On a Family of Random Noble Means Substitutions

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Introduction

This thesis contributes to the extended mathematical description of quasicrystals. In 1982, the materials scientist Dan Shechtman performed experiments with electron diffraction of an $\mathtt{Al}_{86}\mathtt{Mn}_{14}$ alloy. It was expected to observe a diffraction pattern featuring crystallographic point symmetries that correspond to one of the eleven *Laue groups*. In two and three-dimensional space, the periodic structure of crystals is only compatible with d-fold rotational symmetry where $d \in \{1, 2, 3, 4, 6\}$ [Cox61, Sec. 4.5]. Instead, Shechtman observed a diffraction pattern featuring sharp Bragg peaks with a tenfold rotational symmetry from which he inferred the presence of long-range order beyond the realm of perfect crystals. The publication of his results [SBGC84] in 1984 provoked a controversial discussion in the physics community about its reliability and correctness. Nevertheless, other scientists were able to find empirical evidence for the existence of quasicrystal structures, and in 2010 the Icosahedrite [BSYL11], a mineral with the configuration $\mathtt{Al}_{63}\mathtt{Cu}_{24}\mathtt{Fe}_{13}$ which possesses a quasicrystal phase and that was originally discovered in Siberia in 1979, was accepted by the *Internataional Mineralogical Association*. This development culminated in Shechtman being awarded the 1999 Wolf Prize in Physics and the 2011 Nobel Prize in Chemistry.

One important tool in the mathematical modelling of quasicrystals is the theory of tilings of $\mathbb{R}^d$. We refer to [BGM02] for a gentle and illustrative introduction. Here, a tiling is a countable partition $\mathcal{T} = \{\mathcal{T}_i\}_{i \in I}$ of $\mathbb{R}^d$ subject to the conditions that each *tile* $\mathcal{T}_i \subset \mathbb{R}^d$ is closed and $\mathcal{T}_i^\circ \cap \mathcal{T}_j^\circ = \varnothing$. Placing atoms at the vertices of each $\mathcal{T}_i$ leads to a point set of which the Fourier transform yields a good description of the diffraction pattern of a quasicrystal. One of the most prominent instances in two dimensions is the famous *Penrose tiling* which is constructed via two prototiles and that features fivefold rotational symmetry without any translational invariance. A finite patch is illustrated in Figure 0.1.

In general, the structure of systems with pure point diffraction is fairly well understood and a considerable amount of examples is known, including the class of so-called model sets. Nevertheless, there are still unsolved problems such as the famous *Pisot substitution conjecture* [Fog02, Ch. 7], although it is solved for two-letter Pisot substitutions [SS02, Cor. 5.4], or the problem of *homometry*; see [LM11, TB12, Ter13] for background. Less is known in the realm of systems inducing mixed spectra. In this case, the understanding in

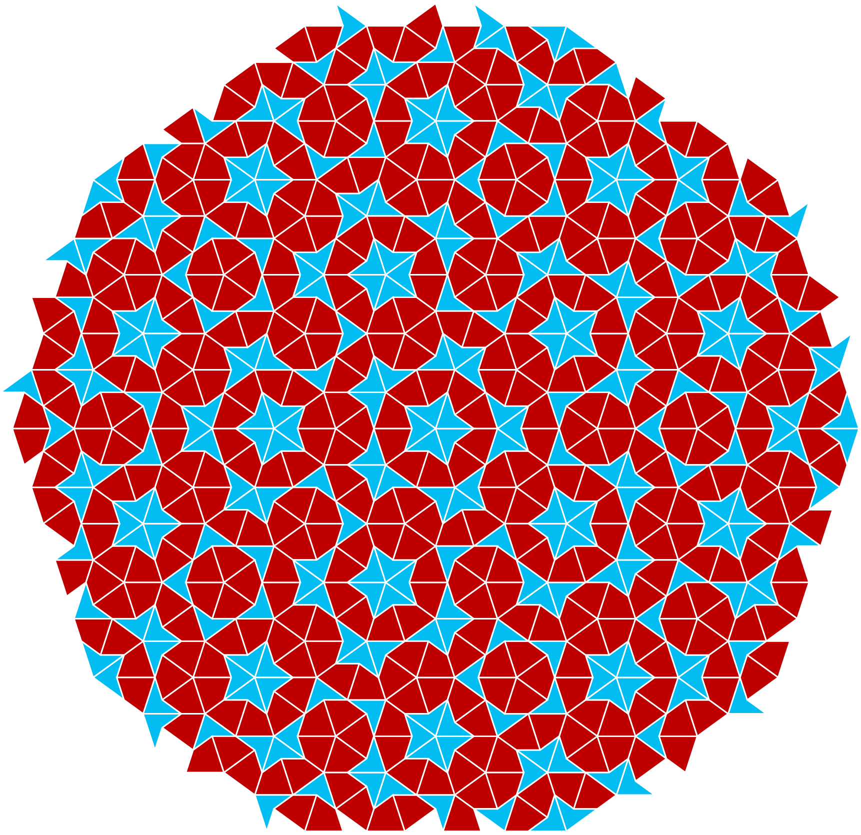

Figure 0.1. A circular patch of the Penrose tiling with *kites* and *darts* as prototiles.

the presence of entropy is only at its beginning [BBM10, BLR07], and it is most desirable to work out particular examples. The mathematical treatment of precisely such an example in one dimension that arises from a substitution dynamical system is the purpose of this thesis.

In a deterministic setting, substitution dynamical systems are well understood and an extensive amount of literature has grown; see [BG13, Fog02, Kit98, LM95, Lot02, Que10] amongst others. In 1989, Godrèche and Luck [GL89] extended the study of conventional substitutions and introduced the notion of local mixtures of substitution rules on the basis of a fixed probability vector along the random Fibonacci substitution. They suggested heuristic considerations for the computation of topological entropy and the spectral type of the diffraction measure. In this thesis, we generalise this analysis by considering the family of random noble means substitutions and offer a mathematical survey of several associated dynamic and analytic aspects.

Outline of the thesis

In *Chapter 1*, we collect the necessary mathematical tools for the treatment of the main content of this thesis. This preliminary part comprises the general mathematical notation that will be applied and relevant concepts from symbolic dynamics, discrete point sets in $\mathbb{R}^d$ and measure theory. We try to be as thorough as possible here, in order to focus on the main matter in the following chapters. Note that we dispense with most proofs in Chapter 1 but present sufficient references for further reading.

In *Chapter 2*, the central families of substitution rules for our concern are introduced. To begin with, the family $\mathcal{N}$ of (deterministic) noble means substitutions is considered and some important properties are gathered. Many of these properties follow from the characterisation of general primitive substitutions which is why we only make explicit the content which is actually relevant in the rest of the treatment. The main part of Chapter 2 is concerned with the extension of $\mathcal{N}$ to the family $\mathcal{R}$ of random noble means substitutions. Here, we derive basic results for the mathematical treatment of the associated dynamical system $(\mathbb{X}_m, S)$ where S denotes the shift and $m \in \mathbb{N}$ is a substitution parameter.

In *Chapter 3*, we study the complexity of random noble means words in terms of the topological entropy $\mathcal{H}_m$. After some initial computations concerning the complexity function and a short discussion of the problems arising here, we give an explicit formula for $\mathcal{H}_m$ and study its behaviour for large m.

Chapter 4 deals with the construction of a translation-invariant (or shift-invariant) measure μ_m on $\mathbb{X}_m$. In the first instance, the known concept of the *induced substitution* is extended to the stochastic regime and elementary properties are derived. In particular, the primitivity of the induced substitution matrix M is an expedient result here. It paves the way for the definition of μ_m via the entries of the statistically normalised right Perron–Frobenius eigenvector of M on the cylinder sets of $\mathbb{X}_m$. Subsequently, we prove the ergodicity of μ_m which facilitates the proper treatment of the diffraction measure of random noble means sets in Chapter 6. The result reads as follows.

Theorem. *For an arbitrary but fixed $m \in \mathbb{N}$, let $\mathbb{X}_m \subset \mathcal{A}_2^{\mathbb{Z}}$ be the two-sided discrete stochastic hull of the random noble means substitution and μ_m be the S-invariant probability measure on $\mathbb{X}_m$ introduced in Eq.* (4.6)*. For any $f \in L^1(\mathbb{X}_m, \mu_m)$ and for an arbitrary but fixed $s \in \mathbb{Z}$, the identity*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=s}^{n+s-1} f(S^i x) = \int_{\mathbb{X}_m} f \, \mathrm{d}\mu_m$$

holds for μ_m-almost every $x \in \mathbb{X}_m$.

In *Chapter 5*, we leave the realm of symbolic dynamics and attend to the description of geometric realisations Λ (and Λ_R in the randomised case) of (random) noble means words as (subsets of) regular model sets. We start with a brief introduction to the theory of iterated function systems and the concept of cut and project schemes. Subsequently, the necessary parameters for the description of Λ as model sets are computed. As the main result of Chapter 5, we prove that each element Λ_R of the continuous stochastic hull features the Meyer property.

In *Chapter 6*, the diffraction measure $\widehat{\gamma_{\Lambda,m}}$ of a typical random noble means set is computed. After a brief introduction to the general theory of the diffraction of discrete point sets, we study a deterministic recursion for the computation of $\widehat{\gamma_{\Lambda,1}}$ that leads to a diffraction spectrum with both a pure point and an absolutely continuous part. Explicitly, we prove that the Lebesgue decomposition in this case reads

$$\widehat{\gamma_{\Lambda,1}} = (\widehat{\gamma_{\Lambda,1}})_{\ominus} + (\widehat{\gamma_{\Lambda,1}})_{\mathsf{pp}} + \phi(k)\lambda,$$

where the precise nature of $(\widehat{\gamma_{\Lambda,1}})_{\ominus}$ stays an open problem. We surmise that $(\widehat{\gamma_{\Lambda,1}})_{\ominus}$ in fact vanishes and present some numerical evidence supporting this conjecture. Finally, an inflation-invariant approach to the pure point part of $\widehat{\gamma_{\Lambda,m}}$ is discussed.

All function plots and numerical calculations in this thesis are produced with the open source computer algebra system *Sage* [Sage]. The typesetting of the text has been done with LaTeX.

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CHAPTER 1

Preliminaries

In Section 1.1 we begin with the determination of the most important general mathematical notation that will be applied throughout the text. There and in the rest of the treatment we will try not to deviate too much from well-established standards in this regard.

In Chapters 1 to 4 we will study the behaviour of symbolic sequences over a finite set of symbols and under the action of certain maps. To this end, we will constitute the necessary terminology, review basic concepts of symbolic dynamics and collect relevant results in Section 1.2. We refer to [BG13, Ch. 4], [LM95, Ch. 1] and [Lot02] for a broader overview in this context.

Section 1.3 is devoted to a short survey concerning basic ideas for the discussion of discrete point sets in $\mathbb{R}^d$ that will be introduced (in fact as subsets of $\mathbb{R}$) in Chapter 5 as an interpretation of symbolic sequences. A detailed derivation can be found in [BG13, Ch. 2].

The purpose of Section 1.4 is to give a brief summary of analytic and measure-theoretic tools that will be applied in Chapters 5 and 6. The general theory can be achieved in standard textbooks like [RS81, Lan93] and especially [Die70, Rud87].

1.1. Notation

We refer to the natural numbers as $\mathbb{N} = \{1, 2, 3, \ldots\}$ and define $\mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}$. As usual, we abbreviate the integers, the rationals, the reals and the complex numbers by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Moreover, we will use the common shorthands $\mathbb{R}_+ \coloneqq \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}_{\geqslant 0} \coloneqq \mathbb{R}_+ \cup \{0\}$.

If S, T are any sets, we denote by $S \,\dot\cup\, T$ the disjoint union of S and T. Let $S \subset \mathbb{R}^d$ where $\subset$ is understood to include equality of sets, then the volume of S, with respect to d-dimensional Lebesgue measure λ, is referred to as $\mathrm{vol}(S)$. The interior of S is written as S° and $\overline{S}$ is its closure with respect to the standard metric topology on $\mathbb{R}^d$ (unless stated otherwise). The open ball of radius r with centre $x \in \mathbb{R}^d$ is denoted by $B_r(x)$, where we will often use the shorthand $B_r \coloneqq B_r(0)$. The standard scalar product in $\mathbb{C}^d$ is denoted as

$$\langle x \mid y \rangle \coloneqq \sum_{i=1}^{d} \overline{x_i} y_i$$

for x, $y \in \mathbb{C}^d$. Note that $\langle \cdot \mid \cdot \rangle$ is linear in the second argument.

The ring of square $(d \times d)$–matrices over a commutative ring R is indicated by $\mathrm{Mat}(d, R)$, where we mostly restrict to the cases $R = \mathbb{Z}$ or $R = \mathbb{R}$, respectively. For $M \in \mathrm{Mat}(d, \mathbb{R})$, by $M \geqslant 0$, $M > 0$ or $M \gg 0$ we mean that M consists of non-negative entries only, non-negative entries with at least one being positive or of solely positive entries, and we will adapt this notation for elements in $\mathbb{R}^d$. We refer to the eigenvalue spectrum of M as $\sigma(M)$ and to the spectral radius of M as $\rho(M)$.

The vector space of continuous functions $X \longrightarrow \mathbb{C}$ is denoted by $\mathcal{C}(X)$ and as usual, $L^1(X, \mu)$ is the vector space of μ-integrable complex-valued functions on X where we will simply write $L^1(X)$ if there is no confusion about the underlying measure μ.

1.2. Symbolic dynamics

1.2.1. Letters and words. For $n \in \mathbb{N}$, let $\mathcal{A}_n := \{a_i \mid 1 \leqslant i \leqslant n\}$ be a set with finitely many distinct elements. We call $\mathcal{A}_n$ an *alphabet* and refer to its elements as *letters*. Any finite string of letters is called a *(finite) word*. If $w = w_0 w_1 \cdots w_{\ell-1}$ is a word then $\widetilde{w} := w_{\ell-1} w_{\ell-2} \cdots w_0$ is the *reversal* of w and we refer to $|w| = \ell = |\widetilde{w}|$ as the *length* of w and $\widetilde{w}$, respectively. In the case of $w = \widetilde{w}$, we call w a (finite) *palindrome*. The empty word is denoted by the symbol ε and we define $|\varepsilon| := 0$. By a *subword* v of w we mean any connected substring of w and write $v \lhd w$ in this case. If we want to emphasise the precise location of a subword, we write $w_{[s,t]} := w_s w_{s+1} \cdots w_{t-1} w_t \lhd w$ for $0 \leqslant s \leqslant t \leqslant \ell - 1$, and we set $w_{[s,t]} := \varepsilon$ in the case of $s > t$. The *occurrence number* of a word v as a subword of w is defined as the cardinality of the index set $\{s \mid v = w_{[s,s+|v|-1]}\}$ and we write $|w|_v$ for this quantity. Furthermore, the set

$$\mathcal{A}_n^\ell := \{a_{j_1} a_{j_2} \cdots a_{j_\ell} \mid 1 \leqslant j_1, j_2, \ldots, j_\ell \leqslant n\}$$

is defined to be the set of all words of length ℓ, including the special case $\mathcal{A}_n^0 := \{\varepsilon\}$. The *Kleene hull*

$$\mathcal{A}_n^* := \bigcup_{\ell \geqslant 0} \mathcal{A}_n^\ell$$

identifies the set of all words over the alphabet $\mathcal{A}_n$, and it forms a free monoid with neutral element ε, where multiplication is given by *concatenation*. The latter means that two arbitrary words $v = v_0 \cdots v_s$ and $w = w_0 \cdots w_t$ in $\mathcal{A}_n^*$ are connected via $vw := v_0 \cdots v_s w_0 \cdots w_t$. Figure 1.1 indicates $\{a, b\}^*$ as a binary tree.

Let $S \subset \mathcal{A}_n^*$ be any non-empty set of words. Then, $\mathcal{F}_\ell(S)$ denotes the set of all subwords of length ℓ of elements in S and we call $\mathcal{F}_\ell(S)$ the ℓth *factor set* of S. If we omit the index, $\mathcal{F}(S)$ is understood to be the set of all subwords of elements in S.

For the extension of the notion of a (finite) word to infinite sequences of letters in $\mathcal{A}_n$, we call $w = (w_i)_{i \in \mathbb{N}_0} \in \mathcal{A}_n^{\mathbb{N}_0}$ a *semi-infinite word*. Accordingly, we

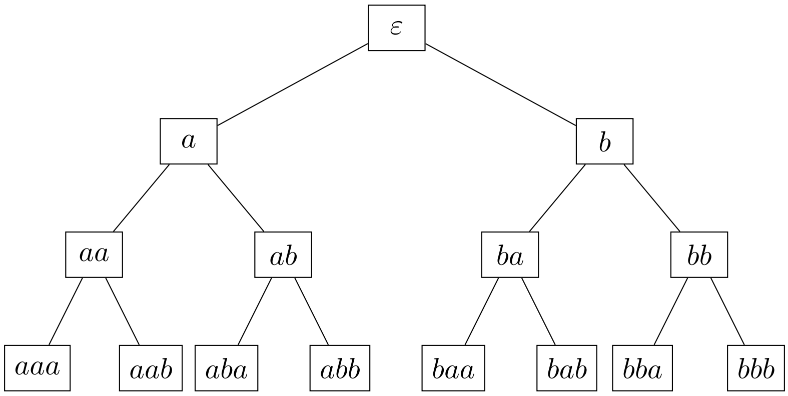

Figure 1.1. All possible words over the alphabet $\mathcal{A}_2 = \{a, b\}$ up to length 3 in lexicographic order.

denote by

$$w = \cdots w_{-3} w_{-2} w_{-1} \,|\, w_0 w_1 w_2 \cdots = (w_i)_{i \in \mathbb{Z}} \in \mathcal{A}_n^{\mathbb{Z}}$$

a *bi-infinite word*, where the vertical bar indicates the reference point. Here, we refer to the set

$$\mathcal{A}_n^{\mathbb{Z}} := \{(w_i)_{i \in \mathbb{Z}} \mid w_i \in \mathcal{A}_n\}$$

as the *full n-shift*. Several of the concepts introduced above for finite words carry over mutatis mutandis to this setting.

The alphabet $\mathcal{A}_n$ is equipped with the discrete topology and is compact in it. Providing $\mathcal{A}_n^{\mathbb{N}_0}$ and $\mathcal{A}_n^{\mathbb{Z}}$ with the product topology, by an application of Tychonoff's theorem [Lan93, Thm. 3.12], both spaces are compact as well. An open, closed and countable basis for the topology of $\mathcal{A}_n^{\mathbb{Z}}$ is given by the class $\mathfrak{Z}(\mathcal{A}_n^{\mathbb{Z}})$ of *cylinder sets*

$$\mathcal{Z}_k(v) := \{w \in \mathcal{A}_n^{\mathbb{Z}} \mid w_{[k, k+\ell-1]} = v\} \tag{1.1}$$

for any $k \in \mathbb{Z}$ and $v \in \mathcal{A}_n^{\ell}$ and we refer to [Bil12, Sec. 2], [LM95, Ch. 6] and [Que10, Ch. 4] for general background. Here, two elements v, $w \in \mathcal{A}_n^{\mathbb{Z}}$ are close if they agree on a large region around the index 0. Therefore, the topology is called the *local topology*. Furthermore, the full n-shift can be equipped with a metric. To begin with, we denote $\mathrm{d}_1(v, w) := |\{i \mid v_i \neq w_i\}|$ as the *Hamming distance* of finite words v, $w \in \mathcal{A}_n^{\ell}$. This concept can be extended to $\mathcal{A}_n^{\mathbb{Z}}$ via

$$\mathrm{d}_2(v, w) := \sum_{\ell \geqslant 0} \frac{\mathrm{d}_1\big(v_{[-\ell,\ell]}, w_{[-\ell,\ell]}\big)}{2^{\ell}} \tag{1.2}$$

for v, $w \in \mathcal{A}_n^{\mathbb{Z}}$, and to $\mathcal{A}_n^{\mathbb{N}_0}$ analogously by considering $\mathrm{d}_1\big(v_{[0,\ell]}, w_{[0,\ell]}\big)$ in Eq. (1.2) instead. This metric generates the local topology introduced above.

For the study of iteration limits of substitution rules (Section 1.2.2 below), it is convenient to consider $\mathcal{A}_n^{\ell}$ as being embedded in $\mathcal{A}_n^{\mathbb{N}_0}$ or $\mathcal{A}_n^{\mathbb{Z}}$; refer to [Fog02, Sec. 1.1.3] in this regard.

1.2.2. Substitution Mappings. A substitution rule, or just a *substitution* for short, on an alphabet $\mathcal{A}_n$ is a monoid endomorphism $\vartheta\colon \mathcal{A}_n^* \longrightarrow \mathcal{A}_n^*$, with the property that $\vartheta(a_i) \neq \varepsilon$ for all $1 \leqslant i \leqslant n$. Thus, a substitution is an element of the monoid of endomorphisms on $\mathcal{A}_n^*$ which we denote by $\mathrm{End}(\mathcal{A}_n^*)$. Later, it will become more convenient to also allow the application of ϑ to words with negative exponents. As this will only be relevant in two situations, we will extend the definition then temporarily. The general theory is not affected by this modification; refer to [BG13, Ch. 4] for a comprehensive treatment of substitution rules in this setting. The endomorphism property $\vartheta(vw) = \vartheta(v)\vartheta(w)$ for any v, $w \in \mathcal{A}_n^*$ ensures that a substitution is completely characterised by the images of the letters in $\mathcal{A}_n$ under ϑ and we will restrict to these simple cases for the definition of all studied substitutions. From now on, we will refer to $\vartheta(w)$ as an *image* of w under ϑ. Note that ϑ extends to mappings $\mathcal{A}_n^{\mathbb{N}_0} \longrightarrow \mathcal{A}_n^{\mathbb{N}_0}$ and $\mathcal{A}_n^{\mathbb{Z}} \longrightarrow \mathcal{A}_n^{\mathbb{Z}}$ via concatenation.

Remark 1.1 (Continuity of substitution mappings)**.** Let ϑ be any substitution on $\mathcal{A}_n$ and v, $w \in \mathcal{A}_n^{\mathbb{Z}}$ with $v \neq w$. Then, there is a $k \in \mathbb{Z}$ with $v_k \neq w_k$ and $|k|$ minimal with this property. If $k = 0$, we have

$$\mathrm{d}_2\big(\vartheta(v), \vartheta(w)\big) \leqslant \sum_{i=0}^{\infty} \frac{2i+1}{2^i} = 3\sum_{i=0}^{\infty} \frac{1}{2^i} \leqslant 3\,\mathrm{d}_2(v,w).$$

If $k \neq 0$, we find

$$v_{[1-|k|,|k|-1]} = w_{[1-|k|,|k|-1]} \quad \text{and} \quad \vartheta(v)_{[\ell,r]} = \vartheta(w)_{[\ell,r]},$$

where $\ell := -\sum_{i=1}^{|k|-1} |\vartheta(v_{-i})|$ and $r := \sum_{i=0}^{|k|-1} |\vartheta(v_i)|$. With $m_k := \min\{|\ell|, r\}$ we obtain

$$\mathrm{d}_2\big(\vartheta(v), \vartheta(w)\big) \leqslant \sum_{i=m_k+1}^{\infty} \frac{2(i-m_k)}{2^i} = \frac{1}{2^{m_k}} \sum_{i=0}^{\infty} \frac{i+1}{2^i} = \frac{1}{2^{m_k-2}}.$$

On the other hand, we have

$$\frac{1}{2^{|k|-2}} = 2\sum_{i=|k|}^{\infty} \frac{1}{2^i} \leqslant 2\sum_{i=|k|}^{\infty} \frac{\mathrm{d}_1\big(v_{[-i,i]}, w_{[-i,i]}\big)}{2^i} = 2\,\mathrm{d}_2(v,w).$$

As $m_k \geqslant |k|$, we finally get

$$\mathrm{d}_2\big(\vartheta(v), \vartheta(w)\big) \leqslant 2\,\mathrm{d}_2(v,w)$$

and therefore the Lipschitz continuity of ϑ on $\mathcal{A}_n^{\mathbb{Z}}$. ♦

A convenient way of simplifying concrete calculations is the assignment of a matrix describing the action of a substitution ϑ in terms of the occurrence numbers of letters. For any such ϑ on the alphabet $\mathcal{A}_n$, we define its *substitution matrix* to be

$$M_\vartheta := \big(|\vartheta(a_j)|_{a_i}\big)_{1\leqslant i,j\leqslant n} \in \mathrm{Mat}(n, \mathbb{Z}).$$

The (Abelianisation) map

$$\phi\colon \mathcal{A}_n^* \longrightarrow \mathbb{Z}^n, \quad w \longmapsto \phi(w) := \big(|w|_{a_1}, \dots, |w|_{a_n}\big)^T,$$

where x^T denotes the transpose of x, keeps track of the occurrence numbers of the letters in a word $w \in \mathcal{A}_n^*$. It is easy to verify that $M_{\vartheta\circ\varphi} = M_\vartheta M_\varphi$ for ϑ, $\varphi \in \operatorname{End}(\mathcal{A}_n^*)$ and $M_\vartheta^k\, \phi(w) = \phi(\vartheta^k(w))$ for any $k \in \mathbb{N}_0$. For a detailed derivation of this process called *Abelianisation*, we refer to [BG13, Ch. 4.1]. Note that there is no one-to-one correspondence between substitutions and their substitution matrices in general, as we will see in Chapter 2 below. Many decisive properties of the substitution itself can nevertheless be derived by analysing the corresponding substitution matrix.

Definition 1.2 (Irreducibility and primitivity of matrices)**.** A non-negative matrix $M \in \operatorname{Mat}(n, \mathbb{R})$ is called *irreducible* if for each pair (i, j) with $1 \leqslant i, j \leqslant n$, there is a power $k \in \mathbb{N}$ such that $(M^k)_{ij} > 0$. The matrix M is called *primitive* if there is a $k \in \mathbb{N}$ such that $M^k \gg 0$.

Definition 1.3 (Irreducibility and primitivity of substitutions)**.** A substitution ϑ on $\mathcal{A}_n$ is called *irreducible* if for each pair (i, j) with $1 \leqslant i, j \leqslant n$, there is a power $k \in \mathbb{N}$ such that a_i is a subword of $\vartheta^k(a_j)$. The substitution ϑ is *primitive* if there is a $k \in \mathbb{N}$ such that a_i is a subword of $\vartheta^k(a_j)$ for all $1 \leqslant i, j \leqslant n$.

It is immediate that a substitution ϑ is irreducible (primitive) if and only if M_ϑ is an irreducible (primitive) matrix.

Primitivity of a substitution enables the application of a powerful tool, central to the complete rest of this text.

Theorem 1.4 (Perron–Frobenius, [Sen06, Que10])**.** *If $M \in \operatorname{Mat}(d, \mathbb{R})$ is a primitive matrix, there exists a simple eigenvalue λ of M with the following properties:*

(a) $\lambda \in \mathbb{R}_+$.
(b) $\lambda = \rho(M)$ and $\lambda > |\mu|$ for any $\mu \in \sigma(M)$ different from λ.
(c) With λ can be associated a left eigenvector $\boldsymbol{L}$ and a right eigenvector $\boldsymbol{R}$, both consisting of strictly positive entries. Those eigenvectors are unique up to scalar multiplication. ∎

This connection of the positivity of a matrix with its spectral properties was proved in 1907 by Oscar Perron [Per07] and extended to irreducible matrices by Georg Frobenius [Fro12] in 1912, wherefore it is also called the Perron–Frobenius theorem.

From now on, the eigenvalue and the eigenvectors stated in Theorem 1.4 are called the PF eigenvalue and the PF eigenvectors.

Remark 1.5 (Perron–Frobenius for irreducible matrices)**.** In the irreducible case, assertion (b) of Theorem 1.4 reduces to $\lambda \geqslant |\mu|$. There are precisely d

eigenvalues λ satisfying $\lambda = |\mu|$ which are

$$\{\lambda\, \mathrm{e}^{2\pi \mathrm{i} k/d} \mid 0 \leqslant k \leqslant d-1\},$$

where $d \geqslant 1$ is the period of M. Even more remarkable is the invariance of the complete eigenvalue spectrum under the action of the cyclic group of order d; refer to [Sen06, Sec. 1.4] for proofs and background information. As a precious source for various aspects and implications of Theorem 1.4 we refer to [Mey00, Ch. 8]. ♦

The primitivity of a substitution ϑ implies that $\lim_{k\to\infty} |\vartheta^k(x)| = \infty$ for any letter $x \in \mathcal{A}_n$, provided $n \geqslant 2$ because if $j \in \mathbb{N}$ is such that $\vartheta^j(x)$ contains all letters of the alphabet, then $|\vartheta^j(x)| \geqslant n$ and consequently $|\vartheta^{ij}(x)| \geqslant n^i$ for all $i \in \mathbb{N}$. One can prove that the letter frequencies, that is the limit

$$\lim_{k\to\infty} \frac{1}{|\vartheta^k(x)|} \phi\bigl(\vartheta^k(x)\bigr),$$

always exist and that they do not depend on the choice of the letter $x \in \mathcal{A}_n$ [Que10, Prop. 5.8].

A convenient normalisation of the right PF eigenvector $\boldsymbol{R} \in \mathbb{R}^d$ of a primitive substitution matrix is

$$\|\boldsymbol{R}\|_1 = \sum_{i=1}^{d} |\boldsymbol{R}_i| = \sum_{i=1}^{d} \boldsymbol{R}_i = 1.$$

Here, the Perron–Frobenius theorem permits us to interpret the entries of $\boldsymbol{R}$ as the frequencies of all letters in $\mathcal{A}_n$ with respect to a fixed point (see below) of the underlying substitution [Que10, Cor. 5.4]. Additionally, the left PF eigenvector can be used to construct a geometric counterpart of a primitive substitution. This concept will be introduced in Chapter 5 for the noble means example.

Definition 1.6 (Legality of words)**.** A word $w \in \mathcal{A}_n^*$ is *legal* with respect to a substitution ϑ, if there is some $k \in \mathbb{N}$ and some $x \in \mathcal{A}_n$ such that $w \lhd \vartheta^k(x)$.

Note that ε is always a legal word. Although this might not seem to be a fruitful observation, we do not want to exclude this case here for technical reasons.

Let $w^{(0)} := w_{-1} \,|\, w_0 \in \mathcal{A}_n^2$ be legal with respect to a substitution ϑ and $w^{(k)} := \vartheta(w^{(k-1)})$ for $k \in \mathbb{N}$. Then, a bi-infinite word $w \in \mathcal{A}_n^{\mathbb{Z}}$ with the property

$$\lim_{k\to\infty} w^{(k)} = w = \vartheta(w) \tag{1.3}$$

is called a bi-infinite *fixed point* of ϑ with legal *seed* $w^{(0)}$, where the limit in Eq. (1.3) is taken in the local topology. Omitting the legality of seeds might lead to contradicting conclusions about the two-sided discrete hull (see below) of a substitution; compare [BG13, Exa. 4.2] in this context. In the following, the term *fixed point* is always understood to be the bi-infinite sequence introduced above. If we restrict to its semi-infinite analogue, we will explicitly emphasise this.

In this regard, the primitivity of a substitution has the following important consequence.

Lemma 1.7 ([BG13, Lem. 4.3]). *If ϑ is a primitive substitution on $\mathcal{A}_n$ for $n \geqslant 2$, there exists some $k \in \mathbb{N}$ and some $w \in \mathcal{A}_n^{\mathbb{Z}}$ with w being a fixed point of ϑ^k.* ∎

Let $S\colon \mathcal{A}_n^{\mathbb{Z}} \longrightarrow \mathcal{A}_n^{\mathbb{Z}}$ be the *shift map*. An application of S to a word w slides all letters one position to the left, that is $S\big((w_k)_{k\in\mathbb{Z}}\big) := (w_{k+1})_{k\in\mathbb{Z}}$. Accordingly, there is an inverse mapping, that shifts all letters one position to the right, that means $S^{-1}\big((w_k)_{k\in\mathbb{Z}}\big) = (w_{k-1})_{k\in\mathbb{Z}}$. Moreover, both S and S^{-1} are continuous, onto and one-to-one on $\mathcal{A}_n^{\mathbb{Z}}$, which turns S into a homeomorphism. This is a convenient advantage over the situation on $\mathcal{A}_n^{\mathbb{N}_0}$, where S is neither invertible nor one-to-one. Compare also [Fog02, Ch. 1] in this regard.

Definition 1.8 (Two-sided discrete hull). Let ϑ be a primitive substitution and $w \in \mathcal{A}_n^{\mathbb{Z}}$ be a fixed point of ϑ^k for some $k \in \mathbb{N}$. The set

$$\mathbb{X}_\vartheta := \overline{\big\{S^j w \mid j \in \mathbb{Z}\big\}} \subset \mathcal{A}_n^{\mathbb{Z}}$$

is called the *two-sided discrete hull* of the substitution ϑ, where the closure is taken in the local topology.

Note that the claimed existence of the fixed point is a consequence of Lemma 1.7 and that the two-sided discrete hull is invariant under the action of both the shift and the substitution. More generally, we call any closed and shift-invariant subset $\mathbb{X} \subset \mathcal{A}_n^{\mathbb{Z}}$ a two-sided *subshift*.

Lemma 1.9. *Let $\mathbb{X} \subset \mathcal{A}_n^{\mathbb{Z}}$ be any subshift. The restriction $S|_{\mathbb{X}}\colon \mathbb{X} \longrightarrow \mathbb{X}$ is a homeomorphism.*

Proof. This is an immediate consequence of S being a homeomorphism on $\mathcal{A}_n^{\mathbb{Z}}$ and $\mathbb{X}$ being shift-invariant. ∎

Remark 1.10 (Two-sided discrete hulls from semi-infinite fixed points). According to Definition 1.8, a bi-infinite fixed point of ϑ^k is needed for the construction of $\mathbb{X}_\vartheta$. As stated in [BG13, Rem. 4.1], it is also possible to build the correct two-sided discrete hull via semi-infinite fixed points. Even if ϑ^k admits the construction of the fixed point $v \,|\, w \in \mathcal{A}_n^{\mathbb{Z}}$, it might happen that there is a $k' < k$ with the property that $\vartheta^{k'}$ possesses either v or w as a semi-infinite fixed point. This constitutes a convenient technical simplification which we will use in Chapter 5 in the noble means case. ♦

In the case of a primitive substitution ϑ, two different fixed points $v, w \in \mathcal{A}_n^{\mathbb{Z}}$ of ϑ^k give rise to the same two-sided discrete hull $\mathbb{X}_\vartheta$; compare [BG13, Lem. 4.2 and Prop. 4.2]. In this sense, $\mathbb{X}_\vartheta$ is uniquely defined [BG13, Thm. 4.1].

Remark 1.11 (Cylinder sets of subshifts)**.** If $\mathbb{X} \subset \mathcal{A}_n^{\mathbb{Z}}$ is any subshift, the class of cylinder sets $\mathfrak{Z}(\mathbb{X})$ is induced by $\mathfrak{Z}(\mathcal{A}_n^{\mathbb{Z}})$ via the subspace topology. Here, we get

$$\mathfrak{Z}(\mathbb{X}) := \{\mathcal{Z} \cap \mathbb{X} \mid \mathcal{Z} \in \mathfrak{Z}(\mathcal{A}_n^{\mathbb{Z}})\}. \qquad \blacklozenge$$

We close this section with some terminology concerning the characterisation of substitutions in the centre of our interest. A bi-infinite word $w \in \mathcal{A}_n^{\mathbb{Z}}$ is called *periodic* if there is some $k \in \mathbb{Z} \setminus \{0\}$, such that $S^k w = w$ and *non-periodic*, when no such k exists. We call a primitive substitution ϑ *aperiodic*, if the uniquely defined two-sided discrete hull of ϑ contains no periodic element. A convenient way of determining the aperiodicity of primitive substitutions is provided by the following result.

Theorem 1.12 ([BG13, Thm. 4.6])**.** *Let ϑ be a primitive substitution with substitution matrix M_ϑ. If the PF eigenvalue of M_ϑ is irrational, then ϑ is aperiodic.* ∎

In Chapter 5, we will offer a geometric interpretation of the PF eigenvalue as the inflation multiplier of an induced geometric inflation rule.

An algebraic integer $\alpha > 1$ is called a *Pisot–Vijayaraghavan number*, or a *PV number* for short, if its algebraic conjugates $\alpha_1, \dots, \alpha_k$, except α itself, comply with $|\alpha_i| < 1$. If α is a PV number and an algebraic unit, we call α a *PV unit*. Last but not least, we refer to a substitution rule ϑ as a *Pisot substitution* if M_ϑ has a largest and simple eigenvalue $\lambda > 1$ and all eigenvalues μ of M_ϑ different from λ satisfy $0 < |\mu| < 1$. For a comprehensive treatment of Pisot substitutions we refer to [Sin06].

1.3. Point sets

Here, we concentrate on point sets in $\mathbb{R}^d$ where the special case of $d = 1$ is considered in Chapters 5 and 6. To begin with, we define for U, $V \subset \mathbb{R}^d$ by

$$U \pm V := \{u \pm v \mid u \in U, v \in V\}$$

the *Minkowski sum (difference)* of the sets U and V. A set consisting of one point is called a *singleton set* and countable unions of singletons are called *point sets*. A point set $P \subset \mathbb{R}^d$ is *discrete* if there is an $r := r(x) > 0$ such that $B_r(x) \cap P = \{x\}$ for each $x \in P$. The set P is *uniformly discrete* if there is an $r > 0$ such that $B_r(x) \cap B_r(y) = \varnothing$, for any distinct x, $y \in P$. Moreover, P is *relatively dense* if there is an $R > 0$ such that $\mathbb{R}^d = P + \overline{B_R(0)}$. We define

$$\begin{aligned} r_{\mathsf{p}}(P) &:= \sup\{r > 0 \mid B_r(x) \cap B_r(y) = \varnothing \text{ for all distinct } x, y \in P\}, \\ R_{\mathsf{c}}(P) &:= \inf\{R > 0 \mid P + \overline{B_R(0)} = \mathbb{R}^d\}. \end{aligned}$$

The number $r_{\mathsf{p}}(P)$ is called the *packing radius* and $R_{\mathsf{c}}(P)$ the *covering radius* of P.

Definition 1.13 (Delone and Meyer set)**.** A point set $P \subset \mathbb{R}^d$ is called a *Delone set* if it is uniformly discrete and relatively dense. Moreover, P is a *Meyer set*, if it is relatively dense and $P - P$ is uniformly discrete.

Let $P \subset \mathbb{R}^d$ be a discrete point set. For $x \in \mathbb{R}^d$ and $r > 0$, we call any set S of the form $B_r(x) \cap P$ a *patch* of P. The point set P is *repetitive* if for any patch S there is a radius $R > 0$ such that $B_R(y)$ contains at least one translate of S for any $y \in \mathbb{R}^d$. Furthermore, P has *finite local complexity* (or is an FLC set) if the set $\{\overline{B_r(x)} \cap P \mid x \in P\}$, for any $r > 0$, contains only finitely many non-empty patches up to translations. Last but not least, a point set P is *locally finite* if $K \cap P$ contains only finitely many elements, for any compact $K \subset \mathbb{R}^d$.

1.4. Functional analysis and measure theory

We start with a brief recollection of basic tools from functional analysis. We denote by

$$\mathcal{S}(\mathbb{R}^d) := \Big\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) \mid \forall \alpha, \beta \in \mathbb{N}_0^d \colon \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)| < \infty \Big\}$$

the *Schwartz space* of functions of rapid decrease. We prefer addressing $\mathcal{S}(\mathbb{R}^d)$ as the space of *test functions* below. Let $f,\, g \in \mathcal{S}(\mathbb{R}^d)$. Then, the *convolution* of f and g is defined by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x-y) g(y) \,\mathrm{d}y = \int_{\mathbb{R}^d} f(y) g(x-y) \,\mathrm{d}y$$

and the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is

$$\widehat{f}(k) := \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} \langle k | x \rangle} f(x) \,\mathrm{d}x.$$

The same definitions apply in the case of $f,\, g \in L^1(\mathbb{R}^d)$. The dual space $\mathcal{S}'(\mathbb{R}^d)$ of linear functionals $T \colon \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathbb{C}$ is the space of *tempered distributions*. The Fourier transform of a tempered distribution T is defined by $\widehat{T}(f) := T(\widehat{f})$ for all test functions $f \in \mathcal{S}(\mathbb{R}^d)$.

Example 1.14 (Dirac distribution)**.** In Chapter 6, we will frequently make use of the *Dirac* distribution (or measure)

$$\delta_x \colon \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathbb{C}, \quad f \longmapsto \delta_x(f) := f(x).$$

Its Fourier transform reads

$$\widehat{\delta_x}(f) = \delta_x(\widehat{f}) = \widehat{f}(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} \langle x | y \rangle} f(y) \,\mathrm{d}y,$$

and we will use the common shorthand $\widehat{\delta_x} = \mathrm{e}^{-2\pi \mathrm{i} \langle x | y \rangle}$. The Dirac distribution also defines a measure, given by

$$\delta_x(B) := \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{otherwise,} \end{cases}$$

for any Borel set $B \subset \mathbb{R}^d$. ♦

1.4.1. Measure theory. In Chapters 4, 5 and 6, measure-theoretic tools are applied to study certain subsets of $\mathbb{R}^d$ and the hull of some substitution rule. Here, we work in the context of a locally compact Hausdorff space X. In this situation, it is convenient to introduce the notion of a measure via linear functionals [Die70, Ch. XIII].

Let $\mathcal{C}_{\mathrm{c}}(X)$ denote the vector space of complex-valued continuous functions with compact support. Now, a complex-valued measure on X is a linear functional

$$\mu\colon \mathcal{C}_{\mathrm{c}}(X) \longrightarrow \mathbb{C}, \quad f \longmapsto \mu(f),$$

with the property that for each compact $K \subset X$, there is a real constant $c_K \geqslant 0$ such that $|\mu(f)| \leqslant c_K \|f\|_\infty$ for all $f \in \mathcal{C}_{\mathrm{c}}(X)$ with support in K. As usual, $\|f\|_\infty := \sup\{|f(x)| \mid x \in K\}$ is the supremum norm of f. The space $\mathcal{M}(X)$ of complex-valued measures is equipped with the *vague (= weak-∗) topology* [Die70, Ch. XIII.4]. Here, a sequence $(\mu_n)_{n\in\mathbb{N}}$ converges vaguely to a measure μ if the sequence $\bigl(\mu_n(f)\bigr)_{n\in\mathbb{N}}$ converges to $\mu(f)$ in $\mathbb{C}$ for all $f \in \mathcal{C}_{\mathrm{c}}(X)$. A suitable decomposition of μ into its real/imaginary and positive/negative parts, along with an application of the Riesz–Markov representation theorem [Rud87, Thm. 2.14] yields a one-to-one correspondence between regular Borel measures on X and the measures defined by the approach via linear functionals [Die70, Ch. XIII.2 and Ch. XIII.3]. This justifies the parallel usage of measures of Borel sets $\mu(B)$ and measures of functions $\mu(f)$. The link between these points of view is established via $\mu(\mathbb{1}_B) = \mu(B)$, where $\mathbb{1}_B$ denotes the characteristic function

$$\mathbb{1}_B\colon X \longrightarrow \{0,1\}, \quad \mathbb{1}_B(x) := \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{otherwise.} \end{cases}$$

As $\mathbb{1}_B \notin \mathcal{C}_{\mathrm{c}}(X)$, one has to apply the regularity of μ to achieve suitable approximations from above and from below; see [Die70, Ch. XIII.7] for background information. The *conjugate* of a measure $\mu \in \mathcal{M}(X)$ is defined by the map $f \longmapsto \overline{\mu(\overline{f})}$ for any $f \in \mathcal{C}_{\mathrm{c}}(X)$. This again defines a measure which is denoted by $\overline{\mu}$. Now, a measure μ is *real* if $\mu = \overline{\mu}$ and a real measure μ is called *positive* if $\mu(f) \geqslant 0$ for all $f \geqslant 0$. The set of real and positive measures on X is abbreviated by $\mathcal{M}^+_{\mathbb{R}}(X)$. A measure $\mu \in \mathcal{M}(X)$ is called *finite* if $|\mu|(X)$ is finite where $|\mu|$ denotes the *total variation of* μ. The latter is the smallest positive measure such that $|\mu(f)| \leqslant |\mu|(f)$ for all non-negative $f \in \mathcal{C}_{\mathrm{c}}(X)$. We define

$$\mathcal{P}(X) := \{\mu \in \mathcal{M}^+_{\mathbb{R}}(X) \mid \|\mu\| = 1\}$$

as the set of *probability measures* on X, where $\|\mu\| := |\mu|(X)$. By the Banach–Alaoglu theorem [RS81, Thm. IV.21], the unit ball $\overline{B_1}$ in $\mathcal{M}(X)$ is compact in the vague topology and $\mathcal{P}(X)$ is a closed subset of $\overline{B_1}$ which then implies the compactness of $\mathcal{P}(X)$. Furthermore, $\mu \in \mathcal{M}(X)$ is *translation bounded* if

$$\sup\{|\mu|(x+K) \mid x \in X\} < \infty$$

for any compact $K \subset X$. Moreover, $\mu \in \mathcal{M}(X)$ is called *positive definite* if $\mu(f * \widetilde{f}) \geqslant 0$ for all $f \in \mathcal{C}_{\mathsf{c}}(X)$, where $\widetilde{f}(x) := \overline{f(-x)}$ is the *reflection* of f. Analogously, we define $\widetilde{\mu}(f) := \overline{\mu(\widetilde{f})}$.

Now, we restrict the exposition explicitly to $X = \mathbb{R}^d$ as we want to study the decomposition of a measure into its pure point and continuous parts with respect to Lebesgue measure. For any $\mu \in \mathcal{M}^+_{\mathbb{R}}(\mathbb{R}^d)$, the set

$$P_\mu := \{x \mid \mu(\{x\}) \neq 0\}$$

is called the set of *pure points* (also the support of *Bragg peaks*) of μ and for any Borel set B we define

$$\mu_{\mathsf{pp}}(B) := \sum_{x \in B \cap P_\mu} \mu(\{x\}) = \mu(B \cap P_\mu)$$

as the *pure point part* of μ. In general, a measure $\mu \in \mathcal{M}^+_{\mathbb{R}}(\mathbb{R}^d)$ is *pure point* if $\mu(B) = \sum_{x \in B} \mu(\{x\})$ for any Borel set B. Furthermore, we define $\mu_{\mathsf{c}} := \mu - \mu_{\mathsf{pp}}$ and observe $\mu_{\mathsf{c}}(\{x\}) = 0$ for all $x \in \mathbb{R}^d$ and say that μ_{c} has no pure points. A measure μ is called *absolutely continuous* with respect to Lebesgue measure if there is a locally integrable function f such that

$$\mu(g) = \int g \, \mathrm{d}\mu = \int g f \, \mathrm{d}\lambda = \lambda(gf),$$

for $g \in \mathcal{C}_{\mathsf{c}}(\mathbb{R}^d)$. Here, f is called the *Radon–Nikodym density* of μ relative to λ. A measure μ is called *singular* relative to Lebesgue measure if and only if $\mu(B) = 0$ for some measurable set $B \subset \mathbb{R}^d$ with $\lambda(\mathbb{R}^d \setminus B) = 0$. A measure that is singular relative to λ without having any pure points is called *singular continuous*.

Theorem 1.15 ([RS81, Thm. I.13 and Thm. I.14])**.** *Any positive, regular Borel measure $\mu \in \mathcal{M}^+_{\mathbb{R}}(\mathbb{R}^d)$ has a unique decomposition*

$$\mu = \mu_{\mathsf{pp}} + \mu_{\mathsf{ac}} + \mu_{\mathsf{sc}},$$

where μ_{pp} is pure point, μ_{ac} is absolutely continuous and μ_{sc} is singular continuous with respect to Lebesgue measure. ∎

The statement of Theorem 1.15 and the preceding discussion can be generalised to complex-valued regular Borel measures μ via a decomposition of μ into its real/imaginary and positive/negative parts; see [Die70, Ch. XIII.16] and [BG13, Prop. 8.4] for background information.

We close this chapter with some basic operations on measures that will be needed in Chapter 6. If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is any finite measure, its Fourier transform reads

$$\widehat{\mu}(k) := \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} \langle k | x \rangle} \, \mathrm{d}\mu(x).$$

Moreover, there is the notion of convolution of finite measures μ, $\nu \in \mathcal{M}(\mathbb{R}^d)$ defined as

$$(\mu * \nu)(f) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x+y) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) \tag{1.4}$$

for any $f \in \mathcal{C}_\mathrm{c}(\mathbb{R}^d)$.

As the convolution of measures in Eq. (1.4) is well-defined only if the involved measures are both finite (or one finite and the other one translation bounded [BG13, Thm. 8.5]), we have to extend the concept in the setting of infinite measures. We define the *Eberlein convolution* of μ, $\nu \in \mathcal{M}(\mathbb{R}^d)$ as

$$\mu \circledast \nu := \lim_{r\to\infty} \frac{\mu_r * \nu_r}{\mathrm{vol}(B_r)}, \tag{1.5}$$

where μ_r and ν_r are the restrictions of μ and ν to $B_r(0)$. The limit in Eq. (1.5) need not exist, but if μ and ν are translation bounded, there is at least one accumulation point [BG13, Prop. 9.1]. This will be the relevant situation in Chapter 6.

CHAPTER 2

Noble means substitutions

In this chapter, we introduce two families of substitutions to which we will restrict ourselves during the rest of this work. We start in Section 2.1 with the family $\mathcal{N}$ of (deterministic) noble means substitutions of which the famous *Fibonacci substitution* is presumably its most prominent and best examined member; compare for example [Sin06, Fog02, BG13]. After a brief collection of some important properties, we extend $\mathcal{N}$ to a one-parameter family $\mathcal{R}$ of random substitutions in Section 2.2, where $\mathcal{R}$ can be regarded as a generalisation of the *random Fibonacci substitution* that was introduced in [GL89, Sec. 5.1].

2.1. The deterministic case

Here, we work over the binary alphabet $\mathcal{A}_2 = \{a, b\}$. For an arbitrary but fixed $m \in \mathbb{N}$ and any $0 \leqslant i \leqslant m$, the substitution $\zeta_{m,i} \colon \mathcal{A}_2^* \longrightarrow \mathcal{A}_2^*$ is defined as

$$\zeta_{m,i} \colon \begin{cases} a & \longmapsto & a^i b a^{m-i}, \\ b & \longmapsto & a, \end{cases} \qquad \text{where} \quad M_m := M_{\zeta_{m,i}} := \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1}$$

is its substitution matrix, which does not depend on i. As $\det(M_m) = -1$, the substitution matrix is *unimodular*. This is a convenient property when the question for a *cut and project* description of associated point sets arises, which will be discussed in Chapter 5. For a detailed description of non-unimodular cases, we refer to [Sin06, Ch. 6.10]. For fixed $m \in \mathbb{N}$, we denote by

$$\mathcal{N} := \mathcal{N}_m := \{\zeta_{m,i} \mid 0 \leqslant i \leqslant m\}$$

the family of *noble means substitutions (NMS)*; see Figure 2.1 for a representation for each of its members as a directed pseudo graph. As $M_m^2 \gg 0$, each substitution in $\mathcal{N}_m$ is primitive, hence there is a power $k \in \mathbb{N}$ such that $\zeta_{m,i}^k$ admits the construction of a bi-infinite fixed point by Lemma 1.7. In the case of $0 < i < m$, all $\zeta_{m,i}$ directly yield one and the same fixed point for all legal seeds $a \,|\, a$, $a \,|\, b$ and $b \,|\, a$, whereas for $i = 0$, we get two different fixed points for $\zeta_{m,i}^2$ with legal seeds $a \,|\, a$ and $a \,|\, b$, and for $i = m$ there are two fixed points with respect to $\zeta_{m,i}^2$ with legal seeds $a \,|\, a$ and $b \,|\, a$. Note that each of the legal seeds $b \,|\, a$, in the case of $i = 0$, and $a \,|\, b$, in the case of $i = m$, leads to the same fixed point as the one with seed $a \,|\, a$. Furthermore, it is clear that $b \,|\, b$ is not a legal seed in the sense of Definition 1.6 for any m or i. Considering $v := a^i b a^{m-i}$, all these assertions

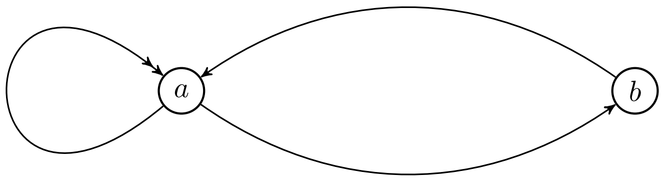

Figure 2.1. The associated pseudo graph for $\zeta_{m,i}$ is independent of i. The loop at a is traversed m times.

can directly be deduced from the first iterations

$$\begin{aligned} a \,|\, a &\xrightarrow{\zeta_{m,i}} v \,|\, v \xrightarrow{\zeta_{m,i}} v^i a v^{m-i} \,|\, v^i a v^{m-i} \\ &\xrightarrow{\zeta_{m,i}} (v^i a v^{m-i})^i \, v (v^i a v^{m-i})^{m-i} \,|\, (v^i a v^{m-i})^i \, v (v^i a v^{m-i})^{m-i} \\ &\xrightarrow{\zeta_{m,i}} \cdots \end{aligned} \tag{2.2}$$

and their analogues for the other legal seeds $a \,|\, b$ and $b \,|\, a$, by distinguishing the relevant cases for $0 \leqslant i \leqslant m$.

Remark 2.1 (The cases $i = 0$ and $i = m$)**.** It is worthwhile to mention that $\zeta_{m,0}$ maps the fixed point $w^{(a|a)}$ of $\zeta_{m,0}^2$ to the fixed point $w^{(a|b)}$ of $\zeta_{m,0}^2$ and vice versa. Here, $w^{(x|y)}$ denotes the fixed point with seed $x \,|\, y$. Analogously, $\zeta_{m,m}$ maps the fixed point $w^{(a|a)}$ of $\zeta_{m,m}^2$ to the fixed point $w^{(b|a)}$ of $\zeta_{m,m}^2$ and also the other way around. This can be understood via Eq. (2.2). Moreover, it is easy to check that $\zeta_{m,i}$ admits the construction of semi-infinite fixed points in these cases because we can take the convergent semi-infinite sequences to the left or right of the reference point, respectively, for this purpose. Recall also Remark 1.10 in this context. ♦

The characteristic polynomial of M_m reads $P_m(x) = x^2 - mx - 1$ and its roots are given by

$$\lambda_m := \frac{m + \sqrt{m^2 + 4}}{2} \quad \text{and} \quad \lambda_m' := \frac{m - \sqrt{m^2 + 4}}{2} = m - \lambda_m,$$

which identifies λ_m as the PF eigenvalue of M_m. Here, we find a justification for the term ‘noble means substitution’. The continued fraction expansion of λ_m reads $[m; m, m, m, \ldots]$. This leads to the *golden mean* $\lambda_1 = [1; 1, 1, 1, \ldots]$, the *silver mean* $\lambda_2 = [2; 2, 2, 2, \ldots]$ and so forth.

Evidently, λ_m is an algebraic unit satisfying $\lambda_m > 1$ for all $m \in \mathbb{N}$ and from $\lambda_m^2 - m\lambda_m - 1 = 0$, we conclude the identity $\lambda_m^{-1} = -\lambda_m'$. As a consequence, we have $-1 < \lambda_m' < 0$ for all $m \in \mathbb{N}$, which finally means that λ_m is a PV unit and each $\zeta_{m,i}$ a Pisot substitution. The sequence $(\lambda_m')_{m \in \mathbb{N}}$ is strictly increasing and tends to 0 as $m \to \infty$. Refer to Table 2.1 for representations of the first five λ_m and λ_m' together with their numerical approximations.

m	λ_m	$N(\lambda_m)$	λ'_m	$N(\lambda'_m)$
1	$\frac{1}{2}(1+\sqrt{5})$	1.618034	$\frac{1}{2}(1-\sqrt{5})$	−0.618034
2	$1+\sqrt{2}$	2.414214	$1-\sqrt{2}$	−0.414214
3	$\frac{1}{2}(3+\sqrt{13})$	3.302776	$\frac{1}{2}(3-\sqrt{13})$	−0.302776
4	$2+\sqrt{5}$	4.236068	$2-\sqrt{5}$	−0.236068
5	$\frac{1}{2}(5+\sqrt{29})$	5.192582	$\frac{1}{2}(5-\sqrt{29})$	−0.192582

Table 2.1. The numbers λ_m, λ'_m and their numerical approximations for $1 \leqslant m \leqslant 5$.

2.1.1. The NMS hull. We refer to $\mathbb{X}_{m,i} := \mathbb{X}_{\zeta_{m,i}}$ as the two-sided discrete hull of $\zeta_{m,i}$ in the sense of Definition 1.8. As λ_m is irrational for all $m \in \mathbb{N}$, we conclude from Theorem 1.12 that $\mathbb{X}_{m,i}$ contains no periodic element. This identifies each $\zeta_{m,i}$ as an aperiodic substitution rule. Moreover, due to [BG13, Prop. 4.5], each $\mathbb{X}_{m,i}$ is a Cantor set, that is a metrisable, compact, perfect and totally disconnected topological space. We refer to [Car00, Ch. 2] and [Els11, Ch. 2.8] for background information.

For notational convenience, we introduce $\mathcal{D}'_m$ (and $\mathcal{D}'_{m,\ell}$) as the set of $\zeta_{m,i}$-legal words (of length ℓ) in the sense of Definition 1.6. The 'primed' versions provide distinguishability with regard to the stochastic situation that we consider from Section 2.2 on.

Lemma 2.2 ([Fog02, Lem. 1.1.2])**.** *For any two sequences v, $w \in \mathcal{A}_n^{\mathbb{Z}}$, the following statements are equivalent:*

(a) $v \in \overline{\{S^k w \mid k \in \mathbb{Z}\}}$.
(b) For every r, $s \in \mathbb{N}$, there is an integer $t(r,s) \in \mathbb{Z}$ with the property $v_{[-r,s]} = w_{[t(r,s),t(r,s)+r+s]}$.
(c) $\mathcal{F}_\ell(\{v\}) \subset \mathcal{F}_\ell(\{w\})$ *for all* $\ell \in \mathbb{N}$. ∎

Let $w \in \mathcal{A}_2^{\mathbb{Z}}$ be a fixed point of some $\zeta_{m,i}$ (or $\zeta_{m,i}^2$). Now, we consider

$$v \in \mathbb{X}_{m,i} = \overline{\{S^k w \mid k \in \mathbb{Z}\}}$$

and apply Lemma 2.2. This immediately yields a characterisation of $\mathbb{X}_{m,i}$ via the sets of $\zeta_{m,i}$-legal words which means that

$$\mathbb{X}_{m,i} = \{w \in \mathcal{A}_2^{\mathbb{Z}} \mid \mathcal{F}(\{w\}) \subset \mathcal{D}'_m\}. \tag{2.3}$$

This constitutes another simplification of the definition of $\mathbb{X}_{m,i}$ in the cases $i = 0$ and $i = m$ (besides the one given in Remark 1.10), respectively. Refer also to Corollary 2.4 below in this context. Obviously, this approach is not restricted to the NMS family.

Our final goal from Section 2.2 on will be the mixture of all noble means substitutions for an arbitrary but fixed natural number m. The substitutions of this family are similar to each other in the following sense.

Let $\mathfrak{F}_n := \langle a_1, \dots, a_n \rangle$ be the free group, generated by the letters of $\mathcal{A}_n$. For an arbitrary substitution ϑ on the alphabet $\mathcal{A}_n$ and a fixed $v \in \mathfrak{F}_n$, we denote by $\vartheta_v(w) := v\vartheta(w)v^{-1}$, for any $w \in \mathcal{A}_n^*$, the *conjugate substitution* to ϑ. As indicated on page 4, we now consider the natural extension of substitution rules in the sense of Section 1.2.2 by considering endomorphisms of $\mathfrak{F}_n$. This can be done by defining $\vartheta(a_i^{-1}) := \big(\vartheta(a_i)\big)^{-1}$ for $1 \leqslant i \leqslant n$; see [Lot02, Sec. 2.3.5] for background information. In this setting, we call a substitution *non-negative* if the images of all a_i consist of letters with exclusively non-negative exponents. For $w = w_0 w_1 \cdots w_\ell \in \mathcal{A}_n^*$, the definition of ϑ_v implies

$$\begin{aligned}
\vartheta_v(w) &= v\vartheta(w)v^{-1} = v\vartheta(w_0)\vartheta(w_1)\cdots\vartheta(w_\ell)v^{-1} \\
&= v\vartheta(w_0)v^{-1}v\vartheta(w_1)v^{-1}v\cdots v^{-1}v\vartheta(w_\ell)v^{-1} \\
&= \vartheta_v(w_0)\vartheta_v(w_1)\cdots\vartheta_v(w_\ell),
\end{aligned}$$

because we have $v^{-1}v = \varepsilon$ in $\mathfrak{F}_n$, wherefore it is enough to test conjugacy on all $a_i \in \mathcal{A}_n$. Applying this to $\vartheta := \zeta_{m,j}$ and $v := a^{i-j}$ for a fixed $m \in \mathbb{N}$ and $0 \leqslant i, j \leqslant m$, we get

$$a^{i-j}\zeta_{m,j}(a)a^{j-i} = a^{i-j}a^j b a^{m-j} a^{j-i} = a^i b a^{m-i} = \zeta_{m,i}(a).$$

The equation for the letter b looks similarly and this implies that all $\zeta_{m,i}$ are pairwise conjugate. Moreover, all $\zeta_{m,i}$ are non-negative substitutions, which means that the agreement of all $\mathbb{X}_{m,i}$ is a special case of the following result.

Proposition 2.3 ([BG13, Prop. 4.6])**.** *Let ϑ be a primitive substitution on the finite alphabet $\mathcal{A}_n$, and let v be a finite word such that ϑ_v is a non-negative substitution as well. Then, ϑ_v is primitive and ϑ and ϑ_v define the same two-sided discrete hull.* ∎

Corollary 2.4. *Let $m \in \mathbb{N}$ be arbitrary and fixed. Then, all members of $\mathcal{N}_m$ define the same two-sided discrete hull.*

Proof. This is an immediate consequence of Proposition 2.3 and the preceding discussion. ∎

As Eq. (2.3) and Corollary 2.4 suggest, it is unnecessary to distinguish the sets of $\zeta_{m,i}$-legal words for different i. That justifies the suppression of i-dependence in the notation of $\mathcal{D}'_m$ and $\mathcal{D}'_{m,\ell}$, respectively. Still in the light of Corollary 2.4, we define $\mathbb{X}'_m$ as *the* noble means hull that is

$$\mathbb{X}'_m := \mathbb{X}_{m,0} = \cdots = \mathbb{X}_{m,m}.$$

For any $w \in \mathcal{A}_n^{\mathbb{Z}}$, the *reflection* of w is defined by $\widetilde{w} = (\widetilde{w}_i)_{i\in\mathbb{Z}} := (w_{-i-1})_{i\in\mathbb{Z}}$, relative to the reference point. Furthermore, w is *quasi-palindromic* if w contains arbitrarily long palindromes as subwords and w is an *infinite palindrome*, if

$\widetilde{w} = S^k w$ for some $k \in \mathbb{Z}$. The two-sided discrete hull $\mathbb{X}_\vartheta$ of a substitution ϑ is called *reflection symmetric*, if $w \in \mathbb{X}_\vartheta$ implies $\widetilde{w} \in \mathbb{X}_\vartheta$ and the same terminology carries over to subsets of $\mathcal{A}_n^*$. If $\mathbb{X}_\vartheta$ contains at least one infinite palindrome, ϑ is called *palindromic*.

Proposition 2.5 ([HKS95, Lem. 3.1]). *If ϑ is a primitive substitution on the alphabet $\mathcal{A}_n = \{a_1, \dots, a_n\}$, with $\vartheta(a_i) = pq_i$ for all $1 \leqslant i \leqslant n$, where p and all q_i are palindromes, then ϑ is palindromic.* ∎

Theorem 2.6 ([Tan07, Thm. 3.13]). *If ϑ is a primitive substitution with fixed point $w \in \mathcal{A}_2^{\mathbb{Z}}$, then $\mathcal{F}(\{w\})$ is reflection symmetric if and only if w is quasi-palindromic.* ∎

The following two properties of the substitutions $\zeta_{m,i}$ are of independent interest.

Lemma 2.7. *For any $m \in \mathbb{N}$, the two-sided discrete hull $\mathbb{X}'_m$ is reflection symmetric.*

Proof. Applying Proposition 2.5 with

$$\begin{array}{llll} p = \varepsilon, \quad q_a = a^{m/2} b a^{m/2} & \text{and} & q_b = a & \text{if } m \text{ is even,} \\ p = a, \quad q_a = a^{(m-1)/2} b a^{(m-1)/2} & \text{and} & q_b = \varepsilon & \text{if } m \text{ is odd,} \end{array}$$

yields the palindromicity of $\zeta_{m,i}$ in the cases of

$$i = \frac{1}{2} m \quad \text{and} \quad i = \frac{1}{2}(m-1) + 1.$$

Note that, in the special case of $m = 1$, we only have a semi-infinite fixed point for $\zeta_{1,1}$ itself but due to Remark 2.1 this suffices to construct the correct two-sided discrete hull. According to [Tan07, Lem. 3.10] the fixed points in these cases are quasi-palindromic which is by Theorem 2.6 equivalent to $\mathcal{D}'_m$ being reflection symmetric. With the characterisation of $\mathbb{X}'_m$ given in Eq. (2.3) combined with Corollary 2.4, this implies the reflection symmetry of $\mathbb{X}'_m$. ∎

Lemma 2.8. *If $w \in \mathcal{A}_2^*$, one has for any $0 \leqslant i \leqslant m$ that*

$$\widetilde{\zeta_{m,i}(w)} a^{m-2i} = a^{m-2i} \zeta_{m,i}(\widetilde{w}).$$

Proof. If $w = w_0 w_1 \cdots w_\ell$ then

$$\begin{aligned} \widetilde{\zeta_{m,i}(w)} &= \big(\zeta_{m,i}(w_0) \cdots \zeta_{m,i}(w_\ell)\big)^{\sim} = \widetilde{\zeta_{m,i}(w_\ell)} \cdots \widetilde{\zeta_{m,i}(w_0)} \\ &= \zeta_{m,m-i}(w_\ell) \cdots \zeta_{m,m-i}(w_0) \\ &= a^{m-2i} \zeta_{m,i}(w_\ell) a^{2i-m} \cdots a^{m-2i} \zeta_{m,i}(w_0) a^{2i-m} \\ &= a^{m-2i} \zeta_{m,i}(\widetilde{w}) a^{2i-m}. \end{aligned}$$ ∎

Let us sum up the properties we have collected about the family $\mathcal{N}_m$ as follows.

Lemma 2.9. *For an arbitrary but fixed $m \in \mathbb{N}$, each member of $\mathcal{N}_m$ is a primitive and aperiodic Pisot substitution with unimodular substitution matrix. Its two-sided discrete hulls $\mathbb{X}_{m,i}$ are uncountable and reflection symmetric, and the $\mathbb{X}_{m,i}$ coincide for $0 \leqslant i \leqslant m$.* ■

2.2. The randomised case

In this section, we establish the fundamentals for the local mixture of substitutions from the noble means family. Here, the term *local* means that, for each letter x of some word w, we randomly apply an element of $\mathcal{N}_m$ to x, for a fixed $m \in \mathbb{N}$. This can be regarded as a generalisation of global mixtures where some element of $\mathcal{N}_m$ is randomly chosen and then applied to all letters in w. The local mixture in the special case of $m = 1$ was first considered in [GL89, Sec. 5.1] and more than twenty years later in [Lüt10]. There, one also finds heuristic considerations concerning the computation of topological entropy and the spectral type of the diffraction measure. We begin this section with the general idea of a random substitution.

Definition 2.10 (Random substitution)**.** A substitution $\vartheta\colon \mathcal{A}_n^* \longrightarrow \mathcal{A}_n^*$ is called *stochastic* or a *random substitution* if there are $k_1, \dots, k_n \in \mathbb{N}$ and probability vectors

$$\Big\{\boldsymbol{p}_i = (p_{i1}, \dots, p_{ik_i}) \mid \boldsymbol{p}_i \in [0,1]^{k_i} \text{ and } \sum_{j=1}^{k_i} p_{ij} = 1,\ 1 \leqslant i \leqslant n\Big\},$$

such that

$$\vartheta\colon a_i \longmapsto \begin{cases} w^{(i,1)}, & \text{with probability } p_{i1}, \\ \quad\vdots & \qquad\quad \vdots \\ w^{(i,k_i)}, & \text{with probability } p_{ik_i}, \end{cases}$$

for $1 \leqslant i \leqslant n$ where each $w^{(i,j)} \in \mathcal{A}_n^*$. The *substitution matrix* is defined by

$$M_\vartheta := \Big(\sum_{q=1}^{k_j} p_{jq} |w^{(j,q)}|_{a_i}\Big)_{ij} \in \mathrm{Mat}(n, \mathbb{Z}).$$

Remark 2.11 (Randomised subword relation)**.** In the stochastic situation we agree on a slightly modified notion of the subword relation. For any v, $w \in \mathcal{A}_n^*$, by $v \triangleleft \vartheta^k(w)$ we mean that v is a subword of *at least* one image of w under ϑ^k for any $k \in \mathbb{N}$. Similarly, by $v \doteq \vartheta^k(w)$ we mean that there is *at least* one image of w under ϑ^k that coincides with v. ♦

Definition 2.12 (Irreducibility and primitivity for random substitutions)**.** A random substitution $\vartheta\colon \mathcal{A}_n^* \longrightarrow \mathcal{A}_n^*$ is *irreducible* if for each pair (i,j) with $1 \leqslant i,j \leqslant n$, there is a power $k \in \mathbb{N}$ such that $a_i \triangleleft \vartheta^k(a_j)$. The substitution ϑ is *primitive* if there is a $k \in \mathbb{N}$ such that $a_i \triangleleft \vartheta^k(a_j)$ for all $1 \leqslant i,j \leqslant n$.

Remark 2.13. As in the deterministic case, a random substitution ϑ is irreducible (primitive) if and only if M_ϑ is an irreducible (primitive) matrix. This follows immediately from Definition 2.12 and Definition 1.2. ♦

Now, let $m \in \mathbb{N}$ and $\boldsymbol{p}_m = (p_0, \dots, p_m)$ be a probability vector that are both assumed to be fixed. That means $\boldsymbol{p}_m \in [0,1]^{m+1}$ and $\sum_{j=0}^{m} p_j = 1$. The random substitution $\zeta_m \colon \mathcal{A}_2^* \longrightarrow \mathcal{A}_2^*$ is defined by

$$\zeta_m \colon \begin{cases} a & \longmapsto \begin{cases} \zeta_{m,0}(a), & \text{with probability } p_0, \\ \quad \vdots & \qquad\qquad \vdots \\ \zeta_{m,m}(a), & \text{with probability } p_m, \end{cases} \\ b & \longmapsto \ a, \end{cases} \tag{2.4}$$

and the one-parameter family $\mathcal{R} = \{\zeta_m\}_{m\in\mathbb{N}}$ is called the family of *random noble means substitutions (RNMS)*. We refer to the p_j as the *choosing probabilities* and call $\zeta_m(w)$ for any $w \in \mathcal{A}_2^*$ an *image* of w under ζ_m. Of course, the deterministic cases of Section 2.1 (choose the corresponding $p_j = 1$) and *incomplete* mixtures, with several $p_j = 0$, are included here, but we are mainly interested in the generic cases where $\boldsymbol{p}_m \gg 0$. This is a standing assumption for the rest of the treatment, where we occasionally comment on the disregarded cases if this seems appropriate.

The substitution matrix of ζ_m in the sense of Definition 2.10 is given by

$$M_m := \begin{pmatrix} \sum_{j=0}^{m} p_j |\zeta_{m,j}(a)|_a & 1 \\ \sum_{j=0}^{m} p_j |\zeta_{m,j}(a)|_b & 0 \end{pmatrix} = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix}.$$

We do not distinguish in notation between the deterministic and the stochastic case here, as the correct meaning will always be clear from the context.

Example 2.14 (Random Fibonacci substitution)**.** One possible image of $a \,|\, a$ under ζ_1^3 can be derived via

$$a \,|\, a \xrightarrow{\zeta_1} ba \,|\, ab \xrightarrow{\zeta_1} aab \,|\, aba \xrightarrow{\zeta_1} abbaa \,|\, baaab \xrightarrow{\zeta_1} \cdots$$

Black letters mean that the according image results from the action of ζ_1 on b whereas red and blue words indicate the two possibilities one has when evaluating ζ_1 on a. Note that the image $w = abbaa \,|\, baaab$ is not legal with respect to any of the two deterministic Fibonacci substitutions $\zeta_{1,0}$ and $\zeta_{1,1}$ because w contains bb as a subword. ♦

Lemma 2.15. *For all $m \in \mathbb{N}$, the substitution ζ_m is primitive.*

Proof. This is an immediate consequence of Definition 2.12. ∎

Remark 2.16 (Local vs. global mixtures)**.** Note that, due to Corollary 2.4, the global mixture of the substitutions in $\mathcal{N}_m$ would not enlarge the two-sided discrete hull, whereas we will see shortly that the local mixture adds considerably to $\mathbb{X}'_m$. ♦

Remark 2.17 (ζ_m as a random variable)**.** Below, it will be convenient to interpret ζ_m as a random variable $\zeta_m \colon \mathcal{A}_2^* \longrightarrow \mathcal{A}_2^*$ with the possible outcomes

$$\zeta_m(a) \in \{a^i b a^{m-i} \mid m \in \mathbb{N},\, 0 \leqslant i \leqslant m\} \quad \text{and} \quad \zeta_m(b) = a.$$

Here, we do not distinguish in notation between the meaning of ζ_m as a substitution rule and as a random variable but we agree on referring to $\zeta_m^k(w)$ as a *realisation* of w with respect to ζ_m^k in this context. ♦

2.2.1. Legal and exact RNMS words. In the stochastic setting, a slightly modified notion of legality of words is needed that incorporates all possible images of b under ζ_m^k in the sense of Remark 2.11.

Definition 2.18 (Legal and exact words)**.** A word $w \in \mathcal{A}_2^*$ is called *legal* (or ζ_m*-legal*) if there is a $k \in \mathbb{N}$ such that $w \triangleleft \zeta_m^k(b)$. For $\ell \geqslant 0$, we define

$$\mathcal{D}_m \coloneqq \{w \in \mathcal{A}_2^* \mid w \text{ is } \zeta_m\text{-legal}\} \quad \text{and} \quad \mathcal{D}_{m,\ell} \coloneqq \{w \in \mathcal{D}_m \mid |w| = \ell\}.$$

If $w \doteq \zeta_m^k(b)$ for some $k \in \mathbb{N}_0$, we refer to w as an *exact substitution word*. We define for any $k \geqslant 1$ the set of exact substitution words (of order k) as

$$\mathcal{G}_{m,k} \coloneqq \{w \in \mathcal{A}_2^* \mid w \doteq \zeta_m^{k-1}(b)\}.$$

Obviously, all subwords of legal words are legal words again. Furthermore, non-empty legal words have the property that they are mapped to non-empty legal words by ζ_m because if $w \in \mathcal{D}_m$, there is a $k \in \mathbb{N}$ with $w \triangleleft \zeta_m^k(b)$. Applying ζ_m once more, immediately leads to $\zeta_m(w) \triangleleft \zeta_m^{k+1}(b)$ which is the legality of $\zeta_m(w)$.

There is an alternative approach to the derivation of $\mathcal{G}_{m,k}$ via the following recursively defined concatenation rule that will become an important tool when studying entropy and diffraction later. For $k \geqslant 3$, we define

$$\mathcal{G}_{m,k} \coloneqq \bigcup_{i=0}^{m} \prod_{j=0}^{m} \mathcal{G}_{m,k-1-\delta_{ij}} \quad \text{with} \quad \mathcal{G}_{m,1} \coloneqq \{b\} \quad \text{and} \quad \mathcal{G}_{m,2} \coloneqq \{a\}\,, \tag{2.5}$$

where δ_{ij} denotes the *Kronecker function*. The product in Eq. (2.5) is understood via the concatenation of words, introduced on page 2. That the substitution-based approach to exact words of Definition 2.18 actually provides the same sets $\mathcal{G}_{m,k}$ as the one via the concatenation rule, follows from an easy proof by induction over k. As all possible images $\zeta_m(a)$ have the same word lengths, we can recursively compute the lengths of exact substitution words. For $k \geqslant 3$, we define

$$\ell_{m,k} \coloneqq m\ell_{m,k-1} + \ell_{m,k-2} \quad \text{with} \quad \ell_{m,1} \coloneqq 1 \quad \text{and} \quad \ell_{m,2} \coloneqq 1. \tag{2.6}$$

Here, we have $|w| = \ell_{m,k}$ for any exact substitution word $w \in \mathcal{G}_{m,k}$. Furthermore, we define $\mathcal{G}_m \coloneqq \lim_{k\to\infty} \mathcal{G}_{m,k}$. Due to [Nil12, Prop. 7], this is a well-defined set because for a fixed $m \in \mathbb{N}$, $k \geqslant 3$ and $r \geqslant 0$ we have

$$\left(\mathcal{G}_{m,k}\right)_{[0,\ell_{m,k}-2]} = \left(\mathcal{G}_{m,k+r}\right)_{[0,\ell_{m,k}-2]}.$$

Remark 2.19 (Legal and exact words)**.** It is immediate that not all ζ_m-legal words can be exact, as one can see from the words aa, $bb \in \mathcal{D}_{m,2} \setminus \mathcal{G}_{m,3}$. That indicates $|\mathcal{G}_{m,k}| < |\mathcal{D}_{m,\ell_{m,k}}|$ for $k \geqslant 3$. Furthermore, one might suspect that the sets of legal words are considerably enlarged when compared to the deterministic situation (regard the word bb again) which will be discussed in Chapter 3 in some detail. ♦

2.2.2. The RNMS hull. In the deterministic setting, we defined the two-sided discrete hull of primitive substitutions via fixed points. In the context of random substitutions, there is no direct analogue of a fixed point, which means that we have to adjust the definition of the hull here.

Definition 2.20 (Two-sided discrete stochastic hull)**.** For an arbitrary but fixed $m \in \mathbb{N}$, define

$$X_m := \Big\{ w \in \mathcal{A}_2^{\mathbb{Z}} \mid w \text{ is an accumulation point of } \big(\zeta_m^k(a \,|\, a)\big)_{k \in \mathbb{N}_0} \Big\}.$$

The *two-sided discrete stochastic hull* $\mathbb{X}_m$ is defined as the smallest closed and shift-invariant subset of $\mathcal{A}_2^{\mathbb{Z}}$ with $X_m \subset \mathbb{X}_m$. Elements of X_m are called *generating random noble means words.*

As $\mathbb{X}_m$ is a closed subset of the compact Hausdorff space $\mathcal{A}_2^{\mathbb{Z}}$, the two-sided discrete stochastic hull is also compact. This implies that each sequence $\big(\zeta_m^k(a \,|\, a)\big)_{k \in \mathbb{N}}$ possesses at least one accumulation point which means that $\mathbb{X}_m$ is non-empty.

Remark 2.21 (The role of $\boldsymbol{p}_m$)**.** In principle, the sets $\mathcal{D}_{m,\ell}$, $\mathcal{G}_{m,k}$ and the stochastic hull $\mathbb{X}_m$ depend on the choice of $\boldsymbol{p}_m$. For example, the word bb is no longer legal if one has $p_0 = 0$ or $p_m = 0$. In contrast, the above sets are invariant under different choices of $\boldsymbol{p}_m$, as long as $\boldsymbol{p}_m \gg 0$. As this is our standing assumption, we do not explicitly emphasise the $\boldsymbol{p}_m$-dependence. ♦

The next result characterises $\mathbb{X}_m$ by the ζ_m-legal subwords as we have similarly seen before in the deterministic case in Eq. (2.3) on page 15. The special case of $m = 1$ can be found in [Lüt10, Rem. 7].

Proposition 2.22. *For any $m \in \mathbb{N}$, we have*

(a) $\mathbb{X}_m = \{ w \in \mathcal{A}_2^{\mathbb{Z}} \mid \mathcal{F}(\{w\}) \subset \mathcal{D}_m \}$,
(b) $\mathbb{X}_m$ *is invariant under* ζ_m,
(c) $\mathbb{X}'_m \subset \mathbb{X}_m$ *and* $\mathbb{X}'_m \neq \mathbb{X}_m$.

Proof. Let $(v^{(n)})_{n \in \mathbb{N}_0}$ be a two-sided growing sequence, that is defined by $v^{(n)} := \zeta_m(v^{(n-1)})$ with any ζ_m-legal seed $v^{(0)}$. As $\mathbb{X}_m$ is a compact space, there exists a convergent subsequence $(v^{(n_i)})_{i \in \mathbb{N}}$ with $\lim_{i \to \infty} v^{(n_i)} = v$, where the limit is taken in the local topology. By the definition of the stochastic hull, we have $v \in \mathbb{X}_m$ and, due to its shift invariance, we also get $S^k v \in \mathbb{X}_m$ for all $k \in \mathbb{Z}$ as well as $\overline{\{S^k v \mid k \in \mathbb{Z}\}} \subset \mathbb{X}_m$, where the closure is again taken in the local topology. By the definition of $\mathbb{X}_m$, we now know that $\mathcal{F}(\overline{\{S^k v \mid k \in \mathbb{Z}\}}) \subset \mathcal{D}_m$. As this reasoning is valid for an arbitrary element in $\mathbb{X}_m$, we find

$$\mathbb{X}_m \subset \{w \in \mathcal{A}_2^{\mathbb{Z}} \mid \mathcal{F}(\{w\}) \subset \mathcal{D}_m\}.$$

Conversely, let $u \in \{w \in \mathcal{A}_2^{\mathbb{Z}} \mid \mathcal{F}(\{w\}) \subset \mathcal{D}_m\}$. Then, $u_{[-\ell,\ell]} \in \mathcal{D}_m$ for all $\ell \in \mathbb{N}$ and by the definition of $\mathcal{D}_m$, there is a sequence $\big(S^{j_\ell}\zeta_m^{k_\ell}(b)\big)_{\ell\in\mathbb{N}}$ with $j_\ell \in \mathbb{Z}$ and $k_\ell \in \mathbb{N}$ such that

$$u_{[-\ell,\ell]} = \big(S^{j_\ell}\zeta_m^{k_\ell}(b)\big)_{[-\ell,\ell]}$$

for all $\ell \in \mathbb{N}$. Then, (a) is implied by $\lim_{\ell\to\infty} S^{j_\ell}\zeta_m^{k_\ell}(b) = u$. Assertion (b) follows by the observation that $\zeta_m(\mathcal{D}_m) \subset \mathcal{D}_m$, as noted on page 20, and an application of (a). For the proof of the first part of (c), it is enough to notice that $\mathbb{X}_m$ contains the fixed points of all $\zeta_{m,i}$ (or $\zeta_{m,i}^2$). That the inclusion is proper can be deduced from $bb \notin \mathcal{D}'_m$ for all $m \in \mathbb{N}$, combined with an application of (a). ∎

Remark 2.23 (Identical realisations)**.** Let

$$r\colon \mathbb{Z} \longrightarrow \mathcal{A}_2^* \times \mathcal{A}_2^*, \quad i \longmapsto \big(\zeta_m^{(i)}(a), \zeta_m^{(i)}(b)\big)$$

be a fixed bi-infinite realisation of the random variable ζ_m, where $\zeta_m^{(i)}$ denotes the realisation at the index i. Then, we define the map

$$R_r\colon \mathcal{A}_2^{\mathbb{Z}} \longrightarrow \mathcal{A}_2^{\mathbb{Z}}, \quad w = (w_i)_{i\in\mathbb{Z}} \longmapsto \big(p_r(w_i)\big)_{i\in\mathbb{Z}},$$

where p_r projects w_i to the first component of r_i if $w_i = a$ and to the second component if $w_i = b$. Now, two elements $v, w \in \mathbb{X}_m$ are called *identically realised* if both v and w are mapped via R_r with respect to the very same realisation r.

This concept carries over to arbitrary random substitutions on $\mathcal{A}_n$ and can be applied to finite and semi-infinite words as well. ♦

Remark 2.24. Note that the substitution ζ_m is not one-to-one on $\mathbb{X}_m$. To see this, let us consider the case $m = 1$. The general case can be treated analogously. Let $w \in \mathcal{A}_2^{\mathbb{Z}}$ be the fixed point of $\zeta_{1,1}^2$ with legal seed $a \,|\, a$. According to Proposition 2.22(c), we know that $w \in \mathbb{X}_1$. By considering the first iterates of $\zeta_{1,1}^2$, it is easy to see that w is of the form $w = Pba \,|\, S$. Due to Remark 2.1, $\zeta_{1,1}(w) = w'$ is the fixed point of $\zeta_{1,1}^2$ with legal seed $b \,|\, a$ and is of the form $w' = Pab \,|\, S \in \mathbb{X}_1$. Applying ζ_1 to w and w' yields

$$\zeta_1(w) = \zeta_1(P)\left\{\begin{array}{c} aab \\ aba \end{array}\right\} | \,\zeta_1(S) \quad \text{and} \quad \zeta_1(w') = \zeta_1(P)\left\{\begin{array}{c} aba \\ baa \end{array}\right\} | \,\zeta_1(S).$$

Here, in the computation of $\zeta_1(w)$ and $\zeta_1(w')$, we assume that P and S are identically realised. As the word aba appears in both images in the middle part, we conclude that although $w \neq w'$ there exist images of w and w' with $\zeta_1(w) = \zeta_1(w')$. ♦

Remark 2.25 (Pre-images in $\mathbb{X}_m$)**.** Restricting to identical realisations in the sense of Remark 2.23, the continuity of ζ_m is just a special case of Remark 1.1.

Now, let $w \in \mathbb{X}_m$ be any element of the two-sided discrete stochastic hull. We want to show that there is at least one $v \in \mathbb{X}_m$ with $\zeta_m(v) \doteq w$. Finding pre-images is obviously connected with the choice of a concrete realisation which means that we can make use of the continuity of ζ_m, that was established in Remark 1.1, here. Recall that, due to Proposition 2.22, all finite subwords of w are ζ_m-legal. This means that there is a $k \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}_0$ the centrally positioned subword $w_{[-\ell,\ell]}$ of w is a subword of $\zeta_m^k(b)$ and we choose k minimal with this property. This implies that there is some image $v_\ell \coloneqq \zeta_m^{k-1}(b)$ with $w_{[-\ell,\ell]} \triangleleft \zeta_m(v_\ell)$ and we choose the indexing of $\zeta_m(v_\ell)$ such that its subword $w_{[-\ell,\ell]}$ is again centrally positioned around the reference point. Due to the compactness of $\mathbb{X}_m$, there is a sequence $(\ell_i)_{i\in\mathbb{N}}$ such that the limit $v \coloneqq \lim_{i\to\infty} v_{\ell_i}$ exists. Putting things together yields

$$w = \lim_{i\to\infty} w_{[-\ell_i,\ell_i]} = \lim_{i\to\infty} \zeta_m(v_{\ell_i}) = \zeta_m\big(\lim_{i\to\infty} v_{\ell_i}\big) = \zeta_m(v).$$

Here, the second equality is the definition of the local topology and the third equality the continuity of ζ_m. As we saw in Remark 2.24, the pre-image of any such w is in general not uniquely defined. This is a further difference to the purely deterministic cases where this uniqueness holds and this property is known as *unique recognisability*; we refer to [Que10, Sec. 5.5.2] and [Fog02, Sec. 7.2.1] for background. ♦

Definition 2.26 (Topologically transitive)**.** Let X be a metric space and $T\colon X \longrightarrow X$ a continuous map. The system (X, T) is *topologically transitive* if, for every pair of non-empty open sets U, $V \subset X$, there is a $k \in \mathbb{Z}$ such that $T^k(U) \cap V \neq \varnothing$.

Theorem 2.27 ([Wal00, Thm. 5.8])**.** *Let X be a compact metric space and $T\colon X \longrightarrow X$ be a homeomorphism. The system (X,T) is topologically transitive if and only if there exists a point $x \in X$ such that its orbit $\{T^k(x) \mid k \in \mathbb{Z}\}$ is dense in X.* ■

The following Proposition is a generalisation of the corresponding result in the special case of $m = 1$ that was proved in [Lüt10, Thm. 5].

Proposition 2.28. *For any $m \in \mathbb{N}$, the system $(\mathbb{X}_m, S)$ is topologically transitive.*

Proof. Consider $v \in \mathbb{X}'_m \subset \mathbb{X}_m$. As $bb \notin \mathcal{D}'_{m,2}$, the letter a is relatively dense in v. We define a sequence $(w^{(k)})_{k\in\mathbb{N}}$ by $w^{(0)} \coloneqq v$ and $w^{(k)} \coloneqq \zeta_m^k(w^{(0)})$ such that the first $|\mathcal{G}_{m,k+2}|$ a's, positioned centrally around the marker of $w^{(0)}$, are bijectively mapped by ζ_m^k onto the words in $\mathcal{G}_{m,k+2}$. This construction directly implies $(w^{(k)})_{k\in\mathbb{N}} \subset \mathbb{X}_m$ by Proposition 2.22(b) and $\mathcal{G}_{m,k+2} \subset \mathcal{F}(\{w^{(k)}\})$. According to [Nil12, Prop. 10], we get $\mathcal{D}_{m,\ell_{m,k+1}} \subset \mathcal{F}(\mathcal{G}_{m,k+2})$ for all $k \geqslant 2$ which implies $\mathcal{D}_{m,q} \subset \mathcal{F}(\mathcal{G}_{m,k+2})$ for all $q \leqslant \ell_{m,k+1}$. We end up with $\mathcal{D}_{m,q} \subset \mathcal{F}(\{w^{(k)}\})$ for all $q \leqslant \ell_{m,k+1}$. As $\mathbb{X}_m$ is a compact metric space, there is a subsequence $(w^{(k_i)})_{i\in\mathbb{N}}$

converging to a word $w \in \mathbb{X}_m$ and $\mathcal{D}_m \subset \mathcal{F}(\{w\})$ which means

$$\overline{\{S^k w \mid k \in \mathbb{Z}\}} = \mathbb{X}_m.$$

The assertion follows by an application of Lemma 1.9 and Theorem 2.27. ∎

The bi-infinite word w, constructed in the proof of Proposition 2.28, is the closest analogon to a fixed point that we can reach in the stochastic setting. In all deterministic cases of $\mathcal{N}$, one knows that for all $w \in \mathbb{X}'_m$, the shift orbit $\{S^k w \mid k \in \mathbb{Z}\}$ is dense in $\mathbb{X}'_m$. This stronger property is called the *minimality* of the system $(\mathbb{X}'_m, S)$ and it applies to (deterministic) primitive substitutions in general [Que10, Prop. 5.5]. Note that the system $(\mathbb{X}_m, S)$ cannot be minimal because of Proposition 2.22(c).

2.2.3. RNMS as a stochastic process. As pointed out on page 18, the locality of ζ_m-applications means that we decide on each occurrence of the letter a separately which of the $m+1$ possible realisations we choose. This amounts to $(m+1)^{|w|_a}$ different realisations of any word $w \in \mathcal{A}_2^*$ under ζ_m. Furthermore, we can attach to each of the possible outcomes of $\zeta_m^k(b)$ for any $k \in \mathbb{N}$ its realisation probability that results from the tree structure that is indicated in Figure 2.2.

Here, we have to cope with a slight technical difficulty. Interpreting successive applications of ζ_m as a graph, Figure 2.2 indicates that the leaves at an arbitrary depth k need not be distinct. This means that in general there is more than one path from the root b to any exact word $w \doteq \zeta_m^k(b)$. Attaching the realisation probabilities to these words, we would rather identify leaves containing equal words and suitably attach a single realisation probability. This can be formalised as follows. Firstly, we introduce a map that identifies equal words and sums their realisation probabilities. To this end, consider a family $S := \{(w_i, p_i)\}_{i \in I}$, for a finite index set I, consisting of pairs that hold finite words in $\mathcal{A}_2^*$ in the first component and real numbers in the second. Now, we define the set $\{w_{i_1}, \dots, w_{i_n}\} := \bigcup_{i \in I}\{w_i\}$ where $\{i_1, \dots, i_n\} \subset I$ and $n \in \mathbb{N}$. Note that we have identified multiple occurrences of words in S and now suitably attach real numbers. Therefore, let

$$q_k := \sum_{\substack{i \in I, \\ w_i = w_k}} p_i \quad \text{for all} \quad k \in \{i_1, \dots, i_n\}.$$

Then, we define a map h via $h(S) := \{(w_{i_1}, q_{i_1}), \dots, (w_{i_n}, q_{i_n})\}$. There are precisely $(m+1)^{|w|_a}$ sequences

$$\boldsymbol{t} := (t_1, t_2, \dots, t_{|w|_a}) \in \{0, \dots, m\}^{|w|_a} =: T(w),$$

corresponding to all possible realisations of any word $w \in \mathcal{A}_2^*$ under ζ_m. For any family $S := \{(w_i, p_i)\}_{i \in I}$, with a finite index set I, we define a new map, again

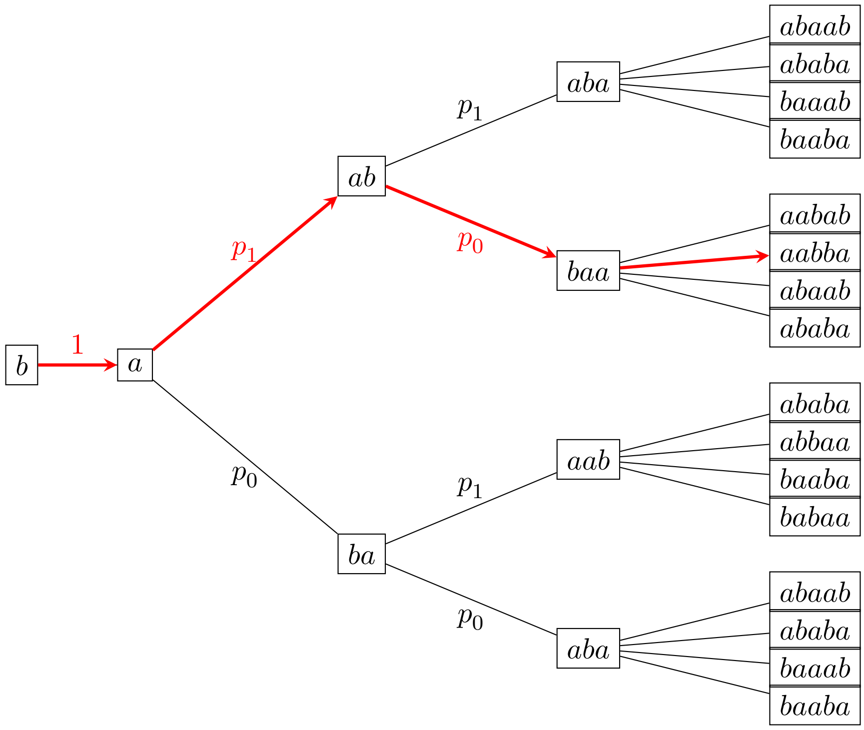

Figure 2.2. The graph shows all possible realisations of $\zeta_1^4(b)$, provided $\boldsymbol{p}_1 \gg 0$. The red path represents one of the 16 possible realisations of $\zeta_1^4(b)$. Note that not all images are distinct and in particular, the realisation probability for the word *aabba* equals $p_0^2 p_1^2$.

denoted as ζ_m, by

$$\zeta_m(S) := h\Bigg\{ \dot{\bigcup_{(w,p_w)\in S}} \Bigg(\dot{\bigcup_{\boldsymbol{t}\in T(w)}} \Big\{ \Big(\zeta_m^{(\boldsymbol{t})}(w), p_w \prod_{k\in \boldsymbol{t}} p_k \Big) \Big\} \Bigg) \Bigg\},$$

where $\zeta_m^{(\boldsymbol{t})}(w)$ refers to the realisation of w with respect to $\boldsymbol{t}$. Note that the disjoint union is meant to preserve the multiplicity of equal elements because h still has to cope with the corresponding probabilities. As the intuition behind this procedure is rather obvious, we do not aim to be more rigorous at this point. Now, we inductively define the attachment of realisation probabilities to exact RNMS words. Provide the word b with the probability 1 and let $\zeta_m(\{(b,1)\}) := \{(a,1)\}$. Now, assume that the family of pairs

$$\zeta_m^{k-1}(\{(b,1)\}) = \{(w_i,p_i)\}_{i\in I}$$

with all w_i distinct is already defined. Then, we set

$$\zeta_m^k(\{(b,1)\}) := \zeta_m\Big(\zeta_m^{k-1}(\{(b,1)\})\Big)$$

for $k \geqslant 2$. This defines a stochastic process (see [DV-J03] for background information) that we we will refer to as $(Z_{m,k})_{k\in\mathbb{N}} := \big(\zeta_m^{k-1}(\{(b,1)\})\big)_{k\in\mathbb{N}}$.

Next, we deduce a stochastic process $(G_{m,k})_{k\in\mathbb{N}}$, based on the concatenation rule of Eq. (2.5). To this end, we introduce the (non-commutative) relation

$$\begin{gathered}\odot\colon (\mathcal{A}_2^* \times [0,1]) \times (\mathcal{A}_2^* \times [0,1]) \longrightarrow \mathcal{A}_2^* \times [0,1],\\ (v,p_v) \odot (w,p_w) \longmapsto (vw, p_v p_w),\end{gathered}$$

which means that we have concatenation of words in the first component and multiplication of real numbers in the second. Moreover, we define the family

$$S \odot T := \{(v,p_v) \odot (w,p_w) \mid (v,p_v) \in S, (w,p_w) \in T\},$$

for families $S := \{(v_i,p_i)\}_{i\in I}$, $T := \{(w_j,p_j)\}_{j\in J}$ with any finite index sets I and J. Based on this, for any $k \geqslant 3$, we define

$$G_{m,k} := h\Big\{\dot{\bigcup_{i=0}^{m}}\Big(\Big(\bigodot_{j=0}^{m} G_{m,k-1-\delta_{ij}}\Big) \odot \{(\varepsilon,p_i)\}\Big)\Big\}, \tag{2.7}$$

with $G_{m,1} := \{(b,1)\}$ and $G_{m,2} := \{(a,1)\}$. The next result validates the intuition that we need not distinguish the stochastic processes $(Z_{m,k})_{k\in\mathbb{N}}$ and $(G_{m,k})_{k\in\mathbb{N}}$.

Lemma 2.29. *The sequences $(Z_{m,k})_{k\in\mathbb{N}}$ and $(G_{m,k})_{k\in\mathbb{N}}$ define identical stochastic processes.*

Proof. First, we show by induction over $n \in \mathbb{N}$ that $\zeta_m(G_{m,n}) = G_{m,n+1}$. This is obvious for $n = 1,2$ and we assume that the claim holds for all $k \leqslant n$. Then,

$$\begin{aligned}\zeta_m(G_{m,n+1}) &= h\Big\{\dot{\bigcup_{i=0}^{m}}\Big(\Big(\bigodot_{j=0}^{m} \zeta_m(G_{m,n-\delta_{ij}})\Big) \odot \{(\varepsilon,p_i)\}\Big)\Big\}\\ &= h\Big\{\dot{\bigcup_{i=0}^{m}}\Big(\Big(\bigodot_{j=0}^{m} G_{m,n+1-\delta_{ij}}\Big) \odot \{(\varepsilon,p_i)\}\Big)\Big\}\\ &= G_{m,n+2},\end{aligned} \tag{2.8}$$

where the first equality is implied by the Bernoulli structure of the process. The second equality, as well as the third, follow by induction and the definition of the set $G_{m,n+2}$. The assertion now follows again by induction because n-fold application of Eq. (2.8) yields

$$Z_{m,n+1} = \zeta_m^n(\{(b,1)\}) = \zeta_m^n(G_{m,1}) = \zeta_m(G_{m,n}) = G_{m,n+1}. \qquad \blacksquare$$

Now, if $w \in \mathcal{G}_{m,k}$, there is a pair $(w,p_w) \in G_{m,k}$ and p_w is called the *realisation probability* of the exact word w induced by the stochastic process $G_{m,k}$.

CHAPTER 3

Topological entropy

This chapter is devoted to the examination of the amount of disorder (or complexity) inherent in the substitution ζ_m. A suitable starting point for the measurement of disorder of an arbitrary substitution rule ϑ on a finite alphabet $\mathcal{A}_n$ is the analysis of the number of legal words with a fixed length. We refer to the map $C\colon \mathbb{N} \longrightarrow \mathbb{N}$, assigning to the natural number ℓ the number of ϑ-legal words of this length, as the *complexity function* of ϑ. In general, we are less interested in the explicit values of this function but rather in the asymptotic behaviour of $C(\ell)$ as $\ell \to \infty$. As $C(\ell) \geqslant \ell + 1$ for all aperiodic substitutions [Que10, Prop. 5.11], the complexity function alone is an insufficient tool when one is interested in more sophisticated assertions on disorder. One reasonable requirement for a decent measure in this regard is that it assigns to any substitution a real number between 0 and a constant $K > 0$. Taking the values of C into account, to structurally easy substitution rules (as periodic ones) should be assigned the value 0 and K to the full n-shift on the alphabet $\mathcal{A}_n$. For other substitutions the measure should increase according to C between 0 and K.

Definition 3.1 (Topological entropy)**.** Let ϑ be a substitution on the alphabet $\mathcal{A}_n$ and C the complexity function of ϑ. Then, we denote by

$$\mathcal{H} := \lim_{\ell\to\infty} \frac{\log(C(\ell))}{\ell} \tag{3.1}$$

the *topological entropy* of ϑ.

Note that the logarithm in Eq. (3.1) is understood to be the natural logarithm with base e. It might seem tempting to decide for a base identical to the number of letters in the considered alphabet, as this would normalise $\mathcal{H}$ to 1 with respect to the full n-shift, but in order to stay consistent with the existing literature [Khi57, GL89] and [Nil12], concerning the noble means setting, we stick to the above choice.

Obviously, the complexity function meets $C(k + \ell) \leqslant C(k)C(\ell)$ for all k, $\ell \in \mathbb{N}$, which makes $\mathcal{H}$ a subadditive function. The existence of the limit in Eq. (3.1) in this setting is a well-known result; see [Gri99, Thm. II.2] and [Hil48, Thm. 6.6.1] for a proof which is based on Fekete's Lemma [Fek23].

For an overview of further concepts of entropy, we refer to [Ber95]. The computation of diverse notions of entropy along several examples of random substitutions has recently been undertaken in [Win11].

We have seen in Chapter 2 that each member of the noble means family is a primitive and aperiodic substitution, which means that each single corresponding complexity function meets $\ell+1 \leqslant C(\ell) \leqslant K\ell$ [Que10, Prop. 5.12] for a constant K that is independent of ℓ. It is not difficult to prove that for any $m \in \mathbb{N}$, the fixed points of all members of $\mathcal{N}_m$ are so-called *Sturmian sequences* which means that the complexity function satisfies $C(\ell) = \ell + 1$ for all $\ell \in \mathbb{N}_0$; see [Lot02, Ch. 2] for general background.

Proposition 3.2. *For any $m \in \mathbb{N}$, each fixed point of any member of $\mathcal{N}_m$ is a Sturmian sequence.*

Proof. Let

$$\tau_0 \colon \begin{cases} a & \longmapsto & b, \\ b & \longmapsto & a, \end{cases} \qquad \tau_1 \colon \begin{cases} a & \longmapsto & ab, \\ b & \longmapsto & a, \end{cases} \qquad \tau_2 \colon \begin{cases} a & \longmapsto & ba, \\ b & \longmapsto & a, \end{cases}$$

be the generators of the Sturmian monoid; see [Lot02, Ch. 2] for background. Then, it is easy to compute that

$$\zeta_{m,i} = \tau_0 \circ \left(\tau_0 \circ \tau_2\right)^{m-i} \circ \left(\tau_0 \circ \tau_1\right)^{i}.$$

This implies that each $\zeta_{m,i}$ is a Sturmian morphism. According to [Lot02, Thm. 2.3.23], the Sturmian morphisms are precisely the non-negative invertible substitutions (see below) on $\mathfrak{F}_2$. As each fixed point of a primitive and invertible substitution is Sturmian [Fog02, Cor. 9.2.7], this implies the assertion. ∎

Extending the notion of substitution rules to $\mathfrak{F}_2$, as we have previously done on page 16, we could have also worked with the inverse of $\zeta_{m,i}$ directly. Here, a substitution $\vartheta \colon \mathfrak{F}_2 \longrightarrow \mathfrak{F}_2$ is *invertible* if there is a map $\varphi \colon \mathfrak{F}_2 \longrightarrow \mathfrak{F}_2$ with $(\vartheta \circ \varphi)(w) = (\varphi \circ \vartheta)(w) = w$ for all $w \in \mathfrak{F}_2$. In the noble means case we find

$$\zeta_{m,i}^{-1} \colon \begin{cases} a & \longmapsto & b, \\ b & \longmapsto & b^{-i}ab^{i-m}, \end{cases} \qquad \text{where} \quad M_m^{-1} = M_{\zeta_{m,i}}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -m \end{pmatrix}$$

is its (generalised) substitution matrix. The detour via the Sturmian monoid and the corresponding decomposition of $\zeta_{m,i}$ is an interesting alternative and plays a role in the computation of *Rauzy fractals* and *cutting sequences*; refer to [BFS12] and references therein.

Corollary 3.3. *The topological entropy of $\zeta_{m,i}$ vanishes for all $m \in \mathbb{N}$ and $0 \leqslant i \leqslant m$.*

Proof. This is an immediate consequence of Proposition 3.2 and Definition 3.1. ∎

These results are not surprising due to the deterministic setting, whereas the picture changes in the stochastic situation. In Section 3.1, we will observe that the computation of the complexity function $C_m \colon \mathbb{N} \longrightarrow \mathbb{N}$, $\ell \longmapsto |\mathcal{D}_{m,\ell}|$ of ζ_m is a difficult problem for general ℓ, and we will give a closed formula for words of length $\ell \leqslant 2m+2$. In Section 3.2, we briefly review an argument by

Nilsson [Nil12] in order to compute the topological entropy of ζ_m without the explicit knowledge of C_m and inspect its behaviour as $m \to \infty$.

3.1. Complexity of the RNMS

The complexity of the substitution ζ_m is measured by the sets of legal words and their cardinalities. Here, we show a simple way of describing the structure of legal words and we specify a closed expression for $|\mathcal{D}_{m,\ell}|$ in the case of $0 \leqslant \ell \leqslant 2m+2$. The explicit computation of the complexity function for arbitrary word lengths stays an open problem. Recall that we assume $\boldsymbol{p}_m \gg 0$, which means that we have to take all $m+1$ possible images of the letter a into account.

We attack the problem via two simple observations. Firstly, the structure of legal words is characterised by the occurrence numbers and legal distributions of the letter b. Secondly, the application of ζ_m to the letter a produces the letter b exactly once. This leads to the following key property.

Lemma 3.4. *Let $m \in \mathbb{N}$ be fixed. For any $0 \leqslant \ell \leqslant 2m+2$, one has $a^\ell \in \mathcal{D}_{m,\ell}$.*

Proof. It is true that $b^2 \in \mathcal{D}_{m,2}$, but b^k with $k \geqslant 3$ is not an element of $\mathcal{D}_{m,k}$. The subword consisting solely of consecutive a's of maximal length can be deduced from

$$ba^{2m+2}b \triangleleft \zeta_m(ab^2a),$$

where it is easy to check that $ab^2a \in \mathcal{D}_{m,4}$ for any $m \in \mathbb{N}$. This implies the assertion. ∎

Lemma 3.5. *Let $m \in \mathbb{N}$ be fixed. For any $0 \leqslant \ell \leqslant m+2$, one has*

$$|\mathcal{D}_{m,\ell}| = 1 + \binom{\ell+1}{2}.$$

Proof. Let $w \in \mathcal{D}_{m,\ell}$, so $|w|_b \leqslant 2$. Assume $|w|_b \geqslant 3$, then there is a legal word v with $|v|_a \geqslant 3$ and $w \triangleleft \zeta_m(v)$. Without loss of generality, let $w \triangleleft \zeta_m(a^3)$ such that $|w|$ is minimal. Then, we get

$$a^m ba^m b^2 a^m \triangleleft \zeta_m(a^3)$$

and consequently $|w| \geqslant m+3$, which is a contradiction to our assumption. Furthermore, it is true that $a^2 \in \mathcal{D}_{m,2}$ and thus all subwords of length ℓ of $\zeta_m(a^2)$ are elements of $\mathcal{D}_{m,\ell}$. It is easy to check that all words $w \in \mathcal{A}_2^\ell$ with $|w|_b \in \{0,1,2\}$ appear as a subword of $\zeta_m(a^2)$. Consequently, we have

$$|\mathcal{D}_{m,\ell}| = \binom{\ell}{0} + \binom{\ell}{1} + \binom{\ell}{2} = 1 + \binom{\ell+1}{2}.$$ ∎

This sequence is listed in the *On-Line Encyclopedia of Integer Sequences* [OEIS] under the reference number $A000124$ and it is known as the *lazy caterer's* *sequence*. It computes the maximal number of pieces formed when slicing a pancake with ℓ cuts.

Lemma 3.6. *Let $m \in \mathbb{N}$ be fixed. For $m+3 \leqslant \ell \leqslant 2m+2$, one has*

$$|\mathcal{D}_{m,\ell}| = \sum_{i=0}^{3} \binom{\ell}{i} - \frac{1}{6} m(m+1)(3\ell - 2m - 4). \tag{3.2}$$

Proof. Analogously to the proof of Lemma 3.5, we conclude that $|w|_b \leqslant 3$ for $|w| \leqslant 2m+2$ by applying ζ_m to the word a^4, which is legal because of Lemma 3.4 and $m \geqslant 1$. Obviously, all words of length ℓ containing the letter b at most twice are again elements of $\mathcal{D}_{m,\ell}$. Between three subsequent b's, the letter a must occur at least m times, as we can see from $a^m b a^i b a^{m-i} b a^m \triangleleft \zeta_m(a^3)$. As the word b^3 is not legal, we have two boundary conditions for the combinatorial inspection of all subwords of $\zeta_m(a^3)$. It is again easy to check that all $w \in \mathcal{A}_2^{\ell}$ with $|w|_b = 3$ and under the two boundary conditions are subwords of $\zeta_m(a^3)$, hence there cannot be more. We may distinguish two possibilities for the distribution of the three b's in a word $w \in \mathcal{D}_{m,\ell}$.

Two b's in a row. Without loss of generality, we may look at a general image of the word a^3 that is

$$a^{m-i} a^i b^2 a^m a^j b a^{m-j} \triangleleft \zeta_m(a^3). \tag{3.3}$$

As we are interested in subwords of length $m+3 \leqslant \ell \leqslant 2m+2$, this leads to the equation $\ell = m+3+(i+j)$ with $i+j \leqslant m-1$. According to Eq. (3.3), the number of possible distributions is obviously symmetric in the position of the word b^2. Consequently, we have exactly

$$2 \sum_{k=1}^{\ell-m-2} k = (\ell - m - 2)(\ell - m - 1)$$

possibilities of constructing a legal word in this case.

Three isolated b's. Let $g(a,b)$ be the occurrence number of the letter a between thrice the letter b. Then, we have $m \leqslant g(a,b) \leqslant \ell - 3$. We list all possible images according to the numbers $g(a,b)$. That means we consider

$$a^{m-i} b a^i a^j b a^{m-j} a^k b a^{m-k} \triangleleft \zeta_m(a^3)$$

and we define $v_{i,j,k} := b a^i a^j b a^{m-j} a^k b$ with $|v_{i,j,k}| = m+3+(i+k)$. By defining $p := i+k$, we get $\ell - m - p - 2$ possible images for fixed p. As we also have $m+p-1$ possibilities to distribute $m+p-1$ times the letter a between three b's, we finally get

$$\sum_{i=0}^{\ell-m-3} (\ell - m - i - 2)(m + i - 1)$$

possibilities of constructing a legal word.

$m \setminus \ell$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	4	7	$\underline{13}$	22	39	67	108	183	305	510	851
2	1	2	4	7	11	19	$\underline{32}$	50	83	136	211	342	549
3	1	2	4	7	11	16	26	42	$\underline{65}$	95	149	234	358
4	1	2	4	7	11	16	22	34	53	80	$\underline{116}$	161	240
5	1	2	4	7	11	16	22	29	43	65	96	137	$\underline{189}$
6	1	2	4	7	11	16	22	29	37	53	78	113	$\underline{159}$
7	1	2	4	7	11	16	22	29	37	46	64	92	$\underline{131}$
8	1	2	4	7	11	16	22	29	37	46	56	76	$\underline{107}$
9	1	2	4	7	11	16	22	29	37	46	56	67	$\underline{89}$
10	1	2	4	7	11	16	22	29	37	46	56	67	$\underline{79}$

Table 3.1. The cardinalities of $\mathcal{D}_{m,\ell}$ for $1 \leqslant m \leqslant 10$ and for $0 \leqslant \ell \leqslant 12$ are shown. In each row, all numbers are computed by the formulas given in Lemma 3.5 and Lemma 3.6 up to the underlined entry. The computation of the remaining entries is based on recursive strategies as described in [Nil12].

Adding the two cases leads to

$$(\ell - m - 2)(\ell - m - 1) + \sum_{i=0}^{\ell-m-3} (\ell - m - i - 2)(m + i - 1)$$

possibilities. Together with Lemma 3.5 that covers the cases $|w|_b \leqslant 2$, and by direct algebraic manipulation we get Eq. (3.2). ■

It is possible to further extend this formula to $2m + 3 \leqslant \ell \leqslant 2m^2 + 3m + 2$ by the same method because

$$(a^m b)^{2m+2}\, ba^m \lhd \zeta_m(a^{2m+3}),$$

which leads to a minimum word length of $(2m + 1)(m + 1) + 2 = 2m^2 + 3m + 3$ for a word v with $|v|_b \geqslant 2m + 3$. Note that this bound is not sharp any more because the word a^{2m+3} is not legal, which means that a legal word, generating a word containing $2m + 3$ times the letter b must have a strictly larger length than $2m + 3$.

Furthermore, the problem of counting legal subwords under the assumption that several of the choosing probabilities vanish, can be treated in the same fashion.

Remark 3.7 (Extension to arbitrary word lengths)**.** Unfortunately, this method is ineligible for deriving the cardinalities of the sets of ζ_m-legal words with an arbitrary word length. Applying the substitution to legal words, which we understand the structure of, in order to control the images and understand their structure, leads to a recursive problem. ♦

$\ell_{1,k}$	2	3	5	8	13	21
$\lvert\mathcal{D}_{1,\ell_{1,k}}\rvert$	4	7	22	108	1356	65800
$\lvert\mathcal{G}_{1,k}\rvert$	2	3	8	30	288	10080

Table 3.2. Comparison of the number of legal and exact words of length $\ell_{1,k}$ for $3 \leqslant k \leqslant 8$.

3.2. Topological entropy of the RNMS

From now on, the topological entropy of ζ_m is denoted by $\mathcal{H}_m$. As we have seen in Section 3.1, we cannot directly compute $\mathcal{H}_m$ via C_m. In [GL89], the authors computed $\mathcal{H}_1$ under the implicit assumption that

$$\lim_{n\to\infty} \frac{\log(C_1(\ell_{1,n}))}{\ell_{1,n}} = \lim_{n\to\infty} \frac{\log(|\mathcal{G}_{1,n}|)}{\ell_{1,n}}. \tag{3.4}$$

As Table 3.2 suggests, the sets of legal and exact words differ significantly in magnitude which makes Eq. (3.4) a non-trivial assertion that was recently proved in the following generality.

Theorem 3.8 ([Nil12, Thm. 3])**.** *For an arbitrary but fixed $m \in \mathbb{N}$, the logarithm of the growth rate of the size of the set of exact random noble means words equals the topological entropy of the random noble means chain, that is*

$$\mathcal{H}_m = \lim_{n\to\infty} \frac{\log(C_m(\ell_{m,n}))}{\ell_{m,n}} = \lim_{n\to\infty} \frac{\log(|\mathcal{G}_{m,n}|)}{\ell_{m,n}}. \qquad \blacksquare$$

Recall that $\lim_{n\to\infty} \mathcal{G}_{m,n}$ was introduced on page 20. This asymptotic identity provides the simplification we need for the computation of $\mathcal{H}_m$ as we can explicitly compute the size of all $\mathcal{G}_{m,n}$.

Proposition 3.9 ([Nil12, Prop. 6])**.** *Let $m \in \mathbb{N}$ be fixed and $n \geqslant 3$. Then,*

$$|\mathcal{G}_{m,n}| = \prod_{i=2}^{n-1} (m(n-i)+1)^{d_{m,i-1}},$$

where $(d_{m,n})_{n\in\mathbb{N}}$ is the sequence defined by $d_{m,n} := m d_{m,n-1} + d_{m,n-2}$ with $d_{m,1} := 1$ and $d_{m,2} := m-1$. $\blacksquare$

Via a standard approach [GKP94, Sec. 7.3] for solving linear recursions, we achieve explicit representations of the sequences $\ell_{m,n}$ and $d_{m,n}$ which read

$$\ell_{m,n} = \frac{(1-\lambda_m')\lambda_m^{n-1} - (1-\lambda_m)(\lambda_m')^{n-1}}{\lambda_m - \lambda_m'},$$
$$d_{m,n} = \frac{(m-1-\lambda_m')\lambda_m^{n-1} - (m-1-\lambda_m)(\lambda_m')^{n-1}}{\lambda_m - \lambda_m'},$$

m	$\mathcal{H}_m$	m	$\mathcal{H}_m$	m	$\mathcal{H}_m$	m	$\mathcal{H}_m$
1	0.444399	6	0.287298	11	0.210664	16	0.168566
2	0.408550	7	0.267301	12	0.200411	17	0.162292
3	0.371399	8	0.250127	13	0.191221	18	0.156524
4	0.338619	9	0.235230	14	0.182933	19	0.151199
5	0.310804	10	0.222185	15	0.175417	20	0.146268

Table 3.3. Topological entropy $\mathcal{H}_m$ for $1 \leqslant m \leqslant 20$.

for fixed $m \in \mathbb{N}$ and all $n \geqslant 1$. We obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{d_{m,n-i}}{\ell_{m,n}} &= \lim_{n\to\infty} \frac{(m-1-\lambda'_m)\lambda_m^{n-1-i} - (m-1-\lambda_m)(\lambda'_m)^{n-1-i}}{(1-\lambda'_m)\lambda_m^{n-1} - (1-\lambda_m)(\lambda'_m)^{n-1}} \\ &= \frac{m-1-\lambda'_m}{1-\lambda'_m} \cdot \frac{1}{\lambda_m^i} = \frac{\lambda_m - 1}{1-\lambda'_m} \cdot \frac{1}{\lambda_m^i}\end{aligned}$$

for all $i \in \mathbb{N}$ because $\lim_{n\to\infty}(\lambda'_m)^n = 0$ for any $m \in \mathbb{N}$. By an application of Theorem 3.8 and of Proposition 3.9, the topological entropy can now be represented as

$$\begin{aligned}\mathcal{H}_m &= \lim_{n\to\infty} \frac{\log(|\mathcal{G}_{m,n}|)}{\ell_{m,n}} = \lim_{n\to\infty} \frac{\log(\prod_{i=2}^{n-1}(m(n-i)+1)^{d_{m,i-1}})}{\ell_{m,n}} \\ &= \lim_{n\to\infty} \sum_{i=2}^{n-1} \frac{d_{m,n-i}}{\ell_{m,n}} \log(m(i-1)+1) = \frac{\lambda_m - 1}{1-\lambda'_m} \sum_{i=2}^{\infty} \frac{\log(m(i-1)+1)}{\lambda_m^i} > 0.\end{aligned}$$

In [GL89], on the (implicit) basis of Theorem 3.8, the entropy per letter for $m = 1$ is computed to be

$$\mathcal{H}_1 = \sum_{i=2}^{\infty} \frac{\log(i)}{\lambda_1^{i+2}} \approx 0.444399 > 0,$$

which can be recovered here because $(\lambda_1 - 1)/(1-\lambda'_1) = 1/\lambda_1^2$. Refer to Table 3.3 for numerical values of $\mathcal{H}_m$ with $1 \leqslant m \leqslant 20$ and to Figure 3.1 for an illustration of the behaviour of $\mathcal{H}_m$ for larger m. These approximations are consistent with the results of Section 3.1. There, within the manageable bounds of word lengths, we have seen that the number of legal subwords of an arbitrary but fixed length decreases if the parameter m grows. This is nothing but a reformulation of decreasing entropy. Independently of Section 3.1, this behaviour is not surprising because the lengths of the images $\zeta_m(a)$ grow strictly in m. Consequently, growing parts of any element in $\mathbb{X}_m$ are completely characterised by $\zeta_m(a)$ which intuitively suggests a decreasing behaviour of C_m in m. This intuition can be made rigorous as follows.

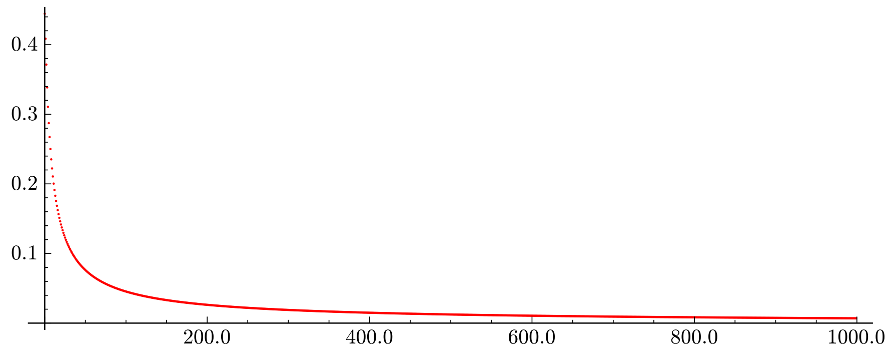

Figure 3.1. Topological entropy $\mathcal{H}_m$ for $1 \leqslant m \leqslant 1000$.

Proposition 3.10. *For any $m \in \mathbb{N}$, one has $\mathcal{H}_m > \mathcal{H}_{m+1}$. In addition, the topological entropy satisfies $\mathcal{H}_m \xrightarrow{m\to\infty} 0$.*

Proof. For the proof of the first part, we represent $\mathcal{H}_m$ in the form

$$\mathcal{H}_m = \frac{\lambda_m - 1}{(1-\lambda'_m)\lambda_m^{3/2}} \sum_{i=1}^{\infty} \frac{\log(mi+1)}{\lambda_m^{i-1/2}}$$

and define

$$\mathcal{P}(m) := \frac{\lambda_m - 1}{(1-\lambda'_m)\lambda_m^{3/2}} \quad \text{and} \quad \mathcal{S}(m) := \sum_{i=1}^{\infty} \mathcal{S}_i(m) := \sum_{i=1}^{\infty} \frac{\log(mi+1)}{\lambda_m^{i-1/2}}.$$

Interpreting $\mathcal{P}$ as a function $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$, its first derivative with respect to m reads

$$\frac{\mathrm{d}}{\mathrm{d}m}\mathcal{P}(m) = -\frac{m-4}{2\sqrt{m^2+4}\,(1-\lambda'_m)^2\lambda_m^{3/2}}$$

and is strictly negative for $m > 4$. Analogously, the first derivative of the ith summand $\mathcal{S}_i(m)$ of the second factor reads

$$\frac{\mathrm{d}}{\mathrm{d}m}\mathcal{S}_i(m) = -\frac{\lambda_m^{1/2-i}\left((2i-1)(im+1)\log(im+1) - 2i\sqrt{m^2+4}\right)}{\sqrt{2}\sqrt{m^2+4}\,(im+1)}.$$

For all $i \in \mathbb{N}$, the sign of $\frac{\mathrm{d}}{\mathrm{d}m}\mathcal{S}_i(m)$ only depends on the sign of the term

$$\mathcal{S}'_i(m) := (2i-1)(im+1)\log(im+1) - 2i\sqrt{m^2+4}.$$

In the case of $m > 5$, we find that $\mathcal{S}'_1(m) > 0$ and $\mathcal{S}'_i(m) < \mathcal{S}'_{i+1}(m)$ for all $i \in \mathbb{N}$. This implies the monotonic behaviour of the factors $\mathcal{P}(m)$ and $\mathcal{S}(m)$ and therefore also of the product $\mathcal{H}_m$ in the case of $m > 5$. The cases $m \in \{1,2,3,4,5\}$ can be estimated separately; see also the numerical values of Table 3.3. For the proof of the second part, recall that differentiation of the geometric series for $x \in \mathbb{R}$

and $|x| < 1$, yields

$$\sum_{i=1}^{\infty} i x^{i-1} = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=0}^{\infty} x^i = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1-x} = \frac{1}{(x-1)^2}. \tag{3.5}$$

Then, for $m \geqslant 1$ and with $\log(x+1) \leqslant \sqrt{x}$ for $x \geqslant 0$, we get

$$\begin{aligned} \frac{\lambda_m - 1}{1 - \lambda'_m} \sum_{i=1}^{\infty} \frac{\log(mi+1)}{\lambda_m^{i+1}} &\leqslant \frac{\lambda_m - 1}{1 - \lambda'_m} \sum_{i=1}^{\infty} \frac{\sqrt{mi}}{m^{i+1}} \\ &\leqslant \frac{\lambda_m - 1}{1 - \lambda'_m} \frac{1}{m} \cdot \frac{1}{\sqrt{m}} \sum_{i=1}^{\infty} i \Big(\frac{1}{m}\Big)^{i-1} \end{aligned}$$

and because of Eq. (3.5) this is

$$= \frac{\lambda_m - 1}{1 - \lambda'_m} \frac{1}{m} \cdot \frac{1}{\sqrt{m}} \frac{1}{(1/m - 1)^2}.$$

As $m \to \infty$, we have

$$\begin{aligned} &\lim_{m\to\infty} \frac{\lambda_m - 1}{1 - \lambda'_m} \frac{1}{m} \cdot \frac{1}{\sqrt{m}} \frac{1}{(1/m - 1)^2} \\ &\qquad = \lim_{m\to\infty} \frac{\lambda_m - 1}{1 - \lambda'_m} \frac{1}{m} \cdot \lim_{m\to\infty} \frac{1}{\sqrt{m}} \frac{1}{(1/m - 1)^2} \\ &\qquad = \lim_{m\to\infty} \frac{1}{1/m^{3/2} - 2/m^{1/2} + m^{1/2}} = 0. \end{aligned}$$

The last equality holds because λ_m/m tends to 1 as $m \to \infty$, and λ'_m to 0 as stated before. Finally, this proves $\lim_{m\to\infty} \mathcal{H}_m = 0$. ∎

CHAPTER 4

Frequency of RNMS subwords

In the context of (deterministic) primitive substitution rules ϑ on $\mathcal{A}_n$, one knows that the frequency of each letter exists and that it is encoded in the statistically normalised right PF eigenvector of M_ϑ. In [Que10, Sec. 5.4.1], the concept of the *induced substitution* ϑ_ℓ is introduced, where ϑ_ℓ is again a primitive substitution [Que10, Lem. 5.3] on the alphabet of ϑ-legal words of length ℓ. Consequently, the statistically normalised right PF eigenvector of the induced substitution matrix holds the frequencies of these legal words [Que10, Cor. 5.4].

In this chapter, we aim at a generalisation of Queffelec's method to the stochastic situation of the random noble means cases. Here, it is not enough to prove the primitivity of some induced substitution matrix but the interpretation of the entries of the statistically normalised right PF eigenvector as the frequencies of legal subwords will require some ergodicity argument.

Section 4.1 is devoted to the essentials about the induced substitution rule concerning the random noble means cases and a remarkable property of the eigenvalue spectrum of the induced substitution matrices is derived that has also been observed for (deterministic) primitive substitution rules [Que10, Prop. 5.19]. In Section 4.2, we will deal with the construction of a shift-invariant probability measure μ_m on $\mathbb{X}_m$ and we will prove the ergodicity of μ_m.

4.1. The induced substitution

In the following, the idea is to consider a substitution $(\zeta_m)_\ell$ that acts on the alphabet $\mathcal{D}_{m,\ell}$ of ζ_m-legal words of length ℓ. Here, we denote $\mathcal{D}^*_{m,\ell}$ as the set of finite words with respect to this alphabet. In order to get the frequencies of finite subwords right, the definition of $(\zeta_m)_\ell$ has to incorporate the different lengths of the images $\zeta_m(a)$ and $\zeta_m(b)$. This guarantees that subwords are neither over- nor undercounted relative to each other. As we deal with the stochastic situation here, we also have to take the realisation probabilities of subwords in the image of some word $w \in \mathcal{D}_{m,\ell}$ under ζ_m into account.

If $w = w_0 w_1 \cdots w_{\ell-1}$ is a word of length ℓ, recall that $w_{[i,j]}$ denotes the subword $w_i \cdots w_j \lhd w$ of length $j - i + 1$ for $0 \leqslant i \leqslant j \leqslant \ell - 1$. In the light of Definition 2.10, we introduce the notion of the substitution $(\zeta_m)_\ell$ induced by ζ_m as follows.

Definition 4.1 (Induced substitution)**.** Let $\ell \in \mathbb{N}$ and $\zeta_m \colon \mathcal{A}_2^* \longrightarrow \mathcal{A}_2^*$ be a random noble means substitution for some fixed $m \in \mathbb{N}$. Then, we refer to

$$(\zeta_m)_\ell \colon \mathcal{D}_{m,\ell}^* \longrightarrow \mathcal{D}_{m,\ell}^*$$

as the *induced substitution* defined by

$$(\zeta_m)_\ell \colon w^{(i)} \longmapsto \begin{cases} u^{(i,1)} := \left(v^{(i,1)}_{[k,k+\ell-1]}\right)_{0 \leqslant k \leqslant |\zeta_m(w_0^{(i)})|-1}, & \text{with probability } p_{i1}, \\ \quad\vdots & \quad\vdots \\ u^{(i,n_i)} := \left(v^{(i,n_i)}_{[k,k+\ell-1]}\right)_{0 \leqslant k \leqslant |\zeta_m(w_0^{(i)})|-1}, & \text{with probability } p_{in_i}, \end{cases}$$

where $w^{(i)} \in \mathcal{D}_{m,\ell}$ and $v^{(i,j)} \in \mathcal{D}_m$ is an image of $w^{(i)}$ under ζ_m with probability p_{ij}.

Note that Definition 4.1 is not restricted to the RNMS cases but works for any random substitution on $\mathcal{A}_n$ when the obvious modifications are considered. In particular, it is compatible with ζ_m^k. Recall that we assume $\boldsymbol{p}_m \gg 0$ which means that the set $\{v^{(i,1)}, \ldots, v^{(i,n_i)}\}$ consists of *all* possible images of $w^{(i)}$ under ζ_m. This leads to a well-defined substitution rule because

$$|\zeta_m(w^{(i)})| = |\zeta_m(w_0^{(i)})\zeta_m(w^{(i)}_{[1,\ell-1]})| \geqslant |\zeta_m(w_0^{(i)})| + \ell - 1.$$

Remark 4.2. For $w \in \mathcal{D}_{m,\ell}^*$, we have

$$(\zeta_m)_\ell(w) = \left((\zeta_m)_\ell(w_i)\right)_{0 \leqslant i \leqslant |w|-1}.$$

Note that all images $(\zeta_m)_\ell(w_i)$ are generated with respect to the same image of w under ζ_m. ♦

Lemma 4.3. *For any m, $\ell \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have $(\zeta_m)_\ell^k = (\zeta_m^k)_\ell$.*

Proof. We perform induction over k. The claim is obvious for $k = 0, 1$ and we assume validity for all $j \leqslant k-1$. Let $w \in \mathcal{D}_{m,\ell}$ and

$$w \xrightarrow{\zeta_m} w^{(1)} \xrightarrow{\zeta_m} w^{(2)} \xrightarrow{\zeta_m} \cdots \xrightarrow{\zeta_m} w^{(k-1)} \xrightarrow{\zeta_m} w^{(k)}$$

such that $w^{(k)}$ is an image of $\zeta_m^k(w)$. Then,

$$\begin{aligned} (\zeta_m)_\ell^k(w) &= (\zeta_m)_\ell\big((\zeta_m)_\ell^{k-1}(w)\big) = (\zeta_m)_\ell\big((\zeta_m^{k-1})_\ell(w)\big) \\ &= \big((\zeta_m)_\ell(w^{(k-1)}_{[i,i+\ell-1]})\big)_{0 \leqslant i \leqslant |\zeta_m^{k-1}(w_0)|-1} \\ &= \big(w^{(k)}_{[i,i+\ell-1]}\big)_{0 \leqslant i \leqslant |\zeta_m^k(w_0)|-1} \\ &= (\zeta_m^k)_\ell(w). \end{aligned}$$

Here, the second equality is implied by the induction hypothesis and the fourth equality follows from Remark 4.2. ■

$w^{(i)} \in \mathcal{D}_{1,2}$	$v^{(i,j)}$	$u^{(i,j)}$	$\mathbb{P}$	$w^{(i)} \in \mathcal{D}_{1,2}$	$v^{(i,j)}$	$u^{(i,j)}$	$\mathbb{P}$
aa	$abab$	$(ab)(ba)$	p_1^2	ab	aba	$(ab)(ba)$	p_1
	$abba$	$(ab)(bb)$	p_0p_1		baa	$(ba)(aa)$	p_0
	$baab$	$(ba)(aa)$	p_0p_1	ba	aab	(aa)	p_1
	$baba$	$(ba)(ab)$	p_0^2		aba	(ab)	p_0
				bb	aa	(aa)	1

Table 4.1. The action of $(\zeta_m)_\ell$ for $m = 1$ and $\ell = 2$ is shown. Recall the notation of Definition 4.1.

Following Definition 2.10 for any $m, \ell \in \mathbb{N}$ and with the notation of Definition 4.1, the *induced substitution matrix* is given by

$$M_{m,\ell} := M_{(\zeta_m)_\ell} := \Big(\sum_{q=1}^{n_j} p_{jq} |u^{(j,q)}|_{w^{(i)}}\Big)_{ij} \in \mathrm{Mat}(|\mathcal{D}_{m,\ell}|, \mathbb{R}), \tag{4.1}$$

where we fix the order of legal subwords $w^{(i)}$ to be lexicographic. Furthermore, the (extended) Abelianisation map is defined by

$$\begin{aligned} \phi_\ell \colon \bigcup_{j\in J} \mathcal{D}_{m,\ell}^* &\longrightarrow \mathbb{R}^{|\mathcal{D}_{m,\ell}|}, \\ w = \{w^{(j)}\}_{j\in J} &\longmapsto \Big(\sum_{j\in J} p_j |w^{(j)}|_{v^{(1)}}, \dots, \sum_{j\in J} p_j |w^{(j)}|_{v^{(k)}}\Big), \end{aligned}$$

where $\{v^{(1)}, \dots, v^{(k)}\} = \mathcal{D}_{m,\ell}$, J is a finite index set and the $\{p_j\}_{j\in J}$ are the probability weights attached to the words $\{w^{(j)}\}_{j\in J}$. Again, we can describe the action of $(\zeta_m)_\ell$ on a finite word $w \in \mathcal{D}_{m,\ell}^*$ in terms of $M_{m,\ell}$ and ϕ_ℓ via $M_{m,\ell}^k \, \phi_\ell(w) = \phi_\ell\big((\zeta_m^k)_\ell(w)\big)$ for any $k \in \mathbb{N}_0$.

Remark 4.4. Recall the notation of Definition 4.1. By construction, the column sums of $M_{m,\ell}$ are either equal to 1 or to $m+1$ because each letter of $u^{(i,j)}$ is counted exactly once, which means that the kth column sum is equal to

$$|(\zeta_m)_\ell(w^{(k)})| (p_{k1} + \dots + p_{kn_k})^{|w^{(k)}|_a} = |\zeta_m(w_0^{(k)})| \in \{1, m+1\}\,. \qquad \blacklozenge$$

Example 4.5 (The action of $(\zeta_1)_2$)**.** For $\ell = 2$ and $m = 1$, the action of $(\zeta_m)_\ell$ is illustrated in Table 4.1. Consequently, the induced substitution matrix in this case reads

$$M_{1,2} = \begin{pmatrix} p_0p_1 & p_0 & 1-p_0 & 1 \\ 1-p_0p_1 & 1-p_0 & p_0 & 0 \\ 1-p_0p_1 & 1 & 0 & 0 \\ p_0p_1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$M_{1,2}\phi_2(aa) = M_{1,2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}p_0p_1\\1-p_0p_1\\1-p_0p_1\\p_0p_1\end{pmatrix} = \phi_2\big((\zeta_1)_2(aa)\big).$$

The characteristic polynomial of $M_{1,2}$ is given by

$$P_{1,2}(x) = (x^2 - x - 1)(x - p_0p_1)(x + p_0),$$

which leads to the eigenvalue spectrum

$$\sigma_{1,2} = \{\lambda_1, \lambda_1', p_0p_1, -p_0\}.$$ ♦

Remark 4.6. Note that $(\zeta_m)_1$ agrees with ζ_m, so that $M_{m,1} = M_1$ but that there is a significant difference between the cases $\ell = 1$ and $\ell \geqslant 2$. Whereas the situation is rather clear for $\ell = 1$, as we saw in Chapter 2, there is no immediate reason why $(\zeta_m)_\ell$ and $M_{m,\ell}$ respectively, should still be primitive for $\ell \geqslant 2$. From Definition 4.1 and Eq. (4.1), we conclude that the local mixing has a direct influence on $M_{m,\ell}$ and in general makes its entries non-integral. ♦

In a preliminary step, we pave the way for the application of Theorem 1.4 to the induced substitution matrices and therefore show the primitivity of all $M_{m,\ell}$. Note that it is clear from the construction that $M_{m,\ell} > 0$.

Proposition 4.7. *For $\boldsymbol{p}_m \gg 0$, the induced substitution $(\zeta_m)_\ell$ is primitive for all m, $\ell \in \mathbb{N}$.*

Proof. Let u, $v \in \mathcal{D}_{m,\ell}$. From the definition of ζ_m, we conclude that there is an image $v' \doteq \zeta_m(v_0)$ such that $v_0' = a$. The legality of u means that there is a power $k_u \in \mathbb{N}$ with $u \triangleleft \zeta_m^{k_u}(a)$ and this implies the existence of an image $w \doteq \zeta_m^{k_u+1}(v)$ such that u is a subword of $w_0 \cdots w_{|\zeta_m^{k_u}(a)|-1}$. Due to Lemma 4.3, we know that

$$w' := \big(w_{[s,s+\ell-1]}\big)_{0\leqslant s\leqslant|\zeta_m^{k_u+1}(v_0)|-1} = (\zeta_m)_\ell^{k_u+1}(v).$$

The letters of this image consist of all subwords of w of length ℓ starting in the word $w_0 \cdots w_{|\zeta_m^{k_u+1}(v_0)|-1}$. We have $\zeta_m^{k_u+1}(v_0) \doteq \zeta_m^{k_u}(v')$ where v' starts with the letter a by construction which implies

$$|\zeta_m^{k_u+1}(v_0)| \geqslant |\zeta_m^{k_u}(a)|.$$

Therefore, we have $|w'|_u > 0$, which settles the irreducibility of $(\zeta_m)_\ell$. Note that the choice of the power k_u is independent of v.

For the proof of primitivity, we consider u, $v \in \mathcal{D}_{m,\ell}$. Along the same lines as in the above discussion, we conclude that

$$u \triangleleft \zeta_m^{n(k_u+1)}(v) \text{ for all } n \in \mathbb{N}$$

and u is a letter of the corresponding image of v under $(\zeta_m)_\ell^{n(k_u+1)}$. Now, let

$$\kappa := \operatorname{lcm}\{k_u + 1 \mid u \in \mathcal{D}_{m,\ell}\}.$$

Then, there is an image $(\zeta_m)_\ell^\kappa(v)$ containing u for all $u,\, v \in \mathcal{D}_{m,\ell}$ which is the primitivity of $(\zeta_m)_\ell$. ∎

Corollary 4.8. *For $\boldsymbol{p}_m \gg 0$, the induced substitution matrix $M_{m,\ell}$ is primitive for all $m,\, \ell \in \mathbb{N}$.*

Proof. This is an immediate consequence of Proposition 4.7 and Remark 2.13. ∎

Remark 4.9 (Primitivity of $(\zeta_m)_\ell$ for $\boldsymbol{p}_m > 0$)**.** The proof of Proposition 4.7 is valid for the induced substitution ϑ_ℓ with respect to any random substitution $\vartheta\colon \mathcal{A}_n^* \longrightarrow \mathcal{A}_n^*$ which meets the following property. There is a letter $a_i \in \mathcal{A}_n$ such that

(a) for all $a_j \in \mathcal{A}_n$, there exists an image $w \doteq \vartheta(a_j)$ with $w_0 = a_i$,
(b) for all ϑ-legal words $w \in \mathcal{A}_n^*$, there is a $k \in \mathbb{N}$ with $w \lhd \vartheta^k(a_i)$.

In the RNMS case, property (b) complies with an arbitrary choice of $\boldsymbol{p}_m$ because of Lemma 2.15 and Lemma 2.9. In contrast, property (a) is incompatible with $p_0 = 1$ because the first letter of $\zeta_{m,0}(a)$ is a b and a switch from $a_i = a$ to $a_i = b$ does not avoid this problem. Nevertheless, also in the deterministic cases, $(\zeta_{m,i})_\ell$ is primitive for all $0 \leqslant i \leqslant m$ [Que10, Lem. 5.3]. In the case of incomplete mixtures, the proof is still valid but one has to be careful about the dimension of $M_{m,\ell}$. Choose $\boldsymbol{p}_m \gg 0$ and $\boldsymbol{p}'_m > 0$ with several zero components. In numerous cases, one has $|\mathcal{D}_{m,\ell}(\boldsymbol{p}_m)| > |\mathcal{D}_{m,\ell}(\boldsymbol{p}'_m)|$ (e.g., choose $p_0 = 0$ or $p_m = 0$ which eliminates bb as a legal subword) and even $M_{m,\ell} \in \mathrm{Mat}(\ell+1, \mathbb{R})$ in the deterministic cases, as we have seen in Proposition 3.2. ♦

The following Proposition is important for two reasons. Firstly, it shows that the induced substitution matrix, in the case of $\ell = 2$, only depends on the three parameters m, p_0 and p_m, although all $m+2$ parameters $m, p_0, \ldots, p_m$ are considered for the construction of $M_{m,2}$. Secondly, the case $\ell = 2$ characterises the eigenvalue spectrum of $M_{m,\ell}$ for any $\ell \geqslant 3$, in the sense that only the eigenvalue 0 is possibly added with increasing word length, as we will see explicitly in Proposition 4.12.

Proposition 4.10. *For an arbitrary but fixed $m \in \mathbb{N}$ and word length $\ell = 2$, the induced substitution matrix reads*

$$M_{m,2} = \begin{pmatrix} m-1+p_0p_m & m-1+p_0 & 1-p_0 & 1 \\ 1-p_0p_m & 1-p_0 & p_0 & 0 \\ 1-p_0p_m & 1 & 0 & 0 \\ p_0p_m & 0 & 0 & 0 \end{pmatrix},$$

with corresponding characteristic polynomial

$$P_{m,2}(x) = (x^2 - mx - 1)(x - p_0p_m)(x + p_0),$$

eigenvalue spectrum

$$\sigma_{m,2} = \{\lambda_m, \lambda'_m, p_0p_m, -p_0\}$$

and statistically normalised right PF eigenvector

$$\boldsymbol{R}_{m,2} = \begin{pmatrix} \frac{2(\lambda_m - 1)}{m(1+p_0p_m)-(2+2\lambda_m-m)(-1+p_0p_m)} \\ \frac{2(1-p_0p_m)}{m(1+p_0p_m)-(2+2\lambda_m-m)(-1+p_0p_m)} \\ \frac{2(1-p_0p_m)}{m(1+p_0p_m)-(2+2\lambda_m-m)(-1+p_0p_m)} \\ \frac{2(1+\lambda'_m)p_0p_m}{m(1+p_0p_m)-(2+2\lambda_m-m)(-1+p_0p_m)} \end{pmatrix}.$$

Proof. We investigate each entry m_{ij} of $M_{m,2}$ combinatorially. The presence of the six 0-entries is obvious. As $(aa) \doteq (\zeta_m)_2(bb)$, it is immediately clear that $m_{1,4} = 1$. The remaining entries can be derived as follows.

- $m_{2,3}$. We have

$$ab \triangleleft \zeta_m(ba) = \zeta_m(b)\zeta_m(a) = a\zeta_m(a),$$

  which implies that we have to choose the image $\zeta_m(a) = ba^m$ here. This image is chosen with probability p_0 and therefore $m_{2,3} = p_0$.
- $m_{1,3}$. Here, we have $aa \triangleleft \zeta_m(ba) = a\zeta_m(a)$, which means that $\zeta_m(a)$ can be chosen as any of the images of the set $\{a^i ba^{m-i} \mid 1 \leqslant i \leqslant m\}$ and therefore aa is chosen with probability $\sum_{j=1}^m p_j = 1 - p_0 = m_{1,3}$.
- $m_{3,2}$. We know that $|\zeta_m(ab)| = m + 2$ and $|\zeta_m(ab)|_b = 1$. Because of $|\zeta_m(a)| = m + 1$, the word ba is included exactly once in each image of $(\zeta_m)_2(ab)$. The probability is therefore $\sum_{j=0}^m p_j = 1 = m_{3,2}$.
- $m_{2,2}$. Here, the argument is similar to the previous case with the exception, that the word $\zeta_m(a)$ starts with the letter b with probability p_0. Therefore, ab cannot be a subword in this case. This implies $m_{2,2} = 1 - p_0$.
- $m_{1,2}$. As $|\zeta_m(ab)| = m + 2$, the letter (aa) occurs at most m times in $(\zeta_m)_2(ab)$ and exactly m times only for $\zeta_m(ab) = ba^{m+1}$ which is chosen with probability p_0. In the remaining possible images, (aa) occurs exactly $m - 1$ times, such that

$$\begin{aligned} m_{1,2} &= mp_0 + (m-1)\sum_{j=1}^m p_j \\ &= mp_0 + (m-1)(1-p_0) = m - 1 + p_0. \end{aligned}$$

- $m_{4,1}$. There is only one possibility for the construction of (bb) in $(\zeta_m)_2(aa)$; on the first a we choose $\zeta_{m,m}$ and on the second one $\zeta_{m,0}$. This leads to $m_{4,1} = p_0p_m$.
- $m_{3,1}$. In fact, this is the case complementary to $m_{4,1}$, because (ba) occurs exactly once in each image of $(\zeta_m)_2(aa)$, with the only exception in the case where we choose $\zeta_{m,m}$ on the first a and $\zeta_{m,0}$ on the second. This happens with probability p_0p_m, which means $m_{3,1} = 1 - p_0p_m$.

- $m_{2,1}$. A simple counting argument similar to the case of $m_{1,2}$ shows that

$$\begin{aligned} m_{2,1} &= \Big(\sum_{i=1}^{m} p_i \sum_{j=1}^{m} p_j\Big) + \Big(2p_0 \sum_{i=1}^{m-1} p_i\Big) + p_0^2 + p_0 p_m \\ &= (1-p_0)^2 + 2p_0(1-p_0-p_m) + p_0^2 + p_0 p_m \\ &= 1 - p_0 p_m. \end{aligned}$$

- $m_{1,1}$. Similarly to $m_{2,1}$, we find

$$\begin{aligned} m_{1,1} = &\Big(mp_0 \sum_{i=1}^{m} p_i\Big) + \Big((m-1) \sum_{i=1}^{m} p_i \sum_{j=1}^{m} p_j\Big) \\ &\quad + \Big((m-2)p_0 \sum_{i=1}^{m-1} p_i\Big) + (m-1)(p_0 p_m + p_0^2) \\ = &\; mp_0(1-p_0) + (m-1)(1-p_0)^2 \\ &\quad + (m-2)p_0(1-p_0-p_m) + (m-1)(p_0 p_m + p_0^2) \\ = &\; m - 1 + p_0 p_m. \end{aligned}$$

These entries constitute the matrix $M_{m,2}$ and therefore determine $P_{m,2}$, $\sigma_{m,2}$ and $\boldsymbol{R}_{m,2}$. ∎

As soon as the complexity function $C_m(\ell)$ and the attributed set of legal words $\mathcal{D}_{m,\ell}$ is known, it is easy to compute the matrices $M_{m,\ell}$ and therefore the right PF eigenvectors by the same method. We proceed with a minor technical preparation for the proof of Proposition 4.11. Here, for some matrix $M \in \mathrm{Mat}(d, \mathbb{R})$, we denote by $\mathcal{N}_M$ the set of non-zero eigenvalues of M.

Lemma 4.11. *Let $A, B \in \mathrm{Mat}(d, \mathbb{R})$. Assume that for all polynomials $q \in \mathbb{R}[x]$ the following properties hold:*

(1) If $q(A) = 0$, then there is a $k \in \mathbb{N}$ with $q(B)B^k = 0$.
(2) If $q(B) = 0$, then there is a $k \in \mathbb{N}$ with $q(A)A^k = 0$.

Then, one has $\mathcal{N}_A = \mathcal{N}_B$.

Proof. Firstly, we consider the minimal polynomial μ_A of A. Here, we have $\mu_A(A) = 0$ and, because of the first property, there exists a $k \in \mathbb{N}$ and a polynomial $p \in \mathbb{R}[x]$ with $p(x)\mu_B(x) = \mu_A(x)x^k$ and μ_B the minimal polynomial of B. Thus, each non-zero eigenvalue of B is also a non-zero eigenvalue of A or $\mathcal{N}_B \subset \mathcal{N}_A$. Interchanging the roles of A and B, along with an application of the second property, then proves the assertion. ∎

The following result is based on [Que10, Prop. 5.10], suitably spelled out for the stochastic situation of $(\zeta_m)_\ell$.

Proposition 4.12. *For any $m \in \mathbb{N}$ and $\ell \geqslant 3$, the eigenvalue spectrum of the matrices $M_{m,\ell}$ is the same as that of $M_{m,2}$, possibly with the additional eigenvalue zero.*

Proof. We fix any $w = w_0 w_1 \cdots w_{\ell-1} \in \mathcal{D}_{m,\ell}$. Based on Definition 4.1, we introduce the following abbreviated notation

$$(\zeta_m^k)_\ell(w) := \left\{ \left(v^{(j)}_{[s,s+\ell-1]} \right)_{0 \leqslant s \leqslant \delta(k)} \right\}_{j \in J},$$

where $\delta(k) := |\zeta_m^k(w_0)| - 1$ and $\{v^{(j)}\}_{j\in J}$, for a finite index set J, is the set of images of w under ζ_m^k. We omitted the notation of the realisation probabilities here because we do not refer to them explicitly in the following. As all possible images of $\zeta_m^k(w_0)$ have the same length and all $\zeta_{m,i}$ are primitive, we can choose $k := k_\ell \in \mathbb{N}$ sufficiently large to ensure that

$$|\zeta_m^k(w_0)| + \ell - 2 < |\zeta_m^k(w_0)| + |\zeta_m^k(w_1)| \tag{4.2}$$

and we fix this choice for ℓ and k for the rest of the proof. Consequently, all images of w under $(\zeta_m^k)_\ell$ are determined by the images $\zeta_m^k(w_0)$ and $\zeta_m^k(w_1)$. We define a map $\pi_2 \colon \mathcal{D}_{m,\ell} \longrightarrow \mathcal{D}_{m,2}$ that sends a legal word of length ℓ to the subword consisting of the first two letters, thus $\pi_2(w_0 \cdots w_{\ell-1}) := w_0 w_1$. Moreover, we consider the map $\pi_{2,\ell,k} \colon \mathcal{D}_{m,2} \longrightarrow \mathcal{D}^*_{m,\ell}$, defined by

$$\pi_{2,\ell,k}(w_0 w_1) := \left\{ \left(v^{(j)}_{[s,s+\ell-1]} \right)_{0 \leqslant s \leqslant \delta(k)} \right\}_{j \in J},$$

for all $w_0 w_1 \in \mathcal{D}_{m,2}$ with

$$\zeta_m^k(w_0 w_1) = \left\{ v_0^{(j)} v_1^{(j)} \cdots v^{(j)}_{|\zeta_m^k(w_0)|-1} v^{(j)}_{|\zeta_m^k(w_0)|} \cdots v^{(j)}_{|\zeta_m^k(w_0 w_1)|-1} \right\}_{j \in J},$$

for a finite index set J. Now, we extend these maps to

$$\pi_2 \colon \mathcal{D}^*_{m,\ell} \longrightarrow \mathcal{D}^*_{m,2} \quad \text{and} \quad \pi_{2,\ell,k} \colon \mathcal{D}^*_{m,2} \longrightarrow \mathcal{D}^*_{m,\ell},$$

via concatenation. From this, we directly deduce $\pi_2 \circ \pi_{2,\ell,k} = (\zeta_m^k)_2$ because

$$\begin{aligned}
(\pi_2 \circ \pi_{2,\ell,k})(w_0 w_1) &= \left\{ \pi_2 \left(v^{(j)}_{[s,s+\ell-1]} \right)_{0 \leqslant s \leqslant \delta(k)} \right\}_{j \in J} \\
&= \left\{ \left(\pi_2 \left(v^{(j)}_{[s,s+\ell-1]} \right) \right)_{0 \leqslant s \leqslant \delta(k)} \right\}_{j \in J} \\
&= \left\{ \left(v^{(j)}_{[s,s+1]} \right)_{0 \leqslant s \leqslant \delta(k)} \right\}_{j \in J} \\
&= (\zeta_m^k)_2(w_0 w_1).
\end{aligned}$$

Similarly, one shows that $\pi_{2,\ell,k} \circ \pi_2 = (\zeta_m^k)_\ell$ and $(\zeta_m)_\ell \circ \pi_{2,\ell,k} = \pi_{2,\ell,k} \circ (\zeta_m)_2$. Finally, we end up with the commutative diagram shown in Figure 4.1. With

$$
\begin{tikzcd}
\mathcal{D}_{m,\ell}^{*} \arrow[r, "(\zeta_m^k)_\ell"] \arrow[d, "\pi_2"'] & \mathcal{D}_{m,\ell}^{*} \arrow[r, "(\zeta_m)_\ell"] & \mathcal{D}_{m,\ell}^{*} \arrow[d, "\pi_2"] \\
\mathcal{D}_{m,2}^{*} \arrow[r, "(\zeta_m)_2"'] \arrow[ur, "\pi_{2,\ell,k}"] & \mathcal{D}_{m,2}^{*} \arrow[r, "(\zeta_m^k)_2"'] \arrow[ur, "\pi_{2,\ell,k}"] & \mathcal{D}_{m,2}^{*}
\end{tikzcd}
$$

Figure 4.1. Relationships between the maps $(\zeta_m)_2$, $(\zeta_m)_\ell$, π_2 and $\pi_{2,\ell,k}$.

$M_{m,\ell}$, M_2 and $M_{2,\ell,k}$ being the matrices corresponding to the mappings $(\zeta_m)_\ell$, π_2 and $\pi_{2,\ell,k}$, this implies

$$M_{2,\ell,k}\,\phi_2(\pi_2(w)) = \phi_\ell((\zeta_m^k)_\ell(w)),$$

for any $w \in \mathcal{D}_{m,\ell}^{*}$ and the corresponding commutative diagram on the level of matrices, with $\xi \coloneqq \pi_2(w)$, is shown in Figure 4.2. From this diagram, we deduce the identity

$$M_{m,\ell}M_{2,\ell,k} = M_{2,\ell,k}M_{m,2}.$$

For any polynomial $q \in \mathbb{R}[x]$ of degree $n \in \mathbb{N}$, we find

$$\begin{aligned} M_{2,\ell,k}\,q(M_{m,2}) &= M_{2,\ell,k}\sum_{i=0}^{n} a_i M_{m,2}^i = \sum_{i=0}^{n} a_i M_{2,\ell,k}M_{m,2}^i \\ &= \sum_{i=0}^{n} a_i M_{m,\ell}^i M_{2,\ell,k} = q(M_{m,\ell})M_{2,\ell,k}. \end{aligned} \tag{4.3}$$

Reapplying the diagram of Figure 4.2, we achieve

$$M_{2,\ell,k}\,q(M_{m,2})M_2 = q(M_{m,\ell})M_{2,\ell,k}M_2 = q(M_{m,\ell})M_{m,\ell}^k. \tag{4.4}$$

Suppose now that $M_{m,2}$ is a root of q. Then, by Eq. (4.4), the polynomial $r(x) = q(x)x^k$ annihilates $M_{m,\ell}$. A further application of Eq. (4.3) provides

$$\begin{aligned} & M_2M_{2,\ell,k}\,q(M_{m,2}) = M_2\,q(M_{m,\ell})M_{2,\ell,k} \\ \Longrightarrow\qquad & M_{m,2}^k\,q(M_{m,2}) = M_2\,q(M_{m,\ell})M_{2,\ell,k}. \end{aligned}$$

Thus, $M_{m,2}$ is a root of r provided $q(M_{m,\ell}) = 0$. According to Lemma 4.11, the matrices $M_{m,2}$ and $M_{m,\ell}$ have the same non-zero eigenvalues. ∎

The key to the proof of Proposition 4.12 clearly was the observation from Eq. (4.2). There, we pointed out that the action of $(\zeta_m)_\ell$ is essentially characterised by the images of words of length 2 under ζ_m^k. At the same time, this is

$$
\begin{tikzcd}
\phi_\ell(w) \arrow[r, "M_{m,\ell}^k"] \arrow[d, "M_2"'] & \phi_\ell\big((\zeta_m^k)_\ell(w)\big) \arrow[r, "M_{m,\ell}"] & \phi_\ell\big((\zeta_m^{k+1})_\ell(w)\big) \arrow[d, "M_2"] \\
\phi_2(\xi) \arrow[r, "M_{m,2}"'] \arrow[ur, "M_{2,\ell,k}"] & \phi_2\big((\zeta_m)_2(\xi)\big) \arrow[r, "M_{m,2}^k"'] \arrow[ur, "M_{2,\ell,k}"] & \phi_2\big((\zeta_m^{k+1})_2(\xi)\big)
\end{tikzcd}
$$

Figure 4.2. Relationships between the matrices $M_{m,2}$, $M_{m,\ell}$, M_2 and $M_{2,\ell,k}$.

an intuitive explanation why the spectrum of $M_{m,\ell}$ is to a great extent fixed by that of $M_{m,2}$.

Corollary 4.13. *If $\boldsymbol{R}_{m,2}$ is a right PF eigenvector of $M_{m,2}$, then $M_{2,\ell,k}\boldsymbol{R}_{m,2}$ is a right PF eigenvector of $M_{m,\ell}$.*

Proof. This is an immediate consequence of

$$M_{m,\ell}M_{2,\ell,k} = M_{2,\ell,k}M_{m,2}. \qquad \blacksquare$$

Example 4.14 (Induced PF eigenvectors)**.** Corollary 4.13 provides a convenient method for the computation of $\boldsymbol{R}_{m,\ell}$ in the case of $\ell \geqslant 3$ which is particularly useful for numerical simulations in Chapter 6. Recall that we get $\boldsymbol{R}_{m,2}$ for all $m \in \mathbb{N}$ from Proposition 4.10. We choose $m = 1$, $\ell = 3$ and $\boldsymbol{p}_1 = (1/4, 3/4)$. From Eq. (4.2), we deduce that we have to consider the images of w under ζ_1^2 for all $w \in \mathcal{D}_{1,2}$. The legal subwords of length 2 and 3 are given by

$$\mathcal{D}_{1,2} = \{aa, ab, ba, bb\} \quad \text{and} \quad \mathcal{D}_{1,3} = \{aaa, aab, aba, abb, baa, bab, bba\}.$$

Now, we have to compute the matrix $M_{2,3,2} \in \mathrm{Mat}(7 \times 4, \mathbb{R})$ and statistically normalise the vector $M_{2,3,2}\boldsymbol{R}_{1,2}$. We find

$$M_{2,3,2} = \begin{pmatrix} \frac{39}{128} & \frac{9}{64} & \frac{3}{64} & 0 \\ \frac{187}{256} & \frac{27}{32} & \frac{5}{32} & \frac{3}{16} \\ \frac{119}{128} & \frac{31}{32} & \frac{21}{32} & \frac{5}{8} \\ \frac{9}{256} & \frac{3}{64} & \frac{9}{64} & \frac{3}{16} \\ \frac{187}{256} & \frac{21}{32} & \frac{11}{32} & \frac{3}{16} \\ \frac{15}{64} & \frac{19}{64} & \frac{33}{64} & \frac{5}{8} \\ \frac{9}{256} & \frac{3}{64} & \frac{9}{64} & \frac{3}{16} \end{pmatrix} \quad \text{and} \quad \boldsymbol{R}_{1,3} \approx \begin{pmatrix} 0.056215 \\ 0.210795 \\ 0.320083 \\ 0.030941 \\ 0.210795 \\ 0.140229 \\ 0.030941 \end{pmatrix}. \qquad \blacklozenge$$

4.2. Ergodicity

In this section, we construct a shift-invariant probability measure μ_m on $\mathbb{X}_m$ and prove its ergodicity. This, in conjunction with Corollary 4.8 and Theorem 1.4, identifies the entries of the statistically normalised right PF eigenvector of $M_{m,\ell}$ as the frequencies of ζ_m-legal words.

4.2.1. A shift-invariant probability measure on the RNMS hull. Let X be a set equipped with the Borel σ-algebra $\mathfrak{B} := \mathfrak{B}_X$ on which a regular Borel measure μ is defined. We call the triple $(X, \mathfrak{B}, \mu)$ a measure space (or a probability space if $\mu \in \mathcal{P}(X)$). If G is a group and $T\colon X \longmapsto X$ a continuous map, one is interested in the behaviour of X under some group action $g \longmapsto T_g$ of G on X. Here, we call T a *transformation* of the space X. For the purpose of this section, we will study the action of $\mathbb{Z}$ on subshifts $\mathbb{X}$ (in particular on $\mathbb{X}_m$) via the shift S and its iterates S^k for $k \in \mathbb{Z}$.

Definition 4.15 (Invariant sets and measures)**.** Let T be a transformation of the probability space $(X, \mathfrak{B}, \mu)$. A Borel set $B \in \mathfrak{B}$ is *T-invariant* if $T^{-1}B = B$. Furthermore, the measure μ is *T-invariant* if $T.\mu(B) := \mu(T^{-1}B) = \mu(B)$ for all Borel sets $B \in \mathfrak{B}$ and we refer to $\mathcal{P}_T(X) := \{\mu \in \mathcal{P}(X) \mid T.\mu = \mu\}$ as the set of *T-invariant probability measures* on X.

Note that the map $\mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, $\mu \longmapsto T.\mu$, in Definition 4.15 is again a continuous map, this time on $\mathcal{P}(X)$ [Wal00, Thm. 6.7].

Now, we consider a transformation T of the measure space $(X, \mathfrak{B}, \mu)$ and a T-invariant Borel set $B \in \mathfrak{B}$. Then, also $T^{-1}(X \setminus B) = X \setminus B$ and it would suffice to restrict the study of T to the cases $T|_B$ and $T|_{X \setminus B}$. If additionally $\mu(B) \in (0, 1)$, the study of the new transformation may be considerably easier. If instead $\mu(B) = 0$ or $\mu(X \setminus B) = 0$, we can neglect the set B or $X \setminus B$. This preliminary line of thought motivates the following Definition.

Definition 4.16 (Ergodicity)**.** If $(X, \mathfrak{B}, \mu)$ is a probability space, then μ is called *ergodic* if all members B of $\mathfrak{B}$ with the property $T^{-1}B = B$ satisfy $\mu(B) = 0$ or $\mu(B) = 1$.

Intuitively, this means that a system cannot be further decomposed into invariant components of positive measure. In the light of the identification of regular Borel measures and linear functionals as introduced in Section 1.4.1, a function $f \in L^1(X, \mu)$ is called *T-invariant* if $T.f(x) := f(T^{-1}x) = f(x)$ for μ-almost every $x \in X$. Here, the measure μ is ergodic if and only if the only invariant functions are constant μ-almost everywhere. One important tool in this context is the following result.

Theorem 4.17 (Birkhoff, [Wal00, Thm. 1.14])**.** *Let $\mathbb{X} \subset \mathcal{A}_n^{\mathbb{Z}}$ be any subshift and μ be an S-invariant regular Borel probability measure on $\mathbb{X}$. If $f \in L^1(\mathbb{X}, \mu)$, then* *the sequence* $\big(\frac{1}{n}\sum_{i=0}^{n-1} f(S^i x)\big)_{n\in\mathbb{N}}$ *converges, for* μ*-almost every* $x \in \mathbb{X}$*, to an* S*-invariant function* $F \in L^1(\mathbb{X},\mu)$ *that satisfies* $\int_{\mathbb{X}} F \,\mathrm{d}\mu = \int_{\mathbb{X}} f \,\mathrm{d}\mu$*. Moreover, if* μ *is an ergodic probability measure, the function* F *is constant* μ*-almost everywhere and*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(S^i x) = \int_{\mathbb{X}} f \,\mathrm{d}\mu \tag{4.5}$$

holds for μ*-almost* $x \in \mathbb{X}$*.* ■

In the situation of Theorem 4.17, we denote $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(S^i x)$ as the *time mean of* f *at* x, provided this limit exists. The *space mean of* f is defined as $\int_{\mathbb{X}} f \,\mathrm{d}\mu$. The significance of Birkhoff's theorem for our concern is that the time means and the space means are equal for μ-almost every $x \in \mathbb{X}$ if and only if μ is ergodic. One direction is the statement around Eq. (4.5). For the converse, assume that Eq. (4.5) holds for μ-almost every $x \in \mathbb{X}$. Then, we find

$$\mu(B) = \int_{\mathbb{X}} \mathbb{1}_B \,\mathrm{d}\mu = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{1}_B(S^i x) \in \{0,1\}$$

for any S-invariant set $B \in \mathfrak{B}$, which is the ergodicity of μ. If the agreement of space and time mean holds for all $x \in \mathbb{X}$, then the measure is *uniquely ergodic* [Que10, Sec. 4.1.2].

Now, we consider the shift S (and the $\mathbb{Z}$-action on $\mathbb{X}_m$ via $k \longmapsto S^k$) on $\mathbb{X}_m$. Recall that S is a homeomorphism of $\mathbb{X}_m$. For any $w \in \mathbb{X}_m$, we define the sequence

$$(\mu_n)_{n\in\mathbb{N}} := \Big(\frac{1}{2n+1}\sum_{i=-n}^{n} S^{-i}.\delta_w\Big)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{X}_m).$$

From the compactness of $\mathcal{P}(\mathbb{X}_m)$, we get the existence of a converging subsequence and $\mu = \lim_{i\to\infty}\mu_{n_i}$ is an S-invariant probability measure by construction. Thus, we find that $\mathcal{P}_S(\mathbb{X}_m) \neq \varnothing$ but this alone is quite unsatisfactory because we strive for a measure that is not only S-invariant but also ergodic. Therefore, we bring Corollary 4.8 and Theorem 1.4 into position in order to define a probability measure on the cylinder sets $\mathfrak{Z}(\mathbb{X}_m)$ that were introduced in Remark 1.11.

Remark 4.18 (The semi-algebra $\mathfrak{Z}(\mathbb{X}_m)$)**.** The class $\mathfrak{Z}(\mathbb{X}_m)$ forms a semi-algebra that generates the Borel σ-algebra $\mathfrak{B}_m$ on $\mathbb{X}_m$. That is

- $\varnothing \in \mathfrak{Z}(\mathbb{X}_m)$,
- $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathfrak{Z}(\mathbb{X}_m) \Longrightarrow \mathcal{Z}_1 \cap \mathcal{Z}_2 \in \mathfrak{Z}(\mathbb{X}_m)$,
- $\mathcal{Z} \in \mathfrak{Z}(\mathbb{X}_m) \Longrightarrow \mathbb{X}_m \setminus \mathcal{Z} = \bigcup_{i=1}^{n}\mathcal{Z}_i$, where each $\mathcal{Z}_i \in \mathfrak{Z}(\mathbb{X}_m)$ and $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ are pairwise disjoint.

This is a well-known fact and the proof is an immediate consequence of the definition of a cylinder set. ♦

Now, let $w \in \mathcal{D}_{m,\ell}$ be any ζ_m-legal word. Then, we define the measure μ_m on $\mathcal{Z}_k(w) \in \mathfrak{Z}(\mathbb{X}_m)$ by

$$\mu_m(\mathcal{Z}_k(w)) := \boldsymbol{R}_{m,\ell}(w), \tag{4.6}$$

for any $k \in \mathbb{Z}$, where $\boldsymbol{R}_{m,\ell}(w)$ is the entry of the statistically normalised right PF eigenvector of $M_{m,\ell}$ with respect to the word w. According to [Que10, Sec. 5.4], this is a consistent definition of a measure on $\mathfrak{Z}(\mathbb{X}_m)$ and there is an extension of μ_m to the Borel σ-algebra $\mathfrak{B}_m$ [Par05, Cor. 2.4.9]. Due to [Par05, Prop. 2.5.1], this extension is unique and we will denote it again as μ_m. Furthermore, $\mu_m \in \mathcal{P}(\mathbb{X}_m)$ because for any $k \in \mathbb{Z}$ and $\ell \in \mathbb{N}$, we have

$$\mu_m(\mathbb{X}_m) = \mu_m\Big(\dot{\bigcup_{w \in \mathcal{D}_{m,\ell}}} \mathcal{Z}_k(w)\Big) = \sum_{w \in \mathcal{D}_{m,\ell}} \mu_m(\mathcal{Z}_k(w)) = \sum_{w \in \mathcal{D}_{m,\ell}} \boldsymbol{R}_{m,\ell}(w) = 1.$$

Moreover, the S-invariance of μ_m is an immediate consequence of the following result.

Theorem 4.19 ([Wal00, Thm. 1.1])**.** *Let T be a transformation of the probability space $(X, \mathfrak{B}, \mu)$. If $\mathfrak{S}$ is a semi-algebra generating $\mathfrak{B}$ such that $T^{-1}M \in \mathfrak{B}$ and $\mu(T^{-1}M) = \mu(M)$ for all $M \in \mathfrak{S}$, then μ is T-invariant.* ∎

One easily shows that $S^{-1}\mathcal{Z}_{k-1}(w) = \mathcal{Z}_k(w)$ and with

$$\mu_m(S^{-1}\mathcal{Z}_k(w)) = \mu_m(\mathcal{Z}_{k+1}(w)) = \boldsymbol{R}_{m,\ell}(w) = \mu_m(\mathcal{Z}_k(w)),$$

we find $\mu_m \in \mathcal{P}_S(\mathbb{X}_m)$ by an application of Theorem 4.19.

4.2.2. Ergodicity of μ_m. We would like to interpret the entries of $\boldsymbol{R}_{m,\ell}$ as the frequencies of ζ_m-legal subwords. This is a known fact in the context of (deterministic) primitive substitutions and therefore also in the case of all members of $\mathcal{N}_m$. Here, one additionally knows that the ergodic S-invariant probability measure [Que10, Thm. 5.6] on $\mathbb{X}_m'$ is unique, so the system is uniquely ergodic. This will no longer be the case in the stochastic setting, but we can prove the ergodicity of μ_m. One important ingredient is Etemadi's formulation of the strong law of large numbers that only requires the sequence of random variables to be pairwise independent.

Theorem 4.20 (Etemadi, [Ete81, Thm. 1])**.** *Let $(X_i)_{i \in \mathbb{N}}$ be a family of pairwise independent, identically distributed, complex random variables with common distribution μ, subject to the integrability condition $\mathbb{E}_\mu(|X_1|) < \infty$. Then,*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[a.s.]{n \to \infty} \mathbb{E}_\mu(X_1) = \int_{\mathbb{R}} x \,\mathrm{d}\mu(x). \qquad ∎$$

Here, $\mathbb{E}_\mu(X)$ denotes the *mean* of the random variable X with respect to the measure μ. The following Proposition constitutes the main result of this chapter.

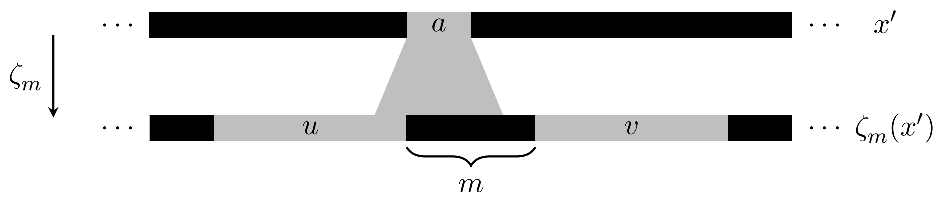

Figure 4.3. The words u, $v \in \mathcal{D}_{m,\ell}$ are independent as of the shift by $\ell + m$ positions. The word $\zeta_m(a)$ can have non-empty overlap with precisely one of the two words.

Proposition 4.21. *For an arbitrary but fixed $m \in \mathbb{N}$, let $\mathbb{X}_m \subset \mathcal{A}_2^{\mathbb{Z}}$ be the two-sided discrete stochastic hull of the random noble means substitution and μ_m be the S-invariant probability measure on $\mathbb{X}_m$ introduced in Eq.* (4.6)*. For any $f \in L^1(\mathbb{X}_m, \mu_m)$ and for an arbitrary but fixed $s \in \mathbb{Z}$, the identity*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=s}^{n+s-1} f(S^i x) = \int_{\mathbb{X}_m} f \,\mathrm{d}\mu_m \tag{4.7}$$

holds for μ_m-almost every $x \in \mathbb{X}_m$.

Proof. Let $x \in \mathbb{X}_m$ be an arbitrary element of the stochastic hull. The idea is to consider the characteristic function $\mathbb{1}_{\mathcal{Z}}$ of some cylinder set $\mathcal{Z} \in \mathfrak{Z}(\mathbb{X}_m)$ and to interpret $X := \big(\mathbb{1}_{\mathcal{Z}}(S^i x)\big)_{i\in\mathbb{N}}$ as a family of μ_m-distributed random variables in order to invoke Theorem 4.20. For this purpose, we have to deal with the pairwise independence of elements in X. In Remark 2.25, we pointed out that there exists an element $x' \in \mathbb{X}_m$ with $\zeta_m(x') \doteq x$ which means that we can study the structure of x that is induced by the action of ζ_m on some element of $\mathbb{X}_m$. For two finite subwords u, $v \in \mathcal{D}_{m,\ell}$ of x, we denote by $u \Cap v$ the overlap of u and v in x and by $|u \Cap v|$ its number of letters. Certainly, u and v cannot be independent if $|u \Cap v| > 0$, but we have to take more into account. Possibly, u and v may contain parts of the image of the same letter under ζ_m. As $|\zeta_m(a)| = m+1 > 1 = |\zeta_m(b)|$, it is sufficient to ensure that at most one of the overlaps $u \Cap \zeta_m(a)$ and $v \Cap \zeta_m(a)$ is non-empty for the very same letter $a \lhd x'$, as illustrated in Figure 4.3. Now, define for any $i \in \mathbb{Z}$, $\ell \in \mathbb{N}$ and a fixed $t \in \mathbb{Z}$, the family

$$(X_{i,k})_{k\in\mathbb{N}_0} := \Big((S^{i+k(\ell+m)} x)_{[t,t+\ell-1]} \Big)_{k\in\mathbb{N}_0}.$$

Then, each $X \in \big\{ (X_{i,k})_{k\in\mathbb{N}_0} \mid s \leqslant i \leqslant \ell + m + s - 1 \big\}$ consists of pairwise independent words in the sense pointed out above. Furthermore, for any $v \in \mathcal{D}_{m,\ell}$, we consider the characteristic function of the cylinder set $\mathcal{Z}_t(v) \in \mathfrak{Z}(\mathbb{X}_m)$, defined by

$$\mathbb{1}_{\mathcal{Z}_t(v)}(x) := \begin{cases} 1, & \text{if } x_{[t,t+\ell-1]} = v, \\ 0, & \text{otherwise.} \end{cases}$$

This leads to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=s}^{n+s-1}\mathbb{1}_{\mathcal{Z}_t(v)}(S^i x)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=s}^{\ell+m+s-1}\sum_{k=0}^{\lfloor\frac{n-1-i}{\ell+m}\rfloor}\mathbb{1}_{\mathcal{Z}_t(v)}\big(S^{i+k(\ell+m)}x\big)$$

$$=\lim_{n\to\infty}\frac{1}{\ell+m}\sum_{i=s}^{\ell+m+s-1}\frac{1}{\lfloor\frac{n-1-i}{\ell+m}\rfloor+1}\sum_{k=0}^{\lfloor\frac{n-1-i}{\ell+m}\rfloor}\mathbb{1}_{\mathcal{Z}_t(v)}\big(S^{i+k(\ell+m)}x\big). \quad (4.8)$$

For $s\leqslant i\leqslant \ell+m+s-1$, we consider the family $\big(\mathbb{1}_{\mathcal{Z}_t(v)}(S^{i+k(\ell+m)}x)\big)_{k\in\mathbb{N}_0}$ and apply Theorem 4.20 to each of the inner sums of Eq. (4.8) separately and appropriately put the resulting means together. Thus, Eq. (4.8) is almost surely

$$=\frac{1}{\ell+m}\sum_{i=s}^{\ell+m+s-1}\mathbb{E}_{\mu_m}\big(\mathbb{1}_{\mathcal{Z}_t(v)}(S^i x)\big)=\mathbb{E}_{\mu_m}\big(\mathbb{1}_{\mathcal{Z}_t(v)}(x)\big)$$

$$=\int_{\mathbb{X}_m}\mathbb{1}_{\mathcal{Z}_t(v)}\,\mathrm{d}\mu_m.$$

Note that the penultimate equality is implied by Theorem 1.4 and the uniqueness of $\boldsymbol{R}_{m,\ell}$ stated therein.

To finish the proof, we need to extend the presented arguments to an arbitrary function in $L^1(\mathbb{X}_m,\mu_m)$. We define

$$\Gamma := \Big\{\sum_{\mathcal{Z}\in S} a_{\mathcal{Z}}\mathbb{1}_{\mathcal{Z}} \mid S\subset\mathfrak{Z}(\mathbb{X}_m)\text{ finite and } a_{\mathcal{Z}}\in\mathbb{C}\Big\}$$

as the set of simple functions on the measure space $(\mathbb{X}_m,\mathfrak{B}_m,\mu_m)$. By linearity, the validity of Eq. (4.7) for $\mathbb{1}_{\mathcal{Z}_t(v)}$ extends to an arbitrary function in Γ. Due to the Stone–Weierstraß theorem [Lan93, Thm. 1.4], Γ is dense in $\mathcal{C}(\mathbb{X}_m)$ and thus also in $L^1(\mathbb{X}_m,\mu_m)$ [Lan93, Thm. 3.1]. This implies the assertion. ∎

Theorem 4.22. *The measure $\mu_m\in\mathcal{P}_S(\mathbb{X}_m)$ is ergodic.*

Proof. This is an immediate consequence of Proposition 4.21 and the discussion following Theorem 4.17. ∎

CHAPTER 5

NMS and RNMS sets as model sets

In this chapter, we want to transfer the inspection of the (random) noble means substitutions from the dynamic to the geometric side. To this end, we follow two different tracks. On the one hand, there is a rather direct derivation of symbolic sequences to point sets in $\mathbb{R}$ where we can make use of our preliminary work in Chapters 1 and 2. In particular, we will benefit once more from Perron–Frobenius theory and make use of the left PF eigenvector that we have disregarded so far. This will be the purpose of Section 5.2. On the other hand, we will derive the parameters for the description of geometric realisations of all members of $\mathcal{N}_m$ (and $\mathcal{R}$) as (subsets of) so-called model sets. This is contained in Section 5.3. As the determination of these parameters is essentially based on the theory of iterated function systems, we briefly collect the basics in this regard in Section 5.1 and also review the results presented in [BM00a] that generalise a famous theorem [Hut81, Thm. 3.1.3] by J. Hutchinson about attractors of contractive iterated function systems. The advantage of this different point of view will be a systematic description of the geometric realisations and a direct proof of their Meyer property.

5.1. Iterated function systems

Let (X, d) be a complete metric space and $\mathcal{K}X$ be the set of non-empty compact subsets of X. As we are mainly interested in $X = \mathbb{R}$, one may think of closed and bounded sets here. For any subset $U \subset X$ and element $x \in X$ we denote by $\mathrm{d}(x, U) := \inf\{\mathrm{d}(x, u) \mid u \in U\}$ the distance of x from U. We wish to interpret the set $\mathcal{K}X$ as a metric space itself and therefore need a notion of distance between compact sets in $\mathcal{K}X$. This is provided by the *Hausdorff metric* [Wic91, Prop. 2.1.3], defined by

$$\mathrm{h}(U, V) := \sup\{\mathrm{d}(x, V), \mathrm{d}(y, U) \mid x \in U, y \in V\},$$

which makes $(\mathcal{K}X, \mathrm{h})$ a complete metric space itself [Wic91, Prop. 2.3.2]. One can prove [Wic91, Note 2.1.6] that for U_i, $V_i \in \mathcal{K}X$, $i \in I$ and any finite index set I, it is true that

$$\mathrm{h}\Big(\bigcup_{i\in I} U_i, \bigcup_{i\in I} V_i\Big) \leqslant \sup\{\mathrm{h}(U_i, V_i) \mid i \in I\}. \tag{5.1}$$

Now, we consider two complete metric spaces (X_1, d_1), (X_2, d_2) and define the set of Lipschitz continuous maps from X_1 to X_2 with Lipschitz constant $\leqslant r$ as $\mathrm{Lip}(r, X_1, X_2)$. If $r < 1$, a Lipschitz function is called a *contraction*.

Given a (finite) family $\mathcal{F} = \{f_1, \ldots, f_n\}$ of maps $f_i \in \mathrm{Lip}(r_{f_i}, X_1, X_2)$, we define two functions

$$\begin{aligned} F\colon X_1 &\longrightarrow \mathcal{K}X_2, \quad x \longmapsto \bigcup_{i=1}^{n} \{f_i(x)\}, \\ F\colon \mathcal{K}X_1 &\longrightarrow \mathcal{K}X_2, \quad U \longmapsto \bigcup_{i=1}^{n} f_i(U). \end{aligned} \tag{5.2}$$

For convenience, we assign the same name to both maps in Eq. (5.2). For any U, $V \in \mathcal{K}X_1$, we get

$$\begin{aligned} \mathrm{h}(F(U), F(V)) &= \mathrm{h}\Big(\bigcup_{i=1}^{n} f_i(U), \bigcup_{i=1}^{n} f_i(V)\Big) \leqslant \sup_{1 \leqslant i \leqslant n} \mathrm{h}\big(f_i(U), f_i(V)\big) \\ &= \sup_{1 \leqslant i \leqslant n} \left\{ \sup_{\substack{x \in U \\ y \in V}} \Big\{ \inf_{y \in V} \big\{\mathrm{d}\big(f_i(x), f_i(y)\big)\big\}, \inf_{x \in U} \big\{\mathrm{d}\big(f_i(x), f_i(y)\big)\big\} \Big\} \right\} \\ &\leqslant \sup_{1 \leqslant i \leqslant n} r_{f_i} \cdot \sup_{\substack{x \in U \\ y \in V}} \big\{\mathrm{d}(x, V), \mathrm{d}(y, U)\big\} = R\, \mathrm{h}(U, V), \end{aligned}$$

with $R \coloneqq \sup\{r_{f_i} \mid 1 \leqslant i \leqslant n\}$ and the first inequality being a consequence of Eq. (5.1). This shows that the Lipschitz continuity of F with respect to h is implied by that of each f_i.

We firstly consider the situation where $X_1 = X_2$. Here, we refer to $\mathcal{F}$ as an *iterated function system* and call $\mathcal{F}$ *contractive* if each $f \in \mathcal{F}$ is a contraction. We have now paved the way for an application of *Banach's contraction mapping principle* to the map F. This leads to the following result.

Theorem 5.1 ([Hut81, Thm. 3.1.3])**.** *Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a contractive iterated function system with $f_i\colon X \longrightarrow X$ and a compact metric space X. Then, there is a unique compact set $A \in \mathcal{K}X$ with the property*

$$A = F(A) = \bigcup_{i=1}^{n} f_i(A).$$

Furthermore, $\big(F^{\ell}(S)\big)_{\ell \in \mathbb{N}}$ converges to A in the Hausdorff metric for any $S \in \mathcal{K}X$ as $\ell \to \infty$. The set A is called the attractor *of $\mathcal{F}$.* ∎

Theorem 5.1 can be generalised to a multi-component situation that will become essential in Section 5.3. Instead of considering a single complete metric space (X, d) and functions $X \longrightarrow X$, we start with finitely many complete metric spaces $(X_1, \mathrm{d}_1), \ldots, (X_n, \mathrm{d}_n)$ and study Lipschitz maps $X_j \longrightarrow X_i$. To this end, we define the product space $X \coloneqq X_1 \times \cdots \times X_n$ and equip it with the metric $\mathrm{d}(x, y) \coloneqq \sup\{\mathrm{d}_i(x_i, y_i) \mid 1 \leqslant i \leqslant n\}$. Here, (X, d) is a again a complete metric space. Let h_i be the Hausdorff metric on $\mathcal{K}X_i$ for each $1 \leqslant i \leqslant n$ and h the Hausdorff metric on $\mathcal{K}X$.

For each pair (i, j) with $1 \leqslant i, j \leqslant n$, let $\mathcal{F}_{ij}$ be a finite set of Lipschitz maps $f\colon X_j \longrightarrow X_i$ where we require that for each i there is at least one j such that $\mathcal{F}_{ij} \neq \varnothing$. In this setting, we call $\mathcal{F} = (\mathcal{F}_{ij})_{1 \leqslant i,j \leqslant n}$ a *generalised iterated function system* and $\mathcal{F}$ is *contractive* if all $f \in \mathcal{F}_{ij}$ are contractions, for all $1 \leqslant i, j \leqslant n$. From Eq. (5.2), for each pair (i, j), we get the map $F_{ij}\colon \mathcal{K}X_j \longrightarrow \mathcal{K}X_i$, where the corresponding Lipschitz constant r_{ij} is given by the maximum of all Lipschitz constants of all mappings in $\mathcal{F}_{ij}$ and $R := \sup\{r_{ij} \mid 1 \leqslant i, j \leqslant n\}$. Now we define the map

$$F_g\colon \mathcal{K}X_1 \times \cdots \times \mathcal{K}X_n \longrightarrow \mathcal{K}X_1 \times \cdots \times \mathcal{K}X_n$$

by

$$\begin{aligned} F_g(U_1, \ldots, U_n) &:= \Big(\bigcup_{j=1}^{n} F_{1j}(U_j), \ldots, \bigcup_{j=1}^{n} F_{nj}(U_j) \Big) \\ &= \Big(\bigcup_{j=1}^{n} \bigcup_{f \in F_{1j}} f(U_j), \ldots, \bigcup_{j=1}^{n} \bigcup_{f \in F_{nj}} f(U_j) \Big), \end{aligned}$$

and note that $\mathcal{K}X_1 \times \cdots \times \mathcal{K}X_n \subsetneq \mathcal{K}X$. With this notation, we get the following result.

Proposition 5.2 ([BM00a, Prop. 4.1])**.** *The mapping F_g is Lipschitz with Lipschitz constant at most R.* ∎

Finally, we return to the setting of Theorem 5.1 and can apply the contraction mapping principle to the map F_g. This implies the existence of a uniquely defined attractor

$$A = A_1 \times \cdots \times A_n \in \mathcal{K}X_1 \times \cdots \times \mathcal{K}X_n$$

for F_g. Consequently, the family $\{A_1, \ldots, A_n\}$ provides a compact solution to the systems

$$A_i = \bigcup_{j=1}^{n} \bigcup_{f \in F_{ij}} f(A_j)$$

for all $1 \leqslant i \leqslant n$. For later reference, we formulate

Corollary 5.3. *Let $\mathcal{F} = (\mathcal{F}_{ij})_{1 \leqslant i,j \leqslant n}$ be a generalised contractive iterated function system. Then, there is a unique compact set $A \in \mathcal{K}X_1 \times \cdots \times \mathcal{K}X_n$ with the property*

$$A = F_g(A) = \Big(\bigcup_{j=1}^{n} \bigcup_{f \in F_{1j}} f(A_j), \ldots, \bigcup_{j=1}^{n} \bigcup_{f \in F_{nj}} f(A_j) \Big). \tag{5.3}$$

Furthermore, the sequence $(F_g^{\ell}(S))_{\ell \in \mathbb{N}}$ converges to A in the Hausdorff metric for any $S \in \mathcal{K}X_1 \times \cdots \times \mathcal{K}X_n$ as $\ell \to \infty$. The set A is called the attractor *of $\mathcal{F}$.* ∎

An even more general situation is considered in [BM00a, Sec. 1], where $\mathcal{F}$ may consist of compact families of contractions. As we will only make use of the finite case, we have restricted our exposition to this situation.

5.1.1. Measures on attractors of iterated function systems. The concepts of Section 5.1 turn out to be even more useful with the observation that the attractor of a (generalised) contractive iterated function system carries a canonically defined invariant measure. We briefly review the basic ideas in the single-component setting and refer to [BM00a, Sec. 4.2] for the generalisation. The measure-theoretic background has been provided in Section 1.4.

Let (X, d) be a compact metric space and $\mathcal{P}(X)$ the space of all probability measures on X. We define the *Kantorovich metric* on $\mathcal{P}(X)$ by

$$\mathrm{k}(\mu, \nu) := \sup\{|\mu(\phi) - \nu(\phi)| \mid \phi \in \mathrm{Lip}(1, X, \mathbb{R})\}$$

and refer to [Rüs07, Ver05] for background information in this regard. A proof that k actually is a metric that induces the vague topology can be found in [BM00a, Prop. 2.2].

Remark 5.4. Hutchinson applies the Kantorovich metric in his pioneering paper on iterated function systems [Hut81] without reference to Kantorovich's work. Most probably, Hutchinson discovered the usefulness of k independently, which rather suggests to denote k as the *Kantorovich–Hutchinson* metric, but for the sake of brevity, we stick to the above choice. ◆

Now, let $\mathcal{F} = \{f_1, \dots, f_n\}$ be a contractive iterated function system with $f_i \colon X \longrightarrow X$ for all i and A the attractor of $\mathcal{F}$. Then, we define a function $\mu \longmapsto f.\mu$ with $f.\mu(\phi) := \mu(\phi \circ f)$ for any $\phi \in \mathcal{C}(A, X)$, which constitutes a map $\mathcal{P}(X) \longrightarrow \mathcal{P}(X)$. Last but not least, we introduce

$$F_{\boldsymbol{s}} \colon \mathcal{P}(X) \longrightarrow \mathcal{P}(X), \quad \mu \longmapsto F_{\boldsymbol{s}}(\mu) := \sum_{i=1}^{n} s_i f_i.\mu$$

for any $\boldsymbol{s} := (s_1, \dots, s_n)$ with $s_i \in (0,1)$ and $\sum_{i=1}^{n} s_i = 1$ and denote as $r := \sup\{r_i \mid 1 \leqslant i \leqslant n\} < 1$ the maximum of all contractivity constants of functions in $\mathcal{F}$. The map $F_{\boldsymbol{s}}$ is a contraction in the Kantorovich metric because, for any $\phi \in \mathrm{Lip}(1, X, \mathbb{R})$, we get

$$\begin{aligned}
\bigl|F_{\boldsymbol{s}}(\mu(\phi)) - F_{\boldsymbol{s}}(\nu(\phi))\bigr| &= \Bigl|\sum_{i=1}^{n} s_i f_i.\mu(\phi) - \sum_{i=1}^{n} s_i f_i.\nu(\phi)\Bigr| \\
&\leqslant r \sum_{i=1}^{n} s_i \bigl|\mu(r^{-1}\phi \circ f_i) - \nu(r^{-1}\phi \circ f_i)\bigr| \\
&\leqslant r\, \mathrm{k}(\mu, \nu),
\end{aligned}$$

where the last inequality is implied by $r^{-1}\phi \circ f_i \in \mathrm{Lip}(1, X, \mathbb{R})$. Finally, we arrive at the following result corresponding to Theorem 5.1.

Theorem 5.5 ([Hut81, Thm. 4.4.1])**.** *The map $F_{\boldsymbol{s}}$ is a contraction in the Kantorovich metric. Furthermore, there exists a unique measure $\mu \in \mathcal{P}(X)$ such that $F_{\boldsymbol{s}}(\mu) = \mu$. If ν is any probability measure on X, then $\big(F^{\ell}_{\boldsymbol{s}}(\nu)\big)_{\ell\in\mathbb{N}}$ converges in the Kantorovich metric to μ as $\ell \to \infty$.* ■

In the multi-component situation, where $\mathcal{F} = (\mathcal{F}_{ij})_{1\leqslant i,j\leqslant n}$ is a generalised iterated function system, each $f \in \mathcal{F}_{ij}$ gives rise to a transformation of measure spaces $\mathcal{M}(X_j) \longrightarrow \mathcal{M}(X_i)$ via $\mu_j \longmapsto f.\mu_j$ and we strive for a family $\{\mu_1, \ldots, \mu_n\}$ that is invariant under the average of all these transformations, which means

$$\mu_i := \sum_{j=1}^{n} \frac{1}{|\mathcal{F}_{ij}|} \sum_{f\in\mathcal{F}_{ij}} f.\mu_j$$

for all $1 \leqslant i \leqslant n$. Here, $|\mathcal{F}_{ij}|$ denotes the number of mappings in $\mathcal{F}_{ij}$ and $1/|\mathcal{F}_{ij}|$ is set to 0 if $\mathcal{F}_{ij} = \varnothing$. More generally, we are free to choose arbitrary weights for each $\mathcal{F}_{ij}$ and will make use of this modification if it seems appropriate. For a justification of these adjustments, we refer to [BM00a, Sec. 4.2]. Note that each μ_i separately may fail to be a probability measure. The result in the more general situation of generalised compact contractive iterated function systems is spelled out in [BM00a]. For later reference, we formulate the corresponding result in the finite case as follows.

Theorem 5.6 ([BM00a, Thm. 4.4])**.** *Let $X_1, \ldots, X_n$ be compact metric spaces and $\mathcal{F} = (\mathcal{F}_{ij})_{1\leqslant i,j\leqslant n}$ be a generalised contractive iterated function system. Assume that each $\mathcal{F}_{ij}$ is equipped with a fixed scalar (same for all) multiple of counting measure η_{ij} and that the matrix*

$$S = \big(\eta_{ij}(\mathcal{F}_{ij})\big)_{ij} \in \mathrm{Mat}(n, \mathbb{R})$$

has a strictly positive right eigenvector $(m_1, \ldots, m_n)^T$, corresponding to the eigenvalue 1*. Then, there exists a unique family $\{\mu_1, \ldots, \mu_n\}$ of measures with $\mu_i \in \mathcal{M}(X_i)$ and the properties*

$$\mu_i := \sum_{j=1}^{n} \frac{1}{\eta_{ij}(\mathcal{F}_{ij})} \sum_{f\in\mathcal{F}_{ij}} f.\mu_j \quad \text{and} \quad \mu_i(X_i) = m_i,$$

for all $1 \leqslant i \leqslant n$. The support of $\boldsymbol{\mu} := \mu_1 \otimes \cdots \otimes \mu_n$ is contained in the attractor of Eq. (5.3)*.* ■

The measure $\boldsymbol{\mu}$ in Theorem 5.6 is called *inflation-invariant* (or *self-similar*). Note that the choice of an appropriate scalar multiple of counting measure, provides the normalisation of the leading eigenvalue to 1.

5.2. From dynamics to geometry

So far, we have only dealt with symbolic sequences, generated by substitutions of the families $\mathcal{N}_m$ and $\mathcal{R}$ respectively, over the binary alphabet $\mathcal{A}_2 = \{a, b\}$. From the primitivity of the substitution matrices M_m, one can derive a geometric

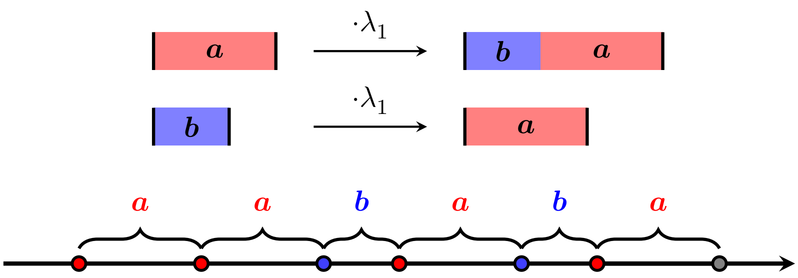

Figure 5.1. The action of the inflation rule $\zeta_{0,1}$ is shown above. The left endpoints of the resulting intervals are chosen as control points below. The colour of the grey-filled dot depends on the letter with which the sequence is continued to the right.

interpretation of symbolic sequences in terms of point sets in $\mathbb{R}$. To this end, we apply once more Theorem 1.4 and make use of the unique existence of strictly positive left and right eigenvectors $\boldsymbol{L}_m$, $\boldsymbol{R}_m$ stated therein. In the following, we use the normalisations

$$\boldsymbol{L}_m = (\boldsymbol{L}_m^{(a)}, \boldsymbol{L}_m^{(b)}) = (\lambda_m, 1) \quad \text{and} \quad \boldsymbol{R}_m = \begin{pmatrix} \boldsymbol{R}_m^{(a)} \\ \boldsymbol{R}_m^{(b)} \end{pmatrix} = \frac{1}{\lambda_m + 1} \begin{pmatrix} \lambda_m \\ 1 \end{pmatrix}, \tag{5.4}$$

where $\|\boldsymbol{R}_m\|_1 = 1$.

Remark 5.7 (Normalisation of $\boldsymbol{L}_m$ and $\boldsymbol{R}_m$)**.** Note that, in the literature, one often finds normalisations of $\boldsymbol{L}_m$ and $\boldsymbol{R}_m$ such that $\langle \boldsymbol{L}_m \,|\, \boldsymbol{R}_m \rangle = 1$. Our choice has the advantage that the Minkowski embedding (see Eq. (5.7) on page 60) of the resulting $\mathbb{Z}$-module (see the subsequent discussion) is particularly convenient to handle. ♦

As stated on page 6, the entries of $\boldsymbol{R}_m$ encode the relative letter frequencies in a fixed point of $\zeta_{m,i}$ (or $\zeta_{m,i}^2$). Now, the vector $\boldsymbol{L}_m$ facilitates a geometric interpretation of all members $\vartheta \in \mathcal{N}_m \cup \mathcal{R}$. We assign a closed interval I_a of length $\boldsymbol{L}_m^{(a)} = \lambda_m$ to the letter a and a closed interval I_b of length $\boldsymbol{L}_m^{(b)} = 1$ to the letter b. The action of ϑ on the intervals can be defined as follows. Firstly, I_a and I_b are scaled by λ_m and secondly, the resulting intervals are dissected according to the substitution rule ϑ. As $\boldsymbol{L}_m$ is an eigenvector of M_m, this can be done consistently. Formally, this defines a new map, acting on $\mathbb{R}$, which will, for convenience, again be denoted by ϑ. In this setting, we refer to ϑ as a one-dimensional *inflation rule*. Note that iterating the inflation rule results in a face-to-face *tiling* of the real line. This is a countable partition $\mathcal{T} = \{\mathcal{T}_i\}_{i \in I}$ such that each $\mathcal{T}_i \subset \mathbb{R}$ is a closed interval with $\mathcal{T}_i^\circ \cap \mathcal{T}_j^\circ = \varnothing$ for all $i, j \in I$ with $i \neq j$ and $\bigcup_{i \in I} \mathcal{T}_i = \mathbb{R}$. Now, we can generate attributed point sets by choosing the left or right endpoints (both possibilities are valid, but once decided on one side, we have to stick to it) of each I_a and I_b as *control points*.

Remark 5.8. We always work with *left* endpoints as control points in this chapter. ♦

The point sets in $\mathbb{R}$ resulting from this procedure are called *geometric realisations* and a sketch of this is given in Figure 5.1 in the case of $\zeta_{0,1}$. Moreover, we denote geometric realisations of fixed points of the substitution $\zeta_{m,i}$ (or $\zeta_{m,i}^2$) as *noble means sets* and refer to them as $\Lambda_{m,i}$. The same method applies to the stochastic situation where the geometric realisations of accumulation points of the sequence $\big(\zeta_m^k(a \,|\, a)\big)_{k\in\mathbb{N}_0}$ are called *generating random noble means sets* and any instance of these is referred to as Λ_m.

Remark 5.9 (Point density of RNMS sets)**.** The choice of the lengths of I_a and I_b according to the entries of $\boldsymbol{L}_m$ implies that for any $0 \leqslant i \leqslant m$, we have

$$\Lambda_{m,i}, \Lambda_m \subset \mathbb{Z}[\lambda_m] = \{a + b\lambda_m \mid a, b \in \mathbb{Z}\} \subset \mathbb{R}.$$

Now, the *point density* of $P \in \{\Lambda_{m,i}, \Lambda_m\}$ for an arbitrary but fixed $m \in \mathbb{N}$ reads

$$\operatorname{dens}(P) = \frac{1}{\langle \boldsymbol{L}_m \,|\, \boldsymbol{R}_m \rangle} = \frac{1 - \lambda_m'}{\sqrt{m^2+4}}. \tag{5.5}$$

We refer to [BG13, Sec. 2.1] for general background concerning the point density of discrete point sets. ♦

5.3. Cut and project

In this section, we want to show that the (random) noble means sets $\Lambda_{m,i}$ (and Λ_m) can be constructed within one and the same *cut and project scheme* as (subsets of) so-called *model sets*.

5.3.1. The general setting.

We begin with a brief introduction to *cut and project schemes* in a general setting and derive the (R)NMS case afterwards.

Definition 5.10 (Cut and project scheme)**.** A *cut and project scheme* is a triple $(\mathbb{R}^d, H, \mathcal{L})$ with a locally compact Abelian group H and a lattice $\mathcal{L} \subset \mathbb{R}^d \times H$. The two canonical projections $\pi_1 \colon \mathbb{R}^d \times H \longrightarrow \mathbb{R}^d$ and $\pi_2 \colon \mathbb{R}^d \times H \longrightarrow H$ are attached and $\mathcal{L}$ is chosen such that $\pi_1\big|_{\mathcal{L}}$ is injective and $\pi_2(\mathcal{L})$ is dense in H. The group $\mathbb{R}^d$ is called the *physical space* and H the *internal space*.

Here, a lattice $\mathcal{L}$ is defined as a discrete subgroup of $\mathbb{R}^d \times H$ such that the factor group $(\mathbb{R}^d \times H)/\mathcal{L}$ is compact. We define $L := \pi_1(\mathcal{L})$ and a mapping

$$\star\colon L \longrightarrow H, \quad x \longmapsto x^\star := \pi_2\big((\pi_1|_{\mathcal{L}})^{-1}\big)(x) \tag{5.6}$$

that is referred to as the *star map*. We can regard $\mathcal{L}$ as the *diagonal* embedding of L, that is $\mathcal{L} := \{(x, x^\star) \mid x \in L\}$; see [BG13, Sec. 3.4] for general background. The situation derived so far is summarised in Figure 5.2.

Given a non-empty and relatively compact set $W \subset H$, we refer to

$$\Theta(W) := \{x \in L \mid x^\star \in W\}$$

$$
\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\
\cup & & \cup & & \cup \text{ dense} \\
\pi_1(\mathcal{L}) & \xleftarrow{1-1} & \mathcal{L} & \longrightarrow & \pi_2(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\quad \star \qquad\quad} & & L^\star
\end{array}
$$

Figure 5.2. General cut and project scheme.

and all its $\mathbb{R}^d$-translates as a *model set* and denote W as its *window*. A model set is *regular* if $\mu_H(\partial W) = 0$ with μ_H being the Haar measure of H and $\partial W = \overline{W} \setminus W^\circ$ the boundary of W. If $L^\star \cap \partial W = \varnothing$, a model set is called *generic*, otherwise it is called *singular*.

It is possible to further generalise the concept of cut and project schemes by considering a σ-compact, locally compact Abelian group as physical space [Schl98, Schl00]. We shall not cover this here, as we only need the special case of point sets in $\mathbb{R}$.

A characterisation of Meyer sets in this setting is provided by the following result.

Theorem 5.11 ([Moo97a, Thm. 9.1(i)]). *A relatively dense point set $P \subset \mathbb{R}^d$ is Meyer if and only if P is a subset of a model set.* ∎

We now want to study whether generating random noble means sets Λ_m are Meyer and embark on the following strategy for a proof. To begin with, we describe the geometric realisations of fixed points of all members of $\mathcal{N}_m$ as regular model sets and therefore calculate the corresponding windows in internal space. Then, we use these windows for the calculation of a 'super window' W with $\Lambda_m \subset \Theta(W)$ and $\mathrm{vol}(W)$ minimal with this property, and finally apply Theorem 5.11.

5.3.2. The NMS case. We start by collecting the necessary data for the description of $\Lambda_{m,i}$ as model sets. We already know that the physical space in Definition 5.10 is $\mathbb{R}$ and even more, we have seen in Section 5.2 that $\Lambda_{m,i} \subset \mathbb{Z}[\lambda_m]$ with $\mathbb{Z}[\lambda_m]$ being dense in $\mathbb{R}$. Now, we define the non-trivial field automorphism on the quadratic field $\mathbb{Q}(\sqrt{m^2+4})$ that is given by

$$
{}': \mathbb{Q}(\sqrt{m^2+4}) \longrightarrow \mathbb{Q}(\sqrt{m^2+4}), \quad x + y\sqrt{m^2+4} \longmapsto x - y\sqrt{m^2+4}
$$

as the star map here, that is $x^\star := x'$. The diagonal (Minkowski) embedding of $\mathbb{Z}[\lambda_m]$ is the lattice

$$
\mathcal{L}_m := \{(x, x^\star) \mid x \in \mathbb{Z}[\lambda_m]\} \subset \mathbb{R} \times \mathbb{R}, \tag{5.7}
$$

$$\begin{tikzcd}
\mathbb{R} & \mathbb{R}\times\mathbb{R} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{R} \\
\mathbb{Z}[\lambda_m] \arrow[u, phantom, "\text{dense }\cup"] \arrow[d, equal] & \mathcal{L}_m \arrow[u, phantom, "\cup"] \arrow[l, "1-1"'] \arrow[r, "1-1"] & \mathbb{Z}[\lambda_m] \arrow[u, phantom, "\cup\text{ dense}"] \arrow[d, equal] \\
L \arrow[rr, "\star"] & & L^\star
\end{tikzcd}$$

Figure 5.3. Cut and project scheme for the noble means sets $\Lambda_{m,i}$.

which leads to $\mathbb{R}$ as internal space. Here, the projection of $\mathcal{L}_m$ into the physical space is also dense in $\mathbb{R}$ as $\pi_1(\mathcal{L}_m) = \mathbb{Z}[\lambda_m]$. In particular, for any lattice $\mathcal{L} \subset \mathbb{R}^d$, there exist linearly independent vectors $b_1, \ldots, b_d \in \mathbb{R}^d$, called the *lattice base*, with

$$\mathcal{L} = \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_d := \Big\{\sum_{i=1}^{d} \alpha_i b_i \mid \alpha_i \in \mathbb{Z}, 1 \leqslant i \leqslant d\Big\}.$$

In the case of $\mathcal{L}_m$, a lattice base is given by $b_1 := (1,1)^T$ and $b_2 := (\lambda_m, \lambda'_m)^T$. For later use, we note the dual lattice of $\mathcal{L}_m$ which is given by

$$\begin{aligned}
\mathcal{L}_m^* &:= \{y \in \mathbb{R}^2 \mid \langle x \,|\, y\rangle \in \mathbb{Z} \text{ for all } x \in \mathcal{L}_m\} \\
&= \left\langle \frac{1}{\sqrt{m^2+4}} \begin{pmatrix} -\lambda'_m \\ \lambda_m \end{pmatrix}, \frac{1}{\sqrt{m^2+4}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle_{\mathbb{Z}}.
\end{aligned} \tag{5.8}$$

The cut and project scheme for the noble means sets is compactly presented in Figure 5.3. The following things are left to be done to complete the picture.

- compute the windows $W_{m,i}$ for $\Lambda_{m,i}$ in internal space
- prove that $\mathrm{vol}(W_{m,i}) > 0$ and $\mathrm{vol}(\partial W_{m,i}) = 0$
- prove that $\mathrm{dens}(\Lambda_{m,i}) = \mathrm{dens}(\Theta(W_{m,i}))$

For the computation of the required windows, we will benefit from the technical preliminaries of Section 5.1. We denote by $\Lambda_{m,i}^{(a)}$ and $\Lambda_{m,i}^{(b)}$ all points of the noble means sets that are generated by the letters a and b, respectively, in the sense of Section 5.2.

Remark 5.12. In the following, we consider iterated function systems resulting from a single application of the substitution $\zeta_{m,i}$. We have seen in Remark 2.1 that we only get semi-infinite fixed points in the cases $i = 0$ and $i = m$ for $\zeta_{m,i}$ itself. Due to Remark 1.10, this is enough to construct the correct two-sided discrete hull wherefore it is sufficient to restrict to this technically more convenient case. ♦

As a result of the action of $\zeta_{m,i}$ on $\mathcal{A}_2$ for any $0 \leqslant i \leqslant m$, we arrive at the following iterated function system in the physical space.

$$\begin{aligned} \Lambda_{m,i}^{(a)} &= \left\{ \bigcup_{j=0}^{i-1} \lambda_m \Lambda_{m,i}^{(a)} + j\lambda_m \right\} \cup \lambda_m \Lambda_{m,i}^{(b)} \cup \left\{ \bigcup_{j=i}^{m-1} \lambda_m \Lambda_{m,i}^{(a)} + j\lambda_m + 1 \right\} \\ \Lambda_{m,i}^{(b)} &= \lambda_m \Lambda_{m,i}^{(a)} + i\lambda_m. \end{aligned} \tag{5.9}$$

With the notation of Section 5.1, we consider the (finite) iterated function system $\mathcal{F} = (\mathcal{F}_{11}, \mathcal{F}_{12}, \mathcal{F}_{21}, \mathcal{F}_{22})$ with the affine map families

$$\begin{aligned} &\mathcal{F}_{11} := \{\lambda_m(x+j) \mid 0 \leqslant j \leqslant i-1\} \cup \{\lambda_m(x+j)+1 \mid i \leqslant j \leqslant m-1\}, \\ &\mathcal{F}_{12} := \{\lambda_m x\}, \quad \mathcal{F}_{21} := \{\lambda_m(x+i)\} \quad \text{and} \quad \mathcal{F}_{22} := \varnothing. \end{aligned}$$

Now, Eq. (5.9) can be represented as

$$\begin{pmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \mathcal{F}_{21} & \mathcal{F}_{22} \end{pmatrix} \begin{pmatrix} \Lambda_{m,i}^{(a)} \\ \Lambda_{m,i}^{(b)} \end{pmatrix} = \begin{pmatrix} \Lambda_{m,i}^{(a)} \\ \Lambda_{m,i}^{(b)} \end{pmatrix}.$$

As $\lambda_m > 1$ for all $m \in \mathbb{N}$, $\mathcal{F}$ is not contractive and therefore in general not uniquely solvable. We define

$$\Gamma_{m,i}^{(a)} := \overline{(\Lambda_{m,i}^{(a)})^\star} \quad \text{and} \quad \Gamma_{m,i}^{(b)} := \overline{(\Lambda_{m,i}^{(b)})^\star}$$

and study the corresponding system in the internal space, which gives

$$\begin{aligned} \Gamma_{m,i}^{(a)} &= \left\{ \bigcup_{j=0}^{i-1} \lambda'_m \Gamma_{m,i}^{(a)} + j\lambda'_m \right\} \cup \lambda'_m \Gamma_{m,i}^{(b)} \cup \left\{ \bigcup_{j=i}^{m-1} \lambda'_m \Gamma_{m,i}^{(a)} + j\lambda'_m + 1 \right\} \\ \Gamma_{m,i}^{(b)} &= \lambda'_m \Gamma_{m,i}^{(a)} + i\lambda'_m. \end{aligned} \tag{5.10}$$

We write $\mathcal{F}^\star$ in this case for the iterated function system.

Proposition 5.13. *For an arbitrary but fixed $m \in \mathbb{N}$ and all $0 \leqslant i \leqslant m$, the attractors of the iterated function system $\mathcal{F}^\star$ are given by the compact intervals*

$$A_{m,i} = A_{m,i}^{(a)} \cup A_{m,i}^{(b)} = i\tau_m + [\lambda'_m, 1],$$

where

$$A_{m,i}^{(a)} = i\tau_m + [0,1] \quad \textit{and} \quad A_{m,i}^{(b)} = i\tau_m + [\lambda'_m, 0],$$

with $\tau_m := -\frac{1}{m}(\lambda'_m + 1)$.

Proof. Plugging the compact sets $A_{m,i}^{(a)}$ and $A_{m,i}^{(b)}$ into Eq. (5.10) leads to

$$\left\{ \bigcup_{j=0}^{i-1} \lambda'_m A_{m,i}^{(a)} + j\lambda'_m \right\} \cup \lambda'_m A_{m,i}^{(b)} \cup \left\{ \bigcup_{j=i}^{m-1} \lambda'_m A_{m,i}^{(a)} + j\lambda'_m + 1 \right\}.$$

As $-1 < \lambda'_m < 0$ and the length of $A^{(a)}_{m,i}$ is equal to 1, all unions of translates of $A^{(a)}_{m,i}$ overlap and consequently

$$
\begin{aligned}
&= [\lambda'_m(i\tau_m + 1) + (i-1)\lambda'_m, \lambda'_m i\tau_m] \cup [\lambda'_m i\tau_m, \lambda'_m(\lambda'_m + i\tau_m)] \\
&\qquad \cup [\lambda'_m(i\tau_m + 1) + (m-1)\lambda'_m + 1, \lambda'_m i\tau_m + i\lambda'_m + 1] \\
&= [\lambda'_m i(\tau_m + 1), \lambda'_m i\tau_m] \cup [\lambda'_m i\tau_m, \lambda'_m(m + i\tau_m) + 1] \\
&\qquad \cup [\lambda'_m(m + i\tau_m) + 1, \lambda'_m i(\tau_m + 1) + 1] \\
&= [\lambda'_m i(\tau_m + 1), \lambda'_m i(\tau_m + 1) + 1] \\
&= i\tau_m + [0, 1] \\
&= A^{(a)}_{m,i}.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\lambda'_m A^{(a)}_{m,i} + i\lambda'_m &= [\lambda'_m(1 + i\tau_m) + i\lambda'_m, \lambda'_m i\tau_m + i\lambda'_m] \\
&= [\lambda'_m + \lambda'_m i(\tau_m + 1), \lambda'_m i(\tau_m + 1)] \\
&= i\tau_m + [\lambda'_m, 0] \\
&= A^{(b)}_{m,i},
\end{aligned}
$$

so that $(A^{(a)}_{m,i}, A^{(b)}_{m,i})$ is a solution to the iterated function system. Since all maps in $\mathcal{F}^*$ are contractions, the assertion now follows from an application of Corollary 5.3. ∎

Lemma 5.14. *For an arbitrary but fixed $m \in \mathbb{N}$ and all $0 \leqslant i \leqslant m$, the point density of the model set $\Theta(A_{m,i})$, within the cut and project scheme of Figure 5.3, is given by*

$$\operatorname{dens}(\Theta(A_{m,i})) = \operatorname{dens}(\Lambda_{m,i}).$$

Proof. To begin with, we order the elements of $\Theta(A_{m,i})$ according to their distance from 0 and assemble them in a sequence $(x_i)_{i\in\mathbb{N}}$ such that $\|x_{i+1}\| \geqslant \|x_i\|$ for all $i \in \mathbb{N}$ and some norm $\|\cdot\|$ in $\mathbb{R}$. From [BG13, Thm. 7.2], we know that the sequence $(x_i^\star)_{i\in\mathbb{N}}$ is uniformly distributed (in the sense of Weyl [Wey16]) in $A_{m,i}$. As the projection $\pi_2(\mathcal{L}_m)$ is dense in the internal space, $\operatorname{vol}(A_{m,i}) < \infty$ and $\operatorname{vol}(\partial A_{m,i}) = 0$, the point density of $\Theta(A_{m,i})$ can be computed via the the formula [Schl98, Thm. 1]

$$\operatorname{dens}(\Theta(A_{m,i})) = \operatorname{dens}(\mathcal{L}_m)\operatorname{vol}(A_{m,i}).$$

As $\operatorname{dens}(\mathcal{L}_m) = |\det(b_1, b_2)|^{-1} = |2\lambda'_m - m|^{-1}$ and $\operatorname{vol}(A_{m,i}) = 1 - \lambda'_m$, the assertion follows from Remark 5.9 and a simple calculation. ∎

Lemma 5.15. *For any $m \in \mathbb{N}$ and $0 \leqslant i \leqslant m$, the set $\Theta(A^\circ_{m,i})$ is repetitive.*

Proof. We consider an arbitrary non-empty patch

$$P := \Theta(A^\circ_{m,i}) \cap I$$

of $\Theta(A^{\circ}_{m,i})$ with a closed interval $I \subset \mathbb{R}$. The definition of the star map implies $P^{\star} \subset A^{\circ}_{m,i}$ and $(-\varepsilon,\varepsilon)+P^{\star} \subset A^{\circ}_{m,i}$ for some $\varepsilon > 0$. Due to the denseness of $\mathbb{Z}[\lambda_m] = \pi_2(\mathcal{L}_m)$ in $\mathbb{R}$, we know that $(-\varepsilon,\varepsilon)\cap\mathbb{Z}[\lambda_m] \neq \varnothing$. For all $t \in \Theta\big((-\varepsilon,\varepsilon)\big)$, the set $t+P$ is a patch of $\Theta(A^{\circ}_{m,i})$ because for any $x \in P$ we get

$$(t+x)^{\star} = t^{\star} + x^{\star} \in (-\varepsilon,\varepsilon) + P^{\star} \subset A^{\circ}_{m,i}.$$

As $\Theta\big((-\varepsilon,\varepsilon)\big)$ is relatively dense, we conclude that P appears with bounded gaps and as $\Theta(A^{\circ}_{m,i})$ has finite local complexity [BG13, Prop. 7.5], the assertion follows. ■

Remark 5.16 (Generic and singular model sets)**.** Proposition 5.13 implies the inclusion $W_{m,i} \subset A_{m,i} = i\tau_m + [\lambda'_m, 1]$. As each $A_{m,i}$ is a compact interval of length $1-\lambda'_m \in \mathbb{Z}[\lambda_m]$, the following short calculation shows that only the cases $i=0$ and $i=m$ lead to singular model sets whereas the intermediate cases are all generic,

$$\lambda'_m + i\tau_m = \frac{(m-i)\lambda'_m}{m} - \frac{i}{m} \begin{cases} \in \mathbb{Z}[\lambda_m], & \text{if } i=0 \text{ or } i=m,\\ \notin \mathbb{Z}[\lambda_m], & \text{if } 0<i<m, \end{cases}$$

and the left endpoint of $A_{m,i}$ is an element of $\mathbb{Z}[\lambda_m]$ if and only if this is also true of the right endpoint. ♦

Corollary 5.17. *For an arbitrary but fixed $m \in \mathbb{N}$ and $0<i<m$, the windows for the noble means sets $\Lambda_{m,i}$, within the cut and project scheme of Figure* 5.3*, are given by the compact intervals $W_{m,i} := A_{m,i}$.*

Proof. From Remark 5.16, it is clear that $\mathcal{L}^{\star}_m \cap \partial A_{m,i} = \varnothing$, so the inclusion $\Lambda_{m,i} \subset \Theta(A^{\circ}_{m,i})$ is obvious. For the converse, we assume there is an element $x \in \Theta(A^{\circ}_{m,i}) \setminus \Lambda_{m,i}$. Then, there exists a $y \in \Lambda_{m,i}$ such that $|x-y| \notin \{1,\lambda_m\}$. Consequently, the elements x and y form a patch in $\Theta(A^{\circ}_{m,i})$ which does not occur in $\Lambda_{m,i}$. As $\Theta(A^{\circ}_{m,i})$ is repetitive by Lemma 5.15, this would imply

$$\operatorname{dens}\big(\Theta(A^{\circ}_{m,i})\big) > \operatorname{dens}(\Lambda_{m,i}),$$

which is a contradiction to Lemma 5.14. ■

For the treatment of the singular cases, let $\Lambda^{(x|y)}_{m,i}$ be the noble means set resulting from the fixed point of $\zeta^2_{m,i}$ with legal seed $x\,|\,y$ in the cases $i=0$ and $i=m$. The corresponding windows for the cut and project description are denoted by $W^{(x|y)}_{m,i}$.

Corollary 5.18. *For any $m \in \mathbb{N}$, the windows for the noble means sets*

$$\Lambda^{(a|a)}_{m,0},\ \Lambda^{(a|b)}_{m,0},\ \Lambda^{(a|a)}_{m,m} \quad \text{and} \quad \Lambda^{(b|a)}_{m,m},$$

within the cut and project scheme of Figure 5.3*, are given by*

$$W^{(a|a)}_{m,0} := [\lambda'_m, 1), \qquad W^{(a|b)}_{m,0} := (\lambda'_m, 1], \tag{5.11}$$

$$W^{(a|a)}_{m,m} := (-1, -\lambda'_m], \quad W^{(b|a)}_{m,m} := [-1, -\lambda'_m), \tag{5.12}$$

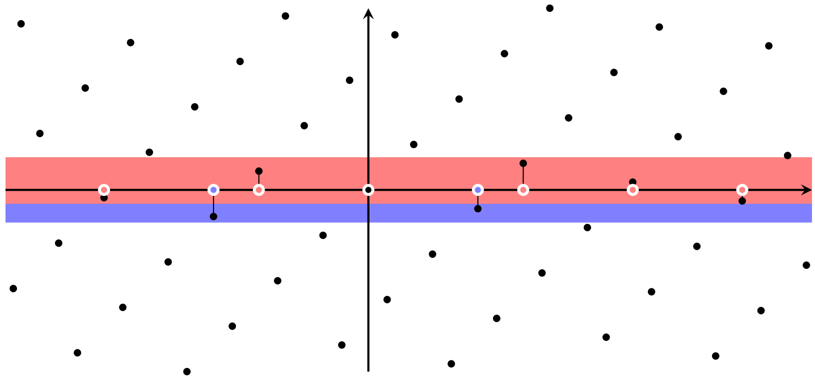

Figure 5.4. Cut and project setting for the substitution $\zeta_{2,1}$. The strip $\mathbb{R} \times W_{2,1}^{(a)}$ is indicated in red, and $\mathbb{R} \times W_{2,1}^{(b)}$ in blue.

distinguished according to the legal two-letter seeds.

Proof. We only prove the first part of Eq. (5.11) as the other identities follow similarly. In this case, it is easy to check that $1 \notin \Lambda_{m,0}^{(a|a)}$ whereas the lattice point $(1,1) \in \mathcal{L}_m$ lies on the boundary of the strip $\mathbb{R} \times [\lambda'_m, 1]$. By Proposition 5.13, this implies

$$\Lambda_{m,0}^{(a|a)} \subset \Theta\big([\lambda'_m, 1)\big).$$

The other inclusion is again implied by Lemma 5.15 in the same way as in the proof of Corollary 5.17. ■

Corollary 5.19. *For an arbitrary but fixed $m \in \mathbb{N}$ and all $0 \leqslant i \leqslant m$, the noble means sets $\Lambda_{m,i}$ are regular model sets.*

Proof. The windows $W_{m,i}$ were computed in Corollary 5.17 and Corollary 5.18 and as the Haar measure on $\mathbb{R}$ is the Lebesgue measure, it is immediate that $\mathrm{vol}(W_{m,i}) > 0$ and $\mathrm{vol}(\partial W_{m,i}) = 0$. Applying Lemma 5.14 proves the assertion. ■

The cut and project setting for the noble means sets is illustrated in Figure 5.4 in the case of $\zeta_{2,1}$.

Corollary 5.20. *For an arbitrary but fixed $m \in \mathbb{N}$ and all $0 \leqslant i \leqslant m$, the noble means sets $\Lambda_{m,i}$ are Meyer.*

Proof. Each $\Lambda_{m,i}$ is relatively dense in $\mathbb{R}$ with covering radius $\lambda_m/2$. The Meyer property is therefore implied by Theorem 5.11. ■

Let us now focus on the description of generating random noble means sets as subsets of model sets. The idea is to use Proposition 5.13 and the thereby implied knowledge of the relative positions of all $W_{m,i}$ to each other to derive an interval W_m with $\Lambda_m \subset \Theta(W_m)$. The special case of $m = 1$ was treated in [Lüt10, Prop. 6].

Proposition 5.21. *Let Λ_m be a generating random noble means set. Then, $\Lambda_m \subset \Theta(W_m)$ with $W_m := [\lambda'_m - 1, 1 - \lambda'_m]$.*

Proof. Assume there is a set $W_m = A \cup B$ in the internal space with the property $\Lambda_m \subset \Theta(W_m) = \Theta(A) \cup \Theta(B)$. Here, the sets $\Theta(A)$ and $\Theta(B)$ denote the left endpoints of intervals generated by the letters a and b, respectively. If Λ_m is a generating random noble means set, the same is true for $\zeta_m(\Lambda_m)$, and the sought-after sets $\Theta(A)$ and $\Theta(B)$ are invariant under ζ_m. Now, consider $x \in \Lambda_m$ and note that the interval $[0, x]$ is always mapped to the interval $\lambda_m \cdot [0, x]$. The sets $\Theta(A)$ and $\Theta(B)$ are consequently invariant under ζ_m if and only if for all $0 \leqslant i \leqslant m$ the inclusions

$$\zeta_{m,i}(\Theta(A)) \subset \Theta(A) \quad \text{and} \quad \zeta_{m,i}(\Theta(B)) \subset \Theta(B)$$

hold. As conditions in the physical space, we get for $0 \leqslant i \leqslant m$ the $m+1$ systems

$$\begin{aligned} &\Theta(A) \supset \left\{ \bigcup_{j=0}^{i-1} \lambda_m \Theta(A) + j\lambda_m \right\} \cup \lambda_m \Theta(B) \cup \left\{ \bigcup_{j=i}^{m-1} \lambda_m \Theta(A) + j\lambda_m + 1 \right\} \\ &\Theta(B) \supset \lambda_m \Theta(A) + i\lambda_m \end{aligned}$$

and in the internal space the corresponding conjugate systems

$$\begin{aligned} &A \supset \left\{ \bigcup_{j=0}^{i-1} \lambda'_m A + j\lambda'_m \right\} \cup \lambda'_m B \cup \left\{ \bigcup_{j=i}^{m-1} \lambda'_m A + j\lambda'_m + 1 \right\} \\ &B \supset \lambda'_m A + i\lambda'_m. \end{aligned} \tag{5.13}$$

As only affine maps appear in Eq. (5.13), it suffices to investigate the extremal cases $i = 0$ and $i = m$. Furthermore, we can assume that A and B are closed intervals, because if $C \in \{A, B\}$ satisfies all conditions of Eq. (5.13) and is no interval, then define $\overline{C} = [\inf C, \sup C]$. As all involved maps are affine, $\overline{C}$ also meets these conditions and we may define $A := [\alpha, \beta]$ and $B := [\gamma, \delta]$. Among the remaining conditions of Eq. (5.13), only the following six are not redundant:

$$\begin{array}{lll} (1)\ \lambda'_m(\beta + (m-1)) \geqslant \alpha & (2)\ \lambda'_m \delta \geqslant \alpha & (3)\ \lambda'_m \gamma \leqslant \beta \\ (4)\ \lambda'_m(\beta + m) \geqslant \gamma & (5)\ \lambda'_m \alpha + 1 \leqslant \beta & (6)\ \lambda'_m \alpha \leqslant \delta. \end{array}$$

Because of Proposition 5.13, we may assert the relative position $\gamma < \alpha \leqslant \delta < \beta$ of A to B. This appears to be a linear optimisation problem, which is not uniquely solvable in general. Consequently, we additionally demand that the interval $W_m = [\gamma, \beta]$ be minimal, which leads to the condition $\lambda'_m(\beta + m) = \gamma$. This equation describes the largest translation to the left and if $\lambda'_m(\beta + m) > \gamma$, the length of W_m was not minimal. By solving the linear optimisation problem of Eq. (5.13) under consideration of all given boundary conditions, we get the intervals

$$A = [-1, 1 - \lambda'_m], \quad B = [\lambda'_m - 1, -\lambda'_m] \quad \text{and} \quad W_m = [\lambda'_m - 1, 1 - \lambda'_m].$$

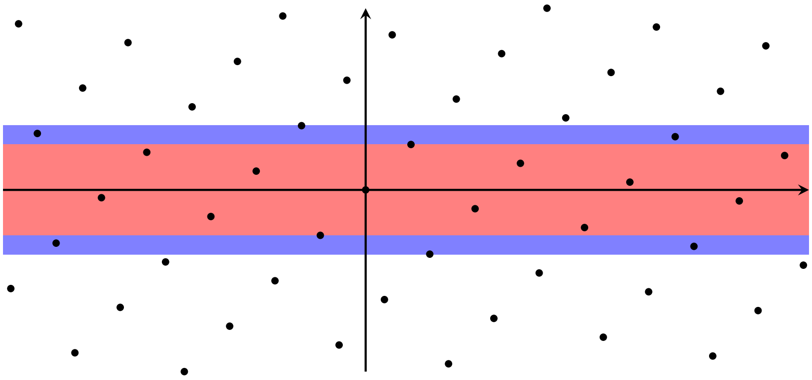

Figure 5.5. The strip $\mathbb{R} \times \bigcup_{i=0}^{m} W_{m,i}$ (red) is strictly contained in the strip $\mathbb{R} \times W_m$ (blue). Here indicated for the case $m = 2$.

These intervals actually satisfy Eq. (5.13), because for $i = m$ we get

$$\begin{aligned}\left\{\bigcup_{j=0}^{m-1} \lambda'_m A + j\lambda'_m\right\} \cup \lambda'_m B &= [-(\lambda'_m)^2 + m\lambda'_m, -\lambda'_m] \\ &\qquad \cup [-1 - m\lambda'_m, 1 + (m-1)\lambda'_m] \\ &= [-1, -\lambda'_m] \cup [-1 - m\lambda'_m, 1 + (m-1)\lambda'_m] \\ &\subset [-1, 1 - \lambda'_m] = A\end{aligned}$$

and

$$\begin{aligned}\lambda'_m A + m\lambda'_m &= [-(\lambda'_m)^2 + (m+1)\lambda'_m, (m-1)\lambda'_m] \\ &= [\lambda'_m - 1, (m-1)\lambda'_m] \\ &\subset [\lambda'_m - 1, -\lambda'_m] = B.\end{aligned}$$

Analogously, we get the corresponding inclusions for $i = 0$. Furthermore, the minimality condition of W_m is fulfilled because

$$\lambda'_m(\beta + m) = \lambda'_m(1 - \lambda'_m + m) = \lambda'_m - 1 = \gamma. \qquad \blacksquare$$

Note that $\bigcup_{i=0}^{m} W_{m,i}$ is a strict subset of W_m, for any $m \in \mathbb{N}$. The situation in the case of $m = 2$ is illustrated in Figure 5.5.

Definition 5.22 (Continuous hull)**.** For an arbitrary but fixed $m \in \mathbb{N}$ and $0 \leqslant i \leqslant m$, we define

$$\mathbb{Y}_{m,i} := \overline{\{t + \Lambda_{m,i} \mid t \in \mathbb{R}\}} \tag{5.14}$$

as the *continuous hull* of the inflation rule $\zeta_{m,i}$.

Note that the closure in Eq. (5.14) is taken with respect to the local topology. Here, two FLC point sets M and N are close if, after a small translation, they agree on a large interval. That is, if

$$M \cap (-1/\varepsilon, 1/\varepsilon) = (-t + N) \cap (-1/\varepsilon, 1/\varepsilon)$$

for some $t \in (-\varepsilon, \varepsilon)$. Accordingly, we get

Definition 5.23 (Continuous stochastic hull)**.** For any $m \in \mathbb{N}$, let

$$Y_m := \Big\{\Lambda_m \mid \Lambda_m \text{ is the geometric realisation of} \\ \text{an accumulation point of } \big(\zeta_m^k(a \,|\, a)\big)_{k\in\mathbb{N}}\Big\}$$

and define the *continuous stochastic hull* $\mathbb{Y}_m$ of the inflation rule ζ_m as the smallest closed and translation-invariant subset of $\mathfrak{D}(\mathbb{R})$ (the set of Delone subsets of $\mathbb{R}$) with $Y_m \subset \mathbb{Y}_m$. The elements of Y_m are called *generating random noble means sets.*

Remark 5.24 (Punctured continuous stochastic hull)**.** Evidently, there is a one-to-one correspondence between all elements of $\mathbb{X}_m$ and those elements of $\mathbb{Y}_m$ that contain $0 \in \mathbb{R}$. The subset

$$\mathbb{Y}_m^{\circledcirc} := \{\Lambda \in \mathbb{Y}_m \mid 0 \in \Lambda\} \subset \mathbb{Y}_m.$$

is called the *punctured continuous stochastic hull.* The elements of $\mathbb{Y}_m^{\circledcirc}$ are precisely the geometric realisations of random noble means words which were considered in Section 5.2. ♦

Equipped with the necessary terminology, we can formulate the following consequence of Proposition 5.21.

Theorem 5.25. *Each random noble means set* $\Lambda \in \mathbb{Y}_m$ *is Meyer.*

Proof. Let Λ_m be a generating random noble means set. Evidently, Λ_m is relatively dense in $\mathbb{R}$ with covering radius $\lambda_m/2$ and, by Proposition 5.21, it is a subset of the model set $\Theta\big([\lambda_m' - 1, 1 - \lambda_m']\big)$. The Meyer property of Λ_m then follows from Theorem 5.11. We know that there is a generating random noble means set whose orbit is dense, Λ_m say. Now, choose an arbitrary random noble means set $\Lambda \in \mathbb{Y}_m$ and a converging sequence $(t_n + \Lambda_m)_{n\in\mathbb{N}}$ with limit Λ. For any $n \in \mathbb{N}$, we find

$$(t_n + \Lambda_m) - (t_n + \Lambda_m) = \Lambda_m - \Lambda_m$$

and therefore $\Lambda - \Lambda \subset \Lambda_m - \Lambda_m$ which means that Λ is uniformly discrete. As the relative denseness of Λ is clear, this proves the assertion. ∎

In general, the Meyer property of a discrete point set is an interesting and desirable structure. For our studies of diffraction in Chapter 6, the Meyer property of all $\Lambda \in \mathbb{Y}_m$ is particularly important because a central result of Strungaru [Str05, Prop 3.12] ensures the presence of an extended pure point component in the diffraction measure of Λ.

CHAPTER 6

Diffraction of the RNMS

In this final chapter, we derive the diffraction measure of some typical element of the continuous RNMS hull. The construction of the ergodic measure μ_m on $\mathbb{X}_m$ provides a convenient basis in this regard because its suspension to a measure ν_m on $\mathbb{Y}_m$ is then also ergodic. The construction of ν_m is the purpose of Section 6.1.1. Ensuing this, we briefly introduce the most important terminology and concepts for the treatment of diffraction of discrete point sets in Section 6.1.2. Recall that the basic measure-theoretic ideas have been provided in Section 1.4.1. For the sake of completeness, we briefly deduce the diffraction measure of any noble means set from general results covering the diffraction of regular model sets in Section 6.2.1 and derive the RNMS cases afterwards in Section 6.2.2. The idea for the there applied recursion-based approach with a suitable split into first and second moments of some complex-valued random variable was pioneered by Godrèche and Luck [GL89, Sec. 5]. We will back up their construction with as much mathematical rigour as possible although some questions have to remain open. Last but not least, we present an inflation-invariant approach to the pure point part of the diffraction measure of the RNMS cases in Section 6.3.

6.1. Basic tools of RNMS diffraction

6.1.1. Suspension of μ_m. From the symbolic dynamics point of view, we so far have investigated the discrete dynamical system $(\mathbb{X}_m, \mathbb{Z}, \mu_m)$, with μ_m being the ergodic measure defined by the frequency of ζ_m-legal words that was constructed in Section 4.2.1. Now, we are interested in an induced continuous dynamical system, for which we can benefit from the ergodicity of μ_m. For the construction of the so-called *special flow*, we follow the exhibition in [CFS82, Ch. 11].

To begin with, consider a μ_m-integrable function $f\colon \mathbb{X}_m \longrightarrow \mathbb{R}_+$ and the space $M_f \subset \mathbb{X}_m \times \mathbb{R}$, defined by

$$M_f := \{(x, y) \mid x \in \mathbb{X}_m, 0 \leqslant y < f(x)\}.$$

We may turn M_f into a measure space itself. To this end, we equip $\mathbb{R}$ with the Borel σ-algebra $\mathfrak{B}_{\mathbb{R}}$ generated by the open intervals (a, b) with a, $b \in \mathbb{Q}$ and provide M_f with the σ-algebra $\mathfrak{B}_f := (\mathfrak{B}_m \otimes \mathfrak{B}_{\mathbb{R}}) \cap M_f$; see [Par05, Prop. 2.1.5] for background in this regard. The basic rectangles R in M_f are of the form $R := B \times (a, b)$ with $B \in \mathfrak{B}_m$ and $a \leqslant \inf\{f(x) \mid x \in B\}$. We define the *suspended* *measure* ν_m^* as the restriction of $\mu_m \otimes \lambda$ to M_f where λ is the Lebesgue measure on $\mathbb{R}$ and the function f is defined by

$$f(x) := \begin{cases} \lambda_m, & \text{if } x_0 = a, \\ 1, & \text{if } x_0 = b. \end{cases} \tag{6.1}$$

This is consistent with the concept of geometric realisations of random noble means words as introduced in Section 5.2. The two values of f match the lengths of the intervals that were chosen according to the normalisation of the left PF eigenvector of M_m specified in Eq. (5.4) on page 58.

Remark 6.1 (Choice of f for the suspension)**.** Note that one often defines f such that $\int f \,\mathrm{d}\mu_m = 1$ in order to immediately make the suspended measure a probability measure. This can be accomplished via the function

$$f(x) := \begin{cases} \lambda_m \operatorname{dens}(\Lambda_m), & \text{if } x_0 = a, \\ \operatorname{dens}(\Lambda_m), & \text{if } x_0 = b. \end{cases}$$

As this definition does not reflect the correct interval lengths of geometric realisations in relation to the Minkowski embedding of Eq. (5.7) and Remark 5.7, we stick to the choice of Eq. (6.1). ♦

Recall that $\mathcal{Z}_0(a)$ and $\mathcal{Z}_0(b)$ are the cylinder sets of the letters a and b at the index 0 in the sense of Remark 1.11 and Eq. (1.1) on page 3. Now, the measure ν_m^* of some basic rectangle R is given by $\nu_m^*(R) = \mu_m(B)(b-a)$ and we find

$$\begin{aligned} \nu_m^*(M_f) &= \int_{\mathbb{X}_m} f \,\mathrm{d}\mu_m = \mu_m(\mathcal{Z}_0(a))\lambda_m + \mu_m(\mathcal{Z}_0(b)) \\ &= \frac{\lambda_m^2}{\lambda_m + 1} + \frac{1}{\lambda_m + 1} = \frac{1}{\operatorname{dens}(\Lambda_m)}, \end{aligned}$$

where $\operatorname{dens}(\Lambda_m) = (1 - \lambda_m')/\sqrt{m^2+4}$, due to Remark 5.9. In order to turn ν_m^* into a probability measure, we set $\nu_m := \operatorname{dens}(\Lambda_m)\nu_m^*$. Furthermore, we define the *special flow* $\{S_t\}$ on the measure space $(M_f, \mathfrak{B}_f, \nu_m)$ by

$$S_t(x,y) := \Big(S^n x, y + t - \sum_{k=0}^{n-1} f(S^k x)\Big)$$

if $t \geqslant 0$. Here, $n \in \mathbb{N}$ is uniquely determined by

$$\sum_{k=0}^{n-1} f(S^k x) \leqslant y + t < \sum_{k=0}^{n} f(S^k x).$$

Moreover, we define

$$S_t(x,y) := \begin{cases} (x, y+t), & \text{if } t < 0 \text{ and } y + t \geqslant 0, \\ \Big(S^{-n} x, y + t + \sum_{k=-n}^{-1} f(S^k x)\Big), & \text{if } t < 0 \text{ and } y + t < 0, \end{cases}$$

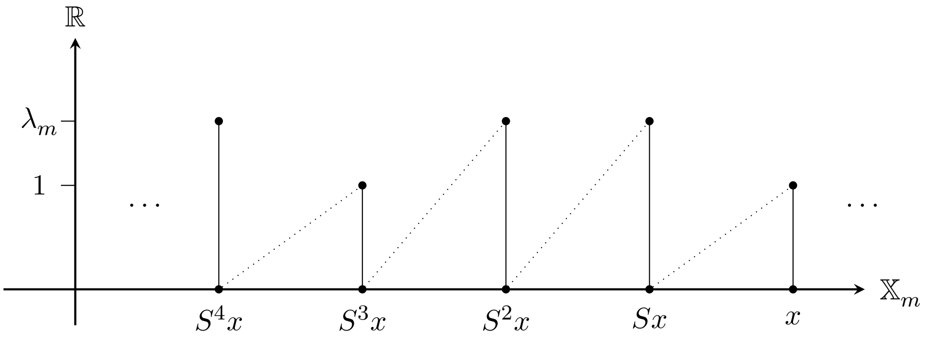

Figure 6.1. Illustration of the action of the special flow in the space $M_f \subset \mathbb{X}_m \times \mathbb{R}$.

where $n \in \mathbb{N}$ is uniquely defined by

$$-\sum_{k=-n}^{-1} f(S^k x) \leqslant y + t < -\sum_{k=-n+1}^{-1} f(S^k x).$$

Finally, we identify $(x, f(x)) \sim (Sx, 0)$ and refer to Figure 6.1 for the interpretation of this construction. Starting at any point $k_0 = (x, 0) \in M_f$, one climbs up the fibre over k_0 until $(x, f(x))$ is reached. This point is identified with $k_1 = (Sx, 0)$ from where the route continues up the fibre over k_1 and so forth. According to [CFS82, Lem. 11.1.1], the measure ν_m is $\{S_t\}$-invariant which means $\nu_m(S_t B) = \nu_m(B)$ for any $t \in \mathbb{R}$ and any $B \in \mathfrak{B}_f$. Following [EW11, Lem. 9.24], the measure ν_m is ergodic if and only if μ_m is ergodic which we have already accomplished in Theorem 4.22.

The preceding discussion reveals the fact that $M_f \cong \mathbb{Y}_m$ and we finally formulate our findings as follows.

Lemma 6.2. *The special flow $\{S_t\}$ is a transformation of the probability space $(\mathbb{Y}_m, \mathfrak{B}_f, \nu_m)$, and the measure ν_m is $\{S_t\}$-invariant and ergodic.* ∎

6.1.2. Essentials and notation for diffraction of discrete point sets. For the computation of the diffraction measure of discrete point sets $P \subset \mathbb{R}^d$, we consider the attributed *Dirac combs*

$$\delta_P := \sum_{x \in P} \delta_x \quad \text{and} \quad \widetilde{\delta_P} := \sum_{y \in P} \widetilde{\delta_y} = \sum_{y \in P} \delta_{-y},$$

and study the properties of the family $\{\gamma_P^{(n)} \mid n > 0\}$ with

$$\gamma_P^{(n)} := \gamma_{\delta_P}^{(n)} := \frac{\delta_{P_n} * \widetilde{\delta_{P_n}}}{\mathrm{vol}(B_n)}, \tag{6.2}$$

where $P_n := P \cap B_n(0)$, as $n \to \infty$. By restricting the intricacy of discrete point sets, one finds the following convenient property of Dirac combs.

Remark 6.3 (Dirac combs of FLC sets)**.** Let $P \subset \mathbb{R}^d$ be an FLC set and consider a weight function $w\colon P \longrightarrow \mathbb{C}$ with $S := \sup\{|w(x)| \mid x \in P\} < \infty$. Then, the Dirac comb

$$\delta_P := \sum_{x \in P} w(x)\delta_x$$

is a translation bounded measure because for any compact $K \subset \mathbb{R}^d$ we get

$$|\delta_P|(y + K) \leqslant S \sum_{x \in P} \delta_x(y + K) \leqslant SN,$$

where $N := \sup\{|P \cap (y + K)| \mid y \in \mathbb{R}^d\}$. As an FLC set in particular is locally finite [BG13, Prop. 2.1], we find $SN < \infty$. ♦

With the obvious modifications, we can consider Eq. (6.2) for an arbitrary translation bounded measure μ. In this more general situation, one knows the following result that has already been observed by Hof [Hof95, Prop. 2.2].

Proposition 6.4 ([BG13, Prop. 9.1])**.** *If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is a translation bounded measure, the family $\{\gamma_\mu^{(n)} \mid n > 0\}$ is precompact in the vague topology. Any accumulation point of this family, of which there is at least one, is translation bounded.* ■

From Remark 6.3 and Proposition 6.4, we conclude that there is at least one accumulation point γ_P of the family $\{\gamma_P^{(n)} \mid n > 0\}$, provided P is an FLC set. We refer to each accumulation point of this family as an *autocorrelation measure* of δ_P or just an *autocorrelation* for short. The approach to diffraction theory via autocorrelation measures goes back to Hof [Hof95] and a detailed exposition can be found in [BG13, Ch. 9].

Corollary 6.5. *For any $\Lambda \in \mathbb{Y}_m$, there is at least one autocorrelation $\gamma_{\Lambda,m}$.*

Proof. Theorem 5.25 gives the Meyer property of Λ. The existence of $\gamma_{\Lambda,m}$ is then a special case of the preceding discussion. ■

Proposition 6.6 ([BF75, Sec. 4])**.** *If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is positive definite, its Fourier transform exists and is a positive and translation bounded measure.* ■

As $\gamma_{\Lambda,m}$ is positive definite by construction for any $\Lambda \in \mathbb{Y}_m$, we get the existence of the Fourier transform $\widehat{\gamma_{\Lambda,m}}$ from Proposition 6.6 and we call $\widehat{\gamma_{\Lambda,m}}$ the *diffraction measure* of Λ. In this context, we refer to [BG13, Ch. 9] for an exposition of theoretical background and numerous examples. Our aim for the rest of this chapter will be the study of $\widehat{\gamma_{\Lambda,m}}$ with a view towards its Lebesgue decomposition that was discussed around Theorem 1.15.

6.2. Diffraction of RNMS sets

6.2.1. The NMS cases. Before we embark on the diffraction of typical RNMS sets, we briefly treat the NMS cases. Concerning regular model sets, of which each noble means set $\Lambda_{m,i} = \Theta(W_{m,i})$ is a particular instance, there is a good understanding both of the autocorrelation and the diffraction measure. For the sake of completeness, we cite the two decisive results in this context. Recall that $\star$ denotes the star map that was introduced in Eq. (5.6) on page 59.

Theorem 6.7 ([BG13, Prop. 9.8]). *Consider the general cut and project scheme of Figure 5.2 and let $P = \Theta(W)$ be a regular model set for it, with compact window $W = \overline{W^{\circ}}$. The autocorrelation measure γ_P of P exists and is a positive and positive definite, translation bounded, pure point measure. It is given by*

$$\gamma_P = \sum_{z \in P-P} \eta(z)\delta_z, \tag{6.3}$$

with the autocorrelation coefficients

$$\begin{aligned}\eta(z) &= \mathrm{dens}(P)\frac{\mu_H(W \cap (z^\star + W))}{\mu_H(W)} \\ &= \frac{\mathrm{dens}(P)}{\mu_H(W)} \int_H \mathbb{1}_W(y)\mathbb{1}_{z^\star + W}(y)\,\mathrm{d}\mu_H(y),\end{aligned}$$

where μ_H is the Haar measure on H. In particular, one has $\eta(0) = \mathrm{dens}(P)$. ■

Theorem 6.8 ([BG13, Thm. 9.4]). *Let $P = \Theta(W)$ be a regular model set for the general cut and project scheme of Figure 5.2 with compact window $W = \overline{W^{\circ}}$ and autocorrelation γ_P as in Eq. (6.3). The diffraction measure $\widehat{\gamma_P}$ is a positive and positive definite, translation bounded, pure point measure. It is given by*

$$\widehat{\gamma_P} = \sum_{k \in \mathcal{L}^{\circledast}} I(k)\delta_k, \tag{6.4}$$

where the diffraction intensities are $I(k) = |A(k)|^2$ with the amplitudes

$$A(k) = \frac{\mathrm{dens}(P)}{\mu_H(W)} \widehat{\mathbb{1}_W}(-k^\star) = \frac{\mathrm{dens}(P)}{\mu_H(W)} \int_H \langle k^\star | y \rangle \,\mathrm{d}\mu_H(y). \qquad ■$$

Remark 6.9 (Fourier module). Here, the *Fourier module* $\mathcal{L}^{\circledast}$ is given by the projection of the dual lattice $\mathcal{L}^*$ to the physical space, that is $\mathcal{L}^{\circledast} := \pi_1(\mathcal{L}^*)$. In the NMS cases, $\mathcal{L}^{\circledast}_m$ is given by

$$\mathcal{L}^{\circledast}_m := \pi_1(\mathcal{L}^*_m) = \mathbb{Z}[\lambda_m]/\sqrt{m^2+4}, \tag{6.5}$$

where $\mathcal{L}^*_m$ is the dual lattice of Eq. (5.8). Simultaneously, this is the pure point part of the dynamical spectrum of $(\mathbb{Y}_m, \{S_t\})$; see [BG13, App. B], [BLvE13] and [Que10, Ch. 3] for background. ♦

Corollary 6.10. *For an arbitrary but fixed $m \in \mathbb{N}$ and $0 \leqslant i \leqslant m$, the diffraction measure of $\Lambda_{m,i}$ is a positive and positive definite, translation bounded, pure point measure. It is explicitly given by*

$$\widehat{\gamma_{\Lambda_{m,i}}} = \sum_{k \in \mathcal{L}^{\circledast}_m} |A_{m,i}(k)|^2\,\delta_k, \tag{6.6}$$

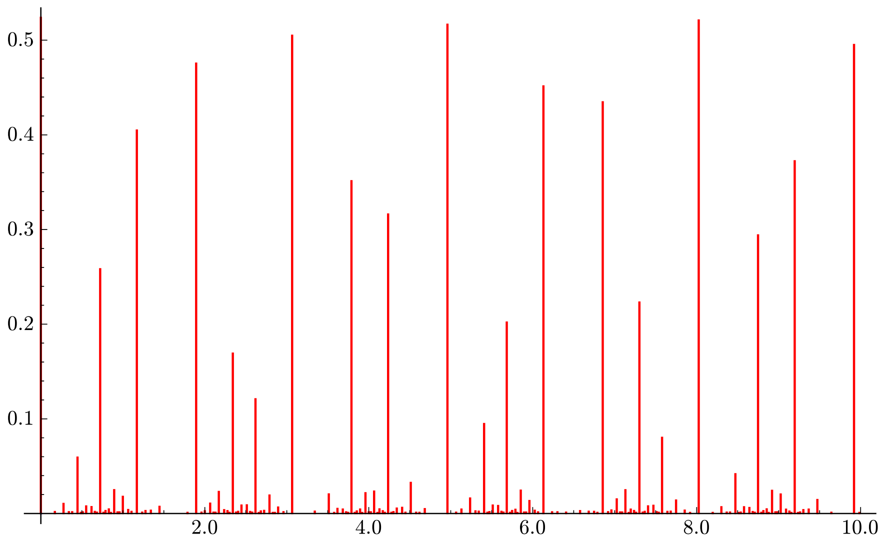

Figure 6.2. Diffraction measure of any $\zeta_{1,i}$. All Bragg peaks at wave number k, with $0 \leqslant k \leqslant 10$, of height $I(k) = |A(k)|^2 \geqslant \mathrm{dens}(\Lambda_{1,i})/1000$ are shown.

with the amplitudes

$$A_{m,i}(k) = \mathrm{dens}(\Lambda_{m,i})\, \mathrm{e}^{-\pi \mathrm{i} k^\star (\lambda'_m + 1)(1-2i/m)}\, \mathrm{sinc}\big(\pi k^\star (1 - \lambda'_m)\big).$$

Proof. To begin with, we note that the Fourier transform of the characteristic function of an interval $[a,b] \subset \mathbb{R}$ can be represented as

$$\widehat{\mathbb{1}_{[a,b]}}(x) = (b-a)\, \mathrm{e}^{-\pi \mathrm{i} x(a+b)}\, \mathrm{sinc}\big(\pi x(b-a)\big), \tag{6.7}$$

where $\mathrm{sinc}(z) \coloneqq \sin(z)/z$. Combining Theorem 6.8 with Proposition 5.13 and Lemma 5.14, we find

$$\begin{aligned} A_{m,i}(k) &= \frac{\mathrm{dens}(\Lambda_{m,i})}{\mathrm{vol}(W_{m,i})} \widehat{\mathbb{1}_{W_{m,i}}}(-k^\star) \\ &= \frac{(1-\lambda'_m)\, \mathrm{e}^{-\pi \mathrm{i} k^\star (\lambda'_m + 1)(1-2i/m)}\, \mathrm{sinc}\big(\pi k^\star (1-\lambda'_m)\big)}{\sqrt{m^2+4}} \\ &= \mathrm{dens}(\Lambda_{m,i})\, \mathrm{e}^{-\pi \mathrm{i} k^\star (\lambda'_m + 1)(1-2i/m)}\, \mathrm{sinc}\big(\pi k^\star (1-\lambda'_m)\big), \end{aligned}$$

by an application of Eq. (6.7). ∎

Remark 6.11 (Role of the amplitudes)**.** Note that $\widehat{\gamma_{\Lambda_{m,i}}}$ in fact does not depend on i because of the absolute square in Eq. (6.6). An illustration of the diffraction measure in the golden mean cases is shown in Figure 6.2. Furthermore, there is an important connection between the amplitudes and the corresponding exponential sums in the following sense. For a fixed $k \in \mathcal{L}_m^{\circledast}$, one can show (the proof of [BG13, Lemma 9.4] for the silver mean case with $i = 1$ can be extended to arbitrary $m \in \mathbb{N}$ and $0 \leqslant i \leqslant m$) that

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \sum_{x\in\Lambda_{m,i,n}} \mathrm{e}^{-2\pi\mathrm{i}kx} = A_{m,i}(k),$$

where $\Lambda_{m,i,n} = \Lambda_{m,i} \cap B_n(0)$. In general, it is a difficult problem to control the growth behaviour of such sums and we will encounter precisely such an instance shortly in the study of the RNMS cases. ♦

6.2.2. The RNMS cases. Also in the stochastic situation, the explicit shape of the autocorrelation measure in Eq. (6.3) can be achieved as follows. We consider the Dirac comb

$$\delta_\Lambda \coloneqq \sum_{x\in\Lambda} \delta_x \quad \text{with} \quad \delta_{\Lambda_n} \coloneqq \sum_{x\in\Lambda_n} \delta_x \quad \text{and} \quad \Lambda_n \coloneqq \Lambda \cap B_n(0),$$

for any random noble means set $\Lambda \in \mathbb{Y}_m$. The approximating autocorrelations in the sense of Proposition 6.4 are denoted as $\gamma_{\Lambda,m}^{(n)}$ and any accumulation point as $\gamma_{\Lambda,m}$. Then, we find

$$\begin{aligned}
\lim_{n\to\infty} \gamma_{\Lambda,m}^{(n)} &= \lim_{n\to\infty} \frac{\delta_{\Lambda_n} * \widetilde{\delta_{\Lambda_n}}}{\mathrm{vol}(B_n)} = \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \Big(\sum_{x\in\Lambda_n} \sum_{y\in\Lambda_n} \delta_{x-y} \Big) \\
&= \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \sum_{z\in\Lambda_n-\Lambda_n} \Big(\sum_{\substack{x,y\in\Lambda_n \\ x-y=z}} 1 \Big) \delta_z \\
&= \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \sum_{z\in\Lambda_n-\Lambda_n} \Big(\sum_{x,x-z\in\Lambda_n} 1 \Big) \delta_z \\
&= \sum_{z\in\Lambda-\Lambda} \Big(\lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \sum_{x,x-z\in\Lambda_n} 1 \Big) \delta_z .
\end{aligned}$$

Now, a closer study of the *autocorrelation coefficients*

$$\eta(z) \coloneqq \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \sum_{x,x-z\in\Lambda_n} 1 = \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} |\Lambda_n \cap (z+\Lambda_n)|$$

is needed. As $\eta(z)$ has the form of an orbit average, it is suggestive to take advantage of the ergodicity of the suspended measure ν_m, derived in the discussion preceding Lemma 6.2. To this end, we proceed with a *regularisation* of $\gamma_{\Lambda,m}^{(n)}$ and introduce some additional notation. In the following, we always assume that $t \in \mathbb{R}$, $\varphi \in \mathcal{C}_\mathsf{c}(\mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R})$. Then, the translation of φ by t is denoted by $\tau_t\varphi(x) \coloneqq \varphi(x-t)$ and similarly $\tau_t\mu(\varphi) \coloneqq \mu(\tau_{-t}\varphi)$. Furthermore, we define the convolution of φ with μ as $(\varphi * \mu)(t) \coloneqq \mu(\tau_t\widetilde{\varphi}) \in \mathcal{C}(\mathbb{R})$ where $\widetilde{\varphi}(x) \coloneqq \overline{\varphi(-x)}$.

In the following, the idea is to consider the convolution of a function $\varphi \in \mathcal{C}_\mathsf{c}(\mathbb{R})$ with δ_Λ where the support of φ lies inside an interval of length $< 2r_\mathsf{p}(\Lambda)$, where $r_\mathsf{p}(\Lambda) = 1/2$ is the packing radius of Λ. The resulting object may be interpreted as a continuous function or as a regular and translation bounded measure; refer to [BLvE13, Sec. 4] or [Schl00] for background information. The ergodicity of ν_m on $\mathbb{Y}_m$, combined with the Ergodic Theorem 6.12 for continuous functions, then yields the decisive advantage of the regularisation process. Finally, a compression of the support of φ gives us the sought-after shape of the autocorrelation. One conceivable choice for the regularisation function is the following. For any $\varepsilon < r_{\mathsf{p}}(\Lambda)$, we define

$$\varphi_\varepsilon \colon \mathbb{R} \longrightarrow \mathbb{R}, \quad \varphi_\varepsilon(t) := \begin{cases} 1 - \frac{|t|}{\varepsilon}, & \text{if } t \in (-\varepsilon, \varepsilon), \\ 0, & \text{otherwise}, \end{cases} \tag{6.8}$$

which is a real-valued map in $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$. Moreover, we set

$$\begin{gathered} F_\varepsilon \colon \mathbb{R} \times \mathbb{Y}_m \longrightarrow \mathbb{R}, \\ F_\varepsilon(t, \Lambda) := (\varphi_\varepsilon * \delta_\Lambda)(t) = \delta_\Lambda\big(\tau_t \widetilde{\varphi_\varepsilon}\big) = \sum_{s \in \Lambda} \varphi_\varepsilon(t - s), \end{gathered}$$

and will make use of the induced function $G_\varepsilon \colon \mathbb{Y}_m \longrightarrow \mathbb{R}$ that is defined by $G_\varepsilon(\Lambda) := (\varphi_\varepsilon * \delta_\Lambda)(0)$. Here, one easily computes that $F_\varepsilon(t, \Lambda) = G_\varepsilon(\Lambda - t)$. The following result may be considered as a continuous analogue of Birkhoff's Ergodic theorem on page 47.

Theorem 6.12 ([MR13, Thm. 2.14])**.** *Let X be any compact metrisable space with the translation action T of $\mathbb{R}^d$ on X and let $\mu \in \mathcal{P}_T(X)$. For any function $f \in L^1(X, \mu)$, the sequence*

$$\big(I_n(\Lambda, f)\big)_{n \in \mathbb{N}} := \Big(\frac{1}{\mathrm{vol}(B_n)} \int_{B_n} f(-t + \Lambda) \, \mathrm{d}t\Big)_{n \in \mathbb{N}}$$

is finite for μ-almost all $\Lambda \in X$ and all $n \in \mathbb{N}$. Furthermore, there is a T-invariant function $F \in L^1(X, \mu)$ such that $\mu(F) = \mu(f)$ and

$$\lim_{n \to \infty} I_n(\Lambda, f) = F(\Lambda), \tag{6.9}$$

for μ-almost all $\Lambda \in X$. Moreover, the measure μ is ergodic if and only if for every $f \in L^1(X, \mu)$, Eq. (6.9) *holds with $F = \mu(f)$ for μ-almost all $\Lambda \in X$.* ■

The treatment of the autocorrelation can now proceed by an application of of Lemma 6.2 and Theorem 6.12. We consider the function

$$\gamma_{\Lambda,\varepsilon}(t) := (\varphi_\varepsilon * \widetilde{\varphi_\varepsilon}) * \big(\delta_\Lambda \circledast \widetilde{\delta_\Lambda}\big)(t) = \lim_{n \to \infty} \frac{(F_\varepsilon * \widetilde{F_\varepsilon})(t, \Lambda)}{\mathrm{vol}(B_n)}$$

and almost surely find

$$\begin{aligned} \gamma_{\Lambda,\varepsilon}(t) &= \lim_{n \to \infty} \frac{1}{\mathrm{vol}(B_n)} \int_{B_n} (\varphi_\varepsilon * \delta_\Lambda)(s + t) \overline{(\varphi_\varepsilon * \delta_\Lambda)(s)} \, \mathrm{d}s \\ &= \lim_{n \to \infty} \frac{1}{\mathrm{vol}(B_n)} \int_{B_n} G_\varepsilon(\Lambda - s - t) \overline{G_\varepsilon(\Lambda - s)} \, \mathrm{d}s \\ &= \int_{\mathbb{Y}_m} G_\varepsilon(\Lambda - t) \overline{G_\varepsilon(\Lambda)} \, \mathrm{d}\nu_m(\Lambda) = \mathbb{E}_{\nu_m}\big(G_\varepsilon(\Lambda - t) \overline{G_\varepsilon(\Lambda)}\big), \end{aligned} \tag{6.10}$$

where the third equality follows almost surely from an application of Theorem 6.12 to the function

$$\mathbb{Y}_m \longrightarrow \mathbb{R}, \quad \Lambda \longmapsto G_\varepsilon(\Lambda - t)\overline{G_\varepsilon(\Lambda)}$$

and with the ergodic measure ν_m on $\mathbb{Y}_m$. Obviously, due to our choice for the regularisation function, one could omit the complex conjugation in the above discussion, but in order to emphasise that, in a more general setting, one is not restricted to φ_ε, we leave it as such. Considering $\varepsilon \searrow 0$ in Eq. (6.10), we almost surely find

$$\gamma_{\Lambda,m} = \mathbb{E}_{\nu_m}(\delta_\Lambda \circledast \widetilde{\delta_\Lambda}).$$

For the computation of $\widehat{\gamma_{\Lambda,m}}$, we define the complex-valued random variable

$$X_n(k) := \sum_{x \in \Lambda_n} \mathrm{e}^{-2\pi \mathrm{i} kx} = \sum_{x \in \Lambda_n} \widehat{\delta_x},$$

with any $\Lambda \in \mathbb{Y}_m$ and $\Lambda_n = \Lambda \cap B_n(0)$. The autocorrelation $\gamma_{\Lambda,m}$ is positive definite by construction and its Fourier transform exists due to Proposition 6.6. We find

$$\begin{aligned}
\widehat{\gamma_{\Lambda,m}} &= \big(\mathbb{E}_{\nu_m}(\delta_\Lambda \circledast \widetilde{\delta_\Lambda})\big)\widehat{\ } = \lim_{n\to\infty} \mathbb{E}_{\nu_m}\Big(\frac{1}{\operatorname{vol}(B_n)} \widehat{\delta_{\Lambda_n}}\widehat{\widetilde{\delta_{\Lambda_n}}}\Big) \\
&= \lim_{n\to\infty} \frac{1}{\operatorname{vol}(B_n)} \mathbb{E}_{\nu_m}\Big(\widehat{\delta_{\Lambda_n}}\overline{\widehat{\delta_{\Lambda_n}}}\Big) = \lim_{n\to\infty} \frac{1}{\operatorname{vol}(B_n)} \mathbb{E}_{\nu_m}(|X_n|^2) \\
&= \lim_{n\to\infty} \frac{1}{\operatorname{vol}(B_n)} |\mathbb{E}_{\nu_m}(X_n)|^2 + \lim_{n\to\infty} \frac{1}{\operatorname{vol}(B_n)} \big(\mathbb{E}_{\nu_m}(|X_n|^2) - |\mathbb{E}_{\nu_m}(X_n)|^2\big) \\
&= \lim_{n\to\infty} \frac{1}{\operatorname{vol}(B_n)} |\mathbb{E}_{\nu_m}(X_n)|^2 + \lim_{n\to\infty} \frac{1}{\operatorname{vol}(B_n)} \mathbb{V}_{\nu_m}(X_n), \qquad (6.11)
\end{aligned}$$

where $\mathbb{V}_{\nu_m}(X_n)$ is the variance of X_n, provided that all limits exist. The idea of breaking up $\widehat{\gamma_{\Lambda,m}}$ according to first and second moments will on the one hand result in $\lim_{n\to\infty} |\mathbb{E}_{\nu_m}(X_n)|^2/\operatorname{vol}(B_n)$ containing the pure point part and on the other hand $\lim_{n\to\infty} \mathbb{V}_{\nu_m}(X_n)/\operatorname{vol}(B_n)$ being the absolutely continuous part of $\widehat{\gamma_{\Lambda,m}}$. In the following, we will restrict to the special case of $m = 1$ and consider suitable subsequences to ensure the convergence in Eq. (6.11). The general case of $m \in \mathbb{N}$ can be treated similarly.

6.2.3. The random Fibonacci case. The observations made in Section 3.2 and Theorem 5.25 identify each random noble means set as an instance of a Meyer set that almost surely has entropy; see [BG12, Sec. 6] for another example. Considering recent developments [BLR07] and taking the following result into account, we expect to find a diffraction spectrum of mixed type in the RNMS cases.

Theorem 6.13 ([Str05, Prop. 3.12])**.** *Let P be a Meyer set and suppose that its autocorrelation γ_P exists. Then, the set of Bragg peaks lies relatively dense.*

Moreover, if P is not pure point diffractive, it has a relatively dense support for the continuous spectrum as well. ■

Here, we restrict to $m = 1$ and consider realisations of the random variable X_n that correspond to exact random Fibonacci words. For $n \geqslant 2$, we define the sequence

$$L_n := L_{n-1} + L_{n-2} \quad \text{with} \quad L_0 := 1 \quad \text{and} \quad L_1 := \lambda_1$$

that possesses the closed form $L_n = \lambda_1^n$ for any $n \in \mathbb{N}$ and furthermore, we set

$$X_n(k) := \begin{cases} X_{n-2}(k) + \mathrm{e}^{-2\pi \mathrm{i} k L_{n-2}}\, X_{n-1}(k), & \text{with probability } p_0, \\ X_{n-1}(k) + \mathrm{e}^{-2\pi \mathrm{i} k L_{n-1}}\, X_{n-2}(k), & \text{with probability } p_1, \end{cases} \tag{6.12}$$

where $X_0(k) := \mathrm{e}^{-2\pi \mathrm{i} k}$ and $X_1(k) := \mathrm{e}^{-2\pi \mathrm{i} k \lambda_1}$. Moreover, we define the sequences

$$(\mathcal{P}_n)_{n\in\mathbb{N}_0} := \Big(\frac{1}{L_n} |\mathbb{E}(X_n)|^2\Big)_{n\in\mathbb{N}_0} \quad \text{and} \quad (\mathcal{S}_n)_{n\in\mathbb{N}_0} := \Big(\frac{1}{L_n} \mathbb{V}(X_n)\Big)_{n\in\mathbb{N}_0}, \tag{6.13}$$

and derive results on the convergence of $(\mathcal{P}_n)_{n\in\mathbb{N}_0}$ and $(\mathcal{S}_n)_{n\in\mathbb{N}_0}$.

Remark 6.14 (Substitution vs. concatenation)**.** As the definition of the random variable X_n in Eq. (6.12) is based on the concatenation rule, defined in Eq. (2.5) on page 20, we have to check that we are still computing the diffraction of a random noble means set, as this was defined via a random substitution rule. This is precisely the purpose of Lemma 2.29. ♦

Remark 6.15 (Restriction to exact RNMS words)**.** Note that it is not immediate that the study of Eq. (6.12) and Eq. (6.13) lead to the correct diffraction measure. Firstly, the random variable X_n disregards all non-exact geometric realisations. This is not a serious problem because the definition of legality of subwords ensures that X_n contains all geometric realisations of legal words, provided the recursion is unwinded far enough. Secondly, the first and second moments in Eq. (6.13) are no longer computed with respect to the measure ν_m but rather based on the realisation probabilities that were introduced after Lemma 2.29 on page 26. The first and second moments of X_n certainly differ, with respect to these distinct measures, for finite $n \in \mathbb{N}$ and although numerical examinations strongly suggest that they coincide as $n \to \infty$, a complete proof is missing at the moment. ♦

Remark 6.16 (Independence of X_n)**.** Consider an arbitrary $n \in \mathbb{N}$. Applying the recursion on X_n once, yields X_{n-1} and X_{n-2} on the right hand side of Eq. (6.12). Now, X_{n-1} depends on the geometric realisations of elements in $\mathcal{G}_{1,n-2}$, but these are obviously independent of the realisations with respect to the above X_{n-2}. In this sense, the random variables X_{n-1} and X_{n-2} are independent and we have $\mathbb{E}(X_{n-1}X_{n-2}) - \mathbb{E}(X_{n-1})\mathbb{E}(X_{n-2}) = 0$. Instead of introducing further notation to emphasise this minor subtlety, we leave it at that because we will use this property only once below. ♦

We proceed with the derivation of recursion formulas for $\mathbb{E}\big(X_n(k)\big)$ and $\mathbb{V}\big(X_n(k)\big)$. For the sake of readability, we introduce the following abbreviations.

$$\begin{aligned} \mathrm{e}_n &:= \mathrm{e}^{-2\pi\mathrm{i}kL_n}, \quad \cos_n := \cos(2\pi k L_n), \quad X_n := X_n(k), \\ \mathbb{E}_n &:= \mathbb{E}\big(X_n(k)\big) \quad \text{and} \quad \mathbb{V}_n := \mathbb{E}\big(\lvert X_n(k)\rvert^2\big) - \lvert\mathbb{E}\big(X_n(k)\big)\rvert^2, \end{aligned} \tag{6.14}$$

for any $n \in \mathbb{N}$ and $k \in \mathbb{R}$. Using the definition of X_n in Eq. (6.12), it is immediate that for $n \geqslant 2$, we have

$$\begin{aligned} \mathbb{E}_n &= \mathbb{E}\big(p_0(X_{n-2} + \mathrm{e}_{n-2}\, X_{n-1}) + p_1(X_{n-1} + \mathrm{e}_{n-1}\, X_{n-2})\big) \\ &= (p_1 + p_0\, \mathrm{e}_{n-2})\mathbb{E}_{n-1} + (p_0 + p_1\, \mathrm{e}_{n-1})\mathbb{E}_{n-2}, \end{aligned} \tag{6.15}$$

where $\mathbb{E}_0 = \mathrm{e}^{-2\pi\mathrm{i}k}$ and $\mathbb{E}_1 = \mathrm{e}^{-2\pi\mathrm{i}k\lambda_1}$. Firstly, we consider the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}_0}$. Applying Eq. (6.15) for any $n \geqslant 2$, we find

$$\begin{aligned} \mathbb{V}_n &= \mathbb{E}\big(p_0 \lvert X_{n-2} + \mathrm{e}_{n-2}\, X_{n-1}\rvert^2 + p_1 \lvert X_{n-1} + \mathrm{e}_{n-1}\, X_{n-2}\rvert^2\big) - \lvert\mathbb{E}_n\rvert^2 \\ &= \mathbb{V}_{n-1} + \mathbb{V}_{n-2} \\ &\qquad + 2p_0p_1\Big\{(1-\cos_{n-2})\lvert\mathbb{E}_{n-1}\rvert^2 + (1-\cos_{n-1})\lvert\mathbb{E}_{n-2}\rvert^2 \\ &\qquad\qquad - \mathrm{Re}\big[(1 - \overline{\mathrm{e}_{n-1}} - \mathrm{e}_{n-2} + \overline{\mathrm{e}_{n-1}}\, \mathrm{e}_{n-2})\mathbb{E}_{n-1}\overline{\mathbb{E}_{n-2}}\big]\Big\} \\ &\qquad + 2\,\mathrm{Re}\big[\big(p_0\, \mathrm{e}_{n-2} + p_1 \overline{\mathrm{e}_{n-1}}\big)\big(\mathbb{E}(X_{n-1}\overline{X_{n-2}}) - \mathbb{E}_{n-1}\overline{\mathbb{E}_{n-2}}\big)\big] \qquad (*) \\ &= \mathbb{V}_{n-1} + \mathbb{V}_{n-2} + 2p_0p_1\Psi_n, \end{aligned}$$

with $\mathbb{V}_0 = \mathbb{V}_1 = 0$ and

$$\begin{aligned} \Psi_n := \Psi_n(k) &:= (1-\cos_{n-2})\lvert\mathbb{E}_{n-1}\rvert^2 + (1-\cos_{n-1})\lvert\mathbb{E}_{n-2}\rvert^2 \\ &\qquad - \mathrm{Re}\big[(1 - \overline{\mathrm{e}_{n-1}} - \mathrm{e}_{n-2} + \overline{\mathrm{e}_{n-1}}\, \mathrm{e}_{n-2})\mathbb{E}_{n-1}\overline{\mathbb{E}_{n-2}}\big] \\ &= \frac{1}{2}\big\lvert(1-\mathrm{e}_{n-2})\mathbb{E}_{n-1} - (1-\mathrm{e}_{n-1})\mathbb{E}_{n-2}\big\rvert^2 \geqslant 0, \end{aligned} \tag{6.16}$$

for any $n \geqslant 2$. We have used that $\mathbb{E}(X_{n-1}\overline{X_{n-2}}) - \mathbb{E}_{n-1}\overline{\mathbb{E}_{n-2}} = 0$ in $(*)$ which was discussed in Remark 6.16. Our study of the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}_0}$ proceeds with some preparing notes on the sequence $(\Psi_n)_{n\geqslant 2}$. The illustrations of Ψ_n, with $n \in \{2,3,4\}$, in Figure 6.3 suggest the boundedness of each Ψ_n and a monotonic behaviour. This can be made rigorous by the following result.

Lemma 6.17. *For all $n \geqslant 2$, the function Ψ_n is real analytic. Moreover, one has $\Psi_n(k) \leqslant 2$ and $\Psi_{n+1}(k) \leqslant \Psi_n(k)$ for all $k \in \mathbb{R}$.*

Proof. The representation of Eq. (6.16) immediately shows the analyticity of Ψ_n because sums and products of trigonometric functions are real analytic. Next, we observe that

$$\begin{aligned} \Psi_2(k) &= \frac{1}{2}\left\lvert(1-\mathrm{e}^{-2\pi\mathrm{i}k})\,\mathrm{e}^{-2\pi\mathrm{i}k\lambda_1} - (1-\mathrm{e}^{-2\pi\mathrm{i}k\lambda_1})\,\mathrm{e}^{-2\pi\mathrm{i}k}\right\rvert^2 \\ &= 1 - \cos\big(2\pi k(1-\lambda_1)\big) \leqslant 2. \end{aligned}$$

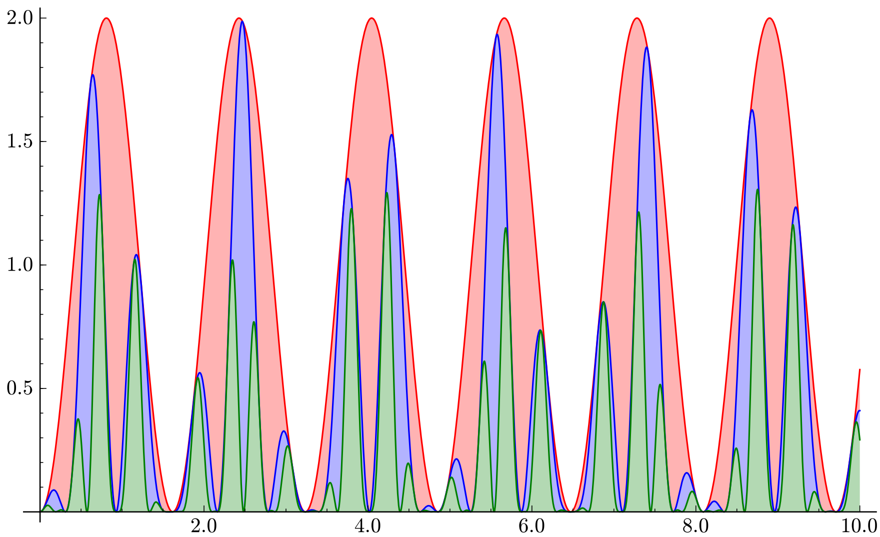

Figure 6.3. The function Ψ_n is shown with $n = 2$ (red), $n = 3$ (blue) and $n = 4$ (green).

Now, for $n \geqslant 2$ we define $\psi_n := \psi_n(k) := (1 - \mathrm{e}_{n-2})\mathbb{E}_{n-1} - (1 - \mathrm{e}_{n-1})\mathbb{E}_{n-2}$. Applying the recursion for $\mathbb{E}_n$ once on the first summand and using the recursion $L_n = L_{n-1} + L_{n-2}$ implies

$$\begin{aligned}
\psi_{n+1} &= (1 - \mathrm{e}_{n-1})\mathbb{E}_n - (1 - \mathrm{e}_n)\mathbb{E}_{n-1} \\
&= (1 - \mathrm{e}_{n-1})\big((p_1 + p_0\,\mathrm{e}_{n-2})\mathbb{E}_{n-1} + (p_0 + p_1\,\mathrm{e}_{n-1})\mathbb{E}_{n-2}\big) \\
&\qquad - (1 - \mathrm{e}_n)\mathbb{E}_{n-1} \\
&= -(p_0 + p_1\,\mathrm{e}_{n-1})\big((1 - \mathrm{e}_{n-2})\mathbb{E}_{n-1} - (1 - \mathrm{e}_{n-1})\mathbb{E}_{n-2}\big) \\
&= -(p_0 + p_1\,\mathrm{e}_{n-1})\psi_n.
\end{aligned} \tag{6.17}$$

This yields the monotonicity of Ψ_n because

$$|\psi_{n+1}| = |p_0\psi_n + p_1\,\mathrm{e}_{n-1}\,\psi_n| \leqslant p_0|\psi_n| + p_1|\psi_n| = |\psi_n|,$$

and therefore

$$\Psi_{n+1}(k) = \frac{1}{2}|\psi_{n+1}(k)|^2 \leqslant \frac{1}{2}|\psi_n(k)|^2 = \Psi_n(k). \qquad \blacksquare$$

Proposition 6.18. *For any $n \in \mathbb{N}_0$, consider the function $\phi_n \colon \mathbb{R} \longrightarrow \mathbb{R}_{\geqslant 0}$, defined by*

$$\phi_n(k) := \frac{1}{L_n}\mathbb{V}(X_n(k)).$$

On $\mathbb{R}$, the sequence $(\phi_n)_{n\in\mathbb{N}_0}$ converges uniformly to the continuous function $\phi \colon \mathbb{R} \longrightarrow \mathbb{R}_{\geqslant 0}$, with

$$\phi(k) := \frac{2p_0p_1\lambda_1}{\sqrt{5}} \sum_{i=2}^{\infty} \lambda_1^{-i}\Psi_i(k). \tag{6.18}$$

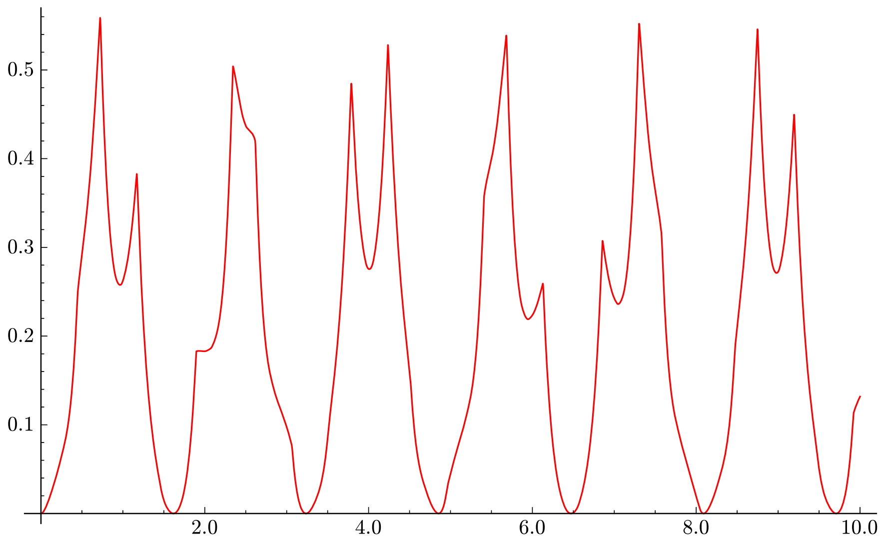

Figure 6.4. The density function ϕ_n is shown with $n = 10$ and $\boldsymbol{p}_1 = (1/2, 1/2)$.

Proof. From the recursion relation $\mathbb{V}_n = \mathbb{V}_{n-1} + \mathbb{V}_{n-2} + 2p_0p_1\Psi_n$, we conclude the representation

$$\lim_{n\to\infty} \phi_n(k) = \lim_{n\to\infty} \frac{2p_0p_1}{L_n} \sum_{i=2}^{n} \ell_{1,n+1-i}\Psi_i(k) = \frac{2p_0p_1\lambda_1}{\sqrt{5}} \sum_{i=2}^{\infty} \lambda_1^{-i}\Psi_i(k),$$

where $\ell_{1,n}$ denotes the nth Fibonacci number as introduced in Eq. (2.6). Next, we observe that ϕ is convergent because an application of Lemma 6.17 yields

$$\phi(k) \leqslant \frac{4p_0p_1\lambda_1}{\sqrt{5}} \sum_{i=0}^{\infty} \lambda_1^{-i-2} = \frac{4p_0p_1\lambda_1}{\sqrt{5}} \leqslant \frac{\lambda_1}{\sqrt{5}}.$$

Thus, ϕ is bounded and the sum consists of non-negative elements only. The uniformity of the convergence is implied by the following short calculation

$$\begin{aligned}
|\phi_n(k) - \phi(k)| &= 2p_0p_1\Big|\sum_{i=2}^{n}\Big(\frac{\ell_{1,n+1-i}}{L_n} - \frac{\lambda_1^{1-i}}{\sqrt{5}}\Big)\Psi_i(k) - \sum_{i=n+1}^{\infty} \frac{\lambda_1^{1-i}}{\sqrt{5}}\Psi_i(k)\Big| \\
&\leqslant 4p_0p_1\Big(\Big|\frac{(\lambda_1')^{n-1}}{\lambda_1^n\sqrt{5}} \sum_{i=0}^{n} (\lambda_1')^{-i}\Big| + \frac{1}{\lambda_1^n\sqrt{5}} \sum_{i=0}^{\infty} \lambda_1^{-i}\Big) \\
&\leqslant \Big|\frac{(\lambda_1')^{n-1} - 1/(\lambda_1')^2}{\lambda_1^n\sqrt{5}(1 - 1/\lambda_1')}\Big| + \frac{1}{\lambda_1^{n-2}\sqrt{5}},
\end{aligned} \tag{6.19}$$

and both summands in the last line converge to zero, as $n \to \infty$. This means that

$$\lim_{n\to\infty} \sup_{k\in\mathbb{R}} |\phi_n(k) - \phi(k)| = 0,$$

which at the same time implies the continuity of ϕ. ∎

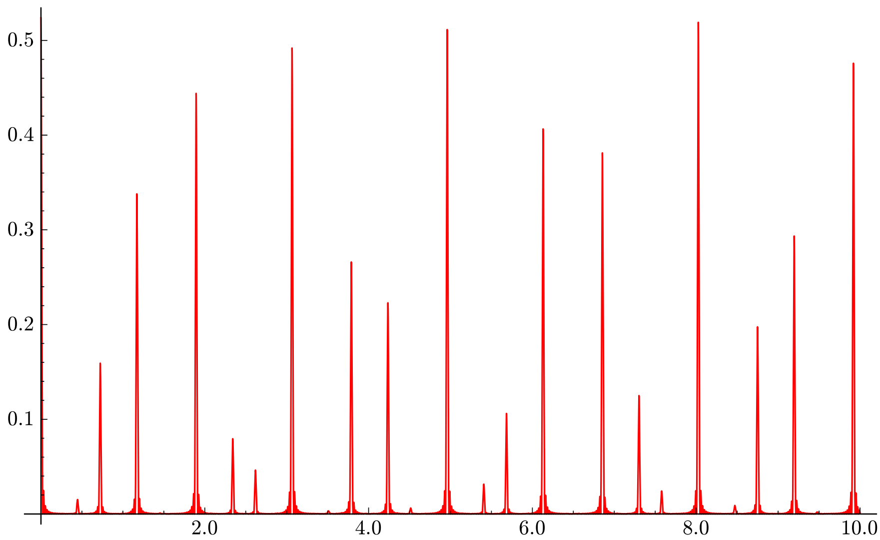

Figure 6.5. An approximation to the pure point part of $\widehat{\gamma_{\Lambda,1}}$, based on the recursion in Eq. (6.15), is shown. Here, it is illustrated with $n = 8$ and $\boldsymbol{p}_1 = (1/2, 1/2)$.

An illustration of ϕ_n for $n = 10$ is shown in Figure 6.4. Even from the (rather mild) bound in Eq. (6.19) one can see that $(\phi_n)_{n\in\mathbb{N}_0}$ converges very quickly such that we get a very accurate picture here.

Corollary 6.19. *The roots of ϕ are precisely the roots of Ψ_2, and they are given by all integer multiples of λ_1.*

Proof. For $n \geqslant 1$, the recursion formula for ψ_n in Eq. (6.17) can be rewritten as

$$\begin{aligned}\psi_{n+1}(k) &= (-1)^{n-1}\psi_2(k)\prod_{j=1}^{n-1}\left(p_0 + p_1\,\mathrm{e}^{-2\pi\mathrm{i}kL_j}\right)\\ &= (-1)^{n-1}\left(\mathrm{e}^{-2\pi\mathrm{i}k\lambda_1} - \mathrm{e}^{-2\pi\mathrm{i}k}\right)\prod_{j=1}^{n-1}\left(p_0 + p_1\,\mathrm{e}^{-2\pi\mathrm{i}kL_j}\right).\end{aligned} \tag{6.20}$$

Considering each factor of the product in Eq. (6.20) separately and including $\Psi_j(k) = |\psi_j(k)|^2/2$ for any $j \geqslant 2$, we explore the function $f_j\colon \mathbb{R} \longrightarrow \mathbb{R}_{\geqslant 0}$ that is defined as

$$f_j(k) := \left|p_0 + p_1\,\mathrm{e}^{-2\pi\mathrm{i}kL_j}\right|^2 = p_0^2 + p_1^2 + 2p_0p_1\cos(2\pi kL_j).$$

Here, for all $j \in \mathbb{N}$, the set of roots of f_j reads

$$R_j = \left\{\frac{\pm\arccos\left(\frac{2p_0p_1-1}{2p_0p_1}\right) + 2\pi q}{2\pi L_j}\,\middle|\, q \in \mathbb{Z}\right\}.$$

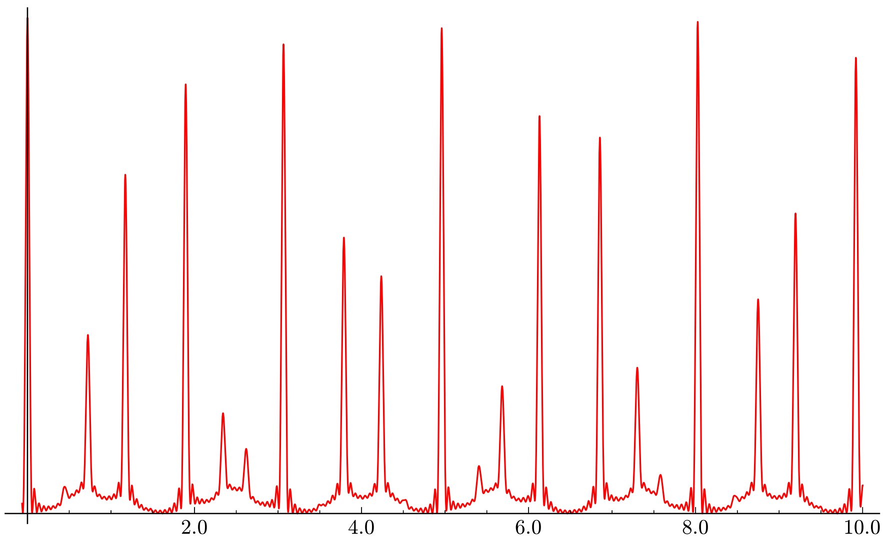

Figure 6.6. The diffraction measure for $m = 1$ and with $\boldsymbol{p}_1 = (1/2, 1/2)$ is illustrated. The approximation is done via the recursion in Eq. (6.11) for $n = 6$.

Moreover, the expression $|\mathrm{e}^{-2\pi\mathrm{i}k\lambda_1} - \mathrm{e}^{-2\pi\mathrm{i}k}|^2 = 2 - 2\cos\bigl(2\pi k(1-\lambda_1)\bigr)$ vanishes on all $k \in \lambda_1\mathbb{Z}$. This implies that

$$\lambda_1\mathbb{Z} \cup \bigcup_{j=1}^{n-1} R_j$$

is the set of roots of Ψ_{n+1} for all $n \geqslant 1$. Because of Lemma 6.17 and the representation of ϕ in Eq. (6.18), this implies that $\lambda_1\mathbb{Z}$ is the set of roots of ϕ. ∎

Finally, Proposition 6.18 implies the vague convergence of the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}_0}$ and the existence of $\widehat{\gamma_{\Lambda,1}}$ immediately yields the vague convergence of $(\mathcal{P}_n)_{n\in\mathbb{N}_0}$. Therefore, we almost surely find that

$$\widehat{\gamma_{\Lambda,1}} = (\widehat{\gamma_{\Lambda,1}})_{\ominus} + (\widehat{\gamma_{\Lambda,1}})_{\mathsf{pp}} + \phi(k)\lambda,$$

where the precise nature of $(\widehat{\gamma_{\Lambda,1}})_{\ominus}$ stays an open question and needs further study in the future (see Remark 6.22 below). Following Hof [Hof95, Thm. 3.2], we find

$$\widehat{\gamma_{\Lambda,1}}(\{k\}) = \lim_{n\to\infty} \frac{1}{L_n^2} \bigl|\mathbb{E}(X_n(k))\bigr|^2,$$

and a sketch of $\widehat{\gamma_{\Lambda,1}}(\{k\})$ and $\widehat{\gamma_{\Lambda,1}}$ is illustrated in Figures 6.5 and 6.6, respectively. Illustrations of $\widehat{\gamma_{\Lambda,1}}$ for different choices of $\boldsymbol{p}_1$ are shown in Appendix A.

Remark 6.20 (Typical realisations)**.** Due to the ergodicity of ν_1, we know that almost all realisations in $\mathbb{Y}_1$ feature the diffraction which we compute via the ensemble. These elements are called *typical*. For any given realisation, we can never say whether it is typical or not. This observation can be pinned down by considering the (deterministic) noble means sets which are included in a ν_1-null set and that yield pure point diffraction measures by Corollary 6.10. ♦

Remark 6.21 (Consistency with deterministic cases)**.** Note that Proposition 6.18 is consistent with what we know about the deterministic cases $\Lambda_{1,0}$ and $\Lambda_{1,1}$ because due to Corollary 6.10, we know that the diffraction measure is pure point there. Setting $p_0 = 0$ or $p_1 = 0$ in the representation of ϕ from Eq. (6.18) indeed shows that $\phi \equiv 0$ which means that the absolutely continuous part vanishes here. ♦

Remark 6.22 (Exclusion of continuous parts)**.** The vague convergence of the sequence $(\mathcal{P}_n)_{n\in\mathbb{N}_0}$ alone does not avoid the presence of continuous components. One idea for a proof of uniform convergence is the reformulation of Eq. (6.15) in terms of a matrix product and the derivation of a sufficiently strong estimate for a suitably chosen matrix norm. To this end, recall the notation of Eq. (6.14) and define for $n \geqslant 1$ the matrices

$$T_n := T_n(k) := \begin{pmatrix} p_1 + p_0\,\mathrm{e}_{n-1} & p_0 + p_1\,\mathrm{e}_n \\ 1 & 0 \end{pmatrix} \tag{6.21}$$

and consider for $n \geqslant 0$ the expression

$$V_n := \begin{pmatrix} \mathbb{E}_{n+1} \\ \mathbb{E}_n \end{pmatrix} = T_n \begin{pmatrix} \mathbb{E}_n \\ \mathbb{E}_{n-1} \end{pmatrix} = \prod_{i=0}^{n-1} T_{n-i} \begin{pmatrix} \mathbb{E}_1 \\ \mathbb{E}_0 \end{pmatrix},$$

where $\prod_{i=0}^{n-1} T_{n-i}$ is the ordered product of the matrices in Eq. (6.21). From the inequality

$$|\mathbb{E}_n| \leqslant \|V_n\|_\alpha \leqslant \Big\| \prod_{i=0}^{n-1} T_{n-i} \Big\|_\beta \|V_0\|_\alpha \leqslant \prod_{i=0}^{n-1} \|T_{n-i}\|_\beta \|V_0\|_\alpha, \tag{6.22}$$

for all $n \in \mathbb{N}$, and the identity $|\mathbb{E}_n| = L_n^{\varepsilon}$ where

$$\varepsilon := \frac{\log(|\mathbb{E}_n|)}{\log(L_n)} \leqslant \frac{1}{n\log(\lambda_1)}\Big(\log(\|V_0\|_\alpha) + \sum_{i=0}^{n-1} \log(\|T_{n-i}\|_\beta)\Big) =: \mathcal{T}_n,$$

for a suitable choice of compatible vector/matrix norms α/β, we conclude that we have to show that $\lim_{n\to\infty} \mathcal{T}_n(k) < 1/2$ for any $k \in \mathbb{R}\setminus\mathcal{L}_1^{\circledast}$ in order to exclude the presence of a singular continuous part. Unfortunately, this estimate via the sub-multiplicativity of the matrix norm is not strong enough, as numerical evaluations of $\mathcal{T}_n$ suggest. Instead, one can study the Lyapunov exponent

$$\mathcal{T}_n' := \frac{\log(\|\prod_{i=0}^{n-1} T_{n-i}\|_\beta \|V_0\|_\alpha)}{n\log(\lambda_1)}$$

resulting from the second inequality of Eq. (6.22) where the sub-multiplicativity of the matrix norm has not been applied yet. In Appendix B, we show approximations of $\mathcal{T}_n'$ where α is the Euclidean vector norm and β the Frobenius matrix norm. Although it is very likely that $\widehat{\gamma_{\Lambda,1}}$ contains no singular continuous component, there is no proof at the moment. Concerning absolutely continuous parts, one would inspect the expression $\mathcal{T}_n''(k) = 2\mathcal{T}_n'(k)$ that one obtains analogously from Eq. (6.22) by considering $|\mathbb{E}_n|^2$. Here, a proof that $\lim_{n\to\infty} \mathcal{T}_n''(k) < 1/2$ for almost all $k \in \mathbb{R}$ is still needed. ♦

The general case for arbitrary $m \in \mathbb{N}$ can be handled analogously, although the treatment of the absolutely continuous part is technically more involved.

6.3. Pure point part of $\widehat{\gamma_{\Lambda,m}}$ via inflation-invariant measures

Here, we access the content of Section 5.1.1 and discuss inflation-invariant (or self-similar) measures on the attractor of the iterated function system of Eq. (5.10) that was computed in Proposition 5.13. These measures constitute a convenient alternative approach to the pure point part of $\widehat{\gamma_{\Lambda,m}}$.

We start with a standard result concerning pushforward measures. In this context, we define $f.\mu(B) := \mu(f^{-1}(B))$ for a Borel set B and a measurable function f, where f^{-1} is the pre-image of B under f. This matches the definition of $f.\mu$ on functions given on page 56 after Remark 5.4.

Theorem 6.23 ([Els11, Thm. 3.1])**.** *Let (X, Σ) and (X', Σ') two measure spaces, $\mu \in \mathcal{M}(X)$ finite and $f \colon X \longrightarrow X'$ a measurable function. Let $f.\mu := \mu \circ f^{-1}$ be the pushforward measure of μ under f and $g \colon X' \longrightarrow \overline{\mathbb{K}}$ integrable with respect to $f.\mu$. Then, $g \circ f \in L^1(X, \mu)$ and*

$$\int_{X'} g \,\mathrm{d}f.\mu = \int_X g \circ f \,\mathrm{d}\mu. \qquad \blacksquare$$

Here, we interpret the Fourier transform of a finite measure μ as a bounded and continuous function [Rud62, Thm. 1.3.3(a)].

Corollary 6.24. *Let $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ be an affine map, defined by $f(x) := ax + b$ for any a, $b \in \mathbb{R}$ and $\mu \in \mathcal{M}(\mathbb{R})$ be a finite measure. Then, we have*

$$\widehat{f.\mu}(k) = \mathrm{e}^{-2\pi \mathrm{i} kb}\, \widehat{\mu}(ak).$$

Proof. Applying Theorem 6.23, we find

$$\widehat{f.\mu}(k) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}kx} \,\mathrm{d}f.\mu(x) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}k(ax+b)} \,\mathrm{d}\mu(x) = \mathrm{e}^{-2\pi\mathrm{i}kb}\, \widehat{\mu}(ak). \qquad \blacksquare$$

Henceforth, we pursue a similar idea as in the treatment of the set-theoretic iterated function systems that were considered in Section 5.3.2. We study the lift of a geometric realisation in the internal space and benefit from λ_m being a PV number. This allows the application of Theorem 5.6 for the solution of the resulting measure-theoretic iterated function system. For notational convenience, we abbreviate $\xi := \lambda_m'$ in the following. For any $0 \leqslant i \leqslant m$, we derive from Eq. (5.10) the following measure-theoretic iterated function system in the sense of Section 5.1.1,

$$\begin{aligned}\mu_{i,a} &= |\xi|\Big(\sum_{j=0}^{i-1} f_j.\mu_{i,a} + \sum_{j=i}^{m-1} g_j.\mu_{i,a} + f_0.\mu_{i,b}\Big)\\ \mu_{i,b} &= |\xi| f_i.\mu_{i,a},\end{aligned} \tag{6.23}$$

with affine functions

$$\begin{aligned} f_j &: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \xi(x+j),\\ g_j &: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \xi(x+j)+1,\end{aligned}$$

which are precisely the contractions that constitute the iterated function system $\mathcal{F}^\star$ in Eq. (5.10). Due to [BM00a, Sec. 7.2], this is the 'non-overlapping' case which means that the measure η_{ij} in Theorem 5.6 is counting measure, scaled by λ_m^{-1}; see property **NO3** in [BM00a, Sec. 7.2]. This yields the matrix

$$S = \begin{pmatrix} m/\lambda_m & 1/\lambda_m \\ 1/\lambda_m & 0 \end{pmatrix} = \begin{pmatrix} -m\xi & -\xi \\ -\xi & 0 \end{pmatrix},$$

which is clearly primitive with PF eigenvalue 1 and corresponding right PF eigenvector $(1, |\xi|)^T$. By Theorem 5.6, the solution of the system in Eq. (6.23) is given by the uniquely defined inflation-invariant measure $\boldsymbol{\mu}_i \coloneqq \mu_{i,a} \otimes \mu_{i,b}$ where

$$\mu_{i,a} \in \mathcal{M}(A^{(a)}_{m,i}) \quad \text{and} \quad \mu_{i,b} \in \mathcal{M}(A^{(b)}_{m,i}).$$

Recall from Proposition 5.13 that

$$A^{(a)}_{m,i} = i\tau_m + [0,1] \quad \text{and} \quad A^{(b)}_{m,i} = i\tau_m + [\xi, 0]$$

are the compact solutions of $\mathcal{F}^\star$. Moreover, this system is consistent in the following sense. One can check that the total mass of $A^{(a)}_{m,i}$ and $A^{(b)}_{m,i}$ under $\mu_{i,a}$ and $\mu_{i,b}$ respectively, is invariant under a single step of the iterated function system. To this end, we observe that

$$\begin{aligned} A^{(a)}_{m,i} &\subset f_j^{-1}(A^{(a)}_{m,i}) \quad \text{for} \quad 0 \leqslant j \leqslant i-1,\\ A^{(a)}_{m,i} &\subset g_j^{-1}(A^{(a)}_{m,i}) \quad \text{for} \quad i \leqslant j \leqslant m-1.\end{aligned}$$

This implies

$$\mu_{i,a}(g_j^{-1}(A^{(a)}_{m,i})) = \mu_{i,a}(f_j^{-1}(A^{(a)}_{m,i})) = \mu_{i,a}(A^{(a)}_{m,i}).$$

Furthermore, we find $A^{(a)}_{m,i} \subset (f_0 \circ f_i)^{-1}(A^{(a)}_{m,i})$ and therefore

$$\mu_{i,a}((f_0 \circ f_i)^{-1}(A^{(a)}_{m,i})) = \mu_{i,a}(A^{(a)}_{m,i}).$$

After decoupling Eq. (6.23), we have

$$\begin{aligned}\mu_{i,a}(A^{(a)}_{m,i}) &= \mu_{i,a}(A^{(a)}_{m,i})|\xi|\Big(\sum_{j=0}^{i-1} 1 + \sum_{j=i}^{m-1} 1 + |\xi|\Big)\\ &= \mu_{i,a}(A^{(a)}_{m,i})(\xi^2 - m\xi),\end{aligned} \tag{6.24}$$

and the assertion follows from $1 = \xi^2 - m\xi$. In the same way, we can compute $A^{(a)}_{m,i} = f_i^{-1}(A^{(b)}_{m,i})$ which means

$$\mu_{i,b}(A^{(b)}_{m,i}) = |\xi|\mu_{i,a}(f_i^{-1}(A^{(b)}_{m,i})) = |\xi|\mu_{i,a}(A^{(a)}_{m,i}). \tag{6.25}$$

In particular, Eq. (6.24) and Eq. (6.25) are valid when $\mu_{i,a}$ and $\mu_{i,b}$ are chosen as the Lebesgue measure on the corresponding windows. With this choice, one can similarly check that Eq. (6.23) is consistent with the evaluation of $\mu_{i,a}$ and $\mu_{i,b}$ on any open interval. An application of Theorem 5.6 then yields the following result.

Corollary 6.25. *The measures $\mu_{i,a} \in \mathcal{M}(A^{(a)}_{m,i})$ and $\mu_{i,b} \in \mathcal{M}(A^{(b)}_{m,i})$ are given by the Lebesgue measure on the windows for the cut and project scheme of Figure 5.3, thus*

$$\mu_{i,a} \coloneqq \mathbb{1}_{A^{(a)}_{m,i}} \lambda \quad \textit{and} \quad \mu_{i,b} \coloneqq \mathbb{1}_{A^{(b)}_{m,i}} \lambda. \qquad \blacksquare$$

The following treatment of the Fourier transform of $\mu_{i,a}$ and $\mu_{i,b}$ can similarly be found for the special case $m = 1$ in [Lüt10].

Proposition 6.26. *Let $\mu_{i,a}$ and $\mu_{i,b}$ be the inflation-invariant measures defined by Eq. (6.23). Then, the vector $\big(\widehat{\mu_{i,a}}(k), \widehat{\mu_{i,b}}(k)\big)^T$ can be arbitrarily well approximated by the product*

$$|\xi|^n \prod_{q=1}^{n} \begin{pmatrix} \sum_{j=0}^{i-1} \mathrm{e}^{-2\pi\mathrm{i}\xi^q kj} + \sum_{j=i}^{m-1} \mathrm{e}^{-2\pi\mathrm{i}\xi^q k(j+1/\xi)} & 1 \\ \mathrm{e}^{-2\pi\mathrm{i}\xi^q ki} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mu_{i,a}}(\xi^n k) \\ \widehat{\mu_{i,b}}(\xi^n k) \end{pmatrix}, \tag{6.26}$$

where the recursion at $k = 0$ leads to the equation

$$\begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{m^2+4} \\ -\xi/\sqrt{m^2+4} \end{pmatrix} = \lambda_m \begin{pmatrix} 1/\sqrt{m^2+4} \\ -\xi/\sqrt{m^2+4} \end{pmatrix}.$$

Proof. By an application of Corollary 6.24, Fourier transform of the iterated function system in Eq. (6.23) yields

$$\widehat{\mu_{i,a}}(k) = |\xi|\Big(\sum_{j=0}^{i-1} \mathrm{e}^{-2\pi\mathrm{i}\xi kj}\, \widehat{\mu_{i,a}}(\xi k) + \sum_{j=i}^{m-1} \mathrm{e}^{-2\pi\mathrm{i}\xi k(j+1/\xi)}\, \widehat{\mu_{i,a}}(\xi k) + \widehat{\mu_{i,b}}(\xi k)\Big)$$

$$\widehat{\mu_{i,b}}(k) = |\xi|\, \mathrm{e}^{-2\pi\mathrm{i}\xi ki}\, \widehat{\mu_{i,a}}(\xi k),$$

which leads to

$$\begin{pmatrix} \widehat{\mu_{i,a}}(k) \\ \widehat{\mu_{i,b}}(k) \end{pmatrix} = |\xi| \begin{pmatrix} \sum_{j=0}^{i-1} \mathrm{e}^{-2\pi\mathrm{i}\xi kj} + \sum_{j=i}^{m-1} \mathrm{e}^{-2\pi\mathrm{i}\xi k(j+1/\xi)} & 1 \\ \mathrm{e}^{-2\pi\mathrm{i}\xi ki} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mu_{i,a}}(\xi k) \\ \widehat{\mu_{i,b}}(\xi k) \end{pmatrix},$$

and after n-fold iteration, we find Eq. (6.26). Now, recall that $|\xi| < 1$ and $(\xi^n k)_{n\in\mathbb{N}}$ tends to zero as $n \to \infty$. With $k = 0$ and $n = 1$, we find

$$\lambda_m \begin{pmatrix} \widehat{\mu_{i,a}}(0) \\ \widehat{\mu_{i,b}}(0) \end{pmatrix} = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mu_{i,a}}(0) \\ \widehat{\mu_{i,b}}(0) \end{pmatrix},$$

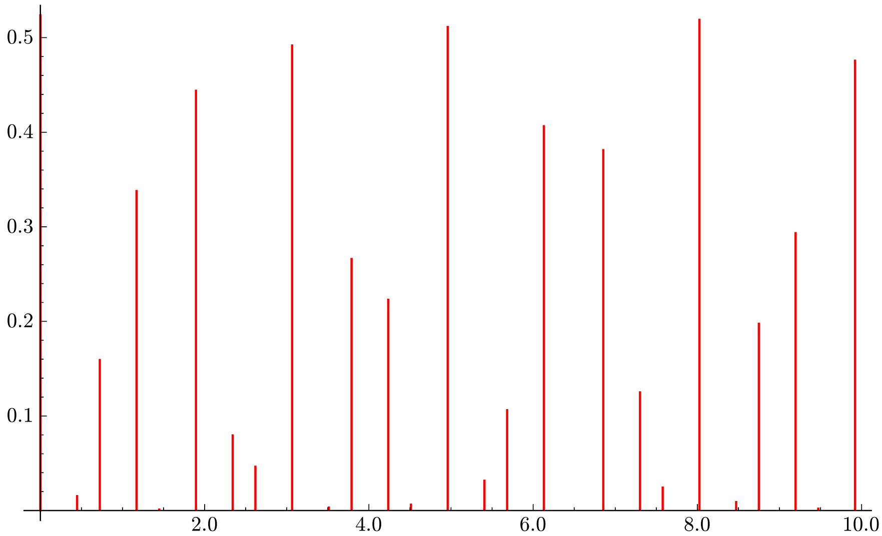

Figure 6.7. Intensities $I(k) = |\widehat{\mu_a}(-k') + \widehat{\mu_b}(-k')|^2$ for the pure point part of $\widehat{\gamma_{\Lambda,1}}$. The case of $\boldsymbol{p}_1 = (1/2, 1/2)$ and $n = 20$ with respect to Corollary 6.27 is illustrated. Compare this with Figure 6.5, based on the recursive approach of Eq. (6.15).

which means that $\left(\widehat{\mu_{i,a}}(0), \widehat{\mu_{i,b}}(0)\right)^T$ is an eigenvector of the substitution matrix M_m. As $\widehat{\mu_{i,a}}(0) + \widehat{\mu_{i,b}}(0)$ must equal the density of a random noble means set (see Theorem 6.7), we fix the recursion via

$$\widehat{\mu_{i,a}}(0) := 1/\sqrt{m^2+4} \quad \text{and} \quad \widehat{\mu_{i,b}}(0) := -\xi/\sqrt{m^2+4}.$$

Due to the compactness of $A^{(a)}_{m,i}$ and $A^{(b)}_{m,i}$, both $\widehat{\mu_{i,a}}$ and $\widehat{\mu_{i,b}}$ are uniformly continuous. Now, define for $q \in \mathbb{N}$ and $0 \leqslant i \leqslant m$ the matrix

$$P_i(\xi^q k) := \begin{pmatrix} \sum_{j=0}^{i-1} \mathrm{e}^{-2\pi \mathrm{i} \xi^q k j} + \sum_{j=i}^{m-1} \mathrm{e}^{-2\pi \mathrm{i} \xi^q k (j+1/\xi)} & 1 \\ \mathrm{e}^{-2\pi \mathrm{i} \xi^q k i} & 0 \end{pmatrix}. \tag{6.27}$$

For each $\varepsilon > 0$, we find a matrix Riesz product representation such that there is an $N \in \mathbb{N}$ with the property

$$\left\| \begin{pmatrix} \widehat{\mu_{i,a}}(k) \\ \widehat{\mu_{i,b}}(k) \end{pmatrix} - |\xi|^n \prod_{q=1}^{n} P_i(\xi^q k) \begin{pmatrix} \widehat{\mu_{i,a}}(0) \\ \widehat{\mu_{i,b}}(0) \end{pmatrix} \right\|_\infty < \varepsilon,$$

for all $n \geqslant N$ and with the supremum norm $\|\cdot\|_\infty$ on $\mathbb{R}^2$. ■

Finally, we use the preceding treatment of the deterministic cases for a similar approach in the RNMS cases. Due to the Bernoulli structure of the stochastic situation, we consider

$$\begin{aligned} \mu_a &= |\xi| \sum_{i=0}^{m} p_i \Big(\sum_{j=0}^{i-1} f_j.\mu_a + \sum_{j=i}^{m-1} g_j.\mu_a + f_0.\mu_b \Big) \\ \mu_b &= |\xi| \sum_{i=0}^{m} p_i f_i.\mu_a, \end{aligned} \tag{6.28}$$

as the suitable iterated function system for the treatment of inflation-invariant measures in the RNMS cases. Along the same lines as in the proof of Proposition 6.26, we find the following result.

Corollary 6.27. *Let μ_a and μ_b be the inflation-invariant measures defined by Eq.* (6.28)*. Then, the vector $\big(\widehat{\mu_a}(k), \widehat{\mu_b}(k)\big)^T$ can be arbitrarily well approximated by the product*

$$|\xi|^n \Big(\prod_{q=1}^{n} \sum_{i=0}^{m} p_i P_i(\xi^q k) \Big) \begin{pmatrix} \widehat{\mu_a}(\xi^n k) \\ \widehat{\mu_b}(\xi^n k) \end{pmatrix},$$

where $P_i(\xi^q k)$ is the matrix defined in Eq. (6.27) *and the recursion at $k = 0$ leads to the equation*

$$\begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{m^2+4} \\ -\xi/\sqrt{m^2+4} \end{pmatrix} = \lambda_m \begin{pmatrix} 1/\sqrt{m^2+4} \\ -\xi/\sqrt{m^2+4} \end{pmatrix}. \qquad \blacksquare$$

The pure point part of $\widehat{\gamma_{\Lambda,m}}$ can now be represented as

$$(\widehat{\gamma_{\Lambda,m}})_{\mathsf{pp}} = \sum_{k \in \mathcal{L}_m^{\circledast}} |\widehat{\mu_a}(-k') + \widehat{\mu_b}(-k')|^2 \delta_k,$$

where $\mathcal{L}_m^{\circledast}$ is the Fourier module of Eq. (6.5). An illustration of the intensities $I(k) = |\widehat{\mu_a}(-k') + \widehat{\mu_b}(-k')|^2$ is shown in Figure 6.7.

Remark 6.28 (Fourier module for local mixtures)**.** Strictly speaking, it is not completely clear that the Fourier module $\mathcal{L}_m^{\circledast}$ is still sufficient after having incorporated the local mixture in Eq. (6.28). Here, we can say the following. The leading eigenvalue λ_m of the substitution matrix M_m is an algebraic unit in the ring $\mathbb{Z}[\lambda_m]$. Now, let $\mathcal{T}$ be the *translation module* (see [BG13, Sec.5.1.2] or [GK97, Sec. 4] for background information) of ζ_m on which M_m acts via multiplication by λ_m. We find that $\lambda_m \mathcal{T} = M_m \mathcal{T} = \mathcal{T}$ and consequently $\lambda_m^{-1} \mathcal{T}^* = (M_m^T)^{-1} \mathcal{T}^* = \mathcal{T}^*$ where $\mathcal{T}^*$ is the *dual module*. Following [GK97, Sec. 4], the Fourier module is then given by $\mathcal{T}^*$. The significance of the leading eigenvalue to be a unit can be seen via the the example of the *period doubling substitution* ($a \longmapsto ab, b \longmapsto aa$). Here, the substitution matrix has PF eigenvalue $\lambda = 2$ and we find [BG13, Sec. 9.4.4] that $\mathcal{L}^{\circledast} = \mathbb{Z}[\frac{1}{2}]$ but $\mathbb{Z}[\frac{1}{2}] \neq \frac{1}{2}\mathbb{Z}$. ♦

Outlook

This thesis establishes a first systematic step into the realm of local mixtures of substitution rules. The choice of the noble means example promised some technical simplifications because all members of $\mathcal{N}_m$ define the same two-sided discrete hull. One obvious extension of the RNMS case can be found in the local mixture of families that do no longer share this property. Concerning the computation of the topological entropy, this has recently been done for some case by Nilsson [Nil13]. More generally, one may raise the question for the properties a family of substitutions must have, in order to preserve the features that we derived in Chapters 2 to 6.

Leaving the realm of symbolic dynamics and one-dimensional inflation rules, one significant enhancement of the theory would be a two or three-dimensional example. The (locally) random Penrose tiling was already discussed by Godrèche and Luck [GL89, Sec. 5.2], although a deeper mathematical analysis is desirable here.

Apart from the aspects of substitution tilings discussed in this work, there is a vital community studying topological invariants in this regard; see amongst others [AP98] and [Sad08]. The cohomology of globally mixed substitution tiling spaces has been recently studied by Gähler and Maloney [GM13]. It would be interesting to compute the cohomology of the RNMS family and explore its connections with other one-dimensional examples. A first step concerning the random Fibonacci case has recently been done in [GP14].

One additional interesting field for further study is the analysis of Schrödinger operators associated with the subshift $\mathbb{X}_m$; refer to [Dam05, DEG12] for background information. Given the invariant measure μ_m and a continuous function $f\colon \mathbb{X}_m \longrightarrow \mathbb{R}$, one defines for every $w \in \mathbb{X}_m$ a potential $U_w\colon \mathbb{Z} \longrightarrow \mathbb{R}$ by $U_w(k) := f(S^k w)$ and a bounded operator H_w that acts on $\ell^2(\mathbb{Z})$ via

$$[H_w\varphi](n) = \varphi(n+1) + \varphi(n-1) + U_w(n)\varphi(n).$$

Here, one is interested in the spectral properties of H. Whereas the situation is rather well understood in the case of $\mathbb{X}'_m$, the RNMS provides a family of examples with positive entropy of which far less is known.

APPENDIX A

Diffraction plots for $\widehat{\gamma_{\Lambda,1}}$

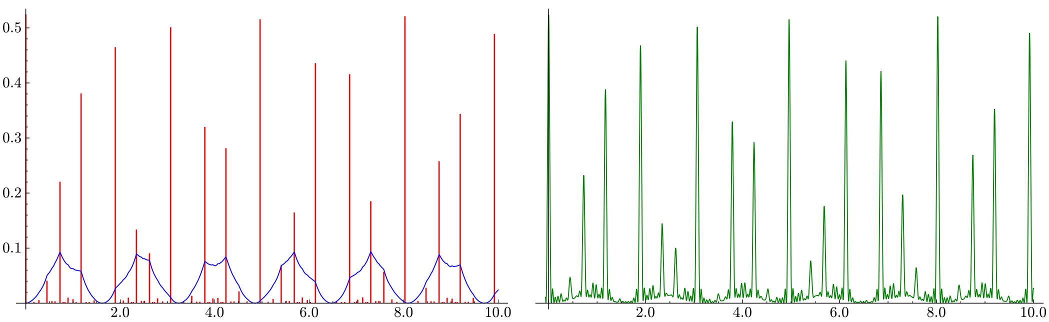

Figure A.1. Pure point (red) and absolutely continuous (blue) part of $\widehat{\gamma_{\Lambda,1}}$ (green), based on the recursion for $\mathbb{E}_n$ and $\mathbb{V}_n$ around Eq. (6.15), for $\boldsymbol{p}_1 = (1/10, 9/10)$.

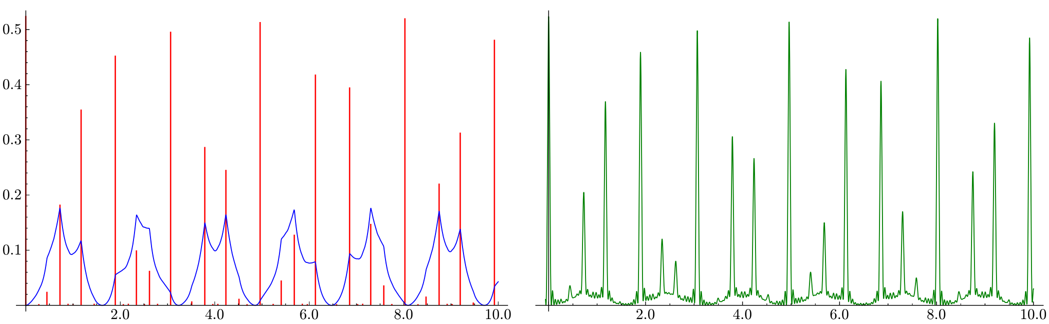

Figure A.2. Pure point (red) and absolutely continuous (blue) part of $\widehat{\gamma_{\Lambda,1}}$ (green), based on the recursion for $\mathbb{E}_n$ and $\mathbb{V}_n$ around Eq. (6.15), for $\boldsymbol{p}_1 = (1/4, 3/4)$.

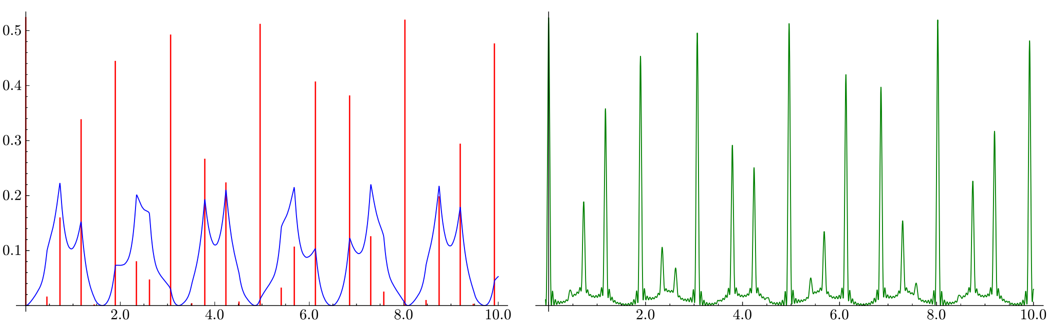

Figure A.3. Pure point (red) and absolutely continuous (blue) part of $\widehat{\gamma_{\Lambda,1}}$ (green), based on the recursion for $\mathbb{E}_n$ and $\mathbb{V}_n$ around Eq. (6.15), for $\boldsymbol{p}_1 = (1/2, 1/2)$.

APPENDIX B

Numerics for the continuous parts of $\widehat{\gamma_{\Lambda,1}}$

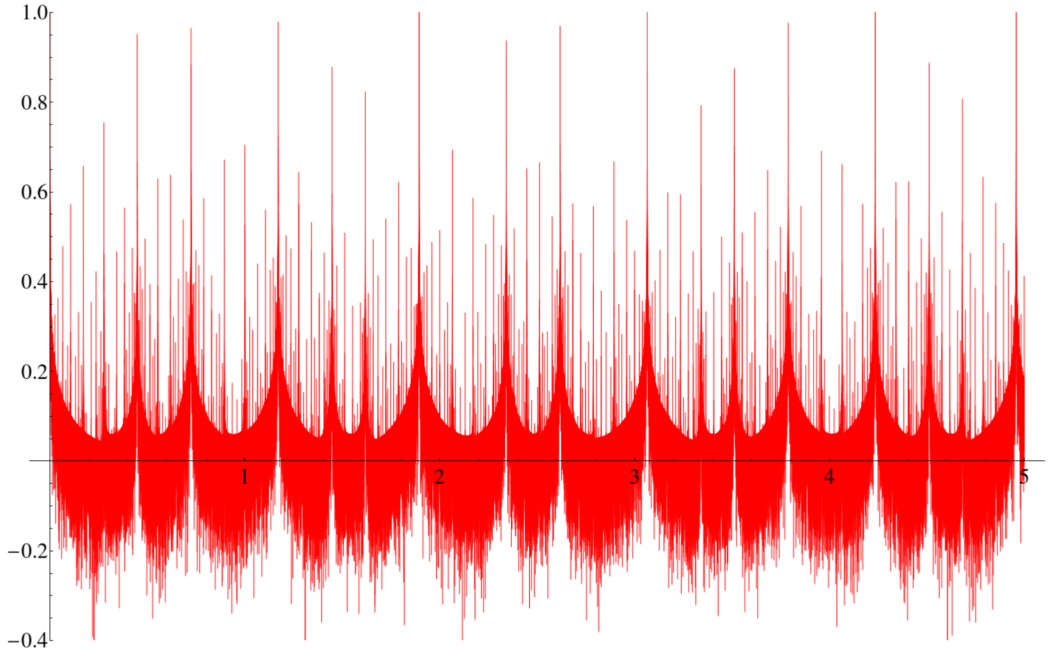

Figure B.1. Approximation of $\mathcal{T}_n'$ from Remark 6.22 on page 84 for $n = 20$ and with vector norm $\alpha = 2$ and matrix norm $\beta =$ Frobenius.

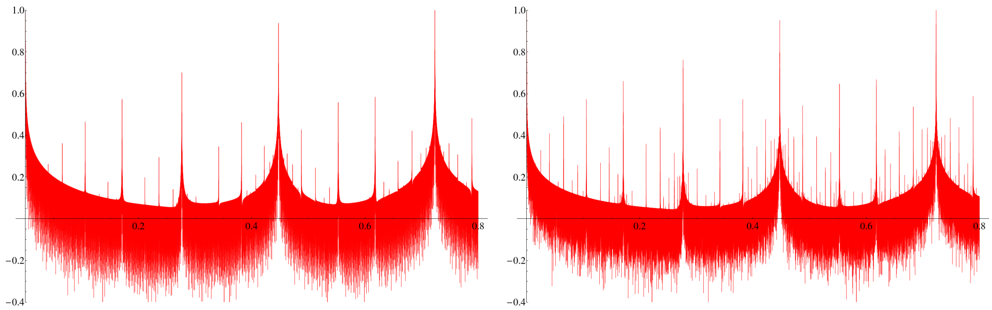

Figure B.2. Approximation of $\mathcal{T}_n'$ from Remark 6.22 on page 84 for $n = 20$ (left), $n = 25$ (right) and with vector norm $\alpha = 2$ and matrix norm $\beta =$ Frobenius. Here, a smaller interval is considered.

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List of Symbols

$\lhd$	subword relation	2
$\doteq$	randomised word equality relation	18
$\dot{\lhd}$	randomised subword relation	18
$\mathbb{1}_B$	characteristic function of the set B	10
$\lvert w\rvert_v$	occurrence number of v in w	2
$\mathcal{A}_n$	alphabet with n letters	2
$\mathcal{A}_n^{\ell}$	words of length ℓ over $\mathcal{A}_n$	2
$\mathcal{A}_n^{*}$	all finite words over $\mathcal{A}_n$	2
$\mathfrak{B}_m$	Borel σ-algebra generated by $\mathfrak{Z}(\mathbb{X}_m)$	48
$\mathcal{C}(X)$	vector space of continuous functions $X \longrightarrow \mathbb{C}$	2
$\mathcal{C}_{\mathrm{c}}(X)$	$\mathcal{C}(X)$-functions with compact support	10
$C_m(\ell)$	complexity function of ζ_m and word length ℓ	28
$\mathcal{D}_m$	set of ζ_m-legal words	20
$\mathcal{D}_{m,\ell}$	set of ζ_m-legal words of length ℓ	20
$\mathcal{D}'_m$	set of $\zeta_{m,i}$-legal words	15
$\mathcal{D}'_{m,\ell}$	set of $\zeta_{m,i}$-legal words of length ℓ	15
$\mathbb{E}_{\mu}(X)$	mean of X with respect to μ	49
$\mathcal{F}(S)$	factor set of S	2
$\mathcal{F}_{\ell}(S)$	ℓth factor set of S	2
$\mathfrak{F}_n$	free group, generated by the letters of $\mathcal{A}_n$	16
ϕ	Abelianisation map	5
$\mathcal{G}_{m,k}$	set of ζ_m-exact substitution words of length $\ell_{m,k}$	20
$\mathcal{H}_m$	topological entropy of ζ_m	32
h	Hausdorff metric on $\mathcal{K}X$	53
$\mathcal{K}X$	set of non-empty compact subsets of X	53
k	Kantorovich metric on $\mathcal{P}(X)$	56
λ_m	inflation multiplier of $\zeta_{m,i}$ and ζ_m	14
λ'_m	algebraic conjugate of λ_m	14
$\ell_{m,k}$	length of exact substitution words in $\mathcal{G}_{m,k}$	20
$\Lambda_{m,i}$	noble means set, generated by the inflation rule $\zeta_{m,i}$	59
Λ_m	generating random noble means set, generated by ζ_m	59
$\mathrm{Lip}(r, X, Y)$	set of Lipschitz functions $X \longrightarrow Y$ with constant $\leqslant$ r	54
$\mathcal{L}_m$	diagonal embedding of $\mathbb{Z}[\lambda_m]$	60
$\mathcal{L}_m^{\circledast}$	Fourier module $\pi_1(\mathcal{L}_m^{*})$	73

$\mathcal{M}(X)$	space of measures on X	10
$\mathrm{Mat}(d, R)$	square $(d \times d)$–matrices over R	2
M_m	substitution matrix of $\zeta_{m,i}$ and ζ_m	13
$M_{m,\ell}$	induced substitution matrix of $\left(\zeta_m\right)_\ell$	39
μ_m	S-invariant probability measure on $\mathbb{X}_m$	49
ν_m	$\{S_t\}$-invariant probability measure on $\mathbb{Y}_m$	70
$\mathcal{P}(X)$	space of probability measures on X	10
$\mathcal{P}_T(X)$	space of T-invariant probability measures on X	47
$\boldsymbol{p}_m$	probability vector for ζ_m	19
$\mathcal{N}_m$	family of noble means substitutions	13
$\mathcal{R}$	family of random noble means substitutions	19
$\mathcal{S}(\mathbb{R}^d)$	Schwartz-space or the space of test functions	9
S	shift map	7
$\mathbb{X}_\vartheta$	two-sided discrete hull, defined by ϑ	7
$\mathbb{X}_m$	two-sided discrete stochastic hull, defined by ζ_m	21
$\mathbb{X}_{m,i}$, $\mathbb{X}'_m$	two-sided discrete hull, defined by $\zeta_{m,i}$	15
$\mathbb{Y}_m$	continuous stochastic hull of ζ_m	68
$\mathbb{Y}_m^{\circledcirc}$	punctured continuous stochastic hull of ζ_m	68
$\mathbb{Y}_{m,i}$	continuous hull of $\zeta_{m,i}$	67
ζ_m	random noble means substitution	19
$\zeta_{m,i}$	noble means substitution	13
$(\zeta_m)_\ell$	ζ_m-induced substitution on $\mathcal{D}_{m,\ell}$	38
$\mathcal{Z}_k(v)$	cylinder set of v at index k	3
$\mathfrak{Z}(\mathbb{X})$	cylinder sets for the product topology of the subshift $\mathbb{X} \subset \mathcal{A}_n^{\mathbb{Z}}$	8

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