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Alleviating the non-ultralocality of the $AdS_5 \times S^5$ superstring

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Abstract: We generalize the initial steps of the Faddeev-Reshetikhin procedure to the $AdS_5 \times S^5$ superstring theory. Specifically, we propose a modification of the Poisson bracket whose alleviated non-ultralocality enables to write down a lattice Poisson algebra for the Lax matrix. We then show that the dynamics of the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring can be naturally reproduced with respect to this modified Poisson bracket. This work generalizes the alleviation procedure recently developed for symmetric space σ-models. It also shows that the lattice Poisson algebra recently obtained for the $AdS_5 \times S^5$ semi-symmetric space sine-Gordon theory coincides with the one obtained by the alleviation procedure.

Keywords: AdS-CFT Correspondence, Integrable Field Theories, Sigma Models

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Contents

1 Introduction

It is well known since the seminal work of Bena, Polchinski and Roiban [1] that classical superstring theory on $\mathrm{AdS}_5 \times S^5$ admits infinitely many conserved charges. It was subsequently shown in [2] that it also has infinitely many conserved charges in involution, thereby establishing the complete classical integrability of the theory. But more importantly, the result of [2] shows that the Poisson bracket of its Lax matrix is of the general form identified in [3, 4] which is parameterized by two matrices r and s. The presence of the matrix s is responsible for the non-ultralocality of this integrable field theory and makes it very problematic to define a corresponding lattice Poisson algebra. Indeed, this serious obstacle has so far precluded the use of the standard Quantum Inverse Scattering Method [5–7] for investigating the quantum integrability of the $\mathrm{AdS}_5 \times S^5$ superstring theory. In light of this shortcoming, the continued string of impressive developments in this field over the past several years (see for instance the review [8]) relied on the implicit assumption of quantum integrability in order to make use of the methods of factorized scattering theory [9].

However, in the case of symmetric space σ-models, we have shown in [10] how the situation may be improved by alleviating their non-ultralocality. This can be seen as a generalization of the first steps of the Faddeev-Reshetikhin procedure [11], developed for the SU(2) principal chiral model, to the case of symmetric space σ-models. Indeed, the

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key advantage of the alleviation procedure is that it enables to write down a quadratic lattice Poisson algebra. The procedure can be broken down into three parts. The first part is achieved by purely algebraic means. It consists in modifying the Poisson bracket of the phase space variables of the theory in such a way that the Poisson bracket of its Lax matrix simplifies greatly. Specifically, although the latter is still non-ultralocal, the kernel of the new matrix s is independent of spectral parameters. Because of this, the Poisson bracket of the Lax matrix can be regularized as in [12] and leads to a well defined lattice Poisson algebra of the general quadratic form in [13, 14]. We shall refer to such a non-ultralocality as being mild. Note that, by construction, the modified Poisson bracket is compatible with the original one. The second part of the procedure concerns the degeneracy of the modified Poisson bracket whose Casimir functions need to be determined and fixed. Indeed, in the spirit of the Faddeev-Reshetikhin procedure, the purpose of the alleviation is to reproduce the dynamics of the σ-model with respect to the modified Poisson bracket. However, since the latter is degenerate, only a reduction of the dynamics may be reproduced. As shown in [10], this reduction coincides exactly with the Pohlmeyer reduction [15] of the symmetric space σ-model. The resulting reduced dynamics is that of the symmetric space sine-Gordon model, the Lagrangian formulation of which is given by a gauged Wess-Zumino-Witten model with an integrable potential [16]. The last part of the procedure consists in showing that the modified Poisson bracket and corresponding Hamiltonian coincide with the canonical Poisson bracket and Hamiltonian stemming from this action.

In view of the possible generalization of the results of [10] to semi-symmetric space σ-models, in [17] we already investigated directly the canonical structure of the semi-symmetric space sine-Gordon model obtained by Pohlmeyer reduction of the $\mathrm{AdS}_5 \times S^5$ superstring [18, 19]. We have shown that the corresponding non-ultralocality is only mild and have given the corresponding lattice Poisson algebra for the discretized Lax matrix. The questions addressed in the present article are the following. Firstly, does the alleviation procedure extend to the $\mathrm{AdS}_5 \times S^5$ superstring theory? Secondly, is this procedure also deeply connected with the Pohlmeyer reduction? We will find that the common answer to both questions is affirmative.

The plan of this article is the following. In section 2, we modify the Poisson bracket of superstring theory on $\mathrm{AdS}_5 \times S^5$ using a simple generalization of the technique presented in [10] to the semi-symmetric space F/G, where the Lie (super)algebras respectively associated with F and G are $\mathfrak{f} = \mathfrak{psu}(2,2|4)$ and $\mathfrak{g} = \mathfrak{so}(4,1) \oplus \mathfrak{so}(5)$. Applying the procedure of [10] simply requires identifying the quartet of algebraic data characterizing the integrability of the $\mathrm{AdS}_5 \times S^5$ superstring at the Hamiltonian level. This quartet is composed of a loop algebra, the Hamiltonian Lax matrix of [2, 20], an R-matrix and an inner product. These elements have already been identified in [21] and therefore the modified Poisson bracket is obtained by a straightforward and direct application of [10], namely by changing the inner product.

Much like in the symmetric space σ-model setting, it turns out that most of the constraints of the $\mathrm{AdS}_5 \times S^5$ superstring are Casimir functions of the modified Poisson bracket. It is therefore natural to set their values to zero. Although some of the constraints of the $\mathrm{AdS}_5 \times S^5$ superstring do not correspond to Casimirs, they may also be put to zero in a natural way. Even after setting all of the constraints to zero, the modified Poisson bracket is still degenerate. All fields take values in $\mathfrak{f}$ but describing the remaining Casimirs requires lifting one field to G. Remarkably, it turns out that these Casimirs correspond to gauge fixing conditions used in the Pohlmeyer reduction of the $\mathrm{AdS}_5 \times S^5$ superstring [18]. We thus set their values accordingly. Details are given in section 3.2. After summarizing the situation in section 3.3, we discuss the reduced theory in section 3.4. First of all, the resulting reduced equations of motion are exactly as in [18] and exhibit a $H_L \times H_R$-gauge invariance where $H_{L,R} \simeq [\mathrm{SU}(2)]^4$. However, they are not Hamiltonian with respect to the modified Poisson bracket but this is remedied by partially fixing the $H_L \times H_R$-gauge invariance to the diagonal subgroup.

We then show that these Hamiltonian equations of motion coincide with those associated with the fermionic extension of the G/H gauged WZW model with an integrable potential as given in [18]. This canonical analysis is presented in section 4.

We conclude by some remarks. There are three appendices. Appendix A contains the table of the modified Poisson bracket. Appendix B recalls some important algebraic properties which are used many times throughout this article. Appendix C contains details of the derivation of the Hamiltonian.

2 Mildly non-ultralocal Poisson bracket

The starting point of the procedure requires identifying the quartet of algebraic data which encodes the integrable structure of the $\mathrm{AdS}_5 \times S^5$ superstring at the Hamiltonian level. This has been done in [21]. For completeness we briefly recall this here and refer the reader to [10] for details regarding the present section. The first element of this quartet is the twisted loop algebra $\widehat{\mathfrak{f}}^\sigma$ defined as follows. One starts from the Lie superalgebra $\mathfrak{f} = \mathfrak{psu}(2,2|4)$. As a vector space, it admits a decomposition into a direct sum $\oplus_{n=0}^3 \mathfrak{f}^{(n)}$ of eigenspaces of a $\mathbb{Z}_4$-automorphism σ satisfying $\sigma^4 = \mathrm{id}$. We denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{f}^{(0)} = \mathfrak{so}(4,1) \oplus \mathfrak{so}(5)$ and by G the corresponding Lie group. The twisted loop algebra $\widehat{\mathfrak{f}}^\sigma$ is then the subalgebra of the loop algebra $\widehat{\mathfrak{f}} = \mathfrak{f} \otimes \mathbb{C}((\lambda))$ consisting of elements $X(\lambda) \in \widehat{\mathfrak{f}}$ which are invariant under the automorphism $\widehat{\sigma}$ of $\widehat{\mathfrak{f}}$ defined by $\widehat{\sigma}(X)(\lambda) = \sigma[X(-i\lambda)]$. The second element has been presented in [2, 20] and is the Hamiltonian Lax matrix $\mathcal{L}(\lambda)$ of the theory. Its expression in terms of the phase space variables $(A^{(i)}, \Pi^{(i)})$ reads

$$\begin{aligned}\mathcal{L}(\lambda) = A^{(0)} &+ \frac{1}{4}(\lambda^{-3} + 3\lambda)A^{(1)} + \frac{1}{2}(\lambda^{-2} + \lambda^2)A^{(2)} + \frac{1}{4}(3\lambda^{-1} + \lambda^3)A^{(3)} \\ &+ \frac{1}{2}(1-\lambda^4)\Pi^{(0)} + \frac{1}{2}(\lambda^{-3} - \lambda)\Pi^{(1)} + \frac{1}{2}(\lambda^{-2} - \lambda^2)\Pi^{(2)} + \frac{1}{2}(\lambda^{-1} - \lambda^3)\Pi^{(3)} . \qquad (2.1)\end{aligned}$$

The next element needed is the R-matrix. It is the standard one defined by $R = \pi_{\geq 0} - \pi_{<0}$ where $\pi_{\geq 0}$ and $\pi_{<0}$ are the projections of $\widehat{\mathfrak{f}}$ onto the subalgebras $\mathfrak{f} \otimes \mathbb{C}[\![\lambda]\!]$ and $\mathfrak{f} \otimes \lambda^{-1}\mathbb{C}[\![\lambda^{-1}]\!]$ respectively. The last element is given by the twist function $\varphi(\lambda) = 4\lambda^{-1}\phi(\lambda)$, where the function $\phi(\lambda)$ obtained in [21] reads, up to an irrelevant overall factor,

$$\phi(\lambda) = \frac{\lambda^4}{(1-\lambda^4)^2} .$$

The twist function uniquely specifies the twisted inner product on $\widehat{\mathfrak{f}}^\sigma$, which is defined for two elements X and Y of $\widehat{\mathfrak{f}}^\sigma$ by computing the residue

$$(X,Y)_\phi = \text{res}_{\lambda=0} d\lambda \frac{4}{\lambda}\phi(\lambda)\langle X(\lambda), Y(\lambda)\rangle \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate invariant graded symmetric bilinear form on $\mathfrak{f}$.

The last two elements of the quartet $(\widehat{\mathfrak{f}}^\sigma, \mathcal{L}, R, \varphi)$, namely the R-matrix and the twist function φ together determine the Poisson bracket of any two functions of the Lax matrix $\mathcal{L}$. For linear functions of $\mathcal{L}$ this reads

$$\{\mathcal{L}_{\underline{\mathbf{1}}}(\sigma), \mathcal{L}_{\underline{\mathbf{2}}}(\sigma')\} = [R_{\underline{\mathbf{12}}}, \mathcal{L}_{\underline{\mathbf{1}}}(\sigma)]\delta_{\sigma\sigma'} - [R^*_{\underline{\mathbf{12}}}, \mathcal{L}_{\underline{\mathbf{2}}}(\sigma)]\delta_{\sigma\sigma'} + (R_{\underline{\mathbf{12}}} + R^*_{\underline{\mathbf{12}}})\delta'_{\sigma\sigma'}\,. \tag{2.3}$$

Its non-ultralocality stems precisely from the twist function φ and the fact that R is not skew-symmetric with respect to (2.2) but instead satisfies

$$R^* = -\tilde{\varphi}^{-1} \circ R \circ \tilde{\varphi} \neq -R\,,$$

where $\tilde{\varphi}$ denotes multiplication by $\varphi(\lambda)$. Finally, as explained in [10], one can recover the Poisson brackets of the fields $(A^{(i)}, \Pi^{(i)})$ appearing in the Lax matrix (2.1) by taking adequate functions of the Lax matrix. The result is

$$\{A^{(i)}_{\underline{\mathbf{1}}}(\sigma), A^{(j)}_{\underline{\mathbf{2}}}(\sigma')\} = 0\,, \tag{2.4a}$$

$$\{A^{(i)}_{\underline{\mathbf{1}}}(\sigma), \Pi^{(j)}_{\underline{\mathbf{2}}}(\sigma')\} = \left[C^{(i\,4-i)}_{\underline{\mathbf{12}}}, A^{(i+j)}_{\underline{\mathbf{2}}}(\sigma)\right]\delta_{\sigma\sigma'} - \delta_{i+j} C^{(i\,4-i)}_{\underline{\mathbf{12}}}\partial_\sigma\delta_{\sigma\sigma'}\,, \tag{2.4b}$$

$$\{\Pi^{(i)}_{\underline{\mathbf{1}}}(\sigma), \Pi^{(j)}_{\underline{\mathbf{2}}}(\sigma')\} = \left[C^{(i\,4-i)}_{\underline{\mathbf{12}}}, \Pi^{(i+j)}_{\underline{\mathbf{2}}}(\sigma)\right]\delta_{\sigma\sigma'}\,, \tag{2.4c}$$

where the Kronecker symbol δ_{i+j} is equal to one if $i+j = 0 \,(\text{mod}\, 4)$ and vanishes otherwise. Here $C^{(i\,4-i)}_{\underline{\mathbf{12}}}$ is the projection onto $\mathfrak{f}^{(i)} \otimes \mathfrak{f}^{(4-i)}$ of the quadratic Casimir $C_{\underline{\mathbf{12}}}$.

The alleviation procedure proposed in [10] now consists in making the following simple change in the above quartet of data

$$(\widehat{\mathfrak{f}}^\sigma, \mathcal{L}, R, 4\lambda^{-1}\phi) \quad \longrightarrow \quad (\widehat{\mathfrak{f}}^\sigma, \mathcal{L}, R, 4\lambda^{-1})\,,$$

where the factors of 4 are introduced for later convenience. In particular, the new quartet has the same Lax matrix as (2.1) but a modified Poisson bracket. The latter is still non-ultralocal as a result of the R-matrix still not being skew-symmetric

$$R^* = -\tilde{\lambda} \circ R \circ \tilde{\lambda}^{-1} \neq -R\,,$$

where $\tilde{\lambda}$ denotes multiplication by λ. However, this non-ultralocality is mild in the sense that the symmetric part $s = \frac{1}{2}(R+R^*)$ of R is a projection onto the constant part $\mathfrak{f}^{(0)}$ of the twisted loop algebra $\widehat{\mathfrak{f}}^\sigma$ [10]. The Poisson brackets between the various phase space fields may be obtained from the new data along the lines of [10]. The resulting non-vanishing Poisson brackets are given in appendix A.

The advantage of having a mild non-ultralocality is that the corresponding Poisson bracket (2.3) can be obtained as the continuum limit of a lattice Poisson bracket constructed

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as follows. An extra piece of data, disappearing in the continuum limit, is a solution α of the modified classical Yang-Baxter equation on $\mathfrak{g}$. In terms of this, the lattice Poisson bracket reads

$$\{\mathcal{L}^n_{\underline{\mathbf{1}}}, \mathcal{L}^m_{\underline{\mathbf{2}}}\} = a_{\underline{\mathbf{12}}} \mathcal{L}^n_{\underline{\mathbf{1}}} \mathcal{L}^m_{\underline{\mathbf{2}}} \delta_{mn} - \mathcal{L}^n_{\underline{\mathbf{1}}} \mathcal{L}^m_{\underline{\mathbf{2}}} d_{\underline{\mathbf{12}}} \delta_{mn} + \mathcal{L}^n_{\underline{\mathbf{1}}} b_{\underline{\mathbf{12}}} \mathcal{L}^m_{\underline{\mathbf{2}}} \delta_{m+1,n} - \mathcal{L}^m_{\underline{\mathbf{2}}} c_{\underline{\mathbf{12}}} \mathcal{L}^n_{\underline{\mathbf{1}}} \delta_{m,n+1},$$

where the matrices a, b, c and d satisfy the conditions of [13, 14] and are given explicitly by [12]

$$a_{\underline{\mathbf{12}}} = (r+\alpha)_{\underline{\mathbf{12}}}, \qquad b_{\underline{\mathbf{12}}} = (-s-\alpha)_{\underline{\mathbf{12}}}, \qquad c_{\underline{\mathbf{12}}} = (-s+\alpha)_{\underline{\mathbf{12}}}, \qquad d_{\underline{\mathbf{12}}} = (r-\alpha)_{\underline{\mathbf{12}}},$$

with $r = \frac{1}{2}(R - R^*)$ the skew-symmetric part of the R-matrix.

3 Modified Poisson bracket and Pohlmeyer reduction

Having defined a new Poisson bracket on the phase space of the $\mathrm{AdS}_5 \times S^5$ superstring, the aim of the present section will be to describe the original dynamics with respect to it. After recalling the Hamiltonian dynamics of the $\mathrm{AdS}_5 \times S^5$ superstring with respect to its original Poisson bracket (2.4), we will show that the modified Poisson bracket is degenerate so that it can only be used to reproduce a reduction of the original dynamics. It will turn out that the Pohlmeyer reduction is essentially forced upon us by the specific form of the Casimirs.

3.1 Original dynamics

To recall the Hamiltonian dynamics of the $\mathrm{AdS}_5 \times S^5$ superstring we closely follow the reference [20]. The phase space is parameterized by the fields $(A^{(i)}, \Pi^{(i)})$ and the Hamiltonian is given by a linear combination of all the first-class constraints, namely

$$H = \int d\sigma \left[\rho^{++}\mathcal{T}_{++} + \rho^{--}\mathcal{T}_{--} - \mathrm{Str}(k^{(3)}\mathcal{K}^{(1)}) - \mathrm{Str}(k^{(1)}\mathcal{K}^{(3)}) - \mathrm{Str}\left((A^{(0)}+\ell)\mathcal{C}^{(0)}\right)\right], \tag{3.1}$$

where the notation is as follows. We have defined

$$\mathcal{T}_{++} = T_{++} - \mathrm{Str}\left(A^{(1)}\mathcal{C}^{(3)}\right), \qquad T_{\pm\pm} = \mathrm{Str}\left(A^{(2)}_\pm A^{(2)}_\pm\right),$$
$$\mathcal{T}_{--} = T_{--} + \mathrm{Str}\left(A^{(3)}\mathcal{C}^{(1)}\right), \qquad A^{(2)}_\pm = \frac{1}{2}\left(\Pi^{(2)} \mp A^{(2)}\right).$$

The full set of constraints are

$$\mathcal{C}^{(0)} \equiv \Pi^{(0)} \approx 0, \tag{3.2a}$$
$$\mathcal{C}^{(1)} \equiv \frac{1}{2}A^{(1)} + \Pi^{(1)} \approx 0, \tag{3.2b}$$
$$\mathcal{C}^{(3)} \equiv -\frac{1}{2}A^{(3)} + \Pi^{(3)} \approx 0, \tag{3.2c}$$
$$T_{\pm\pm} \approx 0. \tag{3.2d}$$

The constraint $\mathcal{C}^{(0)}$ is associated with the G-gauge invariance while (3.2d) are the Virasoro constraints. All these constraints are first-class while the other constraints $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(3)}$ are partly first-class and second-class. One can extract the following first-class constraints

$$\mathcal{K}^{(1)} = 2i\big[A_-^{(2)}, \mathcal{C}^{(1)}\big]_+ \qquad \text{and} \qquad \mathcal{K}^{(3)} = 2i\big[A_+^{(2)}, \mathcal{C}^{(3)}\big]_+ ,$$

which generate κ-symmetry transformations. Finally, the arbitrary functions ℓ, ρ^{++}, ρ^{--}, $k^{(1)}$ and $k^{(3)}$ are Lagrange multipliers associated with the first-class constraints.

The equations of motion for the variables $(A^{(i)}, \Pi^{(i)})$ following from the Hamiltonian (3.1) with respect to the Poisson bracket (2.4) are, up to terms proportional to the constraints,

$$\begin{aligned}
\partial_\tau A^{(0)} - \partial_\sigma(A^{(0)} + \ell) - [A^{(0)} + \ell, A^{(0)}] &= (\rho^{++} + \rho^{--})\Big(\frac{1}{2}[A^{(2)}, \Pi^{(2)}] + [A^{(1)}, A^{(3)}]\Big) \\
&\quad - [A^{(1)}, Q^{(3)}] - [A^{(3)}, Q^{(1)}] , && (3.3a) \\
D_\tau A^{(1)} - D_\sigma\big(\rho^{++} A^{(1)} + Q^{(1)}\big) &= (\rho^{++} + \rho^{--})[A^{(3)}, A_+^{(2)}] - [A^{(2)}, Q^{(3)}] , && (3.3b) \\
D_\tau A_+^{(2)} - D_\sigma\big(\rho^{++} A_+^{(2)}\big) &= [A^{(1)}, Q^{(1)}] , && (3.3c) \\
D_\tau A_-^{(2)} + D_\sigma\big(\rho^{--} A_-^{(2)}\big) &= -[A^{(3)}, Q^{(3)}] , && (3.3d) \\
D_\tau A^{(3)} + D_\sigma\big(\rho^{--} A^{(3)} - Q^{(3)}\big) &= (\rho^{++} + \rho^{--})[A^{(1)}, A_-^{(2)}] - [A^{(2)}, Q^{(1)}] , && (3.3e)
\end{aligned}$$

where the covariant derivatives are defined as

$$D_\tau = \partial_\tau - [A^{(0)} + \ell, \,] \qquad \text{and} \qquad D_\sigma = \partial_\sigma - [A^{(0)}, \,] .$$

Here we have also introduced the fields[1]

$$Q^{(1)} = i[A_+^{(2)}, k^{(1)}]_+ \qquad \text{and} \qquad Q^{(3)} = i[A_-^{(2)}, k^{(3)}]_+ . \tag{3.4}$$

The remaining field equations may be deduced from equations (3.3) by using the constraints (3.2b) and (3.2c). The equations of motion (3.3) are of course invariant under the gauge transformations, which is reflected by their dependence on arbitrary functions of σ and τ.

3.2 Casimirs of the modified Poisson bracket

In order to determine whether the dynamics (3.3) can be reproduced in terms of the modified Poisson bracket given in appendix A, we first need to identify the Casimirs of the latter. Indeed, it will only be possible to reproduce a reduction of the original dynamics where these Casimirs have been set to constants.

To begin with, $\mathcal{C}^{(0)}$ is an obvious Casimir of the modified Poisson bracket. Since it corresponds to a constraint of the superstring, the value of this Casimir is set to zero. It

[1] The fields $Q^{(1)}$ and $Q^{(3)}$ correspond to the fields Q_{1-} and Q_{2+} appearing in the Lagrangian formulation [18]. A consequence of their definitions (3.4) and of the Virasoro constraints (3.2d) is that they are solutions of the algebraic equations $[A_+^{(2)}, Q^{(1)}] = 0$ and $[A_-^{(2)}, Q^{(3)}] = 0$. See also the related analysis in [20].

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then follows that $\mathcal{C}^{(3)}$ is also a Casimir whose value we similarly set to zero. One then finds that $A_+^{(2)}$ becomes a Casimir. This quantity is therefore fixed to a constant by imposing

$$2A_+^{(2)} = \mu_+ T$$

where $\mu_+ \in \mathbb{R}$ is a constant and T is a fixed element of $\mathfrak{f}^{(2)}$. But in order for the Virasoro constraint $\mathrm{Str}(A_+^{(2)} A_+^{(2)}) = 0$ to be satisfied, T has to be taken such that $\mathrm{Str}\, T^2 = 0$. We shall choose[2] the same T as in [18]. Its definition and the fact that it induces a $\mathbb{Z}_2$-grading of $\mathfrak{f}$, denoted $\mathfrak{f}^{[0]} \oplus \mathfrak{f}^{[1]}$, are recalled in appendix B, along with the definitions of some other matrices used below.

Now consider the two remaining constraints of the $\mathrm{AdS}_5 \times S^5$ superstring, namely $\mathcal{C}^{(1)}$ and $T_{--} = \mathrm{Str}(A_-^{(2)} A_-^{(2)})$. Contrary to the previous constraints, these are not Casimirs of the modified Poisson bracket. However, their only non vanishing Poisson brackets are

$$\{\mathcal{C}^{(1)}_{\underline{\mathbf{1}}}(\sigma), A^{(0)}_{\underline{\mathbf{2}}}(\sigma')\}' = -\frac{1}{2}[C^{(13)}_{\underline{\mathbf{12}}}, \mathcal{C}^{(1)}_{\underline{\mathbf{2}}}(\sigma)]\delta_{\sigma\sigma'}\,, \tag{3.5a}$$

$$\{T_{--}(\sigma), A^{(3)}(\sigma')\}' = -\frac{1}{2}[A_-^{(2)}(\sigma), \mathcal{C}^{(1)}(\sigma)]\delta_{\sigma\sigma'}\,. \tag{3.5b}$$

It follows from (3.5a) that any Hamiltonian function will preserve the constraint $\mathcal{C}^{(1)} = 0$ with respect to the modified Poisson bracket. Another way to phrase this is to note that the set of functionals on phase space which vanish when $\mathcal{C}^{(1)}$ does, forms a Poisson ideal. We may therefore restrict ourselves to the Poisson subspace defined by $\mathcal{C}^{(1)} = 0$. In practice, this also means that one can take $A^{(1)}$ as the only dynamical field belonging to $\mathfrak{f}^{(1)}$ and identify $\Pi^{(1)}$ with $-\frac{1}{2}A^{(1)}$ through equation (3.2b). Furthermore, equation (3.5b) shows that T_{--} is a Casimir of the modified Poisson bracket on the subspace defined by $\mathcal{C}^{(1)} = 0$, whose value we set to zero. Finally, one introduces a field $g(\sigma,\tau)$ taking values in G and a function $\mu_-(\sigma,\tau)$ through

$$2A_-^{(2)} = \mu_- g^{-1} T g\,. \tag{3.6}$$

Specifically, the polar decomposition theorem [18, 22] allows us to write $2A_-^{(2)} = g^{-1}(\mu_- T + \widetilde{\mu}_- \widetilde{T})g$. The vanishing of the Casimir T_{--} then requires that either $\mu_- = 0$ or $\widetilde{\mu}_- = 0$. However, $\widetilde{T}$ being conjugate to T by an element of G (see appendix B) equation (3.6) can be taken without loss of generality. We are then led to consider the quantity $\mathrm{Str}(A_-^{(2)} A_-^{(2)} W) = -\frac{1}{2}\mu_-^2$. It is easily checked that, on the subspace just defined this quantity is a Casimir function of the modified bracket and should be put to a constant. Therefore μ_- is a constant and the situation is thus as in [10].

However, this is not the end of the story as there exist two more Casimirs. Indeed, consider the projection $A^{(1)[0]}$ of $A^{(1)}$ to the subalgebra $\mathfrak{f}^{[0]}$. We have

$$\{A^{(1)[0]}_{\underline{\mathbf{1}}}(\sigma), A^{(1)}_{\underline{\mathbf{2}}}(\sigma')\}' = -\frac{1}{2}[C^{(13)[00]}_{\underline{\mathbf{12}}}, A^{(2)}_{+\underline{\mathbf{2}}}]\delta_{\sigma\sigma'} = -\frac{1}{4}\mu_+[C^{(13)[00]}_{\underline{\mathbf{12}}}, T_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'} = 0\,,$$

as any element of $\mathfrak{f}^{[0]}$ commutes with T (see appendix B), and where $C^{(13)[00]}_{\underline{\mathbf{12}}}$ denotes the projection onto $\mathfrak{f}^{(1)[0]} \otimes \mathfrak{f}^{(3)[0]}$ of $C^{(13)}_{\underline{\mathbf{12}}}$. All the other Poisson brackets with $A^{(1)[0]}$ either

[2] As pointed out by one of the referees, another choice of T, living in $\mathfrak{su}(2,2)$, is possible but it will not be studied here.

vanish as well or are proportional to $\mathcal{C}^{(1)}$, which in practice has the same consequence. In other words $A^{(1)[0]}$ is a Casimir. This is a nice result as it corresponds to one of the gauge fixing conditions for the κ-symmetry considered in [18]. The other condition will also be encountered shortly. In order to describe it explicitly we first need to lift the Poisson brackets of $A_-^{(2)}$ to the field g. This lifting is done as follows. The only non-vanishing Poisson bracket of $A_-^{(2)}$ is

$$\{A^{(2)}_{-\underline{\mathbf{1}}}(\sigma), A^{(0)}_{\underline{\mathbf{2}}}(\sigma')\}' = -\frac{1}{2}[C^{(22)}_{\underline{\mathbf{12}}}, A^{(2)}_{-\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,.$$

This may be lifted using (3.6) to a Poisson bracket for g which reads

$$\{g_{\underline{\mathbf{1}}}(\sigma), A^{(0)}_{\underline{\mathbf{2}}}(\sigma')\}' = -\frac{1}{2}g_{\underline{\mathbf{1}}}(\sigma)C^{(00)}_{\underline{\mathbf{12}}}\delta_{\sigma\sigma'}\,,$$

with all the other Poisson brackets of g vanishing. Next, the only non-vanishing Poisson brackets of $A^{(3)}$ are

$$\{A^{(3)}_{\underline{\mathbf{1}}}(\sigma), A^{(0)}_{\underline{\mathbf{2}}}(\sigma')\}' = -\frac{1}{2}[C^{(31)}_{\underline{\mathbf{12}}}, A^{(3)}_{\underline{\mathbf{2}}}(\sigma)]\delta_{\sigma\sigma'}\,,$$
$$\{A^{(3)}_{\underline{\mathbf{1}}}(\sigma), A^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}' = -\frac{1}{2}[C^{(31)}_{\underline{\mathbf{12}}}, A^{(2)}_{-\underline{\mathbf{2}}}(\sigma)]\delta_{\sigma\sigma'}\,.$$

Considering the combination $gA^{(3)}g^{-1}$, a short computation leads to

$$\{(gA^{(3)}g^{-1})_{\underline{\mathbf{1}}}(\sigma), A^{(0)}_{\underline{\mathbf{2}}}(\sigma')\}' = 0\,, \tag{3.7a}$$

$$\{(gA^{(3)}g^{-1})_{\underline{\mathbf{1}}}(\sigma), (gA^{(3)}g^{-1})_{\underline{\mathbf{2}}}(\sigma')\}' = -\frac{1}{2}[C^{(31)}_{\underline{\mathbf{12}}}, (g_{\underline{\mathbf{2}}}A^{(2)}_{-\underline{\mathbf{2}}}g^{-1}_{\underline{\mathbf{2}}})(\sigma)]\delta_{\sigma\sigma'} = -\frac{1}{4}\mu_-[C^{(31)}_{\underline{\mathbf{12}}}, T_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,. \tag{3.7b}$$

As in the case of $A^{(1)[0]}$ above this shows that $(gA^{(3)}g^{-1})^{[0]}$ is a Casimir, which exactly corresponds to the other gauge fixing condition for κ-symmetry considered in [18].

3.3 Pohlmeyer reduction

Let us summarize the situation so far. We have shown that the modified Poisson bracket given in appendix A can be consistently restricted to the constraint surface of the $\mathrm{AdS}_5\times S^5$ superstring defined by (3.2). But this restriction is still degenerate and the form of its Casimirs naturally led us to impose the following further conditions

$$2A^{(2)}_+ = \mu_+ T \qquad \text{and} \qquad 2A^{(2)}_- = \mu_- g^{-1}Tg \tag{3.8a}$$

along with

$$A^{(1)[0]} = 0 \qquad \text{and} \qquad (gA^{(3)}g^{-1})^{[0]} = 0\,. \tag{3.8b}$$

These are exactly the gauge fixing conditions imposed in the Pohlmeyer reduction of the $\mathrm{AdS}_5\times S^5$ superstring [18]. In other words, the modified Poisson bracket naturally restricts to the reduced phase space of the Pohlmeyer reduction of the $\mathrm{AdS}_5\times S^5$ superstring. It is easy to check that the gauge fixing conditions (3.8) are preserved under the dynamics if

$$\rho^{++} = 1\,, \qquad \rho^{--} = 1\,, \qquad Q^{(1)} = 0\,, \qquad Q^{(3)} = 0\,, \qquad \ell(\sigma,\tau) \in \mathfrak{h}\,. \tag{3.9}$$

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These equations are also partial gauge fixing conditions imposed in [18], to which we refer the reader for further detail.

The remaining degrees of freedom are g, $A^{(0)}$, $A^{(1)[1]}$ and $(gA^{(3)}g^{-1})^{[1]}$ and their non-vanishing Poisson brackets read

$$\{g_{\underline{1}}(\sigma), A^{(0)}_{\underline{2}}(\sigma')\}' = -\frac{1}{2} g_{\underline{1}}(\sigma) C^{(00)}_{\underline{12}} \delta_{\sigma\sigma'}\,, \tag{3.10a}$$

$$\{A^{(0)}_{\underline{1}}(\sigma), A^{(0)}_{\underline{2}}(\sigma')\}' = -\frac{1}{2}[C^{(00)}_{\underline{12}}, A^{(0)}_{\underline{2}}(\sigma)]\delta_{\sigma\sigma'} + \frac{1}{2} C^{(00)}_{\underline{12}} \partial_\sigma \delta_{\sigma\sigma'}\,, \tag{3.10b}$$

$$\{A^{(1)[1]}_{\underline{1}}(\sigma), A^{(1)[1]}_{\underline{2}}(\sigma')\}' = -\frac{1}{4}\mu_+ [C^{(13)}_{\underline{12}}, T_{\underline{2}}]\delta_{\sigma\sigma'}\,, \tag{3.10c}$$

$$\{(gA^{(3)}g^{-1})^{[1]}_{\underline{1}}(\sigma), (gA^{(3)}g^{-1})^{[1]}_{\underline{2}}(\sigma')\}' = -\frac{1}{4}\mu_- [C^{(31)}_{\underline{12}}, T_{\underline{2}}]\delta_{\sigma\sigma'}\,. \tag{3.10d}$$

3.4 Reduced equations of motion

Next, we implement the reduction conditions (3.8) together with (3.9) on the equations of motion (3.3) in turn. For the equation (3.3a) of $A^{(0)}$ we find

$$\partial_- A^{(0)} - \partial_\sigma \ell - [\ell, A^{(0)}] = \frac{1}{2}\mu_+\mu_- [g^{-1}Tg, T] + 2[A^{(1)}, A^{(3)}]\,, \tag{3.11}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Equation (3.3d) can be lifted to an equation of motion for g, exactly as in the bosonic case, to give

$$A^{(0)} = \frac{1}{2}\big(-g^{-1}\partial_+ g - \ell + g^{-1}\widetilde{\ell} g\big)\,, \tag{3.12}$$

where the arbitrary function $\widetilde{\ell}$ takes values in $\mathfrak{h}$. On the odd graded part of $\mathfrak{f}$, the equation (3.3b) for $A^{(1)}$ yields

$$\partial_- A^{(1)} = [\ell, A^{(1)}] + \mu_+ [A^{(3)}, T]\,. \tag{3.13}$$

As for the equation of motion (3.3e) of $A^{(3)}$, using (3.12) it may be rewritten as

$$\partial_+(gA^{(3)}g^{-1}) = [\widetilde{\ell}, gA^{(3)}g^{-1}] + \mu_- [gA^{(1)}g^{-1}, T]\,. \tag{3.14}$$

Note that the projections of equations (3.13) and (3.14) to $\mathfrak{f}^{[0]}$ are both trivial, therefore we shall implicitly assume their restrictions to $\mathfrak{f}^{[1]}$ from now on.

The equations of motion (3.11)–(3.14) admit right and left gauge invariances. The right invariance corresponds to those $\mathfrak{g}$-gauge transformations that preserve the reduction conditions. They act as

$$\delta A^{(0)} = \partial_\sigma \alpha_R + [\alpha_R, A^{(0)}]\,, \qquad \delta A^{(1)} = [\alpha_R, A^{(1)}]\,, \qquad \delta A^{(3)} = [\alpha_R, A^{(3)}]\,, \tag{3.15a}$$

$$\delta g = -g\alpha_R\,, \qquad \delta \ell = \partial_- \alpha_R + [\alpha_R, \ell]\,, \tag{3.15b}$$

where $\alpha_R(\sigma,\tau) \in \mathfrak{h}_R$. There is also a left invariance which appears as a result of the lifting to G. It acts only on the fields g and $\widetilde{\ell}$ as

$$\delta g = \alpha_L g \qquad \text{and} \qquad \delta\widetilde{\ell} = \partial_+ \alpha_L + [\alpha_L, \widetilde{\ell}]\,, \tag{3.16}$$

with $\alpha_L(\sigma,\tau) \in \mathfrak{h}_L$.

To obtain equations of motion that are Hamiltonian, one needs to partially gauge fix this $H_L \times H_R$-gauge invariance to the diagonal subgroup. To do this, we introduce

$$J = \partial_\sigma g g^{-1} + g A^{(0)} g^{-1}. \tag{3.17}$$

A short computation shows that J satisfies the following equation of motion

$$\partial_+ J = \partial_\sigma \widetilde{\ell} + [\widetilde{\ell}, J] + \frac{1}{2}\mu_+\mu_-[T, gTg^{-1}] + 2g[A^{(1)}, A^{(3)}]g^{-1},$$

and has the following Poisson brackets

$$\begin{aligned}
\{J_{\underline{\mathbf{1}}}(\sigma), g_{\underline{\mathbf{2}}}(\sigma')\}' &= \frac{1}{2} C^{(00)}_{\underline{\mathbf{12}}} g_{\underline{\mathbf{2}}}(\sigma)\delta_{\sigma\sigma'}\,, \\
\{J_{\underline{\mathbf{1}}}(\sigma), A^{(0)}_{\underline{\mathbf{2}}}(\sigma')\}' &= 0\,, \\
\{J_{\underline{\mathbf{1}}}(\sigma), A^{(1)}_{\underline{\mathbf{2}}}(\sigma')\}' &= 0\,, \\
\{J_{\underline{\mathbf{1}}}(\sigma), (gA^{(3)}g^{-1})_{\underline{\mathbf{2}}}(\sigma')\}' &= 0\,.
\end{aligned}$$

With the help of this field J we may now write the generator of the gauge transformations (3.15) and (3.16) explicitly as follows

$$2\int d\sigma\, \mathrm{Str}\left[\alpha_L\left(J + \frac{1}{\mu_-}[gA^{(3)}g^{-1}, [T, gA^{(3)}g^{-1}]]\right) - \left(A^{(0)} - \frac{1}{\mu_+}[A^{(1)}, [T, A^{(1)}]]\right)\alpha_R\right].$$

We therefore fix the part of the gauge invariance with parameters related through $\alpha_L = -\alpha_R$ by imposing the partial gauge fixing condition

$$J^{[0]} + \frac{1}{\mu_-}[gA^{(3)}g^{-1}, [T, gA^{(3)}g^{-1}]] = A^{(0)[0]} - \frac{1}{\mu_+}[A^{(1)}, [T, A^{(1)}]]\,. \tag{3.18}$$

The residual gauge transformations that preserve this condition are the diagonal transformations for which $\alpha_L = \alpha_R$. Moreover, condition (3.18) is preserved by the dynamics (3.11)–(3.14) provided the arbitrary functions ℓ and $\widetilde{\ell}$ are restricted as

$$\ell - \widetilde{\ell} = -A^{(0)[0]} - J^{[0]} + \frac{1}{\mu_+}[A^{(1)}, [T, A^{(1)}]] - \frac{1}{\mu_-}[gA^{(3)}g^{-1}, [T, gA^{(3)}g^{-1}]]\,. \tag{3.19}$$

Equations (3.18) and (3.19) can be rearranged into the equivalent set of equations

$$\ell = \frac{1}{2}(\ell + \widetilde{\ell}) - A^{(0)[0]} + \frac{1}{\mu_+}[A^{(1)}, [T, A^{(1)}]]\,, \tag{3.20a}$$

$$\widetilde{\ell} = \frac{1}{2}(\ell + \widetilde{\ell}) + J^{[0]} + \frac{1}{\mu_-}[gA^{(3)}g^{-1}, [T, gA^{(3)}g^{-1}]]\,. \tag{3.20b}$$

In other words, after imposing the condition (3.18), the equations of motion no longer depend on the pair of arbitrary functions ℓ and $\widetilde{\ell}$ but only on their sum $\ell + \widetilde{\ell}$. This is a reflection of the fact that the equations of motion are invariant only under the diagonal gauge transformations.

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To implement the partial gauge fixing conditions (3.18) at the level of the equations of motion we simply need to substitute the relations (3.20) for ℓ and $\widetilde{\ell}$. The equations of motion (3.13) and (3.14) for the fermionic fields respectively yield

$$\partial_- A^{(1)} = -\mu_+[T, A^{(3)}] + \left[\frac{1}{2}(\ell + \widetilde{\ell}) - A^{(0)[0]} + \frac{1}{\mu_+}[A^{(1)}, [T, A^{(1)}]], A^{(1)}\right], \quad \text{(3.21a)}$$

$$\partial_+(gA^{(3)}g^{-1}) = -\mu_-[T, gA^{(1)}g^{-1}] + \left[\frac{1}{2}(\ell + \widetilde{\ell}) + J^{[0]} + \frac{1}{\mu_-}[gA^{(3)}g^{-1}, [T, gA^{(3)}g^{-1}]], gA^{(3)}g^{-1}\right]. \quad \text{(3.21b)}$$

For the equation of g we first combine equations (3.12) and (3.17) to get

$$\partial_\tau g g^{-1} + J + g(A^{(0)} + \ell)g^{-1} = \widetilde{\ell}.$$

Then substituting both expressions (3.20b) and (3.20a) into this equation we end up with

$$\partial_\tau g = -gA^{(0)[1]} - J^{[1]}g - g\left(\frac{1}{2}(\ell + \widetilde{\ell}) + \frac{1}{\mu_+}[A^{(1)}, [T, A^{(1)}]]\right) + \left(\frac{1}{2}(\ell + \widetilde{\ell}) + \frac{1}{\mu_-}[gA^{(3)}g^{-1}, [T, gA^{(3)}g^{-1}]]\right)g \quad \text{(3.22)}$$

Finally, the equation of motion (3.11) can be rewritten as

$$\partial_\tau A^{(0)} = \partial_\sigma A^{(0)[1]} + \partial_\sigma\left(\frac{1}{2}(\ell + \widetilde{\ell}) + \frac{1}{\mu_+}[A^{(1)}, [T, A^{(1)}]]\right) + \frac{1}{2}\mu_+\mu_-[g^{-1}Tg, T] + \left[\frac{1}{2}(\ell + \widetilde{\ell}) - A^{(0)[0]} + \frac{1}{\mu_+}[A^{(1)}, [T, A^{(1)}]], A^{(0)}\right] + 2[A^{(1)}, A^{(3)}] \quad \text{(3.23)}$$

where again we have made use of (3.20a).

4 Link with semi-symmetric space sine-Gordon theory

The goal of this section is to establish that the Poisson brackets (3.10) and the constraint (3.18) coincide with the result of the canonical analysis of the $\mathrm{AdS}_5 \times S^5$ semi-symmetric space sine-Gordon theory, defined as a fermionic extension of the G/H gauged WZW with a potential term [18]. In order to make the identification complete, we also indicate the corresponding Hamiltonian which generates the equations of motion (3.21), (3.22) and (3.23).

We shall perform the canonical analysis of the action defined in [18] which reads

$$\begin{aligned}\mathcal{S} = &\frac{1}{2}\int d\tau d\sigma\, \mathrm{Str}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \frac{1}{3}\int d\tau d\sigma d\xi\, \epsilon^{\alpha\beta\gamma}\, \mathrm{Str}(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\partial_\gamma g)\\ &- \int d\tau d\sigma\, \mathrm{Str}(B_+\partial_- g g^{-1} - B_- g^{-1}\partial_+ g + g^{-1}B_+ g B_- - B_+ B_-)\\ &+ \frac{1}{2}\int d\tau d\sigma\, \mathrm{Str}\left(\psi^{(3)}[T, D_+\psi^{(3)}] + \psi^{(1)}[T, D_-\psi^{(1)}]\right)\\ &+ \int d\tau d\sigma \big(\mu^2\, \mathrm{Str}(g^{-1}TgT) + \mu\, \mathrm{Str}(g^{-1}\psi^{(3)} g\psi^{(1)})\big)\,, \end{aligned} \quad \text{(4.1)}$$

where the notation is as follows. Firstly, we take $\epsilon^{\tau\sigma\xi} = 1$. The fields g, $\psi^{(1)}$ and $\psi^{(3)}$ respectively take values in G, $\mathfrak{f}^{(1)[1]}$ and $\mathfrak{f}^{(3)[1]}$, while $B_\pm = B_0 \pm B_1$ are gauge fields taking values in $\mathfrak{h}$. Finally, the covariant derivatives are defined by $D_\pm = \partial_\pm - [B_\pm,]$. We recall the start of the canonical analysis from the results of [17]. The phase space is parametrized by the fields $(g, \mathcal{J}_L, \psi^{(1)}, \psi^{(3)})$ where $\mathcal{J}_L$ takes values in $\mathfrak{g}$, and the non-vanishing Poisson brackets are

$$\{g_{\underline{\mathbf{1}}}(\sigma), \mathcal{J}_{L\underline{\mathbf{2}}}(\sigma')\}' = g_{\underline{\mathbf{1}}} C^{(00)}_{\underline{\mathbf{12}}} \delta_{\sigma\sigma'}\,, \tag{4.2a}$$

$$\{\mathcal{J}_{L\underline{\mathbf{1}}}(\sigma), \mathcal{J}_{L\underline{\mathbf{2}}}(\sigma')\}' = [C^{(00)}_{\underline{\mathbf{12}}}, \mathcal{J}_{L\underline{\mathbf{2}}}]\delta_{\sigma\sigma'} + 2C^{(00)}_{\underline{\mathbf{12}}} \partial_\sigma \delta_{\sigma\sigma'}\,, \tag{4.2b}$$

$$\{\psi^{(1)}_{\underline{\mathbf{1}}}(\sigma), \psi^{(1)}_{\underline{\mathbf{2}}}(\sigma')\}' = [T_{\underline{\mathbf{2}}}, C^{(13)}_{\underline{\mathbf{12}}}]\delta_{\sigma\sigma'}\,, \tag{4.2c}$$

$$\{\psi^{(3)}_{\underline{\mathbf{1}}}(\sigma), \psi^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}' = [T_{\underline{\mathbf{2}}}, C^{(31)}_{\underline{\mathbf{12}}}]\delta_{\sigma\sigma'}\,, \tag{4.2d}$$

together with the gauge fields (B_0, B_1) and their conjuguate momenta[3] (P_0, P_1). There are four constraints,

$$\chi_1 = P_0 \qquad \text{and} \qquad \chi_2 = P_1\,, \tag{4.3a}$$

$$\chi_3 = \mathcal{J}_R^{[0]} + 2B_1 - \frac{1}{2}\big[\psi^{(3)}, [T, \psi^{(3)}]\big]\,, \tag{4.3b}$$

$$\chi_4 = \mathcal{J}_L^{[0]} + 2B_1 + \frac{1}{2}\big[\psi^{(1)}, [T, \psi^{(1)}]\big] \tag{4.3c}$$

where we have defined

$$\mathcal{J}_R = -2\partial_\sigma g g^{-1} + g\mathcal{J}_L g^{-1}.$$

To achieve the comparison with the previous section, we first put strongly to zero the set of second-class constraints χ_2 and χ_3. In addition, we fix the gauge invariance generated by the first-class constraint χ_1 by imposing the condition $B_0 = 0$. All this is done by introducing the corresponding Dirac bracket and by explicitly eliminating the variables (B_1, P_1) and (B_0, P_0). In particular, the elimination of B_1 is realized using the definition (4.3b) of χ_3 to make the replacement

$$B_1 \to -\frac{1}{2}\mathcal{J}_R{}^{[0]} + \frac{1}{4}\big[\psi^{(3)}, [T, \psi^{(3)}]\big]\,. \tag{4.4}$$

The result of this procedure is a straightforward generalization to the case at hand of the result obtained in [23]. The Dirac brackets for the remaining fields $(g, \mathcal{J}_L, \psi^{(1)}, \psi^{(3)})$ are the same as their Poisson brackets. We are left with the single constraint χ_4 which according to the rule (4.4) becomes

$$\chi_4 = \mathcal{J}_L^{[0]} - \mathcal{J}_R^{[0]} + \frac{1}{2}\big[\psi^{(1)}, [T, \psi^{(1)}]\big] + \frac{1}{2}\big[\psi^{(3)}, [T, \psi^{(3)}]\big]\,. \tag{4.5}$$

[3]Their Poisson bracket is canonical, i.e. $\{B_{0\underline{\mathbf{1}}}(\sigma), P_{0\underline{\mathbf{2}}}(\sigma')\}' = C^{(00)[00]}_{\underline{\mathbf{12}}}\delta_{\sigma\sigma'}$ and similarly for B_1 and P_1.

The corresponding Hamiltonian is computed in appendix C and reads

$$\begin{aligned}H' = \int d\sigma\, \mathrm{Str}\, \bigg[&\frac{1}{4}\big(\mathcal{J}_L{}^{[1]}\mathcal{J}_L{}^{[1]} + \mathcal{J}_R{}^{[1]}\mathcal{J}_R{}^{[1]}\big) - \frac{1}{2}\psi^{(3)}[T, \partial_\sigma \psi^{(3)}] + \frac{1}{2}\psi^{(1)}[T, \partial_\sigma \psi^{(1)}] - \mu^2 g^{-1} T g T \\ &- \mu g^{-1}\psi^{(3)} g \psi^{(1)} - \frac{1}{16}\big[\psi^{(3)}, [T, \psi^{(3)}]\big]\big[\psi^{(3)}, [T, \psi^{(3)}]\big] - \frac{1}{16}\big[\psi^{(1)}, [T, \psi^{(1)}]\big]\big[\psi^{(1)}, [T, \psi^{(1)}]\big] \\ &- \frac{1}{4}\mathcal{J}_L{}^{[0]}\big[\psi^{(1)}, [T, \psi^{(1)}]\big] + \frac{1}{4}\mathcal{J}_R{}^{[0]}\big[\psi^{(3)}, [T, \psi^{(3)}]\big] + \lambda \chi_4 \bigg]\end{aligned} \tag{4.6}$$

where λ is a Lagrange multiplier.

In summary, the phase space of the $\mathrm{AdS}_5 \times S^5$ semi-symmetric space sine-Gordon theory may be parametrized by the fields $(g, \mathcal{J}_L, \psi^{(1)}, \psi^{(3)})$ with Poisson brackets given in (4.2) and subject to the first-class constraint (4.5). So we are now in a position to give the sought dictionary between section 3 and the present section. As suggested by the notation, the field g and the constant matrix T are the same in both sections, whereas the remaining fields and parameters are related by

$$\begin{aligned} \mathcal{J}_L &= -2A^{(0)}, & \mathcal{J}_R &= -2J\,, \\ \psi^{(1)} &= \frac{2}{\sqrt{\mu_+}} A^{(1)[1]}, & \psi^{(3)} &= \frac{2}{\sqrt{\mu_-}} (g A^{(3)} g^{-1})^{[1]}, \\ \mu &= -\sqrt{\mu_+ \mu_-}\,, & \lambda &= -\frac{1}{2}(\ell + \widetilde{\ell})\,. \end{aligned} \tag{4.7}$$

One can check that there is perfect agreement, firstly between the Poisson brackets (4.2) and (3.10), secondly between the constraints (4.5) and (3.18), and lastly between the equations of motion generated by the Hamiltonian (4.6) and the equations of motion (3.21), (3.22) and (3.23).

5 Conclusion

Let us start by answering the questions which motivated this work as mentioned in the introduction. We have shown that the alleviation procedure, as developed in [10] for symmetric space σ-models, extends smoothly to the case of the $\mathrm{AdS}_5 \times S^5$ superstring. Moreover, we have found that in this context as well the procedure is tightly linked with Pohlmeyer reduction.

An important point we wish to stress concerns the rigidity of the alleviation procedure. Indeed, at every stage of the procedure there is essentially no freedom. To begin with, the introduction of the modified Poisson bracket is guided by the requirement that its non-ultralocality be only mild. This places severe restrictions on the choice of inner product entering the definition of the Poisson bracket. Subsequently, the degeneracy of the modified Poisson bracket and the specific form of its Casimirs basically compel us to restrict attention to the phase space of the Pohlmeyer reduction of the $\mathrm{AdS}_5 \times S^5$ superstring. The complete procedure therefore leads us very naturally from the $\mathrm{AdS}_5 \times S^5$ superstring theory to the associated semi-symmetric space sine-Gordon theory.

By comparison with our previous work [10] where we were not considering a string theory, let us briefly recall that in the context of the $\mathrm{AdS}_5 \times S^5$ superstring theory, Pohlmeyer reduction corresponds to a reduction of gauge degrees of freedom. The reduction therefore still describes the dynamics of all the physical degrees of freedom of the original $\mathrm{AdS}_5 \times S^5$ superstring. Of course, in the bosonic setting the same interpretation holds if, say, for the σ-model on S^n we consider instead a string theory on $\mathbb{R} \times S^n$ (see for instance [18, 24]).

One could of course take the canonical structure of the $\mathrm{AdS}_5 \times S^5$ superstring and consider its own restriction to the reduced degrees of freedom. In the context of the $\mathrm{AdS}_5 \times S^5$ superstring, this problem has been addressed first in [25] and later in more details in [26, 27]. It turns out that the induced Poisson structure is non-local. This is in stark contrast with the restriction of the modified Poisson bracket to the reduced degrees of freedom as presented in this article. Indeed, the latter is both local and has the property that the corresponding Poisson bracket of the Lax matrix is mildly non-ultralocal.

Evidently, the equivalence between the original $\mathrm{AdS}_5 \times S^5$ superstring and the theory with the modified Poisson bracket describing the Pohlmeyer reduction of the $\mathrm{AdS}_5 \times S^5$ superstring is only classical at this stage. Whether or not this equivalence persists at the quantum level is likely to be a rather delicate issue.[4] Indeed, the corresponding statement for the SU(2) principal chiral model in [11] requires a subtle change of vacuum from the reference state of the Bethe ansatz to the physical ground state given by the Dirac sea of Bethe roots. To further this program, the next challenge would be to find the quantization of the quadratic lattice Poisson algebra of the Lax matrix as described in [17].

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A Modified Poisson bracket

We reproduce below the modified Poisson bracket, which is mildly non-ultralocal. The only Poisson bracket, which involves a derivative of the Dirac δ-function is

$$\{A^{(0)}_{\underline{\mathbf{1}}}(\sigma), A^{(0)}_{\underline{\mathbf{2}}}(\sigma')\}' = -\frac{1}{2}[C^{(00)}_{\underline{\mathbf{12}}}, A^{(0)}_{\underline{\mathbf{2}}} + \frac{1}{2}\mathcal{C}^{(0)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'} + \frac{1}{2}C^{(00)}_{\underline{\mathbf{12}}}\partial_\sigma\delta_{\sigma\sigma'} \,.$$

[4]However, note that what makes the $AdS_5 \times S^5$ superstring special [18] from the point of view of the Pohlmeyer reduction is the fact that conformal invariance holds also at the quantum level.

The complete list of all the other non-vanishing Poisson brackets is

$$
\begin{aligned}
&\{A^{(0)}_{\underline{\mathbf{1}}}(\sigma),A^{(1)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{4}[C^{(00)}_{\underline{\mathbf{12}}},\mathcal{C}^{(1)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{A^{(0)}_{\underline{\mathbf{1}}}(\sigma),A^{(2)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{2}[C^{(00)}_{\underline{\mathbf{12}}},A^{(2)}_{-\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{A^{(0)}_{\underline{\mathbf{1}}}(\sigma),A^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{2}\Big[C^{(00)}_{\underline{\mathbf{12}}},A^{(3)}_{\underline{\mathbf{2}}}+\frac{1}{2}\mathcal{C}^{(3)}_{\underline{\mathbf{2}}}\Big]\delta_{\sigma\sigma'}\,, &&\{A^{(1)}_{\underline{\mathbf{1}}}(\sigma),A^{(1)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{2}[C^{(13)}_{\underline{\mathbf{12}}},A^{(2)}_{+\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{A^{(1)}_{\underline{\mathbf{1}}}(\sigma),A^{(2)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{4}[C^{(13)}_{\underline{\mathbf{12}}},\mathcal{C}^{(3)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{A^{(1)}_{\underline{\mathbf{1}}}(\sigma),A^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{4}[C^{(13)}_{\underline{\mathbf{12}}},\mathcal{C}^{(0)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{A^{(2)}_{\underline{\mathbf{1}}}(\sigma),A^{(2)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{4}[C^{(22)}_{\underline{\mathbf{12}}},\mathcal{C}^{(0)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{A^{(2)}_{\underline{\mathbf{1}}}(\sigma),A^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{4}[C^{(22)}_{\underline{\mathbf{12}}},\mathcal{C}^{(1)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{A^{(3)}_{\underline{\mathbf{1}}}(\sigma),A^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{2}[C^{(31)}_{\underline{\mathbf{12}}},A^{(2)}_{-\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{A^{(0)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(1)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{3}{8}[C^{(00)}_{\underline{\mathbf{12}}},\mathcal{C}^{(1)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{A^{(0)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(2)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{2}[C^{(00)}_{\underline{\mathbf{12}}},A^{(2)}_{-\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{A^{(0)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{4}\Big[C^{(00)}_{\underline{\mathbf{12}}},A^{(3)}_{\underline{\mathbf{2}}}+\frac{1}{2}\mathcal{C}^{(3)}_{\underline{\mathbf{2}}}\Big]\delta_{\sigma\sigma'}\,,\\
&\{A^{(1)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(1)}_{\underline{\mathbf{2}}}(\sigma')\}'=\frac{1}{4}[C^{(13)}_{\underline{\mathbf{12}}},A^{(2)}_{+\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{A^{(1)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(2)}_{\underline{\mathbf{2}}}(\sigma')\}'=\frac{1}{4}[C^{(13)}_{\underline{\mathbf{12}}},\mathcal{C}^{(3)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{A^{(1)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=\frac{3}{8}[C^{(13)}_{\underline{\mathbf{12}}},\mathcal{C}^{(0)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{A^{(2)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(1)}_{\underline{\mathbf{2}}}(\sigma')\}'=\frac{1}{8}[C^{(22)}_{\underline{\mathbf{12}}},\mathcal{C}^{(3)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{A^{(2)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(2)}_{\underline{\mathbf{2}}}(\sigma')\}'=\frac{1}{4}[C^{(22)}_{\underline{\mathbf{12}}},\mathcal{C}^{(0)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{A^{(2)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{8}[C^{(22)}_{\underline{\mathbf{12}}},\mathcal{C}^{(1)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{A^{(3)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(1)}_{\underline{\mathbf{2}}}(\sigma')\}'=\frac{1}{8}[C^{(31)}_{\underline{\mathbf{12}}},\mathcal{C}^{(0)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{A^{(3)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(2)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{4}[C^{(31)}_{\underline{\mathbf{12}}},\mathcal{C}^{(1)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{A^{(3)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{4}[C^{(31)}_{\underline{\mathbf{12}}},A^{(2)}_{-\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{\Pi^{(1)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(1)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{8}[C^{(13)}_{\underline{\mathbf{12}}},A^{(2)}_{+\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{\Pi^{(1)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(2)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{8}[C^{(13)}_{\underline{\mathbf{12}}},\mathcal{C}^{(3)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{\Pi^{(1)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{3}{16}[C^{(13)}_{\underline{\mathbf{12}}},\mathcal{C}^{(0)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{\Pi^{(2)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(2)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{4}[C^{(22)}_{\underline{\mathbf{12}}},\mathcal{C}^{(0)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,, &&\{\Pi^{(2)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{8}[C^{(22)}_{\underline{\mathbf{12}}},\mathcal{C}^{(1)}_{\underline{\mathbf{2}}}]\delta_{\sigma\sigma'}\,,\\
&\{\Pi^{(3)}_{\underline{\mathbf{1}}}(\sigma),\Pi^{(3)}_{\underline{\mathbf{2}}}(\sigma')\}'=-\frac{1}{8}[C^{(31)}_{\underline{\mathbf{12}}},A^{(2)}_{-\underline{\mathbf{2}}}]\,\delta_{\sigma\sigma'}\,.
\end{aligned}
$$

B Additional $\mathbb{Z}_2$-grading

Besides the $\mathbb{Z}_4$-grading of $\mathfrak{f}$ introduced in section 2, throughout the article we make extensive use of an additional $\mathbb{Z}_2$-grading of $\mathfrak{f}$ [18]. We list here its definition and main properties.

We follow the conventions of [18] with regards to the Lie superalgebra $\mathfrak{psu}(2,2|4)$. Defining the matrix

$$T=\frac{i}{2}\mathrm{diag}(1,1,-1,-1,1,1,-1,-1)\,, \tag{B.1}$$

it can be used to define a $\mathbb{Z}_2$-grading $\mathfrak{f}=\mathfrak{f}^{[0]}\oplus\mathfrak{f}^{[1]}$ by setting

$$\mathfrak{f}^{[0]}=\{M\in\mathfrak{f}\,|\,[T,M]=0\}\,,\qquad \mathfrak{f}^{[1]}=\{M\in\mathfrak{f}\,|\,[T,M]_+=0\}\,. \tag{B.2}$$

The projectors onto the respective spaces in (B.2) are given by

$$M^{[0]}=-[T,[T,M]_+]_+\qquad\text{and}\qquad M^{[1]}=-[T,[T,M]]\,. \tag{B.3}$$

Note that $\mathfrak{f}^{[0]}=\mathrm{Ker}(\mathrm{ad}\,T)$ and an alternative characterization of $\mathfrak{f}^{[1]}$ is given by $\mathfrak{f}^{[1]}=\mathrm{Im}(\mathrm{ad}\,T)$. This leads at once to $\mathrm{Str}(\mathfrak{f}^{[0]}\mathfrak{f}^{[1]})=0$.

The subspace $\mathfrak{f}^{(2)[0]}$ is two dimensional, and defining the matrix

$$W=\mathrm{diag}(1,1,1,1,-1,-1,-1,-1)\,,$$

it is spanned by T and $\widetilde{T}=WT$. The matrix $\widetilde{T}$ is conjugate to T by an element of G [22].

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C Derivation of the Hamiltonian

In this appendix we derive the Hamiltonian (4.6) governing the dynamics of the $AdS_5 \times S^5$ semi-symmetric space sine-Gordon theory, after eliminating the constraints χ_2, χ_3 explicitly and gauge fixing the invariance generated by χ_1.

The Hamiltonian obtained from the action (4.1) by Legendre transform reads

$$\begin{aligned} H' = \int d\sigma \,\mathrm{Str}\Big[&\frac{1}{4}(\mathcal{J}_L{}^2 + \mathcal{J}_R{}^2) - \frac{1}{2}\psi^{(3)}[T, \partial_\sigma \psi^{(3)}] + \frac{1}{2}\psi^{(1)}[T, \partial_\sigma \psi^{(1)}] \\ &- \mu^2 g^{-1} T g T - \mu g^{-1}\psi^{(3)} g \psi^{(1)} + \mathcal{J}_R(B_0 + B_1) - \mathcal{J}_L(B_0 - B_1) + 2B_1^2 \\ &+ \frac{1}{2}\psi^{(3)}\left[T, [(B_0 + B_1), \psi^{(3)}]\right] + \frac{1}{2}\psi^{(1)}\left[T, [(B_0 - B_1), \psi^{(1)}]\right]\Big] . \end{aligned} \tag{C.1}$$

One can use the definitions (4.3b) and (4.3c) of the constraints χ_3 and χ_4 to rewrite this as

$$\begin{aligned} H' = \int d\sigma \,\mathrm{Str}\Big[&\frac{1}{4}(\mathcal{J}_L{}^2 + \mathcal{J}_R{}^2) - \frac{1}{2}\psi^{(3)}[T, \partial_\sigma \psi^{(3)}] + \frac{1}{2}\psi^{(1)}[T, \partial_\sigma \psi^{(1)}] - \mu^2 g^{-1} T g T \\ &- \mu g^{-1}\psi^{(3)} g \psi^{(1)} + B_0(\chi_3 - \chi_4) + B_1(\chi_3 + \chi_4 - 2B_1)\Big] . \end{aligned} \tag{C.2}$$

We may add to the Hamiltonian density a term proportional to the square of any constraint since this has no effect on the dynamics along the constraint surface. Adding $-\frac{1}{4}\,\mathrm{Str}(\chi_4^2)$, the last two terms in (C.2) may then be rewritten as

$$B_0(\chi_3 - \chi_4) + B_1(\chi_3 + \chi_4 - 2B_1) - \frac{1}{4}\chi_4^2 = (B_0 + B_1)\chi_3 - B_0\chi_4 - \left(\frac{1}{2}\chi_4 - B_1\right)^2 - B_1^2 .$$

As explained in section 4, we may impose the constraint $\chi_3 = 0$ strongly by introducing a Dirac bracket for the constraints χ_2 and χ_3. Using the explicit expression (4.3c) for χ_4 we have $\frac{1}{2}\chi_4 - B_1 = \frac{1}{2}\mathcal{J}_L{}^{[0]} + \frac{1}{4}\left[\psi^{(1)}, [T, \psi^{(1)}]\right]$. We should then also replace B_1 by the expression in (4.4). Putting all of this together we obtain the Hamiltonian governing the dynamics of the remaining fields

$$\begin{aligned} H' = \int d\sigma \,\mathrm{Str}\Big[&\frac{1}{4}(\mathcal{J}_L{}^{[1]}\mathcal{J}_L{}^{[1]} + \mathcal{J}_R{}^{[1]}\mathcal{J}_R{}^{[1]}) - \frac{1}{2}\psi^{(3)}[T, \partial_\sigma \psi^{(3)}] + \frac{1}{2}\psi^{(1)}[T, \partial_\sigma \psi^{(1)}] - \mu^2 g^{-1} T g T \\ &- \mu g^{-1}\psi^{(3)} g \psi^{(1)} - \frac{1}{16}\left[\psi^{(3)}, [T, \psi^{(3)}]\right]\left[\psi^{(3)}, [T, \psi^{(3)}]\right] - \frac{1}{16}\left[\psi^{(1)}, [T, \psi^{(1)}]\right]\left[\psi^{(1)}, [T, \psi^{(1)}]\right] \\ &- \frac{1}{4}\mathcal{J}_L{}^{[0]}\left[\psi^{(1)}, [T, \psi^{(1)}]\right] + \frac{1}{4}\mathcal{J}_R{}^{[0]}\left[\psi^{(3)}, [T, \psi^{(3)}]\right] - B_0\chi_4\Big] . \end{aligned} \tag{C.3}$$

One can check that it preserves the constraint χ_4. At this point there remains two gauge invariances generated by the first-class constraints χ_1 and χ_4. We therefore add to the Hamiltonian density the linear combination $\mathrm{Str}(v_0\chi_1 + \lambda\chi_4)$ where v_0 and λ are Lagrange multipliers. We fix the invariance generated by χ_1 by imposing the condition $B_0 = 0$. Preserving this constraint requires $v_0 = 0$ and we arrive at the Hamiltonian (4.6).

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