \|\mathbf{x}\| \varphi(\mathbf{x}) d\mathbf{x} < 0.$$

Therefore, the dual optimal value is bounded above by zero; thus it must be achieved at some $\boldsymbol{\xi}^*(\gamma)$. Finally, the fact that $h^\star$ is essentially strictly convex (it is strictly convex over its effective domain) and the set of functions $\{1, \|\mathbf{x}\|\}$ that define $\mathbf{A}$ is linearly independent, implies through Proposition 2, that $\boldsymbol{\xi}^*(\gamma)$ is unique for every $\gamma \in \operatorname{int} \Gamma$. ■

Corollary 2: $\xi_1^*(\gamma) < 0$, *for every* $\gamma \in \operatorname{int} \Gamma$.

Proof: Given that $\boldsymbol{\xi}^* \in \mathcal{M}$, we have $\mathbf{A}^T\boldsymbol{\xi}^*(\gamma)(\mathbf{x}) = \xi_1^*(\gamma) + \xi_2^*(\gamma)\|\mathbf{x}\| < 0$ for all $\mathbf{x} \in \mathcal{S}$. Hence, letting $\mathbf{x} = \mathbf{0}$ renders the result. ■

Based on the preceding theorem, the following proposition characterizes the unique optimal parametric family of spatial densities.

Proposition 3: Consider the optimization problem defined by (21) and (20). Then, for every $\gamma \in \operatorname{int} \Gamma$ *the unique optimal solution is given by,*

$$\varphi_\gamma^*(\mathbf{x}) = \frac{1}{4(\xi_1^*(\gamma) + \xi_2^*(\gamma)\|\mathbf{x}\|)^2}, \quad \textit{for all} \quad \mathbf{x} \in \mathcal{S}. \quad (29)$$

Proof: The function h defined in (24) is both essentially strictly convex and essentially smooth. Indeed, it is strictly convex and differentiable when restricted to its effective domain $[0, \infty)$, and $|h'(x)| = x^{-3/2}$ which tends to ∞ as $x \to 0$. Moreover, h satisfies the growth condition (10) since $\Delta = 0 > \operatorname{ess\,sup}_{\mathbf{x} \in \mathcal{S}} \mathbf{A}^T\boldsymbol{\xi}^*(\mathbf{x})$. Then, invoking Theorem 3 we conclude that for every $\gamma \in \operatorname{int} \Gamma$, the optimal solution to (21) is given by $\varphi_\gamma^*(\mathbf{x}) = (h^\star)'(\mathbf{A}^T\boldsymbol{\xi}^*(\gamma)(\mathbf{x}))$ for all $\mathbf{x} \in \mathcal{S}$, where $\boldsymbol{\xi}^*(\gamma)$ is the unique dual solution determined by Theorem 4. Finally, from (27) we have that $(h^\star)'(x) = 1/4x^2$ for all $x < 0$, and we thus arrive at (29). ■

Remarks:

- Through the use of conjugate duality, Theorem 4, we have transformed the infinite-dimensional optimization problem (21) into a maximization of a strictly concave function over a convex set in $\mathbb{R}^2$, and although the unique solution to (26) cannot be expressed in closed form it can be efficiently found numerically.
- The solution $\varphi_\gamma^*(\mathbf{x})$ obtained in Proposition 3 belongs to $\mathcal{C}(\mathcal{S})$, the set of continuous functions with support over $\mathcal{S}$ which is dense in $\mathcal{L}_1(\mathcal{S})$ and has a positive cone with non-empty interior. We could have chosen $\mathcal{C}(\mathcal{S})$ as the underlying working space and solve (21) through differentiation of the Lagrangian; however, the uniqueness result obtained for $\mathcal{L}_1(\mathcal{S})$ is much stronger.

B. The Optimal Parameter

We now study the optimization over γ in (19), and show that there exists a unique solution γ^*. Since for every $\gamma \in \operatorname{int} \Gamma$ the dual optimum $\boldsymbol{\xi}^*(\gamma)$ is unique, γ^* will determine the unique spatial density $\varphi^* := \varphi_{\gamma^*}^*$ from the family described in (29) that attains the maximum in (16). We start by providing some results concerning the behavior of $\boldsymbol{\xi}^*$ as a function of $\gamma \in \operatorname{int} \Gamma$, that will play a key role in establishing the existence and uniqueness of the solution to (19). Specifically,

Proposition 4: Consider the dual problem defined in (26). Then, the function $\boldsymbol{\xi}^* : \operatorname{int} \Gamma \to \mathcal{M}$ *is differentiable, and* $(\xi_2^*)'(\gamma) < 0$. *Also,* $D'(\gamma) = \langle \boldsymbol{\xi}^*(\gamma), \mathbf{b}'_\gamma \rangle$ *for all* $\gamma \in \operatorname{int} \Gamma$.

Proof: The set $\mathcal{M}$ defined in (28) is open, therefore the following first order condition must be satisfied at $\boldsymbol{\xi}^*$:

$$G(\gamma, \boldsymbol{\xi}^*) = \mathbf{b}_\gamma - \frac{\partial \mathcal{I}_+^\star}{\partial \boldsymbol{\xi}}(\mathbf{A}^T\boldsymbol{\xi}^*) = 0. \quad (30)$$

This equation implicitly defines $\boldsymbol{\xi}^*(\gamma)$ with a Jacobian

$$\frac{\partial G}{\partial \boldsymbol{\xi}} = -\frac{\partial^2 \mathcal{I}_+^\star}{\partial \boldsymbol{\xi}^2},$$

which is negative definite for every $\boldsymbol{\xi} \in \mathcal{M}$ because of the strict convexity of $\mathcal{I}_+^\star$ (strictly convex function composed with a linear function). Thus, it is nonsingular and the implicit theorem function furnishes the differentiability of $\boldsymbol{\xi}^*(\gamma)$. Moreover,

$$(\boldsymbol{\xi}^*)' = \left(\frac{\partial^2 \mathcal{I}_+^\star}{\partial \boldsymbol{\xi}^2}\right)^{-1} \frac{\partial G}{\partial \gamma}. \quad (31)$$

The inverse of the Hessian of $\mathcal{I}_+^\star$ is positive definite, and $\frac{\partial G}{\partial \gamma}$ is the column vector with entries $(0, -\frac{1}{2}e^{-\gamma})$. Hence, left-multiplying (31) by the transpose of $\frac{\partial G}{\partial \gamma}$ yields

$$0 < \left(\frac{\partial G}{\partial \gamma}\right)^T (\boldsymbol{\xi}^*)' = -\frac{1}{2}e^{-\gamma}(\xi_2^*)',$$

and so $(\xi_2^*)' < 0$. Finally, since for every $\gamma \in \operatorname{int} \Gamma$ we have,

$$D(\gamma) = \langle \boldsymbol{\xi}^*(\gamma), \mathbf{b}_\gamma \rangle - \mathcal{I}_+^\star(\mathbf{A}^T\boldsymbol{\xi}^*(\gamma)),$$

it follows that D is differentiable and

$$D'(\gamma) = \langle \boldsymbol{\xi}^*(\gamma), \mathbf{b}'_\gamma \rangle + \left\langle (\boldsymbol{\xi}^*)'(\gamma), \mathbf{b}_\gamma - \frac{\partial \mathcal{I}_+^\star}{\partial \boldsymbol{\xi}}(\mathbf{A}^T\boldsymbol{\xi}^*(\gamma)) \right\rangle;$$

the second term vanishes due to (30). ■

The next theorem in conjunction with Proposition 3 completely characterizes the unique optimal spatial density, solution to problem (16).

Theorem 5: The optimization problem defined by (19) and (18) admits a unique optimal solution $\gamma^* \in \operatorname{int} \Gamma$.

Proof: For all $\gamma \in \Gamma$, define

$$F(\gamma) := \sup_{\varphi \in \mathcal{F}_\gamma} \int_{\mathcal{S}} \sqrt{\varphi(\mathbf{x})} d\mathbf{x},$$

and let $\Psi(\gamma) = \gamma + 2\log F(\gamma)$ denote the objective function in (19). From Theorem 4 we know that $F(\gamma) = -D(\gamma)$ over $\operatorname{int} \Gamma$, and Lemma 2 implies that $\Psi(\gamma) = -\infty$ when γ is at the boundary of the interval Γ. Thus, $\operatorname{dom} \Psi = \operatorname{int} \Gamma$, and Ψ is proper. The function Ψ is also closed; indeed consider any sequence $\{\gamma_k, w_k\} \subset \operatorname{epi} \Psi$ such that $(\gamma_k, w_k) \to (\gamma, w)$. Recall that restricting a function to its effective domain does not affect the epigraph; hence, we can assume that $\{\gamma_k\} \subset \operatorname{dom} \Psi$. Then, by Proposition 4 we know Ψ is continuous over its effective domain, and $\Psi(\gamma) = \lim_{k \to \infty} \Psi(\gamma_k) \geq \lim_{k \to \infty} w_k = w$, which shows that $(\gamma, w) \in \operatorname{epi} \Psi$. Now, since Ψ tends to $-\infty$ at the boundary of its effective domain, we can find a scalar η such that the upper level set $\{\gamma \in \mathrm{dom}\ \Psi : \Psi(\gamma) \geq \eta\}$ is nonempty and bounded. Therefore we can invoke Theorem 1 to conclude that the set of maxima Γ^* is nonempty and compact; moreover, $\Gamma^* \subseteq \mathrm{int}\ \Gamma$.

For every $\gamma^* \in \Gamma^*$ note that since γ^* is an interior point of Γ, the following first order condition must be satisfied:

$$\Psi'(\gamma^*) = 1 + 2\frac{F'(\gamma^*)}{F(\gamma^*)} = 0. \tag{32}$$

Combining Theorem 4 with Proposition 4, we get

$$F'(\gamma^*) = -\left\langle \boldsymbol{\xi}^*(\gamma^*), \mathbf{b}'_{\gamma^*} \right\rangle = \frac{1}{2}\xi_2^*(\gamma^*)e^{-\gamma^*}.$$

Also, Proposition 3 leads to

$$D(\gamma) = \int_{\mathcal{S}} \frac{d\mathbf{x}}{2\mathbf{A}^T\boldsymbol{\xi}^*(\gamma)(\mathbf{x})}, \quad \text{for all} \quad \gamma \in \mathrm{int}\ \Gamma,$$

which implies that $D(\gamma) = 2\left\langle \boldsymbol{\xi}^*(\gamma), \mathbf{b}_\gamma \right\rangle$. Thus, returning to (32) and after some simple algebra we conclude that every $\gamma^* \in \Gamma^*$ must satisfy

$$2\xi_1^*(\gamma^*) = \tau\xi_2^*(\gamma^*), \tag{33}$$

where $\tau > 0$. Let $\tilde{\Gamma} = \{\gamma \in \mathrm{int}\ \Gamma : \xi_2^*(\gamma) < 0\}$, and note from Corollary 2 that $\Gamma^* \subseteq \tilde{\Gamma}$. From Proposition 4 it follows that ξ_2^* is continuous, thus $\tilde{\Gamma}$ is an open set. Inside this set, $(\xi_2^*)'(\gamma) < 0$ and by Proposition 3 it is clear that ξ_1^* should be increasing so that the density defined in (29) integrates to unity over $\mathcal{S}$. Hence, returning to (33) we conclude that the maximizer γ^* has to be unique. ■

Corollary 3: The solution to (16) can be written as

$$\varphi^*(\mathbf{x}) = \frac{K}{(\tau + 2\|\mathbf{x}\|)^2}, \quad \textit{for all} \quad \mathbf{x} \in \mathcal{S}, \tag{34}$$

where $K > 0$ is a normalization constant.

Proof: Letting $K = (\tau/2\xi_1^*(\gamma^*))^2$, the result readily follows by plugging (33) back in (29). ■

Remark: If a demand is placed at location $\mathbf{x}$, then from (12) we note that $\tau + 2\|\mathbf{x}\|$ is the average time the agent has to wait before he can place another demand in $\mathcal{S}$. This is the source of the spatio-temporal dependence between the location and the rate of demands, and not surprisingly, it is reflected on the shape of the optimal spatial density φ^*.

VI. Simulations

In this section we provide simulations that shed light on the theoretical results developed in the previous sections. Let $\mathcal{S}_\rho = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq \rho\}$ be the support of densities, and $\tau = 0.1$. If the physical constraint imposed by the agent carrying and placing the targets on $\mathcal{S}$ were removed and the rate were fixed, then the distribution that attains the maximum system time as $\lambda/v \to \infty$ is uniform; this was proved in [4] using a Hardy-Littlewood-Pólya inequality. However, when the spatio-temporal dependence is introduced, a uniform distribution will induce a rate that is smaller than λ_{φ^*}. This is because φ^* is more concentrated around the depot location than a uniform spatial density and hence the agent has to travel less distance on average between placement of successive targets. Figure 1 shows the plot of the ratio of the system time $\overline{T}^* := \overline{T}(\pi^*, \varphi^*)$ and $\overline{T}_U := \overline{T}(\pi^*, \varphi_{\mathrm{uniform}})$ with respect to increasing size of the region $\mathcal{S}$. As it can be seen, φ^* yields 20% higher system time than a uniform distribution for ρ as low as 1.

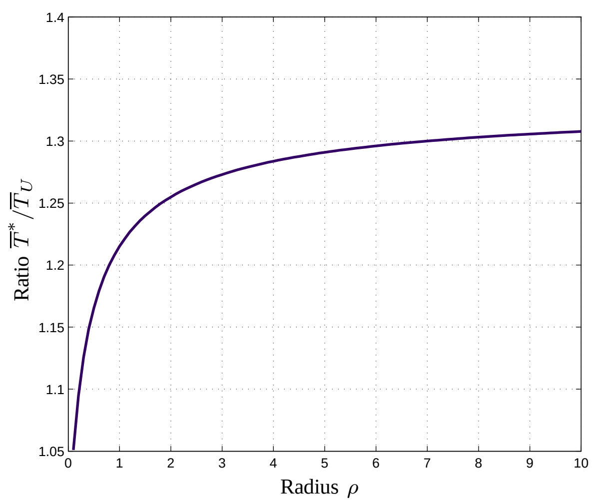

Fig. 1. System time $\overline{T}_U$ with uniform distribution and $\overline{T}^*$.

Recall that the pair (π^*, φ^*) constitutes an equilibrium for the game in the limit as $v \to 0^+$. Therefore, understanding how the relative error between $\overline{T}^*$ and the measured optimal system time $\overline{T}^*_m$ decreases as v becomes closer to zero is an issue of practical significance. To that end, we implemented in Matlab the TSP-based routing policy described in Section IV based on the Lin & Kernighan's algorithm [11]. The results obtained are gathered in Figure 2, where we note that for $v = 0.01$ the relative error $|\overline{T}^* - \overline{T}^*_m|/\overline{T}^*$ is already less than 5%. This observation is actually not surprising, since as implied in [9], the expression for the system time (15) in heavy load is usually a fairly good approximation for the system time under "intermediate" load regimes.

VII. Conclusions

We studied a strategic dynamic vehicle routing problem where demands are place in a bounded region $\mathcal{S}$ by an agent with unitary capacity operating from a depot. We formulated the corresponding complete information zero-sum game, with the average waiting time of a typical demand as the utility function, and showed that an equilibrium in the limiting regime when the vehicle travels arbitrarily slower than the adversarial agent is given by the pair of a TSP-based routing policy and a unique power-law spatial density centered at the depot location. While the TSP based routing policy and its performance analysis has been adopted from [4], [18], the results on the optimal spatial density were rigorously derived using tools from conjugate duality and results concerning the maximization of concave integral functionals subject to linear equality constraints. Remarkably, all the results obtained hold for any bounded region $\mathcal{S}$ with a sufficiently

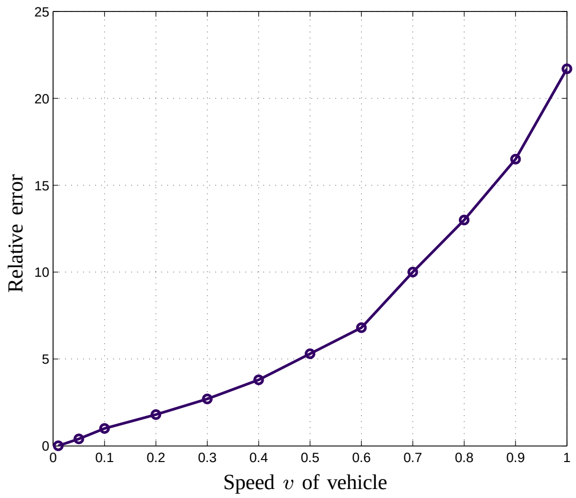

Fig. 2. Relative error of optimal system time $\overline{T}^*$ as a function of the speed v.

smooth boundary, and in particular also for regions with holes. This is an important feature since it allows to introduce support constraints for the spatial distribution adopted by the adversary, which serves as a good abstraction for scenarios involving predator-prey interactions and criminal pursuit. Also note that since lower bounds for the average system time for dynamic vehicle routing under heavy load often take the form of concave integral functionals (see e.g. [8]), the convex analytic approach applied in this paper could be used to formally analyze the performance of policies under worst case scenarios.

Regarding avenues for future research, it would be interesting to relax the complete information assumption. In particular, we are interested in incorporating estimation of φ^* into the strategy set of the system planner. Accordingly, it would be interesting to incorporate estimation cost into the utility function of the game and investigate its effects on the optimal strategies. Such a setup could also provide a natural framework for the formal study of geographic profiling [17], [13], where the objective is to determine the most probable area of a criminal (predator) hideout ("anchor point") based on observed attack locations. It would also be interesting to study strategic dynamic vehicle routing problems involving multiple coordinated adversaries.

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