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Three-loop Debye mass and effective coupling in thermal QCD

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To my sister, Sanda.

Abstract

We determine the three-loop effective parameters of the dimensionally reduced theory of EQCD as matching coefficients to full QCD. The mass parameter m_{E} is interpreted as the high temperature, perturbative contribution to the Debye screening mass of chromo-electric fields and enters the pressure of QCD at the order of g^7. The effective coupling g_{E} can be used to compute the spatial string tension of QCD. However, we suspect that the effective coupling g_{E} obtains through renormalization contributions from higher order operators that have not yet been taken into account. Therefore our result reflects the (still divergent) contribution from the super-renormalizable EQCD Lagrangian. In addition, we present a new method for computing tensor sum-integrals and provide a generalization to the known computation techniques of spectacles-type sum-integrals.

Contents

Chapter 1

Motivation

The theory of strong interactions is well established for roughly fifty years and its validity has been tested many times [1]. It is known that the underlying theory is Quantum Chromodynamics (QCD), a quantum field theory whose degrees of freedom are massive fermions and massless gluons, both subject to the non-abelian $SU(3)$ symmetry group.

Closely related to the the Yang-Mills theory, which is the underlying theory of the gluonic integrations, is the asymptotic freedom of quarks and gluons [2, 3] in the UV and the confinement of quarks at low energies [4].

Technically, QCD can be handled at high energies with the standard quantum field theory approach of a perturbative weak coupling expansion in terms of the QCD coupling, since it is very small in the energy region mentioned, a direct consequence of asymptotic freedom. This method leads to the famous Feynman-diagram machinery of computing physical observables.

However, at low energies, which are the energies of interest, perturbative expansion breaks down, as the strong coupling indeed becomes strong. Physically, the accessible degrees of freedom are not quarks and gluons anymore, but rather mesons and baryons, whose masses are mostly generated by interactions and merely ($\approx 1\%$) by the constituent quark masses [5].

The question of how the transition from low energy hadronic matter to a state of an almost non-interacting gas of quarks and gluons, a quark-gluon plasma (QGP)[6, 7, 8] occurs, is addressed within the framework of statistical mechanics of quantum fields. On the experimental side it was the heavy ion collision programs performed at LBNL (Berkley) and later on at BNL (RHIC), GSI and nowadays also at LHC, that have boosted the research in the field of thermal QCD. On the theoretical side it was the numerical approach within the framework of field theories discretized on a lattice that have provided first results on QGP and on the QCD phase diagram (for a theoretical review on the matter, see [9]). Even though less advanced as in the zero temperature case, analytical computations within thermal field theory have found a wide use not only in particle physics but also on cosmology related problems [10, 11, 12]. Also, they turn out to be very fruitful as they can approach regions in the phase diagram of QCD that are difficult to access with lattice simulations, such as regions with finite chemical potential [13, 14] or even with a magnetic background [15, 16, 17].

There have been many challenges on both the numerical (via lattice simulations) and the analytical (weak-coupling expansion) side so that even after decades of research we are still in the situation in which only limited temperature and density scales can be addressed with any of the approaches and they hardly overlap. Therefore a permanent check with the complementary method has become a widely accepted procedure.

In quantum field theory the complexity of typical calculations shows a rapid grow with every loop order, such that nowadays, when state of the art computations reach even the five-loop order at zero temperature [18], it has become a standard to rely on computer-algebraic tools. The computational procedure has also standardized: the Feynman diagram generation is followed by group-theoretical algebras and scalarization of the integrals. The typically large number of integrals is reduced to a small set of master integrals that are computed analytically or numerically. The last step represents the technically most demanding task and has boosted the development of integral solving techniques.

Multi-loop calculation techniques in zero-temperature field theory are much more advanced as in the case of thermal field theory as their applicability, hence their demand spans over all particle collision related subjects. Some of the most fruitful integral solving methods and mathematical advances in the field include keywords like: Integration by Parts (IBP) [19], difference equations [20], sector decomposition [21], Mellin Barnes transformations [22], differential equations [23], Harmonic Polylogarithms [24]. An introduction to Feynman integral calculus can be found in [25]. Some of the methods are implemented in software packages like Reduze [26] and FIESTA [27].

Due to the finite temperature, quantum field theories exhibit a different analytical structure; we are confronted not with integrals but rather with so-called sum-integrals. This makes a one-to-one transfer of zero-temperature techniques difficult and even makes their feasibility *à priori* uncertain.

Keeping all these ideas in mind, the present thesis intends to provide yet another building block towards multi-loop computations in high temperature QCD. Precisely, we compute two matching coefficients, m_{E} and g_{E}, of the low temperature effective theory of thermal QCD, namely Electrostatic QCD (EQCD). Besides having the aim of a proof-of-principle of the perturbative expansion that in the zero-temperature case works so well, we also have two direct applications of our result. The effective mass m_{E} enters the QCD pressure at $\mathcal{O}(g^7)$ in the coupling. This is the contribution to the first order beyond the famous non-perturbative term $\propto g^6$. The most direct verification of the convergent nature of the perturbative expansion is -precisely in the spirit of testing analytical results against lattice results- the computation of the spatial string tension σ_s, a non-perturbative quantity defined in the framework of lattice QCD and being the subject of investigation ever since.

In addition, this thesis aims to offer a contribution to multi-loop calculation techniques in thermal field theory; once more, borrowing a method from zero-temperature field theory, we provide an adapted method of computing tensor sum-integrals and we generalize the computation procedure first developed by Arnold and Zhai [28] to a broad class of so-called spectacles-type sum-integrals of mass dimension two and zero.

The thesis is structured as follows: The first chapter gives a short introduction on the basic concepts of thermal field theory and of the theory of QCD in the finite temperature picture as the theory of our investigation. From there, the renormalization program for eliminating ultraviolet (UV) divergences and the resummation program for eliminating infrared (IR) divergences for bosonic degrees of freedom are sketched. Finally, we make some general considerations on multi-scale theories and effective theories as a preparation of the second chapter.

In chapter two we provide a possible way out of the IR-divergence problem within the framework of effective field theories by making use of the scale separation in thermal QCD. We then set the matching coefficients to be determined in the physical context of Debye screening and of the spatial string tension. The actual calculations are performed in the background field gauge, since it considerably simplifies the matching computation. Finally we present the technicalities of Feynman diagram generation and their reduction to a small set of master sum-integrals.

The third chapter represents the main part of the thesis. Here we apply Tarasov's method [29] for tensor reduction to the concrete case of a master sum-integral. Afterwards, the general properties of spectacles-type sum-integrals are presented and demonstrated on a concrete example. With the experience gained we generalize the procedure to a set of arbitrary parameters in the constrain of two and zero mass dimensions. Finally, we provide two more concrete computations of sum-integrals that do not completely obey our previously determined generic rules.

In the last chapter, we give the result on the renormalized effective mass to three-loop order and present the results on the effective coupling. As it turns out, in order to complete the computation, renormalization constants from higher order operators are required. Finally we discuss the future computation on the renormalization constants and present an outlook for the present work.

Chapter 2

Introduction

In the following, we give a short introduction on the theory, in which our work is embedded. While making use of the simple model of a scalar field theory, we point out the technical problems that arise in this context and set the stage for a possible solutions presented in chapter 3.

2.1 Thermodynamics of quantum fields

Quantum field theory at finite temperatures is an extension of statistical quantum mechanics to include special relativity. As it describes the thermodynamical properties of relativistic particles, it finds direct use in problems related to the early Universe where thermal aspects of the Standard Model (SM) [30, 31] become important. Throughout this thesis, natural units are employed, $\hbar = c = k_B = 1$.

As in the non-relativistic case, the central quantity is the partition function, the sum over all possible states of the system symbolically written as [32]:

$$\mathcal{Z} = \mathrm{Tr} e^{-\beta \hat{H}} \quad , \quad \beta \equiv 1/T \,. \tag{2.1}$$

Technically, in the case of quantum mechanics it is possible to find a concrete representation of the partition function in terms of a path integral by making use of the position space $|x\rangle$ and the momentum space representation $|p\rangle$. The extension to fields can be performed, if one considers quantum statistical mechanics as a $0+1$ dimensional quantum field theory and extends the theory to $d+1$ dimensions. In that sense, the operator $\hat{x}(t)$ can be regarded as a field at a fixed space point, $\hat{\phi}(\mathbf{0}, t)$. Thus, the partition function in thermal field theory is:

$$\mathcal{Z} = C \int_{\phi(\beta,\mathbf{x}) = \pm\phi(0,\mathbf{x})} \mathcal{D}\phi \exp\left[-\int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^d\mathbf{x} \mathcal{L}_E(\phi, \partial_\mu \phi) \right] \,, \tag{2.2}$$

where the constant C is infinite but will never play a role in actual computations, as seen later on. As a short hand notation, we employ:

$$\mathcal{Z} = C \int \mathcal{D}\phi e^{-S_E} \,, \quad S_E = \int_x \mathcal{L}_E \,, \quad \int_x \equiv \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^d\mathbf{x} \,. \tag{2.3}$$

The τ -direction is bounded and the temperature T enters the partition function via the upper integration limit. Due to the fact that the fields obey (anti-)periodic boundary conditions if they are (fermionic) bosonic, the time component of the momentum in Fourier space is discreet:

$$P \equiv (p_0, \mathbf{p}) \ , \quad p_0 = \begin{cases} 2\pi n T \ , n \in \mathbb{Z} & \text{for bosons} \\ \pi(2n+1)T \ , n \in \mathbb{Z} & \text{for fermions} \end{cases} \ . \tag{2.4}$$

At this point, the formal equivalence between thermal field theory and the path-integral formulation of quantum field theory at zero temperature becomes clear. By starting from the usual generating functional, it is possible to obtain Eq. (2.2) by simply performing a Wick rotation, $t \to \tau \equiv -it$. This leads to a change of the weight inside the path integral, $i \to (-1)$ and of the metric, from Minkovskian to Euclidean, $g_{\mu\nu} = \mathrm{diag}(1,1,1,1)$.

In the following, we keep the Lagrangian as general as possible. It can be split into a kinetic term quadratic in the fields and an interaction term with higher powers in the fields.

$$\begin{aligned} \mathcal{L}_E = \mathcal{L}_0 & \qquad\qquad + \mathcal{L}_I \\ = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} m^2 \phi^2 & \quad + V(\phi) \ , \quad V(\phi) \propto \lambda \phi^{n \geq 2} \ . \end{aligned} \tag{2.5}$$

In order to introduce the mathematical quantities needed later such as propagators, n-point Green's functions or vertex functions, some definitions are needed. The free expectation value of an observable is denoted with:

$$\langle O \rangle_0 = \frac{C \int \mathcal{D}\phi \ O \ e^{-S_0}}{C \int \mathcal{D}\phi \ e^{-S_0}} \ , \tag{2.6}$$

since only the free Lagrangian is used as a weighting factor and loop corrections are exactly zero. The expectation value of two time-ordered fields is the free propagator of the field (cf. Eq. 4.6 later):

$$D_0(x_1, x_2) \equiv \langle 0 | T\{\phi(x_1)\phi(x_2)\} | 0 \rangle_0 = \frac{C \int \mathcal{D}\phi \ \phi(x_1)\phi(x_2) \ e^{-S_0}}{C \int \mathcal{D}\phi \ e^{-S_0}} = \sum\!\!\!\!\!\!\!\int_P \frac{e^{iP(x-y)}}{P^2 + m^2} \ , \tag{2.7}$$

where the integration measure is defined in Eq. (4.5). In momentum-space the propagator is simply[1]

$$D_0(P) = \frac{1}{P^2 + m^2} \ . \tag{2.8}$$

Next, we define the full n-point Green's function as:

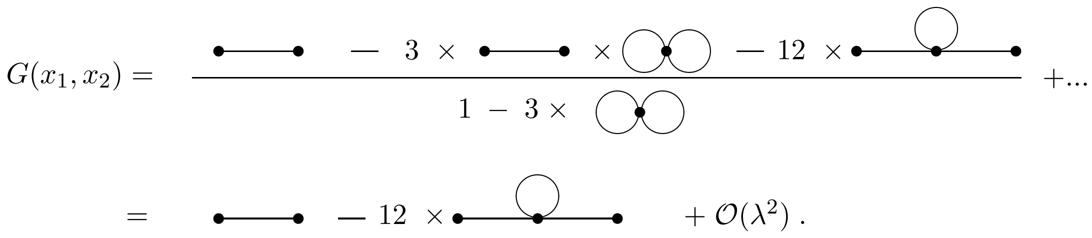
$$G(x_1, ..., x_n) \equiv \langle \phi_1 ... \phi_n \rangle = \frac{1}{\mathcal{Z}} C \int \mathcal{D}\phi \, \phi_1 ... \phi_n e^{-S_E} \ , \quad \phi_i \equiv \phi(x_i) \ . \tag{2.9}$$

In order to compute the Green's function, a Taylor expansion of e^{-S_I} in terms of λ has to be performed, by using the splitting in Eq. (2.5):

$$G(x_1, ..., x_n) = \frac{\int \mathcal{D}\phi \, \phi_1 ... \phi_n e^{-S_0} \sum_{j=0}^{\infty} \frac{(-S_I)^j}{j!}}{\int \mathcal{D}\phi \, e^{-S_E} \sum_{j=0}^{\infty} \frac{(-S_I)^j}{j!}} \ . \tag{2.10}$$

[1] For simplicity, we use the same notation in momentum space as it is always clear from the context which representation is used.

By using Wick's theorem, each new term of the sum generates diagrams according to all the possibilities of contracting the n external fields to the $4 \times j$ internal fields. Due to the numerator, all disconnected diagrams vanish. This is illustrated for the one-loop 2-point function by expanding the denominator as: $\frac{1}{1+x} \approx 1 - x$.



If we modify the partition function, by introducing a source term $J(x)$ as

$$\mathcal{Z}[J] = C \int \mathcal{D}\phi \exp\left[-S_E - \int_x J(x)Q(x)\right] \, , \tag{2.11}$$

we can define the full n-point Green's function in terms of a source derivative:

$$G(x_1, ..., x_n) = \frac{-\delta}{\delta J_1} ... \frac{-\delta}{\delta J_n} W[J=0] \, , \quad J_1 \equiv J(x_1) \, , \tag{2.12}$$

with

$$W[J] \equiv \ln \mathcal{Z}[J] \, . \tag{2.13}$$

By taking the logarithm of $\mathcal{Z}$, the disconnected pieces exactly cancel and only the connected ones remain. Thus, $W[J]$ can be regarded as the generating functional of connected Green's functions.

In the following we perform a Legendre transformation of the form:

$$\Gamma[\bar{\phi}] = W[J] - \int_x J(x)\bar{\phi}(x) \, . \tag{2.14}$$

The new variable $\bar{\phi}$ is the field configuration that minimizes $\Gamma[\bar{\phi}]$ in the limit $J(x) = 0$:

$$\bar{\phi} = -\frac{\delta W[J]}{\delta J(x)} \tag{2.15}$$

as seen from:

$$\frac{\delta \Gamma[\bar{\phi}]}{\delta \bar{\phi}(x)} = \frac{\delta W[J]}{\delta J(x)} \frac{\delta J(x)}{\delta \bar{\phi}(x)} - \frac{\delta J(x)}{\delta \bar{\phi}(x)} \bar{\phi}(x) - J(x) = -J(x) \, . \tag{2.16}$$

By taking the second derivative of Eq. (2.14), we obtain:

$$\frac{\delta^2 \Gamma}{\delta \bar{\phi}^2} = -\frac{\delta J}{\delta \bar{\phi}} = \left[-\frac{\delta \bar{\phi}}{\delta J}\right]^{-1} = -\left[\frac{\delta^2 W}{\delta J^2}\right]^{-1} = -D^{-1} \, , \tag{2.17}$$

where D denotes the full propagator:

$$D(x_1, x_2) \equiv G(x_1, x_2) \, . \tag{2.18}$$

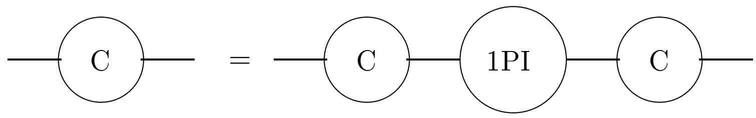


Figure 2.1: Relation between connected and one-particle irreducible two-point functions.

We rewrite Eq. (2.17) as:

$$D \times \left(-\frac{\delta^2\Gamma}{\delta\bar{\phi}^2}\right) \times D = D \ , \tag{2.19}$$

or diagrammatically as in Fig. (2.1).

In conclusion, the vertex functional defined in Eq. (2.14) is the generating functional of one-particle irreducible diagrams. Further, if we define the self-energy Π as:

$$D = \frac{1}{P^2 + m^2 + \Pi(P)} = \frac{1}{D_0^{-1} + \Pi(P)} \ , \tag{2.20}$$

where D_0 is the free propagator, we can relate the self-energy to the two-point vertex function as:

$$\begin{aligned} \Pi(P) &= -P^2 - m^2 - \frac{\delta^2\Gamma}{\delta\bar{\phi}^2} \\ &= -P^2 - m^2 + \left.\frac{\delta^2 \ln \mathcal{Z}[J=0]}{\delta J^2}\right|_{1\mathrm{PI}} \end{aligned} \tag{2.21}$$

Concluding, the self-energy of a field is simply the one-particle irreducible two-point function from which the free propagator has been subtracted. Later on, this will be the starting point of the computation.

Finally, we relate the earlier defined functions to thermodynamical observables by using their definitions from statistical mechanics. In this way, observables such as the free energy, the pressure or the entropy can be obtained:

$$\begin{aligned} f &= -p = \left.\frac{T}{V} \ln \mathcal{Z}[J]\right|_{J(x)=0} \\ s &= -\frac{\partial f}{\partial T} \ . \end{aligned} \tag{2.22}$$

2.2 Path-integral formulation of QCD

So far, we have formulated statistical mechanics in terms of a path-integral of a simple scalar field. In the following, the theory of QCD will be introduced, as a starting point of our calculation. The most important property that the theory of QCD and that of scalars share, is their bosonic nature and therefore the same low energy behavior, which is very different from that of fermionic fields.

Historically, the theory of Quantum Chromodynamics was preceded by Gell-Mann's so-called Eightfold Way, which was an attempt to order the increasing number of newly discovered particles, similar to the previously established $SU(2)$ isospin symmetry of neutrons and protons, in a systematic way.

The proposal to construct the "elementary" particles out of so-called quarks (spin 1/2 and fractional electric charge: $\pm 1/3$, $\pm 2/3$) demanded a new property/charge of the quarks that should take up 3 values in order for the mesons and the baryons to be in concordance with Pauli's exclusion principle[2].

Their mathematical description is grounded on the principle of local gauge invariance of "colored" matter particles that naturally introduces gauge bosons as an intermediating color field.

Consider an n-tuple field in color-space (cf. [33]):

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{N_c} \end{pmatrix} . \tag{2.23}$$

where ϕ may be either a scalar or a spinor field, and N_c is the number of colors.

Next, we construct a generic theory containing these fields, which is Lorentz invariant and invariant under local phase transformations of the fields, that is gauge invariant:

$$\Phi \to \Phi' \equiv V(x)\Phi \Rightarrow \mathcal{L}(\Phi, \partial_\mu \Phi) = \mathcal{L}'(\Phi', \partial_\mu \Phi') \quad , \quad V(x) \equiv e^{iT^a \alpha^a(x)} . \tag{2.24}$$

The $N_c^2 - 1$ matrices $T_a \equiv T^a$ are the generators of the $SU(N_c)$ group under which the Lagrange density is invariant.

In the fundamental representation (T^a is $N_c \times N_c$) we have (together with the vector space spanned by the T^a's) the Lie Algebra:

$$[T^a, T^b] = i f^{abc} T^c , \tag{2.25}$$

with the normalization relation

$$\mathrm{Tr}(T^a T^b) = \frac{\delta^{ab}}{2} . \tag{2.26}$$

"Tr" is the trace of the matrix and f^{abc} are called structure constants and are totally antisymmetric: $f^{abc} = -2i\mathrm{Tr}([T^a, T^b]T^c)$. Another useful representation is the adjoint representation in which the generators T^a are of dimension $(N_c^2 - 1) \times (N_c^2 - 1)$ and:

$$(T_A^b)_{ac} = i f^{abc} \quad , \quad ([T_A^b, T_A^c])_{ae} = i f^{bcd} (T_A^d)_{ae} . \tag{2.27}$$

In this representation, the Casimir quadratic operator is simply the number of colors: $C_A = N_c$.

When allowing independent phase variations of the fields at any space-time point, the derivative term (which is simply the subtraction of the fields at neighboring points) needs to be modified with a scalar quantity that transforms as $U(x,y) \to V(x)U(x,y)V^\dagger(y)$, in order for the derivative to behave properly under phase transformations:

$$n^\mu \partial_\mu \Phi = \lim_{\epsilon \to 0} \frac{\Phi(x + \epsilon n) - \Phi(x)}{\epsilon} \to n^\mu D_\mu \Phi = \lim_{\epsilon \to 0} \frac{\Phi(x + \epsilon n) - U(x + \epsilon n, x)\Phi(x)}{\epsilon} , \tag{2.28}$$

where n^μ is a unit vector and $U(x,y)$ can be expanded in the separation of the two points:

$$U(x + \epsilon n, x) \approx 1 - ig\epsilon n^\mu A_\mu^a T^a \quad , \quad D_\mu \equiv \partial_\mu - igA_\mu^a T^a , \tag{2.29}$$

[2]That is, to allow for the ground state of baryons to exhibit spin 3/2 (e.g. The Δ^{++} baryon).

with g being the coupling constant.

Thus, this new quantity naturally introduces N_c^2-1 vector gauge fields that need to transform as:

$$\begin{aligned} A_\mu^a T^a &\to V(x)\left(A_\mu^a T^a + \frac{i}{g}\partial_\mu\right)V^\dagger(x) \\ &= \left[A_\mu^a + \frac{1}{g}\partial_\mu \alpha^a - f^{abc}\alpha^b A_\mu^c\right]T^a + \mathcal{O}(g^2)\ . \end{aligned} \tag{2.30}$$

so that the Lagrange density containing the covariant derivative remains gauge invariant.

For the theory to be complete, a kinetic term for the newly introduced vector fields is needed. The kinetic term can be obtained constructing a term bilinear in the gauge fields out of the covariant derivative, or using so-called Wilson loops (cf. Chapter 15 of [33]).

Finally, the Lagrangian for the gauge fields, which throughout the thesis will be considered to be the gluonic part of the full QCD Lagrangian, looks like:

$$\mathcal{L}_\mathrm{g} = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F_{\mu\nu} = -\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a\ , \tag{2.31}$$

where the trace is performed in color space and the field strength tensor $F_{\mu\nu} \equiv F_{\mu\nu}^a T^a$ is defined as:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + igf^{abc}A_\mu^b A_\nu^c\ . \tag{2.32}$$

The fermionic part of the thermal QCD Lagrangian contains spinors that solve the Dirac equation in Euclidean metric:

$$(\tilde{\gamma}^\mu \tilde{\partial}_\mu + m)\psi = 0\ , \tag{2.33}$$

where $\tilde{\partial}_0 \equiv \partial_\tau$, $\tilde{\partial}_i = \partial_i$ and $\tilde{\gamma}^\mu$ are the four 4×4 Euclidean gamma matrices. They are four-dimensional objects in Dirac space and anti-commute like Grassmann numbers, $ab=-ba$. The fermionic part of the QCD Lagrangian, constructed to be gauge invariant by substituting the derivative ∂_μ with the covariant derivative D_μ, is:

$$\mathcal{L}_\mathrm{f} = \sum_{f=1}^{N_f}\bar{\psi}_f(\tilde{\gamma}^\mu \tilde{D}_\mu + m)\psi_f\ . \tag{2.34}$$

The sum in Eq. (2.35) is over the fermion flavors N_f and $\bar{\psi} \equiv \psi^\dagger\gamma^0$.

The QCD Lagrangian adds up to:

$$\begin{aligned} \mathcal{L}_\mathrm{QCD} &= \mathcal{L}_\mathrm{f} + \mathcal{L}_\mathrm{g} \\ &= \sum_f^{N_f}\bar{\psi}_f(i\gamma^\mu D_\mu - m)\psi_f - \frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}\ . \end{aligned} \tag{2.35}$$

However, when plugging the Lagrangian (2.35) into the partition function from Eq. (2.2)[3], the quantity is infinite because the integration over the gauge fields runs over all physically equivalent gage configurations. To overcome this problem, the Faddeev-Popov procedure is employed. Integration is restricted only to a gauge-configuration orbit, set by a gauge fixing condition $G(A)=0$ which is chosen to be the generalized Lorentz gauge:

$$G(A) = \partial_\mu A_\mu^a(x) - \omega^a(x)\ , \tag{2.36}$$

[3]The integration measure reads now $\mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi$.

From here the gauge-fixing term in the Lagrangian emerges:

$$\mathcal{L}_{\rm g-f} = -\frac{1}{\xi}\mathrm{Tr}\left[(\partial_\mu A_\mu)^2\right] \ . \tag{2.37}$$

However, this procedure generates a gauge-fixing determinant in the path-integral that explicitly depends on the gauge fields and therefore is expressed as a functional integral over Grassmann fields:

$$\det\left[\frac{\partial_\mu D_\mu}{g}\right] = \int \mathcal{D}c\mathcal{D}\bar{c}\exp\left[-\int_x \bar{c}(\partial_\mu D_\mu)c\right] \ . \tag{2.38}$$

This leads to a term in the Lagrangian containing ghost fields:

$$\mathcal{L}_{\rm ghost} = \partial_\mu \bar{c}^a \partial_\mu c^a + g f^{abc}\partial_\mu \bar{c}^a A_\mu^b c^c \ . \tag{2.39}$$

Finally, the full QCD partition function reads:

$$\begin{aligned}\mathcal{Z} &= C\int_{\rm periodic}\mathcal{D}A\int_{\rm periodic}\mathcal{D}\bar{c}\mathcal{D}c\int_{\rm anti\text{-}periodic}\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \exp\left[-S_{\rm QCD}[A,\bar{\psi},\psi,\bar{c},c]\right] \ , \\ S_{\rm QCD} &= \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2\xi}\left[\partial A^a_\mu\right]^2 + \partial_\mu\bar{c}^a\partial_\mu c^a + g f^{abc}\partial_\mu\bar{c}^a A^b_\mu c^c \ + \sum_f \bar{\psi}_f(\tilde{\gamma}_\mu \tilde{D}_\mu + m)\psi_f \ .\end{aligned} \tag{2.40}$$

Extension to the partition function with a source term $J(x)$ is straightforward.

2.3 Renormalization of ultraviolet divergences

When actually computing physical observables by using structures similar to those in Eq. (2.10), the results are in general infinite due to the large momentum behavior of the integrals (hence ultraviolet divergence). Their divergence is traced back to the fact that the Lagrangian does not contain physical quantities such as physical fields, electric charge or mass, but rather some theoretical (bare) ones (c.f. Ref. [34]).

To overcome this problem, one has to follow three steps. The first step is to regularize the integrals, since technically they are the source of the UV divergences. The second step is to choose some renormalization conditions that set a fixed finite value for the renormalized/physical quantities at a certain energy scale. In the last step, by relating the bare quantities to the renormalized ones, it is possible to absorb all divergences into the renormalization constants of the specific renormalized quantities. Once the renormalization constants are known, all physical quantities are assured to be finite.

In practice, divergences come from structures like:

$$\int_{-\infty}^{\infty} \mathrm{d}^4 p \frac{1}{[p^2+m^2]^2} \ . \tag{2.41}$$

This integral diverges for high enough momentum, as the integrand runs like $1/p$. A straightforward so-called regularization schemes for parameterizing the divergences is the momentum cut-off, in which an upper limit on p^2 is imposed:

$$\int_{-\infty}^{\infty} \mathrm{d}^4 p\frac{1}{[p^2+m^2]^2} = 2\pi^2\int_0^\Lambda \mathrm{d}p\frac{p^3}{[p^2+m^2]^2} = \pi^2\left[\ln\frac{\Lambda^2}{m^2} + \frac{m^2}{\Lambda^2+m^2} + \ln\left(1+\frac{m^2}{\Lambda^2}\right) - 1\right] \ . \tag{2.42}$$

In the final result the momentum cut-off has to be removed, $\Lambda \to \infty$.

A mathematically much more convenient regularization scheme that will be used throughout the thesis, is the so-called dimensional-regularization scheme, in which the dimension of the theory and thus the dimension of the resulting integrals is analytically continued to $d \to d - 2\epsilon$, with $\epsilon > 0$ being a small parameter that is taken to be zero at the and of the calculation. Details on the scheme are to be found for instance in [35]. Since Eq. (2.41) changes its dimensionality to $2 - 2\epsilon$, an arbitrary scale has to be introduced to render its dimension unchanged, thus $\int \to \mu^{2\epsilon} \int$. The divergent integral from Eq. (2.41) becomes:

$$\mu^{2\epsilon} \int_{-\infty}^{\infty} \mathrm{d}^{4-2\epsilon} p \frac{1}{[p^2 + m^2]^2} = \pi^{2-\epsilon} \left(\frac{\mu^2}{m^2} \right)^{\epsilon} \Gamma(\epsilon) = \pi^2 \left[\frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} - \gamma_{\mathrm{E}} - \ln \pi \right] . \tag{2.43}$$

Even though in both equations, (2.42) and (2.43), we encounter the new mass scale within the logarithm, only Λ has the direct physical interpretation of an upper energy scale to which the computation is reliable. As in Eq. (2.43) the divergence comes from $1/\epsilon$ rather than from μ[4], its physical meaning is not obvious from the beginning. However, as it enters in logarithms, $\propto \ln \mu^2/m^2$, their relative contribution to the final result for a fixed energy scale is an indication of the reliability of the result at the given scale[5].

There is a certain freedom in choosing the renormalization constants. Taking for simplicity the scalar field theory, they usually are defined as:

$$\begin{aligned} \phi_B &= \sqrt{Z_\phi} \phi_R \\ \lambda_B &= Z_\lambda \lambda_R \\ m_B^2 &= Z_{m^2} \, m_R^2 \, . \end{aligned} \tag{2.44}$$

Closely related to the regularization scheme is the renormalization scheme. A very useful and the most used scheme is the so-called Minimal Subtraction scheme (MS) [36] and variations thereof, such as the $\overline{\mathrm{MS}}$ -scheme [37] with

$$\mu^2 = \bar{\mu}^2 e^{\gamma_{\mathrm{E}}} / 4\pi \, . \tag{2.45}$$

Since for any modification of the mass scale $\mu^{2\epsilon} \to \mu^{2\epsilon} f(\epsilon)$ the counter-terms remain unchanged, we can write them as:

$$Z = 1 + \sum_{n=1}^{L} \left[\frac{\lambda_R \mu^{-2\epsilon}}{(4\pi)^2} \right]^n \sum_{k=1}^{n} \frac{c_{n,k}}{\epsilon^k} \, , \tag{2.46}$$

with L being the number of loops and $c_{n,k}$ are complex numbers.

The $\overline{\mathrm{MS}}$ procedure is the following. By inserting Eq. (2.44) into the ϕ^4 Lagrangian for instance, it is possible to split it into the part in which all quantities have been replaced by the renormalized ones and a counter-term piece:

$$\begin{aligned} \mathcal{L}_{\mathrm{ren}} = & \frac{1}{2} (\partial_\mu \phi_R)^2 + \frac{1}{2} m_R^2 \phi_R^2 + \frac{\lambda_R \mu^{2\epsilon}}{4} \phi_R^4 \\ & + \frac{1}{2} (Z_\phi - 1)(\partial_\mu \phi_R)^2 + \frac{1}{2} (Z_\phi Z_{m^2} - 1) m_R^2 \phi_R^2 + (Z_\lambda Z_\phi^2 - 1) \frac{\lambda_R \mu^{2\epsilon}}{4} \phi_R^4 \, . \end{aligned} \tag{2.47}$$

[4]The scale μ is kept arbitrary but finite.

[5]For a concrete example on this matter, see chapter 5.

The counter terms are all at least of $\mathcal{O}(\lambda_R)$ (cf. Eq. 2.46) and do not enter tree-level computations, as should be the case.

The coefficients $c_{n,k}$ from Eq. (2.46) are determined by calculating the renormalization conditions with the renormalized Lagrangian, Eq. (2.47), and by absorbing order by order the divergences into the renormalization constants.

If the renormalization constants are known to a given order in λ_R, then any other physical quantity can be computed in this way. However, new interactions with new Feynman rules emerge from the counter-terms. Therefore, this procedure is tedious due to the large number of diagrams that arise.

The second method that will also be used in this thesis is simply to compute quantities with the original Lagrangian containing only bare quantities $\mathcal{L}(\phi_B, m_B)$. In the divergent result these quantities are then replaced by the renormalized ones with Eq. (2.44). In this way a finite result is assured.

In principle, all physical quantities are renormalization prescription independent. However, since in practice the perturbative expansion is truncated at a finite order, the renormalization prescription enters the physical (renormalized) quantities through an arbitrary mass scale (such as $\bar{\mu}$ for $\overline{\text{MS}}$). The equation that describes the change of the renormalized parameters with respect to the change of the mass scale, is called renormalization group equation (RGE). For a single-mass theory and for a mass-independent scheme (such as $\overline{\text{MS}}$) it looks like:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} + \gamma_m(\lambda_R)m_R\frac{\partial}{\partial m_R} - n\gamma(\lambda_R)\right]\Gamma^n_R(p, \lambda_R, m_R, \mu) = 0\ . \tag{2.48}$$

A very important quantity of the previous equation is the so-called beta function $\beta(\lambda_R)$ that describes the change of the coupling with the change of the scale:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\lambda_R = \beta(\lambda_R)\ . \tag{2.49}$$

In general the beta function is expressed as a perturbative expansion in the renormalized coupling:

$$\beta(\lambda_R) = \beta_0\frac{\lambda_R^2}{16\pi^2} + \beta_1\left(\frac{\lambda_R^2}{16\pi^2}\right)^2 + \dots\ . \tag{2.50}$$

The sign of the beta coefficients β_i determine the strength of the coupling at high energies; positive coefficients such as those of quantum electrodynamics make sure that at high energies the coupling strength grows. QCD has negative coefficients and this leads to its famous property of being asymptotically free.

By plugging in the first term on the right hand side (rhs.) of Eq. (2.50) into Eq. (2.49) we obtain

$$\lambda_R(\mu) = -\frac{16\pi^2}{\beta_0}\frac{1}{\ln\frac{\mu}{\mu_0}} \tag{2.51}$$

as a leading order approximation for the running of the coupling with the energy scale. From here, the QCD renormalization is straightforward.

2.4 Resummation of infrared divergences

In the following, we present the infrared problem, which is typical to any Yang-Mills theory. Given its bosonic nature, we compute the free energy density of a scalar field and take the limit $m \to 0$ in the end, in order to illustrate the procedure.

Considering a massive scalar field theory with a ϕ^4 interaction: $V(\phi) = \frac{\lambda}{4}\phi^4$, the naive free energy density is [32]:

$$
\begin{aligned}
f &= -\frac{T}{V}\ln \mathcal{Z} \\
&= f_0(m,T) + \frac{T}{V}\langle S_I \rangle_0 - \frac{T}{2V}\langle S_I^2 \rangle_{0,\text{connected}} + \ldots
\end{aligned}
\tag{2.52}
$$

The definition in Eq. (2.6) for the expectation value and the Taylor expansion of $\ln(1-x) \approx -x - \frac{x^2}{2}...$ have been used in order to generate only connected diagrams.

The term f_0 is the (scalar field version of the) Stefan-Boltzmann law,

$$
f_0(m,T) = -\frac{\pi^2 T^4}{90} + \mathcal{O}(m^2 T^2)\ , \tag{2.53}
$$

and it contains also a divergent factor that can be removed by renormalization. However, this is beyond the purpose of this example.

The first correction to f_0 is:

$$
\begin{aligned}
f_1(m,T) &= \lim_{V\to\infty} \frac{T}{V} \frac{\int \mathcal{D}\phi \int_x \frac{\lambda}{4}\phi(x)^4\ e^{-S_0}}{\int \mathcal{D}\phi\ e^{-S_0}} \\
&= \frac{\lambda}{4} \underbrace{\int_x}_{=\beta V} \underbrace{\lim_{V\to\infty} \frac{\int \mathcal{D}\phi \phi(x)\phi(x)\phi(x)\phi(x)\ e^{-S_0}}{\int \mathcal{D}\phi\ e^{-S_0}}}_{3[\langle \phi(0)\phi(0)\rangle_0]^2} \\
&= \frac{3\lambda}{4}[D_0(0)]^2 = \frac{3\lambda}{4}\left[\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2+m^2} \right]^2
\end{aligned}
\tag{2.54}
$$

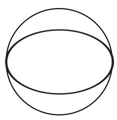
Since the the propagator $D_0(x,y)$ depends only on $x-y$, terms of the form $D_0(x,x)$ are due to translational invariance $D_0(0,0)$. The factor 3 comes from applying Wick's theorem that states that the free expectation value of an n-point function can be expressed in terms of products of two-point functions:

$$
\langle \phi(x_1)\phi(x_2)...\phi(x_{n-1})\phi(x_n)\rangle_0 = \sum_{\text{all comb.}} \langle \phi(x_1)\phi(x_2)\rangle_0 ... \langle \phi(x_{n-1})\phi(x_n)\rangle_0\ . \tag{2.55}
$$

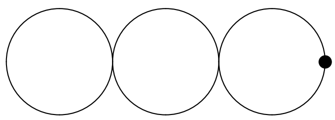
The last term reads:

$$
f_2(m,T) = -\lim_{V\to\infty} \frac{T}{2V}\frac{\lambda^2}{16}\left[\int_{x,y} \langle \phi(x)^4 \phi(y)^4 \rangle_0 - \left(\int_x \langle \phi(x)^4\rangle_0 \right) \right]
$$

$$
= -\frac{\lambda^2}{16}\Bigg[12 \times
$$



$$
+ 36
$$



$$
\Bigg] \tag{2.56}
$$

The dot on the loop denotes an extra power on the propagator, thus $1/[P^2+m^2]^2$.

The first diagram in Eq. (2.56) does not cause divergences in the limit $m \to 0$, therefore it will not enter the discussion. IR divergences in the limit $m \to 0$ are caused only by the second diagram as will be shown shortly.

For that consider the most general one-loop sum-integral:

$$\begin{aligned} J_A(m,T) &\equiv \sum\!\!\!\!\!\!\!\int_P \frac{1}{[P^2+m^2]^A} \\ &= T\int_p \frac{1}{[p^2+m^2]^A} + \sum\!\!\!\!\!\!\!\int_P{}' \frac{1}{[P^2+m^2]^A} \, , \end{aligned} \tag{2.57}$$

where the sum was split into the Matsubara zero mode, $p_0 = 0$, and the non-zero modes. For the zero-mode the integral has a simple expression (cf. Appendix B for details on solving such integrals):

$$J_A^0(m,T) = T\int_p \frac{1}{[p^2+m^2]^A} = \frac{T}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2}-A)}{\Gamma(A)} \frac{1}{[m^2]^{A-\frac{d}{2}}} \, . \tag{2.58}$$

For the non-zero mode part, a Taylor expansion for small m is performed and we obtain a solution in terms of an infinite series as:

$$\begin{aligned} J_A^{'}(m,T) &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{1}{[P^2+m^2]^A} \\ &= \frac{2T}{(4\pi)^{\frac{d}{2}}(2\pi T)^{2A-d}} \sum_{i=1}^{\infty} \left[\frac{-m^2}{(2\pi T)^2}\right]^i \frac{\Gamma(A+i-\frac{d}{2})}{\Gamma(i+1)\Gamma(A)} \zeta(2A+2i-d) \, . \end{aligned} \tag{2.59}$$

Thus, the zero-mode part generates only terms with an odd power in m (as we consider $d = 3$), whereas the non-zero mode part generates only terms with even power in m. Moreover, the non-zero modes part also generates divergences that are removed by renormalization.

So, with the definitions at hand, we can compute the first two corrections to the free energy. The following result excludes the divergent part:

$$\begin{aligned} f_1(m,T) &= \frac{3\lambda}{4}\left[\frac{T^2}{12} - \frac{mT}{4\pi} + \mathcal{O}(m^2)\right]^2 = \frac{3\lambda}{4}\left[\frac{T^4}{144} - \frac{mT^3}{24\pi} + \mathcal{O}(m^2T^2)\right] \\ f_2(m,T) &= -\frac{9\lambda^2}{4}\frac{T^4}{144}\frac{T}{8\pi m} + \mathcal{O}(m) \, . \end{aligned} \tag{2.60}$$

It becomes clear that the first divergence in the limit $m \to 0$ is coming from f_2, more precisely from the following piece of the diagram:

$$\left[\sum\!\!\!\!\!\!\!\int_P{}' \frac{1}{P^2+m^2}\right]^2 \times T\int_p \frac{1}{[p^2+m^2]^2} \, . \tag{2.61}$$

It turns out that such combinations of odd powers of m coming from the zero-mode pieces of the sum-integrals generate IR divergences. So, to the n-th order the diagram that generates the divergence is the product of $n+1$ one-loop diagrams of which n pieces are non-zero modes and one piece is a zero-mode integral with the propagator to the n-th power:

$$\frac{(-1)^{n+1}}{n!}\langle S_I^n\rangle_{0,\mathrm{IR}} \to \frac{(-1)^{n+1}}{n!}\left(\frac{\lambda}{4}\right)^n 6^n 2^{n-1}(n-1)! \left[\frac{T^2}{12}\right]^n T\int_p \frac{1}{[p^2+m^2]^n} \, . \tag{2.62}$$

The term $6^n 2^{n-1}(n-1)!$ is the symmetry factor coming from the $4n$ field contractions:

$$\langle \phi_1 \underbrace{\phi_1\phi_1}_{6} \overbrace{\phi_1\phi_2}^{2(n-1)} \underbrace{\phi_2\phi_2}_{6} \overbrace{\phi_2\phi_3}^{2(n-2)} \phi_3 \ldots \rangle_0 \ , \tag{2.63}$$

and $T/12$ is the leading term from the non-zero modes.

Further, by writing the zero-mode term as a derivative with respect to the mass, we obtain:

$$J_1^0(m,T) = \int_p \frac{1}{p^2+m^2} = \frac{-m}{4\pi} = \frac{\mathrm{d}}{\mathrm{d}m^2}\left(\frac{-m^3}{6\pi}\right) \ . \tag{2.64}$$

The integral in Eq. (2.62) can be re-expressed as:

$$\int_p \frac{1}{[p^2+m^2]^n} = \frac{(-1)^n}{(n-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}m^2}\right)^n \left(\frac{m^3}{6\pi}\right) \ . \tag{2.65}$$

To this point, we have the generic n-loop term that gives rise to a divergence in the limit $m \to 0$. By summing up all the pieces (cf. Fig. (2.2)), we obtain:

$$\sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\lambda T^2}{4}\right)^n \left(\frac{\mathrm{d}}{\mathrm{d}m^2}\right)^n \left(\frac{m^3 T}{6\pi}\right) = -\frac{T}{12\pi}\left(m^2 + \frac{\lambda T^2}{4}\right)^{\frac{3}{2}} \ . \tag{2.66}$$

The lhs. of Eq. (2.66) is simply the Taylor expansion of the rhs. around $\lambda T^2/4$. It becomes clear that, by summing up the leading IR divergent contributions to all orders in λ, we generate a term that permits taking the limit $m \to 0$. It also changes the weak coupling expansion qualitatively, by introducing a term of the form $\lambda^{3/2}$.

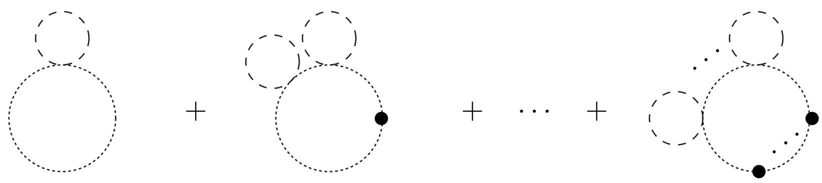


Figure 2.2: Diagrammatic resummation of the infrared divergence. The dotted loops denote zero-mode contribution whereas a black dot means an additional power on the propagator. The dashed lines denote non-zero mode loops.

Thus, the free energy density of a massless scalar field to the resumed one-loop order is:

$$f(T) = -\frac{\pi^2 T^4}{90}\left[1 - \frac{15\lambda}{32\pi^2} + \frac{15\lambda^{3/2}}{16\pi^3} + \mathcal{O}(\lambda^2)\right] \tag{2.67}$$

Higher orders for the free energy density are known up to $\mathcal{O}(\lambda^{5/2}\ln\lambda)$. (c.f. [38]).

Physically, the massless fields acquire an effective thermal mass (similar to the Debye mass in a QED plasma), hence the zero-modes cannot propagate beyond a length proportional to m_{eff}^{-1}. An alternative approach is by starting with a Lagrange density in which a mass term for the Matsubara zero-modes is added to the free part and the same amount is subtracted from the interaction part. A calculation using this technique to four-loop order can be found in [39, 40]. Of course non-zero modes are screened as well, but in the weak coupling expansion their effective mass contribution plays a sub-dominant role ($\lambda T \ll 2\pi T$).

Fermionic fields do not generate IR divergences since their zero-mode contribution is of the form:

$$\int_p \frac{1}{(\pi T)^2 + m^2 + \mathbf{p}^2} \propto \sqrt{(\pi T)^2 + m^2}\ , \tag{2.68}$$

and hence finite in the limit $m \to 0$.

2.5 Effective field theories

As seen previously, even though the original massless scalar field theory contains only one scale coming from the Temperature T, a second scale of the order of $\sqrt{\lambda}T$ emerges through the resummation of the soft modes. This phenomenon is specific to bosonic fields, which are the only fields to exhibit a zero Matsubara mode. Since the fermionic fields are IR safe, their acquired thermal mass is negligible close to the original scale T.

From here, the question arises of how to handle in a systematic way the soft scale and any other scale that might arise at higher loops. At one-loop order, the prescription states to add up only products of one-loop integrals in which the momentum flow factorizes. However, at higher loop orders the situation aggravates, since also diagrams with no factorable momentum flow may contribute and keeping track of all possible contributions becomes cumbersome.

An alternative approach in computing IR safe observables is the effective field theory method. The reasoning is that only at the level of physical observables the dynamical screening of soft modes sets in, but not at the level of the theory itself. Therefore, it is not important which theory works as input in the partition function, as long as the physical outcome is the same.

In order to adopt the effective field theory approach here, the decoupling theorem [41] has to hold. That is, all the effects depending on the higher scale can be absorbed into the redefinition of the renormalized parameters of the effective theory[6]. In addition, the requirement that the energy scales are well separated, $\sqrt{\lambda T} \ll 2\pi T$, should be fulfilled.

Thus, from the starting point of a generic two mass scale theory, $m \ll M$, an effective low-energy field theory can be generated in the spirit of [42]. By aiming at the reorganization of the effective theory operators in terms of $1/M^2$, the effective theory will generate new point interactions by integrating out the heavy scale (cf. Fig. (2.3)).

Moreover, higher order operators containing only light fields and fulfilling the symmetries of the original theory need to be added to the effective Lagrangian. The operators can be classified according to their UV and IR importance. There are marginal operators that are equally important to any scale of the effective field theory, such as the kinetic part of the Lagrangian. Relevant operators are those that contribute only at low energies and have a negligible effect in the UV regime. Such an operator is the effective mass operator ($\propto \phi^2_{\text{light}}$). Finally, irrelevant operators are those that have a vanishing contribution in the low energy regime and are of the order $\mathcal{O}(m^2/M^2)$. Higher loop contributions to the coefficients are to be computed by a perturbative matching of n - point vertex functions with the requirement that they coincide

[6]In fact, this requirement is mandatory for any effective theory and it lies in the very nature of the Standard Model (SM) that the physics at higher scales, such as the Planck scale, is encoded via renormalization.

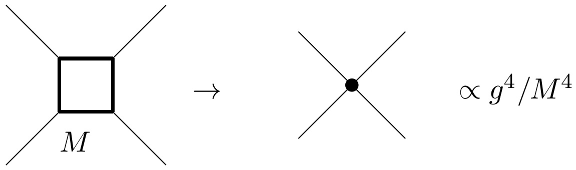


Figure 2.3: Generation of effective vertices by integrating out heavy loops. The original coupling is taken to be $\propto g\phi^2_{\text{heavy}}\phi_{\text{light}}$.

up to a given order in the coupling in the low energy regime. Therefore, renormalization is equally important in defining an effective field theory as well as the Lagrangian itself.

The momentum cutoff regularization introduces a mass-dependent subtraction scheme. Therefore, the counter-terms and with them the β-coefficients of the coupling depend explicitly on the heavy mass of the original theory. In this situation the UV cutoff Λ is consistent with the physical interpretation of the effective theory. It indeed is the upper energy bound at which the effective theory is reliable. Nevertheless, Lorentz and gauge invariance are broken in this case. A more important disadvantage is that beyond tree-level, loop corrections may come with a relative contribution of $\mathcal{O}(1)$, indicating a breakdown of the perturbative expansion.

The more convenient renormalization program is the mass-independent scheme introduced by dimensional regularization of the integrals. The arbitrary scale μ occurs only in logarithms, $\ln(M/\mu)$, and does not introduce explicit powers such as M^2/μ^2. Therefore, truncation of the effective Lagrangian to a given order still renders a loop expansion convergent. Higher order operators can be added gradually according to the aimed accuracy of the matching.

Chapter 3

Setup

In this chapter we first implement the effective theory approach for thermal QCD as a possible solution to its multi-scale nature. Before starting the matching computation for the parameters of the effective theory of QCD, we embed them in the picture of physical quantities of a QCD gas via the Debye mass, the QCD pressure and the spatial string tension. In the remaining part of the chapter we introduce the computational framework, more specifically the background field method, the matching computation, the diagram generation and the reduction of the matching parameters in terms of a few master sum-integrals.

3.1 Electrostatic and Magnetostatic QCD

The particular example of the resummation of the free energy density of a scalar field presented in section 2.4 can be extended to a generic prescription of whether a Yang-Mills theory is IR safe or not. Linde and Gross et al. (Ref. [43, 44]) have argued that for a massless bosonic field theory at n-loop order the most IR sensitive part of the free energy density $f(T)$ is the zero Matsubara mode. If one takes into account the thermal mass generation, so that the bosonic propagator looks like $1/[(2\pi nT)^2 + \mathbf{p}^2 + m^2(T)]$, the IR sensitive part of $f(T)$ is (with g being the strong coupling and $\mathbf{q}_i$ some linear combination of $\mathbf{p}_i$):

$$f(T) \propto (2\pi T)^{n+1}(g^2)^n \int \mathrm{d}^3\mathbf{p}_1 ... \mathrm{d}^3\mathbf{p}_{n+1} \prod_{i=1}^{2n} \frac{1}{\mathbf{q}_i^2 + m^2(T)} \approx g^6 T^4 \left[\frac{g^2 T}{m(T)}\right]^{n-3} . \tag{3.1}$$

In the case of gluons it turns out that the temporal component A_0^a behaves differently from the spatial components A_i^a. The former one exhibits a thermal mass starting from the first loop order. This comes from the fact that the $\Pi_{00}(\mathbf{p}^2)$ component of the self-energy tensor $\Pi_{\mu\nu}$ does not vanish, whereas the spatial components do. Therefore, in the spirit of a QED plasma, the screening of the color-electric field was called Debye screening, with the QCD Debye mass computed first by Shuryak [45]:

$$m_D^2(T) = \frac{g^2 T^2}{3}\left(N_c + \frac{N_f}{2}\right) . \tag{3.2}$$

By plugging this result into Eq. (3.1), we see that the perturbative expansion still is convergent but the thermal mass generates a qualitative change in the perturbative series of thermodynamic quantities in terms of a contribution of the form $(g^2)^{\text{half-integer}}$.

Unlikely to QED, where magnetic fields are not screened due to the nonexistence of magnetic monopoles ($\partial_i B_i = 0$), in QCD a field configuration can be found in which the divergence of the chromo-magnetic field is not zero, since the corresponding equation involves the covariant derivative: $D_i B_i = 0$.

There is strong evidence that the first contribution to a screening mass of the spatial components of the fields is of order $g^2 T$ (cf. section 3.2). By using the argument stated by Linde and Eq. (3.1), it becomes clear that, when trying to go beyond first order all other contributions become equally important as they are of $\mathcal{O}(1)$. In conclusion, thermal effects induce a third scale related to magnetic screening, which is purely non-perturbative.

Being confronted with three scales, a hard scale $\propto 2\pi T$, a soft scale $\propto gT$ and an ultrasoft scale $\propto g^2 T$, a straightforward approach in QCD computations is to isolate each scale and perform the computations independently. Obviously, the non-perturbative scale calls for alternative methods such as lattice QCD. In the end the contributions have to be summed up. A different and successful scheme of integrating out the hard scale is the hard thermal loop framework pioneered in [46, 47] and pushed towards three-loop accuracy [48, 49, 50].

The scale separation has been first done in [51, 52, 53] and extended later to higher order operators in [54]. In these works, the hard scale was directly integrated out generating an effective Lagrangian in which the spatial and the temporal vector field components decouple. Nevertheless, determining the new parameters of the theory beyond leading order is in general difficult as it is necessary to keep track of diagrams with mixed propagators of zero and non-zero modes, very similar to the discussion in section 2.5.

Another method was employed later on in [31, 55, 56] that will be used also here and later for the calculation. The procedure is to construct a general Lagrangian considering the symmetries and properties of the original theory and to perform a matching between them in order to determine the parameters of the new theory.

Since we are interested in generating an effective theory describing the soft modes that do not depend on the temporal coordinate τ, the procedure is called dimensional reduction and the emerging effective field theory is called Electorstatic QCD (EQCD). The symmetries involved are spatial rotations and translations (as Lorentz invariance is broken by the heat bath), the gauge symmetry of the original Lagrangian and the symmetry under $A_0 \to -A_0$. Moreover, the fields do not depend on τ, so its integration will generate a simple T^{-1}-factor in the action.

The gauge transformations of the fields differs for A_i and A_0:

$$\begin{aligned}\tilde{A}_i \to & V\tilde{A}_i V^{-1} + \frac{i}{g} V \partial_i V^{-1} ,\\ \tilde{A}_0 \to & V\tilde{A}_0 V^{-1} .\end{aligned} \tag{3.3}$$

The spatial components transform like gauge fields, whereas the temporal component transforms like a scalar field in the adjoint representation. The fields change as well. At tree-level we have $A_\mu = \sqrt{T}^{-1}\tilde{A}_\mu + \mathcal{O}(g^2)$ and at higher loop order they obtain even gauge-dependent contributions [30, 57]. However, in the following we drop the tilde on the fields to simplify the notation.

As the time derivative is also zero $\partial_0 \to 0$, the gluonic part of the QCD Lagrangian in Eq. (2.31) becomes:

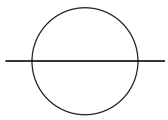
$$\begin{aligned}\mathcal{L}^0_{\mathrm{EQCD}} &= \frac{1}{2}\mathrm{Tr}\{F_{ij}F_{ij}\} + \mathrm{Tr}\{[D_i, A_0][D_i, A_0]\} ,\\ D_i &= \partial_i + ig_{\mathrm{E}} A_i ,\end{aligned} \tag{3.4}$$

where $[A, B] = AB - BA$ is the commutator of A and B. Besides the part coming directly from the original QCD Lagrangian, there are in principle infinitely many other operators that are allowed by symmetries and thus can be included. The effective theory is non-renormalizable. Nevertheless, for the purpose of this work, we restrict ourselves to the operators up to dimension 4. The action of the EQCD theory reads:

$$S_{\mathrm{EQCD}} = \frac{1}{T}\int \mathrm{d}^d\mathbf{x}\left\{\mathcal{L}^0_{\mathrm{EQCD}} + m_{\mathrm{E}}^2\mathrm{Tr}[A_0^2] + \lambda^{(1)}(\mathrm{Tr}[A_0^2])^2 + \lambda^{(2)}\mathrm{Tr}[A_0^4]\right\} . \tag{3.5}$$

The low energy regime of the QCD Lagrangian is described by a pure gauge theory coupled to a massive scalar field in the adjoint representation and lives in a three-dimensional (3d) volume (hence dimensional reduction). Without referring specifically to the finite temperature aspects of the problem, the UV properties of this theory can be drawn.

Truncated up to the operators shown in Eq. (3.5), this theory is super-renormalizable [58], so there is only a finite number of ultraviolet divergent diagrams, specifically with the topology shown in Fig. (3.1):



$$\overset{d=3-2\epsilon}{\longrightarrow} \frac{1}{\epsilon} + \text{finite terms} .$$

Figure 3.1: The topology of the integrals that exhibit UV divergences and hence contribute to the mass counter-term.

They enter the mass term of the A_0-filed to two-loop order, thus it is the only parameter that requires renormalization [30, 59]:

$$\begin{aligned} m_{\mathrm{B}}^2 &= m_{\mathrm{R}}^2(\bar{\mu}_3) + \delta m^2 , \\ \delta m^2 &= 2(N_c^2+1)\frac{1}{(4\pi)^2}\frac{\mu_3^{-4\epsilon}}{4\epsilon}(-g_{\mathrm{E}}^2\lambda C_A + \lambda^2) . \end{aligned} \tag{3.6}$$

Here, the parameter $\lambda^{(2)}$ was set to 0 and $\lambda^{(1)} \equiv \lambda$ because the quartic terms in A_0 in the Lagrangian are independent only for $N_c \geq 4$.

The mass parameter μ_3 is the arbitrary scale introduced through the $\overline{\mathrm{MS}}$ renormalization scheme in the effective theory and it is independent of the mass scale μ of full QCD, which enters the expression in Eq. (3.6) after matching (cf. chapter 5). Since the fields and the effective coupling do not require renormalization, they are renormalization group invariant (e.g. $\mu_3\partial_{\mu_3}g_{\mathrm{E}}^2 = 0$). On dimensional grounds the relation between the effective coupling in 3d and the coupling in 4d is:

$$g_{\mathrm{E}}^2 = T[g^2(\bar{\mu}) - \beta_0\ln(\bar{\mu}/c_{g^2}T)] . \tag{3.7}$$

Hence, the effective coupling depends on the arbitrary $\overline{\mathrm{MS}}$ -scale μ of full QCD only and the coefficients in front of the logarithm are to any loop-order entirely determined by the beta function (cf. Eq. (2.50)) of the QCD coupling. The coefficient c_{g^2} can be determined by a matching as seen later.

In order to describe the thermal effects of the theory, the matching to the full QCD theory of the so far undetermined parameters has to be performed. For EQCD, the hard scale $\approx 2\pi T$

is entirely encoded in the parameters $g_{\rm E}$, $m_{\rm E}$ and λ. This can be seen through their dependence solely on $(g^2)^{\rm integer}$. The most recent results on $m_{\rm E}$ and $g_{\rm E}$ are to be found in [60].

It is possible to go a step further and to integrate out also the soft scale gT, hence to eliminate A_0. The procedure is similar to the QCD-EQCD reduction; the most general Lagrangian that satisfies the properties of the underlying theories is simply a $SU(N_c)$ gauge theory living in three dimensions. It is called Magnetostatic QCD (MQCD):

$$\mathcal{L}_{\rm MQCD} = \frac{1}{2}{\rm Tr}F_{ij}F_{ij} + ... \; , \quad D_i = \partial_i + ig_{\rm M}A_i \; . \tag{3.8}$$

The equality between the EQCD and the MQCD gauge fields is only at tree-level: $\tilde{A}_i = \tilde{\tilde{A}}_i + \mathcal{O}(g)$. Nevertheless, we drop the double tilde for simplicity.

The magnetic coupling can be computed by matching to the theory of EQCD and is expressed as a function of the EQCD parameters $g_{\rm M}(g_{\rm E}, m_{\rm E}, \lambda^{(1)}, \lambda^{(2)}..)$. To tree level the relation is trivial: $g_{\rm M} = g_{\rm E}$. The coupling has been computed to two-loop order in [61].

As the expansion of $g_{\rm M}$ is rather in g and not in g^2 (cf. section 3.3), it becomes clear that both, the hard scale and the soft scale, enter the MQCD theory via its parameters.

In conclusion, one isolates the non-perturbative ultra-soft scale, which is related to the magnetic screening, in a simple three-dimensional gauge theory, whereas the hard and the soft scales are treated analytically through the matching to full QCD.

At this point, it is possible to use this theory in numerical lattice computation in order to extract physical observables [9, 62, 63, 64]. This can be done, if the parameters of the 4d continuous theory of QCD are properly mapped onto the parameters of the equivalent 3d theory discretized on the lattice. This non-trivial task has been extensively addressed in [65, 66, 67, 68, 69].

3.2 Debye screening

The Debye mass is a fundamental property of a plasma. It quantifies to which extent fields are screened due to thermal effects. It is well known that in usual QED plasmas only electric fields are screened ($\nabla\mathbf{B} = 0$), whereas magnetic fields are not. In the non-abelian case magnetic screening is present due to the self-interaction of gluons.

In the case of a non-abelian plasma the situation is much more complicated, as for a long time it was not even clear what the mathematically correct definition of the screening mass is. Taking the straightforward definition of QED, as to what constitutes a screening mass of electric fields, namely to the first loop order this is simply the longitudinal part of the gluon self-energy (polarization tensor) in the static regime ($p_0 = 0$) and in the limit of vanishing spatial momentum [45], we obtain:

$$m_D^2 = \lim_{\mathbf{k}\to 0} \Pi_{00}(p_0 = 0, \mathbf{k}^2) \; . \tag{3.9}$$

The transverse part of the polarization tensor Π_{ij} is zero to this loop-order.

On the other hand, first estimates on the possible magnitude of the magnetic screening mass came from [70, 71]. However, soon it became clear that the screening of chromo-magnetic fields is a purely non-perturbative effect that scales like $m_{\rm magn} \propto g^2T$. Moreover, definition (3.9) does not hold at next-to-leading order for the chromo-electric screening due to the explicit gauge dependence of the electric screening mass [72].

After further investigations on this matter a more sensible definition was proposed, so that the Debye mass is both, gauge independent and infrared safe [73, 74, 75, 76, 77, 78]. The Debye mass is defined in terms of the pole of the static gluon propagator:

$$\mathbf{p}^2 + \Pi(p_0 = 0, \mathbf{p}^2)\big|_{\mathbf{p}^2=-m_D^2} = 0. \tag{3.10}$$

A more subtle definition is found in [79].

At next-to-leading order the computation of the Debye mass requires regularization by explicitly introducing the magnetic screening mass. Therefore, it acquires a non-perturbative contribution from the ultra-soft sector that can be determined only via non-perturbative methods [80, 79]. Some numerical studies even suggest that the image, in which the magnetic screening mass is much smaller than the electric screening holds only at very high temperatures [81].

Thus, to the first non-perturbative terms, the Debye mass is up [75, 81, 82]

$$m_D = m_D^{\rm LO} + \frac{Ng^2T}{4\pi}\ln\frac{m_D^{\rm LO}}{g^2T} + c_N g^2 T + d_{N,N_f} g^3 T + \mathcal{O}(g^4 T) \ , \tag{3.11}$$

where the g^2T term in the logarithm comes precisely from the magnetic screening mass: $m_{\rm magn} = c_{\text{non-pert}} \times g^2T$. The term $m_D^{\rm LO}$ is the leading-order term of the Debye mass, Eq. (3.2). The coefficients c_N and d_{N,N_f} are non-perturbative and are to be determined via lattice QCD [81] or even analytically [79].

Given the definition in Eq. (3.11), the Debye mass can be related to the mass parameter of EQCD $m_{\rm E}$. First of all, $m_{\rm E}$ is a bare parameter that requires renormalization (cf. Eq. (3.6)). The renormalized parameter $m_{\rm E,ren}$ is the high-temperature perturbative contribution to Eq. (3.11), as it contains only the hard scale. Thus, when referring to $m_{\rm E}$ as being the Debye mass, the perturbative contribution thereof should be understood.

Further, the mass parameter $m_{\rm E}$ enters the pressure of QCD at $\mathcal{O}(g^7)$. The investigation of the pressure of a hot gas of quarks and gluons traces back to the seventies. It represents the equation of state of thermal QCD and is therefore essential in understanding the phase diagram of QCD (in particular the high temperature and the finite density [11] region).

Closely related to the previous section, the pressure acquires contributions from all three scales $2\pi T$, gT and g^2T. Starting from the leading order, resummation needs to be done in order to remove infrared divergences. However, the famous Linde problem sets in at three-loop order $\propto \mathcal{O}(g^6)$ rendering a breakdown of the perturbative expansion. Thus, resummation changes the analytic behavior of the pressure:

$$p(T) = T^4\left(c_0 + c_2 g^2 + c_3 g^3 + c_4{}' g^4 \ln(1/g) + c_4 g^4 + c_5 g^5 + c_6{}' g^6 \ln(1/g) + c_6 g^6\right) \ . \tag{3.12}$$

The first three coefficients c_0 [45, 83], c_3 [84] and $c_4{}'$ [85] were computed in the classical picture by tedious diagram resummation. Merely the following two coefficients c_4 [28] and c_5 [86] where computed by using a modified Lagrangian that explicitly includes the EQCD mass parameter $m_{\rm E}$, as pioneered in [39]. Braaten finally introduced the method of effective field theories in the computation of the pressure by individually calculating its contributions coming from the three different scales by using their according effective theories (QCD, EQCD, MQCD). After having determined the parameters of the theories by matching (cf. section 3.5 later) to the desired order, all the contributions can be summed up [55, 56]. Finally, the last perturbative coefficient $c_6{}'$ was computed in [59, 87, 88], whereas the coefficient c_6, which contains both perturbative and non-perturbative contributions was determined only partly up to now [89, 90].

Despite the fact that, in the end infrared divergences can be handled systematically up to the non-perturbative scale, the convergence of the perturbative expansion down to temperatures of interest still remains an open issue [88, 91].

In this spirit, the pressure reads:

$$
\begin{aligned}
p_{\mathrm{QCD}}(T) &\equiv \lim_{V\to\infty}\frac{T}{V}\ln\int DA_\mu^a D\psi D\bar{\psi}\ \exp\left[-S_{\mathrm{QCD}}\right] \\
&= p_E(T) + \lim_{V\to\infty}\frac{T}{V}\ln\int DA_i^a DA_0^a\ \exp\left[-S_{\mathrm{EQCD}}\right] \\
&= p_E(T) + p_M(T) + \lim_{V\to\infty}\frac{T}{V}\ln\int DA_i^a\ \exp\left[-S_{\mathrm{MQCD}}\right] \\
&= p_E(T) + p_M(T) + p_G(T)\ .
\end{aligned}
\tag{3.13}
$$

Eq. (3.13) summarizes the effective theory procedure in computing thermodynamical quantities. This procedure ensures that the final quantity does not require infrared resummation, since this is accounted for through the parameters of the low energy effective theories.

In particular, the soft-scale contribution of the pressure p_M is expressed as an expansion in the EQCD parameters as:

$$
p_M(T) = Tm_{\mathrm{E}}^3\left[b_1 + \frac{g_{\mathrm{E}}^2}{m_{\mathrm{E}}}\left(b_2\ln\frac{\mu}{m_{\mathrm{E}}} + b_2'\right) + b_3\left(\frac{g_{\mathrm{E}}^2}{m_{\mathrm{E}}}\right)^2 + \mathcal{O}(\lambda^{(1)}, \lambda^{(2)}, g_{\mathrm{E}}^6/m_{\mathrm{E}}^3)\right]\ . \tag{3.14}
$$

The next perturbative contribution beyond the last result known in literature is of order $\mathcal{O}(g^7)$. As it has an odd power in g, it is a contribution from the soft scale, thus from the $p_M(T)$ term[1]. Investigating Eq. (3.14) more closely by explicitly plugging in the effective parameters $m_{\mathrm{E}}(g)$ and $g_{\mathrm{E}}(g)$, the g^7 contribution to the pressure comes only from the b_1 coefficient $\propto m_{\mathrm{E}}^3$. Taking the notation from Eqs. (4.1) and (5.2) from [88] and Eq. (5.7), we obtain:

$$
\frac{p_M(T)}{T\mu^{-2\epsilon}} \ni \frac{d_A m_{\mathrm{E}}^3}{12\pi} + \mathcal{O}(\epsilon) = \frac{d_A T^3}{8(4\pi)^5}\left(\frac{\alpha_{\mathrm{E6}}^2}{\sqrt{\alpha_{\mathrm{E4}}}} + 4\sqrt{\alpha_{\mathrm{E4}}}\alpha_{\mathrm{E8}}\right)g^7 + \mathcal{O}(\epsilon)\ . \tag{3.15}
$$

In Eq. (5.13) the coefficient is computed explicitly.

3.3 Spatial string tension

The most important phenomenological application of the effective coupling of EQCD g_{E} is related to the so-called spatial string tension, $\sigma_s(T)$ of QCD. Since it is a non-perturbative quantity, it has been determined with lattice simulations for quite some time [62, 92, 93] and recently even using novel theoretical approaches such as the AdS/CFT duality [94].

It is obtained from a rectangular Wilson loop $W_s(R_1, R_2)$ in the (x_1, x_2)-plane of size $R_1\times R_2$. Given the Wilson loop, the potential V_s is defined as:

$$
V_s(R_1) = -\lim_{R_2\to\infty}\frac{1}{R_2}\ln W_s(R_1, R_2)\ . \tag{3.16}
$$

[1]Since g_{M} contains through its matching to EQCD both hard ($2\pi T$) and soft (gT) scales, a contribution to the pressure at $\mathcal{O}(g^7)$ comes also from $p_G(T)$ and it is multiplied by the non-perturbative constant coming from $\mathcal{O}(g^6)$.

The spatial string tension σ_s is defined as the asymptotic behavior of the potential:

$$\sigma_s = \lim_{R_1\to\infty} \frac{V_s(R_1)}{R_1} \ . \tag{3.17}$$

It has the dimensionality of $[\mathrm{GeV}]^2$ and thus expressed in lattice calculations in terms of a dimensionless function of the normalized temperature [62]:

$$\frac{\sqrt{\sigma_s}}{T} = \phi\left(\frac{T}{T_c}\right) , \tag{3.18}$$

where T_c is the QCD transition temperature ($T_c \approx 150 - 160\mathrm{MeV}$ [95, 96]).

The spatial string tension can also be determined in a three-dimensional pure Yang-Mills theory such as MQCD as repeatedly confirmed [97, 98, 99]. As in this theory the magnetic coupling g_M is the only scale and it has energy dimension one, it is possible to relate the spatial string tension to the coupling by a non-perturbative constant $\sigma_s = c \times g_\mathrm{M}^4$. The constant was computed in [100] for $N_c = 3$

$$\frac{\sqrt{\sigma_s}}{g_\mathrm{M}^2} = 0.553(1) \ . \tag{3.19}$$

This value is remarkably close to the theoretical prediction $\sqrt{\sigma_s}/g_\mathrm{M}^2 = 1/\sqrt{\pi}$.

On the other hand, the magnetic coupling g_M has an analytic expression in terms of both, the QCD coupling g (via g_E) and the QCD scale in the $\overline{\mathrm{MS}}$ scheme $\Lambda_{\overline{\mathrm{MS}}}$. According to Eq. (3.18), the relation between T_c and $\Lambda_{\overline{\mathrm{MS}}}$ is needed for a comparison to lattice results.

On the analytical side, the relation between g_M and g_E is known up to the second loop-order [61]:

$$g_\mathrm{M}^2 = g_\mathrm{E}^2 \left[1 - \frac{1}{48}\frac{g_\mathrm{E}^2 C_A}{\pi m_\mathrm{E}} - \frac{17}{4608}\left(\frac{g_\mathrm{E}^2 C_A}{\pi m_\mathrm{E}}\right)^2\right] , \tag{3.20}$$

where the contributions coming from $\lambda^{(1,2)}$ are omitted [60]:

$$\frac{\delta g_\mathrm{M}^2}{g_\mathrm{E}^2} = -g_\mathrm{E}^2 C_A \frac{2(C_A C_F + 1)\lambda^{(1)} + (6C_F - 1)\lambda^{(2)}}{384(\pi m_\mathrm{E})^2} \tag{3.21}$$

since they contribute, in terms of the 4d coupling only to order $\mathcal{O}(g^6)$ and are numerically insignificant.

However, as lattice computations constantly increase their accuracy and their predictive potential, it is worth looking at higher order corrections on g_E, coming from the matching to QCD. For instance at $T \approx 10T_c$ and using the μ_opt-scale as defined in [58], the last term in (3.20) gives a correction of $\approx 20\%$ relative to the second term and even at $T = 1000T_c$ the correction is still of 14%.

This suggests that both higher order corrections in m_E and g_E may give a noticeable contribution to g_M but also higher order terms in the $(g_\mathrm{E}^2/m_\mathrm{E})$-expansion certainly contribute. A rough estimate on the third expansion term in $g_\mathrm{M}(g_\mathrm{E}, m_\mathrm{E})$, namely $g_\mathrm{E}^6/m_\mathrm{E}^3$ shows that at the order g^5 in $g_\mathrm{M}(g)$ the contributions coming from the coefficients of $m_\mathrm{E}(g)$ and $g_\mathrm{E}(g)$ are $\approx 60\%$ of the coefficient standing in front of $g_\mathrm{E}^6/m_\mathrm{E}^3$. This suggests that at higher orders both, the expansion of g_M as well as the higher orders in g_E and m_E are important.

The task to relate the theoretical prediction from EQCD and MQCD to the lattice computations translates into the determination of $T_c/\Lambda_{\overline{\mathrm{MS}}}$. This has been rigorously done in [60] in two manners: via the zero temperature string tension $\sqrt{\sigma_s}$ and via the so-called Sommer parameter r_0 [101].

For the first method, results for the ratio $T_c/\sqrt{\sigma_s}$ are taken from [102] and combined with the ratio $\Lambda_{\overline{\text{MS}}}/\sqrt{\sigma_s}$ from [103], to obtain the needed relation $T_c/\Lambda_{\overline{\text{MS}}}$.

The second method makes use of the result of $r_0 T_c$ from [104] to combine it with $r_0\Lambda_{\overline{\text{MS}}}$ [105] to again obtain the desired ratio. A discrepancy with the range $T_c/\Lambda_{\overline{\text{MS}}} = 1.15 \ldots 1.25$ was found.

Meanwhile, it is expected that further studies in lattice simulations lead to more reliable results, for instance for the Sommer parameter and the QCD scale [106, 107]. This would definitely narrow the uncertainty of $T_c/\Lambda_{\overline{\text{MS}}}$ on the numerical side and thus justify the need for higher order corrections on the theoretical side.

The remarkable good agreement between the numerical and the theoretical studies given in [60] support this idea:

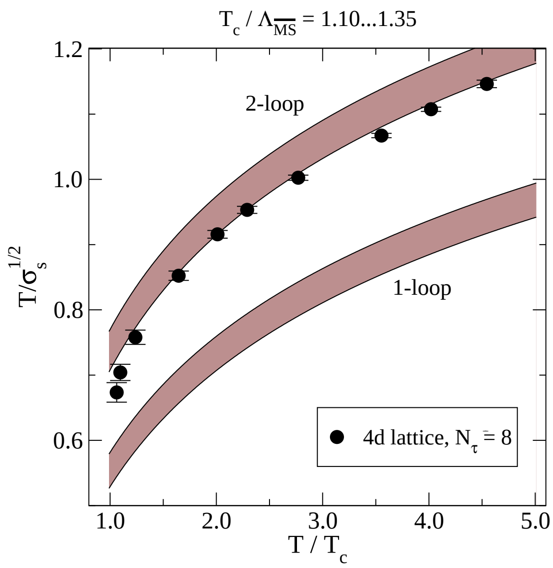


Figure 3.2: The lattice data comes from [62], whereas the theoretical curves represent the one- and two-loop results with a variation of $T_c/\Lambda_{\overline{\text{MS}}} = 1.10$ to $T_c/\Lambda_{\overline{\text{MS}}} = 1.35$.

3.4 Background field method

In general terms, for performing a matching computation (very similar to the computation of renormalization constants), n-point vertex functions need to be computed in both theories, having as external legs the same fields that the coupling multiplies in the Lagrangian. For instance in determining the effective mass parameter m_{E}, the A_0 self-energy needs to be computed. Similarly, in computing the effective coupling it is in principle possible to choose between computing 3-point or 4-point vertex functions with the gauge fields as external lines.

However, by making use of the so-called background field method, first developed in [108], it is possible to achieve quite a simplification: only self-energies need to be computed. In the following, the line of argument from [108] is used to shortly present the properties and benefits of the background field method.

Starting with the effective action from Eq. (2.40)[2], we define a new quantity by shifting the gauge fields only in the gauge action with a so-called background field, $A_\mu^a \to A_\mu^a + B_\mu^a$:

$$
\begin{aligned}
\tilde{\mathcal{Z}}[J,B] &= \int \mathcal{D}A \det\left[\frac{\delta \tilde{G}^a}{\delta\alpha^a}\right] \exp\left[-S[A+B] - \int_x \frac{1}{2\xi}\tilde{G}^a\tilde{G}^a + \int_x JA\right] \\
&= \int \mathcal{D}(A+B) \det\left[\frac{\delta \tilde{G}^a}{\delta\alpha^a}\right] \exp\left[-S[A+B] - \int_x \frac{1}{2\xi}\tilde{G}^a\tilde{G}^a + \int_x J(A+B)\right] e^{-JB} \\
&= \mathcal{Z}[J]e^{-JB}
\end{aligned}
\tag{3.22}
$$

with $JA \equiv J_\mu^a A_\mu^a$.

The term $\delta\tilde{G}^a/\delta\alpha^a$ is the derivative of the gauge fixing term with respect to the gauge transformation:

$$
A_\mu^a \to A_\mu^{a\prime} = A_\mu^a - f^{abc}\alpha^b(A_\mu^c + B_\mu^c) + \frac{1}{g}\partial_\mu\alpha^a \,. \tag{3.23}
$$

With this definition, we have a new generating functional for connected Green's functions:

$$
\tilde{W}[J,B] = \ln\tilde{\mathcal{Z}}[J,B] = W[J] - \int_x JB \,, \tag{3.24}
$$

and by defining

$$
\tilde{A}_\mu^a = \frac{\delta\tilde{W}[J,B]}{\delta J_\mu^a} = \frac{\delta(W[J]-JB)}{\delta J_\mu^a} = \frac{\delta W[j]}{\delta J_\mu^a} - B_\mu^a = \bar{A}_\mu^a - B_\mu^a \,, \tag{3.25}
$$

we perform a Legendre transformation in order to obtain a modified effective action (the generator of 1PI functions):

$$
\begin{aligned}
\tilde{\Gamma}[\tilde{A},B] &= \tilde{W}[J,B] - \int_x J\tilde{A} \\
&= \tilde{W}[J,B] - \int_x J\tilde{A} = W[J] - \int_x JB - \int_x J(\bar{A}-B) \\
&= W[J] - \int_x J\bar{A} = \Gamma[\bar{A}] = \Gamma[\tilde{A}+B] \,.
\end{aligned}
\tag{3.26}
$$

In the end we set $\tilde{A}=0$ and obtain:

$$
\tilde{\Gamma}[0,B] = \Gamma[B] \,. \tag{3.27}
$$

The last equation shows that $\tilde{\Gamma}[0,B]$ contains all 1PI functions generated by $\Gamma[B]$. Since the 1PI functions are generated by computing derivatives with respect to $\tilde{A}$, which here is 0, it means that $\Gamma[B]$ is the sum of all vacuum 1PI graphs in the presence of B.

There are two methods of computing $\tilde{\Gamma}[0,B]$. The first one treats B exactly in such a way that it directly enters the propagators and the vertices in the Feynman rules. This is difficult to perform in practice.

[2]Note that here, the ghost fields do not enter yet as the gauge-fixing determinant is still in the path- integral.

The most convenient method is to treat B perturbatively, that is to split the action in the following way:

$$S[A+B] = S_0[A] + S_{\text{int}}[A,B] \ . \tag{3.28}$$

The part containing the original A-fields is taken to be the free Lagrangian, thus propagators are as usual, whereas the remaining part containing the B-fields represents the interactions. Furthermore, as $\tilde{\Gamma}$ generates only vacuum diagrams, the original A-fields enter the diagrams only as internal lines.

The effective action Γ is in general not gauge-invariant due to the source term J^a_μ. Only for observables computed on the mass shell, $\delta\Gamma/\delta\bar{A} = 0$, the independence on the gauge fixing term is recovered. The advantage of the background field method is that it retains explicit gauge invariance for the background field. A specific gauge fixing term $\tilde{G}^a$ exists that ensures gauge invariance of $\tilde{\Gamma}[0,B]$ with respect to B. In other words, instead of computing 1PI n-point functions in a theory with explicit gauge invariance breaking $\Gamma[B]$, we rather compute 1PI vacuum diagrams in a modified theory $\tilde{\Gamma}[0,B]$ that is still gauge invariant with respect to B. In practice, the B-fields will enter only as external lines in the diagrams, whereas the A-fields will enter only as internal lines.

The gauge fixing term that ensures gauge invariance under a B-field variation is simply the background field covariant derivative of A in the adjoint representation:

$$\tilde{G}^a = \partial_\mu A^a_\mu + g f^{abc} A^b_\mu B^c_\mu = D^a_\mu(B) A^a_\mu \ . \tag{3.29}$$

By performing an adjoint group rotation on the source term and on the original gauge field

$$A^a_\mu \to A^a_\mu - f^{abc}\alpha^b A^c_\mu \ , \quad J^a_\mu \to J^a_\mu - f^{abc}\alpha^b J^c_\mu \ , \tag{3.30}$$

the gauge invariance of $\tilde{\Gamma}[0,B]$ under

$$B^a_\mu \to B^a_\mu - f^{abc}\alpha^b B^c_\mu + \frac{1}{g}\partial_\mu \alpha^a \tag{3.31}$$

can be confirmed.

The explicit gauge invariance of the action $\Gamma[B]$ connects the renormalization constant of the coupling and that of the fields due to the following reasoning. The gauge invariant action needs to take the form of divergent constant$\times(F_{\mu\nu})^2$. According to Eq (2.44) this would be:

$$(F_{\mu\nu})_{\text{ren}} = Z_A^{1/2}[\partial_\mu B^a_\nu - \partial_\nu B^a_\mu + Z_g Z_A^{1/2} i g f^{abc} B^b_\mu B^c_\nu] \ , \tag{3.32}$$

thus imposing:

$$Z_g = Z_A^{-1/2} \ . \tag{3.33}$$

Since the Lagrangian has been changed through the addition of the B-field, also the Feynman rules will change. They can be found in [108, 109].

3.5 Parameter matching

After establishing the framework for performing the computation, the matching procedure is initiated. For that, some preparations are needed. First of all, the computation is performed in the static regime so that external momenta are taken purely spatial, $p_0 = 0$. In fact, the limit for vanishing spatial external momenta $\mathbf{p} \to 0$ is considered as well, as will be motivated later on.

Even though the background field B has no gauge parameter, it is introduced by hand as a cross-check of the validity of the final result: $\xi_{\text{here}} = 1 - \xi_{\text{standard}}$. The gluon propagator becomes:

$$\langle B_\mu^a(p) B_\nu^b(-p) \rangle = \delta_{ab} \left[\frac{\delta_{\mu\nu}}{P^2} - \xi \frac{P_\mu P_\nu}{(P^2)^2} \right] . \tag{3.34}$$

In the following the gluon self-energy is split into temporal and spatial components since we already know that the effective mass is related only to the A_0 fields, hence to Π_{00}. The tensor structure is separated from the self-energy by making use of all symmetric combinations of vectors and rank-two tensors that can generate the same tensor structure as in $\Pi_{\mu\nu}$. These are: $g_{\mu\nu}$, $p_\mu p_\nu$, $P_\mu u_\nu + P_\nu u_\mu$, where $u_\mu = (1, \mathbf{0})$ is the rest frame of the heat bath which is orthogonal to the static external momentum, $u_\mu P_\mu = 0$. The components Π_{0i} and Π_{i0} vanish identically and only three independent component remain:

$$\begin{aligned} \Pi_{00}(\mathbf{p}) &= \Pi_{\text{E}}(\mathbf{p}^2) , \\ \Pi_{ij}(\mathbf{p}) &= \left(\delta_{ij} - \frac{p_i p_j}{\mathbf{p}^2} \right) \Pi_{\text{T}}(\mathbf{p}^2) + \frac{p_i p_j}{\mathbf{p}^2} \Pi_{\text{L}}(\mathbf{p}^2) . \end{aligned} \tag{3.35}$$

It turns out that the longitudinal part Π_{L} vanishes order by order in the loop-expansion.

For the matching computation of the mass parameter m_{E} we use the definition of the Debye mass in Eq. (3.10), disregarding the fact that the actual Debye mass contains non-perturbative terms. We are merely interested in its magnitude $\propto gT$.

On the full QCD, side the Eq. (3.10) looks like:

$$\mathbf{p}^2 + \Pi_{\text{E}}(\mathbf{p}^2) \big|_{\mathbf{p}^2 = -m_D^2} = 0 . \tag{3.36}$$

On the EQCD side, we have:

$$\mathbf{p}^2 + m_{\text{E}}^2 + \Pi_{\text{EQCD}}^{A_0}(\mathbf{p}^2) \Big|_{\mathbf{p}^2 = -m_D^2} = 0 . \tag{3.37}$$

In the following we perform a twofold expansion in terms of the external momentum $\mathbf{p}^2$ and in terms of the coupling, since the self-energies contain at this point the contributions from all orders in g. The expansion in the external momentum is justified by the fact that it is evaluated at the scale of $\mathcal{O}(gT)$, which by definition is a soft scale:

$$\Pi(\mathbf{p}^2) = \sum_{n=1}^{\infty} \Pi_n(0)(g^2)^n + \mathbf{p}^2 \sum_{n=1}^{\infty} \Pi_n{}'(0)(g^2)^n + \ ... \ + (\mathbf{p}^2)^j \sum_{n=1}^{\infty} \Pi_n^{(j)}(0)(g^2)^n . \tag{3.38}$$

On the EQCD side, as all vacuum diagrams are scaleless, they vanish identically in the dimensional regularization scheme that we employ, $\Pi_{\text{EQCD}}(0) = 0$. Thus, from equation (3.37) we are left with the identity:

$$m_{\text{E}}^2 = m_D^2 . \tag{3.39}$$

Eq. (3.36) however needs to be solved iteratively for every loop order. Recall that any $\mathbf{p}^2$ accounts for a $g^2 T^2$ term. Thus, at one-loop we break the Taylor expansion in $\mathbf{p}^2$ at the first term (cf. Eq. (3.38)):

$$m_{\text{E}_{1-\text{loop}}}^2 = \Pi_{\text{E}_{1-\text{loop}}}(0) = \Pi_{\text{E}1}(0) g^2 . \tag{3.40}$$

At two-loop we go to order $\mathbf{p}^2$ in the expansion and substitute the external momentum with Eq. (3.40):

$$\begin{aligned} m^2_{\mathrm{E}_{2-\mathrm{loop}}} &= \Pi_{\mathrm{E}_{2-\mathrm{loop}}}(0) - m^2_{\mathrm{E}_{1-\mathrm{loop}}} \Pi'_{\mathrm{E}_{1-\mathrm{loop}}}(0) \\ &= \Pi_{\mathrm{E}1}(0) g^2 + \left[\Pi_{\mathrm{E}2}(0) - \Pi_{\mathrm{E}1}(0)\Pi'_{\mathrm{E}1}(0)\right] g^4 \ . \end{aligned} \tag{3.41}$$

And finally to three-loop, we have:

$$\begin{aligned} m^2_{\mathrm{E}} \equiv m^2_{\mathrm{E}_{3-\mathrm{loop}}} &= \Pi_{\mathrm{E}_{3-\mathrm{loop}}}(0) - m^2_{\mathrm{E}_{2-\mathrm{loop}}} \Pi'_{\mathrm{E}_{1-\mathrm{loop}}}(0) + (-m^2_{\mathrm{E}_{1-\mathrm{loop}}})^2 \Pi''_{\mathrm{E}_{1-\mathrm{loop}}}(0) \\ &= \Pi_{\mathrm{E}1}(0) g^2 + \left[\Pi_{\mathrm{E}2}(0) - \Pi_{\mathrm{E}1}(0)\Pi'_{\mathrm{E}1}(0)\right] g^4 + \left[\Pi_{\mathrm{E}3}(0) - \Pi_{\mathrm{E}1}(0)\Pi'_{\mathrm{E}2}(0) \right. \\ &\left. -\Pi'_{\mathrm{E}1}(0)\Pi_{\mathrm{E}2}(0) + \Pi_{\mathrm{E}1}(0)(\Pi'_{\mathrm{E}1}(0))^2 + (\Pi_{\mathrm{E}1}(0))^2 \Pi''_{\mathrm{E}1}(0)\right] g^6 \ . \end{aligned} \tag{3.42}$$

In order to compute the effective coupling, merely the self-energy of the gauge field is needed. Consider the general structure of the gauge part of the EQCD Lagrangian:

$$\mathcal{L}_{\mathrm{EQCD}} \approx c_2 B_i^2 + c_3 g B_i^3 + c_4 g^2 B_i^4 \ . \tag{3.43}$$

The coefficient c_2 is the field normalization in the effective theory and can be simply absorbed by a redefinition of the field: $B_i \to \sqrt{c_2} B_i$, thus having:

$$\mathcal{L}_{\mathrm{EQCD}} \approx B_i^2 + c_3 c_2^{-3/2} g B_i^3 + c_4 c_2^{-2} g^2 B_i^4 \ . \tag{3.44}$$

From here we can read off the effective coupling: $g_{\mathrm{E}} = c_3 c_2^{-3/2} g = c_4^{1/2} c_2^{-1} g$. However, due to the background field arguments of gauge symmetry, it is required that the coefficients are identical $c_2 = c_3 = c_4$. Finally, transforming the 3d notation via scaling $B \to \sqrt{T} B$ and comparing it to the full QCD Lagrangian, we obtain:

$$g_{\mathrm{E}} = T^{1/2} c_2^{-1/2} g \ . \tag{3.45}$$

In conclusion, in order to obtain the effective coupling within the framework of the background field theory, it is necessary to compute the field normalization c_2.

This can be done through the effective potential:

$$V(\bar{B}) = \sum_{n=2}^{\infty} \frac{1}{n!} \int_{p_1} \dots \int_{p_n} \bar{B}^{a_1}_{\mu_1}(p_1) \dots \bar{B}^{a_n}_{\mu_n}(p_n) \Gamma^{a_1 \dots a_n}_{\mu_1 \dots \mu_n} \left[\bar{B}(p_1), \dots, \bar{B}(p_n)\right] \ . \tag{3.46}$$

As it is connected to the effective action, the computation can as well be translated into matching the terms of the potential which are quadratic in the fields. Note that to the lowest order the term proportional to the quadratic field is the inverse gluon propagator. It contains two terms, $\propto \delta_{ij}$ and $\propto p_i p_j$. They both will lead to the same result as it should be, but for simplicity we take only the δ_{ij} term:

$$\begin{aligned} \left[\mathbf{p}^2 + \Pi_{\mathrm{T}}(\mathbf{p}^2)\right] B^2_{\mathrm{QCD}} &= [\mathbf{p}^2 + \Pi^{B_i}_{\mathrm{EQCD}}(\mathbf{p}^2)] B^2_{\mathrm{EQCD}} \\ &= [\mathbf{p}^2 + \Pi^{B_i}_{\mathrm{EQCD}}(\mathbf{p}^2)] c_2 B^2_{\mathrm{QCD}} \ . \end{aligned} \tag{3.47}$$

Again, after expanding the self-energies, the EQCD self-energy vanishes identically and we are left with:

$$\begin{aligned} &\mathbf{p}^2 + \Pi_{\mathrm{T}}(0) + \mathbf{p}^2 \Pi'_{\mathrm{T}}(0) + \dots = c_2 \mathbf{p}^2 \\ &\Rightarrow c_2 = 1 + \Pi'_{\mathrm{T}}(0) \ , \end{aligned} \tag{3.48}$$

Note that the equation in the last line is exact. We have picked out only the terms proportional to $\mathbf{p}^2$, therefore a further expansion in $\mathbf{p}^2$ is not needed. The constant term and the higher terms in $\mathbf{p}^2$ are expected to contribute to an effective mass in the EQCD theory, which theoretically would be the thermal mass of the chromo-magnetic fields. But we already know that it is of non-perturbative nature and in particular that it vanishes at order $\mathcal{O}(gT)$. This is indeed the case, since the calculations later will show that $\Pi_{\mathrm{T}}(0) = 0$ at least to three-loop order.

Finally, expanding the self-energy in terms of g^2,

$$\Pi'_{\mathrm{T}}(0) = g^2\Pi'_{\mathrm{T}1}(0) + g^4\Pi'_{\mathrm{T}2}(0) + g^6\Pi'_{\mathrm{T}3}(0) \; , \tag{3.49}$$

we obtain for the effective coupling:

$$\begin{aligned} g_{\mathrm{E}}^2 = Tg^2 \Big\{ & 1 - g^2\Pi'_{\mathrm{T}1} + g^4 \left[\left(\Pi'_{\mathrm{T}1}(0)\right)^2 - \Pi'_{\mathrm{T}2}(0) \right] \\ & - g^6 \left[\left(\Pi'_{\mathrm{T}1}(0)\right)^3 - 2\Pi'_{\mathrm{T}1}(0)\Pi'_{\mathrm{T}2}(0) + \Pi'_{\mathrm{T}3}(0) \right] \Big\} \; . \end{aligned} \tag{3.50}$$

3.6 Automatized sum-integral reduction

The remaining task is to compute the QCD gluon self-energy to three-loop order. The fermion masses are neglected throughout the calculation, as their contribution is sub-leading.

At this order in the coupling approximately 500 Feynman diagrams should be generated. As the encountered task is tremendous, a computer-algebraic approach is needed. The following project builds upon a two-loop calculation [60] and its extension is described in detail in [109, 110]. Here, merely a summary is given.

The diagrams are generated with QGRAF [111] and further manipulated with FORM [112] and FERMAT [113]. The preparation of the generated diagrams consists of decoupling the tensor structures (scalarization), decoupling the external momentum (Taylor expansion), the color sums of the $SU(N)$-algebra and performing the traces over gamma matrices. The $\mathcal{O}(10^7)$ generated sum-integrals can be parametrized as:

$$\bar{M}^{s_7 s_8 s_9}_{s_1 s_2 s_3 s_4 s_5 s_6; c_1 c_2 c_3} = \sum\!\!\!\!\!\!\!\int_{PQR} \frac{p_0^{s_7} q_0^{s_8} r_0^{s_9}}{[P^2]^{s_1}[Q^2]^{s_2}[R^2]^{s_3}[(P+Q)^2]^{s_4}[(P+R)^2]^{s_5}[(Q-R)^2]^{s_6}} \; , \tag{3.51}$$

where the fermion signature is encoded as: $P^2 = [(2n + c_i)\pi T]^2 + \mathbf{p}^2$, with $c_i = 0(1)$ for bosons(fermions). As later on only pure bosonic sum-integrals are used, we adopt the simplification

$$\bar{M}^{s_7 s_8 s_9}_{s_1 s_2 s_3 s_4 s_5 s_6; 000} \equiv M^{s_7 s_8 s_9}_{s_1 s_2 s_3 s_4 s_5 s_6} \tag{3.52}$$

The non-trivial topologies are shown in Fig. (3.3):

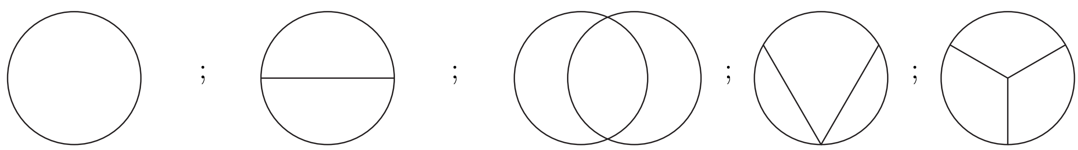


Figure 3.3: Non-trivial topologies at one-, two-, and three-loop order. The two-loop sum-integral is called sunset-type. The three-loop ones are of basketball-, spectacles-, and mercedes-type.

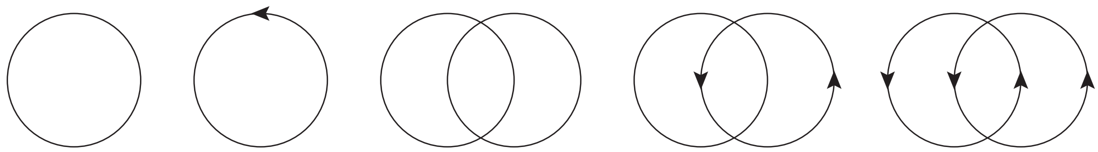


Figure 3.4: Fermion signatures of the one- and and three-loop order master sum-integrals. There arrow defines a fermion propagator and the simple lines define a bosonic one.

The essential task is to reduce the resulting sum-integrals to a small number of master integrals [114]. This has been done by implementing Laporta's algorithm [20] of Integration by Parts (IBP) [19].

The IBP procedure generates algebraic relations between master sum-integrals of different topologies and different exponent parameters by using the d-dimensional divergence theorem [115, 116] (Gauss' law):

$$\sum\!\!\!\!\!\!\!\int_{PQR} \partial_{\mathbf{s}} \left[\mathbf{t} \frac{p_0^{s_7} q_0^{s_8} r_0^{s_9}}{[P^2]^{s_1}[Q^2]^{s_2}[R^2]^{s_3}[(P+Q)^2]^{s_4}[(P+R)^2]^{s_5}[(Q-R)^2]^{s_6}} \right] = 0 \,, \tag{3.53}$$

where $\mathbf{s}$ and $\mathbf{t}$ are linear combinations of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$.

Additional relations among the sum-integrals are generated by performing momentum shifts. These shifts reduce also the fermion signatures to a total number of three (cf. Fig. (3.4)). The so generated under-determined system of equations is solved by using a so-called lexicographic ordering in order to express the most "difficult" sum-integrals in terms of the most "simple" ones. The "simplicity" of a sum-integral depends in general terms on the power of its propagators.

The essential difference to integrals encountered in zero-temperature physics is the fact that here the momentum derivatives specific to the IBP algorithm act only on spatial components, leaving the Matsubara modes untouched [116] (cf. appendix D).

The IBP reduction generates one- and three-loop master sum-integrals. The general structure of the self-energy is therefore:

$$\Pi_3 = \sum_j a_j A_j + \sum_j b_j B_j \,, \tag{3.54}$$

where $A_l = (\text{one-loop})^3$ are products of sum-integrals of the first two types in Fig. (3.4) and B_j are sum-integrals of the last three types in Fig. (3.4). The coefficients a_j and b_j are ratios of polynomial in $d = 3 - 2\epsilon$.

As will be extensively presented in the following chapter, state of the art techniques for sum-integral calculations offer exact analytic solutions only for the one-loop and (via IBP reduction) for the two-loop cases. Methods for solving basketball- and spectacles-type sum-integrals are based on an extensive procedure of subtraction of divergent parts and a numerical calculation of the finite remainder. It is rather a case-by-case analysis that permits computation only up to the constant term. Unfortunately, the IBP reduction generated terms that diverge in the limit $\epsilon \to 0$ as $1/\epsilon$ in the case of $\Pi_{\mathrm{E}3}$ and as $1/\epsilon^2$ for $\Pi_{\mathrm{T}3}$. Therefore, a change of basis is required in order to proceed with the sum-integral evaluation.

3.7 Basis transformation

The basis transformation is performed only for the gluonic part of the self-energy, thus permitting us to solve only for the pure gluonic case.

The task of finding a suitable basis of master sum-integrals that do not have divergent pre-factors consists in reverse engineering the IBP reduction process. This task is demanding in two ways. First of all, there is no prescription to trace the algorithm back and this translates into a manual search for a suitable basis. From here the second difficulty arises, namely to find the balance between finite pre-factors and yet simple enough master sum-integrals that can be computed with today's techniques.

There is no doubt that the possible choices are numerous, however, we have orientated our search to find sum-integrals that have already been computed, or at least that are parametrically close to the known sum-integrals.

Starting with Eqs. (C.14) and (C.15) from [110], which represent the $\Pi_{\mathrm{E}3}(0)$ and the $\Pi'_{\mathrm{T}3}(0)$ contributions and by using the IBP relations from appendix F provided by Jan Möller [117], we obtain the following expressions:

$$\begin{aligned} C_A^{-3}\Pi_{\mathrm{E}3} = &-\frac{(d-7)(d-3)(d-1)^2}{2}M^{000}_{31111-2} - \frac{(d-3)(d-1)^2(7d-13)}{4}V^{000}_{111110} \\ &- 8(d-4)(d-3)(d-1)^2 V^{020}_{211110} + \frac{720-13912d+35443d^2-34716d^3}{30(d-7)d}[I^0_1]^2 I^0_3 \\ &+ \frac{15515d^4-3440d^5+417d^6-28d^7+d^8}{30(d-7)d}[I^0_1]^2 I^0_3 \\ &+ \frac{3024-48076d+168800d^2-261896d^3+214359d^4-99892d^5}{36(d-7)(d-5)(d-2)d} I^0_1[I^0_2]^2 \\ &+ \frac{28027d^6-4824d^7+509d^8-32d^9+d^10}{36(d-7)(d-5)(d-2)d} I^0_1[I^0_2]^2 \end{aligned} \tag{3.55}$$

and

$$\begin{aligned} C_A^{-3}\Pi'_{\mathrm{T}3} = &r_1(d)M^{000}_{121110} + r_2(d)M^{000}_{211110} + r_3(d)M^{002}_{221110} + r_4(d)M^{020}_{311110} + r_5(d)M^{022}_{411110} \\ &+ r_6(d)M^{000}_{310011} + r_7(d)M^{000}_{114000} + r_8(d)M^{000}_{123000} + r_9(d)M^{000}_{222000}\,, \end{aligned} \tag{3.56}$$

with

$$\begin{aligned} r_1(d) &= \frac{107662-196843d+138960d^2-48945d^3+9198d^4-837d^5+20d^6+d^7}{8d(d-5)(d-2)(d-1)}\,, \\ r_2(d) &= \frac{94896-215472d+201560d^2-101965d^3+30585d^4-5566d^5+606d^6-37d^7+d^8}{8d(d-5)(d-2)(d-1)}\,, \\ r_3(d) &= \frac{-62-717d+876d^2-330d^3+42d^4-d^5}{d(d-2)}\,, \\ r_4(d) &= \frac{1440-7876d+7801d^2-3004d^3+526d^4-40d^5+d^6}{3d(d-2)}\,, \\ r_5(d) &= \frac{4(-186+65d+37d^2-13d^3+d^4)(d-6)}{d(d-2)}\,, \end{aligned}$$

$$r_6(d) = \frac{(d-31)(d-1)^2}{4d}\,,$$
$$r_7(d) = \frac{-25568+22382d-3253d^2-1932d^3+806d^4-122d^5+7d^6}{2d(d-9)(d-2)}\,,$$
$$r_8(d) = \frac{110760-151302d+74899d^2-11395d^3-1654d^4+632d^5-53d^6+d^7}{3d(d-9)(d-7)(d-2)}\,,$$
$$r_9(d) = \frac{964718-2366265d+2451867d^2-1335353d^3}{24d(d-7)(d-5)^2(d-2)^2(d-1)}\,, + \frac{397943d^4-61043d^5+3225d^6+229d^7-25d^8}{24d(d-7)(d-5)^2(d-2)^2(d-1)}\,. \tag{3.57}$$

Note that the sum-integrals of the form M^{000}_{abc000} are products of one-loop tadpoles and therefore known analytically (cf. Eq. (B.3)). The sum-integrals M^{000}_{111110} and M^{000}_{310011} have already been calculated in [40] and [118], respectively.

Finally, we present the two matching coefficients to three-loop order in d dimensions. Note the gauge independent result.

$$m_{\mathrm{E}}^2 = \frac{g^2T^2I_1^0C_A}{(d-1)^2}\left[1+g^2C_A\frac{(d^2-11d+46)I_2^0}{6}+g^4C_A^2\left(r_{m,1}(d)I_1^0I_3^0+r_{m,2}(d)I_1^0[I_2^0]^2 - \frac{(d-3)(d-7)M^{000}_{31111-2}}{2I_1^0}-\frac{(d-3)(7d-13)M^{000}_{111110}}{4I_1^0}-\frac{8(d-3)(d-4)M^{020}_{211110}}{I_1^0}\right)\right] \tag{3.58}$$

with

$$r_{m,1}(d) = \frac{720-12472d+9779d^2-2686d^3+364d^4-26d^5+d^6}{30d(d-7)}\,,$$
$$r_{m,2}(d) = \frac{3024-42028d+81720d^2-56428d^3+19783d^4-3898d^5+448d^6-30d^7+d^8}{36d(d-2)(d-5)(d-7)} \tag{3.59}$$

and

$$\begin{aligned} g_{\mathrm{E}}^2 = g^2T\Bigg[&1-g^2C_A\frac{d-25}{6}I_2^0-g^4C_A^2\left(\frac{(d-1)^2(d^2-31d+144)I_1^0I_3^0}{3d(d-7)}\right.\\ &\left.+\frac{(d-25)^2[I_2^0]^2}{36}+\frac{72(d-3)(d-4)(4d^2-21d-7)[I_2^0]^2}{36d(d-2)(d-5)(d-7)}\right)+g^3C_A^3\Bigg(-r_1(d)M^{000}_{121110}\\ &-r_2(d)M^{000}_{211110}-r_3(d)M^{002}_{221110}-r_4(d)M^{020}_{311110}-r_5(d)M^{022}_{411110}-r_6(d)M^{000}_{310011}\\ &-r_7(d)[I_1^0]^2I_4^0-\frac{(d-1)^2(d-25)(d^2-31d+144)+9d(d-7)r_8(d)}{9d(d-7)}I_1^0I_2^0I_3^0\\ &+\left(\frac{144(d-3)(d-4)(d-25)(4d^2-21d-7)}{216d(d-2)(d-5)(d-7)}-\frac{(d-25)^3}{216}-r_9(d)\right)[I_2^0]^3\Bigg)\Bigg]\,. \end{aligned} \tag{3.60}$$

The non-trivial sum-integrals in Eq. (3.55) are multiplied by a factor $(d-3)$, meaning that only the divergent pieces of the sum-integrals have to be determined. The remaining task is to compute the 7 yet unknown non-trivial master sum-integrals, $M^{000}_{31111-2}$, V^{020}_{211110}, M^{000}_{121110}, M^{000}_{211110}, M^{002}_{221110}, M^{020}_{311110} and M^{022}_{411110}.

Chapter 4

Master sum-integrals

This is the main part of the work and it deals with techniques of solving 3-loop sum-integrals for so-called spectacles type (cf. Fig (4.42)). The main ideas are based on the paper by Arnold and Zhai, [28] and they have been extended to the spectacles type topology. In addition, we adopt a technique for manipulating tensor sum-integrals, originally developed for zero temperature integrals, in Ref. [29]. This method turns out to be very fruitful since the standard technique for tensor structure manipulation was shown to lead outside the usual classes of sum-integrals.

The main feature that sets sum-integrals apart from integrals encountered in zero temperature field theory is the sum over Matsubara modes. There are several complications that come along with this new analytic structure. First of all, the summation over the temporal component of the vectors breaks the rotational invariance of the integrand. Moreover, these temporal components act like masses for the space-like ones in the propagators, such that, for a l-loop sum-integral this fact translates into an l-scale problem. Assuming that by any given technique it is possible to give a (numerical) result (ϵ-expansion) in terms of these l scales, still the summation over these "masses" has to be performed. Since the mass dimension of sum-integrals is usually 0 (for coupling matching), 2 (for mass matching) or 4 (for pressure/free energy computations), the summations over the Matsubara modes typically necessitate regularization[1].

The above mentioned particularities of sum-integrals make it difficult to automatize their evaluation. Up to this point, the methods presented in this chapter, are state of the art and are based in principle on a case by case analysis of the sum-integrals involved.

This chapter is structured as follows: First, Tarasov's method for dealing with tensor integrals is presented and applied to our particular case of the master sum-integral $M^{000}_{31111-2}$. Afterwards, we present the general properties of spectacles type sum-integrals and their splitting in order to make them accessible for the succeeding computation. We then demonstrate the solving techniques on the simple example of M^{020}_{211110}. A detailed presentation of this computational method is also to be found in Refs. [38, 40, 118, 119]. With the experience gained from this particular case, we proceed to the generalization of the method and pave the way for the semi-automatized computation of (almost all) remaining master sum-integrals. All the results on the sum-integrals can be found in section 4.3, in subsections 4.5.1 and 4.5.2 and in appendix A.

[1] Similar to the zeta function regularization: $\sum_{n=1}^{\infty} n = \zeta(-1) = -1/12$.

Notations and conventions

In the following we slightly modify the notation convention the generic sum-integral in Eq. (3.51) like:

$$M^{0s_6s_7}_{s_1s_2s_3s_4s_50} \equiv V(d; s_1s_2s_3s_4s_5; s_6s_7) \;, \tag{4.1}$$

and in particular:

$$M^{000}_{31111-2} \equiv M_{3,-2} \tag{4.2}$$

d	parameters	$\mathcal{V}_i$	parameters	$\mathcal{Z}_i$
3	21111;20	2	12111;02	1
	31111;22	3	12111;20	2
			12211;22	3
			12121;22	4
5	31122;11	4		
7	32222;00	5	23222;00	5
	52211;00	6	23231;00	6
	42221;00	7	23321;00	7
	43211;00	8		
	33221;00	9		
	33212;00	10		
	33311;00	11		
3	12111;00	12	12111;00	8
	21111;00	13	12121;02	9
	22111;02	14	12121;20	10
	31111;20	15	12211;02	11
	41111;22	16	12221;22	12
			13111;02	13
			02221;02	14
			03121;02	15

Figure 4.1: Convention for denoting the sum-integrals and their finite parts

In order to ensure a smooth reading, we number the sum-integrals, as well as the zero-mode sum-integrals as shown in Fig. (4.1). The same convention will be used to denote the numerical, finite pieces of each sum-integral, as in:

$$\begin{aligned} V(d; s_1s_2s_3s_4s_5; s_6s_7) &= V_i \\ V^{\mathrm{f},\#}(d; s_1s_2s_3s_4s_5; s_6s_7) &= \frac{T^{3d+3-2s_{12345}+s_{67}}}{(4\pi)^{2s_{12345}-s_{67}-\frac{3}{2}(d+1)}}\mathcal{V}_{i,\#} \;, \end{aligned} \tag{4.3}$$

with the convention:

$$s_{123\ldots} \equiv s_1 + s_2 + s_3 + \ldots \;. \tag{4.4}$$

The sum-integral measure is defined as:

$$\sum\!\!\!\!\!\!\!\int_P \equiv T \sum_{p_0=-\infty}^{\infty} \int_p \qquad \text{with} \qquad \int_p \equiv \mu^{2\epsilon} \int_{-\infty}^{\infty} \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \;. \tag{4.5}$$

This measure is split into:

$$
\begin{aligned}
\sum\!\!\!\!\!\!\!\int_P &= \sum\!\!\!\!\!\!\!\int_P{}' + \sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \\
&\equiv \sum_{p_0 \neq 0} \int_p + T \int_p \ ,
\end{aligned}
\tag{4.6}
$$

the parts denoting the omission of the $p_0 = 0$ Matsubara mode and the corresponding remainder.

In addition, we have the integral measure:

$$
\int_P \equiv \mu^{2\epsilon} \int_{-\infty}^{\infty} \frac{\mathrm{d}^{d+1}\mathbf{P}}{(2\pi)^{d+1}} \ , \tag{4.7}
$$

which is the integral measure defined in Eq. (4.5), with the additional shift $d \to d+1$.

4.1 Taming tensor structures

This section follows the presentation given in Ref. [120].

4.1.1 Reduction of 3-loop massive tensor integrals in Euclidean metric

In the following, the technique based on Ref. [29] is used to reduce tensor sum-integrals typically encountered in thermal field theory. We treat only a particular case of spectacles type integrals, needed in the calculation for $M_{3,-2}$. A more generalized approach can be found in [120].

First, the 3-loop massive spectacles-type integral with Euclidean metric is defined as (c.f. Fig. (4.2)):

$$
S^d_{\nu_1\nu_2\nu_3\nu_4\nu_5} \equiv \int_{pqr} \frac{1}{(p^2+m_1^2)^{\nu_1}(q^2+m_2^2)^{\nu_2}(r^2+m_3^2)^{\nu_3}((p+q)^2+m_4^2)^{\nu_4}((p+r)^2+m_5^2)^{\nu_5}} \ . \tag{4.8}
$$

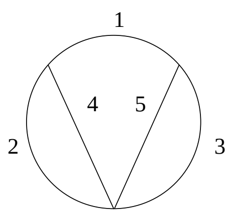


Figure 4.2: The generalized massive spectacles-type integral.

Using the parametric representation via a Laplace transformation

$$
\frac{1}{(p^2+m^2)^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}\alpha \, \alpha^{\nu-1} e^{-\alpha(p^2+m^2)} \ , \tag{4.9}
$$

we obtain

$$
S^d_{\nu_1\nu_2\nu_3\nu_4\nu_5} = C \int_{pqr} \int_0^\infty \prod_{i=1}^{5} \left(\mathrm{d}\alpha_i \alpha_i^{\nu_i - 1} \right) e^{-\alpha_1(p^2+m_1^2)} \cdot \ldots \cdot e^{-\alpha_5((p+r)^2+m_5^2)} \ , \tag{4.10}
$$

with $C = \prod_{i=1}^{5} \Gamma(\nu_i)^{-1}$.

The rearrangement of the exponents in order to obtain Gaussian integrands in the momenta and their subsequent integration lead to:

$$S^d = \frac{C}{\left[(4\pi)^{\frac{d}{2}}\right]^3} \int_0^\infty \prod_{i=1}^{5} \mathrm{d}\alpha_i \alpha_i^{\nu_i - 1} \frac{1}{[D(\alpha)]^{\frac{d}{2}}} e^{-\sum_{i=1}^{5} \alpha_i m_i^2} , \tag{4.11}$$

where

$$D(\alpha) = \alpha_1(\alpha_2 + \alpha_4)(\alpha_3 + \alpha_5) + (\alpha_3 + \alpha_5)\alpha_2\alpha_4 + (\alpha_2 + \alpha_4)\alpha_3\alpha_5 , \tag{4.12}$$

is the so-called Symanzik polynomial and can be obtained from graph theory (c.f. [121]).

This particular representation of loop integrals is useful since the dimension is encoded only as the exponent of the $D(\alpha)$ polynomial, besides some unimportant pre-factors.

Two types of tensor integrals needed in further calculations are defined as:

$$T^d_{1\,\mu\nu}(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) = \frac{\mathrm{d}^2}{\mathrm{d}a_{1\mu} \mathrm{d}a_{2\nu}} \left(\tilde{S}^d_{\nu_1\nu_2\nu_3\nu_4\nu_5} \right)_{\substack{a_1=0 \\ a_2=0}} , \tag{4.13}$$

$$T^d_{2\,\mu\nu\rho\sigma}(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) = \frac{\mathrm{d}^4}{\mathrm{d}a_{1\mu} \mathrm{d}a_{1\nu} \mathrm{d}a_{2\rho} \mathrm{d}a_{2\sigma}} \left(\tilde{S}^d_{\nu_1\nu_2\nu_3\nu_4\nu_5} \right)_{\substack{a_1=0 \\ a_2=0}} , \tag{4.14}$$

with

$$\tilde{S}^d_{\nu_1\nu_2\nu_3\nu_4\nu_5} \equiv \int_{pqr} \frac{e^{a_1 \cdot q} e^{a_2 \cdot r}}{(p^2 + m_1^2)^{\nu_1} (q^2 + m_2^2)^{\nu_2} (r^2 + m_3^2)^{\nu_3} ((p+q)^2 + m_4^2)^{\nu_4} ((p+r)^2 + m_5^2)^{\nu_5}} . \tag{4.15}$$

Using the α-parameterization, Eq.(4.15) becomes:

$$\tilde{S}^d = \frac{C}{\left[(4\pi)^{\frac{d}{2}}\right]^3} \int_0^\infty \prod_{i=1}^{5} \mathrm{d}\alpha_i \alpha_i^{\nu_i - 1} \frac{1}{[D(\alpha)]^{\frac{d}{2}}} e^{\frac{1}{4D(\alpha)}\left(\beta_1 a_1^2 + \beta_2 a_2^2 + 2\beta_3 a_1 \cdot a_2\right)} e^{-\sum_{i=1}^{5} \alpha_i m_i^2} . \tag{4.16}$$

On the one hand, using the representation of Eq. (4.15) we get for Eq. (4.13):

$$T^d_{1\,\mu\nu} = \int_{pqr} \frac{q_\mu r_\nu}{(p^2 + m_1^2)^{\nu_1} (q^2 + m_2^2)^{\nu_2} (r^2 + m_3^2)^{\nu_3} ((p+q)^2 + m_4^2)^{\nu_4} ((p+r)^2 + m_5^2)^{\nu_5}} . \tag{4.17}$$

On the other hand, using the representation of Eq. (4.16), we have:

$$T^d_{1\,\mu\nu} = \frac{\delta_{\mu\nu} C}{\left[(4\pi)^{\frac{d}{2}}\right]^3} \int_0^\infty \prod_{i=1}^{5} \mathrm{d}\alpha_i \alpha_i^{\nu_i - 1} \frac{\alpha_4 \alpha_5}{2[D(\alpha)]^{\frac{d+2}{2}}} e^{\frac{1}{4D(\alpha)}\left(\beta_1 a_1^2 + \beta_2 a_2^2 + 2\beta_3 a_1 \cdot a_2\right)} e^{-\sum_{i=1}^{5} \alpha_i m_i^2} , \tag{4.18}$$

with

$$\begin{aligned} \beta_1 &= \alpha_1(\alpha_3 + \alpha_5) + \alpha_3(\alpha_4 + \alpha_5) + \alpha_4\alpha_5 , \\ \beta_2 &= \alpha_1(\alpha_2 + \alpha_4) + \alpha_2(\alpha_4 + \alpha_5) + \alpha_4\alpha_5 , \\ \beta_3 &= \alpha_4\alpha_5 . \end{aligned} \tag{4.19}$$

Except for some pre-factors, this integral seems to be a scalar integral in $d + 2$ dimensions and of the same topology as the one in Eq. (4.8) containing two propagators raised to a higher power. Therefore, by simply adjusting the pre-factors and reading off the dimension and the different powers of the propagators, we obtain:

$$T^d_{1\,\mu\nu}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) = \delta_{\mu\nu}\frac{(4\pi)^3}{2}\nu_4\nu_5 S^{d+2}_{\nu_1\nu_2\nu_3\nu_4+1\nu_5+1}\ . \tag{4.20}$$

In conclusion, using the α-parameterization of the loop-integral it is possible to rewrite a tensor integral as a scalar integral of higher dimension but with the same topology and with the tensor structure appearing as a pre-factor.

In order to determine a similar relation for Eq. (4.14), we perform the following calculation, with $\partial_{i\mu} \equiv \frac{\partial}{\partial a_{i_\mu}}$:

$$\begin{aligned}&\partial_{1\mu}\partial_{1\nu}\partial_{2\rho}\partial_{2\sigma}e^{\frac{1}{4D(\alpha)}(\beta_1 a_1^2+\beta_2 a_2^2+2\beta_3 a_1\cdot a_2)}\Big|_{\substack{a_1=0\\a_2=0}}\\ &= \frac{1}{[D(\alpha)]^2}\left[\beta_1\beta_2\delta_{\mu\nu}\delta_{\rho\sigma}+\beta_3^2(\delta_{\mu\sigma}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\sigma})\right]\ .\end{aligned} \tag{4.21}$$

This result shows that the tensor integral in Eq. (4.14) is expressed as a linear combination of 26 scalar integrals of dimension $d+4$. Instead of showing the result, we will exploit some symmetries of a particular integral, in which $m_2 = m_3 = m_4 = m_5$. This particular choice of the masses ensures that the massive integral (4.8) will have the same symmetries as the massless sum-integrals we are interested in to compute.

The procedure is to group all terms of $\beta_1\beta_2$ that lead to identical massive scalar integrals after several changes of the integration variables. For instance, the term $\alpha_1^2\alpha_3\alpha_4$ will generate an integral of the form $S^d_{\nu_1+2\nu_2\nu_3+1\nu_4+1\nu_5}$ which, after a momentum translations $q \to -q-p$ becomes $S^d_{\nu_1+2\nu_2+1\nu_3+1\nu_4\nu_5}$. In this way it is possible to group all 25 terms of $\beta_1\beta_2$ into 7 terms of which everyone generates one scalar integral.

After some calculations, we obtain for the second tensor integral (4.14):

$$\begin{aligned}T^d_{2\,\mu\nu\rho\sigma} = (4\pi)^6\Big\{&\Big[b_1 S^{d+4}_{\nu_1+2\nu_2+1\nu_3+1\nu_4\nu_5}+b_2 S^{d+4}_{\nu_1+1\nu_2+1\nu_3+1\nu_4+1\nu_5}+b_3 S^{d+4}_{\nu_1\nu_2+1\nu_3+1\nu_4+1\nu_5+1}\\ &+b_4 S^{d+4}_{\nu_1+1\nu_2+2\nu_3+1\nu_4\nu_5}+b_5 S^{d+4}_{\nu_1\nu_2+2\nu_3+1\nu_4+1\nu_5}+b_6 S^{d+4}_{\nu_1\nu_2+2\nu_3+1\nu_4\nu_5}\Big]\delta_{\mu\nu}\delta_{\rho\sigma}\\ &+b_7 S^{d+4}_{\nu_1\nu_2+2\nu_3+2\nu_4\nu_5}(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\sigma})\Big\}\ ,\end{aligned} \tag{4.22}$$

with

$$\begin{aligned}b_1 &= \nu_1(\nu_1+1)(\nu_2+\nu_4)(\nu_3+\nu_5)\ ,\\ b_2 &= 2\nu_1[\nu_2\nu_4(\nu_3+\nu_5)+\nu_3\nu_5(\nu_2+\nu_4)]\ ,\\ b_3 &= 2\nu_2\nu_3\nu_4\nu_5\ ,\\ b_4 &= \nu_1[\nu_5(\nu_5+1)(\nu_2+\nu_4)+\nu_4(\nu_4+1)(\nu_3+\nu_5)]\ ,\\ b_5 &= \nu_4(\nu_4+1)\nu_2(\nu_3+\nu_5)+\nu_5(\nu_5+1)\nu_3(\nu_2+\nu_4)\ ,\\ b_6 &= \nu_4\nu_5[\nu_3(\nu_4+1)+\nu_2(\nu_5+1)]\ ,\\ b_7 &= \nu_4(\nu_4+1)\nu_5(\nu_5+1)\ .\end{aligned} \tag{4.23}$$

In principle each new tensor index will raise the dimension of the scalar integral with 2 and the tensor structure will appear as a pre-factor of all possible combinations of Kronecker deltas.

4.1.2 Lowering dimension of scalar integrals

By projecting out the tensor structure of massive integrals in Euclidean metric, higher dimensional scalar integrals are obtained, as was shown above. It is also possible to find a relation for lowering the dimension of scalar integrals, as will be shown in the following.

To this end, we define the operator:

$$D(\partial) \equiv D\left(\alpha_i \to \frac{\partial}{\partial m_i^2}\right) = \frac{\partial}{\partial m_1^2}\frac{\partial}{\partial m_2^2}\frac{\partial}{\partial m_3^2} + \dots \tag{4.24}$$

Applying this operator to Eq. (4.11), we obtain:

$$\begin{aligned} D(\partial)S^d_{\nu_1\nu_2\nu_3\nu_4\nu_5} &= \frac{C}{\left[(4\pi)^{\frac{d}{2}}\right]^3}\int_0^\infty \prod_{i=1}^5 \mathrm{d}\alpha_i\alpha_i^{\nu_i-1}\frac{1}{[D(\alpha)]^{\frac{d}{2}}}D\left(\frac{\partial}{\partial m_i^2}\right)e^{-\sum_{i=1}^5\alpha_i m_i^2} \\ &= -\frac{1}{(4\pi)^3}S^{d-2}_{\nu_1\nu_2\nu_3\nu_4\nu_5}\ , \end{aligned} \tag{4.25}$$

where we have used that $D(\alpha)$ (c.f. Eq. (4.12)) is a homogeneous polynomial of degree three.

On the other hand, applying $D(\partial)$ on Eq. (4.8) we get terms of the form:

$$\frac{\partial}{\partial m_i^2}S^d_{\dots\nu_i\dots} = -\nu_i S^d_{\dots\nu_i+1\dots}\ . \tag{4.26}$$

For our particular choice of $D(\partial)$ we will obtain a linear combination of eight scalar integrals. We choose the masses to be $m_2 = m_3 = m_4 = m_5$ in order to relate this integral to our needed sum-integrals later on. After some momentum translation, it is possible to rewrite all integrals in terms of only two.

$$\begin{aligned} D(\partial)S^d_{\nu_1\nu_2\nu_3\nu_4\nu_5} =& -\nu_1(\nu_2+\nu_4)(\nu_3+\nu_5)S^d_{\nu_1+1\nu_2+1\nu_3+1\nu_4\nu_5} \\ &- [(\nu_2\nu_4(\nu_3+\nu_5) + (\nu_2+\nu_4)\nu_3\nu_5]S^d_{\nu_1\nu_2+1\nu_3+1\nu_4+1\nu_5}\ . \end{aligned} \tag{4.27}$$

Combining Eq. (4.25) and Eq. (4.27), we get an expression that relates scalar integrals of different dimension:

$$S^d_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \frac{(4\pi)^{-3}}{b_8}S^{d-2}_{\nu_1\nu_2-1\nu_3-1\nu_4-1\nu_5} - \frac{b_9}{b_8}S^d_{\nu_1+1\nu_2\nu_3\nu_4-1\nu_5}\ . \tag{4.28}$$

where

$$\begin{aligned} b_8 &= (\nu_2-1)(\nu_4-1)(\nu_3+\nu_5-1) + (\nu_3-1)(\nu_2+\nu_4-2)\nu_5\ , \\ b_9 &= \nu_1(\nu_2+\nu_4-2)(\nu_3+\nu_5-1)\ . \end{aligned} \tag{4.29}$$

These relations can be used to obtain the result in Ref. [120] as an alternative to Eq. (4.30). In fact, we have used the different results as a cross check of Tarasov's method for sum-integrals.

4.1.3 Rearrangement of $M_{3,-2}$

In the following Tarasov's method (Ref. [29]) is applied to the master sum-integral $M_{3,-2}$. For that, $M_{3,-2}$ is rearranged by exploiting its $R \leftrightarrow Q$ symmetry and by expanding the numerator:

$$
\begin{aligned}
M_{3,-2} &= \sum\!\!\!\!\!\!\int_{PQR} \frac{\left[(Q-R)^2\right]^2}{P^6 Q^2 R^2 (P+Q)^2 (P+R)^2} \\
&= 2\sum\!\!\!\!\!\!\int_{PQR} \frac{Q^2}{P^6 R^2 (P+Q)^2 (P+R)^2} + 4\sum\!\!\!\!\!\!\int_{PQR} \frac{(Q\cdot R)^2}{P^6 Q^2 R^2 (P+Q)^2 (P+R)^2} \\
&- 8\sum\!\!\!\!\!\!\int_{PQR} \frac{Q\cdot R}{P^6 R^2 (P+Q)^2 (P+R)^2} + 2\sum\!\!\!\!\!\!\int_{PQR} \frac{1}{P^6 (P+Q)^2 (P+R)^2} \\
&\equiv 2M_a + 4M_b - 8M_c + 2M_d \ .
\end{aligned} \tag{4.30}
$$

M_b and M_c are tensor sum-integrals. Nevertheless, M_c has a simple enough structure to be computed via usual projection techniques as will be shortly presented. For M_b it is necessary to apply Tarasov's method.

M_a, M_c and M_d

In order to calculate these sum integrals, we first exploit their symmetries with respect to the integration variables, the property of dimensionless integrals to be 0 in dimensional regularization and the fact that integrals of the type $\sum\!\!\!\!\!\int_Q Q_\mu/[(Q^2)^n]$ are 0 due to oddness of the integrand.

After momentum translation, we get:

$$
M_a \overset{Q\to Q-P}{=} I_1^0 \sum\!\!\!\!\!\!\int_{PQ} \frac{1}{P^4 Q^2 (P+Q)^2} = I_1^0 L^d(211;00) \ , \tag{4.31}
$$

where I and L are defined in appendix B, Eqs. (B.3, B.6) and for these particular values, L is given in Eq. (B.9).

Similarly, M_d is merely a product of tadpole integrals (cf. Eq. B.3):

$$
M_d = [I_1^0]^2 I_3^0 \ . \tag{4.32}
$$

The sum-integral M_c is written in terms of its tensor components:

$$
M_c = g_{\mu\nu} \sum\!\!\!\!\!\!\int_P \frac{1}{P^6} V_\mu(P) \tilde{V}_\nu(P) \ , \tag{4.33}
$$

with

$$
\tilde{V}_\mu(P) = \sum\!\!\!\!\!\!\int_Q \frac{Q_\mu}{(P+Q)^2} \overset{Q\to Q-P}{=} -I_1^0 P_\mu \ . \tag{4.34}
$$

and

$$
\begin{aligned}
V_\mu(P) &= \sum\!\!\!\!\!\!\int_Q \frac{Q_\mu}{Q^2 (P+Q)^2} = \sum\!\!\!\!\!\!\int_Q \frac{-Q_\mu - P_\mu}{(P+Q)^2 Q^2} \\
\Rightarrow 2\sum\!\!\!\!\!\!\int_Q \frac{Q_\mu}{Q^2(P+Q)^2} &= -P_\mu \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2 (P+Q)^2} \Rightarrow V_\mu(P) = -\frac{1}{2} P_\mu \Pi_{110}(P) \ ,
\end{aligned} \tag{4.35}
$$

with Π_{110} defined in Eq. (4.43).

Eq. (4.33) yields:

$$
M_c = \frac{1}{2} I_1^0 \sum\!\!\!\!\!\!\int_P \frac{1}{P^4} \Pi_{110} = \frac{1}{2} I_1^0 L^d(211;00) \ . \tag{4.36}
$$

4.1.4 Reduction of tensor sum-integral M_b

In order to reduce M_b, we split the numerator of the second term in Eq. (4.30) into the temporal and the spatial components:

$$(R\cdot Q)^2=(r_0q_0+r_iq_i)(r_0q_0+r_jq_j)=r_0^2q_0^2+2r_0q_0(r\cdot q)+(r\cdot q)^2 \tag{4.37}$$

so that we obtain:

$$\begin{aligned} M_b=&\underbrace{\sum\!\!\!\!\!\!\!\int_{PQR}\frac{r_0^2q_0^2}{P^6Q^2R^2(P+Q)^2(P+R)^2}}_{\equiv V(3;31111;22)}+2\sum\!\!\!\!\!\!\!\int_{PQR}\frac{r_0q_0(r\cdot q)}{P^6Q^2R^2(P+Q)^2(P+R)^2}\\ &+\sum\!\!\!\!\!\!\!\int_{PQR}\frac{(r\cdot q)^2}{P^6Q^2R^2(P+Q)^2(P+R)^2}\ , \end{aligned} \tag{4.38}$$

The first term in Eq. (4.38) is a regular scalar sum-integral whereas the last two terms contain a tensor structure that will be treated with the methods shown previously.

The second term can be identified with (cf. Eq. 4.20):

$$\begin{aligned} \sum\!\!\!\!\!\!\!\int_{PQR}\frac{r_0q_0(r\cdot q)}{P^6Q^2R^2(P+Q)^2(P+R)^2}&=T^3\sum_{p_0q_0r_0}r_0q_0T1^d_{ii}(3,1,1,1,1)\Big|_{\substack{m_1=p_0\\ \dots\\ m_5=p_0+r_0}}\\ &=\frac{(4\pi)^3}{2}\delta_{ii}T^3\sum_{p_0q_0r_0}r_0q_0S^{d+2}_{31122}\Big|_{\substack{m_1=p_0\\ \dots\\ m_5=p_0+r_0}}\\ &=d\frac{(4\pi)^3}{2}\sum\!\!\!\!\!\!\!\int^{\{d+2\}}_{PQR}\frac{r_0q_0}{P^6Q^2R^2(P+Q)^4(P+R)^4}\\ &\equiv d\frac{(4\pi)^3}{2}V(5;31122;11)\ . \end{aligned} \tag{4.39}$$

This manipulation is possible due to the fact that we have identified the masses of the integral of Eq. (4.20) with the Matsubara modes which we sum over. By using the constraint on the masses (c.f. $m_2=...=m_5$), we ensure that both the sum-integrals and the massive integrals exhibit the same symmetries.

Finally the third term of Eq. (4.38) is written as (cf. Eq. (4.22)):

$$\begin{aligned} &\sum\!\!\!\!\!\!\!\int_{PQR}\frac{(r\cdot q)^2}{P^6Q^2R^2(P+Q)^2(P+R)^2}=T^3\sum_{p_0q_0r_0}T^d_{2\,ijij}(3,1,1,1,1)\Big|_{\substack{m_1=p_0\\ \dots\\ m_5=p_0+r_0}}\\ &=\frac{(4\pi)^6}{4}T^3\sum_{p_0q_0r_0}\Big[\delta_{ij}\delta_{ij}\Big(2S^{d+4}_{32222}+48S^{d+4}_{52211}+24S^{d+4}_{42221}+24S^{d+4}_{43211}\\ &+8S^{d+4}_{33221}+4S^{d+4}_{33212}+3S^{d+4}_{33311}\Big)+\ (\delta_{ij}\delta_{ij}+\delta_{ii}\delta_{jj})4S^{d+4}_{33311}\Big]\Big|_{\substack{m_1=p_0\\ \dots\\ m_5=p_0+r_0}}\\ &=\frac{(4\pi)^6}{4}d\Bigg[2\sum\!\!\!\!\!\!\!\int^{\{d+4\}}_{PQR}\frac{1}{P^6Q^4R^4(P+Q)^4(P+R)^4}+48\sum\!\!\!\!\!\!\!\int^{\{d+4\}}_{PQR}\frac{1}{P^{10}Q^4R^4(P+Q)^2(P+R)^2} \end{aligned}$$

$$
\begin{aligned}
&+ 24\sum\!\!\!\!\!\!\!\int_{PQR}^{\{d+4\}} \frac{1}{P^8Q^4R^4(P+Q)^4(P+R)^2} + 24\sum\!\!\!\!\!\!\!\int_{PQR}^{\{d+4\}} \frac{1}{P^8Q^6R^4(P+Q)^2(P+R)^2} \\
&+ 8\sum\!\!\!\!\!\!\!\int_{PQR}^{\{d+4\}} \frac{1}{P^6Q^6R^4(P+Q)^4(P+R)^2} + 4\sum\!\!\!\!\!\!\!\int_{PQR}^{\{d+4\}} \frac{1}{P^6Q^6R^4(P+Q)^2(P+R)^4} \\
&\left. +4(2+d)\sum\!\!\!\!\!\!\!\int_{PQR}^{\{d+4\}} \frac{1}{P^6Q^6R^6(P+Q)^2(P+R)^2} \right] .
\end{aligned}
\tag{4.40}
$$

We have managed to express Eq. (4.38) as a sum of nine 3-loop scalar sum-integrals of different dimensions [2]. A different basis set for $M_{3,-2}$ can be found in [120]:

$$
\begin{aligned}
M_{3,-2} &= 2[I_1^0]^2 I_3^0 - 2I_1^0 L^d(211;00) + 4V(3;31111;22) + 4d(4\pi)^3 V(5;31122;11) \\
&\quad + d(4\pi)^6 \left[2V(7,32222;00) + 48V(7,52211;00) + 24V(7;42221;00) \right. \\
&\quad \left. + 24V(7;43211;00) + 8V(7;33221;00) + 4V(7;33212;00) + 4(d+2)V(7;33311;00)\right] .
\end{aligned}
\tag{4.41}
$$

4.2 Properties of spectacles-types. Splitting

The goal is to express the sum-integrals as a Laurent expansion in ϵ up to $\mathcal{O}(\epsilon^0)$. Experience shows that every loop can exhibit at most a pole of order 1, $1/\epsilon$, so that the highest degree of divergence encountered is $1/\epsilon^3$. The basic idea in computing three-loop sum-integrals originates from the paper of Arnold and Zhai [28]. It is based on two essential properties of the sum-integrals.

The first property is related to the topology. For three-loop sum-integrals of basketball and spectacles type it is possible to perform a decomposition into one-loop structures, by cutting the diagram as demonstrated in Fig (4.3). In this way, the one-loop structures are treated separately and plugged in the overall integration.

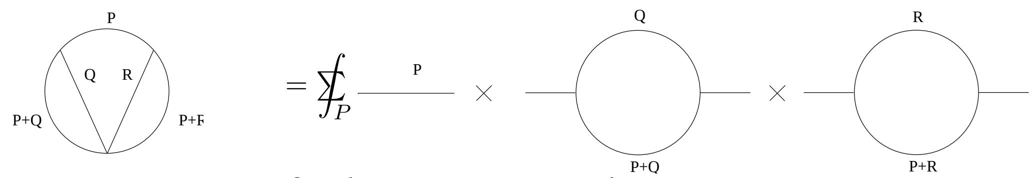


Figure 4.3: One-loop substructures of V-type sum-integrals.

The second property originates in the fact that the propagator (and therefore the one-loop generalized 2-point function) has a simpler structure in configuration space rather then in momentum space, if the structure itself is finite when setting $\epsilon = 0$. Therefore, the idea is to subtract from the one-loop substructures terms which generate divergences and then to express the finite remainder via a Fourier transformation in configuration space.

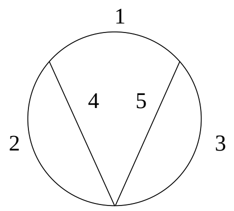
In order to perform a proper splitting of the sum-integral, the origin of divergences should be investigated. We call ultraviolet (UV) divergences those that arise from the limit of high momenta, $\mathbf{p} \to \infty$ and $p_0 \to \infty$. The infrared (IR) divergences refer here to those arising whenever some Matsubara-mode is set to zero, $p_0 = 0$. There is no need to distinguish between

[2] Notice that $d = 3 - 2\epsilon$ is still valid, and that deviating values are explicitly denoted in the notation of the integral as: $V(d + 2\epsilon, ...)$

these divergences in the final result (that is, to explicitly use ϵ_{IR} and ϵ_{UV}) since the matching procedure in the effective theory setup presented in section 3.5 is taking care of separating them.

Thus, the most general form of a spectacles-type sum-integral is rewritten as:

$$V(d; s_1, ..., s_5; s_6 s_7) \equiv$$



$$= \sum\!\!\!\!\!\!\!\int_P \frac{\Pi_{s_2 s_4 s_6}\Pi_{s_3 s_5 s_7}}{[P^2]^{s_1}} \ , \tag{4.42}$$

with the generalized one-loop 2-point function:

$$\Pi_{abc} = \sum\!\!\!\!\!\!\!\int_Q \frac{q_0^c}{[Q^2]^a[(P+Q)^2]^b} \ . \tag{4.43}$$

In the following, we investigate the analytic behavior of the substructures and separate the contributions that give rise to (IR and UV) divergences. This splitting is kept as general as possible. As it turns out, it can be applied to almost all sum-integrals encountered in our computation.

4.2.1 UV divergences

First, we concentrate on the UV divergences of the substructures and the UV divergences they may generate in the overall integration. It is possible to isolate them into three terms.

For that, notice that the sum in the 2-point function increases its complexity considerably and therefore it is replaced by an shifted integral into the complex plane as in the thermal sum formula [122, 123]:

$$\begin{aligned} S \equiv T\sum_{p_0} f(p_0) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi} f(p_0) + \int_{-\infty-i0^+}^{\infty-i0^+} \frac{\mathrm{d}p_0}{2\pi}\left[f(p_0)+f(-p_0)\right] n_B(ip_0) \\ &= S^0 + S^T \ , \end{aligned} \tag{4.44}$$

with

$$n_B(ip_0) = \frac{1}{e^{ip_0/T}-1} \ . \tag{4.45}$$

The function f needs to be analytic in the complex plane and regular on the real axis. In addition it should grow slower then $e^{\beta|p|}$ at large $|p|$, so that the contour of integration can be closed in the complex plane, as shown later.

The first part of the expression is the zero temperature limit of the sum (obvious by the explicit lack of the T-parameter and denoted by Π^B, c.f. Eq. (4.46) below), whereas the second term is the thermal remainder of the sum.

$$\Pi^B_{abc} = \int_Q \frac{q_0^c}{[Q^2]^a[(P+Q)^2]^b} \ . \tag{4.46}$$

The latter part of the sum is UV finite. However, it might happen that by plugging this remainder in the overall P-integration, it will generate some further divergences. Therefore, the recipe is to subtract from the remainder Π^{0-B} as many leading terms as necessary in its asymptotic expansion of P (c.f. subsection 4.2.3). These leading UV terms are denoted with Π^C and their concrete definition is given in Eq. (4.110).

There is one additional term needed for a proper expansion, denoted as Π^D. It is related to the zero-temperature term, Π^B. Since the zero temperature term exhibits an ϵ divergence (c.f. Eq. (4.108, B.2) below), the term is constructed just to cancel this divergence by introducing an arbitrary parameter α_i which will cancel in the end:

$$\Pi^D_{abc} = \frac{(P^2)^\epsilon}{(\alpha_i T^2)^\epsilon} \Pi^B_{abc} \; . \tag{4.47}$$

In this way, the combination Π^{B-D}_{abc} can always be constructed to be finite, (of $\mathcal{O}(\epsilon^0)$).

With these ideas in mind, we can now perform a preliminary splitting of the product of two such one-loop structures[3], denoting for brevity $\Pi_i \equiv \Pi_{abc}$, $i = 1, 2$:

$$\Pi_1\Pi_2 = \frac{1}{2}\Pi_1^{0-B}\Pi_2^{0-B} + \Pi_1^{B-D}\Pi_2^{0-B} + \Pi_1^{D}\Pi_2^{0-B} + \frac{1}{2}\Pi_1^{B}\Pi_2^{B} + (1 \leftrightarrow 2) \; . \tag{4.48}$$

The first two terms, as well as the first two terms from $(1 \leftrightarrow 2)$ are (in principle) finite when plugged in into Eq. (4.42) and can be computed in configuration space, whereas all the other terms are expressed in momentum space via zeta and gamma functions.

This splitting is to be understood as a guideline. In general there are some conditions for how many terms should be subtracted from the Π's. They are related to the superficial degree of divergence of the sum-integral (c.f. subsection 4.2.3).

4.2.2 IR divergences

There are two sources of divergences that may occur. The first one is related to the zero Matsubara-mode of the overall integration variable $p_0 = 0$ and the other may occur within the one-loop 2-point function $\Pi_{s_1 s_2 s_3}$.

In the latter case, the IR divergence is coming from the zero Matsubara mode, thus we define that particular contribution as [4]:

$$\Pi^E_{ab0} = \int_q \frac{1}{[q^2]^{s_1}[(p+q)^2 + p_0^2]^{s_2}} \; . \tag{4.49}$$

Note that only $s_3 = 0$ gives a finite contribution to the IR part. This integral can now be simply regarded as a massive one-loop tadpole in d dimensions. Thus, it can be manipulated with standard zero temperature techniques such as Integration by Parts (IBP) relations (c.f. appendix D).

There are two situations in which the IR-sensitive part has to be subtracted. The first scenario occurs whenever the following condition is true:

$$\max(2s_1 - s_3, 2s_2) > d \; . \tag{4.50}$$

[3]The factor 1/2 avoids over counting in $(1 \leftrightarrow 2)$.

[4]There is a second IR divergence arising when s_2 is sufficiently high. In that situation, the mode $q_0 = -p_0$ will generate an IR divergence, that will occur even for non-vanishing s_3. Nevertheless, this case does not arise here.

In this case $\Pi_{s_1 s_2 s_3}$ is IR divergent and the following splitting should be performed: $\Pi = \Pi^{0-E} + \Pi^{E}$, otherwise the finite piece of the 2-point function yields a contribution of the form: $\frac{1}{q_0=0}$. In both cases, $\Pi_{abc}(p_0 = 0)$ and $\Pi_{abc}(p_0 \neq 0)$, this behavior has to be accounted for. In addition, a $\Pi^{E}(p_0 = 0)$ may have to be subtracted even if the condition is not fulfilled in order to render the overall integration over $\mathbf{p}$ IR safe. In this situation one may think of Π^{E} as the IR counterpart of Π^{C}.

4.2.3 Splitting

Considering the most general spectacles type sum-integral from Eq. (4.42), after subtracting the divergent zero temperature parts from the sub-loops the integrand runs as (c.f. leading UV behavior of the sub-loops, Eq. 4.110):

$$\frac{\Pi^{0-B}_{s_2 s_4 s_6} \Pi^{0-B}_{s_3 s_5 s_7}}{[P^2]^{s_1}} = [P^2]^{-s_1 - \min(s_2 - s_6/2, s_4) - \min(s_3 - s_7/2, s_5)} \, . \tag{4.51}$$

Therefore, the condition for the first term of Eq. (4.48) to be UV finite, reads:

$$-2s_1 - \min(2s_2 - s_6, 2s_4) - \min(2s_3 - s_7, 2s_5) \leq -d - 2 \, , \tag{4.52}$$

or

$$c_0 = \theta(d - 2(s_1 - 1) - \min(2s_2 - s_6, 2s_4) - \min(2s_3 - s_7, 2s_5)) \, . \tag{4.53}$$

Should this condition not be fulfilled, leading UV terms (Π^{C}) have to be subtracted, as seen through the Heaviside theta function. However, it turns out that for sum-integrals of mass dimension 2 only basketball type sum-integrals may exhibit a non-vanishing c_0, whereas for the spectacle-types encountered here, it varies between -4 and 0.

Further, since Π^{B-D}_{abc} goes as $\ln P^2/[P^2]^{a+b-\frac{c}{2}-\frac{d+1}{2}}$, the condition for the second term in Eq. (4.48) to be UV finite, reads:

$$-2s_{124} + (d+1) + s_6 - \min(2s_3 - s_7, 2s_5) \leq -2 - d \, , \tag{4.54}$$

or

$$c_2 = \theta(2d + 3 - 2s_{124} + s_6 - \min(2s_3 - s_7, 2s_5)) \, . \tag{4.55}$$

In this way, a better splitting is:

$$\begin{aligned} \Pi_1 \Pi_2 = {} & \frac{1}{2}\left[\Pi_1^{0-B} - c_0 \Pi_1^{C}\right]\left[\Pi_2^{0-B} - c_0 \Pi_2^{C}\right] + \Pi_1^{B-D}\left[\Pi_2^{0-B} - c_2 \Pi_2^{C}\right] + c_0 \Pi_1^{C} \Pi_2^{0-B} \\ & + \Pi_1^{D}\left[\Pi_2^{0-B} - c_2 \Pi_2^{C}\right] + c_2 \Pi_1^{B} \Pi_2^{C} + \frac{c_0}{2} \Pi_1^{C} \Pi_2^{C} + \frac{1}{2} \Pi_1^{B} \Pi_2^{B} + (1 \leftrightarrow 2) \, . \end{aligned} \tag{4.56}$$

In the case where the Matsubara mode of the overall integration variable is zero, $p_0 = 0$, the only potentially divergent terms are the zero temperature and the zero-mode parts, $\Pi^{\{B,E\}}$. We have determined an empirical rule for deciding which part should be subtracted and which not and it is related to the superficial degree of divergence of the substructures. The following rule works for all $\Pi_{s_1 s_2 s_3}(p_0 = 0)$ except for Π_{110}, for which both terms need to be subtracted.

$$\Pi^{A}_{abc} = \theta(d + 2 - 2a - 2b + c)\Pi^{B}_{abc} + \theta(2a + 2b - c - 2 - 2d)\Pi^{E}_{abc} \, . \tag{4.57}$$

Thus, for the $p_0 = 0$ part, the separation of the substructures looks like:

$$\Pi_1(p_0=0)\Pi_2(p_0=0) = \frac{1}{2}\Pi_1^{0-A}(p_0=0)\Pi_2^{0-A}(p_0=0) + \Pi_1^{A}(p_0=0)\Pi_2^{0}(p_0=0) + (1 \leftrightarrow 2)\ . \tag{4.58}$$

Finally, summarizing all divergences, the generic splitting becomes:

$$\begin{aligned}
V(d; s_1, ..., s_5; s_6 s_7) &= \sum\!\!\!\!\!\!\!\int_P \frac{\Pi_{s_2s_4s_6}\Pi_{s_3s_5s_7}}{[P^2]^{s_1}} \\
&= \sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\frac{1}{2}\Pi^{0-A}_{s_2s_4s_6}\Pi^{0-A}_{s_3s_5s_7} + \Pi^{A}_{s_2s_4s_6}\Pi_{s_3s_5s_7} + (246 \leftrightarrow 357)}{[P^2]^{s_1}}\Bigg|_{\mathrm{IBP}} \\
&+ \sum\!\!\!\!\!\!\!\int_P{}' \frac{\frac{1}{2}\Pi^{0-B-c_0C}_{s_2s_4s_6}\Pi^{0-B-c_0C}_{s_3s_5s_7} + \Pi^{B-D}_{s_2s_4s_6}\Pi^{0-B-c_2C}_{s_3s_5s_7} + (246 \leftrightarrow 357)}{[P^2]^{s_1}} \\
&+ \Bigg\{ \sum\!\!\!\!\!\!\!\int_P{}' \frac{c_0\Pi^{C}_{s_2s_4s_6}\Pi^{0-B}_{s_3s_5s_7} + \Pi^{D}_{s_2s_4s_6}\Pi^{0-B-c_2C}_{s_3s_5s_7} + c_2\Pi^{B}_{s_2s_4s_6}\Pi^{C}_{s_3s_5s_7} + (246 \leftrightarrow 357)}{[P^2]^{s_1}} \\
&+ \sum\!\!\!\!\!\!\!\int_P{}' \frac{\frac{c_0}{2}\Pi^{C}_{s_2s_4s_6}\Pi^{C}_{s_3s_5s_7} + \frac{1}{2}\Pi^{B}_{s_2s_4s_6}\Pi^{B}_{s_3s_5s_7} + (246 \leftrightarrow 357)}{[P^2]^{s_1}} \Bigg\} \\
&\equiv V^{\mathrm{z(ero-mode)}} + V^{\mathrm{f(inite)}} + \left\{ V^{\mathrm{d(ivergent)}} \right\}\ .
\end{aligned} \tag{4.59}$$

Again, V^{f} contains all necessary subtractions to make the integrals finite, and V^{d} all subtracted terms. More explicitly, we have:

$$\begin{aligned}
&\sum\!\!\!\!\!\!\!\int_P{}' \frac{\frac{1}{2}\Pi^{0-B-c_0C}_{s_2s_4s_6}\Pi^{0-B-c_0C}_{s_3s_5s_7} + (246 \leftrightarrow 357)}{[P^2]^{s_1}} \equiv V_i^{\mathrm{f},1} \\
&\sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi^{B-D}_{s_2s_4s_6}\Pi^{0-B-c_2C}_{s_3s_5s_7} + (246 \leftrightarrow 357)}{[P^2]^{s_1}} \equiv V_i^{\mathrm{f,2a}} + V_i^{\mathrm{f,2b}} \overset{\text{if } a=b}{=} V_i^{\mathrm{f},2}\ .
\end{aligned} \tag{4.60}$$

The zero-mode contribution V^{z} is a special case for itself, since in general it is not possible to eliminate all divergences by only subtracting contributions $\Pi^{\{B,E\}}$ from the one-loop structures. In that sense, the divergences are much too "high". To "lower" them, IBP relations are used (only for V^{z}!):

$$\begin{aligned}
\frac{\partial}{\partial p_i} p_i \circ \sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{s_2s_4s_6}\Pi_{s_3s_5s_7}}{[P^2]^{s_1}} &\to \#_1 \sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{s_2s_4s_6}\Pi_{s_3s_5s_7}}{[P^2]^{s_1-1}} \\
+ \#_2 \sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{s_2-1s_4s_6}\Pi_{s_3s_5s_7}}{[P^2]^{s_1}} &+ \#_3 \sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{s_2s_4s_6}\Pi_{s_3-1s_5s_7}}{[P^2]^{s_1}} + ...
\end{aligned} \tag{4.61}$$

Therefore, only after IBP reduction, the splitting program of the sub-loops can be used. The zero mode divergent and respectively finite parts are denoted with Z^{d} and Z^{f}. Details on IBP zero-mode reduction are to be found in appendix D.

Finally, in Fig. (4.4) we provide the splitting coefficients from Eqs. (4.53) and (4.55), for all the sum-integrals that obey this generic separation procedure.

d	V(s_1,...,s_8)	c_0	c_1	c_2
3	21111;020	0	1	1
	31111;022	0	1	1
	21111;000	0	0	0
	31111;020	0	0	0
	41111;022	0	0	0
5	31122;011	0	1	1
7	32222;000	0	0	0
	52211;000	0	0	0
	42221;000	0	0	0
	43211;000	0	1	0
	33221;000	0	1	0
	32212;000	0	1	0
	33311;000	0	1	1

Figure 4.4: Splitting coefficients of the sum-integrals.

4.3 A first example, $V(3; 21111; 20)$

In the following, we demonstrate both the splitting procedure and the actual computation of each term in part on one of the two non-trivial master sum-integrals that enters the three-loop term of the mass parameter (3.58), $V(3, 21111; 20)$ [124] (c.f. Fig. (4.5)). Based on this example it is possible to generalize the computation to a generic set of parameters. Thus, the remaining sum-integrals are to be treated in a similar way, by using the formulas from section 4.4. The corresponding result can be found in Appendix A.

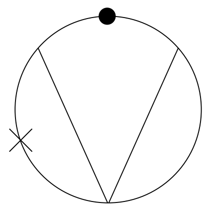


Figure 4.5: Sum-integral $V(3, 21111; 20)$. The dot denotes an extra power on the propagator and the "$\times$" denotes a quadratic Matsubara-mode in the numerator.

The sum-integral is split as:

$$
\begin{aligned}
V(3; 21111; 20) = & \sum\!\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{112}^{0-B}\Pi_{110}^{0-B} + \Pi_{112}^{B-D}\Pi_{110}^{0-B-C} + \Pi_{110}^{B-D}\Pi_{112}^{0-B-C}}{[P^2]^2} \\
& + \sum\!\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{112}^{D}\Pi_{110}^{0-B-C} + \Pi_{112}^{B}\Pi_{110}^{C} + \Pi_{110}^{D}\Pi_{112}^{0-B-C} + \Pi_{110}^{B}\Pi_{112}^{C} + \Pi_{112}^{B}\Pi_{110}^{B}}{[P^2]^2} \\
& + \sum\!\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{112}\Pi_{110}}{[P^2]^2} \, .
\end{aligned}
\tag{4.62}
$$

The prescription is to compute the finite parts first and the divergent terms in the end. The zero-mode contribution generates two additional sum-integrals via IBP transformations.

4.3.1 Building blocks of the sum-integral

Before starting the actual computation, we define all necessary pieces in both momentum space and in configurations space. For a detailed presentation of their computation, see section 4.4. The zero temperature piece of Π_{110} is according to Eq. (4.46):

$$\Pi^B_{110} = \int_Q \frac{1}{Q^2(P+Q)^2} = \frac{g(1,1,d+1)}{[P^2]^{2+\frac{d+1}{2}}} = \frac{\mu^{2\epsilon}\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)[P^2]^{\epsilon}}\ , \tag{4.63}$$

where the integral can be solved via Feynman parameters (c.f. Eq. (B.1)) and g is defined in Eq. (B.2).

The zero temperature part of Π^0_{112} is:

$$\Pi^B_{110} = \int_Q \frac{q_0^2}{Q^2(P+Q)^2} = u_\mu u_\nu \int_Q \frac{Q_\mu Q_\nu}{Q^2(P+Q)^2} \quad u_\mu = (1,\mathbf{0})\ . \tag{4.64}$$

The tensor integral is solved by using the standard projection technique with the ansatz:

$$\int_Q \frac{Q_\mu Q_\nu}{Q^2(P+Q)^2} = AP^2 g_{\mu\nu} + BP_\mu P_\nu \Rightarrow \begin{cases} \int_Q \frac{\overbrace{g_{\mu\nu}Q_\mu Q_\nu}^{=Q^2}}{Q^2(P+Q)^2} & = 0 = P^2((d+1)A+B) \\ \int_Q \frac{\overbrace{P_\mu P_\nu Q_\mu Q_\nu}^{=\frac{1}{4}[(P+Q)^2-P^2-Q^2]^2}}{Q^2(P+Q)^2} & = \frac{P^4}{4}\Pi^B_{110} = P^4(A+B) \end{cases}\ , \tag{4.65}$$

where we have used the property $g_{\mu\nu}g_{\mu\nu} = d+1 = 4-2\epsilon$. By solving the system of equations, we obtain:

$$\Pi^B_{112} = \frac{g(1,1,d+1)}{4d}\frac{(d+1)p_0^2 - P^2}{[P^2]^{2+\frac{d+1}{2}}}\ . \tag{4.66}$$

As presented in more detail in section (4.4), the leading UV contributions Π^C are simply obtained by adding up the contributions of Π with the external momentum flow (P) going through each propagator in the limit $P \to \infty$:

$$\begin{aligned}
\Pi^C_{110} &= \lim_{P\to\infty}\left[\sum\!\!\!\!\!\!\!\int_P \frac{1}{Q^2(P+Q^2)} + \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(P+Q^2)}\right] = \frac{2}{P^2}\sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2} = \frac{2I^0_1}{P^2} \\
\Pi^C_{112} &= \lim_{P\to\infty}\left[\sum\!\!\!\!\!\!\!\int_P \frac{1}{Q^2(P+Q^2)} + \sum\!\!\!\!\!\!\!\int_Q \frac{(-q_0-p_0)^2}{Q^2(P+Q^2)}\right] = \frac{2}{P^2}\sum\!\!\!\!\!\!\!\int_Q \frac{q_0^2}{Q^2} + \frac{p_0^2}{P^2}\sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2} = \frac{2I^1_1 + p_0^2 I^0_1}{P^2}\ .
\end{aligned} \tag{4.67}$$

For defining these quantities in configuration space, we use the inverse Fourier transform of the propagator (c.f. Eq. (4.112)) $\frac{1}{P^2} = \frac{1}{4\pi}\int \mathrm{d}^3\mathbf{r}\frac{e^{i\mathbf{pr}}}{r}e^{-|p_0|r}$. In this way, Π_{110} and Π_{112} become:

$$\begin{aligned}
\Pi_{110} &= T\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\int \mathrm{d}^3\mathbf{r}\int \mathrm{d}^3\mathbf{s}\frac{e^{i\mathbf{qr}+i(\mathbf{q}+\mathbf{p})\mathbf{s}}}{16\pi^2 rs}\sum_{q_0=-\infty}^{\infty} e^{-|q_0|r-|p_0+q_0|s} \\
&= T\int \mathrm{d}^3\mathbf{r}\int \mathrm{d}^3\mathbf{s}\,\delta^{(3)}(\mathbf{r}+\mathbf{s})\frac{e^{i(\mathbf{p})\mathbf{s}}}{16\pi^2 rs}\sum_{q_0=-\infty}^{\infty} e^{-|q_0|r-|p_0+q_0|s} \\
&= \frac{T^3}{4}\int \mathrm{d}^3\mathbf{r}\frac{e^{i\mathbf{pr}}}{\bar{r}^2}e^{-|p_0|r}f_{3,110}(\bar{r},|\bar{p}_0|)\ ,
\end{aligned} \tag{4.68}$$

and similarly

$$\Pi_{112} = \pi^2 T^5 \int \mathrm{d}^3\mathbf{r} \frac{e^{i\mathbf{pr}}}{\bar{r}^2} e^{-|p_0|r} f_{3,112}(\bar{r}, |\bar{p}_0|) \ , \tag{4.69}$$

with

$$f_{3,110}(x,n) = \sum_{m=-\infty}^{\infty} e^{-(|m|+|m+n|-|n|)x} = \coth x + |n| \ ; \ \ f_{3,112}(x,n) = \sum_{m=-\infty}^{\infty} m^2 e^{-(|m|+|m+n|-|n|)x} \ , \tag{4.70}$$

given in Eqs. (E.5 and E.6).

In a similar way, we compute $\Pi^{\{B,C\}}_{\{110,112\}}$. Notice that the $\Gamma(\epsilon)$ term from Eq. (4.63), that exhibits a $1/\epsilon$ pole, cancels the $\Gamma(\epsilon)$ term from the denominator of Eq. (4.112), and renders the quantity finite:

$$\Pi^{\{B,C\}}_{\{110,112\}} = \frac{T^3}{4} \int \mathrm{d}^3\mathbf{r} \frac{e^{i\mathbf{pr}}}{\bar{r}^2} e^{-|p_0|r} \left\{ f^{B,C}_{3,110}(x,y), (2\pi T)^2 f^{B,C}_{3,112}(x,y) \right\} \ , \tag{4.71}$$

with the definitions of f to be found in appendix E.

4.3.2 Finite parts

With the previously calculated building blocks, we can compute all pieces in the splitting of Eq. (4.62).

The first finite piece of the sum-integral reads:

$$\begin{aligned}
V^{\mathrm{f},1}(3;21111;20) &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi^{0-B}_{112}\Pi^{0-B}_{110}}{P^4} \\
&= \frac{\pi^2 T^9}{4} \sum_{p_0}{}' \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{s} \frac{e^{-|p_0|(r+s)}}{\bar{r}^2\bar{s}^2} f^{0-B}_{3,112}(\bar{r},|\bar{p}_0|) f^{0-B}_{3,110}(\bar{s},|\bar{p}_0|) \left[\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}(\mathbf{r}+\mathbf{s})}}{P^4} \right] \\
&= \frac{T^2}{2(4\pi)^4} \sum_{p_0}{}' \int_0^\infty \mathrm{d}\bar{r}\mathrm{d}\bar{s}\, e^{-|p_0|(r+s)} f^{0-B}_{3,112}(\bar{r},|\bar{p}_0|) f^{0-B}_{3,110}(\bar{s},|\bar{p}_0|) \int_0^\pi \mathrm{d}\theta \sin\theta \frac{e^{-|p_0|\sqrt{r^2+s^2+2rs\cos\theta}}}{|\bar{p}_0|} \\
&= \frac{T^2}{(4\pi)^4} \int_0^\infty \mathrm{d}x\mathrm{d}y \frac{1}{xy} \sum_{n=1}^{\infty} \frac{e^{-n(x+y)}}{n^3} \left[e^{-n|x-y|}(1+n|x-y|) - e^{-n(x+y)}(1+n(x+y)) \right] \\
&\times \left[\frac{1}{2}(n+\coth x)\mathrm{csch}^2 x + \frac{n^2}{2}\coth x + \frac{n}{6} - \frac{1}{2x^3} - \frac{n}{2x^2} - \frac{n^2}{2x} \right] \times \left[\coth y - \frac{1}{y} \right] \\
&= \frac{T^2}{(4\pi)^4} \times 0.014356026(1) \ .
\end{aligned} \tag{4.72}$$

In the second line of the previous equation, we have simply plugged in the 2-point functions and have performed the Fourier transform (Eq. (4.112)). Afterwards we have rescaled the integrand and have performed the configuration space angular integrations by choosing the spherical coordinates such that: $|\mathbf{r}+\mathbf{s}| = \sqrt{r^2+s^2+2rs[\text{polar angle}]}$. The remaining angular integration becomes trivial. Finally, the sum can be performed analytically. The two dimensional integration is performed with `Mathematica` [125] numerically.

The second finite piece consists of two terms:

$$V^{\rm f,2a}(3;21111;20)+V^{\rm f,2b}(3,21111,20)=\sum\!\!\!\!\!\!\!\int_P{}' \left[\frac{\Pi^{B-D}_{112}\Pi^{0-B-C}_{110}}{[P^2]^2}+\frac{\Pi^{B-D}_{110}\Pi^{0-B-C}_{112}}{[P^2]^2}\right] . \tag{4.73}$$

Expanding the terms Π^{B-D} in ϵ, we obtain:

$$\Pi^{B-D}_{110}\frac{1}{(4\pi)^2}=\ln\frac{\alpha_1T^2}{P^2}\quad\text{and}\quad\Pi^{B-D}_{112}=\frac{4p_0^2-P^2}{12(4\pi)^2}\ln\frac{\alpha_2T^2}{P^2} . \tag{4.74}$$

As α_i is a dummy variable that cancels in the final result, it can be chosen to simplify the computation. For $\alpha_i=4\pi^2$ no term of the form ln(const) will appear in the integral and the first term of Eq. (4.74) simply reads:

$$\begin{aligned}
\sum\!\!\!\!\!\!\!\int_P{}'\frac{\Pi^{B-D}_{112}\Pi^{0-B-C}_{110}}{[P^2]^2} &= \frac{T^4}{48(4\pi)^2}\sum_{p_0}{}'\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{(P^2-4p_0^2)\ln\bar P^2}{P^4}\int\mathrm{d}^3\mathbf{r}\frac{e^{i\mathbf{pr}}}{\bar r^2}e^{-|p_0|r}f^{0-B-C}_{3,110}(\bar r,|\bar p_0|)\\
&= \frac{T^2}{48(2\pi)^5}\sum_{p_0}{}'\int_0^\infty\mathrm{d}\bar r\,e^{-|p_0|r}f^{0-B-C}_{3,110}(\bar r,|\bar p_0|)\int_0^\infty\mathrm{d}\bar p\frac{p^2(4\bar p_0^2-\bar P^2)}{\bar P^4}\ln\bar P^2\frac{\sin pr}{pr}\\
&= -\frac{T^2}{6(4\pi)^4}\sum_{n=1}^\infty\int_0^\infty\mathrm{d}x\left[\coth x-\frac{1}{x}-\frac{x}{3}\right]\frac{e^{-2nx}}{x}\\
&\times\left[\gamma_{\rm E}(1-2nx)+2nx+e^{2nx}(1+2nx)\mathrm{Ei}(-2nx)-(1-2nx)\ln\frac{2n}{x}\right]\\
&= \frac{T^2}{(4\pi)^4}\times(0.001351890(1)) .
\end{aligned} \tag{4.75}$$

In the second line we have rescaled the integral and have performed the angular integration of $\mathbf{r}$ and $\mathbf{p}$ (c.f. Eqs (C.4, C.6)). Afterwards, the momentum integration was performed as in Eq. (C.12).

The second term is computed in a similar way:

$$\begin{aligned}
\sum\!\!\!\!\!\!\!\int_P{}'\frac{\Pi^{B-D}_{110}\Pi^{0-B-C}_{112}}{[P^2]^2} &= -\frac{T^2}{(4\pi)^4}\sum_{n=1}^\infty\int_0^\infty\mathrm{d}x\left[\frac{1}{2}(n+\coth x)\mathrm{csch}^2x+\frac{n^2}{2}\coth x\right.\\
&\left.+\frac{n}{6}-\frac{1}{2x^3}-\frac{n}{2x^2}-\frac{n^2}{2x}+\frac{x}{30}-\frac{n^2x}{6}\right]\frac{e^{-2nx}}{n}\left[1-\gamma_{\rm E}+e^{2nx}\mathrm{Ei}(-2nx)+\ln\frac{2n}{x}\right]\\
&= \frac{T^2}{(4\pi)^4}\times(-0.006354602(1)) .
\end{aligned} \tag{4.76}$$

A good consistency check is not to set α to a particular value and to check that in the finial result of the sum-integral is α-independent. The numerical value is obtained by performing the integral numerically for the first 10000 terms of n. In order to get an estimate of the remainder, we have fitted the the integral between $n=10.001$ and $n=100.000$ to a power law $f(n)=an^{-b}$ and have performed the summation $n=10.001...\infty$ analytically. Thus, by chopping the sum at $n=10.000$ we get a relative error of $\mathcal{O}(10^{-10})$.

4.3.3 Divergent parts

According to the splitting in Eq. (4.56) there are for the non-zero modes $p_0 \neq 0$ five divergent terms. Keeping in mind that terms like $\sum\!\!\!\!\!\!\int_P \Pi^0/[P^2]$ are two-loop sum-integrals (c.f. Eq. B.6 and for fixed parameters, they are in appendix B) and by using the definitions in Eqs. (4.6, B.3, B.5), we obtain:

$$\begin{aligned}
\sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi^D_{112}\Pi^{0-B-C}_{110}}{P^4} &= \frac{g(1,1,d+1)}{4d(\alpha T^2)^{2-\frac{d+1}{2}}} \sum\!\!\!\!\!\!\!\int_P{}' \frac{(d+1)p_0^2 - P^2}{P^4}\left[\Pi^0_{110} - \frac{g(1,1,d+1)}{[P^2]^{2-\frac{d+1}{2}}} - \frac{2I_1^0}{P^2}\right] \\
&= \frac{g(1,1,d+1)}{4d(\alpha T^2)^{2-\frac{d+1}{2}}}\Big[(d+1)\Big(L^d(211;20) - g(1,1,d+1)I^1_{4-\frac{d+1}{2}} - 2I_1^0 \times I_3^1\Big) \\
&- L^d(111;00) + J^d(111;0) + g(1,1,d+1)I^0_{3-\frac{d+1}{2}} + 2I_1^0 \times I_2^0\Big] \\
&= \frac{T^2}{(4\pi)^4}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon}\left(-\frac{7}{72} + \ln G - \frac{\ln 2\pi}{12}\right) + \mathcal{O}(\epsilon^0)\right] ,
\end{aligned} \tag{4.77}$$

$$\begin{aligned}
\sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi^B_{112}\Pi^C_{110}}{P^4} &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{g(1,1,d+1)}{4d}\frac{(d+1)p_0^2 - P^2}{[P^2]^{2-\frac{d+1}{2}}}\frac{2I_1^0}{[P^2]^3} \\
&= \frac{g(1,1,d+1)I_1^0}{2d}\left[(d+1)I^1_{5-\frac{d+1}{2}} - I^0_{4-\frac{d+1}{2}}\right] \\
&= \frac{T^2}{48(4\pi)^4}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon} + \frac{1}{2} + \gamma_{\rm E} + 24\ln G + \mathcal{O}(\epsilon)\right] ,
\end{aligned} \tag{4.78}$$

$$\begin{aligned}
\sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi^D_{110}\Pi^{0-B-C}_{112}}{P^4} &= \frac{g(1,1,d+1)}{(\alpha T^2)^{2-\frac{d+1}{2}}} \sum\!\!\!\!\!\!\!\int_P{}' \left[\frac{\Pi^0_{112}}{P^4} - \frac{(d+1)p_0^2 - P^2}{4d}\frac{g(1,1,d+1)}{[P^2]^{2-\frac{d+1}{2}}}\frac{1}{P^4}\right. \\
&\left. - \left(I_1^0\frac{p_0^2}{P^2} + \frac{2I_1^1}{P^2}\right)\frac{1}{P^4}\right] \\
&= \frac{g(1,1,d+1)}{(\alpha T^2)^{2-\frac{d+1}{2}}}\left[L^d(211;02) - J^d(211;1) - \frac{g(1,1,d+1)}{4d}\right. \\
&\left. \times\left((d+1)I^1_{4-\frac{d+1}{2}} - I^0_{3-\frac{d+1}{2}}\right) - I_1^0 \times I_3^1 - 2I_1^1 \times I_3^0 \right] \\
&= \frac{T^2}{(4\pi)^4}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon}\left(-\frac{5}{48} - \frac{\gamma_{\rm E}}{24} + \frac{\ln G}{2} + \frac{\zeta(3)}{120}\right) + \mathcal{O}(\epsilon^0)\right] ,
\end{aligned} \tag{4.79}$$

$$\begin{aligned}
\sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi^B_{110}\Pi^C_{112}}{P^4} &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{g(1,1,d+1)}{[P^2]^{2-\frac{d+1}{2}}}\frac{1}{P^4}\left(I_1^0\frac{p_0^2}{P^2} + \frac{2I_1^1}{P^2}\right) \\
&= g(1,1,d+1)\left[I_1^0 \times I^1_{5-\frac{d+1}{2}} + 2I_1^1 \times I^0_{5-\frac{d+1}{2}}\right] \\
&= \frac{T^2}{96(4\pi)^4}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{9}{2} + \gamma_{\rm E} + 24\ln G - \frac{4\zeta(3)}{5}\right) + \mathcal{O}(\epsilon^0)\right] ,
\end{aligned} \tag{4.80}$$

and

$$\sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi^B_{112}\Pi^B_{110}}{P^4} = \frac{g(1,1,d+1)^2}{4d}\sum\!\!\!\!\!\!\!\int_P{}' \frac{(d+1)p_0^2 - P^2}{[P^2]^{2-\frac{d+1}{2}}}\frac{1}{[P^2]^{2-\frac{d+1}{2}}}\frac{1}{P^4}$$
$$= \frac{g(1,1,d+1)^2}{4d}\left[(d+1)I^1_{5-d} - I^0_{4-d}\right]$$
$$= \frac{-1}{48(4\pi)^4}\left(\frac{\mu^2 e^{-\gamma_{\rm E}}}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2} + \frac{72\ln G - 1}{\epsilon} + 2 + \frac{7\pi^2}{4} + 360\ln G - 216\zeta''(-1) + \mathcal{O}(\epsilon)\right] . \tag{4.81}$$

$G \approx 1.2824$ is the Glaisher constant. And the $\gamma_{\rm E}$ constant, along with the Stieltjes constants are defined via:

$$\zeta(1+x) = \frac{1}{x} + \gamma_{\rm E} + \sum_{i=1}^{\infty}\gamma_i\frac{(-x)^i}{i!} . \tag{4.82}$$

The generic formulas for the divergent parts are listed in Eqs. (4.137, 4.138, 4.139).

4.3.4 Zero-modes

The zero-mode component of $V(3;21111;20)$ exhibits an IR divergence that cannot be cured by simply subtracting different terms from the one-loop substructures, as in Eq. (4.57). This behavior stems from the $\delta_{p_0}/[P^2]^2$ factor in the overall integration. Therefore, we use IBP relations to lower the exponent of P and afterwards to apply proper splitting to the sum-integrals emerging from the IBP reduction:

$$\partial_{\mathbf{p}}\mathbf{p}\circ\sum\!\!\!\!\!\!\!\int_P \delta_{p_0}\frac{\Pi_{112}\Pi_{110}}{[P^2]^2} = 0 . \tag{4.83}$$

Using Eq. (D.1), we obtain:

$$\mathbf{p}\partial_{\mathbf{p}}\delta_{p_0}\Pi_{110} = \left[-\mathbf{p}^2\Pi_{210} - \Pi_{110} + I^0_2\right]\delta_{p_0} \tag{4.84}$$

and

$$\mathbf{p}\partial_{\mathbf{p}}\delta_{p_0}\Pi_{112} = \left[-\mathbf{p}^2\Pi_{212} - \Pi_{112} + I^1_2\right]\delta_{p_0} . \tag{4.85}$$

With the product rule in Eq. (4.83) and by plugging in the relations for the substructures, we obtain:

$$\sum\!\!\!\!\!\!\!\int_P \delta_{p_0}\frac{\Pi_{112}\Pi_{110}}{P^4} = \frac{1}{d-6}\sum\!\!\!\!\!\!\!\int_P \delta_{p_0}\left[\frac{\Pi_{210}\Pi_{112}}{P^2} + \frac{\Pi_{212}\Pi_{110}}{P^2} - I^0_2\frac{\Pi_{112}}{P^4} - I^1_2\frac{\Pi_{110}}{P^4}\right]$$
$$= \frac{Z(3;12111;02) + Z(3;12111;20) - I^0_2\times J^d(211;1) - I^1_2\times J^d(211;0)}{d-6} . \tag{4.86}$$

The last two terms are of the form 1loop×2loop, and are trivial to compute (cf. Eqs. (B.3,B.5)). The first two terms are zero-mode 3-loop sum-integrals that need further manipulation.

4.3.5 Zero-mode masters

For $V(3;21111;20)$, we encountered two non-trivial zero-mode sum-integrals that are calculated now in detail. In addition, the definitions of two new 2-point functions (and their divergent pieces) are needed here, both in momentum space and in configuration space. Since the method of computing them was presented in subsection 4.3.1, we simply refer to their definitions in appendix E.

For $Z(3;12111;02)$ we have the splitting:

$$Z(3;12111;02) = \sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{1}{P^2} \left[\Pi_{210}^{0-E} \Pi_{112}^{0-B} + \Pi_{210} \Pi_{112}^{B} + \Pi_{210}^{E} \Pi_{112} \right] . \tag{4.87}$$

The first term is finite and according to Eqs.(E.16, E.6, E.24), it looks like:

$$\begin{aligned}
\sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{210}^{0-E}\Pi_{112}^{0-B}}{P^2} &= \frac{T^7}{32} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\mathbf{p}^2} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{s} \frac{e^{i\mathbf{p}(\mathbf{r}+\mathbf{s})}}{\bar{r}\bar{s}^2} f_{3,210}^{0-E}(\bar{r},0) f_{3,112}^{0-B}(\bar{s},0) \\
&= \frac{T^2}{4(2\pi)^5} \int_0^\infty \mathrm{d}\bar{r}\mathrm{d}\bar{s}\, \bar{r} f_{3,210}^{0-E}(\bar{r},0) f_{3,112}^{0-B}(\bar{s},0) \left[\int_0^\infty \mathrm{d}\bar{p} \frac{\sin pr}{pr} \frac{\sin ps}{ps} \right] \\
&= -\frac{4T^2}{(4\pi)^4} \int_0^\infty \mathrm{d}x \mathrm{d}y \frac{x \ln\left(1-e^{-2x}\right)}{x+y+|x-y|} \left(\frac{\coth y \,\mathrm{csch}^2 y}{2} - \frac{1}{2y^3} \right) \\
&= \frac{T^2}{(4\pi)^4} \int_0^\infty \mathrm{d}y \frac{1-y^3 \coth y \,\mathrm{csch}^2 y}{12y^4} \left[+4y^3 - 2\pi^2 y + 3(2i\pi y^2 + \mathrm{Li}_3\left(e^{2y}\right)) - 3\zeta(3) \right] \\
&= \frac{T^2}{(4\pi)^4} \times -0.02850143769881264033(1) \, .
\end{aligned} \tag{4.88}$$

In the second line we have performed all angular integrations (c.f. Eqs. (C.4, C.6)). Afterwards integration over p was done as in Eq. (C.8). In the second last line of Eq. (4.88) the integration over x was performed by separating the interval into $[0,y]$ and $[y,\infty]$ and solving the parts individually. The generic formula for the finite term can be found in 4.131.

At last, the divergent, analytic terms read (cf. Eqs (4.49, B.2, B.5)):

$$\begin{aligned}
&\sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{1}{P^2} \left[\Pi_{210} \Pi_{112}^{B} + \Pi_{210}^{E} \Pi_{112} \right] \\
&= -\frac{g(1,1,d+1)}{4d} J^d(2-\frac{d+1}{2},2,1;0) + Tg(2,1,d) J^d(4-\frac{d}{2},1,1;1) \, .
\end{aligned} \tag{4.89}$$

The second zero-mode sum-integral, $Z(3;12111;20)$ is split as:

$$Z(3;12111;20) = \sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{1}{P^2} \left[\Pi_{212}^{0-B} \Pi_{110}^{0-B-E} + \Pi_{212} \Pi_{112}^{B+E} + \Pi_{212}^{B} \Pi_{110} \right] . \tag{4.90}$$

The finite term requires the same steps as Eq. (4.88):

$$\begin{aligned}
\sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{212}^{0-E}\Pi_{112}^{0-B-E}}{P^2} &= \frac{2T^2}{(4\pi)^4} \int_0^\infty \mathrm{d}x \mathrm{d}y \frac{x}{x+y+|x-y|} \left(\frac{\mathrm{csch}^2 x}{2} - \frac{1}{2x^2} \right) \left(\coth y - \frac{1}{y} - 1 \right) \\
&= \frac{T^2}{(4\pi)^4} \int_0^\infty \mathrm{d}y \frac{(1+y-y\coth y)(y+\ln y + \ln \mathrm{csch}\, y)}{2y^2} \\
&= \frac{T^2}{(4\pi)^4} \times 1.197038267(1) \, .
\end{aligned} \tag{4.91}$$

The divergent, analytic terms read:

$$
\begin{aligned}
&\sum\!\!\!\!\!\!\int_P \delta_{p_0}\frac{1}{P^2}\left[\Pi_{212}\Pi_{112}^{B+E}+\Pi_{212}^{B}\Pi_{110}\right]=g(1,1,d+1)J^d(3-\frac{d+1}{2},2,1;1)\\
&+Tg(1,1,d)J^d(3-\frac{d}{2},2,1,1)+\frac{2g(1,1,d+1)-g(2,1,d+1)}{4d}J^d(3-\frac{d+1}{2},1,1;0)\ .
\end{aligned}
\tag{4.92}
$$

Finally, we sum up all terms:

$$
Z(3;12111;02)=-\frac{T^2}{24(4\pi)^4}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon}+\left(\frac{16}{3}-5\gamma_{\mathrm{E}}+2\ln 2+5\ln\pi-24\mathcal{Z}_1\right)+\mathcal{O}(\epsilon^0)\right]\ ,
\tag{4.93}
$$

$$
\begin{aligned}
Z(3;12111;20)=&-\frac{T^2}{16(4\pi)^4}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(8-\gamma_{\mathrm{E}}+\ln\frac{\pi}{4}\right)+(24-16\gamma_{\mathrm{E}}\right.\\
&+24\ln 2\left(1-\gamma_{\mathrm{E}}+\ln\frac{\pi}{4}\right)-\frac{27{\gamma_{\mathrm{E}}}^2}{2}+\frac{37\pi^2}{12}+36\ln^2 2+(16-9\gamma_{\mathrm{E}})\ln\frac{\pi}{4}\\
&\left.+\frac{9}{2}\ln^2\frac{\pi}{4}-36\gamma_1-16\mathcal{Z}_2\right)+\mathcal{O}(\epsilon^0)\Bigg]\ .
\end{aligned}
\tag{4.94}
$$

4.3.6 Results

In the end, we add up the terms of the previous section and obtain the ϵ-expansion up to $\mathcal{O}(\epsilon)$ of the second master sum-integral of the effective mass parameter.

$$
V(3;21111;20)=\frac{1}{96}\frac{T^2}{(4\pi)^4}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{55}{6}+\gamma_{\mathrm{E}}+24\ln G\right)+v_2+\mathcal{O}(\epsilon)\right]\ .
\tag{4.95}
$$

$$
\begin{aligned}
v_2=&-\frac{673}{36}+\ln 2\left(-\frac{68}{3}-8\gamma_{\mathrm{E}}+288\ln G+\frac{8\zeta(3)}{5}\right)+\gamma_{\mathrm{E}}\left(\frac{79}{6}-72\ln G\right)\\
&-\frac{31{\gamma_{\mathrm{E}}}^2}{2}+\frac{143\pi^2}{36}-8\ln^2 2+300\ln G+16\ln\pi+8\ln^2\pi-48\gamma_1\\
&-\frac{2\zeta(3)}{5}-\frac{8\zeta'(3)}{5}+72\zeta''(-1)+n_2\\
&\approx 93.089439628(1)\ .
\end{aligned}
\tag{4.96}
$$

And

$$
n_2=+96\mathcal{V}_2-32(\mathcal{Z}_1+\mathcal{Z}_2)\approx -36.495260342(1)\ .
\tag{4.97}
$$

By using the definition in Eq. (4.30) and the results from appendix A for the component spectacles sum-integrals, the last building block of the effective mass becomes[5]:

$$
M_{3,-2}=\frac{-5}{36}\frac{T^2}{(4\pi)^4}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{11}{30}+\gamma_{\mathrm{E}}+24\ln G\right)+m+\mathcal{O}(\epsilon)\right]\ ,
\tag{4.98}
$$

[5]The constant term m shows a deviation from the result in [120] of 1%, most probably due to the poor numerics. However, at this point we could not locate the exact error source. Fortunately the term m is not needed in the final computation.

with

$$\begin{aligned}
m =& -\frac{491}{225} + \ln 2\left(-12 + \frac{16\gamma_{\rm E}}{5} + \frac{384\ln G}{5} - \frac{2\zeta(3)}{25}\right) + \ln G\left(\frac{2506}{25} - \frac{168\gamma_{\rm E}}{5}\right) \\
&+ \frac{536\gamma_{\rm E}}{75} - \frac{99{\gamma_{\rm E}}^2}{10} + \frac{377\pi^2}{150} - \frac{24\ln^2 2}{5} - \frac{48\ln\pi}{5} + \frac{24\ln^2\pi}{5} - \frac{112\gamma_1}{5} - \frac{23\zeta(3)}{150} \\
&- \frac{72}{5}\zeta''(-1) + \frac{\zeta'(3)}{25} + n_1 \\
&\approx 42.1672(1)
\end{aligned} \tag{4.99}$$

with

$$\begin{aligned}
n_1 =& -\frac{144\mathcal{V}_3}{5} - \frac{432\mathcal{V}_4}{5} - \frac{216\mathcal{V}_5}{5} - \frac{5184\mathcal{V}_6}{5} - \frac{2592\mathcal{V}_7}{5} - \frac{2592\mathcal{V}_8}{5} - \frac{864\mathcal{V}_9}{5} \\
&- \frac{432\mathcal{V}_{10}}{5} - 432\mathcal{V}_{11} - \frac{144(\mathcal{Z}_1 + \mathcal{Z}_2)}{25} - \frac{1188\mathcal{Z}_3}{5} + 432\mathcal{Z}_4 - \frac{1512\mathcal{Z}_5}{5} \\
&\approx -2.17594(1)\ .
\end{aligned} \tag{4.100}$$

4.4 Generalizing the sum-integral computation

In subsection 4.2.3 we have determined a general separation of the sum-integral such that any piece can be computed in part with certain methods. Moreover, the concrete example of $V(3;21111;20)$ indicates that generalization of the computation procedure can be achieved to a certain extent.

In order to proceed with the computation of the finite and divergent parts, the different building blocks of the one-loop structures need to be determined in more detail than previously in subsection 4.3.1.

In Eq. (4.46), the definition for the zero-temperature part of the 2-point function in Eq. (4.43) is given. It is an analytic function in P^2 and p_0 but a closed formula for generalized parameters is yet unknown. Therefore, only the needed cases $s_3 = 0, 1, 2$ are explicitly calculated. For $s_3 = 0$, we have an integrand with rotational invariance and the integral can be solved in $d+1$-dimensional spherical coordinates (c.f. Eq. B.2).

For the remaining cases, $s_3 = 1, 2$, the most general (tensor) zero-temperature part of Eq. (4.43) is defined as:

$$\int_Q \frac{Q_{\mu_1} \times \ldots \times Q_{\mu_n}}{[Q^2]^a[(P+Q)^2]^b}\ . \tag{4.101}$$

The common projection technique expresses the tensor integral as a linear combination of all possible tensors made out of the metric tensor, $g_{\mu_1\mu_2}$ and out of the external momentum P_μ. They are the basis vectors of the tensor space.

$$\int_Q \frac{Q_{\mu_1} \times \ldots \times Q_{\mu_n}}{[Q^2]^a[(P+Q)^2]^B} = \sum_{\sigma(n)} \sum_{j=0}^{[n/2]} \tilde{B}^{\sigma(n)}_{n,j+1} \left[\prod_{i=1}^{[n/2]-j} g_{\mu_{2i-1}\mu_{2i}} \times P^2\right] \times \left[\prod_{i=2([n/2]-j)+1}^{n} P_{\mu_i}\right]\ , \tag{4.102}$$

where $[n]$ denotes the integer part of n and in particular, whenever we have a combination including an ϵ-term, we always consider:

$$[\text{number} \pm c \times \epsilon] \equiv [\text{number}]\ . \tag{4.103}$$

The first sum $\sum_{\sigma(n)}$ denotes the sum over all possible combinations of $\mu_1, ..., \mu_n$ taking into account the $g_{\mu\nu} = g_{\nu\mu}$ symmetry and the commutativity of $g_{\mu\nu}$ and P_μ. The coefficients are computed by contracting each side of the equation with every "basis vector" and by solving then the system of equations (having a unique solution) with respect to the coefficients $\tilde{B}^{\sigma(n)}_{n,j+1}$. Our concrete case however simplifies, since we need only the $q_0 = U_\mu Q_\mu$ case with $U_\mu \equiv (1, \mathbf{0})$. In this situation, the sum over all possible combinations of the term $g_{\mu_1\mu_2} \times ... \times g_{\mu_{2i+1}\mu_{2i}} \times P_{\mu_{2i+1}} \times ... \times P_{\mu_n}$ simply reduce upon contraction with $U_{\mu_1}...U_{\mu_n}$ to (symm. fact.) $\times\, p_0^{n-2i+1}$. We therefore absorb the symmetry factor into the coefficient and redefine: (symm. fact.) $\times\, \tilde{B}^{\sigma(n)}_{n,j+1} \equiv B_{n,j+1}$. In the end we have:

$$\begin{aligned}
\int_Q \frac{q_0^c}{[P^2]^a[(P+Q)^2]^b} &= \sum_{j=0}^{[c/2]} B_{c,j+1} \left[\prod_{i=2([c/2]-j)+1}^{c} p_0 \right] [P^2]^{[c/2]-j} \\
&= \sum_{j=0}^{[c/2]} B_{c,j+1} \times p_0^{\{c\}+2j} \times [P^2]^{[c/2]-j} \ ,
\end{aligned} \tag{4.104}$$

where $\{c\} = c - 2\,[c/2] \equiv \frac{1}{2}[1-(-1)^c] = \begin{cases} 0, & c \ \text{even} \\ 1, & c \ \text{odd} \end{cases}$.

The coefficients B are assumed to be known, since they are simply the solution of the system of $[c/2]$ equations. Their general structure is a linear combination of scalar integrals of the form

$$\frac{1}{[P^2]^x} \int_Q \frac{1}{[Q^2]^{a+y}[(P+Q)^2]^{b-x-y}} \ . \tag{4.105}$$

Since these scalar integrals are all proportional to $1/[P^2]^{a+b-\frac{d+1}{2}}$, we redefine the general expression of the zero-temperature part of Π as:

$$\int_Q \frac{q_0^c}{[P^2]^a[(P+Q)^2]^b} = \sum_{j=0}^{[c/2]} A_{c,j+1} \frac{p_0^{\{c\}+2j} \times [P^2]^{[c/2]-j}}{[P^2]^{a+b-\frac{d+1}{2}}} \ , \tag{4.106}$$

A key ingredient in determining the coefficients $A_{c,j}$ is to rewrite scalar products as $2(PQ) = (P+Q)^2 - P^2 - Q^2$ when generating the system of $[c/2]$ equations. In the end we have ($a \equiv s_1$, $b \equiv s_2$):

$$\begin{aligned}
A_{1,1} &= \frac{1}{2}\left[g(s_1, s_2-1, d+1) - g(s_1, s_2, d+1) - g(s_1-1, s_2, d+1)\right] \\
A_{2,1} &= \frac{1}{4d}\left[2g(s_1-1, s_2, d+1) + 2g(s_1, s_2-1, d+1) + 2g(s_1-1, s_2-1, d+1)\right. \\
&\quad \left. - g(s_1, s_2-2, d+1) - g(s_1-2, s_2, d+1) - g(s_1, s_2, d+1)\right] \\
A_{2,2} &= g(s_1-1, s_2, d+1) - (d+1)A_{2,1} \ ,
\end{aligned} \tag{4.107}$$

with $g(a,b,d)$ defined in Eq. (B.2). The dependence of A on $\{s_1, s_2, d\}$ is implied. In summary:

$$\int_Q \frac{\{1, q_0, q_0^2\}}{[Q^2]^{s_1}[(P+Q)^2]^{s_2}} = \frac{\{g(s_1, s_2, d+1), p_0 A_{1,1}, A_{2,1}P^2 + A_{2,2}p_0^2\}}{[P^2]^{s_{12}-\frac{d+1}{2}}} \ . \tag{4.108}$$

For the leading UV part of Π^{0-B}_{abc}, Eq.(4.44) is used:

$$\begin{aligned}\Pi^{0-B}_{abc} = \int_q \int_{-\infty-i0^+}^{\infty-i0^+} \frac{\mathrm{d}q_0}{2\pi} \frac{q_0^c}{[q_0^2+\mathbf{q}^2]^a} \left[\frac{1}{[(q_0+p_0)^2+(\mathbf{q}+\mathbf{p})^2]^b} \right. \\ \left. + \frac{1}{[(q_0-p_0)^2+(\mathbf{q}+\mathbf{p})^2]^b} \right] \frac{1}{e^{iq_0/T}-1} \,. \end{aligned} \tag{4.109}$$

The integration over q_0 has to be performed using the residue theorem. In order to extract the leading UV piece out of the integral, an asymptotic expansion in terms of $\mathbf{p}^2$ is carried out. In this way the integrand simplifies and integration over $\mathbf{q}$ can be performed for each term of the expansion in part. These terms represent the leading UV pieces of Π^{0-B}.

There is a much simpler way to extract (at least the first two) leading UV contributions (Π^C_{abc}) out of the thermal part of Π_{abc}, as will be discussed now [28].

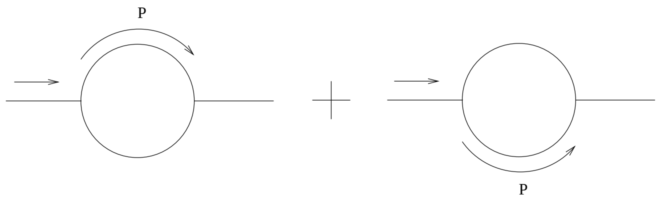


Figure 4.6: Extraction of leading UV piece out of $\Pi(P)$.

By adding up the contributions in which the external momentum flows through both loop lines and by taking the limit $P \to \infty$, one obtains the leading momentum behavior multiplied by some one-loop tadpole.

$$\begin{aligned}\Pi^C_{abc} &= \lim_{P\to\infty} \sum\!\!\!\!\!\!\int_Q \frac{q_0^c}{[Q^2]^a[(P+Q)^2]^b} + \lim_{P\to\infty} \sum\!\!\!\!\!\!\int_Q \frac{(-q_0-p_0)^c}{[(P+Q)^2]^a[Q^2]^b} \\ &= \left[\sum\!\!\!\!\!\!\int_Q \frac{q_0^c}{[Q^2]^a} \right] \frac{1}{[P^2]^b} + (-1)^c \sum_{n=0}^{c} \binom{c}{n} \frac{p_0^{c-n}}{[P^2]^a} \left[\sum\!\!\!\!\!\!\int_Q \frac{q_0^n}{[Q^2]^b} \right] \\ &= \frac{\eta_c I_a^{c/2}}{[P^2]^b} + (-1)^c \sum_{n=0}^{c} \binom{c}{n} \frac{\eta_n p_0^{c-n}}{[P^2]^a} I_b^{n/2} \,, \end{aligned} \tag{4.110}$$

where $\eta_i = \frac{1+(-1)^{[i]}}{2} = \begin{cases} 0, & i \text{ odd} \\ 1, & i \text{ even} \end{cases}$.

The term η_i needs to be included as I_a^b contains by definition only even powers in the Matsubara mode. Thus, we have to make sure that terms with odd powers in the Matsubara modes vanish. The second term of the right hand side (rhs) of Eq. (4.110) is obtained by performing a momentum translation, $Q \to -Q - P$.

The last term needed for the computation of the sum-integral is $\Pi^E(p_0 = 0)$. It is simply a generalized one-loop self-energy in d dimensions. This case is largely used for the zero modes:

$$\Pi^E_{ab0}(p_0 = 0) = \frac{g(a,b,d)}{(p^2)^{a+b-d/2}} \,. \tag{4.111}$$

With these definitions, the finite and divergent parts can finally be computed.

4.4.1 Finite parts

This setup follows the one in [120], but with a slightly different approach on the finite pieces. The central formula for performing the finite terms in configuration space is the inverse Fourier transformation of the propagator:

$$\frac{1}{[P^2]^s} = \frac{2^{1-s}}{(2\pi)^{d/2}\Gamma(s)} \int \mathrm{d}^3\mathbf{r} e^{i\mathbf{pr}} \left(\frac{p_0^2}{r^2}\right)^{\frac{d-2s}{4}} K_{\frac{d}{2}-s}(|p_0|r) \, , \tag{4.112}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind, Eq. (C.1).

Based on this definition the general one-loop function can be computed in configuration space. By plugging Eq. (4.112) into Eq. (4.43) and by performing the momentum integration, we get:

$$\begin{aligned} \Pi_{s_1 s_2 s_3} = & \frac{T 2^{2-s_{12}}}{(2\pi)^d \Gamma(s_1)\Gamma(s_2)} \sum_{q_0} q_0^{s_3} \int \mathrm{d}^d\mathbf{r}\, r^{2s_{12}-d} e^{i\mathbf{pr}} [q_0^2]^{\frac{d-2s_1}{4}} [(q_0+p_0)^2]^{\frac{d-2s_2}{4}} \\ & \times K_{\frac{d}{2}-s_1}(|q_0|r) K_{\frac{d}{2}-s_2}(|q_0+p_0|r) \, . \end{aligned} \tag{4.113}$$

Using the definition in Eq. (C.2) that explicitly determines the Bessel function of half-integer argument, we obtain:

$$\Pi_{s_1 s_2 s_3} = \frac{(2\pi T)^{2d+1-2s_{12}+s_3}}{2^{s_{12}}(2\pi)^d \Gamma(s_1)\Gamma(s_2)} \int \mathrm{d}^d\mathbf{r}\, \bar{r}^{s_{12}-d-1} e^{i\mathbf{pr}} e^{-|p_0|r} f_{d,s_1 s_2 s_3}(\bar{r}, |\bar{p_0}|) \, , \tag{4.114}$$

with

$$\begin{aligned} f_{d,s_1 s_2 s_3}(x,n) = & e^{|n|x} \sum_{m=-\infty}^{\infty} \sum_{i=0}^{||\frac{d}{2}-s_1|-\frac{1}{2}|} \sum_{j=0}^{||\frac{d}{2}-s_2|-\frac{1}{2}|} \frac{(i+|\frac{d}{2}-s_1|-\frac{1}{2})!}{i!\,(-i+|\frac{d}{2}-s_1|-\frac{1}{2})!} \\ & \times \frac{(j+|\frac{d}{2}-s_2|-\frac{1}{2})!}{j!\,(-j+|\frac{d}{2}-s_2|-\frac{1}{2})!} \frac{m^{s_3}|m|^{\frac{d-1}{2}-s_1-i}|n+m|^{\frac{d-1}{2}-s_2-j}}{(2x)^{i+j}} e^{-x(|m|+|n+m|)} \, . \end{aligned} \tag{4.115}$$

The function $f_{d,s_1 s_2 s_3}(x,n)$ is in general some function $f(\coth x, |n|)$ and specific values are explicitly shown in appendix E:

The zero temperature part Π^B is in general a product of a simple propagator-like structure of the form $[P^2]^{-\epsilon}$ and a divergent pre-factor. Therefore, the $\Gamma(\epsilon)^{-1}$ from Eq. (4.112) cancels the divergence of the pre-factor and renders the zero-temperature piece in configuration space finite [6].

$$\Pi^B_{s_1 s_2 s_3} = \frac{(2\pi T)^{2d+1-2s_{12}+s_3}}{2^{s_{12}}(2\pi)^d \Gamma(s_1)\Gamma(s_2)} \int \mathrm{d}^d\mathbf{r}\, \bar{r}^{s_{12}-d-1} e^{i\mathbf{pr}} e^{-|p_0|r} f^B_{d,s_1 s_2 s_3}(\bar{r}, |\bar{p_0}|) \, , \tag{4.116}$$

with

$$\begin{aligned} f^B_{d,s_1 s_2 0} = & \frac{(4\pi)^{\frac{d+1}{2}} g(s_1,s_2,d+1)\Gamma(s_1)\Gamma(s_2)}{\Gamma\left(s_{12}-\frac{d+1}{2}\right)} |n|^{d-s_{12}} \\ & \times \sum_{j=0}^{||\frac{1}{2}-s_{12}+d|-\frac{1}{2}|} \frac{(j+|\frac{1}{2}-s_{12}+d|-\frac{1}{2})!}{j!(-j+|\frac{1}{2}-s_{12}+d|-\frac{1}{2})!} (2|n|x)^{-j} \, . \end{aligned} \tag{4.117}$$

[6] It is not the case for Π_{210} for which only Π^{B+C}_{210} is finite in $d=3$.

The definitions for $s_3 = 1, 2$ are straightforward.

For the leading UV part, we simply plug Eq. (4.110) into Eq. (4.112):

$$\Pi^C_{s_1 s_2 s_3} = \frac{(2\pi T)^{2d+1-2s_{12}+s_3}}{2^{s_{12}}(2\pi)^d \Gamma(s_1)\Gamma(s_2)} \int \mathrm{d}^d\mathbf{r}\, \bar{r}^{s_{12}-d-1} e^{i\mathbf{p}\mathbf{r}} e^{-|p_0|r} f^C_{d,s_1 s_2 s_3}(\bar{r}, |\bar{p}_0|) \,, \tag{4.118}$$

with

$$\begin{aligned} f^C_{d,s_1 s_2 s_3}(x,n) &= \frac{\eta_{s_3}(2\pi)^{\frac{d+1}{2}} 2^{s_1}\Gamma(s_1) I_{s_1}^{s_3/2}}{(2\pi T)^{d+1-2s_1+s_3}} x^{\frac{d+1}{2}-s_1} |n|^{\frac{d+1}{2}-s_2} \kappa_{\frac{d}{2}-s_2}(|n|x) \\ &+ (-1)^{s_3} \sum_{i=0}^{s_3} \eta_i \binom{s_3}{i} \frac{(2\pi)^{\frac{d+1}{2}} 2^{s_2}\Gamma(s_2) I_{s_2}^{i/2}}{(2\pi T)^{d+1-2s_2+i}} x^{\frac{d+1}{2}-s_2} |n|^{\frac{d+1}{2}-s_1} n^{s_3-i} \kappa_{\frac{d}{2}-s_1}(|n|x) \,. \end{aligned} \tag{4.119}$$

where κ is defined in Eq. (C.2).

After defining the building blocks of the computation in configuration space, all three generic types of finite pieces can be determined.

We start with the term:

$$V^{\mathrm{f},1}(d; s_1 s_2 s_3 s_4 s_5; s_6 s_7) \equiv \sumint_P{}' \frac{\Pi^{0-B}_{s_2 s_4 s_6} \Pi^{0-B}_{s_3 s_5 s_7}}{[P^2]^{s_1}} \,. \tag{4.120}$$

There is a specific ordering in the computation that guarantees the analytic manipulation of this sum-integral to at most a double integral from 0 to ∞. The prescription is the following:

- Plug in the definitions from Eq.(4.114, 4.116) by first explicitly computing the sums in Eq.(4.115, 4.117).

- Perform the momentum integration/Fourier transformation, Eq. (4.114):

$$\begin{aligned} \int_p \frac{e^{i\mathbf{p}(\mathbf{r}+\mathbf{s})}}{[P^2]^a} &= \frac{2^{-a}}{(2\pi)^{\frac{d-1}{2}}\Gamma(a)} e^{-|p_0||\mathbf{r}+\mathbf{s}|} |p_0|^{\frac{d-1}{2}-a} |\mathbf{r}+\mathbf{s}|^{\frac{d-1}{2}-a} \\ &\times \sum_{j=0}^{||\frac{d}{2}-a|-\frac{1}{2}|} \frac{(j+|\frac{d}{2}-a|-\frac{1}{2})!}{j!\,(-j\,|\frac{d}{2}-a|-\frac{1}{2})!} (2|p_0||\mathbf{r}+\mathbf{s}|)^{-j} \,. \end{aligned} \tag{4.121}$$

- Perform the angular integrations of the configuration space variables. Notice that, by conveniently choosing the axes in such a way that the angle between $\mathbf{r}$ and $\mathbf{s}$ is the polar angle, all remaining angular integrations become trivial (c.f. Eq. (C.4)). The angle integration generates the function $h_{a,b}(r, s, |p_0|)$, Eq. (C.9).

- Perform summation over p_0. It is not always worthwhile for a numerical integration to perform it. It turns out that for terms in higher dimensions it is more efficient to sum up the first few terms of the numerically integrated double integral.

- rescaling (to avoid a dimension-full integrand) can be performed at any stage of the computation.

The generic result is:

$$
\begin{aligned}
&V^{\mathrm{f},1}(d;s_1s_2s_3s_4s_5;s_6s_7) = \\
&\frac{T^{3(d+1)-2s_{12345}+s_{67}}}{2^{\frac{5}{2}(d-1)-s_{12345}+s_{46}}\left[\prod_{i=1}^5\Gamma(s_i)\right]\pi^{-1/2}\Gamma(\frac{d}{2})\Gamma(\frac{d-1}{2})(4\pi)^{2s_{12345}-s_{67}-\frac{3}{2}(d+1)}} \\
&\times\int_0^\infty \mathrm{d}x_1\mathrm{d}x_2\, x_1^{s_{24}-2}x_2^{s_{35}-2}\sum_{n=1}^\infty \frac{e^{-n(x_1+x_2)}}{n^{a-\frac{d-1}{2}}} f^{0-B}_{s_2s_4s_6}(x_1,n)f^{0-B}_{s_3s_5s_7}(x_2,n) \\
&\times\sum_{j=0}^{\left|\left|\frac{d}{2}-s_1\right|-\frac{1}{2}\right|}\frac{(j+\left|\frac{d}{2}-s_1\right|-\frac{1}{2})!}{j!\left(-j+\left|\frac{d}{2}-s_1\right|-\frac{1}{2}\right)!}\frac{h_{-\frac{d+1}{2}+s_1-j,\frac{d-3}{2}}(x_1,x_2,n)}{(2n)^j}\ .
\end{aligned}
\tag{4.122}
$$

The second class of finite terms is of the form:

$$
V^{\mathrm{f},2}(d;s_1s_2s_3s_4s_5;s_6s_7) \equiv \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi^{B-D}_{s_2s_4s_6}\Pi^{0-B-C}_{s_3s_5s_7}}{[P^2]^{s_1}}\ . \tag{4.123}
$$

Due to the $B-D$ term, whose ϵ-expansion generates a term of the type $\ln P^2+\ln\,\alpha\times\mathrm{const}$, the integral will contain a trivial part which is merely a two-loop sum-integral (to be computed analytically) and a complicated term that necessitates a numerical evaluation.

$$
\begin{aligned}
\Pi^B_{abc}-\Pi^D_{abc} &= \left([P^2]^{-\epsilon}-[\alpha T^2]^{-\epsilon}\right)\frac{\sum_j^{[c/2]}A_{c,j+1}[P^2]^{[c/2]-j}p_0^{\{c\}+2j}}{[P^2]^{\left[a+b-\frac{d+1}{2}\right]}} \\
&= \sum_j^{[c/2]}\underbrace{(-\epsilon A_{c,j+1})}_{\mathcal{O}(\epsilon^0)}\frac{[P^2]^{[c/2]-j}p_0^{\{c\}+2j}}{[P^2]^{\left[a+b-\frac{d+1}{2}\right]}}\left[\ln\frac{P^2}{4\pi^2}-\ln\frac{\alpha T^2}{4\pi^2}\right]+\mathcal{O}(\epsilon)\ .
\end{aligned}
\tag{4.124}
$$

The trivial part contains the dummy variable α and makes sure that it cancels the α-dependence in the divergent piece so that in the end the sum-integral is indeed α-independent.
By plugging in the second term in Eq. (4.124) in Eq. (4.123), we obtain:

$$
V^{\mathrm{f},2}\big|_{\ln\alpha} = \ln\frac{\alpha T^2}{4\pi^2}\sum_j^{[\frac{s_6}{2}]}(\epsilon A_{s_6,j+1})\sum\!\!\!\!\!\!\!\int_P{}'\frac{[P^2]^{[s_6/2]-j}p_0^{\{s_6\}+2j}}{[P^2]^{\left[s_{24}-\frac{d+1}{2}\right]+s_1}}\Pi^{0-B-C}_{s_3s_5s_7}\ . \tag{4.125}
$$

Terms of the form $\sum\!\!\!\!\!\!\int_P{}'\frac{p_0^{s_5}}{[P^2]^{s_1}}\Pi_{s_2s_3s_4}$ are simply two-loop sum-integrals (with the $p_0=0$ mode subtracted) that can be reduced via IBP relations to products of one-loop tadpoles (c. f. Eq. (B.6)):

$$
\sum\!\!\!\!\!\!\!\int_P{}'\frac{p_0^{s_5}}{[P^2]^{s_1}}\Pi_{s_2s_3s_4} = L^d(s_1s_2s_3;s_4s_5)-[1-\delta_{0,s_5}]J^d(s_1s_2s_3;s_4)\ . \tag{4.126}
$$

The delta function takes care that the zero-mode subtraction makes sense only for $s_5=0$ for which a zero-mode is existent. All the other terms are simply products of one-loop tadpoles defined in Eq. (B.3). So by plugging in Eqs. (4.46, 4.110) into Eq. (4.125), we get:

$$
\begin{aligned}
V^{\mathrm{f},2}\big|_{\ln\alpha} = \ln\frac{\alpha T^2}{4\pi^2}\sum_{j=0}^{\left[\frac{s_6}{2}\right]}\overbrace{(\epsilon A_{s_6,j+1})}^{\epsilon\to 0}\Bigg[& L^d\left(\left[s_{24}-\frac{d+1}{2}\right]-\left[\frac{s_6}{2}\right]+s_1+j,s_3,s_5;\{s_6\}+2j,s_7\right)\\
& -\delta_{0,\{s_6\}+2j}\times\eta_{s_7}\times J^d\left(\left[s_{24}-\frac{d+1}{2}\right]-\left[\frac{s_6}{2}\right]+s_1+j,s_3,s_5;\frac{s_7}{2}\right)\\
& -\sum_{i=0}^{\left[\frac{s_7}{2}\right]}A_{s_7,i+1}\times\eta_{\{s_7\}+\{s_6\}+2(i+j)}\times I^{\frac{\{s_7\}+\{s_6\}}{2}+i+j}_{s_{135}-\frac{d+1}{2}+\left[s_{24}-\frac{d+1}{2}\right]-\left[\frac{s_6}{2}\right]-\left[\frac{s_7}{2}\right]+i+j}\\
& -\eta_{s_7}\times\eta_{\{s_6\}+2j}\times I^{\frac{s_7}{2}}_{s_3}\times I^{\frac{\{s_6\}}{2}+j}_{s_{15}+\left[s_{24}-\frac{d+1}{2}\right]-\left[\frac{s_6}{2}\right]+j}\\
& -(-1)^{s_7}\sum_{i=0}^{s_7}\binom{s_7}{i}\times\eta_i\times\eta_{s_7+\{s_6\}-i+2j}\times I^{\frac{i}{2}}_{s_5}\times I^{\frac{s_7}{2}+\frac{\{s_6\}}{2}-\frac{i}{2}+j}_{s_{13}+\left[s_{24}-\frac{d+1}{2}\right]-\left[\frac{s_6}{2}\right]+j}\Bigg]\,.
\end{aligned}
\tag{4.127}
$$

The first term in Eq. (4.123) is now:

$$
V^{\mathrm{f},2}\big|_{\ln P^2} = \sum\!\!\!\!\!\!\!\int_P{}'\sum_{j=0}^{\left[\frac{s_6}{2}\right]}(-\epsilon A_{s_6,j+1})\,\frac{p_0^{\{s_6\}+2j}}{[P^2]^{s_1+\left[s_{24}-\frac{d+1}{2}\right]-\left[\frac{s_6}{2}\right]+j}}\ln\frac{P^2}{4\pi^2}\Pi^{0-B-C}_{s_3s_5s_7}\,. \tag{4.128}
$$

The ordering of integration is:

- Perform angular integration of configuration space variable, Eq. (C.5, C.6)
- Angular integration of the momentum space variable becomes trivial.
- Perform integration of radial component of momentum variable, which generates the function $l_{a,d}$, Eq. (C.12).
- Rescaling can be performed at any stage.

The outcome is

$$
\begin{aligned}
V^{\mathrm{f},2}\big|_{\ln P^2} = & \frac{\sqrt{\pi}}{\Gamma(\frac{d}{2})\Gamma(s_3)\Gamma(s_5)}\frac{T^{2d-2s_{135}+s_{67}+2-2\left[s_{24}-\frac{d+1}{2}\right]}}{2^{\frac{3d-1}{2}-1-s_{35}+s_7}(4\pi)^{2s_{135}-s_{67}+2\left[s_{24}-\frac{d+1}{2}\right]-d-1}}\\
& \times\sum_{n=1}^{\infty}\int_0^{\infty}\mathrm{d}x\,x^{s_{35}-2}e^{-nx}f^{0-B-C}_{d,s_3s_5s_7}(x,n)\sum_{j=0}^{\left[\frac{s_6}{2}\right]}\frac{-\epsilon A_{s_6,j+1}\times n^{s_6-2\left[\frac{s_6}{2}\right]+2j}}{2^{s_6-\left[\frac{s_6}{2}\right]-s_1-\left[s_{24}-\frac{d+1}{2}\right]+j}}\\
& \times\frac{l_{s_1+\left[s_{24}-\frac{d+1}{2}\right]-\left[\frac{s_6}{2}\right]+j,d}(x,n)}{\Gamma\left(s_1+\left[s_{24}-\frac{d+1}{2}\right]-\left[\frac{s_6}{2}\right]+j\right)}\,.
\end{aligned}
\tag{4.129}
$$

The last class of finite integrals is part of the zero-mode contribution $V^{\mathrm{z,f}}$.

The IBP zero-mode reduction program proceeds this step (c.f. appendix D). It works in such a way, that the new obtained zero-mode sum-integrals can be calculated with a similar splitting procedure as for the non-zero case. That means, that potentially IR divergent pieces stem only from the one-loop substructures, Π. The proper subtraction should render the remainder finite (c.f. 48).

The generic finite part is then:

$$V^{\mathrm{z,f}}(d;s_1s_2s_3s_4s_5;s_6s_7) = \sum\!\!\!\!\!\!\!\int_P \delta_{p_0}\frac{1}{[P^2]^{s_1}}\Pi^{0-A}_{s_2s_4s_6}\Pi^{0-A}_{s_3s_5s_7}\ . \tag{4.130}$$

Similarly to the first two finite parts, there is some prescription of the ordering in which the integrals should be performed. During integration, two different cases need to be taken into account: $s_1 \neq 0$ and $s_1 = 0$.

- $s_1 \neq 0$ case:
  - Perform the angular integration of the configuration space variables, Eq. (C.5).
  - Perform the integration over the radial part of the momentum variable, Eq. (C.8). It's angular integral is trivial.
- $s_1 = 0$ case:
  - The momentum integral is the integral representation of the Dirac delta function. It generates a $\delta^{(3)}(\mathbf{r}+\mathbf{s})$ and eliminates directly one of the configuration space integrals.
  - Perform angular integration of the configuration space variable, Eq. (C.4).
- Rescale the integral.
- For a certain combination of exponent parameters, the integration can be reduced to 1 dimension. E.g. $Z(3;1212;20)$.

The general result for the $s_1 \neq 0$ case is:

$$\begin{aligned}
V^{\mathrm{z,f}} &= \sum\!\!\!\!\!\!\!\int_P \delta_{p_0}\frac{\Pi^{0-A}_{s_2s_4s_6}\Pi^{0-A}_{s_3s_5s_7}}{[P^2]^{s_1}} \\
&= \frac{\sqrt{\pi}\,\Gamma(\frac{d}{2}-s_1)}{2^{d-1-s_{2345}+2s_1+s_{67}}\Gamma(\frac{d}{2})^2\left[\prod_{i=1}^{5}\Gamma(s_i)\right]}\frac{T^{3(d+1)-2s_{12345}+s_{67}}}{(4\pi)^{-\frac{3}{2}(d+1)+2s_{12345}-s_{67}}} \\
&\times \int_0^\infty \mathrm{d}x_1\mathrm{d}x_2\, x_1^{s_{24}-2}x_2^{s_{35}-2} f^{0-A}_{d,s_2s_4s_6}(x_1,0) f^{0-A}_{d,s_3s_5s_7}(x_2,0) a_{s_1,d}(x_1,x_2)\ .
\end{aligned} \tag{4.131}$$

Since Eq. (4.57) shows that in any sum-integral of interest, there is (at least) one substructure of the form Π^{0-E}, that is, for which summation over the Matsubara modes need not to be performed a priori, we give an alternative expression for this substructure (c.f Eq. (E.4)), in particular for the case $s_1 = 0$:

$$\begin{aligned}
V^{\mathrm{z,f}} &= \sum\!\!\!\!\!\!\!\int_P \delta_{p_0}\Pi^{0-E}_{s_2s_4s_6}\Pi^{0-B}_{s_3s_5s_7} \\
&= \frac{\sqrt{\pi}}{2^{2d-1-s_{2345}+s_{67}}\Gamma(\frac{d}{2})\left[\prod_{i=2}^{5}\Gamma(s_i)\right]}\frac{T^{3(d+1)-2s_{2345}+s_{67}}}{(4\pi)^{-\frac{3}{2}(d+1)+2s_{2345}-s_{67}}} \\
&\times \int_0^\infty \mathrm{d}x x^{s_{2345}-d-3} f^{0-B}_{d,s_3s_5s_7}(x,0) \sum_{i=0}^{||\frac{d}{2}-s_3|-\frac{1}{2}|}\frac{(i+|\frac{d}{2}-s_3|-\frac{1}{2})!}{i!\,(-i+|\frac{d}{2}-s_3|-\frac{1}{2})!} \\
&\times \sum_{j=0}^{||\frac{d}{2}-s_5|-\frac{1}{2}|}\frac{(j+|\frac{d}{2}-s_5|-\frac{1}{2})!}{j!\,(-j+|\frac{d}{2}-s_5|-\frac{1}{2})!}\frac{\mathrm{Li}_{s_{24}-s_6-d+1+i+j}\left(e^{-2x}\right)}{(2x)^{i+j}}
\end{aligned} \tag{4.132}$$

with

$$\mathrm{Li}_s\left(x\right) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \ . \tag{4.133}$$

4.4.2 Divergent parts

With the expressions of the divergent pieces $\Pi^{\{B,C,E\}}$ in Eqs. (4.108, 4.110, 4.49, 4.111) and through their simple propagator-type structure, it becomes clear that the divergent parts of the sum-integral are formed of at most two-loop sum-integrals, but predominantly of some factorized one-loop structures.

There are in principle two types of divergent structures that occur. The first one is of the form:

$$\Pi_1^D \Pi_2^0, \text{ or } \Pi_1^C \Pi_2^0 \ . \tag{4.134}$$

These have the property that all propagators are of the form $1/[P^2]^{\mathbb{Z}}$ and the two-loop sum-integrals that they form, can be systematically reduced to a product of one-loop by IBP. By recalling the definition of a two-loop sum-integral, Eq. (B.6), and also the fact that the zero Matsubara mode is omitted in the outermost integration, we have:

$$\SumInt_P{}' \frac{p_0^a}{[P^2]^b} \Pi^0_{cfg} = \SumInt_P \frac{p_0^a}{[P^2]^b} \Pi^0_{cfg} - \SumInt_P \delta_{p_0} \frac{p_0^a}{[P^2]^b} \Pi^0_{cfg} \ . \tag{4.135}$$

The first term on the right hand side is a standard two-loop sum-integral, ($L^d(bcf;ag)$) and the second term is -only if $a = 0$- a special two-loop sum-integral defined in Eq. (B.5), $J^d(bcf;g)$ and otherwise 0.

The other case occurs if we are dealing with any combination of $\Pi^{\{B,C,D,E\}}$. In that case, it is not excluded to obtain some combinations of propagators of the form $1/[P^2]^{\mathbb{Z}+\epsilon}$. The result will be some product of simple one-loop tadpoles or J^d (in the case of Π^E). In addition, the omission of zero Matsubara mode is irrelevant, since it gives rise to a scaleless integral, with in dimensional regularization is 0:

$$\SumInt_P \delta_{p_0} \frac{1}{[P^2]^a} = 0 \ . \tag{4.136}$$

With these ideas in mind, we have 5 types of divergent structures (omitting those multiplied by c_0). The first divergent part is of the form:

$$\begin{aligned}
&V^{\mathrm{d},1}(d;s_1s_2s_3s_4s_5;s_6s_7) = \sumint_P{}' \frac{\Pi^D_{s_2s_4s_6}\Pi^{0-B-c_2C}_{s_3s_5s_7}}{[P^2]^{s_1}} + (246 \leftrightarrow 357) \\
&= \frac{1}{(\alpha T^2)^\epsilon} \sum_{i=0}^{[s_6/2]} A_{s_6,i+1} \Bigg[L^d\left(s_{124} - \frac{[d]+1}{2} - \left[\frac{s_6}{2}\right] + 1, s_3, s_5; \{s_6\}+2i, s_7\right) \\
&- \delta_{0,\{s_6\}+2i} \times \eta_{s_7} J^d\left(s_{124} - \frac{[d]+1}{2} - \left[\frac{s_6}{2}\right] + 1, s_3, s_5; \frac{s_7}{2}\right) \\
&- \sum_{j=0}^{[s_7/2]} A_{s_7,j+1}\eta_{\{s_6\}+\{s_7\}+2(i+j)} \times I^{\frac{\{s_6\}+\{s_7\}}{2}+i+j}_{s_{12345}-1-\frac{d+[d]}{2}-[\frac{s_6}{2}]-[\frac{s_7}{2}]+i+j} \\
&- c_2\eta_{s_7} \times \eta_{\{s_6\}+2i} \times I^{\frac{s_7}{2}}_{s_3} \times I^{\frac{\{s_6\}}{2}+i}_{s_{1245}-\frac{[d]+1}{2}-[\frac{s_6}{2}]+i} \\
&- c_2(-1)^{s_7}\sum_{k=0}^{s_7}\binom{s_7}{k}\eta_k\eta_{\{s_6\}+2i+s_7-k} \times I^{\frac{k}{2}}_{s_5} \times I^{\frac{\{s_6\}+s_7-k}{2}+i}_{s_{1234}-\frac{[d]+1}{2}-[\frac{s_6}{2}]+i}\Bigg] + (246 \leftrightarrow 357)\,.
\end{aligned} \tag{4.137}$$

Recall that $[x]$ means the integer part of x, particularly $[3-2\epsilon]=3$.

The second term is of the form:

$$\begin{aligned}
&V^{\mathrm{d},2}(d;s_1s_2s_3s_4s_5;s_6s_7) = \sumint_P{}' \frac{\Pi^B_{s_2s_4s_6}\Pi^C_{s_3s_5s_7}}{[P^2]^{s_1}} + (246 \leftrightarrow 357) \\
&= \sum_{i=0}^{[s_6/2]} A_{s_6,i+1}\Bigg[\eta_{s_7} \times \eta_{\{s_6\}+2i} \times I^{\frac{s_7}{2}}_{s_3} \times I^{\frac{\{s_6\}}{2}+i}_{s_{1245}-\frac{d+1}{2}-[\frac{s_6}{2}]+i} \\
&+ (-1)^{s_7}\sum_{j=0}^{s_7}\binom{s_7}{j}\eta_j\eta_{\{s_6\}+2i+s_7-j} \times I^{\frac{j}{2}}_{s_5} \times I^{\frac{\{s_6\}+s_7-j}{2}+i}_{s_{1234}-\frac{d+1}{2}-[\frac{s_6}{2}]+i}\Bigg] + (246 \leftrightarrow 357)\,.
\end{aligned} \tag{4.138}$$

Finally, the last divergent part of the non-zero modes is:

$$\begin{aligned}
&V^{\mathrm{d},3}(d;s_1s_2s_3s_4s_5;s_6s_7) = \sumint_P{}' \frac{\Pi^B_{s_2s_4s_6}\Pi^B_{s_3s_5s_7}}{[P^2]^{s_1}} \\
&= \sum_{i=0}^{[s_6/2]}\sum_{j=0}^{[s_7/2]} A_{s_6,i+1} \times A_{s_7,j+1} \times \eta_{\{s_6\}+\{s_7\}+2(i+j)} \times I^{\frac{\{s_6\}+\{s_7\}}{2}+i+j}_{s_{12345}-d-1-[\frac{s_6}{2}]-[\frac{s_7}{2}]+i+j}\,.
\end{aligned} \tag{4.139}$$

The divergent parts that may occur in the zero-mode integrals are:

$$\begin{aligned}
V^{\mathrm{d,z},1}(d;s_1s_2s_3s_4s_5;s_6s_7) &= \sumint_P \delta_{p_0}\frac{\Pi^E_{s_2s_4s_6}\Pi_{s_3s_5s_7}}{[P^2]^{s_1}} \\
&= \delta_{0,s_6} \times \eta_{s_7} \times Tg(s_2,s_4,d) \times J^d\left(s_{124}-\frac{d}{2}, s_3, s_5; \frac{s_7}{2}\right)\,,
\end{aligned} \tag{4.140}$$

and

$$\begin{aligned}
V^{\mathrm{d,z},2}(d;s_1s_2s_3s_4s_5;s_6s_7) &= \sumint_P \delta_{p_0}\frac{\Pi^B_{s_2s_4s_6}\Pi_{s_3s_5s_7}}{[P^2]^{s_1}} \\
&= \delta_{0,\{s_6\}} \times \eta_{s_7} \times A_{s_6,1} \times J^d\left(s_{124}-\frac{d+1}{2}-\left[\frac{s_6}{2}\right], s_3, s_5; \frac{s_7}{2}\right)\,.
\end{aligned} \tag{4.141}$$

Concrete examples of divergent part calculations are given in the next two sections.

4.5 Dimension zero sum-integrals

All the sum-integrals of mass dimension zero enter the three-loop term of the effective coupling in Eq. (3.60). They are different from the previous class because, firstly not all of them can be treated the splitting prescription given in Eq. (4.56) and they are the first known three-loop sum-integrals that exhibit the full power of divergence, namely ϵ^{-3}.

In this section the calculation of two sum-integrals for which splitting differs from Eq. (4.56) is presented in more detail. The remaining sum-integrals are to be found in Appendix A.

4.5.1 Example 1: $V(3;12111;00)$

This subsection follows the computation in Ref. [126]. In order to improve legibility, we will denote for the moment this sum-integral from Fig. (4.7) as: $V_{12} \equiv V(3;12111;00)$, according to the convention of naming the finite parts (c.f. Table 4.1).

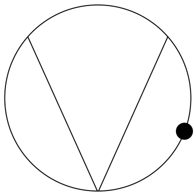


Figure 4.7: Sum-integral $V(3,12111;00)$.

It is due to the explicit $p_0 = 0$ mode in Π_{210} expressed in configuration space (and that would generate a $1/0$ term), that one has to subtract the IR part rather than the zero temperature part. In this sense, we have the following splitting:

$$\begin{aligned}
V_{12} &\equiv \sum\!\!\!\!\!\!\!\int_P \frac{\Pi_{210}\Pi_{110}}{P^2} \\
&= \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{210}^{0-E}\Pi_{110}^{0-B}}{P^2} + \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{210}^{E}\Pi_{110}^{0-B}}{P^2} + \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{110}^{B-D}\Pi_{210}^{0-E-B-C}}{P^2} + \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{110}^{D}\Pi_{210}^{0-E}}{P^2} \\
&+ \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{110}^{B-D}\Pi_{210}^{B+C}}{P^2} + \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{210}^{E}\Pi_{110}^{B}}{P^2} + \sum\!\!\!\!\!\!\!\int_P \delta_{p_0}\frac{\Pi_{210}\Pi_{110}}{P^2} \\
&= V_{12}^{\mathrm{f},1} + V_{12}^{\mathrm{f},2} + V_{12}^{\mathrm{f},3} + V_{12}^{\mathrm{d},1} + V_{12}^{\mathrm{d},2} + V_{12}^{\mathrm{d},3} + V_{12}^{\mathrm{z}} \,,
\end{aligned} \tag{4.142}$$

with $V_{12}^{\mathrm{f}} = V_{12}^{\mathrm{f},1} + V_{12}^{\mathrm{f},2} + V_{12}^{\mathrm{f},3}$.

Finite parts

Using the definitions in Eqs. (4.114, E.5, E.23) we get for the first part of V_1^{f}:

$$\begin{aligned}
V_{12}^{\mathrm{f},1} &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{(\Pi_{210}-\Pi_{210}^{E})(\Pi_{110}-\Pi_{110}^{B})}{P^2} \\
&= T\sum_{p_0}{}' \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{P^2}\frac{T}{2(4\pi)^2}\int \mathrm{d}^3\mathbf{r}\frac{1}{\bar{r}}e^{i\mathbf{pr}}\sum_{q_0}{}'\frac{1}{|\bar{q}_0|}e^{-|q_0|r-|q_0+p_0|r}\frac{T^3}{4} \\
&\cdot \int \mathrm{d}^3\mathbf{s}\frac{1}{\bar{s}^2}e^{i\mathbf{ps}}\left(\coth\bar{s}-\frac{1}{\bar{s}}\right)e^{-|p_0|s} \,.
\end{aligned} \tag{4.143}$$

Notice, that the sum in Π_{210} is not performed yet; it turns out that the sum-integral can be reduced to a one-dimensional integral if the order of summation and integration are switched (opposed to the prescription from page 62).

After performing first the momentum integration and afterwards averaging over the angles in configuration space, Eq. (C.9), we obtain:

$$\begin{aligned} V_{12}^{\mathrm{f},1} = & \frac{2}{(4\pi)^6} {\sum_{n,m}}' \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \frac{1}{y} \left(\coth y - \frac{1}{y}\right) \frac{1}{|n||m|} e^{-|n|y-(|m|+|m+n|)y} \\ & \cdot \left(e^{-|n||x-y|} - e^{-|n|(x+y)}\right) . \end{aligned} \tag{4.144}$$

We now perform the integration over x. The case for which $|m|+|m+n|-|n|=0$ has to be treated separately. After carefully splitting the interval over m accordingly, and performing the x-integration, we get:

$$\begin{aligned} V_{12}^{\mathrm{f},1} = & \frac{2}{(4\pi)^6} {\sum_{n}}' \int_0^\infty \mathrm{d}y \left(\coth y - \frac{1}{y}\right) \frac{e^{-2|n|y}}{|m|} H_{|n|} + \\ & + \frac{2}{(4\pi)^6} {\sum_{n}}' \sum_{\substack{m\in(-\infty,-|n|-1) \\ \cup(1,\infty)}} \int_0^\infty \mathrm{d}y \frac{1}{y} \left(\coth y - \frac{1}{y}\right) \frac{e^{-2|n|y}}{|n||m|} \frac{1-e^{-|m|y-|m+n|y+|n|y}}{|m|+|m+n|-|n|} \\ & + \frac{2}{(4\pi)^6} {\sum_{n,m}}' \int_0^\infty \mathrm{d}y \frac{1}{y} \left(\coth y \frac{1}{y}\right) \frac{1}{|n||m|} \frac{e^{-|m|y-|m+n|y-|n|y} - e^{-2|n|y}}{|m|+|m+n|+|n|} , \end{aligned} \tag{4.145}$$

where H_n is the harmonic number of n: $H_n = \sum_{i=1}^n 1/i$.

By explicitly resolving the summation intervals and using symmetry transformations of the form $m \to -m$ and $m \to m+|n|$, we can rewrite the term as:

$$\begin{aligned} V_{12}^{\mathrm{f},1} = & \frac{4}{(4\pi)^6} \sum_{n=1}^\infty \int_0^\infty \mathrm{d}y \left(\coth y - \frac{1}{y}\right) \frac{e^{-2ny}}{n} H_n \\ & + \frac{2}{(4\pi)^6} \sum_{n=1}^\infty \int_0^\infty \mathrm{d}y \frac{1}{y} \left(\coth y - \frac{1}{y}\right) \frac{e^{-2ny}}{n} \sum_{m=1}^\infty \left(1-e^{-2my}\right) \left(\frac{1}{m^2} - \frac{1}{(m+n)^2}\right) . \end{aligned} \tag{4.146}$$

Performing first the integration over m:

$$\begin{aligned} & \sum_{m=1}^\infty \frac{1}{m^2} = \zeta(2) = \frac{\pi^2}{6}, \quad \sum_{m=1}^\infty \frac{e^{-2my}}{m^2} = \mathrm{Li}_2\left(e^{-2y}\right) , \\ & \sum_{m=1}^\infty \frac{1}{(m+n)^2} = \psi^{(1)}(n+1), \quad \sum_{m=1}^\infty \frac{e^{-2my}}{(m+n)^2} = e^{-2y} \Phi(e^{-2y}, 2, 1+n) , \end{aligned} \tag{4.147}$$

$\mathrm{Li}_2(x)$ being the Dilog function, $\psi^{(1)}$ the Polygamma function and Φ the Hurwitz Lerchphi function, we obtain:

$$\begin{aligned} V_{12}^{\mathrm{f},1} = & \frac{2}{(4\pi)^6} \int_0^\infty \mathrm{d}y \frac{1}{y} \left(\coth y - \frac{1}{y}\right) \left[\left(\frac{\pi^2}{6} - \mathrm{Li}_2(e^{-2y})\right) \sum_{n=1}^\infty \frac{e^{-2ny}}{n} + 2y \sum_{n=1}^\infty \frac{e^{-2ny}}{n} H_n \right. \\ & \left. - \sum_{n=1}^\infty \frac{e^{-2ny}}{n} \psi^{(1)}(n+1) + e^{-2y} \sum_{n=1}^\infty \frac{e^{-2ny}}{n} \Phi(e^{-2y}, 2, n+1) \right] . \end{aligned} \tag{4.148}$$

It turns out that the sums over n can be performed analytically, by using the integral representations of the special functions involved:

$$\sum_{n=1}^{\infty} \frac{e^{-2ny}}{n} = \mathrm{Li}_1\left(e^{-2y}\right) = -\ln(1-e^{-2y}) \,. \tag{4.149}$$

For the second sum in Eq. (4.148), we use the identity $H_n = \psi(n+1) + \gamma_{\mathrm{E}}$ and the integral representation of ψ:

$$\psi(z) = \int_0^{\infty} \mathrm{d}t \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) \,. \tag{4.150}$$

By first performing the sum, we get:

$$\sum_{n=1}^{\infty} \frac{e^{-2ny}}{n}\psi(n+1) = -\int_0^{\infty} \mathrm{d}t\, e^{-t}\left[\frac{\ln(1-e^{-2y})}{t} - \frac{\ln(1-e^{-2y-t})}{1-e^{-t}}\right] \,. \tag{4.151}$$

We now turn our attention to the second term in the brackets and by using the transformation $1-e^{-t}=u$, we obtain:

$$\begin{aligned}
\int_0^{\infty} \mathrm{d}t \frac{\ln(1-e^{-2y-t})}{1-e^{-t}}e^{-t} &= \int_0^1 \mathrm{d}u \frac{\ln(1-e^{-2y}+e^{-2y}u)}{u} \\
&= \int_0^1 \frac{\mathrm{d}u}{u}\ln(1-e^{-2y}) + \int_0^1 \mathrm{d}u \frac{\ln\left(1-\frac{e^{-2y}}{e^{-2y}-1}u\right)}{u} \\
&= \ln(1-e^{-2y})\int_0^1 \frac{\mathrm{d}u}{u} - \mathrm{Li}_2\left(\frac{e^{-2y}}{e^{-2y}-1}\right) \,.
\end{aligned} \tag{4.152}$$

We rewrite now the following integrals as:

$$\int_0^{\infty} \mathrm{d}t \frac{e^{-t}}{t} \overset{e^{-t}\to v}{=} -\int_0^1 \frac{\mathrm{d}v}{\ln v}; \quad \int_0^1 \frac{\mathrm{d}u}{u} \overset{u\to 1-v}{=} \int_0^1 \frac{\mathrm{d}v}{1-v} \,. \tag{4.153}$$

Next, we make use of the definition of the γ_{E} constant, $\gamma_{\mathrm{E}} = \int_0^1 \mathrm{d}v \left(\frac{1}{\ln v} + \frac{1}{1-v}\right)$ and obtain:

$$\sum_{n=1}^{\infty} \frac{e^{-2ny}}{n} H_n = -\mathrm{Li}_2\left(\frac{e^{-2y}}{e^{-2y}-1}\right) \,. \tag{4.154}$$

For the sum involving $\psi^{(1)}$ and Φ, we use their integral representations and employ the same approach as above.

$$\psi^{(1)}(z) = \int_0^{\infty} \mathrm{d}t \frac{te^{-zt}}{1-e^{-t}}; \; \Phi(z,s,a) = \frac{1}{\Gamma(s)}\int_0^{\infty} \mathrm{d}t \frac{t^{s-1}e^{-at}}{1-ze^{-t}} \,. \tag{4.155}$$

Finally, we get:

$$\begin{aligned}
V_{12}^{\mathrm{f},1} &= \frac{1}{(4\pi)^6}\int_0^{\infty} \mathrm{d}y \frac{1-y\coth y}{3y^2}\Big[8y(\pi^2-3i\pi y+y^2) + \ln(e^{2y}-1)\big(-2\pi^2+24i\pi y+12y^2 \\
&\quad +3\ln(e^{2y}-1)(-2i\pi-4y+\ln(e^{2y}-1))\big) - 3(\pi+i\ln(e^{2y}-1))^2 \ln\frac{1+\coth y}{2} \\
&\quad -6\ln(1-e^{-2y})\,\mathrm{Li}_2\left(e^{-2y}\right) - 12y\,\mathrm{Li}_2\left(\frac{\coth y+1}{2}\right) + 6\,\mathrm{Li}_3\left(-2e^{y}\sinh y\right)\Big] \\
&= \frac{1}{(4\pi)^6}\mathcal{V}_{12,1} = \frac{1}{(4\pi)^6}\cdot 0.68647205936406189 54(1) \,.
\end{aligned} \tag{4.156}$$

The second finite part of Eq. (4.142), according to the IBP reduction of Π_{210}^E in Eq. D.5:

$$\begin{aligned} V_{12}^{\mathrm{f},2} &= \sum\!\!\!\!\!\!\!\int_P{}' \, \Pi_{210}^E \frac{\Pi_{110} - \Pi_{110}^B}{P^2} \\ &= T \sum_{p_0}{}' \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(-\frac{T}{4\pi} \frac{|p_0|}{P^4} \right) \frac{1}{P^2} \frac{T^3}{4} \int \mathrm{d}^3\mathbf{r} \frac{1}{\bar{r}^2} e^{i\mathbf{pr}} \left(\coth \bar{r} - \frac{1}{\bar{r}} \right) e^{-|p_0| r} \, . \end{aligned} \tag{4.157}$$

Momentum integration and rescaling are performed:

$$V_{12}^{\mathrm{f},2} = -\frac{2}{(4\pi)^6} \sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}x \left(\frac{x}{n} + \frac{1}{n^2} \right) \left(\coth x - \frac{1}{x} \right) e^{-2nx} \, . \tag{4.158}$$

First the integration and afterwards the summation can be done analytically:

$$\begin{aligned} V_{12}^{\mathrm{f},2} &= \frac{2}{(4\pi)^6} \sum_{n=1}^{\infty} \frac{3 + 2n - 4n \ln n + 4n\psi(n) - 2n^2 \psi^{(1)}(n)}{4n^3} \\ &= -\frac{1}{(4\pi)^6} \left((2\pi)^2 \ln G - \frac{\pi^2}{3} \ln 2\pi - \frac{\pi^2}{6} - \frac{3}{2}\zeta(3) \right) \, . \end{aligned} \tag{4.159}$$

For computing:

$$V_{12}^{\mathrm{f},3} = \sum\!\!\!\!\!\!\!\int_P{}' \, \frac{\Pi_{110}^{B-D} \Pi_{210}^{0-E-B-C}}{P^2} \, , \tag{4.160}$$

the generic formula in Eq. (4.129) can be used, noticing that only the combination Π_{210}^{B+C} makes sense in $d = 3$ dimension, since every piece is divergent individually and only their sum is finite. Therefore we have, Eqs. (E.15, E.25):

$$\begin{aligned} V_{12}^{\mathrm{f},3} = &-\frac{T}{8\pi^2} \sum_{p_0}{}' \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(\frac{\ln \bar{P}^2}{P^2} - \ln \frac{\alpha_1}{4\pi^2} \frac{1}{P^2} \right) \frac{T}{2(4\pi)^2} \int \mathrm{d}^3\mathbf{r} \frac{1}{\bar{r}} e^{i\mathbf{pr}} \\ &\times \left[f_{3,210}^{0-E}(\bar{r}, |\bar{p}_0|) - f_{3,210}^{B+C}(\bar{r}, |\bar{p}_0|) \right] \, , \end{aligned} \tag{4.161}$$

Finally, we obtain:

$$\begin{aligned} V_{12}^{\mathrm{f},3} &= \ln \frac{\alpha_1}{4\pi^2} \frac{2 - 2{\gamma_{\mathrm{E}}}^2 - 4\gamma_1 - \frac{\zeta(3)}{6}}{(4\pi)^6} \\ &\quad - \frac{4}{(4\pi)^6} \sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}x e^{-2nx} \Big[e^{2nx} B(e^{-2x}, n+1, 0) + H_n - \ln(1 - e^{-2x}) \\ &\quad - \left(\gamma_{\mathrm{E}} - e^{2nx} \mathrm{Ei}(-2nx) + \ln \frac{n}{2x} + \frac{x}{6n} \right) \Big] \cdot \left[\left(\ln \frac{2n}{x} - \gamma_{\mathrm{E}} \right) - \mathrm{Ei}(-2nx) e^{2nx} \right] \\ &= \frac{1}{(4\pi)^6} \left[\ln \frac{\alpha_1}{4\pi^2} \left(2 - 2{\gamma_{\mathrm{E}}}^2 - 4\gamma_1 - \frac{\zeta(3)}{6} \right) + \mathcal{V}_{1,3} \right] \, , \end{aligned} \tag{4.162}$$

with $\mathcal{V}_{12,3} = -3.202(1)$.

The summation over the second part converges very slowly and the evaluation of the integrand itself is tedious since it contains special functions. The summation was done up to $n = 7000$ with a relative error of $\mathcal{O}(10^{-3})$ and beyond that a power-law function ($a \cdot n^{-b}$, $b \approx 1.89$) was used to fit the data in the interval $[9000, 19000]$. The analytic summation $n = 9001...\infty$ gives an error of $\mathcal{O}(10^{-4})$.

Divergent parts

The divergent parts are:

$$
\begin{aligned}
V_{12}^{\mathrm{d},1} &= \sum\!\!\!\!\!\!\!\int_P{}' \Pi_{110}^{D} \frac{\Pi_{210} - \Pi_{210}^{E}}{P^2} \\
&= \frac{g(1,1,d+1)}{(\sqrt{\alpha_1}T)^{3-d}} \left[L^d(121;00) - J^d(121;0) - J^d(211;0) \right] \\
&= \frac{1}{2(4\pi)^6} \left(\frac{\mu^2 e^{\frac{\gamma_\mathrm{E}}{3} - \frac{\ln(4\pi\alpha_1)}{3}}}{T^2} \right)^{3\epsilon} \left[\frac{1}{\epsilon^3} + \frac{3}{\epsilon^2} + \frac{1}{\epsilon}\left(9 - 4{\gamma_\mathrm{E}}^2 + \frac{\pi^2}{12} - 8\gamma_1 \right) \right. \\
&+ \left(23 - 12{\gamma_\mathrm{E}}^2 + \frac{16{\gamma_\mathrm{E}}^3}{3} - \frac{5\pi^2}{12} + 16\pi^2 \ln G - \frac{4\pi}{3}\ln 2\pi \right. \\
&\left. \left. - 24\gamma_1 + 16\gamma_\mathrm{E}\gamma_1 + 8\gamma_2 - 7\zeta(3) \right) + \mathcal{O}(\epsilon) \right] ,
\end{aligned}
\tag{4.163}
$$

$$
\begin{aligned}
V_{12}^{\mathrm{d},2} &= \sum\!\!\!\!\!\!\!\int_P{}' (\Pi_{110}^{B} - \Pi_{110}^{D}) \frac{\Pi_{210}^{C}}{P^2} \\
&= g(1,1,d+1) \sum\!\!\!\!\!\!\!\int_P{}' \left(\frac{1}{(P^2)^{2-\frac{d+1}{2}}} - \frac{1}{(\alpha_1 T^2)^{2-\frac{d+1}{2}}} \right) \left[\frac{g(2,1,d+1)}{(P^2)^{3-\frac{d+1}{2}}} + \frac{I_2^0}{P^2} + \frac{I_1^0}{P^4} \right] \frac{1}{P^2} \\
&= g(1,1,d+1) \left[g(2,1,d+1) I_{5-d}^0 + I_2^0 I_{4-\frac{d+1}{2}}^0 + I_1^0 I_{5-\frac{d+1}{2}}^0 \right. \\
&- \left. \frac{g(2,1,d+1) I_{4-\frac{d+1}{2}}^0 + [I_2^0]^2 + I_1^0 I_3^0}{(\alpha_1 T^2)^{2-\frac{d+1}{2}}} \right] \\
&= -\frac{1}{3(4\pi)^6} \left(\frac{\mu^2}{T^2} \right)^{3\epsilon} \left[\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2}\left(3 - \frac{3\ln\alpha_1}{2} \right) + \frac{1}{\epsilon}\left(7 - \frac{9{\gamma_\mathrm{E}}^2}{2} + \frac{\pi^2}{4} + 3\gamma_\mathrm{E}\ln 4\pi \right. \right. \\
&- \left. \left. \frac{3}{2}[\ln 4\pi]^2 - 6\gamma_1 - \ln\alpha_1 \left(\frac{9}{2} + \frac{3}{2}\gamma_\mathrm{E} + \frac{3}{2}\ln 4\pi \right) + \frac{3[\ln\alpha_1]^2}{4} \right) + \mathcal{O}(\epsilon^0) \right] ,
\end{aligned}
\tag{4.164}
$$

and

$$
\begin{aligned}
V_{12}^{\mathrm{d},3} &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{110}^{B}\Pi_{210}^{E}}{P^2} = g(1,1,d+1) J^d\left(2, 3 - \frac{d+1}{2}, 1; 0 \right) \\
&= -\frac{1}{96(4\pi)^4} \left(\frac{\mu^2}{T^2} \right)^{3\epsilon} \left[\frac{1}{\epsilon} + \left(\frac{7}{2} + 3\gamma_\mathrm{E} - 72\ln G + 3\ln\pi \right) \right] + \mathcal{O}(\epsilon) .
\end{aligned}
\tag{4.165}
$$

Notice that:

$$
\begin{aligned}
V_{12}^{\mathrm{d},1} + V_{12}^{\mathrm{d},2} = \frac{1}{6(4\pi)^6} \left(\frac{\mu^2}{T^2} \right)^{3\epsilon} \frac{1}{\epsilon^3} & \left[1 + [...]\cdot\epsilon + [...]\cdot\epsilon^2 + \right. \\
& \left. + \left(... + \ln\frac{\alpha_1}{4\pi^2}(-12 + 12{\gamma_\mathrm{E}}^2 + 24\gamma_1 + \zeta(3)) \right) \cdot \epsilon^3 \right] .
\end{aligned}
\tag{4.166}
$$

From here we could read off the analytic expression for Eq. (4.162).

Zero mode-master

Here we briefly present the calculation of one of the zero-mode masters coming from $V(3;12111;00)$. As already mentioned on page 48, from Π_{110} both the zero temperature and the zero-mode contribution have to be subtracted, such that we end with:

$$
\begin{aligned}
Z(3;12111;00) &= \sum\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{210}\Pi_{110}}{P^2} \\
&= \sum\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{210}^{0-E}\Pi_{110}^{0-B-E}}{P^2} + \sum\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{210}^{E}\Pi_{110}}{P^2} + \sum\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{210}\Pi_{110}^{B+E}}{P^2} \ .
\end{aligned} \tag{4.167}
$$

Thus, the finite part is:

$$
\begin{aligned}
&Z^{\mathrm{f}}(3;12111;00) = \sum\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{210}^{0-E}\Pi_{110}^{0-B-E}}{P^2} \\
&= -\frac{4}{(4\pi)^6}\int_0^\infty \mathrm{d}x\mathrm{d}y \frac{x+y-|x-y|}{y}\ln(1-e^{-2x})\left(\coth y - \frac{1}{y} - 1\right) \\
&= -\frac{2}{3(4\pi)^6}\int_0^\infty \mathrm{d}y \frac{y\coth y - y - 1}{y^2}\left(-2\pi^2 y + 6i\pi y^2 + 4y^3 + 3\,\mathrm{Li}_3\left(e^{2y}\right) - 3\zeta(3)\right) \\
&= \frac{\mathcal{Z}_f}{(4\pi)^6} = \frac{1}{(4\pi)^6}\times(-5.16622349123187417171(1)) \ .
\end{aligned} \tag{4.168}
$$

The divergent terms are:

$$
\begin{aligned}
&\sum\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{210}^{E}\Pi_{110} + \Pi_{210}\Pi_{110}^{B+E}}{P^2} = Tg(2,1,d)J^d(4-\frac{d}{2},1,1;0) \\
&+ g(1,1,d+1)J^d(3-\frac{d+1}{2},2,1;0) + Tg(1,1,d)J^d(3-\frac{d}{2},2,1;0) \ .
\end{aligned} \tag{4.169}
$$

And finally:

$$
\begin{aligned}
Z(3;12111;00) = \frac{1}{(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon} \Bigg[&\frac{1}{\epsilon}\left(\frac{\pi^2}{3} - \frac{\zeta(3)}{2}\right) + \left(\gamma_{\mathrm{E}} + \ln\pi + \frac{4}{3}\right)\Bigg(\pi^2 + \frac{3\zeta(3)}{2} \\
&+ \zeta(3)\left(3\ln 2 - \frac{11}{3}\right) - 24\pi^2\ln G - 3\zeta'(3) + \mathcal{Z}_8\Bigg)\Bigg] \ .
\end{aligned} \tag{4.170}
$$

4.5.2 Example 2: $V(3;22111;02)$

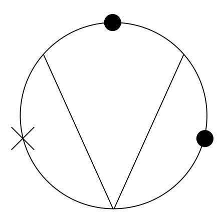


Figure 4.8: Sum-integral $V(3,22111;02)$.

This is the last sum-integral (cf. Fig. (4.8)) for which the splitting of Eq. (4.56) does not apply:

$$\begin{aligned} V_{14} &= \sum\!\!\!\!\!\!\!\int_P \frac{\Pi_{210}\Pi_{112}}{P^4} \\ &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{210}^{0-E}\Pi_{112}^{0-B}}{P^4} + \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{210}^{E}\Pi_{112}^{0-B}}{P^4} + \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{112}^{B-D}\Pi_{210}^{0-E-B-C}}{P^4} \\ &+ \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{112}^{D}\Pi_{210}^{0-E}}{P^4} + \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{112}^{B-D}\Pi_{210}^{B+C}}{P^4} + \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{210}^{E}\Pi_{112}^{B}}{P^4} + \sum\!\!\!\!\!\!\!\int_P \delta_{p_0} \frac{\Pi_{210}\Pi_{112}}{P^4} \\ &= V_{14}^{\mathrm{f},1} + V_{14}^{\mathrm{f},2} + V_{14}^{\mathrm{f},3} + V_{14}^{\mathrm{d},1} + V_{14}^{\mathrm{d},2} + V_{14}^{\mathrm{d},3} + V_{14}^{\mathrm{z}} \,. \end{aligned} \tag{4.171}$$

Finite part

The first finite part, we treat in the same way as Eq. (4.143); we first perform $\mathbf{p}$ integration, Eq. (C.9) and rescale the integral.

$$\begin{aligned} V_{14}^{\mathrm{f},1} &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{210}^{0-E}\Pi_{112}^{0-B}}{P^4} \\ &= \frac{1}{(4\pi)^6} \sum_{m,n}{}' \int_0^\infty \mathrm{d}x\mathrm{d}y \frac{1}{y} \frac{e^{-|m|x-|m+n|x-|n|y}}{|n|^3|m|} f_{3,112}^{0-B}(y,|n|) \\ &\times \left[e^{-|n||x-y|}(1-|n||x-y|) - e^{-|n|(x+y)}(1-|n|(x+y))\right] \,. \end{aligned} \tag{4.172}$$

It becomes clear that integration over x and summation over m is much more demanding in this case. Therefore, we use the generalized formulas in Eqs. (C.19, C.18) and obtain:

$$\begin{aligned} V_{14}^{\mathrm{f},1} &= \frac{1}{144(4\pi)^6} \sum_{n=1}^\infty \int_0^\infty \mathrm{d}y \frac{e^{-2ny}}{n^4 y^4} \left[3y^3 e^{-2y}\left(n^2\coth y + (n+\coth y)\mathrm{csch}^2 y\right) - 3\right. \\ &- \left. ny(3+3ny-y^2)\right] \times \left\{-6n\left[-3\Phi(e^{-2y},2,1+n) + n\Phi(e^{-2y},3,1+n)\right]\right. \\ &+ \left[(3-2n)n\pi^2 + 6\left[\gamma_{\mathrm{E}} + \psi_0(1+n)\right]\left[1+2n\left(y(3+ny)-1\right)\right]\right. \\ &+ 12n(y-1)(\ln[1-e^{-2y}] + \Phi(e^{-2y},1,1+n)) + 3n\left((-6+4ny)\psi_1(1+n)\right. \\ &- \left.\left.\left. n\psi_2(1+n) + (4n-4ny-6)\,\mathrm{Li}_2\left(e^{-2y}\right) - 2n\,\mathrm{Li}_3\left(e^{-2y}\right) + 2n\zeta(3)\right)\right]\right\} \\ &= \frac{1}{(4\pi)^6} \times (0.1544(1)) \,. \end{aligned} \tag{4.173}$$

The second finite term, is after momentum integration and scaling simply:

$$\begin{aligned} V_{14}^{\mathrm{f},2} &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{210}^{E}\Pi_{112}^{0-B}}{P^4} \\ &= -\frac{1}{3(4\pi)^6} \sum_{n=1}^\infty \int_0^\infty \mathrm{d}x \frac{x^2}{n^3}\left(1+\frac{3}{nx}+\frac{3}{n^2x^2}\right) f_{3,112}^{0-B} e^{-2nx} \\ &= \frac{1}{(4\pi)^6} \times (-0.101108838933043(1)) \,. \end{aligned} \tag{4.174}$$

The sum has been done analytically and the integration numerically.

The third finite part is according to Eq. (4.129):

$$
\begin{aligned}
V_{14}^{\mathrm{f},3} &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{112}^{B-D}\Pi_{210}^{0-E-B-C}}{P^4} \\
&= \frac{1}{3(4\pi)^6}\int_0^\infty \mathrm{d}x x \sum_{n=1}^{\infty} f_{3,210}^{0-E-B-C}(x,n)e^{-nx}\left[l_{3,1}(x,n) - 2n^2 l_{3,2}(x,n)\right] \\
&= -\frac{1}{3(4\pi)^6}\sum_{n=1}^{\infty}\int_0^\infty \mathrm{d}x e^{-2nx}\Bigg[e^{2nx}B(e^{-2x},1+n,0) + e^{2nx}\mathrm{Ei}(-2nx) + H_n - \frac{x}{6} \\
&-\ln(1-e^{-2x}) - \ln\frac{ne^{\gamma_{\mathrm{E}}}}{2x}\Bigg] \times \left[2nx + e^{2nx}(1+2nx)\mathrm{Ei}(-2nx) + (2nx-1)\ln\frac{2n}{xe^{\gamma_{\mathrm{E}}}}\right] \\
&= -0.162(1)\ .
\end{aligned}
\tag{4.175}
$$

These finite parts are time consuming as they contain special functions. Moreover their convergence is very low. For the last finite part, the integral has to be evaluated up to $n = 150.000$ in order to obtain an relative error of $\mathcal{O}(10^{-9})$.

Divergent parts

The divergent parts are simply:

$$
\begin{aligned}
V_{14}^{\mathrm{d},1} &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{112}^{D}\Pi_{210}^{0-E}}{P^4} = \sum\!\!\!\!\!\!\!\int_P{}' \frac{(d+1)p_0^2 - P^2}{4d}\frac{g(1,1,d+1)}{(\alpha_2 T^2)^{2-\frac{d+1}{2}}}\frac{\Pi_{210}^{0-E}}{P^4} \\
&= \frac{g(1,1,d+1)}{4d(\alpha_2 T^2)^{2-\frac{d+1}{2}}}\Bigg[(d+1)\sum\!\!\!\!\!\!\!\int_P{}' \frac{p_0^2\Pi_{210}^0}{P^4} - \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{210}^0}{P^4} \\
&-(d+1)\sum\!\!\!\!\!\!\!\int_P{}'\sum\!\!\!\!\!\!\!\int_Q \delta_{q_0}\frac{p_0^2}{P^4Q^4(P+Q)^2} + \sum\!\!\!\!\!\!\!\int_P{}'\sum\!\!\!\!\!\!\!\int_Q \delta_{q_0}\frac{1}{P^2Q^4(P+Q)^2}\Bigg] \\
&= \frac{g(1,1,d+1)}{4d(\alpha_2 T^2)^{2-\frac{d+1}{2}}}\Bigg[(d+1)L^d(221;20) - \Big(L^d(121;00) - J^d(121;0)\Big) \\
&-(d+1)J^d(221;1) + J^d(211;0)\Bigg]\ ,
\end{aligned}
\tag{4.176}
$$

$$
\begin{aligned}
V_{14}^{\mathrm{d},2} &= \sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi_{112}^{B-D}\Pi_{210}^{B+C}}{P^4} \\
&= \frac{g(1,1,d+1)}{4d}\sum\!\!\!\!\!\!\!\int_P{}' \frac{(d+1)p_0^2 - P^2}{P^4}\left[\frac{1}{(P^2)^{2-\frac{d+1}{2}}} - \frac{1}{(\alpha_2 T^2)^{2-\frac{d+1}{2}}}\right] \\
&\times\left[\frac{g(2,1,d+1)}{(P^2)^{3-\frac{d+1}{2}}} + \frac{I_2^0}{P^2} + \frac{I_1^0}{P^4}\right] \\
&= \frac{g(1,1,d+1)}{4d}\Bigg[(d+1)\Big(g(2,1,d+1)\hat{I}^1_{5-\frac{d+1}{2}}(\alpha_2) + I_2^0\hat{I}_3^1(\alpha_2) + I_1^0\hat{I}_4^1(\alpha_2)\Big) \\
&-\Big(g(2,1,d+1)\hat{I}^0_{4-\frac{d+1}{2}}(\alpha_2) + I_2^0\hat{I}_2^0(\alpha_2) + I_1^0\hat{I}_3^0(\alpha_2)\Big)\Bigg]\ ,
\end{aligned}
\tag{4.177}
$$

$$
\begin{aligned}
V_{14}^{\mathrm{d},3} &= \sum\!\!\!\!\!\!\!\int_{P}{}' \frac{\Pi_{210}^{E}\Pi_{112}^{B}}{P^4} = \sum\!\!\!\!\!\!\!\int_{P}{}' \sum\!\!\!\!\!\!\!\int_{Q} \delta_{q_0} \frac{1}{Q^4(P+Q)^2} \frac{g(1,1,d+1)}{4dP^4} \frac{(d+1)p_0^2 - P^2}{(P^2)^{2-\frac{d+1}{2}}} \\
&= \frac{g(1,1,d+1)}{4d} \left[(d+1) J^d\left(2, 4 - \frac{d+1}{2}, 1; 1\right) - J^d\left(2, 3 - \frac{d+1}{2}, 1; 0\right) \right] .
\end{aligned}
\tag{4.178}
$$

Zero-mode

The zero-mode contribution simplifies via IBP reduction to (c.f. Eq. (D.30)):

$$
V_{14}^{\mathrm{z}} = \frac{Z(3; 12211; 02) + Z(3; 12121; 02) - I_2^1 \times J^d(221; 0)}{d-5} . \tag{4.179}
$$

The explicit values for the zero-mode masters are in appendix (A).

Summing up, we notice that the generalization of the splitting procedure and the generic formulas of the individual pieces simplify the work considerably. Even in the two concrete example presented at last, we have partially borrowed some of the generic results from section 4.4.

Chapter 5

Results

At this point we have computed all necessary pieces in order to determine m_{E} and g_{E}. We plug in the computed sum-integrals into Eqs. (3.58) and (3.60) and provide the final, renormalized results on the matching coefficients. We find out that in the case of the effective coupling we still need to consider operators of higher dimension in the Lagrangian from Eq. (3.5). Further, we discuss in which way the higher order operators will enter our computation by pointing out the UV properties of the diagrams that contain the new interactions. We end the current thesis with an outlook on this work, specifically on finishing the computation of g_{E} and on a wider prospective: on the need of new techniques for solving sum-integrals.

5.1 Debye mass

With the master sum-integrals at hand $M^{000}_{31111-2}$, M^{000}_{11110} and M^{020}_{211110}, we are able to express Eq. (3.58) in $d = 3 - 2\epsilon$ dimensions. However, the mass parameter is expressed in terms of the 4d bare coupling g^2 and thus it requires renormalization to render the parameter finite.

For that, we recall the relation between the bare and the renormalized coupling from Eq. (2.44), here rewritten as: $g_B^2 = Z_g g^2(\bar{\mu})$ and the combination $\mu^{-2\epsilon} g^2(\bar{\mu})$ is dimensionless. We have dropped the subscript R for simplicity. The renormalization constant Z_g has to be known to three-loop order in terms of the beta coefficients of QCD.

By starting from the condition that the bare coupling g_B^2 should not depend on the mass scale μ, we obtain the RGE equation for the renormalized coupling and relate it afterwards to the renormalization constant

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_B^2 \overset{!}{=} 0 = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z_g g^2(\bar{\mu}) \,, \tag{5.1}$$

with Z_g taking the general form (2.46):

$$\begin{aligned} Z_g = 1 &+ \frac{g(\bar{\mu})^2 \mu^{-2\epsilon}}{(4\pi)^2} \frac{c_{1,1}}{\epsilon} + \left[\frac{g(\bar{\mu})^2 \mu^{-2\epsilon}}{(4\pi)^2} \right]^2 \left(\frac{c_{2,2}}{\epsilon^2} + \frac{c_{2,1}}{\epsilon} \right) \\ &+ \left[\frac{g(\bar{\mu})^2 \mu^{-2\epsilon}}{(4\pi)^2} \right]^3 \left(\frac{c_{3,3}}{\epsilon^3} + \frac{c_{3,2}}{\epsilon^2} + \frac{c_{3,1}}{\epsilon} \right) + \mathcal{O}(g^8) \,. \end{aligned} \tag{5.2}$$

As we intend to obtain the RGE equation in terms of the beta coefficients in the limit[1] $\epsilon \to 0$,

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g^2(\bar{\mu}) = \frac{\beta_0}{(4\pi)^2} g^4(\bar{\mu}) + \frac{\beta_1}{(4\pi)^4} g^6(\bar{\mu}) + \frac{\beta_2}{(4\pi)^6} g^8(\bar{\mu}) + \mathcal{O}(g^{10}) \,, \tag{5.3}$$

we relate the unknown coefficients $c_{i,j}$ to β_i and obtain:

$$Z_g = 1 + \frac{g(\bar{\mu})^2 \mu^{-2\epsilon}}{(4\pi)^2} \frac{\beta_0}{2\epsilon} + \left[\frac{g(\bar{\mu})^2 \mu^{-2\epsilon}}{(4\pi)^2}\right]^2 \left(\frac{\beta_1}{4\epsilon} + \frac{\beta_0^2}{4\epsilon^2}\right) + \left[\frac{g(\bar{\mu})^2 \mu^{-2\epsilon}}{(4\pi)^2}\right]^3 \left(\frac{\beta_2}{6\epsilon} + \frac{7\beta_0\beta_1}{24\epsilon^2} + \frac{\beta_0^3}{8\epsilon^3}\right) \,, \tag{5.4}$$

were

$$\beta_0 = \frac{-22C_A}{3} \,, \quad \beta_1 = \frac{-68C_A^2}{3} \,, \quad \beta_2 = \frac{-2857C_A^3}{27} \,, \tag{5.5}$$

are the first three beta coefficients for the pure gluonic QCD [5].

In addition, we recall that EQCD is a super-renormalizable theory with the mass-term as the only parameter requiring renormalization (cf. Eq. (3.6)). To obtain the correct counter-term, we need the tree-level EQCD matching coefficients λ and g_E with respect to the renormalized 4d coupling, $g(\bar{\mu})$ [127]:

$$\begin{aligned}
\delta m_{\mathrm{E}}^2 &= 2(N_c^2+1)\frac{1}{(4\pi)^2}\frac{\mu_3^{-4\epsilon}}{4\epsilon}(-g_E^2\lambda C_A + \lambda^2) \\
&= 2(N_c^2+1)\frac{1}{(4\pi)^2}\frac{\mu_3^{-4\epsilon}}{4\epsilon}(-g(\bar{\mu})^2 T)\left(\frac{20}{3}\frac{C_A^2}{N_c^2+1}\frac{g(\bar{\mu})^4\mu^{-2\epsilon}T}{(4\pi)^2}\right) C_A + \mathcal{O}(g^8) \\
&= -\frac{10C_A^3}{3\epsilon}\frac{T^2}{(4\pi)^4} g(\bar{\mu})^6 \mu_3^{-4\epsilon}\mu^{-2\epsilon} + \mathcal{O}(g^8) \,.
\end{aligned} \tag{5.6}$$

Due to reasons of dimensionality, the matching introduces an extra term $\mu^{-2\epsilon}$ as only the combination $g^2(\mu)\mu^{-2\epsilon}$ is dimensionless. Hence m_{E}^2 is of dimension two. By using Eq. (5.6) for the definition of the renormalized mass, the divergence in Eq. (3.58) is exactly cancelled and we obtain the renormalized effective mass in EQCD to three-loop order:

$$m_{\mathrm{E,ren}}^2 = T^2 \left[g^2(\bar{\mu})(\alpha_{\mathrm{E4}} + \epsilon\alpha_{\mathrm{E5}}) + \frac{g^4(\bar{\mu})}{(4\pi)^2}(\alpha_{\mathrm{E6}} + \epsilon\beta_{\mathrm{E2}}) + \frac{g^6(\bar{\mu})}{(4\pi)^4}\alpha_{\mathrm{E8}}\right] + \mathcal{O}(g^8(\bar{\mu})) \,. \tag{5.7}$$

The known coefficients α_{E4}, α_{E5}, α_{E6} and β_{E2} from [60] (an the references therein) are recovered. The constant terms in ϵ are

$$\alpha_{\mathrm{E4}} = \frac{C_A}{3} \,, \quad \alpha_{\mathrm{E6}} = -\beta_0 C_A L + \frac{5C_A^2}{9} \,, \tag{5.8}$$

where

$$L \equiv \ln \frac{\bar{\mu} e^{\gamma_{\mathrm{E}}}}{4\pi T} \,. \tag{5.9}$$

Finally defining

$$L_3 \equiv \ln \frac{\mu_3^2 e^{z_1}}{4\pi T^2} \quad ; \quad z_1 = \frac{\zeta'(-1)}{\zeta(-1)} \,, \tag{5.10}$$

[1] Before taking $\epsilon \to 0$, note that Eq. (5.3) contains on the rhs. terms like $[g^2(\bar{\mu})\mu^{-2\epsilon}]^n$ to match the dimensionality with the lhs.

we obtain for the last coefficient the following expression:

$$\alpha_{\mathrm{E}8} = \frac{C_A^3}{27}\left[484L^2 + 244L - 180L_3 + \frac{1091}{2} - \frac{56\zeta(3)}{5}\right] . \tag{5.11}$$

Note that the coefficients in front of the logarithm L are entirely determined by the beta coefficients and the effective mass coefficients of lower order: e.g. $484C_A^3/27 = -\beta_0^3$. The three-loop result depends on two arbitrary scales $\bar{\mu}$ and μ_3, the first one coming from the 4d renormalization, whereas the second scale enters through the 3d renormalization.

For plotting the result, the concrete running of the 4d coupling with respect to the energy scale is needed. For that, Eq. (5.3) is solved iteratively to three-loop order [5, 128]

$$\frac{g^2(\bar{\mu})}{(4\pi)^2} = -\frac{1}{\beta_0 t} - \frac{\beta_1 \ln t}{\beta_0^3 t^2} - \frac{1}{\beta_0^3 t^3}\left(\frac{\beta_1^2}{\beta_0^2}(\ln^2 t - \ln t - 1) + \beta_2\right) , \tag{5.12}$$

with $t = \ln[\bar{\mu}/\Lambda_{\overline{\mathrm{MS}}}]$, and $\Lambda_{\overline{\mathrm{MS}}}$ is the QCD scale defined in the $\overline{\mathrm{MS}}$ scheme [37, 128].

There is a freedom in choosing the arbitrary mass scales μ and μ_3. We employ the standard procedure to choose the "optimal" value that minimizes the one-loop effective coupling g_{E} [58, 129]: $\mu_{\mathrm{opt}} = 4\pi e^{-\gamma_{\mathrm{E}}-1/22}T$. In order to inspect the sensitivity of the result with respect to the arbitrary scale, we vary it in the range $\mu_{\mathrm{opt}}/2$ and $2\mu_{\mathrm{opt}}$. As we have no information on an optimal scale for μ_3, we simply set it equal to μ.

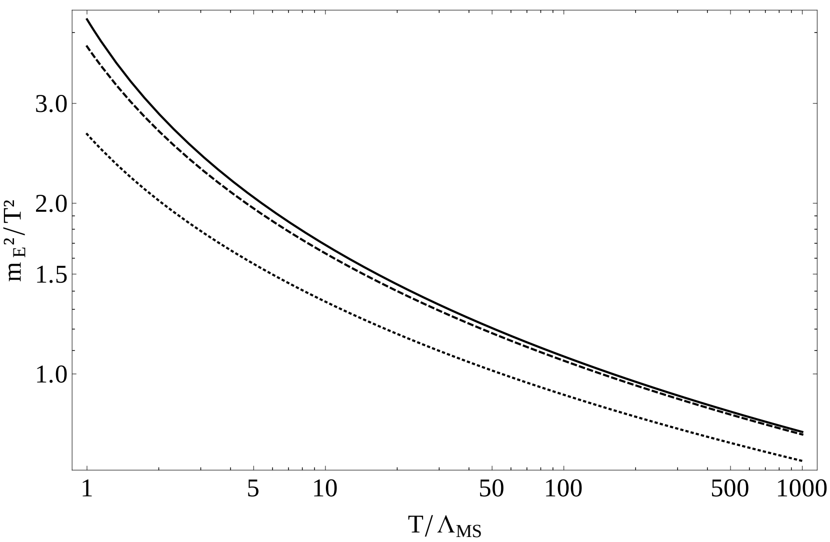


Figure 5.1: The normalized mass parameter m_{E}^2/T^2 up to one-, two- and three loops (dotted, dashed and continuous lines) as a function of $T/\Lambda_{\overline{\mathrm{MS}}}$.

The plot in Fig. (5.1) shows the mass parameter up to the one-, two- and three-loop order and with the fixed μ_{opt}. It shows a slight increase of the three-loop result with respect to the lower loops. Moreover, the plot indicates the convergence of the perturbative expansion to a limiting value, as the correction to the two-loop result is much smaller then the correction from one to two loops. Remarkably, the convergence shows to hold up to temperatures near the QCD scale.

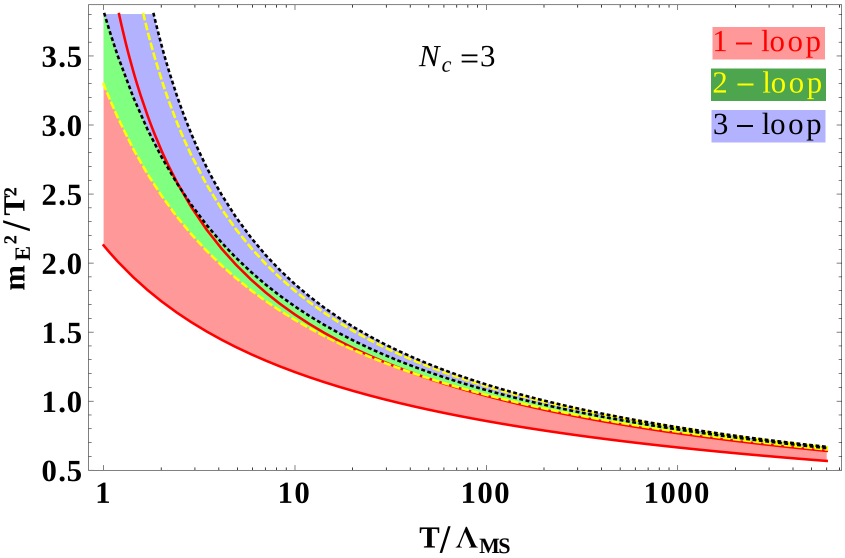


Figure 5.2: The normalized mass parameter $m_{\rm E}^2/T^2$ up to one-, two- and three loops (continuous (red), dashed (yellow) and dotted (black) lines) as a function of $T/\Lambda_{\overline{\rm MS}}$. The colored bands come due to the variation of the optimal scale $\mu_{\rm opt}$.

In Fig. (5.2), we plot the mass parameter to increasing loop order and with the variation of the arbitrary scale. Indeed, the convergence is confirmed again as the sensitivity on the arbitrary scale is slightly smaller than for the one- and two-loop cases.

In the end, we provide the renormalized contribution to the QCD pressure to $\mathcal{O}(g^7)$, coming from the mass parameter (cf. Eq. (3.15) and [88]):

$$p_M(T)|_{m^3_{\rm E,ren}} = \frac{54 d_A T^4 C_A^{7/2}}{\sqrt{3}(4\pi)^5}\left[605\ln^2\frac{\bar{\mu}e^{\gamma_{\rm E}}}{4\pi T} + 299\ln\frac{\bar{\mu}e^{\gamma_{\rm E}}}{4\pi T} - 180\ln\frac{\mu_3^2 e^{z_1}}{4\pi T^2} + \frac{2207}{4} - \frac{56\zeta(3)}{5}\right] g^7 . \tag{5.13}$$

5.2 Effective coupling

For computing the effective coupling, we perform the same steps as for the mass parameter; we plug in the master sum-integrals computed in the previous chapter into Eq. (3.60) and perform the renormalization of the QCD coupling. The outcome is:

$$g_{\rm E}^2 = T\left[g^2(\bar{\mu}) + \frac{g^4(\bar{\mu})}{(4\pi)^2}(\alpha_{\rm E7}+\epsilon\beta_{\rm E3}) + \frac{g^6(\bar{\mu})}{(4\pi)^4}(\gamma_{\rm E1}+\epsilon\beta_{\rm E5}) + \frac{g^8(\bar{\mu})}{(4\pi)^6}\alpha'_{\rm E9} + \mathcal{O}(g^{10})\right] . \tag{5.14}$$

The coefficients $\alpha_{\rm E7}$, $\beta_{\rm E3}$ and $\gamma_{\rm E1}$ can be found in [60] and contain also the fermionic degrees of freedom. Here, we present merely the gluonic pieces:

$$\alpha_{\rm E7} = -\beta_0 L + \frac{C_A}{3} , \quad \gamma_{\rm E1} = -\beta_1 L + \left[\beta_0 L - \frac{C_A}{3}\right]^2 + \frac{C_A^2}{18}(341 - 20\zeta(3)) . \tag{5.15}$$

And finally:

$$
\begin{aligned}
\alpha'_{\mathrm{E}9} = & -\frac{61\zeta(3)C_A^3}{5\epsilon} \\
& + \left\{ \frac{10648}{27}L^3 + \frac{1408}{3}L^2 + \left(\frac{14584}{27} - \frac{4394\zeta(3)}{45}\right) L + \frac{9187}{36} - \frac{1136\gamma_{\mathrm{E}}}{9} - 68{\gamma_{\mathrm{E}}}^2 - \frac{1064{\gamma_{\mathrm{E}}}^3}{9} \right. \\
& - \frac{10333\pi^2}{1134} - \frac{188\gamma_{\mathrm{E}}\pi^2}{25} + \frac{124\pi^4}{2835} - \frac{4216\pi^2 z_1}{189} + \frac{1136\ln 2}{9} - \frac{104{\gamma_{\mathrm{E}}}^2\ln 2}{3} + \frac{1844\pi^2\ln 2}{63} \\
& + \frac{4216\pi^2\ln\pi}{189} - 136\gamma_1 - \frac{1168\gamma_{\mathrm{E}}\gamma_1}{3} - \frac{208\gamma_1\ln 2}{3} - \frac{688\gamma_2}{3} + \frac{1503337\zeta(3)}{9450} \\
& \left. + \frac{3214\gamma_{\mathrm{E}}\zeta(3)}{45} + \frac{28z_1\zeta(3)}{3} - \frac{410\zeta(3)\ln 2}{27} + \frac{29\zeta(5)}{81} - \frac{8852\zeta'(3)}{135} + 303.8(1) \right\} C_A^3 \,.
\end{aligned}
\tag{5.16}
$$

The most striking property is the presence of a divergent term, even after renormalization. The terms in front of the logarithms L are entirely determined by the beta coefficients and the coefficients of the lower loop-order of g_{E}. The numerical term 303.8(1) has a very low accuracy mostly due to the finite terms in Eqs. (4.162) and (4.175).

In the following we discuss the divergent term in Eq. (5.16). Recall that the Lagrangian used in this computation (Eq. (3.5)) is super-renormalizable and thus the 3d effective coupling does not exhibit any divergent counter-terms that could cancel the leftover divergence. Obviously, we have overlooked something.

The possibility of a technical error in evaluating the master sum-integrals or in performing the IBP reduction is very low, since the master sum-integrals were cross checked independently[2], and the same IBP reduction was used for the mass parameter. Moreover, the explicit gauge independent result consolidates our arguments.

Therefore, we inspect the idea that higher order operators in the effective Lagrangian may contribute with a divergent factor to the effective coupling g_{E} at the order g^8.

5.3 Higher order operators

The hint that we may not have considered operators of high enough dimension comes precisely from the $\zeta(3)$-term multiplying the divergent piece in Eq. (5.16), as the same term is found for all the tree-level matching coefficients of the dimension six operators in [54]:

$$
\begin{aligned}
\Delta\mathcal{L}_{\mathrm{EQCD}} = & -\frac{g^2\zeta(3)N_c}{32\pi^4T^2}\mathrm{Tr}\left\{ \frac{ig}{90}F_{\mu\nu}F_{\nu\rho}F_{\rho\mu} - \frac{19}{90}(D_\mu F_{\mu\nu})^2 - \frac{19ig}{15}F_{0\mu}F_{\mu\nu}F_{\nu 0} \right. \\
& \left. + \frac{1}{30}(D_\mu F_{\mu 0})^2 - \frac{6ig}{5}A_0(D_\mu F_{\mu\nu})F_{0\nu} + \frac{11}{6}g^2A_0^2F_{\mu\nu}^2 \right\} .
\end{aligned}
\tag{5.17}
$$

Here, we employ the fundamental representation and $\partial_0 = 0$. By adding Eq. (5.17) to the original Lagrangian, the theory becomes non-renormalizable. Therefore, we expect to find renormalization constants for the fields and for the effective coupling starting with $\mathcal{O}(g^8)$. To find diagrams, which are potentially divergent at $\mathcal{O}(g^8)$ and have the correct structure in order

[2]The most complicated sum-integral $M^{000}_{311111-2}$ was expressed in terms of a different set of higher dimensional sum-integrals as in Ref. [120] and the results agree.

to be regarded as renormalization counter-terms, the new vertices have to be extracted from the Lagrangian.

As a simple exercise, we read off the possible vertices that emerge from the operators, without explicitly performing the Lorentz-index symmetrization or the color algebra. We are merely interested in the power of the 4d coupling g and in the power of the momentum that multiplies the vertices. Fig. (5.3) shows all possible vertices. The general structure of a n_L-particle vertex is found to be $v_L \propto g^{n_L} k^{6-n_L}$.

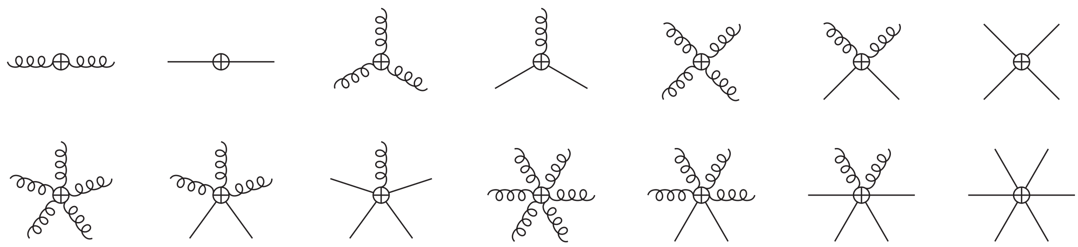
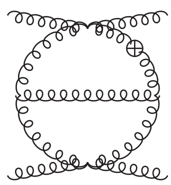


Figure 5.3: The vertices emerging from the dimension six operators (5.17). The curly lines are the gauge fields A_i^a and the full lines are the adjoint scalar fields A_0^a.



$\equiv g^8 \mathcal{G}(d, \mathbf{p}^2)$

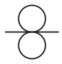
Figure 5.4: A generic 2-loop integral with a dimension 6 vertex that may contribute to the effective coupling renormalization at $\mathcal{O}(g^8)$.

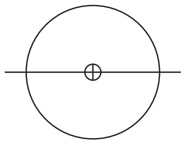
A direct consequence of $d = 3 - 2\epsilon$ is the fact that divergent integrals arise only in integrals with an even number of loops . Hence, we look for a simple and divergent integral to contribute to the effective coupling counter-term. As it turns out, one of the simplest divergent sum-integrals in which one of the inner lines is the first dimension-six vertex from Fig. (5.3) is $\mathcal{G}(d, \mathbf{p}^2)$ (cf. Fig. (5.4)). A quick inspection shows that it behaves due to dimensional reasons like:

$$\mathcal{G}(3 - 2\epsilon, \mathbf{p}^2) =\propto \frac{1}{\epsilon} + \ln \frac{\mathbf{p}^2}{\mu^2} + \text{finite} . \tag{5.18}$$

This kind of divergence would precisely account for the coupling renormalization constant to $\mathcal{O}(g^8)$.

However, there are other possible new diagrams that appear already at $\mathcal{O}(g^6)$, such as shown in Fig. (5.5). They are of dimension two and naturally would be proportional to m^2. These potentially account for the mass renormalization already at $\mathcal{O}(g^6)$, an order that is entirely determined by the super-renormalizable Lagrangian. In fact, having a closer look on the integral $\mathcal{M}(d, m^2)$, it becomes clear that the dimension-six vertex in the diagram has the role of contracting the propagator to a point. Thus, we obtain in fact no contribution to any renormalization constant, because the two-loop integral factorizes into a product of two one-loop integrals:

$$\mathcal{M}(3-2\epsilon, m^2) \propto$$  $$= \text{finite} \; . \tag{5.19}$$



$$\equiv g^6 \mathcal{M}(d, m^2)$$

Figure 5.5: A generic two-loop integral with a dimension-six vertex, that may contribute already at $\mathcal{O}(g^6)$ to the mass renormalization. As the mass counter-term is already determined precisely within the super-renormalizable theory, all contributions of this form should cancel.

In conclusion, we expect that, by adding the dimension-six operators to the Lagrangian in Eq. (3.5), as the theory becomes non-renormalizable, counter-terms emerge precisely to $\mathcal{O}(g^8)$ to cancel the divergence in Eq. (5.16).

Therefore, the remaining task is to determine all the renormalization constants of the fields A_i^a and A_0^a, and of the effective coupling g_E by a standard procedure of computing two- and four-point functions to two-loop order in $d = 3-2\epsilon$ dimensions. As the background field method can be applied in this situation as well, the procedure reduces merely to a two-point function computation.

In addition, no further diagrams need to be evaluated in EQCD since the matching procedure of section 3.5 employed a Taylor expansion in the external momentum, making all loop integrals on the EQCD side to vanish identically. Thus, in this computation the only contribution from the new operators is through the renormalization constants.

5.4 Outlook

In the present thesis, the mass parameter m_{E} and the effective coupling g_{E} of EQCD have been computed to three-loop order as matching coefficients to full QCD. The usual technique of computing three- or four-point functions was simplified to computing only two-point functions by applying the background field method. The task was simplified further by computing the vertex functions in the limit of vanishing external momenta that led to identically vanishing integrals on the EQCD side.

The demanding task was handled with computer algebraic software. To three-loop order ≈ 500 Feynman diagrams were generated and reduced via IBP relations to a set of a few tens of master sum-integrals. As their pre-factors diverge in $d = 3-2\epsilon$ dimensions, a clever basis transformation was performed that could eliminate the divergent pre-factors at the price of introducing sum-integrals of a higher complexity.

Finally, the master sum-integrals have been solved by partly generalizing the known techniques [28] and by borrowing a method from zero-temperature field theory of tensor integral manipulation [29].

The mass parameter contributes to the computed order to the QCD pressure starting with order g^7. We did not manage to finalize the computation on the effective coupling as it turns out that contributions from higher order operators enter our result through the renormalization of the fields and of the coupling, starting with $\mathcal{O}(g^8)$. After determining these renormalization constants, the result on the effective coupling can be used in determining the spatial string tension of QCD, as already done in [60].

The remaining task is to compute the renormalization constants to $\mathcal{O}(g^8)$, which involves a two-loop computation of two-point functions within the framework of the modified, non-renormalizable Lagrangian of EQCD. This computation is not expected to be mathematically demanding but rather demanding on the organizational side, as we have 14 new types of interactions.

Once the effective coupling is completed to three-loop order, a computation of the magnetic coupling g_{M} would be desirable for a more accurate determination of the spacial string tension (cf. section 3.3).

An extension of the present results to the fermionic sector is indicated due to the reasons of completeness. However, we do not expect a quantitative change in Fig. (5.1). For that, a similar basis transformation needs to be performed for the master sum-integrals with a fermionic signature. There are reason to believe that a suitable basis does exist with both, finite pre-factors and simple enough master sum-integrals as to be manageable with the present techniques.

On a wider perspective, the present computation has shown that state of the art techniques for solving sum-integrals are pushed to their limit. There is certainly a need for new methods that permit the computation of sum-integrals in principle to arbitrarily high order in ϵ and for a wider class of topologies. A future computation of the complete QCD pressure to $\mathcal{O}(g^6)$ involves four-loop sum-integrals of which only few topologies can be handled with the methods presented here. Any diagram that contains a mercedes-type subdiagram is in principle unsolvable yet. Also, at $\mathcal{O}(g^6)$ three-loop sum-integrals have to be known up to $\mathcal{O}(\epsilon)$ due to renormalization. At last, one could mention the extension from a scenario of massless fermions to the massive case and ideally to the case with finite chemical potential. All these additions would change the analytical structure of the sum-integrals. These technicalities have to be overcome eventually if we want to push reliable analytical results closer towards the non-perturbative region in the QCD phase diagram.

Appendix A

Integrals

In this appendix, we gather all pieces for the remaining sum-integrals that are the building blocks of $M_{3,-2}$ and of Π_T and that were not explicitly computed in the main text.

A.1 Finite parts

For a large number of these integrals, the last summation over Matsubara-modes was not performed analytically as this usually generated integrands containing hundreds of terms. Instead, we have evaluated the integral numerically for every mode individually and have truncated the sum such that the remainder would not exceed a fixed relative contribution usually taken $\mathcal{O}(10^{-9})$. The relative contribution of the remainder was determined by interpolating the the sum with a power-law $f(n) = an^{-b}$ and by performing the summation from the truncated term to ∞ analytically. As some of the pieces showed a very low convergence, we truncated the sums to a relative error of $\mathcal{O}(10^{-5})$.

A.1.1 First finite piece

These terms are of the form:

$$\sum\!\!\!\!\!\!\!\int_P{}' \frac{\Pi^{0-B}_{s_2 s_4 s_6} \Pi^{0-B}_{s_3 s_5 s_7}}{[P^2]^{s_1}} , \tag{A.1}$$

and their generic result is Eq. (4.122):

$$V^{\mathrm{f},1}(3; 31111; 22) = \frac{T^2}{(4\pi)^4}\mathcal{V}_{3,1} = \frac{T^2}{(4\pi)^4} \times (0.0046390318(1)) , \tag{A.2}$$

$$V^{\mathrm{f},1}(5; 31122; 11) = \frac{T^2}{(4\pi)^7}\mathcal{V}_{4,1} = \frac{T^2}{(4\pi)^7} \times (0.00199480835(1)) , \tag{A.3}$$

$$V^{\mathrm{f},1}(7; 32222; 00) = \frac{T^2}{(4\pi)^{10}}\mathcal{V}_{5,1} = \frac{T^2}{(4\pi)^{10}} \times (5.495(1) \times 10^{-7}) , \tag{A.4}$$

$$V^{\mathrm{f},1}(7; 52211; 00) = \frac{T^2}{(4\pi)^{10}}\mathcal{V}_{6,1} = \frac{T^2}{(4\pi)^{10}} \times (3.5741(1) \times 10^{-7}) , \tag{A.5}$$

$$V^{\mathrm{f},1}(7; 42221; 00) = \frac{T^2}{(4\pi)^{10}}\mathcal{V}_{7,1} = \frac{T^2}{(4\pi)^{10}} \times (4.2900(1) \times 10^{-7}) , \tag{A.6}$$

$$V^{\mathrm{f},1}(7;43211;00) = \frac{T^2}{(4\pi)^{10}}\mathcal{V}_{8,1} = \frac{T^2}{(4\pi)^{10}} \times (8.8987(1) \times 10^{-7}) \;, \tag{A.7}$$

$$V^{\mathrm{f},1}(7;33221;00) = \frac{T^2}{(4\pi)^{10}}\mathcal{V}_{9,1} = \frac{T^2}{(4\pi)^{10}} \times (8.277(1) \times 10^{-7}) \;, \tag{A.8}$$

$$V^{\mathrm{f},1}(7;33212;00) = \frac{T^2}{(4\pi)^{10}}\mathcal{V}_{10,1} = \frac{T^2}{(4\pi)^{10}} \times (0.0000119229(1)) \;, \tag{A.9}$$

$$V^{\mathrm{f},1}(7;33311;00) = \frac{T^2}{(4\pi)^{10}}\mathcal{V}_{11,1} = \frac{T^2}{(4\pi)^{10}} \times (0.0003192203(1)) \;, \tag{A.10}$$

$$V^{\mathrm{f},1}(3;21111;00) = \frac{1}{(4\pi)^{6}}\mathcal{V}_{13,1} = \frac{1}{(4\pi)^{6}} \times (0.09378301925(1)) \;, \tag{A.11}$$

$$V^{\mathrm{f},1}(3;31111;20) = \frac{1}{(4\pi)^{6}}\mathcal{V}_{15,1} = \frac{1}{(4\pi)^{6}} \times (0.02978074457(1)) \;, \tag{A.12}$$

$$V^{\mathrm{f},1}(3;41111;22) = \frac{1}{(4\pi)^{6}}\mathcal{V}_{16,1} = \frac{1}{(4\pi)^{6}} \times (0.01099409787(1)) \;. \tag{A.13}$$

A.1.2 Second finite piece

The generic result of these integrals is Eq. (4.123):

$$V^{\mathrm{f},2}(3;31111;22) = \frac{T^2}{(4\pi)^{4}}\mathcal{V}_{3,2} = \frac{T^2}{(4\pi)^{4}} \times (-0.000721758(1)) \;, \tag{A.14}$$

$$V^{\mathrm{f},2}(5;31122;11) = \frac{T^2}{(4\pi)^{7}}\mathcal{V}_{4,2} = \frac{T^2}{(4\pi)^{7}} \times (-0.000106808(1)) \;, \tag{A.15}$$

$$V^{\mathrm{f},2}(7;32222;00) = \frac{T^2}{(4\pi)^{10}}\mathcal{V}_{5,2} = \frac{T^2}{(4\pi)^{10}} \times (-0.00028062(1)) \;, \tag{A.16}$$

$$V^{\mathrm{f},2}(7;52211;00) = \frac{T^2}{(4\pi)^{10}}\mathcal{V}_{6,2} = \frac{T^2}{(4\pi)^{10}} \times (7.8965(1) \times 10^{-5}) \;, \tag{A.17}$$

$$\begin{aligned} V^{\mathrm{f},2\mathrm{a}+2\mathrm{b}}(7;42221;00) &= \frac{T^2}{(4\pi)^{10}}(\mathcal{V}_{7,2a} + \mathcal{V}_{7,2b}) \\ &= \frac{T^2}{(4\pi)^{10}} \times [(-7.8964(1) \times 10^{-5}) + (7.0158(1) \times 10^{-5})] \;, \end{aligned} \tag{A.18}$$

$$\begin{aligned} V^{\mathrm{f},2\mathrm{a}+2\mathrm{b}}(7;43211;00) &= \frac{T^2}{(4\pi)^{10}}(\mathcal{V}_{8,2a} + \mathcal{V}_{8,2b}) \\ &= \frac{T^2}{(4\pi)^{10}} \times [(-7.8964(1) \times 10^{-5}) + (-3.5715(1) \times 10^{-5})] \;, \end{aligned} \tag{A.19}$$

$$\begin{aligned} V^{\mathrm{f},2\mathrm{a}+2\mathrm{b}}(7;33221;00) &= \frac{T^2}{(4\pi)^{10}}(\mathcal{V}_{9,2a} + \mathcal{V}_{9,2b}) \\ &= \frac{T^2}{(4\pi)^{10}} \times [0 + (-7.606(1) \times 10^{-5})] \;, \end{aligned} \tag{A.20}$$

$$V^{\mathrm{f},2\mathrm{a}+2\mathrm{b}}(7;33212;00) = \frac{T^2}{(4\pi)^{10}}(\mathcal{V}_{10,2a} + \mathcal{V}_{19,2b})$$

$$= \frac{T^2}{(4\pi)^{10}} \times [(-1.4031(1) \times 10^{-4}) + (7.1430(1) \times 10^{-5})] \,, \tag{A.21}$$

$$V^{\mathrm{f},2}(7;33311;00) = \frac{T^2}{(4\pi)^{10}} \mathcal{V}_{11,2} = \frac{T^2}{(4\pi)^{10}} \times (1.4286(1) \times 10^{-4}) \,, \tag{A.22}$$

$$V^{\mathrm{f},2}(3;21111;00) = \frac{1}{(4\pi)^6} \mathcal{V}_{13,2} = \frac{1}{(4\pi)^6} \times (-0.9507801527(1)) \,, \tag{A.23}$$

$$\begin{aligned} V^{\mathrm{f},2\mathrm{a}+2\mathrm{b}}(3;31111;20) &= \frac{1}{(4\pi)^6} (\mathcal{V}_{15,2a} + \mathcal{V}_{15,2b}) \\ &= \frac{1}{(4\pi)^6} \times [(-0.0169179735(1)) + (-0.08467859163(1))] \,, \end{aligned} \tag{A.24}$$

$$V^{\mathrm{f},2}(3;41111;22) = \frac{1}{(4\pi)^6} \mathcal{V}_{16,2} = \frac{1}{(4\pi)^6} \times (-0.015384387080(1)) \,. \tag{A.25}$$

A.1.3 Finite parts for the zero-modes

Here, we list the concrete results for Eqs. (4.131, 4.132):

$$Z^{\mathrm{f}}(3;12211;22) = \frac{T^2}{(4\pi)^4} \mathcal{Z}_3 = \frac{T^2}{(4\pi)^4} \times (0.190165350(1)) \,, \tag{A.26}$$

$$Z^{\mathrm{f}}(3;12121;22) = \frac{T^2}{(4\pi)^4} \mathcal{Z}_4 = \frac{T^2}{(4\pi)^4} \times (-0.012563934311(1)) \,, \tag{A.27}$$

$$Z^{\mathrm{f}}(7;23222;00) = \frac{T^2}{(4\pi)^{10}} \mathcal{Z}_5 = \frac{T^2}{(4\pi)^{10}} \times (0.000275985995(1)) \,, \tag{A.28}$$

$$Z^{\mathrm{f}}(7;23231;00) = \frac{T^2}{(4\pi)^{10}} \mathcal{Z}_6 = \frac{T^2}{(4\pi)^{10}} \times (0.0000305224843(1)) \,, \tag{A.29}$$

$$Z^{\mathrm{f}}(7;23321;00) = \frac{T^2}{(4\pi)^{10}} \mathcal{Z}_7 = \frac{T^2}{(4\pi)^{10}} \times (0.00208559268(1)) \,, \tag{A.30}$$

$$Z^{\mathrm{f}}(3;12121;02) = \frac{1}{(4\pi)^6} \mathcal{Z}_9 = \frac{1}{(4\pi)^6} \times (-0.0417499660(1)) \,, \tag{A.31}$$

$$Z^{\mathrm{f}}(3;12121;20) = \frac{1}{(4\pi)^6} \mathcal{Z}_{10} = \frac{1}{(4\pi)^6} \times (-2.0660279047(1)) \,, \tag{A.32}$$

$$Z^{\mathrm{f}}(3;12211;02) = \frac{1}{(4\pi)^6} \mathcal{Z}_{11} = \frac{1}{(4\pi)^6} \times (-0.5170838408(1)) \,, \tag{A.33}$$

$$Z^{\mathrm{f}}(3;12221;22) = \frac{1}{(4\pi)^6} \mathcal{Z}_{12} = \frac{1}{(4\pi)^6} \times (-0.2399902511(1)) \,, \tag{A.34}$$

$$Z^{\mathrm{f}}(3;13111;02) = \frac{1}{(4\pi)^6} \mathcal{Z}_{13} = \frac{1}{(4\pi)^6} \times (-0.04487446214(1)) \,, \tag{A.35}$$

$$Z^{\mathrm{f}}(3;02221;02) = \frac{1}{(4\pi)^6} \mathcal{Z}_{14} = \frac{1}{(4\pi)^6} \times (-0.07420667719(1)) \,, \tag{A.36}$$

$$Z^{\mathrm{f}}(3;03121;02) = \frac{1}{(4\pi)^6} \mathcal{Z}_{15} = \frac{1}{(4\pi)^6} \times (-0.01111795886(1)) \,. \tag{A.37}$$

A.2 Zero-modes results

Finally, we gather all remaining sum-integrals. Besides the zero-mode sum-integrals encountered in Eqs. (4.93, 4.94, 4.170), also the following ones are required. They are built up from the finite pieces listed above and from divergent pieces from Eqs. (4.140, 4.141) and by using the generic splitting of Eq. (4.58):

$$Z(3;12211;22) = \frac{-1}{8}\frac{T^2}{(4\pi)^4}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon}+4-3\gamma_{\rm E}+3\ln\pi-8\mathcal{Z}_3+\mathcal{O}(\epsilon)\right] , \tag{A.38}$$

$$Z(3;12121;22) = \frac{1}{48}\frac{T^2}{(4\pi)^4}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon}+\frac{2}{3}-3\gamma_{\rm E}+3\ln\pi+48\mathcal{Z}_4+\mathcal{O}(\epsilon)\right] , \tag{A.39}$$

$$\begin{aligned} Z(3;12121;02) &= \\ &= \frac{1}{(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[-\frac{\pi^2}{36}\frac{1}{\epsilon}+\pi^2\left(-\frac{1}{54}-\frac{\gamma_{\rm E}}{12}+2\ln G-\frac{\ln\pi}{12}\right)-\frac{3\zeta(3)}{20}+\mathcal{Z}_9+\mathcal{O}(\epsilon)\right] , \end{aligned} \tag{A.40}$$

$$\begin{aligned} Z(3;12121;20) = \frac{1}{(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}&\left[\left(\frac{\pi^2}{12}-\frac{\zeta(3)}{4}\right)\frac{1}{\epsilon}+\frac{\pi^2}{4}\left(3+\gamma_{\rm E}-24\ln G+\ln\pi\right)\right. \\ &\left. +\,\zeta(3)\left(\frac{-5}{3}+\frac{3\gamma_{\rm E}}{4}+\frac{3\ln 4\pi}{4}\right)-\frac{3\zeta'(3)}{2}+\mathcal{Z}_{10}+\mathcal{O}(\epsilon)\right] , \end{aligned} \tag{A.41}$$

$$\begin{aligned} Z(3;12211;02) &= \\ &= \frac{1}{(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{\pi^2}{12}\frac{1}{\epsilon}+\pi^2\left(\frac{1}{3}+\frac{\gamma_{\rm E}}{4}-4\ln G+\frac{\ln(\pi/4)}{12}\right)-\frac{7\zeta(3)}{4}+\mathcal{Z}_{11}+\mathcal{O}(\epsilon)\right] , \end{aligned} \tag{A.42}$$

$$\begin{aligned} Z(3;12221;22) &= \\ &= \frac{1}{(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{\pi^2}{48}\frac{1}{\epsilon}+\frac{\pi^2}{16}(3+\gamma_{\rm E}-24\ln G+\ln\pi)+\mathcal{Z}_{12}+\mathcal{O}(\epsilon)\right] , \end{aligned} \tag{A.43}$$

$$\begin{aligned} Z(3;13111;02) &= \\ &= \frac{1}{(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{\pi^2}{72}\frac{1}{\epsilon}+\pi^2\left(\frac{13}{108}+\frac{\gamma_{\rm E}}{24}-\ln G+\frac{\ln(\pi)}{24}\right)-\frac{\zeta(3)}{40}+\mathcal{Z}_{13}+\mathcal{O}(\epsilon)\right] , \end{aligned} \tag{A.44}$$

$$\begin{aligned} Z(3;02221;02) &= \\ &= \frac{1}{(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{\pi^2}{12}\frac{1}{\epsilon}+\pi^2\left(\frac{\gamma_{\rm E}}{4}-6\ln G+\frac{\ln(\pi)}{4}\right)+\frac{3\zeta(3)}{4}+\mathcal{Z}_{14}+\mathcal{O}(\epsilon)\right] , \end{aligned} \tag{A.45}$$

$$Z(3;03121;02) = \frac{1}{(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[0\times\frac{1}{\epsilon}+\frac{\pi^2}{6}-\frac{\zeta(3)}{4}+\mathcal{Z}_{15}+\mathcal{O}(\epsilon)\right] , \tag{A.46}$$

$$Z(7;23222;00) = \frac{-1}{720}\frac{T^2}{(4\pi)^{10}}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{16}{15}-\gamma_{\rm E}+\ln\frac{\pi}{4}\right)+\mathcal{O}(\epsilon^0)\right] , \tag{A.47}$$

$$Z(7;23231;00) = \frac{1}{960}\frac{T^2}{(4\pi)^{10}}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{2}{5}-\gamma_{\rm E}+\ln\frac{\pi}{4}\right)+\mathcal{O}(\epsilon^0)\right] , \tag{A.48}$$

$$Z(7;23222;00) = \frac{-1}{720}\frac{T^2}{(4\pi)^{10}}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{16}{15}-\gamma_{\rm E}+\ln\frac{\pi}{4}\right)+\mathcal{O}(\epsilon^0)\right] . \tag{A.49}$$

A.3 Remaining sum-integral results

The remaining sum-integrals entering the definition of $M_{3,-2}$ (c.f. Eq. (4.30)) and also the sum-integrals with mass dimension zero entering Π_T are listed below. The results for the 7-dimensional sum-integrals are shown only up to the constant term, because they will not enter the final result in the mass parameter.

$$V(3;31111;22) = \frac{1}{288}\frac{T^2}{(4\pi)^4}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{1}{12}+\gamma_{\rm E}+24\ln G\right)+\mathcal{O}(\epsilon^0)\right] , \tag{A.50}$$

$$V(5;31122;11) = \frac{-1}{162}\frac{T^2}{(4\pi)^7}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[0\times\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right] , \tag{A.51}$$

$$V(7;32222;00) = \frac{-7}{4320}\frac{T^2}{(4\pi)^{10}}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{17}{105}+\gamma_{\rm E}+24\ln G\right)+\mathcal{O}(\epsilon^0)\right] , \tag{A.52}$$

$$V(7;52211;00) = \frac{-19}{17280}\frac{T^2}{(4\pi)^{10}}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{61}{285}+\gamma_{\rm E}+24\ln G\right)+\mathcal{O}(\epsilon^0)\right] , \tag{A.53}$$

$$V(7;42221;00) = \frac{19}{2160}\frac{T^2}{(4\pi)^{10}}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{1747}{2280}-\frac{25\gamma_{\rm E}}{38}+\frac{78\ln G}{19}+\frac{63\ln 2\pi}{19}\right)+\mathcal{O}(\epsilon^0)\right] , \tag{A.54}$$

$$V(7;43211;00) = \frac{1}{640}\frac{T^2}{(4\pi)^{10}}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{281}{810}+\gamma_{\rm E}+24\ln G\right)+\mathcal{O}(\epsilon^0)\right] , \tag{A.55}$$

$$V(7;33221;00) = \frac{7}{17280}\frac{T^2}{(4\pi)^{10}}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{13}{30}+\gamma_{\rm E}+24\ln G\right)+\mathcal{O}(\epsilon^0)\right] , \tag{A.56}$$

$$V(7;33212;00) = \frac{-1}{576}\frac{T^2}{(4\pi)^{10}}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{1}{18}+\gamma_{\rm E}+24\ln G\right)+\mathcal{O}(\epsilon^0)\right] , \qquad (A.57)$$

$$V(7;33311;00) = \frac{-1}{540}\frac{T^2}{(4\pi)^{10}}\left(\frac{\mu^2}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{7}{20}+\gamma_{\rm E}+24\ln G\right)+\mathcal{O}(\epsilon^0)\right] , \qquad (A.58)$$

$$\begin{aligned}
&V(3;12111;00) = \\
&= \frac{1}{6(4\pi)^6}\left(\frac{\mu^2 e^{\gamma_{\rm E}}}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^3}+\frac{3}{\epsilon^2}+\frac{1}{\epsilon}\left(13-6{\gamma_{\rm E}}^2+\frac{3\pi^2}{4}-12\gamma_1-3\zeta(3)\right)\right. \\
&+\left(51-42{\gamma_{\rm E}}^2+4\pi^2\left(\frac{19}{16}+\ln 2\pi-12\ln G\right)+2\ln 2\left(12-12{\gamma_{\rm E}}^2-24\gamma_1-\zeta(3)\right)\right. \\
&\left.\left.+\ \gamma_E\left(-24-24\gamma_1+18\zeta(3)\right)-84\gamma_1-36\gamma_2+\frac{25\zeta(3)}{2}-16\zeta^{'}(3)+6(\mathcal{V}_{12}+\mathcal{Z}_8)\right)\right]+\mathcal{O}(\epsilon) \\
&= \frac{1}{6(4\pi)^6}\left[\frac{\mu^2}{T^2}\right)^{3\epsilon}\left(\frac{1}{\epsilon^3}-\frac{2.86143}{\epsilon^2}+\frac{15.2646}{\epsilon}-45.2(1)+\mathcal{O}(\epsilon)\right] ,
\end{aligned} \qquad (A.59)$$

with

$$\mathcal{V}_{12,1}+\mathcal{V}_{12,3}+\mathcal{Z}_8 = -7.68(1) . \qquad (A.60)$$

The largest uncertainty comes from $\mathcal{V}_{12,3}$.

$$\begin{aligned}
&V(3;21111;000) = \\
&= \frac{1}{3(4\pi)^6}\left(\frac{\mu^2 e^{\gamma_{\rm E}}}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^3}+\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(8-6{\gamma_{\rm E}}^2+\frac{3\pi^2}{4}-12\gamma_1+\zeta(3)\right)\right. \\
&+\left(16+24{\gamma_{\rm E}}^2\ln 2+\frac{\pi^2}{6}(7+96\ln\ G-8\ln\ 2\pi)+24\gamma_1(\gamma_{\rm E}+2\ln 2)+36\gamma_2\right. \\
&\left.\left.-\ \left(\frac{43}{3}+6\gamma_{\rm E}\right)\zeta(3)+6\zeta^{'}(3)+(3\mathcal{V}_{13}-2\mathcal{Z}_8)\right)+\mathcal{O}(\epsilon)\right] \\
&= \frac{1}{3(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^3}-\frac{3.86}{\epsilon^2}+\frac{20.93}{\epsilon}-60.38+\mathcal{O}(\epsilon)\right] ,
\end{aligned} \qquad (A.61)$$

with

$$\mathcal{V}_{13,1}+\mathcal{V}_{13,2}-\frac{2}{3}\mathcal{Z}_8 \approx 2.5870483449(1) . \qquad (A.62)$$

$$
\begin{aligned}
&V(3;22111;002) = \\
&= -\frac{5}{48(4\pi)^6}\left(\frac{\mu^2 e^{\gamma_{\rm E}}}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon} + \left(\frac{209}{30} + \pi^2\left(\frac{5}{9} + \frac{4\gamma_{\rm E}}{5} - \frac{4\ln 2}{5}\right)\right.\right. \\
&+ \zeta(3)\left(\frac{4\ln 2}{15} - \frac{2087}{225}\right) - \frac{28\gamma_{\rm E}}{5} - \frac{24{\gamma_{\rm E}}^2}{5} - \frac{\pi^4}{225} + \frac{28\ln 2}{5} - \frac{48\gamma_1}{5} - \frac{4\zeta'(3)}{15} \\
&\left.\left. - \frac{48}{5}\mathcal{V}_{14} + \frac{24}{5}\mathcal{Z}_9\right) + \mathcal{O}(\epsilon)\right] \\
&= -\frac{5}{48(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon} - 5.04 + \mathcal{O}(\epsilon)\right]
\end{aligned}
\tag{A.63}
$$

with

$$
\mathcal{V}_{14,1} + \mathcal{V}_{14,2} + \mathcal{V}_{14,3} - \frac{1}{2}(\mathcal{Z}_9 + \mathcal{Z}_{11}) = 0.17(1)\ , \tag{A.64}
$$

with the largest error coming from $\mathcal{V}_{14,3}$.

$$
\begin{aligned}
&V(3;31111;020) = \\
&= \frac{1}{8(4\pi)^6}\left(\frac{\mu^2 e^{\gamma_{\rm E}}}{4\pi T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{13}{3} + \frac{2\zeta(3)}{15}\right) + \left(\frac{140}{9} - \frac{20\gamma_{\rm E}}{3} + \frac{20\ln 2}{3}\right.\right. \\
&- \frac{\pi^4}{900} + \pi^2\left(\frac{35}{36} + \frac{5\gamma_{\rm E}}{9} - \frac{7\ln 2}{15} - \frac{16\ln G}{15} + \frac{4\ln\pi}{45}\right) \\
&+ \zeta(3)\left(-\frac{107}{50} - \frac{4\gamma_{\rm E}}{5} + \frac{2\ln 2}{3} + 8\ln G\right) - 10{\gamma_{\rm E}}^2 - 20\gamma_1 - \frac{8\zeta'(3)}{15} \\
&\left.\left. +8\mathcal{V}_{15} + \frac{4}{5}(\mathcal{Z}_9 + \mathcal{Z}_{10})\right) + \mathcal{O}(\epsilon)\right] \\
&= \frac{1}{8(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^2} - \frac{1.36}{\epsilon} + 10.80 + \mathcal{O}(\epsilon)\right]\ ,
\end{aligned}
\tag{A.65}
$$

with

$$
\mathcal{V}_{15,1} + \mathcal{V}_{15,2} + \mathcal{V}_{15,3} + \frac{1}{10}(\mathcal{Z}_9 + \mathcal{Z}_{10}) = -0.2825936076(1)\ . \tag{A.66}
$$

$$
\begin{aligned}
&V(3;41111;022) = \\
&= \frac{1}{432(4\pi)^6}\left(\frac{\mu^2 e^{\gamma_{\rm E}}}{4\pi T^2}\right)^{3\epsilon}\Bigg[\frac{1}{\epsilon^3}+\frac{5}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{341}{18}-6{\gamma_{\rm E}}^2+\frac{3\pi^2}{4}-12\gamma_1+5\zeta(3)\right) \\
&+\Bigg(\frac{985}{18}+6\gamma_{\rm E}+\pi^2\left(\frac{107}{84}-\gamma_{\rm E}-\frac{43\ln 2}{35}+\frac{936\ln G}{35}-\frac{78\ln\pi}{35}\right) \\
&+\zeta(3)\left(\frac{1243}{70}-30\gamma_{\rm E}+10\ln 2+120\ln G\right)+\gamma_1\left(-52+24\gamma_{\rm E}+48\ln 2\right) \\
&+2{\gamma_{\rm E}}^2(12\ln 2-13)+\frac{23\pi^4}{1050}-8\ln 2+36\gamma_1^2+10\zeta^{'}(3)+432\mathcal{V}_{16} \\
&+\ \frac{54}{35}(-6\mathcal{Z}_9-44\mathcal{Z}_{12}-4\mathcal{Z}_{13}+\mathcal{Z}_{14}+4\mathcal{Z}_{15})\Bigg)+\mathcal{O}(\epsilon^4)\Bigg] \\
&= \frac{1}{432(4\pi)^6}\left(\frac{\mu^2}{T^2}\right)^{3\epsilon}\left[\frac{1}{\epsilon^3}-\frac{0.86}{\epsilon^2}+\frac{19.10}{\epsilon}+18.89+\mathcal{O}(\epsilon)\right]\ ,
\end{aligned}
\tag{A.67}
$$

with

$$
\mathcal{V}_{16,1}+\mathcal{V}_{16,2}+\frac{1}{280}(4\mathcal{Z}_{15}+\mathcal{Z}_{14}-4\mathcal{Z}_{13}-44\mathcal{Z}_{12}-6\mathcal{Z}_9)=0.0344343(1)\ . \tag{A.68}
$$

Appendix B

Analytic functions

Here we collect some analytic functions that are the main building blocks of the divergent pieces of the master integrals. They are obtained by either expressing the integral in spherical coordinates or by using the Feynman parameterization of the form:

$$\frac{1}{A_1^{a_1}}\cdots\frac{1}{A_n^{a_n}}=\frac{\Gamma(a_1+\cdots+a_n)}{\Gamma(a_1)\ldots\Gamma(a_n)}\int_0^\infty \mathrm{d}x_1\ldots\int_0^\infty \mathrm{d}x_n\frac{\delta(1-x_1-\cdots-x_n)x_1^{a_1-1}\ldots x_n^{a_n-1}}{[x_1A_1+\ldots x_nA_n]^{a_1+\ldots a_n}}\,. \tag{B.1}$$

The massless one-loop generalized propagator at zero temperature is:

$$\mu^{-2\epsilon}g(s_1,s_2,d)=\mu^{-2\epsilon}(\mathbf{p}^2)^{s_{12}-\frac{d}{2}}\int_q\frac{1}{(\mathbf{q}^2)^{s_1}[(\mathbf{p}+\mathbf{q})^2]^{s_2}}=\frac{\Gamma(\frac{d}{2}-s_1)\Gamma(\frac{d}{2}-s_2)\Gamma(s_{12}-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(s_1)\Gamma(s_2)\Gamma(d-s_{12})}\,. \tag{B.2}$$

The generalized one-loop tadpole is:

$$\mu^{-2\epsilon}I_b^a\equiv\mu^{-2\epsilon}\sum\!\!\!\!\!\!\!\int_P\frac{(p_0^2)^a}{[P^2]^b}=\frac{2T(2\pi T)^{d+2a-2b}\Gamma(b-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(b)}\zeta(-d-2a+2b)\,, \tag{B.3}$$

and a variation thereof is:

$$\hat{I}_b^a(\alpha)=I^a_{b+2-\frac{d+1}{2}}-\frac{I_b^a}{(\alpha T^2)^{2-\frac{d+1}{2}}}\,. \tag{B.4}$$

A special two-loop tadpole is:

$$\begin{aligned}\mu^{-4\epsilon}J^d(s_1s_2s_3;s_4)&=\mu^{-4\epsilon}\sum\!\!\!\!\!\!\!\int_{PQ}\delta_{p_0}\frac{(p_0^2)^{s_4}}{[P^2]^{s_1}[Q^2]^{s_2}[(P+Q)^2]^{s_3}}\\&=\frac{2T^2(2\pi T)^{2d-2s_{123}+2s_4}}{(4\pi)^d}\frac{\Gamma(\frac{d}{2}-s_1)\Gamma(s_{12}-\frac{d}{2})\Gamma(s_{13}-\frac{d}{2})\Gamma(s_{123}-d)}{\Gamma(\frac{d}{2})\Gamma(s_2)\Gamma(s_3)\Gamma(s_{1123}-d)}\zeta(2s_{123}-2s_4-2d)\,.\end{aligned} \tag{B.5}$$

The generalized two-loop sum-integral

$$L^d(s_1s_2s_3;s_4s_5)=\sum\!\!\!\!\!\!\!\int_{PQ}\frac{p_0^{s_4}q_0^{s_5}}{[P^2]^{s_1}[Q^2]^{s_2}[(P+Q)^2]^{s_3}} \tag{B.6}$$

can always be reduced via IBP relations to a product of one-loop tadpole integrals multiplied by a ratio over polynomials in d. The explicit reduction of all two-loop sum-integrals that are needed in the calculations, are:

$$L^d(111;00) = 0\ , \tag{B.7}$$

$$L^d(211;20) = 0\ , \tag{B.8}$$

$$L^d(211;00) = -\frac{1}{(d-2)(d-5)}[I_2^0]^2\ , \tag{B.9}$$

$$L^d(311;22) = \frac{-d^3+12d^2-51d+76}{4(d-5)(d-7)}I_1^0 \times I_2^0\ , \tag{B.10}$$

$$L^d(211;02) = \frac{d-3}{d-5}I_1^0 \times I_2^0\ , \tag{B.11}$$

$$L^d(221;20) = -\frac{(d-1)(d-4)}{2(d-2)(d-5)(d-7)}[I_2^0]^2\ , \tag{B.12}$$

$$L^d(311;20) = \frac{(d-4)^2}{(d-2)(d-5)(d-7)}[I_2^0]^2\ , \tag{B.13}$$

$$L^d(311;02) = \frac{(d-3)(d-4)}{2(d-2)(d-5)(d-7)}[I_2^0]^2 + \frac{d-4}{d-7}I_1^0 \times I_3^0\ , \tag{B.14}$$

$$L^d(411;22) = -\frac{(d-4)^2(d-5)}{8(d-2)(d-7)(d-9)}[I_2^0]^2 - \frac{(d-5)(d-6)}{6(d-9)}I_1^0 \times I_3^0\ , \tag{B.15}$$

$$L^d(421;00) = \frac{-6(d(d-13)+28)}{(d-2)(d-4)(d-9)(d-11)}I_3^0 \times I_4^0\ , \tag{B.16}$$

$$L^d(322;00) = \frac{12(d-8)(d-5)}{(d-2)(d-4)(d-9)(d-11)}I_3^0 \times I_4^0\ , \tag{B.17}$$

$$L^d(312;11) = -\frac{(d-3)(d-5)(d-6)}{2(d-2)(d-7)(d-9)}I_2^0 \times I_3^0\ , \tag{B.18}$$

$$L^d(331;00) = \frac{-12(d-8)(d-5)}{(d-2)(d-4)(d-9)(d-11)}I_3^0 \times I_4^0\ , \tag{B.19}$$

$$L^d(232;00) = \frac{12(d-8)(d-5)}{(d-2)(d-4)(d-9)(d-11)}I_3^0 \times I_4^0\ . \tag{B.20}$$

Appendix C

Configuration space definitions

Here we give the remaining building blocks for the finite pieces of the sum-integrals. For writing the propagators in configuration space, a central ingredient is the modified Bessel function of the second kind with half-integer order:

$$K_\nu(z) = \sqrt{\frac{\pi}{2}}\frac{e^{-z}}{\sqrt{z}}\kappa_\nu(z) \; , \tag{C.1}$$

with

$$\kappa_\nu(z) = \sum_{j=0}^{\left||\nu|-\frac{1}{2}\right|} \frac{\left(j+|\nu|-\frac{1}{2}\right)!}{j!\left(-j+|\nu|-\frac{1}{2}\right)!}(2z)^{-j}, \; \nu - \frac{1}{2} \in \mathbb{Z} \; . \tag{C.2}$$

The Bessel function enters the Fourier transformation of the propagator:

$$\int_p \frac{e^{-i\mathbf{pr}}}{[P^2]^a} = \frac{2^{1-a}}{(2\pi)^{d/2}\Gamma(a)}\left(\frac{p_0^2}{r^2}\right)^{\frac{d-2a}{4}} K_{\frac{d}{2}-a}(\sqrt{p_0^2 r^2}) \; . \tag{C.3}$$

The integrals in configuration space are expressed in terms of spherical coordinates. The d-dimensional integration measure is:

$$\begin{aligned}\int \mathrm{d}\Omega_d &\equiv \int_0^\pi \cdots \int_0^{2\pi} \sin^{d-2}\phi_1 \mathrm{d}\phi_1 \sin^{d-3}\phi_2 \mathrm{d}\phi_2 \cdots \sin\phi_{d-2}\mathrm{d}\phi_{d-2}\mathrm{d}\phi_{d-1} \\ &= 2\prod_{i=0}^{d-2}\frac{\sqrt{\pi}\Gamma(1/2+i/2)}{\Gamma(1+i/2)} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)} \; .\end{aligned} \tag{C.4}$$

If the integrand contains a scalar product of the form $\mathbf{p}\cdot\mathbf{r}$, it is always possible to choose the orientation of the coordinate system in such a way that $\mathbf{p}\cdot\mathbf{r} = pr\cos\phi_1$. Therefore, we have:

$$\int \mathrm{d}\Omega_d e^{i\mathbf{pr}} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)}\, {}_0F_1\left(\frac{d}{2}, -\frac{(pr)^2}{4}\right) \; , \tag{C.5}$$

with ${}_0F_1$ being the confluent Hypergeometric function. Concretely:

$$\int \mathrm{d}\Omega_{\{3,5,7\}} e^{i\mathbf{pr}} = 4\pi\left\{\frac{\sin pr}{pr}, 2\pi\frac{\sin pr - pr\cos pr}{(pr)^3}, (2\pi)^2\frac{3\sin pr - 3pr\cos pr - (pr)^2\sin pr}{(pr)^5}\right\} \; . \tag{C.6}$$

These definitions are used to compute the generic integral:

$$\int \mathrm{d}^d\mathbf{p}\frac{1}{[\mathbf{p}^2]^n}\int \mathrm{d}\Omega_{\mathbf{r}}\mathrm{d}\Omega_{\mathbf{s}}e^{i\mathbf{p}(\mathbf{r}+\mathbf{s})} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\frac{2^{1-4n}(4\pi)^d\Gamma(\frac{d}{2}-n)}{\Gamma(n)\Gamma(\frac{d}{2})}\times a_{n,d}(r,s)\ , \tag{C.7}$$

with

$$a_{n,d} = \frac{{}_2F_1(1-n,\frac{d}{2}-n,\frac{d}{2},\frac{(r+s-|r+s|)^2}{(r+s+|r-s|)^2})}{(r+s+|r-s|)^{d-2n}}\ . \tag{C.8}$$

The integration was performed by going to spherical coordinates in the p-variable and by choosing the orientation of the coordinates system such that: $\mathbf{p}(\mathbf{r}+\mathbf{s}) = pr\cos\phi_{r,1} + ps\cos\phi_{s,1}$. In this way the other two angular integration become trivial. ${}_2F_1$ is the Gauss Hypergeometric function. This integral is used for the zero-mode finite piece calculation (Eqs. 4.131, 4.132).

The next generic angular integration needed in the computation of the first finite piece, Eq. (4.122), is of the form:

$$\begin{aligned} h_{a,b}(x,y,n) &= \int_0^\pi e^{-n|\mathbf{x}+\mathbf{y}|}|\mathbf{x}+\mathbf{y}|^a \sin^{2b+1}\theta\mathrm{d}\theta \\ &= \frac{1}{nxy}\sum_{i=0}^{b}\binom{b}{i}\sum_{j=0}^{2i}(-1)^{i+j}\binom{2i}{j}\Gamma(4i-2j+a+2)\frac{(x^2+y^2)^j}{(2xy)^{2i}} \\ &\times \sum_{k=0}^{4i-2j+a+1}\frac{e^{-n|x-y|}|x-y|^{4i-2j+a-k+1} - e^{-n|x+y|}|x+y|^{4i-2j+a-k+1}}{n^k\Gamma(4i-2j+a-k+2)}\ . \end{aligned} \tag{C.9}$$

where $n>0$, $a\geq -1$, $b\geq 0$ and $\mathbf{x}$ and $\mathbf{y}$ are vectors with $\mathbf{xy} = xy\cos\theta$.

In order to compute the second finite piece, Eq. (4.123), we need an integral of the form:

$$\begin{aligned} &\int_0^\infty \mathrm{d}p\frac{p^{d-1}}{[P^2]^a}\ln P^2\int \mathrm{d}\Omega_d e^{i\mathbf{pr}} \\ &= -\partial_a\frac{p^{d-1}}{[P^2]^a}\times\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}{}_0F_1(\frac{d}{2},-\frac{(pr)^2}{4}) \\ &= -2\pi^{\frac{d}{2}}\partial_a\left[\Gamma^{-1}(a)\left(\frac{2|p_0|}{r}\right)^{\frac{d}{2}-a}K_{a-\frac{d}{2}}(|p_0|r)\right]\ . \end{aligned} \tag{C.10}$$

After performing the derivative with respect to a, we define the integral as:

$$\int_0^\infty \mathrm{d}p\frac{p^{d-1}}{[P^2]^a}\ln P^2\int \mathrm{d}\Omega_d e^{i\mathbf{pr}} = 2^{-a}(2\pi)^{\frac{d+1}{2}}\Gamma^{-1}(a)l_{a,d}(r,|p_0|)\ , \tag{C.11}$$

with

$$l_{a,d}(x,n) = \sqrt{\frac{\pi}{2}}\left(\frac{n}{x}\right)^{\frac{d}{2}-a}\left[\left(\psi(a)+\ln\frac{2n}{x}\right)K_{a-\frac{d}{2}}(nx) - \partial_a K_{a-\frac{d}{2}}(nx)\right]\ . \tag{C.12}$$

Eq. (10.2.34) of Ref. [130] gives a relation between the derivative of $K_n(x)$ at $n=\pm 1/2$ and the function $\mathrm{Ei}(x)$. From this relation we derive all other derivatives of higher $|n|$. So, starting from Eq. (10.2.34) of Ref.[130]:

$$\left.\frac{\partial}{\partial\nu}K_\nu(x)\right|_{\nu=\pm\frac{1}{2}} = \mp\sqrt{\frac{\pi}{2x}}\mathrm{Ei}(-2x)e^x\ , \tag{C.13}$$

and using the recursion formula:

$$f_{n-1}(z) - f_{n+1}(z) = (2n+1)z^{-1} f_n(z),\; f_n(z) = (-1)^{n+1}\sqrt{\pi/(2z)}K_{n+\frac{1}{2}}(z)\;, \tag{C.14}$$

we obtain:

$$\frac{\partial}{\partial\nu}K_\nu(x)\bigg|_{\nu=\pm\frac{3}{2}} = \pm\sqrt{\frac{\pi}{2x}}\left[\mathrm{Ei}(-2x)e^x\left(1-\frac{1}{x}\right)+\frac{2}{x}e^{-x}\right]\;, \tag{C.15}$$

$$\frac{\partial}{\partial\nu}K_\nu(x)\bigg|_{\nu=\pm\frac{5}{2}} = \pm\sqrt{\frac{\pi}{2x}}\left[\mathrm{Ei}(-2x)e^x\left(-1+\frac{3}{x}-\frac{3}{x^2}\right)+\frac{2}{x}\left(1+\frac{4}{x}\right)e^{-x}\right]\;. \tag{C.16}$$

The following generic formula is used in calculating the first finite piece of $V(3;12111;00)$ and $V(3;22111;02)$. It shows how to perform the integration in configuration space before performing the sum of one of the Π's. There are two cases:

$$s^-_{a,b,c}(y,\alpha) = \sum_{m,n}{}' \frac{e^{-|n|y}}{|n|^a|m|^b}\int_0^\infty \mathrm{d}x e^{-(|m|+|m+n|)x}e^{-|n||x-y|}\alpha^c|x-y|^c. \tag{C.17}$$

α is simply a control parameter. The prime in the sum denotes omission of the zero-mode. After splitting the integration interval into $[0,y]$ and $[y,\infty)$ and carefully splitting the summation intervals, so that for any interval the integration is finite, we obtain:

$$\begin{aligned} s^-_{a,b,c}(y,\alpha) = \sum_{n=1}^\infty\sum_{m=1}^\infty &\Bigg\{2\alpha^c\sum_{k=0}^c\binom{c}{k}y^{c-k}(-1)^k\frac{e^{-2ny}}{n^2}\left(\frac{1}{m^b}+\frac{1}{(m+n)^b}\right) \\ &\times\left(\frac{\Gamma(k+1)}{(2m)^{k+1}}-\frac{e^{-2my}}{2m}\sum_{i=0}^k\frac{y^{k-i}}{(2m)^i}\frac{\Gamma(k+1)}{\Gamma(k-i+1)}\right)+2\alpha^c\delta_{m,1}\frac{y^{c+1}}{1+c}\frac{e^{-2ny}}{n^a} \\ &+2\alpha^c\sum_{k=0}^c\binom{c}{k}(-y)^k\sum_{i=0}^{c-k}y^{c-k-i}\frac{\Gamma(c-k+1)}{\Gamma(c-k-i+1)} \\ &\times\left[\delta_{m,1}\frac{e^{-2ny}}{n^a}\frac{H_{b,n}}{(2n)^{i+1}}+\frac{e^{-(2m+2n)y}}{n^a}\left(\frac{1}{m^b}+\frac{1}{(m+n)^b}\right)\frac{1}{(2m+2n)^{i+1}}\right]\Bigg\}\;. \end{aligned} \tag{C.18}$$

The second case is:

$$\begin{aligned} s^+_{a,b,c}(x,\alpha) &= \sum_{m,n}{}' \frac{e^{-|n|y}}{|n|^a|m|^b}\int_0^\infty \mathrm{d}x e^{-(|m|+|m+n|)x}e^{-|n|(x+y)}\alpha^c(x+y)^c \\ &= \sum_{n=1}^\infty\sum_{m=1}^\infty\Bigg\{2\alpha^c\sum_{k=0}^c\binom{c}{k}y^{c-k}\Gamma(k+1)\frac{e^{-2ny}}{n^a}\Bigg[\delta_{m,1}\frac{H_{b,n}}{(2n)^{k+1}} \\ &+\left(\frac{1}{m^b}+\frac{1}{(m+n)^b}\right)\frac{1}{(2m+2n)^{k+1}}\Bigg]\Bigg\}\;. \end{aligned} \tag{C.19}$$

Here, $H_{b,n} = \sum_{i=1}^n n^{-b}$ is the Harmonic number of order b, and the factor $\delta_{m,1}$ states that the summation over m should be omitted.

Appendix D

IBP reduction for zero-modes

In this appendix we present the strategy for mapping the zero-mode sum-integrals to others that can be computed by using the separation of Eq. (4.58). For that, the derivative of the propagator is needed:

$$\begin{aligned}
\mathbf{p}\partial_{\mathbf{p}}\frac{\delta_{p_0}}{[P^2]^n} &= -2n\frac{\delta_{p_0}}{[P^2]^n}\ , \\
\mathbf{q}\partial_{\mathbf{p}}\frac{\delta_{p_0}}{[P^2]^n} &= -n\delta_{p_0}\frac{(P+Q)^2-P^2-Q^2}{[P^2]^{n+1}}\ , \\
\mathbf{p}\partial_{\mathbf{p}}\frac{\delta_{p_0}}{[(P+Q)^2]^n} = \mathbf{p}\partial_{\mathbf{q}}\frac{\delta_{p_0}}{[(P+Q)^2]^n} &= -n\delta_{p_0}\frac{(P+Q)^2+P^2-Q^2}{[(P+Q)^2]^{n+1}}\ , \\
\mathbf{q}\partial_{\mathbf{q}}\frac{\delta_{p_0}}{[(P+Q)^2]^n} = \mathbf{q}\partial_{\mathbf{p}}\frac{\delta_{p_0}}{[(P+Q)^2]^n} &= -n\delta_{p_0}\frac{(P+Q)^2-P^2+Q^2-2q_0^2}{[(P+Q)^2]^{n+1}}\ .
\end{aligned} \tag{D.1}$$

Before turning the attention to the zero-mode sum-integrals, we shortly present the IBP reduction of Π^E_{210}, needed in the computation of $V(3;12111;00)$. According to Eq. (D.1), we can write down two relations for Π^E_{ab0} as:

$$\partial_{\mathbf{q}}\mathbf{q}\circ \sum\!\!\!\!\!\!\!\int_Q \delta_{q_0}\frac{1}{[Q^2]^a[(P+Q)^2]^b} = 0 \Rightarrow \left[(d-2a-b) - b\mathbf{1}^-\mathbf{2}^+ + bP^2\mathbf{2}^+\right]\Pi^E_{ab0} = 0\ , \tag{D.2}$$

and

$$\partial_{\mathbf{q}}\mathbf{p}\circ\Pi^E_{ab0} = 0 \Rightarrow \left[(a-b) - a\mathbf{1}^+\mathbf{2}^- + b\mathbf{1}^-\mathbf{2}^+ + aP^2\mathbf{1}^+ - bP^2\mathbf{2}^+ + 2bp_0^2\mathbf{2}^+\right]\Pi^E_{ab0} = 0\ , \tag{D.3}$$

with $\mathbf{I}^\pm Z(d;...s_i...,s_6s_7) = Z(d;...s_i\pm 1...,s_6s_7)$. Applying it on the concrete case of Π^E_{110}, we obtain:

$$\Pi^E_{120} = \frac{1}{P^2}\left[\Pi^E_{020} - (d-3)\Pi^E_{110}\right];\ \Pi^E_{210} = \frac{1}{P^2}\left[(P^2-2p_0^2)\Pi^E_{120} - \Pi^E_{020}\right]\ , \tag{D.4}$$

leading to:

$$\Pi^E_{210} = (d-3)\frac{2p_0^2-P^2}{P^4}\Pi^E_{110} - \frac{2p_0^2}{P^4}\Pi^E_{020}\ . \tag{D.5}$$

The first term vanishes in the limit of $d = 3$ and the second term is simply:

$$\Pi^E_{020} = T \int_q \frac{1}{q^2 + p_0^2} = \frac{T}{8\pi |p_0|} \ . \tag{D.6}$$

Now we turn our attention to the general IBP relation for a zero-mode sum-integral:

$$\begin{aligned} \partial_{\mathbf{p}} \mathbf{p} \circ Z(d; s_1 s_2 s_3 s_4 s_5; s_6 s_7) = 0 \\ \left[(d - 2s_1 - s_4 - s_5) + s_4 \mathbf{4}^+ \left(\mathbf{2}^- - \mathbf{1}^-\right) + s_5 \mathbf{5}^+ \left(\mathbf{3}^- - \mathbf{1}^-\right)\right] Z(d; s_1 s_2 s_3 s_4 s_5; s_6 s_7) = 0 \ , \end{aligned} \tag{D.7}$$

In addition to this equation, also a boundary condition is needed for the first parameter s_1 of Z. When looking at Eq. (C.8) the obvious condition for that, in order to have a converging momentum integral, is $d{-}1 \geq 2s_1$. This means for instance, that zero-modes of the form $Z(7; 3...)$ should be manageable. It turns out however that for some combination of the other parameters, in other words due to the particular form of $f^{0-A}_{7,abc}$, the integral cannot be rendered convergent through subtraction of simple terms. This implies that only for $Z(7; 2...)$ all combinations of parameters lead to IR "manageable" results.

In the following the case of $Z(3; 21111; 20)$ is skipped, since it is treated in the main text. For the case of $Z(5; 31122; 11)$, we have the relation:

$$Z(5; 31122; 11) = 0 \ , \tag{D.8}$$

due to the fact that the substructures Π_{121} are identically zero for $p_0 = 0$. Due to the summation the terms cancel exactly:

$$\Pi_{ab1}(p_0 = 0) = \sum\!\!\!\!\!\!\int_Q \frac{q_0}{(Q^2)^a [q_0^2 + (\mathbf{p} + \mathbf{q})^2]^a} \overset{q_0 \to -q_0}{=} -\Pi_{ab1}(p_0 = 0) \ . \tag{D.9}$$

D.1 IBP for $Z(3; 31111; 22)$

Applying Eq. (D.7) on the sum-integral in case, we obtain:

$$(d - 8) Z(3; 31111; 22) - 2Z(3; 22111; 22) + I^1_2 J^d(311; 1) = 0 \ . \tag{D.10}$$

Note that we have used the property of the one-loop structure: $\Pi_{abc}(p_0 = 0) = \Pi_{bac}(p_0 = 0)$ and the factorization $I \times J^d$ in case one parameter is equal to zero. The zero-mode $Z(3; 22111; 22)$ does not fulfill the above mentioned condition, so a second IBP reduction needs to be performed:

$$(d - 5) Z(3; 22111; 22) - Z(3; 12121; 22) - Z(3; 12211; 22) + I^1_2 \times J^d(221; 1) = 0 \ , \tag{D.11}$$

so that in the end we have:

$$Z(3; 31111; 22) = \frac{2}{d - 8} \left[\frac{Z(3; 12211; 22) + Z(3; 12121; 22) - I^1_2 \times J(221; 1)}{d - 5} - I^1_2 \times J^d(311; 1) \right] \ . \tag{D.12}$$

D.2 IBP for zero-modes in $d = 7 - 2\epsilon$

This set of sum-integrals turns out to have a common basis set of only three sum-integrals. The procedure is to start from the sum-integral with the highest parameter s_1 and reduce it step by step via IBP to the desired order of $Z(7; 2...)$. So we have:

$$(d-10)Z(7;52211;00) - 2Z(7;42221;00) = 0\ . \tag{D.13}$$

This requires:

$$(d-10)Z(7;42221;00) - Z(7;32222;00) - 2Z(7;33221;00) + 2Z(7;43211;00) = 0\ . \tag{D.14}$$

From here, we need three more relations:

$$\begin{aligned}(d-10)Z(7;32222;00) - 4Z(7;23222;00) + 4Z(7;33212;00) &= 0\\ (d-6)Z(7;33221;00) - Z(7;23222;00) - 2Z(7;23231;00) &= 0\\ (d-9)Z(7;43211;00) - Z(7;33212;00) - Z(7;33221;00) + Z(7;42221;00) &= 0\ .\end{aligned} \tag{D.15}$$

Up to this point, we still need to reduce the following zero-mode:

$$(d-8)Z(7;33311;00) - 2Z(7;23321;00) + Z(7;33212;00) = 0\ . \tag{D.16}$$

And finally, from the previous equation, we still need to reduce:

$$\begin{aligned}(d-9)Z(7;33212;00) - Z(7;23222;00) - 2Z(7;23321;00) +&\\ Z(7;32222;00) + 2Z(7;33311;00) = 0\ .&\end{aligned} \tag{D.17}$$

From this system of equations, we obtain the following solutions for our zero-modes:

$$\begin{aligned}
Z(7;32222;00) &= 4\frac{(d(d-18)+76)\,Z_5 - 2(d-10)Z_7}{(d-6)(d-9)(d-12)}\ ,\\
Z(7;52211;00) &= \frac{4(3d-32)(d-8)Z_5 + 8(d-9)(d-12)Z_6 - 4(6d-56)Z_7}{(d-6)(d-8)(d-9)(d-11)(d-12)}\ ,\\
Z(7;42221;00) &= \frac{d-10}{2}Z(7;52211;00)\ ,\\
Z(7;43211;00) &= \frac{(d-8)\,((2d-53)d+340)\,Z_5 + 2(d-12)^2(d-9)Z_6}{(d-6)(d-8)(d-9)(d-11)(d-12)}\\
&\quad + \frac{2(d-10)\,((d-20)d+104)\,Z_7}{(d-6)(d-8)(d-9)(d-11)(d-12)}\ ,\\
Z(7;33221;00) &= \frac{Z_5 + 2Z_6}{d-6}\ ,\\
Z(7;33212;00) &= \frac{(d-8)(d-14)Z_5 + 2(d-10)^2 Z_7}{(d-6)(d-9)(d-12)}\ ,\\
Z(7;33311;00) &= \frac{2(14-d)Z_5 + 2(d(d-21)+106)Z_7}{(d-6)(d-9)(d-12)}
\end{aligned} \tag{D.18}$$

where we have denoted (table 4.1):

$$Z(7;23222;00) = Z_5;\ Z(7;23231;00) = Z_6;\ Z(7;23321;00) = Z_7\ . \tag{D.19}$$

D.3 IBP reduction for the master-integrals of mass dimension zero

In this case, the reduction generates a larger basis set as the original one, this being due to the presence of Matsubara modes in the numerator of some propagators, which has the consequence that the new basis sets overlap only sparsely.

The approach is the same as previously; the first zero-mode $Z(3;12111;00)$ does not need further reduction. In the following we generate IBP relations as:

$$(d-6)Z(3;21111;00) - 2Z(3;12111;00) + 2Z(3;22101;00) = 0\ , \tag{D.20}$$

$$(d-5)Z(3;22111;02) - Z(3;12121;02) - Z(3;12211;02) + Z(3;22210;02) = 0\ . \tag{D.21}$$

This relations are not coupled and already give the needed result. In the following, we proceed to the last two zero-modes sum-integrals, that require several steps of IBP reduction.

First:

$$\begin{aligned}(d-8)Z(3;31111;20) - Z(3;22111;02) - Z(3;22111;20) + Z(3;32101;02) + \\ Z(3;32101;20) = 0\ .\end{aligned} \tag{D.22}$$

This equation calls for two other equations:

$$(d-5)Z(3;22111;02) - Z(3;12121;02) - Z(3,12211;02) + Z(3;22210;02) = 0\ , \tag{D.23}$$

and

$$(d-5)Z(3;22111;20) - Z(3;12121;20) - Z(3,12211;02) + Z(3;22210;20) = 0\ . \tag{D.24}$$

Finally we have:

$$(d-10)Z(3;4111;22) - 2Z(3;32111;22) + 2Z(3;42101;22) = 0 \tag{D.25}$$

and the subsequent relations are:

$$\begin{aligned}(d-7)Z(3;32111;22) - Z(3;22121;22) - Z(3;22211;22) + Z(3;32210;22) = 0\ ,\\ (d-7)Z(3;22121;22) - Z(3;12221;22) - 2Z(3;13121;22) + \\ +2Z(3;23111;22) + Z(3;22220;22) = 0\ ,\\ (d-4)Z(3;22211;22) - 2Z(3;12221;22) = 0\ ,\\ (d-6)Z(3,23111;22) - Z(3;13121;22) - Z(3;13211;22) + \\ +Z(3;22121;22) + Z(3;23210;22) = 0\ .\end{aligned} \tag{D.26}$$

Thus, the new basis set is much larger than the initial one. In order to avoid new substructures, such as Π_{322} and Π_{312}, we use an additional IBP relation to transform then into already known substructures:

$$
\begin{aligned}
&\partial_{\mathbf{q}}\mathbf{q}\circ\Pi_{ij0}(p_0=0)=0\\
\Rightarrow\ &(d-2i-j)\Pi_{ij0}+2i\Pi_{i+1,i,2}-j\Pi_{i-1,j+1,0}+j\mathbf{p}^2\Pi_{i,j+1,0}+2j\Pi_{i,j+1,2}=0\ ,
\end{aligned}
\tag{D.27}
$$

leading to:

$$
\begin{aligned}
\Pi_{322}&=-\frac{1}{8}\left[(d-6)\Pi_{220}-2\Pi_{310}+2\mathbf{p}^2\Pi_{320}\right]\\
\Pi_{312}&=-\frac{1}{4}\left[(d-6)\Pi_{210}+2\Pi_{222}+\mathbf{p}^2\Pi_{220}\right]\ .
\end{aligned}
\tag{D.28}
$$

Thus, we obtain the following result in terms of the new eight basis zero-mode masters:

$$
Z(3;21111;00)=\frac{2Z(3;12111;00)-2I_2^0J^d(211;0)}{d-6}\ , \tag{D.29}
$$

$$
Z(3;22111;02)=\frac{Z(3;12211;02)+Z(3;12121;02)-I_2^1J^d(221;0)}{d-5}\ , \tag{D.30}
$$

$$
\begin{aligned}
Z(3;31111;20)=&\frac{Z(3;12121;02)+Z(3;12121;20)+2Z(3;12211;02)}{(d-8)(d-5)}\\
&-\frac{I_2^1J^d(221;0)+I_2^0J^d(221;1)+(d-5)(I_2^0J^d(311;1)+I_2^1J^d(311;0))}{(d-5)(d-8)}\ ,
\end{aligned}
\tag{D.31}
$$

$$
\begin{aligned}
Z(3;41111;22)=&\frac{Z(3;02221;02)-I_2^1J^d(122;0)+(d-6)\left[Z(3;12211;02)+I_2^1J^d(221;0)\right]}{(d-10)(d-8)(d-7)(d-5)}\\
&+\frac{2Z(3;13111;02)-(d-6)Z(3;12121;02)}{2(d-10)(d-8)(d-5)}+\frac{Z(3;03121;02)}{400-170d+23d^2-d^3}\\
&+\frac{(6d-40)Z(3;12221;22)-2(d-4)I_2^1J^d(222;1)}{(d-10)(d-8)(d-7)(d-4)}-\frac{2I_2^1J^d(321;1)}{70-17d+d^2}\\
&-\frac{2I_2^1J^d(411;1)}{d-10}\ .
\end{aligned}
\tag{D.32}
$$

Appendix E

Configuration space definitions for Π_{abc}

This appendix provides details about the concrete form of all the needed one-loop structures Π_{abc} in configuration space used in the finite pieces of the master integrals.

Recall formula (4.114):

$$\Pi^{0,B,C,E}_{s_1 s_2 s_3} = \frac{(2\pi T)^{2d+1-2s_{12}+s_3}}{2^{s_{12}}(2\pi)^d \Gamma(s_1)\Gamma(s_2)} \int \mathrm{d}^d\mathbf{r}\, \bar{r}^{s_{12}-d-1} e^{i\mathbf{pr}} e^{-|p_0|r} f^{0,B,C,E}_{d,s_1 s_2 s_3}(\bar{r}, |\bar{p_0}|) \ . \tag{E.1}$$

In the following all the f's are provided. First we start with $f^0_{d,abc}$. It is in general a function of $f(\coth x, |n|)$, according to:

$$\sum_{m=-\infty}^{\infty} e^{-(|m|+|n+m|-|n|)} = |n| + \coth x \tag{E.2}$$

and, for any polynomial in $|m|$:

$$\sum_{m=-\infty}^{\infty} e^{-(|m|+|n+m|-|n|)} p(|m|) = \sum_{m=0}^{|n|} p(|m|) + p(-\partial_{2r}) \frac{1}{e^{2r}-1} + e^{2|n|r} p(-\partial_{2r}) \frac{e^{-2|n|r}}{e^{2r}-1} \ . \tag{E.3}$$

In addition, note that for $p_0 = 0$, $f_{d,abc}(x, 0)$ reduces to a sum of polylogarithms according to:

$$\sum_{n=1}^{\infty} \frac{e^{-2nx}}{n^a} = \mathrm{Li}_a\left(e^{-2x}\right) \ . \tag{E.4}$$

Denoting $c \equiv \coth x$, we have:

$$f^0_{3,110}(x,n) = c + |n| \ , \tag{E.5}$$

$$f^0_{3,112}(x,n) = \frac{|n| + 3c|n|^2 + 2|n|^3 + 3(c^2-1)(c+|n|)}{6} \ , \tag{E.6}$$

$$f^0_{3,212}(x,n) = \frac{-1 + c^2 + |n|(c+|n|)}{2} \ , \tag{E.7}$$

$$f^0_{5,121}(x,n) = -\frac{n(3(c+|n|) + (-2 + 3c^2 + 3c|n| + 2|n|^2)x)}{6x} \ , \tag{E.8}$$

$$f^0_{7,210}(x,n) = \frac{18(c^2 + c|n| + |n|^2 - 1)x + (12c(c^2-1) + 2(6c^2-7)|n| + 3c|n|^2 + 5|n|^3)x^2}{6x^3} \ , \tag{E.9}$$

$$+\frac{36(c+|n|)+(3+9c^4-9c|n|+9c^3|n|-4|n|^2+|n|^4+3c^2(|n|^2-4))x^3}{12x^3}\ , \quad \text{(E.10)}$$

$$f^0_{7,220}(x,n)=\frac{6c(1+|n|x)+3x(c^2-1)(2+(c+|n|)x)+|n|(6+6|n|x+(|n|^2-1)x^2)}{6x^2}\ , \quad \text{(E.11)}$$

$$f^0_{7,310}(x,n)=\frac{3x(c^2-1)(3+cx+nx)+3c(6+nx(3+nx))+n(18+x(x+n(9+2nx)))}{6x^2}\ , \quad \text{(E.12)}$$

$$f^0_{7,320}(x,n)=\frac{(c^2-1)x+(c+|n|)(2+|n|x)}{2x}\ , \quad \text{(E.13)}$$

$$f^0_{7,330}(x,n)=c+|n|\ . \quad \text{(E.14)}$$

Next, we provide the f's with specific parameters that may demand an a priori subtraction of the Π^E pieces to avoid 1/0 terms in the summation. These pieces are needed only for the zero-mode parts, so that we set $n=0$:

$$f^{0-E}_{3,210}(x,n)=e^{2|n|x}B(e^{-2x},|n|+1,0)+H_{|n|}-\ln(1-e^{-2x})\ , \quad \text{(E.15)}$$

$$f^{0-E}_{3,210}(x,0)=-2\ln(1-e^{-2x})\ , \quad \text{(E.16)}$$

$$f^{0-E}_{3,220}(x,0)=2\,\mathrm{Li}_2\left(e^{-2x}\right)\ , \quad \text{(E.17)}$$

$$f^{0}_{3,222}(x,0)=c-1\ , \quad \text{(E.18)}$$

$$f^{0-E}_{3,310}(x,0)=2\,\mathrm{Li}_2\left(e^{-2x}\right)+\frac{2\,\mathrm{Li}_3\left(e^{-2x}\right)}{x}\ , \quad \text{(E.19)}$$

$$f^{0-E}_{3,320}(x,0)=2\,\mathrm{Li}_3\left(e^{-2x}\right)+\frac{2\,\mathrm{Li}_4\left(e^{-2x}\right)}{x}\ , \quad \text{(E.20)}$$

$$f^{0-E}_{3,330}(x,0)=2\,\mathrm{Li}_4\left(e^{-2x}\right)+\frac{4\,\mathrm{Li}_5\left(e^{-2x}\right)}{x}+\frac{2\,\mathrm{Li}_6\left(e^{-2x}\right)}{x^2}\ , \quad \text{(E.21)}$$

where

$$B(x,a,b)=\int_0^x \mathrm{d}t t^{a-1}(1-t)^{b-1}\ , \quad \text{(E.22)}$$

is the incomplete Beta function.

Now we provide the zero temperature definitions of f[1]:

$$f^B_{3,110}(x,n)=|n|+\frac{1}{x}\ , \quad \text{(E.23)}$$

$$f^B_{3,112}(x,n)=\frac{|n|^3}{3}+\frac{1}{2x^3}+\frac{|n|}{2x^2}+\frac{n^2}{2x}\ , \quad \text{(E.24)}$$

$$f^{B+C}_{3,210}(x,n)=\gamma_{\mathrm{E}}+e^{2|n|x}\mathrm{Ei}(-2|n|x)+\ln\frac{|n|}{2x}+\frac{x}{6|n|}\ , \quad \text{(E.25)}$$

$$f^B_{3,212}(x,n)=\frac{1}{2x^2}+\frac{|n|}{x}+|n|^2\ , \quad \text{(E.26)}$$

$$f^B_{5,121}(x,n)=-\frac{n}{x^2}-\frac{n|n|}{x}-\frac{n^3}{3}\ , \quad \text{(E.27)}$$

$$f^B_{7,210}(x,n)=\frac{35}{4x^4}+\frac{35|n|}{4x^3}+\frac{15|n|^2}{4x^2}+\frac{5|n|^3}{6x}+\frac{|n|^4}{12}\ , \quad \text{(E.28)}$$

[1]Note that only the combination Π^{B+C}_{210} is finite in $d=3$ dimensions.

$$f^B_{7,220}(x,n) = \frac{5}{2x^3} + \frac{5|n|}{2x^2} + \frac{|n|^2}{x} + \frac{|n|^3}{6} \ , \tag{E.29}$$

$$f^B_{7,310}(x,n) = \frac{5}{x^3} + \frac{5|n|}{x^2} + \frac{2|n|^2}{x} + \frac{|n|^3}{3} \ , \tag{E.30}$$

$$f^B_{7,320}(x,n) = \frac{3}{2x^2} + \frac{3|n|}{2x} + \frac{|n|^2}{2} \ . \tag{E.31}$$

$$\tag{E.32}$$

And finally, we show the necessary leading UV terms:

$$f^C_{3,112}(x,n) = \frac{|n|^2 x}{6} - \frac{x}{30} \ , \tag{E.33}$$

$$f^C_{5,121}(x,n) = -\frac{n}{6} - \frac{n|n|x}{6} \ , \tag{E.34}$$

$$f^C_{7,310}(x,n) = \frac{1}{2x} + \frac{|n|}{2} + \frac{|n|^2 x}{6} + \frac{x^3}{1890} \ , \tag{E.35}$$

$$f^C_{7,320}(x,n) = \frac{1}{6} + \frac{|n|x}{6} + \frac{x^2}{180} \ . \tag{E.36}$$

Appendix F

IBP relations for the basis changes

In this appendix, we give the needed IBP relation in $d = 3 - 2\epsilon$ dimensions, in order to perform the suitable basis transformation of (C.14) and (C.15) from [110], to render coefficients finite in the limit $\epsilon \to 0$. These relations are part of a large database of IBP relations that was provided by Jan Möller [117]. The relations presented here were chosen in the spirit described in section 3.7.

For Π_{E3}, they are:

$$M^{000}_{111110} = \frac{148 - 60d + 6d^2}{3(d-5)(d-4)^2} M^{000}_{210011} + \frac{16}{3(d-4)^2} M^{000}_{310011} \,, \tag{F.1}$$

$$\begin{aligned} M^{020}_{211110} = &\frac{-240 + 134d - 21d^2 + d^3}{2(d-7)(d-6)(d-5)(d-4)^2} M^{000}_{113000} \\ &+ \frac{896 - 446d + 73d^2 - 4d^3}{2(d-7)(d-6)^2(d-5)(d-4)} M^{000}_{122000} \\ &+ \frac{299728 - 275712d + 100444d^2 - 18108d^3 + 1615d^4 - 57d^5}{12(d-7)(d-6)(d-5)^2(d-4)^2} M^{000}_{210011} \\ &+ \frac{-24 + 3d}{2(d-7)(d-5)(d-4)} M_{220011}002 \\ &+ \frac{-80 + 18d - d^2}{2(d-7)(d-6)(d-5)(d-4)} M^{020}_{310011} \\ &+ \frac{51232 - 23128d + 3470d^2 - 173d^3}{6(d-7)(d-6)(d-5)(d-4)^2} M^{200}_{310011} \\ &+ \frac{512}{(d-6)(d-5)(d-4)^2} M^{600}_{510011} \,, \end{aligned} \tag{F.2}$$

$$\begin{aligned} M^{000}_{31111-2} = &\frac{p_{1,1}(d)}{2(d-8)(d-7)(d-6)(d-4)^2(d-3)(d-2)(3d-20)} M^{000}_{113000} \\ &- \frac{p_{1,2}(d)}{2(d-8)^2(d-7)(d-6)^2(d-4)^2(d-3)(d-2)(3d-20)} M^{000}_{122000} \\ &+ \frac{p_{1,3}(d)}{12(d-8)(d-7)(d-6)(d-5)(d-4)^2(d-3)(d-2)(3d-20)} M^{000}_{210011} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{p_{1,4}(d)}{2(d-8)(d-7)(d-6)(d-4)^2(d-3)(d-2)(3d-20)} M^{002}_{220011} \\
&- \frac{-18367680 + 12301488d - 3275264d^2 + 433278d^3 - 28477d^4 + 744d^5}{2(d-8)(d-7)(d-6)(d-4)(d-3)(d-2)(3d-20)} M^{020}_{310011} \\
&- \frac{72(-12+d)(-10+d)}{(d-8)(d-7)(d-4)(d-3)(d-2)(3d-20)} M^{130}_{410011} \\
&- \frac{1024(2691 - 777d + 56d^2)}{(d-8)(d-6)(d-4)^2(d-3)(d-2)} M^{600}_{510011} \\
&+ \frac{98304}{(d-8)(d-6)(d-4)^2(d-3)(d-2)} M^{640}_{530011} \ , \qquad \text{(F.3)}
\end{aligned}
$$

with

$$
\begin{aligned}
p_{1,1}(d) &= 59336640 - 63078320d + 28473920d^2 - 7490234d^3 + 1336901d^4 - 174277d^5 \\
&\quad + 16160d^6 - 928d^7 + 24d^8 \ , \\
p_{1,2}(d) &= 2855308800 - 3237773312d + 1591362144d^2 - 442880256d^3 + 76305228d^4 \\
&\quad - 8326788d^5 + 561189d^6 - 21310d^7 + 348d^8 \ , \\
p_{1,3}(d) &= 26632233600 - 35846652912d + 20928142876d^2 - 6924466708d^3 \\
&\quad + 1420678191d^4 - 185150687d^5 + 14974094d^6 - 687360d^7 + 13716d^8 \ , \\
p_{1,4}(d) &= 108234240 - 89067072d + 30550848d^2 - 5590308d^3 + 575340d^4 \\
&\quad - 31557d^5 + 720d^6 \ . \qquad \text{(F.4)}
\end{aligned}
$$

For Π'_{T3}, we have:

$$
\begin{aligned}
M^{022}_{411110} = &\frac{p_{2,1}(d)}{8(d-7)(d-8)(d-9)(d-4)(d-5)^2(d-6)^2(d-3)^2} M^{000}_{114000} \\
&+ \frac{p_{2,2}(d)}{4(d-8)(d-9)(d-4)(d-5)^2(d-6)^2(d-7)^2(d-3)^2} M^{000}_{123000} \\
&+ \frac{p_{2,3}(d)}{256(d-7)(d-8)(d-9)(d-4)(d-5)^2(d-6)^2(d-3)^2} M^{000}_{220011} \\
&+ \frac{p_{2,4}(d)}{384(d-4)(d-8)(d-9)(d-2)(d-6)^2(d-7)^2(d-3)^2(d-5)^3} M^{000}_{222000} \\
&+ \frac{-p_{2,5}(d)}{128(d-7)(d-8)(d-9)(d-4)(d-5)^2(d-6)^2(d-3)^2} M^{000}_{310011} \\
&+ \frac{p_{2,6}(d)}{4(d-7)(d-8)(d-9)(d-4)(d-5)^2(d-6)^2(d-3)^2} M^{002}_{320011} \\
&+ \frac{-p_{2,7}(d)}{8(d-7)(d-8)(d-9)(d-4)(d-5)^2(d-6)^2(d-3)^2} M^{020}_{410011} \\
&+ \frac{-24(269d^4 - 8320d^3 + 95490d^2 - 479860d + 888021)}{(d-6)(d-7)(d-9)(d-4)(d-5)^2(d-3)^2} M^{220}_{510011} \\
&+ \frac{8(8047d^4 - 247972d^3 + 2849686d^2 - 14485888d + 27502047)}{(d-6)(d-7)(d-9)(d-4)(d-5)^2(d-3)^2} M^{400}_{510011} \\
&+ \frac{-30720(49d^2 - 711d + 2623)}{(d-6)(d-7)(d-9)(d-4)(d-5)^2(d-3)^2} M^{800}_{710011}
\end{aligned}
$$

$$+\frac{-368640}{(d-6)(d-7)(d-9)(d-4)(d-5)^2(d-3)^2}M^{730}_{730-111}$$

$$+\frac{-1290240}{(d-6)(d-7)(d-9)(d-4)(d-5)^2(d-3)^2}M^{820}_{820-111}\ , \tag{F.5}$$

with

$$\begin{aligned}
p_{2,1}(d) =&(d-2)(8d^9-368d^8+7504d^7-89111d^6+683254d^5-3608255d^4\\
&+14256518d^3-46293495d^2+116182716d-152317971)\ ,\\
p_{2,2}(d) =&4d^{10}-204d^9+4803d^8-72029d^7+809167d^6-7339813d^5+52777475d^4\\
&-278994927d^3+986568461d^2-2048698083d+1870829946\ ,\\
p_{2,3}(d) =&79869d^8-4037326d^7+87620700d^6-1061968604d^5+7812514612d^4\\
&-35356262430d^3+94351342448d^2-130527026040d+64326147651\ ,\\
p_{2,4}(d) =&96d^{11}-86421d^{10}+5375292d^9-153305147d^8+2586291594d^7\\
&-28437667028d^6+212405026246d^5-1089048350724d^4\\
&+3775673987846d^3-8444268898919d^2+10951876160526d-6212405250801\ ,\\
p_{2,5}(d) =&168943d^8-9557016d^7+235488122d^6-3301073030d^5+28792746146d^4\\
&-160004011372d^3+553185231922d^2-1087803532662d+931423919427\ ,\\
p_{2,6}(d) =&11d^7+1645d^6-89139d^5+1794069d^4-18776435d^3+109500887d^2\\
&-339147237d+436796199\ ,\\
p_{2,7}(d) =&3(2171d^7-113535d^6+2540413d^5-31533199d^4+234594837d^3\\
&-1046679269d^2+2595120339d-2760725997)\ ,
\end{aligned} \tag{F.6}$$

$$\begin{aligned}
M^{020}_{311110} =&\frac{p_{3,1}(d)}{4(d-5)(d-6)(d-7)(d-8)(d-4)(d-3)^2}M^{000}_{114000}\\
&+\frac{-p_{3,2}(d)}{2(d-4)(d-6)(d-8)(d-3)(d-7)^2(d-5)^2}M^{000}_{123000}\\
&+\frac{-p_{3,3}(d)}{128(d-6)(d-7)(d-8)(d-4)(d-5)^2(d-3)^2}M^{000}_{220011}\\
&+\frac{p_{3,4}(d)}{64(d-4)(d-6)(d-8)(d-2)(d-7)^2(d-3)^2(d-5)^3}M^{000}_{222000}\\
&+\frac{p_{3,5}(d)}{64(d-6)(d-7)(d-8)(d-4)(d-5)^2(d-3)^2}M^{000}_{310011}\\
&+\frac{3(d+1)(3d-19)(d-9)}{2(d-4)(d-6)(d-8)(d-3)(d-5)^2}M^{002}_{320011}\\
&+\frac{-3(d-9)(53d^3-1123d^2+7891d-18437)}{4(d-4)(d-6)(d-7)(d-8)(d-3)(d-5)^2}M^{020}_{410011}\\
&+\frac{-288(d-9)}{(d-4)(d-7)(d-3)^2(d-5)^2}M^{220}_{510011}\\
&+\frac{-96(57d^2-876d+3355)}{(d-7)(d-4)(d-5)^2(d-3)^2}M^{400}_{510011}
\end{aligned}$$

$$+ \frac{122880}{(d-7)(d-4)(d-5)^2(d-3)^2} M^{800}_{710011} \ , \tag{F.7}$$

with

$$\begin{aligned}
p_{3,1}(d) =& 3(d-9)(5d^2-42d+1)(d-2) \ , \\
p_{3,2}(d) =& 8d^6 - 273d^5 + 3858d^4 - 29038d^3 + 123676d^2 - 285521d + 281754 \ , \\
p_{3,3}(d) =& 3(1199d^6 - 41508d^5 + 578831d^4 - 4114536d^3 + 15388613d^2 \\
& - 27364596d + 15595005) \ , \\
p_{3,4}(d) =& 1143d^8 - 55758d^7 + 1176024d^6 - 13988986d^5 + 102457978d^4 \\
& - 471885410d^3 + 1329287528d^2 - 2080966998d + 1371885519 \ , \\
p_{3,5}(d) =& 3(2357d^6 - 96694d^5 + 1642835d^4 - 14793300d^3 \\
& + 74449531d^2 - 198518262d + 219089661) \ ,
\end{aligned} \tag{F.8}$$

$$\begin{aligned}
M^{002}_{221110} =& \frac{3(d-2)}{2(d-5)(d-6)(d-3)} M^{000}_{114000} + \frac{-d^2+9d-24}{(d-6)(d-3)(d-5)^2} M^{000}_{123000} \\
& + \frac{-(11d^2-72d+141)(3d-19)}{64(d-6)(d-3)(d-5)^2} M^{000}_{220011} \\
& + \frac{9d^4-136d^3+684d^2-1368d+1131}{96(d-3)(d-6)(d-2)(d-5)^3} M^{000}_{222000} \\
& + \frac{-(3d-19)(7d^2-102d+339)}{32(d-6)(d-3)(d-5)^2} M^{000}_{310011} \\
& + \frac{3d-19}{(d-6)(d-5)^2} M^{002}_{320011} + \frac{-9(3d-19)(d-7)}{2(d-6)(d-3)(d-5)^2} M^{020}_{410011} \\
& + \frac{-96}{(d-3)(d-5)^2} M^{220}_{510011} + \frac{32}{(d-3)(d-5)^2} M^{400}_{510011} \ ,
\end{aligned} \tag{F.9}$$

$$\begin{aligned}
M^{000}_{211110} =& \frac{7d^3-97d^2+405d-459}{(d-5)(d-6)(d-4)(d-3)^2} M^{000}_{220011} \\
& + \frac{-2(5d^4-102d^3+748d^2-2322d+2511)}{3(d-4)(d-6)(d-2)(d-5)^2(d-3)^2} M^{000}_{222000} \\
& + \frac{-2(7d^2-84d+249)(d-7)}{(d-5)(d-6)(d-4)(d-3)^2} M^{000}_{310011} + \frac{512}{(d-5)(d-4)(d-3)^2} M^{400}_{510011} \ ,
\end{aligned} \tag{F.10}$$

and

$$M^{000}_{121110} = \frac{1}{d-3} M^{000}_{220011} - \frac{1}{3(d-3)(d-2)} M^{000}_{222000} \ . \tag{F.11}$$

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Selbstständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbstständig und ohne fremde Hilfe verfasst und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Bielefeld, 28.10.2013