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Asymptotic equivalence of differential equations with piecewise constant argument

Na Song[1] and Hong-Xu Li[1,*]

[1] *Department of Mathematics, Sichuan University, Chengdu, Sichuan 610 064, P. R. China*

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Abstract. By using the method of investigation of differential equations with piecewise constant argument and some integral inequalities of Gronwall type we obtain some results of asymptotic equivalence of the stable solutions of some differential equations with piecewise constant argument. Our results generalize and improve some recent results.

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1. Introduction

Differential equations with piecewise constant argument (EPCA) describe hybrid dynamical systems. These equations combine the properties of both differential and difference equations. A systematic study of theoretical and practical problems of scalar EPCA of type

$$u' = g(t, u(t), u(\gamma(t))) \tag{1}$$

with $\gamma(t)$ some step function was initiated by Cooke, Wiener and Shah in the early 80's of the last century [10,20,22]. Since then, EPCAs have attracted great attention from the researchers in mathematics, biology, engineering and other fields (see e.g. [3,6,7,9,12,17,23] and references therein).

In this paper, we study the asymptotic equivalence of stable solutions of the following EPCAs:

$$x'(t) = A(t)x(t) + B(t)x(\gamma(t)), \tag{2}$$

$$y'(t) = A(t)y(t) + f(t, y(t), y(\gamma(t))), \tag{3}$$

where $x(t), y(t) \in \mathbb{C}^n$, $A(t), B(t)$ are $n \times n$ complex continuous matrixes on $I \triangleq [0, \infty)$, $\gamma(t)$ is a step function of general type which was introduced by Akhmet [2], and $f : I \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is a continuous function such that $f(\cdot, 0, 0) \in L_1(I)$ and for $t \in I, x_1, x_2, y_1, y_2 \in \mathbb{C}^n$,

$$\|f(t, x_1, y_2) - f(t, x_2, y_2)\| \le \lambda_1(t)\|x_1 - x_2\| + \lambda_2(t)\|y_1 - y_2\|, \tag{4}$$

*Corresponding author. *Email addresses:* `songna021@163.com` (N. Song), `hoxuli@sohu.com` (H. X. Li)

where $\lambda_1, \lambda_2 \in L_1(I)$ are positive on I.

The study of asymptotic behavior of equations is an important topic in the theory of differential equations, which could date back to 1948 Levinson [15]. Especially, to the asymptotic equivalence of the stable solutions of (2) and (3), some results are presented in [18] for the case when $B(t) = 0$ and $\lambda_1(t) = \lambda_2(t)$. If, in addition, $A(t)$ and $\lambda_1(t) = \lambda_2(t)$ are constant, the same problem was studied in [1]. So our results can be regarded as an extension of the results in [1,18]. For more results concerning the asymptotic behavior of equations, we refer the readers to [5,8,11,13,14,16,19,21].

The main purpose of this paper is to obtain a theorem of asymptotic equivalence of the stable solutions of equations (2) and (3) by using some integral inequalities of Gronwall type and the method of investigation of EPCA introduced by Akhmet [2–4]. Our results generalize and improve some results (see Remark 1).

The paper is organized as follows. In Section 2, we give some integral inequalities of Gronwall type and some preliminary results including a theorem for the asymptotic behavior of the general equation of the form (1) (see Theorem 1). The main theorem of the asymptotic equivalence of equations (2) and (3) is presented in Section 3.

2. Preliminary results

Throughout this paper, $\mathbb{N}$ and $\mathbb{C}$ denote the sets of all natural and complex numbers, respectively. Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{C}^n$. Fix two real valued non-negative sequences $\{t_i\}, \{\xi_i\}, i \in \mathbb{N}$ such that $t_i < t_{i+1}, t_i \le \xi_i \le t_{i+1}$ and $t_i \to \infty$ as $i \to \infty$. Denote $I_{-1} = [0, t_0), I_i = [t_i, t_{i+1}), i \in \mathbb{N}$ and $I = \bigcup_{i=-1}^{\infty} I_i$. The step function γ in (2) and (3) is defined by $\gamma : I \to I$, $\gamma(t) = t, t \in I_{-1}$ and $\gamma(t) = \xi_i, t \in I_i, i \in \mathbb{N}$.

To study (2) and (3), we first study the following general equation:

$$u' = g(t, u(t), u(\gamma(t))), \tag{5}$$

where $g : I \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is a continuous function satisfying $g(t, 0, 0) \in L_1(I)$ and for $t \in I$, $x_1, x_2, y_1, y_2 \in \mathbb{C}^n$,

$$\|g(t, x_1, y_1) - g(t, x_2.y_2)\| \le \eta_1(t)\|x_1 - x_2\| + \eta_2(t)\|y_1 - y_2\| \tag{6}$$

with positive functions $\eta_1, \eta_2 \in L_1(I)$ such that

$$\int_{t_i}^{\xi_i} (\eta_1(s) + \eta_2(s))ds \le v < 1 \quad \text{for some } v > 0, i \in \mathbb{N}. \tag{7}$$

In the sequel, we denote

$$\eta(t) = \eta_1(t) + \frac{\eta_2(t)}{1 - v}, \quad t \in I. \tag{8}$$

The integrable condition $f(t, 0, 0) \in L_1(I)$ for (3) can be reduced to $f(t, 0, 0) \equiv 0$. In fact, as in [18], we may add $f(t, 0, 0)$ to the right-hand side of (2) and translate

f in (3) to $f - f(t, 0, 0)$. Then the same discussions can be done. Similar reduction can be taken for g in (5). So in the sequel we always assume that

$$f(t, 0, 0) = 0, \quad g(t, 0, 0) = 0.$$

Definition 1. *A solution of (5) is a continuous function u on I such that*

(i) The derivative $u'(t)$ exists at each point $t \in I$ with the possible exception of the points $t_i, i \in \mathbb{N}$, where the one-side derivatives exist;

(ii) (5) is satisfied for u on each interval $(t_i, t_{i+1}), i \in \mathbb{N}$, and it holds for the right derivative at the points $t_i, i \in \mathbb{N}$.

The solution of (2) and (3) can be defined similarly.

Definition 2. *(2) and (3) are called equivalent if there exists a homeomorphism between the sets of solutions $x(t)$ and $y(t)$; and (2) and (3) are called asymptotically equivalent if, in addition, $x(t) - y(t) \to 0$ as $t \to \infty$.*

By an argument similar to the proof of [18, Lemma 1], we can get the following result of inequalities of Gronwall type, and we omit the details of the proof.

Lemma 1. *Let $u, \eta_1, \eta_2 : I \to I$ be functions such that u is continuous, η_1, η_2 are locally integrable and satisfy* (7)*, and*

$$u(t) \leq u(\tau) + \int_\tau^t (\eta_1(s)u(s) + \eta_2(s)u(\gamma(s)))ds, \quad t \in I_\tau \triangleq [\tau, \infty), \tau \in I.$$

Then for $t \in I_\tau$,

$$u(t) \leq u(\tau) \exp\left(\int_\tau^t \eta(s)ds\right), \quad u(\gamma(t)) \leq \frac{u(\tau)}{1-v} \exp\left(\int_\tau^t \eta(s)ds\right).$$

Now we give the existence and uniqueness result of the solution for (5).

Lemma 2. *Assume that* (6) and (7) *hold. Then for every $(\tau, u_0) \in I \times \mathbb{C}^n$ there exists a unique solution $u = u(t, \tau, u_0), t \in I_\tau$ of* (5) *such that $u(\tau) = u_0$. Moreover, the solution of* (5) *is stable.*

Proof. Let u_1 and u_2 be two solutions of (5) on I_τ. By (6),

$$\begin{aligned}\|u_1(t) - u_2(t)\| \leq &\|u_1(\tau) - u_2(\tau)\| \\ &+ \int_\tau^t (\eta_1(s)\|u_1(s) - u_2(s)\| + \eta_2(s)\|u_1(\gamma(s)) - u_2(\gamma(s))\|)ds.\end{aligned}$$

Then it follows from Lemma 1 that the solution of (5) is stable and $u_1(t) \equiv u_2(t)$ if $u_1(\tau) = u_2(\tau)$.

To prove the existence of the solution of (5) on I_τ, we first prove the local result, namely, for $(\tau, u_0) \in \bar{I}_i \times \mathbb{C}^n$, there exists a unique solution $u = u(t, \tau, u_0)$ of (5) on $\bar{I}_i$.

If $\tau \in I_{-1}$, by the theory of ordinary differential equations (see e.g [14]), we can get the existence of the solution of (5) on I_{-1}. For $i \geq 0$, decompose $\bar{I}_i = I_i^+ + I_i^-$ in the advanced part $I_i^+ = [t_i, \xi_i]$ and in the delayed part $I_i^- = [\xi_i, t_{i+1}]$. Assume that $\tau \in \bar{I}_i$. Let $u_0(t) \equiv u_0$ and

$$u_{k+1}(t) = u_0 + \int_\tau^t g(s, u_k(s), u_k(\gamma(s)))ds, \quad k \geq 0, t \in \bar{I}_i.$$

We declare that for $k \geq 0$,

$$\|u_{k+1}(t) - u_k(t)\| \leq \begin{cases} \dfrac{\|u_0\|}{(k+1)!}\left(\displaystyle\int_{t_i}^t (\eta_1(s) + \eta_2(s))ds\right)^{k+1}, & \tau, t \in I_i^-, \\ v^{k+1}, & \tau, t \in I_i^+. \end{cases} \tag{9}$$

In fact, by (6) and (7), it is easy to check that (9) holds for $k = 0$. Suppose that (9) holds for $k = n$. Then by (6), for $\tau, t \in I_i^-$,

$$\begin{aligned}
\|u_{n+2}(t) - &u_{n+1}(t)\| \\
&\leq \int_\tau^t (\eta_1(s)\|u_{n+1}(s) - u_n(s)\| + \eta_2(s)\|u_{n+1}(\gamma(s)) - u_n(\gamma(s))\|)ds \\
&\leq \frac{\|u_0\|}{(n+1)!}\int_\tau^t \left(\eta_1(s)\left(\int_{t_i}^s (\eta_1(r) + \eta_2(r))dr\right)^{n+1}\right. \\
&\quad \left. +\eta_2(s)\left(\int_{t_i}^{\gamma(s)} (\eta_1(r) + \eta_2(r))dr\right)^{n+1}\right) ds \\
&\leq \frac{\|u_0\|}{(n+1)!}\int_\tau^t (\eta_1(s) + \eta_2(s))\left(\int_{t_i}^s (\eta_1(r) + \eta_2(r))dr\right)^{n+1} ds \\
&= \frac{\|u_0\|}{(n+2)!}\left(\left(\int_{t_i}^t (\eta_1(s) + \eta_2(s))ds\right)^{n+2} - \left(\int_{t_i}^\tau (\eta_1(s) + \eta_2(s))ds\right)^{n+2}\right) \\
&\leq \frac{\|u_0\|}{(n+2)!}\left(\int_{t_i}^t (\eta_1(s) + \eta_2(s))ds\right)^{n+2}.
\end{aligned}$$

Meanwhile, if $\tau, t \in I_i^+$, by (6) and (7),

$$\begin{aligned}
\|u_{n+2}(t) - u_{n+1}(t)\| & \\
&\leq \int_\tau^t (\eta_1(s)\|u_{n+1}(s) - u_n(s)\| + \eta_2(s)\|u_{n+1}(\gamma(s)) - u_n(\gamma(s))\|)ds \\
&\leq \int_\tau^t (\eta_1(s)v^{n+1} + \eta_2(s)v^{n+1})ds \leq v^{n+2}.
\end{aligned}$$

Therefore, (9) holds for $k = n+1$, and then (9) is true for all $k \geq 0$. It follows from (9) that $u(t) = \sum_{k=0}^\infty (u_{k+1}(t) - u_k(t))$ is well defined on $\bar{I}_i$ and is a solution of the integral equation

$$u(t) = u_0 + \int_\tau^t g(s, u(s), u(\gamma(s)))ds.$$

This implies that $u(t)$ is a solution of (5) on I_i with $u(\tau) = u_0$.

Now the global existence of the solution $u(t)$ of (5) on I_τ follows from the above local one immediately since $I = \bigcup_{-1}^{\infty} I_i$. The proof is complete. $\square$

Denote by $\mathcal{U}$ the set of all stable solutions $u(t, \tau, u_0)$ of (5) on I_τ. Then we have the following result.

Theorem 1. *Assume that* (6) and (7) *hold. Then there exists a bijective mapping* $\varphi : \mathcal{U} \to \mathbb{C}^n$, $u \mapsto u_\infty$ *such that*

$$u(t) = u_\infty + \|u(\tau)\| o\left(\exp\left(\int_t^\infty \eta(s)ds\right) - 1\right) \to u_\infty \quad \text{as } t \to \infty. \tag{10}$$

Moreover, φ *is an asymptotic equivalence for* τ *big enough.*

Proof. It follows from the proof of Lemma 2 that the mapping $\varphi_1 : \mathcal{U} \to \mathbb{C}^n$, $u(t, \tau, u_0) \mapsto u_0$ is a homeomorphism. From (6),

$$\|u(t)\| \le \|u_0\| + \int_\tau^t (\eta_1(s)\|u(s)\| + \eta_2(s)\|u(\gamma(s))\|)ds, \quad t \in I_\tau.$$

Together with Lemma 1 this implies that $u(t)$ is bounded on I_τ and

$$g(t, u(t), u(\gamma(t))) \in L_1(I_\tau).$$

Define $\varphi_2 : \mathbb{C}^n \to \mathbb{C}^n$, $u_0 \mapsto u_\infty$ by

$$u_\infty = u_0 + \int_\tau^\infty g(s, u(s), u(\gamma(s)))ds. \tag{11}$$

Then for $t \in I_\tau$,

$$u(t) = u_\infty - \int_t^\infty g(s, u(s), u(\gamma(s)))ds.$$

It is clear that φ_2 is bijective. Then $\varphi = \varphi_2\varphi_1$ is bijective. By (6), (8) and Lemma 1,

$$\begin{aligned}
&\left\|\int_t^\infty g(s, u(s), u(\gamma(s)))ds\right\| \\
&\qquad \le \int_t^\infty (\eta_1(s)\|u(s)\| + \eta_2(s)\|u(\gamma(s))\|)ds \\
&\qquad \le \int_t^\infty \left(\eta_1(s)\|u(\tau)\| \exp\left(\int_\tau^s \eta(r)dr\right) + \frac{\eta_2(s)}{1-v}\|u(\tau)\| \exp\left(\int_\tau^s \eta(r)dr\right)\right) ds \\
&\qquad = \|u(\tau)\| \int_t^\infty \eta(s) \exp\left(\int_\tau^s \eta(r)dr\right) ds \\
&\qquad = \|u(\tau)\| \left(\exp\left(\int_\tau^\infty \eta(s)ds\right) - \exp\left(\int_\tau^t \eta(s)ds\right)\right) \\
&\qquad = \|u(\tau)\| o\left(\exp\left(\int_t^\infty \eta(s)ds\right) - 1\right) \quad \text{as } t \to \infty.
\end{aligned} \tag{12}$$

This leads to (10). Moreover, let $\alpha(t) = \exp(\int_\tau^\infty \eta(s)ds) - \exp(\int_\tau^t \eta(s)ds)$. For $u_0^i \in \mathbf{C}^n, i = 1, 2$ and $u_i = u_i(t, \tau, u_0^i)$, as in (12), we can get that

$$\left\| \int_\tau^\infty [g(s, u_1(s), u_1(\gamma(s))) - g(s, u_2(s), u_2(\gamma(s)))] ds \right\| \le \|u_1(\tau) - u_2(\tau)\| \alpha(\tau). \quad (13)$$

If τ is so large that $\alpha(\tau) < 1$, from (11) and (13), it is easy to obtain that

$$(1 - \alpha(\tau))\|u_0^1 - u_0^2\| \le \|u_\infty^1 - u_\infty^2\| \le \|u_0^1 - u_0^2\|(1 + \alpha(\tau)),$$

which implies that φ_2 is a homeomorphism. Then φ is an asymptotic equivalence for τ big enough. □

3. Asymptotic equivalence

Let $X(t, s)$ be the fundamental matrix solution of

$$z'(t) = A(t)z(t)$$

such that $X(t, t) = J$, the identity matrix. We also denote $X(t) = X(t, \tau)$ for $t \in I_\tau$. Let

$$M_i(t) = X(t, \xi_i) + \int_{\xi_i}^t X(t, s)B(s)ds, \quad t \in \bar{I}_i, i \ge 0.$$

We need the following assumption:

(A_1) Functions $\eta_1(t) = 0, \eta_2(t) = \|X^{-1}(t)\|\|B(t)\|\|X(\gamma(t))\|) \in L_1(I)$ satisfy (7).

Lemma 3. *Assume* (A_1). *Then* (2) *has a unique solution* $x(t) = x(t, \tau, x_0)$ *on* I_τ *for each* $(\tau, x_0) \in I \times \mathbb{C}^n$. *Moreover,* $x(t)$ *is stable and*

$$\det[M_i(t)] \ne 0, \quad t \in \bar{I}_i, i \ge 0. \quad (14)$$

Proof. Noticing that $A(t), B(t)$ are $n \times n$ complex continuous matrixes on I, by [3, Theorem 2.1] (We note that the system in [3, Theorem 2.1] is real; however, this does not affect the applicability of [3, Theorem 2.1] for complex system), we have the following assertion: (2) *has a unique solution* $x(t) = x(t, \tau, x_0)$ *for each* $(\tau, x_0) \in I \times \mathbb{C}^n$ *if and only if* (14) *holds.* So it is sufficient to prove that (2) has a unique solution $x(t) = x(t, \tau, x_0)$ for each $(\tau, x_0) \in I \times \mathbb{C}^n$ and $x(t)$ is stable. In fact, if we make $x(t) = X(t)u(t)$ in (2), u satisfies

$$u'(t) = \tilde{g}(t, u(t), u(\gamma(t))), \quad (15)$$

where $\tilde{g}(t, u(t), u(\gamma(t))) = X^{-1}(t)B(t)X(\gamma(t))u(\gamma(t))$. Then

$$\|\tilde{g}(t, u_1(t), u_1(\gamma(t))) - \tilde{g}(t, u_2(t), u_2(\gamma(t)))\| \le \bar{\eta}_2(t)\|u_1(\gamma(t)) - u_2(\gamma(t))\|. \quad (16)$$

It is easy to see that $x(t) = X(t)u(t)$ is a solution of (2) if and only if $u(t)$ is a solution of (15). Therefore, by (A_1), (16) and Lemma 2, for each $(\tau, x_0) \in I \times \mathbb{C}^n$, (2) has a unique solution $x(t) = x(t, \tau, x_0)$, and it is stable. The proof is complete. □

If (A_1) holds, by Lemma 3 and the theory in [3, Chapter 2], we can get the expression of the fundamental matrix $Y(t)$ of the solution of (2): for $\tau \in \bar{I}_i, t \in \bar{I}_l, l > i$,

$$Y(t) = M_l(t) \prod_{k=l}^{i+1} (M_k^{-1}(t_k) M_{k-1}(t_k)) M_i^{-1}(\tau).$$

The following assumption is needed:

(A_2) The functions $\eta_1(t), \eta_2(t)$ given below satisfy (7):

$$\begin{aligned}\eta_1(t) &= \|Y^{-1}(t)\|(\lambda_1(t)\|Y(t)\| + \|B(t)\|\|Y(\gamma(t))\|) \in L_1(I),\\ \eta_2(t) &= \|Y^{-1}(t)\|\lambda_2(t)\|Y(\gamma(t))\| \in L_1(I).\end{aligned}$$

Now we are in a position to give our main result, where η is the one in (8) with η_1, η_2 given in (A_2) and we denote $\bar{\eta}(t) = \lambda_1(t)\|Y(t)\| + (\|B(t)\| + \lambda_2(t))\|Y(\gamma(t))\| = \|Y^{-1}(t)\|^{-1}(\eta_1(t) + \eta_2(t))$ with λ_1, λ_2 given in (4).

Theorem 2. *Assume that* (4), (A_1) *and* (A_2) *hold. Then* (2) (*resp.* (3)) *has a unique solution $x = x(t, \tau, x_0)$* (*resp. $y = y(t, \tau, y_0)$*) *on I_τ for each $(\tau, x_0) \in I \times \mathbb{C}^n$* (*resp. $(\tau, y_0) \in I \times \mathbb{C}^n$*). *Furthermore, assume that τ is big enough, then* (2) *and* (3) *are equivalent and there exists $\nu \in \mathbb{C}^n$ such that*

$$y(t) = Y(t)(\nu + \epsilon(t)) \quad \text{as } t \to \infty, \tag{17}$$

where the error function $\epsilon(t) = o\left(\exp\left(\int_t^\infty \eta(s)ds\right) - 1\right)$. So $x(= Y\mu) \leftrightarrow y \leftrightarrow \nu$ are equivalences.

Moreover, if $\epsilon_0(t) \triangleq \int_t^\infty \|Y(t,s)\|\bar{\eta}(s)ds \to 0$ as $t \to \infty$, (2) *and* (3) *are asymptotically equivalent and we have the asymptotic formula*

$$y(t) = Y(t)\nu + o(\epsilon_0(t)) \quad \text{as } t \to \infty, \tag{18}$$

for some constant vector $\nu \in \mathbb{C}^n$. So $x \leftrightarrow y \leftrightarrow \nu$ are asymptotic equivalences.

Proof. Lemma 3 implies that (2) has a unique solution $x = x(t, \tau, x_0)$ on I_τ for each $(\tau, x_0) \in I \times \mathbb{C}^n$, and x is stable. Meanwhile, (14) implies that $Y(t)$ is well defined. As in the proof of Lemma 3, if we make $y(t) = Y(t)u(t)$ in (3), u satisfies

$$u'(t) = \hat{g}(t, u(t), u(\gamma(t))), \tag{19}$$

where

$$\hat{g}(t, u(t), u(\gamma(t))) = Y^{-1}(t)[f(t, Y(t)u(t), Y(\gamma(t))u(\gamma(t))) - B(t)Y(\gamma(t))u(t)]. \tag{20}$$

From (4),

$$\begin{aligned}&\|\hat{g}(t, u_1(t), u_1(\gamma(t))) - \hat{g}(t, u_2(t), u_2(\gamma(t)))\|\\ &\qquad \le \|Y^{-1}(t)\|(\lambda_1(t)\|Y(t)\| + \|B(t)\|\|Y(\gamma(t))\|)\|u_1(t) - u_2(t)\|\\ &\qquad\quad + \|Y^{-1}(t)\|\lambda_2(t)\|Y(\gamma(t))\|\|u_1(\gamma(t)) - u_2(\gamma(t))\|\\ &\qquad = \eta_1(t)\|u_1(t) - u_2(t)\| + \eta_2(t)\|u_1(\gamma(t)) - u_2(\gamma(t))\|.\end{aligned}$$

It is easy to see that $y(t) = Y(t)u(t)$ is a solution of (3) if and only if $u(t)$ is a solution of (19). Then (A_2) and Lemma 2 yield that (3) has a unique solution $y = y(t, \tau, y_0)$ on I_τ for each $(\tau, y_0) \in I \times \mathbb{C}^n$, and y is stable.

Meanwhile, by an argument similar to the first paragraph in the proof of Theorem 1, we can see that $u(t)$ is bounded, i.e. $\|u(t)\| \leq M, t \in I_\tau$ for some $M > 0$, and $\hat{g}(t, u(t), u(\gamma(t)) \in L_1(I_\tau)$. Let

$$\nu = u_\infty = u_0 + \int_\tau^\infty \hat{g}(s, u(s), u(\gamma(s)))ds, \quad \epsilon(t) = -\int_t^\infty \hat{g}(s, u(s), u(\gamma(s)))ds.$$

Then

$$u(t) = \nu + \epsilon(t), \quad t \in I_\tau.$$

As in (12), we can get that

$$\left\| \int_t^\infty \hat{g}(s, u(s), u(\gamma(s)))ds \right\| \leq \|u_0\| \exp\left(\int_\tau^t \eta(s)ds \right) \left(\exp\left(\int_t^\infty \eta(s)ds \right) - 1 \right).$$

This implies that $\epsilon(t) = o\left(\exp\left(\int_t^\infty \eta(s)ds\right) - 1\right)$ as $t \to \infty$, and then (17) holds. Moreover, by (20),

$$\begin{aligned}
&\left\| Y(t) \int_t^\infty \hat{g}(s, u(s), u(\gamma(s)))ds \right\| \\
&\quad = \left\| \int_t^\infty Y(t,s)[f(s, Y(s)u(s), Y(\gamma(s))u(\gamma(s))) - B(s)Y(\gamma(s))u(s)]ds \right\| \\
&\quad \leq \int_t^\infty \|Y(t,s)\| [(\lambda_1(s)\|Y(s)\| + \|B(s)\| \|Y(\gamma(s))\|)\|u(s)\| \\
&\qquad + \lambda_2(s)\|Y(\gamma(s))\| \|u(\gamma(s))\|]ds \\
&\quad \leq M \int_t^\infty \|Y(t,s)\| \bar{\eta}(s)ds = M\epsilon_0(t),
\end{aligned}$$

which yields that (18) holds if $\epsilon_0(t) \to 0$ as $t \to \infty$.

Assume that τ is big enough. Since all the hypotheses of Theorem 1 are satisfied with $\hat{g}$ replacing g, $\nu \leftrightarrow u$ is an asymptotic equivalence by Theorem 1. Meanwhile, it is clear that $\nu \leftrightarrow x (= Y\nu)$ is a homeomorphism, and (17) implies that $x \leftrightarrow y$ is also a homeomorphism. Notice that $x = Y\nu$ is the unique solution of (2) with $x(\tau) = \nu$. Thus (2) and (3) are equivalent, and $x \leftrightarrow y \leftrightarrow \nu$ are equivalences. Moreover, if $\epsilon_0(t) \to 0$ as $t \to \infty$, (18) implies that (2) and (3) are asymptotically equivalent, and $x \leftrightarrow y \leftrightarrow \nu$ are asymptotic equivalences. This completes the proof. □

Remark 1. *Theorem 2 generalizes and improves the main result of Theorem 4 in [18]. In fact, let $B(t) = 0$. Then $Y(t) = X(t)$ and $\eta_1 = \eta_2 = 0$. Thus assumption (A_1) is satisfied automatically and (A_2) becomes:*

(A_2') $\eta_1(t) = \|X^{-1}(t)\|\lambda_1(t)\|X(t)\|$ and $\eta_2(t) = \|X^{-1}(t)\|\lambda_2(t)\|X(\gamma(t))\|$ satisfy (7).

If, in addition, $\lambda_1 = \lambda_2$, the assumption (A_2) is the same as (H2) in [18], and Theorem 2 becomes [18, Theorem 4]. However, [18, Theorem 4] may not be applicable for the case $\lambda_1 \neq \lambda_2$ (see the example below).

Example 1. *For simplicity, assume that $\xi_i - t_i = 1, i \geq 0$. Consider the following scalar equations:*

$$x'(t) = -x(t), \tag{21}$$
$$y'(t) = -y(t) + f(t, y(t), y(\gamma(t))), \tag{22}$$

where f satisfies (4) with

$$\begin{cases} \lambda_1(t) = pe^{\xi_i - t}, \quad \lambda_2(t) = qe^{\xi_i - t}, \quad t \in I_i, i \geq 0, \\ p > (e+1)^{-1}/2, \quad 0 < q < 1 + p(1-e). \end{cases}$$

It is easy to see that $X(t) = e^{\tau - t}$ is the fundamental solution of (21). Then for $i \geq 0$,

$$\begin{aligned} \int_{t_i}^{\xi_i} (\eta_1(t) + \eta_2(t))dt &= \int_{t_i}^{\xi_i} \|X^{-1}(t)\|(\lambda_1(t)\|X(t)\| + \lambda_2(t)\|X(\gamma(t))\|)dt \\ &= \int_{t_i}^{\xi_i} (pe^{\xi_i - t} + q)dt = p(e-1) + q < 1. \end{aligned}$$

This means that assumption (A_2') holds. So Theorem 2 is applicable to equations (21) and (22) by Remark 1.

However, by the hypothesis on f above, we can get that f satisfies condition (H1) in [18] with $\lambda(t) = \max\{\lambda_1(t), \lambda_2(t)\} = pe^{\xi_i - t}$, and the function $\eta(t)$ in (H2) is given by

$$\eta(t) = |X(t)||X^{-1}(t)|(1 + X^{-1}(t, \gamma(t)))\lambda(t) = p(e^{\xi_i - t} + 1).$$

Then

$$2\int_{t_i}^{\xi_i} \eta(t)dt = 2p(e+1) > 1, \quad i \geq 0.$$

This implies that condition (H2) in [18] is not satisfied. Therefore, [18, Theorem 4] is not applicable here.

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References

[1] M. U. Akhmet, *Asymptotic behavior of solutions of differential equations with piecewise constant arguments*, Appl. Math. Lett. **21**(2008), 951–956.

[2] M. U. Akhmet, *Integral manifolds of differential equations with piecewise constant argument of generalized type*, Nonlinear Anal. **66**(2007), 367–383.

[3] M. U. Akhmet, *Nonlinear Hybrid Continuous/Discrete-Time Models*, Atlantis Press, Paris, 2011.

[4] M. U. Akhmet, H. Oktem, S. W. Pickl, G. W. Weber, *An anticipatory extension of Malthusian model*, in: *AIP Conference Proceedings 839*, (D. M. Dubois, Ed.), The American Institute of Physics, New York, 2006, 260–264.

[5] M. U. AKHMET, M. TLEUBERGENOVA, A. ZAFER, *Asymptotic equivalence of differential equations and asymptotically almost periodic solutions*, Nonlinear Anal. **67**(2007), 1870–1877.

[6] S. BUSENBERG, K. L. COOKE, *Models of vertically transmitted diseases with sequential-continuous dynamics*, in: *Nolinear phenomena in Mathematical Science*, (V. Lakshmikantham, Ed.), Academic Press, New York, 1982, 179–187.

[7] S. BUSENBERG, K. L. COOKE, *Vertically Transmitted Diseases*, Springer-Verlag, Berlin, 1993.

[8] S. K. CHOI, Y. H. GOO, N. J. KOO, *Asymptotic equivalence between two linear differential systems*, Ann. Differential Equations **13**(1997), 44–52.

[9] K. L. COOKE, J. WIENER, *A survey of differential equations with piecewise continuous arguments*, Lecture Notes in Math., Vol. 1475, Springer-Verlag, Berlin, 1991.

[10] K. L. COOKE, J. WIENER, *Retarded differential equations with piecewise constant delays*, J. Math. Anal. Appl. **99**(1984), 265–297.

[11] W. A. COPPEL, *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, 1965.

[12] I. GYORI, G. LADAS, *Oscillation Theory of Delay Differential Equations. With Applications*, Oxford University Press, New York, 1991.

[13] J. R. HADDOCK, T. KRISZTIN, J. H. WU, *Asymptotic equivalence of neutral and infinite retarded differential equations*, Nonlinear Anal. **14**(1990), 369–377.

[14] P. HARTMAN, *Ordinary Differential Equations*, John Wiley and Sons, Inc., New York, London, Sydney, 1964.

[15] N. LEVINSON, *The asymptotic nature of solutions of linear systems of differential equations*, Duke Math. J. **15**(1948), 111–126.

[16] V. V. NEMYTSKII, V. V. STEPANOV, *Qualitative Theory of Differential Equations*, Princeton University Press, New Jersey, 1966.

[17] G. PAPASCHINOPOULOS, *Linearization near the integral manifold for a system of differential equations with piecewise constant argument*, J. Math. Anal. Appl. **215**(1997), 317–333.

[18] M. PINTO, *Asymptotic equivalence of nonlinear and quasi linear differential equations with piecewise constant arguments*, Math. Comput. Modelling **49**(2009), 1750–1758.

[19] S. SAITO, *Asymptotic equivalence of quasilinear ordinary differential systems*, Math. Japan **37**(1992), 503–513.

[20] S. M. SHAH, J. WIENER, *Advanced differential equations with piecewise constant argument deviations*, Int. J. Math. Math. Sci. **6**(1983), 671–703.

[21] M. SVEC, *Asymptotic relationship between solutions of two systems of differential equations*, Czechoslovak Math. J. **29**(1974), 44–58.

[22] J. WIENER, *Differential equations with piecewise constant delays*, in: *Trends in the Theory and Practice of Nonlinear Differential Equations*, (V. Lakshmikantham, Ed.), Marcel Dekker, New York, 1983, 547–580.

[23] J. WIENER, *Pointwise initial value problems for functional-differential equations*, in: *Differential Equations*, (I. W. Knowles and R. T. Lewis, Eds.), North-Holland, New York, 1984, 571–580.