# CROSSED PRODUCT ORDERS OVER VALUATION RINGS II: TAMELY RAMIFIED CROSSED PRODUCT ALGEBRAS

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Abstract

Let $V$ be a commutative valuation domain of arbitrary Krull-dimension, with quotient field $F$, and let $K$ be a finite Galois extension of $F$ with group $G$, and $S$ the integral closure of $V$ in $K$. Suppose one has a 2-cocycle on $G$ which takes values in the group of units of $S$. Then one can form the crossed product of $G$ over $S$, $S * G$, which is a $V$-order in the central simple $F$-algebra $K * G$. If we assume $S * G$ is a Dubrovin valuation ring of $K * G$, then the main result of this paper is that, given a suitable definition of tameness for central simple algebras, $K * G$ is tamely ramified and defectless over $F$ if and only if $K$ is tamely ramified and defectless over $F$. We also study the residue structure of $S * G$, as well as its behaviour upon passage to Henselization.

## *Introduction*

This paper is a sequel to [**8**], whose results we shall use freely. If $R$ is a ring, then $\overline{R}$ will denote its quotient modulo the Jacobson radical, $Z(R)$ its centre and, if $X$ is a subset of $R$, then $C_R(X)$ will denote the centraliser of $X$ in $R$. In the case of fields, all valuation-theoretic notions and terminology are as defined in [**2**]. The rest of the terminology used, if not defined, has been described in [**8**]. Let $V$ be a commutative valuation domain with quotient field $F$, and let $K$ be a finite Galois extension of $F$ with group $G$. Let $S$ be the integral closure of $V$ in $K$. Given a two-cocycle which takes values in the group of units of $S$, we can always form the crossed product $S * G = \sum_{\sigma \in G} S x_\sigma$, which is a $V$-order in the central simple $F$-algebra $K * G = \sum_{\sigma \in G} K x_\sigma$. This object has been studied in [**5**, **13**], among other places, assuming that $V$ is a DVR. Recently it has been studied in [**8**], for an arbitrary valuation ring $V$.

Let $Q$ be a central simple $F$-algebra, and let $B$ be a subring of $Q$. Then $B$ is called a *Dubrovin valuation ring* of $Q$ if it is a semihereditary order in $Q$ and $\overline{B}$ is a simple Artinian ring. For properties of such rings, the reader may consult [**1**, **3**, **4**, **10**, **12**]. Suppose B is a Dubrovin valuation ring of $Q$ with centre $V$. Associated with the pair $(B, V)$, we have, according to [**12**], the *value group* of $B$, $\Gamma_B = \mathrm{st}(B)/U(B)$, where $\mathrm{st}(B) = \{x \in U(Q) \mid xBx^{-1} = B\}$ and $U(\cdot)$ is the group of multiplicative units of a given ring; the *ramification index* of $B$ over $V$, $e(B \mid V) = [\Gamma_B : \Gamma_V]$, where $\Gamma_V$ is the value group of $V$; and the *residue degree* of $B$ over $V$, $f(B \mid V) = [\overline{B} : \overline{V}]$. If $\overline{p}$ is the characteristic exponent of $\overline{V}$, that is, $\overline{p} = \max\{\mathrm{char}(\overline{V}), 1\}$, it was shown in [**12**, Theorem C] that $[Q : F] = e(B \mid V) f(B \mid V) \eta^2 \overline{p}^a$ for some positive integer $\eta$ and non-negative integer $a$. We say that $(Q, B)$ is *defectless* over $(F, V)$ when $[Q : F] = e(B \mid V) f(B \mid V)$. The number $\eta$ is called the *extension number* of $V$ to $Q$, described in [**3**]. By [**12**, Theorem F], $B$ is integral

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over $V$ if and only if $\eta = 1$. We observe that, if $S * G$ is a Dubrovin valuation ring, which is the assumption for a greater part of this paper, then it is integral over $V$.

In the commutative case, we have the following situation. We fix once and for all an extension $W$ of $V$ to $K$. Let $n = [K : F] = \mid G \mid$; $\Gamma_W$ the value group of $W$; $e = [\Gamma_W : \Gamma_V]$, the ramification index of $W$ over $F$; $f = [\overline{W} : \overline{V}]$, the residue degree of $W$ over $F$; and $g$ be the number of extensions of $V$ to $K$. It is known that $n = efg\overline{p}^d$ in this case, where $d$ is a non-negative integer. Following [**2**], we say that $(K, W)$ is defectless over $(F, V)$ if $n = efg$, and we say that $(K, W)$ is tamely ramified over $(F, V)$ if $\mathrm{char}(\overline{V})$ does not divide $e$ and $\overline{W}$ is separable over $\overline{V}$.

When $B$ is a Dubrovin valuation ring of a central simple $F$-algebra $Q$, we will therefore say that $(Q, B)$ is *tamely ramified* over $(F, V)$ if $Z(\overline{B})$ is separable over $\overline{V}$ and $\mathrm{char}(\overline{V})$ does not divide $e(B \mid V)$. We do not assume that $\overline{p}$ is co-prime to $[Q : F]$. However, our definition of tameness is in some sense stronger than that of [**6**], which was applied only to division algebras with invariant valuation rings. The main result of this paper justifies our choice of the definition of tameness. We readily see that any Azumaya algebra over a valuation ring is tamely ramified and defectless, by [**10**, Proposition 3.2] and [**12**, Corollary 3.4].

In section 1, we prove the main result of the paper, which states that, assuming $S * G$ is a Dubrovin valuation ring, then $K * G$ is tamely ramified and defectless over $F$ if and only if $K/F$ is tamely ramified and defectless. If we assume that $K/F$ is tamely ramified and defectless then, by [**8**, Theorem 2], $J(S * G) = J(S) * G$ and therefore $\overline{S * G} \cong M_g(\overline{W} * G^Z)$ by [**8**, Lemma 2], where $G^Z$ is the decomposition group of $W$ over $F$. Let $G^T$ be the inertia group of $W$ over $F$. In section 2, it is shown that when $S * G$ is a Dubrovin valuation ring and $K/F$ is tamely ramified and defectless, then $\overline{W} * G^Z$ is a generalized crossed product of $G^Z/G^T$ over $\overline{W} * G^T$. A necessary and sufficient condition is given for the generalized crossed product to become a classical crossed product algebra. In section 3, we show that, if $(K_h, W_h)$ is a Henselization of $(K, W)$, and $(F_h, V_h)$ a Henselization of $(F, V)$, then $(S * G) \otimes_V V_h \cong M_g(W_h * G^Z)$. The value function associated with an integral Dubrovin valuation ring described in [**10**] easily materializes in $(K * G) \otimes_F F_h$ whenever $K/F$ is tamely ramified and defectless and $S * G$ is a Dubrovin valuation ring.

## 1. *Tamely Ramified Dubrovin Crossed Products*

In this section, we prove the main result of this paper.

Lemma 1. *Let $C = Z(\overline{W} * G^Z)$. Then*
(i) $C_{\overline{W} * G^Z}(\overline{W}) = \overline{W} * G^T$,
(ii) $\overline{W} C \subseteq Z(\overline{W} * G^T)$,
(iii) $C_{\overline{W} * G^Z}(\overline{W} C) = \overline{W} * G^T$,
(iv) *$\overline{W} * G^Z$ is a simple ring if and only if $\overline{W} * G^T$ is a simple ring. When this happens, $Z(\overline{W} * G^T) = \overline{W} C$ and $S * G$ is a Dubrovin valuation ring of $K * G$.*

*Proof.* (i). It is clear that $\overline{W} * G^T \subseteq C_{\overline{W} * G^Z}(\overline{W})$. Now, let $z \in C_{\overline{W} * G^Z}(\overline{W})$. One can write $z = \sum_{\sigma \in G^Z} w_\sigma x_\sigma$. Fix $\tau \in G^Z$. Since $zw = wz$ $\forall w \in \overline{W}$, we must have $w_\tau \tau(w) = w_\tau w$ $\forall w \in \overline{W}$. Thus either $w_\tau = 0$ or $\tau \in G^T$. This shows that $z \in \overline{W} * G^T$.

(ii). We know $C \subseteq C_{\overline{W}*G^Z}(\overline{W}) = \overline{W} * G^T$. Therefore, $\overline{W}C \subseteq \overline{W} * G^T$. Clearly, we have $\overline{W}C \subseteq Z(\overline{W} * G^T)$, since $\overline{W} \subseteq Z(\overline{W} * G^T)$.

(iii). Since $\overline{W}C \subseteq Z(\overline{W} * G^T)$, we have $C_{\overline{W}*G^Z}(\overline{W}C) \supseteq \overline{W} * G^T$. On the other hand, $C_{\overline{W}*G^Z}(\overline{W}C) \subseteq C_{\overline{W}*G^Z}(\overline{W}) = \overline{W} * G^T$. So $C_{\overline{W}*G^Z}(\overline{W}C) = \overline{W} * G^T$.

(iv). Suppose $\overline{W} * G^Z$ is a simple ring. Since $C_{\overline{W}*G^Z}(\overline{W}C) = \overline{W} * G^T$, $\overline{W} * G^T$ must be a simple ring. On the other hand, suppose $\overline{W} * G^T$ is a simple ring. We will use a minimal length argument to show that $\overline{W} * G^Z$ is a simple ring. Observe that, since $G^T \trianglelefteq G^Z$, $\overline{W} * G^Z \cong (\overline{W} * G^T) * (G^Z/G^T)$. Let $I$ be a two-sided ideal of $(\overline{W} * G^T) * (G^Z/G^T)$, $I \neq (\overline{W} * G^T) * (G^Z/G^T)$. Hence $I \cap (\overline{W} * G^T) = \{0\}$, since $\overline{W} * G^T$ is simple. Let $x = a_{\overline{\sigma}_1} x_{\overline{\sigma}_1} + \cdots + a_{\overline{\sigma}_r} x_{\overline{\sigma}_r} \in I$, with $r$ minimal. Since $G^Z/G^T \cong \mathrm{Aut}_{\overline{V}}(\overline{W})$ by [**2**, Theorem 19.6], if $r > 1$, $\exists\ w \in \overline{W}$ such that $\overline{\sigma}_1(w) \neq \overline{\sigma}_2(w)$. Since $\overline{W} \subseteq Z(\overline{W} * G^T)$, $\overline{\sigma}_1(w)$, $\overline{\sigma}_2(w)$ are central units of $\overline{W} * G^T$, and hence $x - \overline{\sigma}_1(w)^{-1} x w$ is a shorter length element of $I$. This is impossible, and hence $r = 1$ and $x = a_{\overline{\sigma}_1} x_{\overline{\sigma}_1}$. But then, $x x_{\overline{\sigma}_1}^{-1} = a_{\overline{\sigma}_1} \in I \cap (\overline{W} * G^T) = \{0\}$. So $x = 0$ and hence $I = \{0\}$. Therefore $\overline{W} * G^Z$ is a simple ring.

Since $C_{\overline{W}*G^Z}(\overline{W}C) = \overline{W} * G^T$, by the double centralizer property, $\overline{W}C = C_{\overline{W}*G^Z}(\overline{W} * G^T) \supseteq Z(\overline{W} * G^T)$. Hence we have equality $\overline{W}C = Z(\overline{W} * G^T)$. When $\overline{W} * G^Z$ is a simple ring, then $S * G$ is a Dubrovin valuation ring by [**8**, Lemma 2].

Let $G^V$ be the ramification group of $W$ over $F$. It is known that $G^V$ is the only $\overline{p}$-Sylow subgroup of $G^T$, and a normal subgroup of $G^Z$. Let $f = f_0 \overline{p}^s$, where $f_0$ is the degree over $\overline{V}$ of the maximal separable extension of $\overline{V}$ in $\overline{W}$. Let $e = e_0 \overline{p}^t$, with $\overline{p}$ co-prime to $e_0$. Since $\mid G^T \mid = e \overline{p}^s \overline{p}^d$, we have $\mid G^V \mid = \overline{p}^{t+s+d}$. We also know that $\mid G^Z \mid = e f \overline{p}^d$.

Theorem 1. *Suppose $S * G$ is a Dubrovin valuation ring. Then $(K * G, S * G)$ is tamely ramified and defectless over $(F, V)$ if and only if $(K, W)$ is tamely ramified and defectless over $(F, V)$. When this happens, $f(S * G | V) = e f^2 g^2$, and $e(S * G | V) = e$.*

*Proof.* Suppose $K * G$ is tamely ramified and defectless over $F$. When $\mathrm{char}(\overline{V}) = 0$, then $K/F$ is tamely ramified and defectless. So we will assume $\overline{p} > 1$. We will first show that $J(S * G) = J(S) * G$.

Since by [**11**, Theorem 4.2] $J(W) * G^V \subseteq J(W * G^V)$, we have $\overline{W * G^V} = \overline{\overline{W} * G^V}$, and hence $\overline{W * G^V}$ is a purely inseparable field extension of $\overline{W}$, by [**11**, Lemma 16.3]. But the canonical ring epimorphism $\overline{W} * G^V \mapsto \overline{W * G^V}$ is $\overline{W}$-linear. Hence

$$\dim_{\overline{W}} (\overline{W * G^V}) \leqslant \dim_{\overline{W}} (\overline{W} * G^V) = \mid G^V \mid = \overline{p}^{t+s+d},$$

and thus $[\overline{W * G^V} : \overline{W}] = \overline{p}^r$, for some $0 \leqslant r \leqslant s + t + d$.

Since $G^T \trianglelefteq G^Z$, $J(\overline{W} * G^T) * (G^Z/G^T)$ is an ideal of $(\overline{W} * G^T) * (G^Z/G^T) = \overline{W} * G^Z$. Further,

$$\frac{\overline{W} * G^Z}{J(\overline{W} * G^T) * (G^Z/G^T)} \cong \frac{(\overline{W} * G^T) * (G^Z/G^T)}{J(\overline{W} * G^T) * (G^Z/G^T)} \cong$$

$$(\overline{\overline{W} * G^T}) * (G^Z/G^T) \cong (\mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_k) * (G^Z/G^T),$$

where $\mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_k$ is a finite direct sum of finite dimensional simple rings. Note that $\overline{W} \subseteq Z(\mathcal{R}_i)$ for all $i$, since $G^T$ acts trivially on $\overline{W}$. Let $H_i = \{\overline{\sigma} \in G^Z/G^T \mid \mathcal{R}_i^{\overline{\sigma}} = \mathcal{R}_i$ and $\overline{\sigma}$ is inner on $\mathcal{R}_i\}$. Let $N_i$ be a Sylow $\overline{p}$-subgroup of $H_i$. Using a minimal length argument like the one employed in the proof of Lemma 1(iv) above, we can show that $\mathcal{R}_i * N_i$ is simple Artinian for $i = 1, \ldots, k$. This shows that $(\overline{\overline{W} * G^T}) * (G^Z/G^T)$ is semisimple, by [**14**, Theorem 2.4(i)]. This implies that $J(\overline{W} * G^Z) = J((\overline{W} * G^T) * (G^Z/G^T)) = J(\overline{W} * G^T) * (G^Z/G^T)$.

Since $G^V \trianglelefteq G^T$ and $\mid G^T/G^V \mid^{-1} \in \overline{W} * G^V$, we have $J(\overline{W} * G^T) = J((\overline{W} * G^V) * (G^T/G^V)) = J(\overline{W} * G^V) * (G^T/G^V)$ by [**11**, Theorem 4.2], and so $J(\overline{W} * G^Z) = J(\overline{W} * G^T) * (G^Z/G^T) = J(\overline{W} * G^V) * (G^Z/G^V)$, since $G^V \trianglelefteq G^Z$.

Therefore,

$$\overline{W * G^Z} = (\overline{W * G^V}) * (G^Z/G^V).$$

Since $S * G/(J(S) * G) \cong M_g(\overline{W} * G^Z)$ by [**8**, Lemma 2] and $J(S) * G \subseteq J(S * G)$, we have $\overline{S * G} \cong M_g(\overline{\overline{W} * G^Z}) \cong M_g(\overline{W * G^Z})$, which implies that

$$f(S * G \mid V) = [\overline{W * G^V} : \overline{W}][\overline{W} : \overline{V}] \mid G^Z/G^V \mid g^2 =$$

$$(\overline{p}^r)(f)(e_0 f_0) g^2 = \frac{\overline{p}^r f e_0 \overline{p}^t f_0 \overline{p}^s g^2}{\overline{p}^t \overline{p}^s} = \frac{e f^2 \overline{p}^r g^2}{\overline{p}^t \overline{p}^s}.$$

Since $K * G$ is defectless, $e(S * G \mid V) f(S * G \mid V) = n^2$. Therefore, $e(S * G \mid V) \dfrac{e f^2 \overline{p}^r g^2}{\overline{p}^t \overline{p}^s} = e^2 f^2 g^2 \overline{p}^{2d}$, and so we have $e(S * G \mid V)\overline{p}^r = e\overline{p}^d \overline{p}^{t+s+d}$. Since $r \leqslant t + s + d$, we can write $e(S * G \mid V) = e\overline{p}^d \overline{p}^{t+s+d-r}$, where all exponents of $\overline{p}$ are non-negative. But $\overline{p}$ is co-prime to $e(S * G \mid V)$, by assumption. Thus $d = 0$ and $r = t + s$. Therefore,

$$f(S * G \mid V) = \frac{e f^2 \overline{p}^r g^2}{\overline{p}^t \overline{p}^s} = e f^2 g^2.$$

Now consider the canonical ring epimorphism

$$\frac{S * G}{J(S) * G} = M_g(\overline{W} * G^Z) \mapsto \overline{S * G}.$$

This is $\overline{V}$-linear. Since $[\dfrac{S * G}{J(S) * G} : \overline{V}] = e f^2 g^2 \overline{p}^d = e f^2 g^2 = [\overline{S * G} : \overline{V}]$, the map must be a bijection. Hence $\dfrac{S * G}{J(S) * G}$ is semisimple, and so $J(S * G) = J(S) * G$.

Thus by [**8**, Lemma 2], $\overline{S * G} = M_g(\overline{W} * G^Z)$, which is a finite dimensional simple algebra, since $S * G$ is a Dubrovin valuation ring of the finite dimensional simple algebra $K * G$. Hence $\overline{W} * G^Z$ is also a finite dimensional simple algebra with the same center, say $C$. By Lemma 1, $\overline{W} * G^T$ is a central-simple $\overline{W}C$-algebra. But $C$ is Galois over $\overline{V}$, by [**12**, Corollary B]. Hence, by Galois theory, $\overline{W}C$ is Galois over $\overline{W}$, and thus $\overline{W} * G^T$ is a separable $\overline{W}$-algebra. By [**5**, Lemma 4], this means $(\mid G^T \mid, \overline{p}) = 1$. Hence $K/F$ is tamely ramified and defectless, by [**8**, Lemma 1].

Now assume that $K/F$ is tamely ramified and defectless. We have $(\overline{p}, \mid G^T \mid) = 1$, by [**8**, Lemma 1]. Let $C = Z(\overline{S * G})$. By [**8**, Theorem 2(a) & Lemma 2], $\overline{S * G} = M_g(\overline{W} * G^Z)$, and so we have that $C = Z(\overline{W} * G^Z)$ and $f(S * G \mid V) = e f^2 g^2$. By Lemma 1, $C \subseteq Z(\overline{W} * G^T)$. But in this case $\overline{W} * G^T$ is separable over $\overline{W}$ by [**13**, Theorem 1.1], and in turn $\overline{W}$ is separable over $\overline{V}$. Therefore $C$ is separable over $\overline{V}$.

Since the Dubrovin valuation ring $S * G$ is integral over $V$, the extension number of $V$ to $Q$ is 1. Therefore by [**12**, Theorem C], $n^2 = e(S*G\,|\,V)f(S*G\,|\,V)\overline{p}^a$, where $a$ is some non-negative integer. Since we know that $f(S*G|V) = ef^2g^2$, we see that $e = e(S*G\,|\,V)\overline{p}^a$. But $(\overline{p}, e) = 1$ by assumption. Therefore $\overline{p}^a = 1$, that is, $S*G$ is defectless over $V$. Further, $(\overline{p}, e(S*G\,|\,V)) = 1$. The first statement is therefore proved.

The other assertions are now self-evident from the proof.

We will see later in Section 3 that, with the assumption contained in Theorem 1, we have, in addition, the result that $\Gamma_{S*G} \cong \Gamma_W$.

Remark. The analogue of Theorem 1 does not hold for Dubrovin valuation rings in general. For let $K/F$ be a Galois extension that is *not* tamely ramified and defectless, for example, the ones in Examples 1 and 2 of [**8**]. Consider a crossed product algebra $K*G$, where the 2-cocycle is trivial. Although $K/F$ is not tamely ramified and defectless, any Dubrovin valuation ring $B$ of $K*G$ lying over $V$ is Azumaya over $V$, as $K*G$ is split, and hence $B$ is tamely ramified and defectless over $F$ by [**10**, Proposition 3.2] and [**12**, Corollary 3.4]. But then, $B$ cannot be isomorphic to $S*G$ in this case, by Theorem 1 or [**8**, Theorem 3]. Thus the requirement in Theorem 1 that $S*G$ be a Dubrovin valuation ring of $K*G$ cannot be dropped.

Examples of tamely ramified and defectless Dubrovin crossed products can easily be constructed using [**8**, Theorem 3]. In Examples 1 and 2 of [**8**], we encounter Dubrovin crossed products that are not tamely ramified, although they are defectless.

Proposition 1. *Suppose $K/F$ is tamely ramified and defectless. Then the following are equivalent:*

(i) *$S*G$ is a Dubrovin valuation ring of $K*G$.*
(ii) *$W*G^Z$ is a Dubrovin valuation ring of $K*G^Z$.*
(iii) *$W*G^T$ is a Dubrovin valuation ring of $K*G^T$.*
(iv) *$\overline{W}*G^Z$ is a simple ring.*
(v) *$\overline{W}*G^T$ is a simple ring.*

*Proof.* Since $K/F$ is tamely ramified and defectless, then, by [**8**, Theorem 2], $S*G$ is semihereditary and $\overline{S*G} = M_g(\overline{W}*G^Z)$. Thus $S*G$ is Dubrovin valuation ring if and only if $\overline{W}*G^Z$ is a simple ring and, by Lemma 1, this is true if and only if $\overline{W}*G^T$ is a simple ring.

Now let $F^T$ (resp. $F^Z$) be the inertia (resp. decomposition) field of $W$ over $F$. Both $K/F^T$ and $K/F^Z$ are tamely ramified and defectless, by [**2**, 22.1 & 22.3], since $K/F$ is. Further, we know that both $W\cap F^T$ and $W\cap F^Z$ are indecomposed in $K$, by [**2**, Theorem 15.7]. Hence, by [**8**, Theorem 2(a)], both $W*G^T$ and $W*G^Z$ are semihereditary, and $J(W*G^T) = J(W)*G^T$, $J(W*G^Z) = J(W)*G^Z$. Therefore, $W*G^T$ is a Dubrovin valuation ring if and only if $\overline{W}*G^T$ is a simple ring, if and only if $\overline{W}*G^Z$ is a simple ring by Lemma 1(iv), if and only if $W*G^Z$ is a Dubrovin valuation ring. The proposition is thus proved.

Another condition equivalent to $S*G$ being a Dubrovin valuation ring will be given in Theorem 3(ii).

## 2. *The Residue Structure of $S * G$*

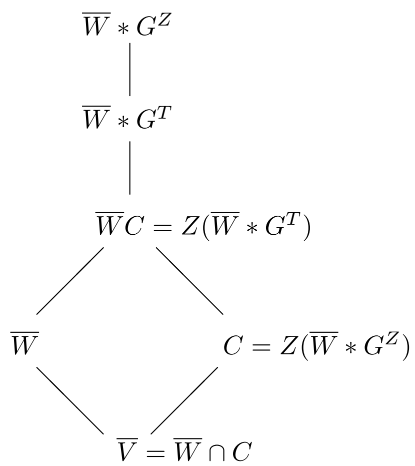
If $S*G$ is a tamely ramified and defectless Dubrovin valuation ring, then $\overline{S * G} \cong M_g(\overline{W} * G^Z)$, and hence, to study the structure of $\overline{S * G}$, one need only consider $\overline{W} * G^Z$.

Let $Q$ be a central simple $F$-algebra. It is not always possible to find a maximal subfield of $Q$ which is a Galois extension of $F$, that is, $Q$ need not be a "classical" crossed product algebra. What is often the case is that there exists a subfield $L$ of $Q$ which is Galois over $F$ but $[L : F]^2 < [Q : F]$. If $A$ is the centralizer of $L$ in $Q$, and $H$ is the Galois group of $L$ over $F$, then $Q$ is said to be a *generalized crossed-product of $H$ over $A$*. In case $L$ is a maximal subfield of $Q$, we will say that $Q$ is a *classical crossed product algebra*. Recall that if $B$ is a Dubrovin valuation ring of $Q$ with center $V$, then each $a \in \mathrm{st}(B)$ induces a ring automorphism of $\overline{B}$ via conjugation. In fact, Wadsworth [**12**, Corollary B] showed that this map induces a surjection $\omega : \Gamma_B/\Gamma_V \mapsto \mathrm{Aut}_{\overline{V}}(Z(\overline{B}))$ (see also [**3**, Corollary 4.4(i)]). When $K/F$ is tamely ramified and defectless and $S * G$ is a Dubrovin valuation ring, then $S * G$ is tamely ramified and defectless, by Theorem 1, and hence $Z(\overline{S * G})$ is Galois over $\overline{V}$, by [**12**, Corollary B].

Theorem 2. *Suppose $S * G$ is a tamely ramified and defectless Dubrovin valuation ring of $K * G$. Let $C = Z(\overline{W} * G^Z)$. Then:*

(i) a. *We have that $\overline{W}C = Z(\overline{W} * G^T)$,*
b. *further, $\overline{W}C$ is Galois over both $\overline{W}$ and $C$,*
c. *$Gal(\overline{W}C/C) \cong Gal(\overline{W}/\overline{V}) \cong G^Z/G^T$ and $Gal(\overline{W}C/\overline{W}) \cong Gal(C/\overline{V})$,*
d. *$\overline{W} * G^Z$ is a generalized crossed product of $G^Z/G^T$ over $\overline{W} * G^T$.*

*We have the following diagram:*




(ii) *The Wadsworth map, $\omega$, is a bijection if and only if $\overline{W} * G^T$ is commutative. When this happens, $\overline{W} * G^T$ is a maximal subfield of $\overline{W} * G^Z$, it is Galois over $C$, and $\overline{W} * G^Z$ is a classical crossed product algebra of $G^Z/G^T$ over $\overline{W} * G^T$.*

*Proof.* By Lemma 1, $\overline{W}C = Z(\overline{W} * G^T)$. Now, we know that $\overline{W}$ is Galois over $\overline{V}$, with group $G^Z/G^T$, since $K/F$ is tamely ramified. Also, as we noted before the theorem, $C$ is Galois over $\overline{V}$. We claim that $\overline{W} \cap C = \overline{V}$. To see this, note that if

$w \in \overline{W} \cap C$, then $wx_\sigma = x_\sigma w \ \forall\ \sigma \in G^Z \Rightarrow \overline{\sigma}(w) = w \ \forall\ \overline{\sigma} \in G^Z/G^T$. This means $w \in \overline{V}$, since $\overline{W}/\overline{V}$ is Galois with group $G^Z/G^T$. The first part of the theorem now follows from Galois theory and Lemma 1(iii).

As for second part, note that

$$e = [\overline{W} * G^T : \overline{W}C][\overline{W}C : \overline{W}] = [\overline{W} * G^T : \overline{W}C][C : \overline{V}].$$

From Theorem 1, we know that $[\Gamma_{S*G} : \Gamma_V] = e$. Thus $\omega$ is a bijection if and only if $[C : \overline{V}] = e$, if and only if $[\overline{W} * G^T : \overline{W}C] = 1$, if and only if $\overline{W} * G^T$ is commutative. When this happens, $\overline{W} * G^T$ is a field. Further, we now also have

$$[\overline{W} * G^T : C]^2 = [\overline{W}C : C]^2 = [\overline{W} : \overline{V}]^2 = f^2$$

$$= [\overline{W} * G^Z : \overline{W} * G^T][\overline{W}C : C] = [\overline{W} * G^Z : C].$$

Therefore, $\overline{W} * G^T$ is a maximal subfield of $\overline{W} * G^Z$, and a Galois extension of $C$ by the first part of the theorem. The proof is now complete.

Remark. From a purely ring-theoretic point of view, the distinction between generalized crossed products and classical crossed products is superfluous in this case: since $G^T \trianglelefteq G^Z$, $\overline{W} * G^Z \cong (\overline{W} * G^T) * (G^Z/G^T)$, and hence $\overline{W} * G^Z$ is always a crossed product of $G^Z/G^T$ over $\overline{W} * G^T$.

The following corollary now follows easily from Theorem 1 and its proof, and from Proposition 1 and Theorem 2.

Corollary 1. *The order $S * G$ is a tamely ramified and defectless Dubrovin valuation ring if and only if $\overline{W} * G^T$ is a simple separable $\overline{W}$-algebra; it is a tamely ramified and defectless Dubrovin valuation ring and the Wadsworth map is a bijection if and only if $\overline{W} * G^T$ is a separable field extension of $\overline{W}$.*

When $K/F$ is tamely ramified and defectless, then $(|G^T|, \overline{p}) = 1$, by [**8**, Lemma 1]. Therefore, by [**2**, Corollary 20.10(b), Corollary 20.12, & 20.2], $G^T \cong \Gamma_W/\Gamma_V$, and so $G^T$ is abelian. Thus, if $f$ is the 2-cocycle (not to be confused with the residue degree of $W$ over $F$), then $\overline{W} * G^T$ is commutative if and only if $f(\sigma, \tau) - f(\tau, \sigma) \in J(W) \ \forall \sigma, \tau \in G^T$. But this characterization of the commutativity of $\overline{W} * G^T$ is hardly illuminating; indeed, an example of a tamely ramified and defectless Dubrovin valuation ring $S * G$ with $\overline{W} * G^T$ non-commutative is unknown to us, and may well not exist! However, when $V$ is a DVR or, more generally, when $G^T$ is cyclic, then $\overline{W} * G^T$ is commutative. This is the essence of the following proposition.

Recall that the initial index of $W$ over $F$, $\epsilon(W | F)$, or just $\epsilon$ when the context is clear, is the number of elements in the set $\{\delta \in \Gamma_W \mid 0 \leqslant \delta < \gamma$ for all positive $\gamma \in \Gamma_V\}$. It is known that $\epsilon \leqslant e$.

Lemma 2. *The ring $S * G$ is finitely generated over $V$ if and only if $S$ is finitely generated over $V$, if and only if $K/F$ is defectless and $\epsilon = e$.*

*Proof.* Suppose $S * G$ is finitely generated over $V$, with a generating set $\{y_i\}_{i=1}^l$, say. Write $y_i = \sum_{\sigma \in G} s_\sigma^{(i)} x_\sigma$. Then $\{s_1^{(i)}\}_{i=1}^l$ is a generating set for $S$ over $V$. The converse is obvious.

On the other hand, $S$ is finitely generated over $V$ if and only if $K/F$ is defectless and $\epsilon = e$, by [**2**, Theorem 18.6]

PROPOSITION 2. *Suppose $S * G$ is a tamely ramified and defectless Dubrovin valuation ring of $K * G$. When $e(B \mid V)$ is square-free, or when $S * G$ is finitely generated over $V$, then the Wadsworth map is a bijection.*

*Proof.* We will show that, in either case, $G^T$ is cyclic. We know that $K/F$ is tamely ramified and defectless, and hence $G^T \cong \Gamma_W/\Gamma_V$, as we already noted above. Since $\mid G^T \mid = e = e(B \mid V)$, if $e(B \mid V)$ is square free then $G^T$ is cyclic, since it is abelian.

Now suppose $S*G$ is a finitely generated Dubrovin valuation ring. We will show that $G^T$ is again cyclic. Let $E = \{\delta \in \Gamma_W \mid 0 \leqslant \delta < \gamma \text{ for all positive } \gamma \in \Gamma_V\}$. Then $E = \{\delta_0, \delta_1, \ldots, \delta_{\epsilon-1}\}$, where $0 = \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_{\epsilon-1}$. We claim that $\delta_i = i\delta_1$, $0 \leqslant i \leqslant \epsilon - 1$. This clearly holds for $i = 0, 1$. Assume $\delta_j = j\delta_1$ for all $0 \leqslant j < k \leqslant \epsilon - 1$, for some $k$. If $\delta_k > k\delta_1$, then $\delta_{k-1} = (k-1)\delta_1 < k\delta_1 < \delta_k$, contradicting the fact that, by definition of $E$, $k\delta_1 \in E$. On the other hand, if $\delta_k < k\delta_1$, then we have $(k-1)\delta_1 = \delta_{k-1} < \delta_k < k\delta_1 \Rightarrow 0 < \delta_k - \delta_{k-1} < k\delta_1 - (k-1)\delta_1 = \delta_1$, again contradicting the fact that $\delta_k - \delta_{k-1} \in E$. So we must have $\delta_i = i\delta_1$ for all $0 \leqslant i \leqslant \epsilon - 1$. But by Lemma 2, we have that $e = \epsilon$. Hence $\Gamma_W/\Gamma_V$, and therefore $G^T$, is a cyclic group, by the foregoing argument.

Since $G^T$ is a cyclic group, and it acts trivially on $\overline{W}$, $\overline{W} * G^T$ must be commutative. The result now follows from Theorem 2.

REMARKS. It appears that the inertia group, $G^T$, plays a critical role in the behaviour of $S * G$. To start with, when $\overline{W} * G^T$ is a simple ring, then $S * G$ is a Dubrovin valuation ring. The converse is false: in both Examples 1 and 2 of [**8**], $G = G^T$ and $J(S * G) \supset J(S) * G$ and hence $\overline{W} * G^T$ $(= \overline{S} * G = S * G/(J(S) * G))$ is not a simple ring.

Further, when $\overline{W} * G^T$ is $\overline{W}$-separable, then $S * G$ is semihereditary, from the proof of [**8**, Theorem 2]. When $\overline{W} * G^T$ is a simple separable $\overline{W}$-algebra, then $S*G$ is a tamely ramified and defectless Dubrovin valuation ring and conversely. If $S * G$ is a tamely ramified and defectless Dubrovin valuation ring and $G^T$ is cyclic or, more generally, when $\overline{W} * G^T$ is a separable field extension of $\overline{W}$, then the Wadsworth map is a bijection.

## 3. *The Henselization of $S * G$*

Let $B$ be an integral Dubrovin valuation ring of a central simple $F$-algebra $Q$. In [**10**], we encounter a *value function* $\mathbf{\Phi} : Q \mapsto \Gamma_B \cup \{\infty\}$ associated with $B$, which is a surjection, and has the following defining properties: for all $x, y \in Q$, we have

(i) $\mathbf{\Phi}(x) = \infty$ if and only if $x = 0$,

(ii) $\mathbf{\Phi}(x + y) \geqslant \min\{\mathbf{\Phi}(x), \mathbf{\Phi}(y)\}$,

(iii) $\mathbf{\Phi}(xy) \geqslant \mathbf{\Phi}(x) + \mathbf{\Phi}(y)$,

(iv) $B = \{x \in Q \mid \mathbf{\Phi}(x) \geqslant 0\}$ and $J(B) = \{x \in Q \mid \mathbf{\Phi}(x) > 0\}$,

(v) $\mathbf{\Phi}(Q) = \mathbf{\Phi}(\mathrm{st}(\mathbf{\Phi})) \cup \{\infty\}$, where $\mathrm{st}(\mathbf{\Phi}) = \{x \in U(Q) \mid \mathbf{\Phi}(x^{-1}) = -\mathbf{\Phi}(x)\}$.

Let $(K_h, W_h)$ be a Henselization of $(K, W)$ (see [**2**, §17] for definition). Their value groups are the same, that is, $\Gamma_{W_h} = \Gamma_W$. We let $\boldsymbol{v}$ be a valuation on $K_h$ corresponding to $W_h$. Let $(F_h, V_h)$ be the unique Henselization of $(F, V)$ contained in $(K_h, W_h)$ [**2**, Theorem 17.11]. By [**2**, 17.16 & Theorem 17.11], we see that $K_h = KF_h$. Note that $W \cap (K \cap F_h)$ is indecomposed in $K$, since $V_h$ is indecomposed in $K_h$. Hence, by [**2**, Theorem 15.7], we have $F^Z \subseteq K \cap F_h$, where $F^Z$ is the decomposition field of $W$ over $F$. But since $K/F$ is a finite Galois extension, [**2**, Theorem 17.7] implies $[K : F] = [K_h : F_h]g$, hence $[K \cap F_h : F] = g$, and so we must have $F^Z = K \cap F_h$. The Galois extension $K_h/F_h$ has therefore group $G^Z$. Also, a routine argument verifies that $(K, W)$ is tamely ramified and defectless over $(F^Z, W \cap F^Z)$ precisely when $(K_h, W_h)$ is tamely ramified and defectless over $(F_h, V_h)$.

Any $\sigma \in G$ can be considered as a ring automorphism on $K \otimes_F F_h$ via the action $\sigma(k \otimes u) = \sigma(k) \otimes u$, for $k \in K, u \in F_h$. Also, if $f$ is the 2-cocycle, then $f(\sigma, \tau)$ can be identified with $f(\sigma, \tau) \otimes 1 \in U(S \otimes_V V_h)$, for all $\sigma, \tau \in G$. The restriction of $\sigma$ to $S \otimes_V V_h$ is again an automorphism. Therefore there is a canonical $F_h$-algebra isomorphism from $(K * G) \otimes_F F_h$ to $(K \otimes_F F_h) * G$ mapping $kx_\sigma \otimes u$ to $(k \otimes u)x_\sigma$, which restricts to an isomorphism between $(S * G) \otimes_V V_h$ and $(S \otimes_V V_h) * G$.

Theorem 3. *We have*

(i) $(K * G) \otimes_F F_h \cong M_g(K_h * G^Z)$ *and* $(S * G) \otimes_V V_h \cong M_g(W_h * G^Z)$,

(ii) *the order* $S*G$ *is a Dubrovin valuation ring of* $K*G$ *if and only if* $W_h*G^Z$ *is a Dubrovin valuation ring of the central simple* $F_h$*-algebra* $K_h * G^Z$,

(iii) *if* $S*G$ *is a tamely ramified and defectless Dubrovin valuation ring of* $K*G$, *then the map* $\boldsymbol{\phi}$ *from* $K_h * G^Z$ *to* $\Gamma_W \cup \{\infty\}$ *given by* $\boldsymbol{\phi}(\sum_{\sigma \in G^Z} k_\sigma x_\sigma) = \min_{\sigma \in\ G^Z}\{\boldsymbol{v}(k_\sigma)\}$ *is a value function corresponding to the Dubrovin valuation ring* $W_h * G^Z$.

*Proof.* Since $K \otimes_F F_h \cong \oplus_{i=1}^g K_h$ and, by [**4**, Proposition 11], $S \otimes_V V_h \cong \oplus_{i=1}^g W_h$, we see that $(K * G) \otimes_F F_h \cong (K \otimes_F F_h) * G \cong (\oplus_{i=1}^g K_h) * G$ and $(S * G) \otimes_V V_h \cong (S \otimes_V V_h) * G \cong (\oplus_{i=1}^g W_h) * G$. The first statement now follows from [**14**, Proposition 2.3].

We know that $S*G$ is semihereditary if and only if $(S*G)\otimes_V V_h$ is semihereditary by [**7**, Theorem 3.4] or [**9**, Theorem 2.7]. Moreover, $\overline{S * G}$ is a simple ring if and only if $\overline{(S * G) \otimes_V V_h}$ is a simple ring by [**7**, Lemma 3.1(5)]. Therefore, the order $S * G$ is a Dubrovin valuation ring if and only if $(S * G) \otimes_V V_h$ is a Dubrovin valuation ring and, by [**1**, Theorem 7, §1], $(S * G) \otimes_V V_h$ is a Dubrovin valuation ring if and only if the Morita-equivalent ring, $W_h * G^Z$, is a Dubrovin valuation ring.

By Theorem 1, $K/F$ is tamely ramified and defectless hence, as in the proof of Proposition 1, we have that $K/F^Z$ is tamely ramified and defectless. Therefore $K_h/F_h$ is also tamely ramified and defectless, and so $J(W_h * G^Z) = J(W_h) * G^Z$ by [**8**, Theorem 2(a)] . Hence condition (iv) for a value function holds. The other conditions are easily seen to hold as well. The last statement is thus proven.

Corollary 2. *Suppose* $S * G$ *is a tamely ramified and defectless Dubrovin valuation ring of* $K * G$. *Then we have*

(i) $\Gamma_{S*G} \cong \Gamma_W$,

(ii) *the Wadsworth map is a bijection if and only if Gal*$(Z(\overline{S * G})/\overline{V}) \cong G^T$.

*Proof.* The map $\boldsymbol{\Phi}$ from $M_g(K_h * G^Z)$ to $\Gamma_W \cup \{\infty\}$ given by $\boldsymbol{\Phi}((x_{ij})) = \min_{i,j}\{\boldsymbol{\phi}(x_{ij})\}$ is a value function corresponding to the Dubrovin valuation ring $M_g(W_h * G^Z)$. Hence $\Gamma_{(S*G)\otimes_V V_h} \cong \Gamma_W$, by [**10**, Theorem 2.3]. Since $(S * G) \otimes_V V_h$

is an immediate extension of $S*G$ by [**12**, §4], the first result follows. We conclude from the proof of [**10**, Lemma 2.2], and from [**10**, Theorem 2.3], that there is a surjective homomorphism from $\mathrm{st}(S*G)$ to $\Gamma_W/\Gamma_V$ whose kernel is $U(S*G)\cdot U(F)$, and so we have $\Gamma_{S*G}/\Gamma_V \cong \Gamma_W/\Gamma_V$. Therefore, if the Wadsworth map is a bijection, then $\mathrm{Gal}(Z(\overline{S*G})/\overline{V}) \cong G^T$. On the other hand, if $\mathrm{Gal}(Z(\overline{S*G})/\overline{V}) \cong G^T$ then $\mid \Gamma_{S*G}/\Gamma_V \mid = \mid \mathrm{Gal}(Z(\overline{S*G})/\overline{V}) \mid$ by Theorem 1, and it immediately follows that the Wadsworth map is a bijection.

Remark. Recall that $W$ is said to be unramified over $F$ if $e = 1$ and $\overline{W}$ is separable over $\overline{V}$. We now can obtain a different proof of the fact that, if $S*G$ is Azumaya over $V$, then $K/F$ is unramified and defectless [**8**, Theorem 3]: since $S*G$ is Azumaya over $V$, it is a tamely ramified and defectless Dubrovin valuation ring, and hence $K/F$ is tamely ramified and defectless by Theorem 1. In addition, from [**12**, Corollary 3.4] we have that $\Gamma_{S*G} = \Gamma_V$. It follows from Corollary 2 that $\Gamma_W = \Gamma_V$, hence $e = 1$ and $K/F$ is unramified and defectless. The converse of [**8**, Theorem 3] is well known.

Just as we have been able to define tame central simple algebras, we now define inertial central simple algebras. In the commutative case, $W$ is said to be inertial over $F$ if $[K : F] = f$ and $\overline{W}$ is separable over $\overline{V}$. Therefore, given an arbitrary Dubrovin valuation ring $B$ of $Q$, we will say that $(Q, B)$ is *inertial* over $(F, V)$ if $[Q : F] = f(B|V)$ and $Z(\overline{B})$ is separable over $\overline{V}$, following the terminology used if $Q$ were a division algebra with $B$ as an invariant valuation ring.

In the case when $V$ is Henselian and $Q$ is a division ring with $B$ as an invariant valuation ring, it was shown in [**6**, §2] that $B$ is inertial over $F$ if and only if it is Azumaya over $V$. We now easily generalize this result. We hasten to point out that [**6**] is a far-reaching account of division algebras over Henselian fields.

Proposition 3. *A Dubrovin valuation ring $B$ of $Q$ is inertial over $F$ if and only if it is Azumaya over $V$.*

*Proof.* Suppose $B$ is inertial over $F$. First, assume $V$ is Henselian. Then $B \cong M_l(\Delta)$, where $\Delta$ is an invariant valuation ring of the underlying division algebra part of $Q$. Clearly, $\Delta$ is also inertial over $V$, and hence it is Azumaya over $V$, by [**6**, Lemma 2.2]. Therefore $B$ is Azumaya over $V$.

For an arbitrary $V$, first note that by [**10**, Proposition 3.2], $B$ has to be integral over $V$, as well as finitely generated. Hence $B \otimes_V V_h$ is also a Dubrovin valuation ring, since $B$ is integral. Since $B \otimes_V V_h$ is an immediate extension of $B$, $B \otimes_V V_h$ is inertial over $F_h$. Thus $B \otimes_V V_h$ is Azumaya over $V_h$, and hence $Z(\overline{B \otimes_V V_h}) = \overline{V_h} = \overline{V}$. But all this means that $B$ is a finitely generated Dubrovin valuation ring, and $\overline{B}$ is a central simple $\overline{V}$-algebra, hence $B$ is Azumaya over $V$.

Suppose $B$ is Azumaya over $V$. Then it is finitely generated and hence by [**10**, Proposition 3.2], it is defectless. But $e(B \mid V) = 1$, by [**12**, Corollary 3.4]. Thus $[Q : F] = f(B|V)$ and, since $Z(\overline{B}) = \overline{V}$, $B$ is inertial over $F$.

By [**8**, Theorem 3], we immediately have the following.

Corollary 3. *Let $K/F$ be an arbitrary Galois extension.*

(i) *The $V$-order $S * G$ is a Dubrovin valuation ring inertial over $F$ if and only if $K/F$ is unramified and defectless.*

(ii) *If $V$ is Henselian, then $S * G$ is a Dubrovin valuation ring inertial over $F$ if and only if $K/F$ is an inertial extension.*

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