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EFFECTS OF NON-LINEARITIES AND DISORDER IN SYSTEMS WITH MULTIPLE ABSORBING STATES

A perspective for modeling the dynamics of complex ecosystems

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All that I am, or hope to be,
I owe to my angel mother.

— Abraham Lincoln

Dedicated to the loving memory of my mother
April 1953 – March 2010

...and father

and to my father, who supported me in every choice.

ABSTRACT

Interacting particle systems are a particular class of stochastic processes where single degrees of freedom interact through probabilistic rules defined over a graph which reflects the spatial topology of the model. From the statistical mechanics point of view these models are of particular interest since they are genuinely out-of-equilibrium processes and introduce new universality classes and dynamical phase transitions. Among these processes, systems with absorbing states are characterized by points in the state-space in which the dynamics becomes trivial and, once reached, cannot be leaved. Because of the several possible interpretations, these models have found many applications in different areas of science: from condensed matter Physics to Biology, from Ecology to Sociology and Finance and also, in their quantum versions, to quantum control theory. Despite their importance for possible applications, though, a unified understanding of these systems is still lacking.

In theoretical ecology, many open fundamental questions about the dynamics of ecosystems provide the cue for a further development of the theory of interacting particle systems. In particular, in this thesis we will address three main topics: *i) Spontaneous neutral symmetry breaking*. A central problem in ecology is the elucidation of the mechanisms responsible for biodiversity and stability. Neutral theory provides gross patterns in accord with empirical observations, but its validity is still highly debated. In particular, it is not clear how this theory can originate the observed non-neutral dynamics. Within a completely species-symmetric theory, we demonstrate that nonlinear dynamics can lead to a stationary state characterized by both stability and biodiversity by spontaneously breaking the neutral symmetry. *ii) Habitat heterogeneities.* It is known that habitat can have a great impact on the dynamics of species. In its most basic level of abstraction, its effects can be mimicked by an interacting particle system in a quenched random external field that locally breaks the species symmetry. We propose here an effective solution of the model in the long-times limit. *iii) Role of boundary conditions.* For non-equilibrium systems near a critical point, little is known about the role of the boundary conditions to the global phase diagram of the system. We analyze here a paradigmatic non-equilibrium critical model with mixed symmetry-preserving boundary conditions.

RIASSUNTO

I sistemi di particelle interagenti sono una particolare classe di processi stocastici in cui singoli gradi di libertà interagiscono secondo leggi probabilistiche su di un grafo che definisce la particolare topologia spaziale del modello. Dal punto di vista meccanico - statistico questi modelli sono particolarmente interessanti in quanto sono genuinamente fuori equilibrio ed introducono nuove classi di universalità e transizioni di fase dinamiche. Tra questi processi, i sistemi con stati assorbenti sono caratterizzati da punti nello spazio delle fasi in cui la dinamica diventa banale e che una volta visitati non possono essere abbandonati. Date le numerose possibili interpretazioni, questi modelli hanno trovato numerose applicazioni in aree differenti: dalla Fisica alla Biologia, dall'Ecologia alla Sociologia e la Finanza, fino, nelle loro versioni quantistiche, alla teoria del controllo quantistico. Tuttavia, nonostante la loro importanza per le loro possibili applicazioni, è ancora carente una comprensione teorica unificata di questi sistemi.

In Ecologia teorica, molte domande fondamentali sulla dinamica degli ecosistemi forniscono lo spunto per uno sviluppo ulteriore della teoria dei sistemi di particelle interagenti. In particolare, in questa tesi affronteremo i seguenti argomenti: *i) Rottura spontanea della simmetria neutrale.* Un problema centrale in ecologia è la spiegazione dei meccanismi responsabili della biodiversità e della stabilità. La teoria neutrale fornisce risultati in accordo con le osservazioni sperimentali, ma la sua validità è ancora fortemente dibattuta. In particolare, non è chiaro come essa possa produrre gli effetti non neutrali osservati. In una teoria completamente specie-simmetrica, dimostriamo che dinamiche non lineari posso produrre uno stato stazionario caratterizzato da stabilità ed una ricca biodiversità tramite la rottura spontanea della simmetria neutrale. *ii) Habitat eterogeneo.* È noto che l'habitat può influenzare grandemente la dinamica di un ecosistema. In prima approssimazione, questi effetti possono essere mimati introducendo un campo esterno aleatorio di tipo «quenched» che rompe localmente la simmetria tra specie. Proponiamo qui una soluzione efficace di questo problema nel limite di tempi lunghi. *iii) Ruolo delle condizioni al contorno.* Per i sistemi fuori dall'equilibrio vicino a punti critici si conosce poco sul ruolo delle condizioni al contorno sul diagramma di fase del sistema. Noi studiamo un importante modello critico fuori dall'equilibrio con condizioni miste al contorno che preservano la simmetria globale del sistema.

PUBLICATIONS

Some ideas and figures have appeared previously in the following publications:

• C. Borile, M. A. Muñoz, S. Azaele, Jayanth R. Banavar, and A. Maritan,
Spontaneously broken neutral symmetry in an ecological system,
Phys. Rev. Lett., **109**:038102, Jul 2012.

• C. Borile, A. Maritan and M. A. Muñoz,
Spatial disorder in the Voter Model,
Submitted to J. Stat. Mech., 2013.

• C. Borile and A. Maritan,
Role of boundaries in a critical nonequilibrium model (tentative title),
In preparation, to be submitted to Phys. Rev. Lett., 2013.

Be courteous to all, but intimate with few,
and let those few be well tried before
you give them your confidence.

— George Washington

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CONTENTS

Part I

MOTIVATIONS AND A MATHEMATICAL PREAMBLE

An interesting theoretical question on Nature often comes with the concomitant problem of choosing the right language, that is, the best mathematical framework to translate ideas into formulas we can handle. In this first part of the thesis we introduce the questions and the language.

1 INTRODUCTION

> We build too many walls and not enough bridges.
> — ***Sir Isaac Newton***

Taking the cue from the famous proverb 'There is nothing new under the sun', we can certainly say that 'there is nothing *sure* under the sun'. Despite of all the neat, elegant and –relatively– simple formulation of the fundamental laws governing the physical universe, that is ought to be the entire universe we live in, in the human-sized everyday world nothing goes as it should be. The great majesty of the galaxies gently moving in a curved spacetime, or the astonishing oddity of the microscopic world where symmetry is the queen of hearts find no equivalent in the world we can directly feel or touch. All the variety of phenomena and structures of a plant, a cell, or the brain is difficult to explain with nothing but four fundamental forces.

If we think about sciences like Biology, Ecology, Economics or Sociology what comes to mind is a huge series of veiled and highly interconnected processes that produce the phenomena we observe and describe. Here, 'description' is a key word since our predictive power in these fields is very poor. This fact is frustrating; Should we abandon the idea of a 'physics of the living world'? Hopefully, the answer is no. When a problem does not admit a solution by means of known methods and principles, new principles and methods have to be set up. Perhaps an entire new field of science collecting different perspective can emerge, as in the case of what it is now known as the *science of complexity*.

1.1 COMPLEX SYSTEMS

To be naïve and as general as possible, a system is *complex* when «more is different», as the Nobel laureate P. Anderson titled[1] his famous paper in Science (1972) [1]. A space rocket is an

1 Opposed to the reductionist idea, commonly accepted in Physics, that «less is more».

incredibly complicated object, but it is exactly the sum of all the small components it is made of. Thus, a space rocket is not complex. Instead, complexity deals with simple objects interacting in a simple way, but that when considered as a whole, in a big picture, produce patterns and behaviors that are neither ordered nor completely random, and "the whole is more than the sum of its parts"[2]. This phenomenon is broadly called *emergence*. Quoting Lewes: "Every resultant is either a sum or a difference of the co-operant forces; their sum, when their directions are the same – their difference, when their directions are contrary. Further, every resultant is clearly traceable in its components, because these are homogeneous and commensurable. It is otherwise with emergents, when, instead of adding measurable motion to measurable motion, or things of one kind to other individuals of their kind, there is a co-operation of things of unlike kinds. The emergent is unlike its components insofar as these are incommensurable, and it cannot be reduced to their sum or their difference" [2].

Complexity can generate either from dynamical or static processes, and from deterministic and probabilistic rules, classical or quantum.

What is intriguing from a theoretical point of view is that disparates events with very different fundamental interactions and constituents exhibits anyway common properties and statistical characteristics that indicate a sort of "universality principle", where details are effectively averaged out when going from the micro to the macro, and general principles, laws and models can hold. This is why it is now a common belief that complex systems can be collected in an unified theoretical framework able to make predictions and to quantify processes that appear to be purely random.

As physicists, we look for fundamental principles sustaining and guiding hypothesis and models. What has been found so far can be summarized in three basic ideas: *Noise matters* – exceptional events driven by random forces are plausible–, *no fine-tuning* –Nature does not want to care about parameters– and *information transmission* –it is all about how a single element perceives the global state of the system–.

2 Aristotle, *Metaphysica*.

1.2 A PARADIGM OF COMPLEXITY IN NATURE: ECOLOGICAL SYSTEMS

An *ecological system* is defined as a set (community) of individuals –different one another in nature and behavior– interacting with each other and with the surrounding environment as a system [3]. Complexity rise first of all from the intrinsic probabilistic nature of biological, thus not precisely quantifiable and maybe also variable in time and/or in space, interactions among individuals and from the topology of the network of interactions itself. It acts simultaneously on several different scales in space and time, and for the time being a unique model capable of describing quantitatively a whole ecosystem is by far beyond our theoretical capabilities. What can be done by now is to concentrate on specific aspects of an ecosystem that can be safely isolated from the rest and abstracted to simpler processes. This is not a reductionist approach since even the simpler models are still complex processes involving many "elementary" building blocks and must be treated probabilistically, but it may be the case that these *null models* can capture the essentials of the problem of interest while maintaining the possibility of a complete mathematical treatment. Therefore, in this thesis we will deal with null-models of ecosystems. As a corollary this null-models will necessitate a further development of our present knowledge on the statistical mechanics of out-of-equilibrium particle systems, from which the more fundamental theoretical interest of this work.

1.2.1 *Dynamics*

In ecology, contrarily to other fields like biology, mathematical modeling of evolutionary dynamics has a long history the dates back to the end of the eighteenth century, when the first differential equation aimed at describing the change of a population in time due to natality and mortality was proposed[3]. Verhulst later proposed an improvement of the Malthus' equation that

3 R. Malthus in *An essay of the principle of the population, as it affects the future improvement of society.* (1798)

is now known as the *logistic equation*, that for a population of $N(t)$ individuals at time t reads

$$\frac{dN(t)}{dt} = \lambda N(t)[1 - \frac{N(t)}{K}]. \tag{1}$$

This simple[4] equation represents the almost free exponential growth of a population up to a "carrying capacity" of the habitat, when it saturates to a constant value. Obviously, this is just a rudimentary approximation of what the real process might be, but it gives a first glance into what is now a very florid branch of science that is population dynamics. In 1925, Lotka and Volterra proposed independently an equation for the prey-predator dynamics of two species, one predating on the other. Their equation was very successful since it predicts the periodic oscillation of the two species with a little lag of the predator behind the prey, fact that was experimentally observed in ecological communities from rabbits and lynxes to bacteria and yeast. Their equations for the density of the two populations x and y are

$$\begin{cases} \dot{x} = ax - bxy \\ \\ \dot{y} = -cy + dxy. \end{cases} \tag{2}$$

Also in the '20s the idea of demographic stochasticity and random dispersal as a core part of the systems' dynamics was introduced and revealed fundamental in the further development of a mathematical theory of ecosystems. The major revolution, though, was made in the '70s when a systematic and formal theory of dynamical systems and stochastic processes was developed. In that decades, moreover, the concept of neutral evolution started to diffuse in the ecology community, reaching its maximum expression if the 2000s with the monograph by the ecologist S. P. Hubbell [5] and successive works.

Since the last decades, the field of population dynamics has been continuously growing, mainly because of the establishment of the new field of "complex science", that is linking many different areas of science into a common line of thinking, with common denominator the idea that the key to understand the behavior of complex systems is to concentrate on the emergent

4 Even if apparently simple, this equation still reserves some surprises: for λ large enough the map becomes chaotic and nothing exact can be said about its future evolution [4].

properties deriving from simple microscopic models, dropping all the unnecessary details, and many macro- or micro- ecological systems often provide an useful test bench for mathematical models.

1.2.2 *Stylized facts and static distributions*

As it is always the case when a theory or a model about some physical phenomenon is proposed, also models of complex systems –in our case ecological systems– need experimental observations able to confirm or to confute a prediction or a description. This can be difficult when the system under analysis cannot be reproduced in a laboratory or when one has only a single realization of the process at his disposal. Furthermore, in ecology it is often difficult to have access to observables with sufficient accuracy or for enough time.

So, usually one looks at static distributions that are meaningful for characterizing the underlying dynamical process. The main quantities of interest are the relative species abundance (RSA), defined as the distribution of the number of species having a given number n of individuals in the sample ecosystem, the species-area relationship (SAR), that is, the distribution of the number of species in a given area, or the β-diversity, that represents the probability of finding two individuals at a distance r apart each other to be of the same species. Quantities more directly related to the dynamics and for this reason more difficult to measure are the resilience and persistence of an ecosystem, defined as the capability of responding and recovering quickly from external perturbations, so it is a measure of robustness, and the resistance to external perturbations respectively.

In this thesis we will be interested mainly in the fine properties of the RSA distribution and in the resilience and persistence of an ecosystem as macroscopic consequences of microscopic interactions or external perturbations starting from individual-based models of ecological systems.

1.3 GENERAL ORGANIZATION OF THE THESIS

This manuscript is organized as follows:

In chapter 2 we will give a brief introduction to the mathematical techniques and tools that we will use extensively in the rest of the thesis. In particular, topics in nonequilibiurm statistical physics, stochastic processes and interacting particle systems will be presented.

Chapter 3 will be dedicated to neutral/symmetric models of interacting particles, where there will be discussed the possible consequences of non-linear interaction terms in the global phase of the system and in particular the conditions for a spontaneous breaking of the neutral symmetry. Albeit rather generic and applicable to many possible phenomena, we will concentrate on the possible implications in theoretical ecology, where this kind of models are commonly used and have already proved to produce results in good agreement with experimental observations.

In chapters 4 and 5 we will concentrate on a particular model of genuinely out-of-equilibrium interacting spin systems and give some general results on its long time behavior. Chapter 4 will focus on the effects of quenched disorder on its macroscopic phase while chapter 5 will investigate the role of mixed mild boundary conditions on a model posed at criticality and discuss the possible applications in conservation ecology.

Finally, in chapter 6 we will report some concluding remark and future perspectives of this work.

During the development of this manuscript it will be frequently used a terminology derived from the ecology community besides the physics jargon; this is due both because of the motivations at the basis of this works and because it offers a clearer and more concrete picture of the hypotheses and global settings.

2

INTERACTING PARTICLE SYSTEMS AND STOCHASTIC DYNAMICS

> One must still have chaos in oneself
> to be able to give birth to a dancing star.
> — ***F. Nietzsche***

In modern theoretical sciences it is evident that *determinism*, even in principle, is nothing but a mere artifact introduced in the hope that the exact prediction of the future behavior of natural, or in some cases even human, phenomena were possible. Nevertheless, this idea withstood all refutations up to the end of the nineteenth century when discoveries as quantum mechanics or deterministic chaos mined the foundations of that time science. Our universe is not exactly predictable, and this is now also conceptually clear. Quantum mechanics provided the methods to quantitatively study, even if in probabilistic terms, the microscopic world, but it is not applicable to the phenomenology of the meso- or macro-scopic. At these scales it is very often the case, from chemical reactions of molecules to the dynamics of populations of animals or bacteria, that underlying deterministic laws of motion arise, but with some kind of *noise* acting on it. Indeed, life is nor mechanistic nor entirely chaotic. We refer to this class of systems as models of *limited predictability* [6], and it is for these kind of models that stochastic processes and the theory of probability revealed fundamental. This manuscript deals with these concepts.

In this chapter we introduce some basic notions of the mathematical theory of stochastic processes that will be useful in the remainder of the thesis.

2.1 STATISTICAL PHYSICS OUT OF EQUILIBRIUM

Quoting [7], "Statistical physics is an unusual branch of science. It is not defined by a specific subject per se, but rather by ideas and tools that work for an incredibly wide range of problems. Statistical physics is concerned with interacting systems that

consist of a huge number of building blocks –particles, spins, agents, etc. The local interactions between these elements lead to emergent behaviors that can often be simple and clean [...]". In particular, non-equilibrium statistical physics describes the time evolution of many-particle systems that are open systems, coupled to an external environment that allows for a constant flux of energy, particles or other quantities. When this is the case, the consequence at the microscopic level is that there are non-vanishing currents of probability in the phase space of the system between two different states, and the *detailed balance* condition is no longer satisfied. It is, in fact, this last condition that determines the relaxation toward an equilibrium state described by a unique Gibbs measure[1]. We broadly speak of *non-equlibrium* when the probability distribution of the microscopic states of the system is not the Gibbs measure.

Loosely speaking, while the core of the theoretical foundations of equilibrium statistical physics is the Hamiltonian (energy function) $H(k)$ relative to the microscopic state k of the system and the probability of observing the system in that state $P_{eq}(k) = \mathcal{Z}^{-1} e^{-\beta H(k)}$, in nonequilibrium systems there is still a complete lack of a "canonical" theoretical framework and in spite of the growing effort in the recent years there is not a clear comprehension of how to obtain general macroscopic observables for an arbitrary system out of equilibrium, or even how to exactly approach it. This is mainly because the probability distribution for the microstates of the system can depend strongly on the specific interactions between the particles. This is why most of the theoretical work on these kind of systems is formulated in terms of *phenomenological models*, e.g. by considering a stochastic Langevin equation in a coarse-grained picture of the system (see next section).

Despite of the conceptual difficulties, it is promising that many of the phenomenological aspects of equilibrium systems find an almost exact counterpart for non-equilibrium. In particular, the most important property of statistical models is *Universality* in phase transitions: Near a phase transition the microscopic details of the system are irrelevant and different models can be cast into –relatively few– *universality classes*. The specific universality class a model should belong to depends only

1 Or by a linear combination of different Gibbs measures if the system is not ergodic.

on general properties of the system like symmetries or dimensionality. This concept is fundamental since it states that in the vicinity of a phase transition we can study the phenomenology of complicated systems by considering other simpler models belonging to the same universality class. The concept of universality and criticality could be more than a simple mathematical abstraction in many different natural phenomena, and in chapter 5 we will report some evidences that many macroscopic phenomena are (self-)tuned near a generalized critical point. In the next chapters we will be strongly interested in phase transitions in nonequilibrium models, especially we will consider mechanisms inducing transitions from a fluctuating (active) phase to a so-called absorbing phase, where the dynamics eventually lead the system to one absorbing state of the model, a particular state of the system where fluctuations are suppressed and the dynamics becomes trivial [8]. Systems with these kind of states are important, among all possible fields of application, in Chemistry, since they model chemical reactions, or in Biology and in life sciences in general as, for example, they could model the evolution of a population or the spreading of a disease. In these case the absorbing state would represent the extinction of the population or the disappearing of the disease, respectively.

All these phenomena are characterized by –intrinsic or extrinsic– stochasticity and must be described in probabilistic terms. For this purpose we will introduce briefly the notion of stochastic processes and interacting particle systems in the next sections of this chapter.

2.2 A TOOLBOX FOR STOCHASTIC PROCESSES

2.2.1 *Some definitions*

A *stochastic process* is simply defined as 'a collection of random variables indexed by time' [9]. This definition is so simple that could sound a little pointless, but captures the essence of the problem. The extremely vast zoology of stochastic processes and the gigantic theoretical interest and possible applications coming from this simple statement lies in the words "random" and "time". Time can be discrete or continuous, and provides the notion of "previous" and "next" for the state of the system, that is, for the value taken by the random variables. The

state of the system at a successive time may or may not depend on the history of the process, can couple together different variables, can follow different distributions, and all these possible choices end up in very different phenomenologies. Formally, a stochastic process is then a collection $\{X_t\}_{t\in T}$, $T = \mathbb{Z}, \mathbb{R}$, where the random variable X_t follows the probability distribution $P(X_t = x|x_0, t_0; x_1, t_1; \ldots; x_{t-1}, t_{t-1})$.

Libraries could be filled with only books on stochastic processes, so that here we will just report some results that will be useful in the next chapters (for a self-contained and general description of stochastic processes in physics see for example [6, 10]).

First of all we will deal only with *Markov processes*, that are a particular kind of stochastic processes characterized by the property of "no memory", that is, if X_t, $t \in \mathbb{N}$, is our process then $\forall\, t > 0$

$$P(X_{t+1} = x|X_t = y_t, \ldots, X_0 = y_0) = P(X_{t+1} = x|X_t = y). \quad (3)$$

This is an assumption of "independence in time" in the sense that "conditional on the present state of the system, its future and past are independent"[2]. This property, in physical terms, is equivalent to implicitly assume that we are separating the evolution of the system into three different timescales, the first is fast and corresponds to the fundamental processes that determine the origin of an effective randomness at a second, slower, timescale, that is the one of interest. Lastly, one has to assume that still this second timescale is much faster than the typical time of observation of the system or duration of the process. At the first timescale one cannot assume a strict markovianity, that instead effectively appears at the second. A clear example is the case of Brownian motion, paradigm of all stochastic process, where the motion of a pollen in suspension on water moves incessantly in a irregular and chaotic motion. This random motion is due to the continuous collisions of the molecules of water with the pollen. These single molecules' trajectories depend on their previous history, but the net effect for large quantities of water's molecules at the 'pollen-scale' is perfectly Markovian.

Given the Markov assumption, then, a specific model is completely defined by: *i)* The set of all possible states of the process Ω, that can be countable or finite, e.g. $\Omega = \{0, 1, \ldots, N\}$, or a

2 Taken from Encyclopedia Britannica.

general continuous manifold, e.g. $\Omega = \mathbb{R}^n$, and either *iia)* the *transition rates*, if time is continuous, from one state to another[3], $w_{y\to x}$, defined as the quantity such that, if $x \neq y$,

$$P(x, t+h \mid y, t) = w_{y\to x} h + o(h), \tag{4}$$

or *iib)* the *transition probabilities* $p(x, t+1|y, t)$ if time is discrete. Both the transition rates or probabilities can be collectively cast in a transition operator Q or matrix P, respectively.

So far so good, but, beyond the simple definition of the specific model, how to extract useful informations about the system? As we said before there is not a "standard" approach, but there are different powerful tools we can use. In the next section we will briefly introduce some of these methods, that will be used heavily in the remainder of the thesis.

2.2.2 *Fokker-Planck and Langevin equations*

A Markov process with a countable set of states $i, j, \dots \in \Omega$ and transition operator Q can be easily represented by its Master Equation (ME)

$$\dot{P}(i, t) = \sum_{j\in\Omega} \left[w_{j\to i} P(j, t) - w_{i\to j} P(i, t) \right], \tag{5}$$

representing the time evolution of the probability density function for the system in terms of the current of probability in the phase space. Often the ME can be effectively approximated (see for example Appendix A.1) by the relative Fokker-Planck, or Kolmogorov, equation (FPE) –assuming that the states of the system admit a proper "continuum limit" in D dimensions–, that in its general form reads

$$\partial_t P(x, t) = -\partial_i \left[\mathcal{A}_i(x) P(x, t) \right] + \frac{1}{2} \partial_i \partial_j \left[\mathcal{B}_{ij}(x) P(x, t) \right], \tag{6}$$

where $\partial_i \equiv \partial/\partial x_i, i = 1, \dots, D$, and sum over repeated indices is understood. The first term in the r.h.s. of the equation represents the *deterministic drift* part of the process, while the second term represents the stochastic, *diffusive* part.

3 If Ω is not countable the definition is slightly more complicated since one needs a whole probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is a σ-algebra of Ω, and a filtration $\mathcal{F}_t$, but the intuitive substance does not change.

Furthermore, it can be shown that equation (6) is equivalent, in the Ito prescription, to a stochastic differential form

$$dx = \mathcal{A}(x)dt + g(x) \cdot d\mathbf{W}(t) \tag{7}$$

where[4] $g(x) \equiv \sqrt{\mathcal{B}(x)}$, and $\mathbf{W}(t)$ is a standard, in general multidimensional, Wiener process. In Physics the above equation is usually written as a stochastic differential equation for the time derivative of x,

$$\dot{x} = \mathcal{A}(x) + g(x)\xi(t) \tag{8}$$

with $\xi(t)$ gaussian white noise δ-correlated in time, that is, a random term following a normal distribution of unit variance whose correlators are $\langle \xi \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$, where $\langle \bullet \rangle \equiv \int \mathcal{D}\xi P[\xi] \bullet$. These kind of equations are called *Langevin equations*.

From a physical point of view it is clear that equation (8) represents the equations of motion for the generic quantity x subject to both a deterministic, $\mathcal{A}(x)$, and a stochastic, $g(x)\xi(t)$, 'force'. For example x could be the position of a brownian particle, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, or could be a generic scalar or vectorial field $x = \phi(\vec{x}, t)$.

In general, studying and solving a stochastic process reduces to computing *averages* over some observable:

$$\langle \mathcal{O} \rangle = \int_\Omega dx P(x)\mathcal{O}(x). \tag{9}$$

From whatever of the above equivalent equations for the evolution of the system, one can still rarely solve the full motion, but at least there are several possible methods for studying asymptotic behaviors, average quantities, steady states or the presence of different phases in the system.

The most basic quantity one can look for, and that we will be interested in, is the *stationary distribution* of a stochastic process, that for a Markov process with transition operator (or matrix) $P_t(x, y)$ can be defined as the probability measure $\pi(x)$ on the state space, if it exists, that satisfies the condition

$$\pi(y) = \sum_x \pi(x)P_t(x, y), \ t > 0. \tag{10}$$

4 Assuming that the square root is well defined, namely $\mathcal{B}$ is (semi)positive defined.

This in equivalent to say that the joint probability distribution $\pi(x_1, t_1, x_2, t_2, \ldots, x_n, t_n)$ satisfy the time translation invariance, i.e.,

$$\pi(x_1, t_1, \ldots, x_n, t_n) = \pi(x_1, t_1 + \varepsilon, \ldots, x_n, t_n + \varepsilon) \; \forall \varepsilon, \tag{11}$$

and in particular we will have

$$\begin{aligned} \pi(x, t) &= \pi(x) \\ \pi(x_1, t_1, x_2, t_2) &= \pi(x1, t_1 - t_2; x_2, 0). \end{aligned} \tag{12}$$

In the Fokker-Planck picture, a stationary distribution $\pi(x)$ will satisfy

$$\partial_t P(x, t)|_\pi \equiv 0. \tag{13}$$

There is a particular case for which the stationary distribution can be easily computed exactly, that is, when the process satisfies the *potential condition* (or *zero-current condition*), that is, referring to equation (6), the probability current $J_i(x, t) \equiv \mathcal{A}_i(x)P(x, t) + \frac{1}{2}\partial_j \left[\mathcal{B}_{ij}(x)P(x, t)\right]$ vanishes identically for all $x \in \Omega$ in the stationary state. This will be the case if we can rewrite the FPE (6) in the equivalent form

$$\begin{aligned} \partial_i \log[\pi(x)] &= B_i^{-1} k(x) \left[2A_k(x) - \partial_j B_{kj}(x)\right] \\ &\equiv Z_i[A, B, x], \end{aligned} \tag{14}$$

end the condition for the vanishing curl $\vec{\partial} \wedge Z \equiv 0$ is satisfied. This conditions are called "potential" since in this case the stationary distribution can be written as deriving from a potential $V(x)$,

$$\begin{aligned} \pi(x) &= e^{\int^x dy Z[A,B,y]} \\ &= e^{-V(x)}. \end{aligned} \tag{15}$$

Usually this is not the case for general problems, as we will see in the next chapters, and different techniques will be needed. Nevertheless, since a FPE in one dimension always satisfies the potential condition, in some cases we will try to reduce a multidimensional problem to a 1-dimensional FPE for a relevant observable and find the associate potential.

In the next chapters we will be concentrated in models presenting multiple stationary states, in general non-ergodic, and in particular types of stationary states that impose stronger conditions on the stationary distribution function. These states are called absorbing.

2.2.3 *Absorbing states and absorbing phase*

Among all stochastic processes, we will be interested in models presenting multiple absorbing states. Given a Markov chain with state space Ω, an *absorbing state* is a state $x_0 \in \Omega$ for which $P_{x_0 \to x_0} \equiv 1$. In the Langevin picture of Eq. (8), an absorbing state x_0 is defined as a particular state in which

$$\dot{x}|_{x_0} \equiv 0. \tag{16}$$

This means that both the deterministic and the stochastic forces are zero in the absorbing point, that once reached cannot be leaved. In the FPE picture this will translate in a stationary distribution that is a δ-function $\pi(x) = \delta(x_0)$ giving mass only to that absorbing state. We note that in presence of absorbing states the stationary distribution will be singular and in general *non normalizable*. For this reason some trick will be needed to obtain a normalizable probability distribution function.

If a system has one or more absorbing states, the first question that can be asked is whether this state will be reached in a finite time if a dynamics starts from an arbitrary different state, perhaps depending on the parameters of the model. If this is the case we will say that the system is in an absorbing phase, opposed to the *active* one where the system keeps fluctuating indefinitely around some average value or oscillates periodically, but still the state is continuously changing in time due to the stochastic noise. When there is a sharp transition between the two regimes for some value of the parameters defining the model, we will speak of a active-to-absorbing phase transition. As mentioned previously, phase transitions in non-equilibrium systems show universality and are thus particularly appealing for physicists since simple models, analytically tractable, can be studied and represent some kind of "Ising model" for non-equilibrium. Beside this, they find many applications in many interdisciplinary problems of population dynamics, spreading, coarsening, reaction-diffusion models and others.

2.3 INTERACTING PARTICLE SYSTEMS

The class of stochastic processes known as *interacting particle systems* [9, 11] started to develop about 1970 and is still heavily developing, also because since the first mathematical and

abstract works it became more and more evident that these systems present often numerous possible "applications" in largely different fields, as condensed matter Physics, Biology, Neuroscience, Ecology, Sociology, Finance, etc. to name a few. In general models of this kind are used when there is an evolution in time of individual "particles" through a network of interactions.

In a nutshell, usually an interacting particle system is a Markov process, precisely a Feller process, with transition operator (matrix) $Q(t)$ defined on a totally disconnected space $\{0,\dots,q\}^{\Lambda}$ where Λ is a countable set. For concreteness, the most typical case especially in Statistical Mechanics is that of *spin systems*, that is, a set of coupled binary variables where the interactions are probabilistic and defined over the (discrete) topology of an underlying graph. A *graph* $\mathcal{G} = (V, E)$ is the couple defined by a set of *vertices* V and a set of *edges* $E \subset V \times V$ joining pair of vertices.

The peculiarity of these processes is that each transition involves only one vertex at a time, and while the global infinite system is markovian, this is generally not true for the evolution of single particles.

2.3.1 *Some notations and definitions*

Let $\Omega = \{0,1\}^{V}$ be the state space of the system, we will (usually) denote with $i, j, \dots \in V$ the vertices of the graph and by $\sigma, \eta, \dots \in \Omega$ the states of the system. The neighborhood of a vertex i is denoted by ∂i, that is, $\partial i = \{j \in V \,:\, (i,j) \in E\}$.

Call η^{i} the state obtained by "flipping" the spin at position i, $\eta(i)$, to $1-\eta(i)$ leaving all the other spins unchanged, then the model is defined by the function $c(i,\eta) : V \times \Omega$ that gives the rate of flipping for the spin at i,

$$c(i,\eta) = w_{\eta\to\eta^{i}}, \tag{17}$$

and the *probability generator* for the process is

$$\mathcal{L}f(\eta) = \sum_{i} c(i,\eta)\left[f(\eta^{i}) - f(\eta)\right]. \tag{18}$$

$\mathcal{L}$ is called a generator because it can be shown that the Markov transition operator of the process is formally given by $Q(t) =$

$e^{t\mathcal{L}}$ and it encodes the information about the transition rates of the model.

The function $c(i,\eta)$ usually defines *local* interactions among different spins, that is, the rate of flipping for a spin at vertex i depends on the nearest neighbors spins $j \in \partial i$. In this notation, an absorbing state is a state η_0 such that $\mathcal{L}f(\eta_0) \equiv 0$ and the relative measure on Ω is the pointmass δ_{η_0}, that gives measure 1 to the absorbing state and 0 to all other states. Absorbing states are a particular case of *stationary distribution*, i.e. a distribution π that is invariant for the dynamics, $\pi Q(t) = \pi \; \forall \; t \geqslant 0$. It is possible to prove [9] that a Feller process, so also an interacting particle system, has always at least one stationary measure[5]. If a stationary distribution π is unique and is the limit of any distribution under the dynamics, $\lim_{t\to\infty} \nu Q(t) = \pi \; \forall \; \nu$, then the system is said to be *ergodic*.

To conclude this introductory section, we mention a tool that is particularly important for particle systems, and even if we will not use it directly, it is fundamental for the exact solution of the voter model, that will be the basis for all the models analysed in the following chapters. Suppose two particle systems with generators $\mathcal{L}_1$ and $\mathcal{L}_2$ and a function H on $\Omega_1 \times \Omega_2$ such that $\mathcal{L}_1 H(\cdot,\eta_2)(\eta_1)$ and $\mathcal{L}_2 H(\eta_1,\cdot)(\eta_2)$ are well defined and $\mathcal{L}_1 H(\cdot,\eta_2)(\eta_1) = \mathcal{L}_2 H(\eta_1,\cdot)(\eta_2) \; \forall \, \eta_1, \eta_2$, then we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *dual* with respect to H. If two systems are dual for some functions, than it is possible to perform computations on one system in terms of the other, perhaps simpler.

In the next section we introduce the voter model, that will be useful in the following chapters, and we point out the possibility, common in the physics community, of connecting the microscopic description of the model to a coarse-grained description in terms of phenomenological Langevin equations for some order parameters.

2.3.2 *An example: The voter model*

The voter model (VM) is the spin system on $\Omega = \{-1,1\}^\Lambda$ defined by the rates

$$c(i,\sigma) = \frac{1}{2z_i} \sum_{j\in\partial i} (1 - \sigma_i \sigma_j), \tag{19}$$

5 Here we use "measure" and "distribution" as synonyms.

where $\sigma_i = \sigma(i)$ is the value of the spin at vertex i and $z_i = |\partial i|$ is the degree of vertex i.

The first fact to notice is that the system presents a global $\mathbb{Z}_2$ symmetry, since the rates are invariant under the transformation $\sigma \to -\sigma$, and that it is never ergodic since we there are two absorbing states $\sigma_1 \equiv 1$ and $\sigma_{-1} \equiv -1$ and the respective pointmasses δ_1, δ_{-1} are stationary. The first question is thus if these are the only stationary distributions. First, without any proof, we note that if Λ is finite, that is, the total number of spins is finite, say N, then the only possible stationary distributions are the pointmasses on the absorbing states since at some time a large fluctuation will almost surely take the system to one of the absorbing states, where the dynamics will stop. Therefore, in general, finite-size systems are always in the absorbing phase, strictly speaking. Nevertheless we can speak of active stationary distributions for infinite systems, or of *quasi-stationary distributions* for finite systems, that are defined as stationary distribution of the system conditioned to have not reached the absorbing states.

It can be shown (see Appendix A.2) that the VM is dual to a class of coalescing random walks going backward in time and many of the exact results on its limiting behaviors can be computed via this duality.

Some of the most important results on the VM can be summarized as follows: In lattices of dimension $D \leqslant 2$ the infinite system reaches one absorbing state almost surely[6] and the distributions $\delta_{\pm 1}$ are the only stationary distributions, while for $D > 2$ there is a 1-parameter family of stationary distributions depending on the initial conditions and the system persists indefinitely in the active phase. Consistently, the 2-points correlation function $G(r,t) \equiv \langle \sigma_i(t)\sigma_{i+\vec{r}}(t)\rangle$ has the following limiting behavior for large times [7]

$$G(r,t) \underset{t\to\infty}{\sim} \begin{cases} 1 - \dfrac{r}{\sqrt{Dt}} & \text{if } D = 1 \\ 1 - \dfrac{\log(r/a)}{\log(\sqrt{Dt}/a)} & \text{if } D = 2 \\ \left(\dfrac{a}{r}\right)^{D-2} & \text{if } D > 2, \end{cases} \tag{20}$$

with a, D constants. It can be seen that below the upper critical dimension $D_c = 2$ the system shows *coarsening*, that is,

6 i.e., with probability one.

domains of aligned spins become larger with time and the system eventually reach one of the two absorbing states. Exactly at the upper critical dimension the coarsening is logarithmic, while for $D > D_c$ the correlation function approaches a non-trivial limiting value. Furthermore, another interesting quantity is the mean time needed to reach one of the two absorbing states when the size of the system is $N < \infty$, call it $T_N \equiv \langle \min_t\{t : \sigma(t) \equiv \pm 1\}\rangle_{\text{realizations}}$. It can be shown that

$$T_N \sim \begin{cases} N^2 & \text{if } D = 1 \\ N \log N & \text{if } D = 2 \\ N & \text{if } D > 2. \end{cases} \tag{21}$$

In the physics community, the VM is usually mapped to an effective field theory for the magnetization field $\phi(\vec{x}, t)$, in the continuum-space limit. The corresponding action for ϕ and its conjugated response field ψ in the Doi-Peliti [12] path-integral formalism is [13]

$$\mathcal{S}[\phi, \psi] = \int d^D x dt \{\psi(\partial t - \lambda\nabla^2)\phi - \frac{\sigma^2}{2}\psi^2(1 - \phi^2)\} \tag{22}$$

corresponding to the Langevin equation

$$\dot{\phi}(x, t) = \lambda\nabla^2\phi(x, t) + \sigma\sqrt{1 - \phi^2(x, t)}\xi(x, t). \tag{23}$$

Looking at the Langevin equation above it is clear that the equation of motion for the magnetization field is a purely diffusive term plus a multiplicative noise that vanishes at the absorbing boundaries. The power-law behavior of T_N is Eq. (21) is characteristic of the absence of surface tension and for the voter model it is a direct consequence of the recurrence properties of the random walk, to which it is dual. The VM has no order parameter and no phase transitions, but later it was realized that the VM is the critical point of a larger class of models characterizing the non-equilibrium models with two symmetric absorbing states and no surface tension at the boundaries between different domains. We will briefly give an introduction to this important class of models in the next section.

2.3.3 *Nonlinearity and the generalized voters*

Thinking about how the VM is defined, it seems to be a very peculiar model, lacking of any characteristic scale or parameter. The first question can be then on how to introduce an order parameter to pass from a disordered phase to an ordered one. The first possible answer is to introduce a finite-temperature noise on the system (the VM is implicitly defined at $T = 0$), but any finite temperature destroys the ordered phase and $T_c = 0$. More interesting is the case of *interfacial noise*: a non-trivial transition appears when noise is added only at the boundaries between different domains, and from the study of the critical exponents a new universality class emerged, the *voter universality class* [14], that in $D = 1$ is equivalent to the parity-conserving (PC) class.

In 2003 a paper by M. Droz, A. L. Ferreira and A. Lipowski reported a computational analysis of a "voter Potts model" [15] in which they found that a rather general Potts Hamiltonian $H = -\sum_{(i,j)} \delta(\sigma_i, \sigma_j)$, endowed with some Metropolis nonequilibrium dynamics imposing two symmetric absorbing states, presents two different critical temperatures: The first one causes the spontaneous breaking of the Ising ($\mathbb{Z}_2$) symmetry, while the second one determines the transition from the disordered to the ordered phase, and belongs to the DP universality class. It became clear that the voter universality class was by far richer of phenomenology than expected.

Later it was proposed a minimal, in principle not unique but derived by symmetry constraints, field theory aimed at explaining the *generalized voter* models [16]. This can be written as a Langevin equation for the magnetization field

$$\begin{aligned} \dot{\phi}(x,t) \quad &= \nabla^2\phi(x,t) + (a\phi - b\phi^3)(1-\phi^2) + \\ &\quad + \sigma\sqrt{1-\phi^2}\,\xi(x,t). \end{aligned} \tag{24}$$

The equation above is reminiscent of equation (23) but with a additive "force" term $\nu[\phi; a, b] = (a\phi - b\phi^3)(1-\phi^2)$ that can be thought as derived from a ϕ^6 potential

$$V_{GV}[\phi; a, b] = -\frac{a}{2}\phi^2 + \frac{a+b}{4}\phi^4 - \frac{b}{6}\phi^6, \tag{25}$$

where the two parameters a, b are needed to include the scenario described by Droz *et al.* of two different phase transitions.

The (a, b) phase space can be divided into several different phases:

1. $b \leqslant 0$, *Unique GV phase transition*: If $b < 0$ than the potential has an extremal point at $\phi = 0$, that is a minimum for $a < 0$ and a maximum for $a > 0$. In general for $a, b < 0$ there are two maxima at $\phi = \pm\sqrt{a/b}$ if they lie in the interval $[-1, 1]$ and the origin and the two absorbing states are locally stable. When a increases the maxima approaches the origin and the local minimum at $\phi = 0$ progressively becomes a global maximum when $a = 0$. General considerations [16], anyway, lead to argue that the situation for $b < 0$ reduces to the case $b = 0$ when fluctuations are included. For all these cases then the transition from absorbing to active phase is in the voter universality class.

2. $b > 0$, *Separate Ising and DP phase transitions*: When $b > 0$ and $a < 0$ the stable state is for $\phi = 0$. When $0 < a < b$ the origin and the absorbing states are unstable and the Ising symmetry is broken, but with the system still in the active phase. For $a \geqslant b$ the system is in the absorbing phase and the transition should be in the DP universality class.

Note that the VM point is for $a = b = 0$, when $V_{GV} = 0$. After its introduction, a large number of microscopic spin models were found to be well described, in a coarse-grained effective picture, by equation (24) [17, 18, 19].

In fact, the characteristic behavior of the standard VM turns out to be very fragile with respect to changes in the microscopic definition of the model, as for example the absence of surface tension (and thus the logarithmic coarsening in $D = 2$) disappears as soon as some nonlinear interaction is turned on.

Part II

MODELS OF ECOLOGIES

This part containes the principal original work carried out by the author of this manuscript. In here, different mathematical models of interacting particle systems are presented and studied with a special attention to the possible applications in theoretical ecology.

3

NEUTRAL VS. NON-NEUTRAL

Simplicity does not precede complexity,
but follows it.
— ***A. Perlis***

One of the most striking fact on earth is life, and a strictly connected problem is the explanation of the extremely large variety of living organisms. But what do we know about this latter question? *Biodiversity* –its origin, maintenance and loss– is a major issue in science that might shed light in our comprehension of evolution and life, that is still a highly controversial topic. Are we made up by pure chance? Is natural selection always the sole evolutionary driving force? How do we define selection? Similar species are effectively equivalent? Is an ecological community stable, and what does stability means? In the remainder of this chapter we will address some of these issues, in particular we will concentrate on the last two questions from a dynamical point of view and for an ecological system formed of species in the same trophic level[1]

3.1 A THEORY FOR BIODIVERSITY: NEUTRALITY

The idea of *neutral evolution* was first introduced in genetics, where it was introduced by Kimura [20] in the '60s as a tentative approach to the problem of mutations that do not modify the fitness of an individual in the environment. The main idea is that the great majority of mutations at the level of DNA does not modify effectively the individual's fitness. In the modern genetics terminology, the fitness niches are 'broad' and inside them the 'fitness landscape' is flat, that is, the random diffusive motion on this landscape determines the formation of genealogies. Only occasionally a mutation modifies substantially an

1 The term *trophic level* denotes the position occupied by an individual in the food chain. So, when considering a population of individuals in the same trophic level there are not prey-predators, mutualistic, parasite-host interactions etc.

individual's identity. Later, ecologist S. P. Hubbell "transposed" this approach to ecology [5]. In a nutshell, we speak of neutrality in ecology when *every individual in the same trophic level is functionally equivalent (i.e. undergoes the same dynamical rules) regardless of the species it belongs to* [21]. This means that we make the assumption of flat fitness and that the ecosystem's biodiversity is driven only by the intrinsic stochasticity of the rates of birth, death, diffusion, speciation, etc. at an individual level.

It is clear that this is only a null hypothesis, as real ecosystems are not strictly neutral [22], but it turned to be a powerful tool for a gross modeling of ecosystems, and one could be surprised of how good neutral models often fit to data [23, 24, 25].

In order to have a more realistic description of real ecosystems, one could be tempted of relaxing the original definition of neutrality given by Hubbell adding the possibility of a global or local density dependence, but always with the same parameters for all the species. We will speak in that case of a *Neutral Symmetric* (or generalized neutral) theory. This assumption has to be introduced in order to enhance the ecosystem's stability [26] and reproduce static quantities observed in real ecosystems [24].

3.2 CAN A NEUTRAL THEORY DIFFERENTIATE THE SPECIES?

Suppose that we want to build a mathematical model for describing an ecosystem's dynamics where a large number S of species coexist up to speciation or immigration timescales that eventually avert monodominance or extinction and where a species can be distinguishable from the others for some observable. Obviously a nonneutral model would do the job, since individuals belonging to different species would have different birth, death, diffusion, speciation rates etc., but it would take a large number of parameters, think for example to the Lotka-Volterra equation (2) that for only two species has three free parameters, that could hide the emergent stylized properties of a general ecosystem that we want to understand. On the other hand, a neutral model will be defined by a small number of parameters and in a simple conceptual way, but when all individuals are identical we do not expect nonneutral behavior to be observable, apart from trivial statistical fluctuations. So we have to face a problem of choosing the right approach for what

we want to study. Since we are not interested in analysing a particular system and its details, and we don't know *a priori* of what species we are talking about, we want a model that ideally has no free parameters but that can grasp the essentials of a general ecological system, *including non-neutral phenomenology*, so that we can see only the effects of the emerging, self-organized and universal properties.

In this section we set up a model that aims at reconciling these two approach, keeping the simplicity of a neutral/symmetric modeling with the possibility of nonneautral behavior spontaneously emerging from the dynamics. It is also worth saying that it is formally elegant and theoretically intriguing the idea that the variety of a rich biodiversity is the expression of a unique symmetric equation governing the "motion" of an ecological system. We will speak then of *spontaneously broken neutral symmetry*.

3.3 SPONTANEOUS NEUTRAL SYMMETRY BREAKING

We begin by recalling in a colloquial way our starting point: the voter model (VM), introduced formally in section 2.3.2. The VM is our most basic intuition of a completely neutral dynamics; it is a null model that nevertheless captures qualitatively, and to some extent quantitatively, the basic ideas of an ecological system and has the rare quality of being also an exactly solvable model. So, on each vertex i (the voter) of a regular lattice in D dimensions or of a network $\mathcal{G}$ define a binary variable $\sigma_i = 0, 1$ (the opinion) and apply the following dynamical rules:

- Choose a voter at random,
- replace its value by the value of a randomly chosen n.n.,
- repeat *ad infinitum* or until consensus (all voters with the same opinion) is reached.

From the terminology used above it is clear that the first possible interpretation of the VM is a null model of the spreading of opinions in a two-parties election when individuals have no memory and no partisanship, but here we interpret the model in a slightly different way: Call the voters 'individuals' and call the opinions 'species', then when a individual is chosen at random it represents the death (at rate 1) of that individual, and

its patch in the territory –that we suppose saturated– is free to be occupied by a new individual by reproduction of one of the neighbors (at a rate that depends linearly on the density of individuals of a certain species in the neighborhood). This is the most basic model of an ecosystem: Two species that compete for a territory only by diffusion and random deaths. Consensus here means that one of the two species underwent extinction and the other reached the monodominance of the territory.

In this perspective, the first obvious extension we need is to increase the number of total species from only two to a generic number S, in principle much larger. The dynamics remains unchanged apart from the fact that now we have S symmetric absorbing points.

The main problem with this model is that it has only two phases depending on the spatial dimension of the system (recall that the model has no free parameters): An absorbing ($D \leqslant 2$) and an active phase ($D > 2$). The ecologically relevant dimension $D = 2$ is exactly the upper critical dimension of the model and the system is in the absorbing phase, that means that the system will always fall in the absorbing state and, due to the particular dynamics without surface tension at the boundaries between different domains of opinions (there is no bulk noise) [14], this will typically happen in a power-law time in the system's size. Something slightly more involved is thus needed to describe our ecological system. In the next section we will introduce a generalization of the VM that, thanks to nonlinearities in the transition rates presents different nontrivial (quasi-)stationary regimes in every spatial dimension, maintaining the global $\mathbb{Z}_S$ symmetry of the multispecies VM, characterizing broadly the class of ecological neutral theories.

3.3.1 *A symmetric nonlinear model*

Consider a regular lattice of N vertices in D dimensions, let us call it $\Lambda \subset \mathbb{Z}^D$, and interpret each vertex as a patch of a certain fixed area containing M individuals, *i.e.*, take a coarse-grained view of the total system. At each vertex i reside M variables $\sigma_i^a \in \{1, \dots, S\}$, with $i = 1, \dots, N$ and $a = 1, \dots, M$. Now, consider the equivalent 'population number variables' representation of the system, with $n_i^\alpha = 1, \dots, M$ representing the total

number of individuals of the species $\alpha = 1, \ldots, S$ at site i. Obviously $\sum_{i=1}^{N}(\sum_{\alpha=1}^{M} n_i^\alpha) = NM$ at every time.

Following the VM updating rules of the lattice, we choose at time t a site k at random and an individual at that site, let us suppose that it is of species β. Now, from site l chosen randomly in its neighborhood we choose another individual, say of species γ. Then, a generic n_i^α will evolve according to

$$n_i^\alpha(t) \to n_i^{'\alpha}(t+1) = n_i^\alpha(t) + \delta_{ik}(\delta^{\alpha\gamma} - \delta^{\alpha\beta}). \tag{26}$$

This means that the transition probability $P(n_k^\beta \to n_k^{'\beta})$ of colonization of site k is given in the most general form by

$$P(n_k^\beta \to n_k^{'\beta}) = K_{kl}^{\beta\gamma} n_k^\beta n_l^\gamma, \tag{27}$$

where for example if $K_{kl}^{\beta\gamma} \equiv 1/M\mu$, with μ lattice coordination number –or in general the degree of the considered node–, we obtain the standard (linear) multispecies voter behavior. In general K can be a more involved function of the individuals of the lattice. For example, it can represent the ability of a species to colonize a patch depending on the density of individuals in the neighborhood. In fact, *density dependence* effects are observed in real ecosystems: The Jansen-Connell effect states that a species has a greater probability of reproduction when the seeds are spread far away from the parent plant, so that the reproduction rate of a given species decreases with its local population size, or the Allee effect, a positive density dependence for small densities.

We will consider from now on the more appropriate continuous time version of the model. Let $n \equiv \{n_i^\alpha\}_{i,\alpha}$ be the vector of all the population numbers, then the transition rates $W(n \to n')$ of the global markov process that represents our model is given by

$$W(n \to n') = \frac{1}{NM} \sum_{i\in\Lambda} \sum_{j\in\partial i} \sum_{\alpha=1}^{S} \sum_{\beta=1}^{S} K_{ij}^{\alpha\beta} n_i^\alpha n_j^\beta \Delta_{ij}^{\alpha\beta} \tag{28}$$

where $\Delta_{ij}^{\alpha\beta} \equiv \delta_{n_i^{'\alpha}, n_i^\alpha - 1} \delta_{n_i^{'\beta}, n_i^\beta + 1} \prod_{\gamma\neq\alpha,\beta} \delta_{n_i^{'\gamma}, n_i^\gamma} \prod_{k\neq i} \delta^S_{n_k, n_k}$.

From equation (28) we can write the (deterministic) evolution equation for the joint probability distribution $P(n, t)$ describing the probability of having at time t a population vector n (see

Chapter 2). This is given in general by the *Master Equation* (ME), equation (5), that in this case becomes

$$\partial_t P(n,t) = \sum_{n'} \left[W(n' \to n) P(n',t) - W(n \to n') P(n,t) \right]. \quad (29)$$

Since equation (29) cannot be solved exactly, as a first step to get some insight into the model we perform a Kramers-Moyal (KM) expansion of the ME and discard all the terms above the second order, that is, we consider the Fokker-Planck approximation of the ME. The leading order, corresponding to the deterministic terms, reads

$$\begin{aligned} A_i^\alpha(n) &= \sum_{n'} W(n \to n')(n_i'^\alpha - n_i^\alpha) \\ &= \sum_j \sum_\beta \left[K_{ij}^{\beta\alpha} n_i^\beta n_j^\alpha - K_{ij}^{\alpha\beta} n_i^\alpha n_j^\beta \right]. \end{aligned} \quad (30)$$

Consequently, the deterministic equations of motion are

$$\dot{n}_i^\alpha = A_i^\alpha(n), \quad i \in \Lambda, \ \alpha = 1, \dots, S. \quad (31)$$

Now, the symmetry property (neutrality) that we want to impose to the model constrain the function K in two ways: *i)* it cannot depend explicitly on the species' labels, and *ii)* it can at best depend only on the density of species β and γ. For simplicity here we will assume that it is a scalar function only of the density of the invading species from site l, $\rho_l^\gamma \equiv n_l^\gamma / N$, namely $K_{kl}^{\beta\gamma} = K_{kl}(\rho_l^\gamma)$. Equation (31), written in terms of population densities –that from now on we assume to be in the continuous limit $N \gg 1$– becomes, after a proper rescaling of time $t \to t/N^2$,

$$\dot{\rho}_i^\alpha = \sum_j \sum_\beta \left[K_{ij}(\rho_j^\alpha) \rho_i^\beta \rho_j^\alpha - K_{ij}(\rho_j^\beta) \rho_i^\alpha \rho_j^\beta \right]. \quad (32)$$

3.3.2 *Infinite dispersal approximation*

Infinite dispersal[2] means that we drop the notion of space and every individual is connected with all the other individuals of the system, so that the density of a species at a site corresponds

2 For physicists it is a synonym of *complete graph* or *mean field*.

to the global density of that species in the whole system. Thus, $\rho_i^\alpha \to \rho^\alpha$ and we can write

$$K_{ij}(\rho) = \begin{cases} K(\rho) \text{ if } i \neq j \\ 0 \text{ otherwise.} \end{cases} \tag{33}$$

In this approximation, equation (32) takes the simpler form

$$\rho^\alpha = (N-1)\rho^\alpha \sum_\beta \left\{ \rho^\beta \left[K(\rho^\alpha) - K(\rho^\beta) \right] \right\}. \tag{34}$$

Starting from these equations of motion we can study the different stationary states dependent on the the single scalar function $K(z)$.

3.3.3 *Deterministic dynamics: Stationary solutions and stability*

Consider equation (34), then it is easy to see that there are always at least $S+1$ stationary systems' configurations independently of the specific choice of $K(\rho)$, namely

$$\begin{aligned} &\rho_{sym} = (\tfrac{1}{S}, \ldots, \tfrac{1}{S}), \\ &\rho_{abs} = (\underbrace{0, \ldots, 0}_{\nu_0 \text{ times}}, 1, 0, \ldots, 0), \quad \nu_0 \in \{0, \ldots, S\}. \end{aligned} \tag{35}$$

They correspond to the symmetric state where all the species coexist with the same average density and the S different absorbing states. This clearly reflects the symmetry of the problem. What changes with respect to K is the stability of these fixed points (see Appendix B for a brief introduction to the linear stability analysis).

Another possibility we want to check, and the most interesting one for what we want to demonstrate, is the case of one species having a density $\varphi > 1/S$ and all the others a symmetric density $\zeta = (1-\varphi)/(S-1)$. This is the simplest case of broken symmetry in the active phase. In general we could consider the case of $\varphi_1, \ldots, \varphi_S$ different densities.

The former case is equivalent to

$$\rho_{bs} = (\varphi, \zeta, \ldots, \zeta) \tag{36}$$

or any other permutation of the vector entries. The stationarity of this 'broken symmetry' state is guaranteed if we set

$$\begin{aligned} 0 = \dot{\rho}^1 &= \varphi(1-\varphi)\left[K(\varphi) - K(\zeta)\right] \\ 0 = \dot{\rho}^\nu &= -\varphi\zeta\left[K(\varphi) - K(\zeta)\right], \ \nu = 2, \ldots, S. \end{aligned} \tag{37}$$

Thus, the first condition we have to impose is

$$K(\varphi) = K(\zeta). \tag{38}$$

More conditions on K are given by imposing the stability conditions of the stationary points. In general we have to compute the Jacobian matrix $J_\mu{}^\nu[A] = \partial_\mu A^\nu$ of $A^\nu = (N-1)\rho^\nu \sum_\mu [K(\rho^\nu) - \rho^\mu K(\rho^\mu)]$. Recalling the saturation constraint $\rho^S = 1 - \sum_{\mu=1}^{S-1} \rho^\mu$ we have

$$\begin{aligned} J_\mu{}^\nu[A] =& \rho^\nu \Big[-K(\rho^\mu) - \rho^\mu \partial_\mu K(\rho^\mu) + K(\rho^S) + \rho^S \partial_S K(\rho^S) \Big], \\ & \text{for } \mu \neq \nu \end{aligned} \tag{39}$$

and

$$\begin{aligned} J_\nu{}^\nu[A] =& \Big[K(\rho^\nu) - \sum_\mu \rho^\mu K(\rho^\mu) \Big] + \rho^\nu \Big[\partial_\nu K(\rho^\nu) + \\ & - K(\rho^\nu) - \rho^\nu \partial_\nu K(\rho^\nu) + \rho^S \partial_S K(\rho^S) + K(\rho^S) \Big]. \end{aligned} \tag{40}$$

The stability of the considered fixed point ρ_* is given evaluating the jacobian matrix, J, at ρ_* and imposing the condition $J|_{\rho_*} < 0$ (Appendix B.2). For the absorbing fixed point we have $\rho_*^\nu = \delta^{\nu,1}$, that corresponds to the diagonal matrix

$$J|_{\rho_{abs}} = \left(\begin{array}{c|ccc} K(0) - K(1) & 0 & \cdots & 0 \\ \hline 0 & K(0) - K(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K(0) - K(1) \end{array} \right) \tag{41}$$

and thus the absorbing states[3] are – at least local – attractors of the dynamics if

$$K(1) > K(0) \text{ (stability of the absorbing states).} \tag{42}$$

Consider now the symmetric coexistence fixed point with state vector ρ_{sym} and call $K'(\rho) \equiv \partial_\nu K(\rho^\nu)$, then the jacobian in that point, $J|_{\rho_{sym}}$, is given by

$$J|_{\rho_{sym}} = \left(\begin{array}{c|ccc} \frac{1}{S} K'(\frac{1}{S}) & 0 & \cdots & 0 \\ \hline 0 & \frac{1}{S} K'(\frac{1}{S}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{S} K'(\frac{1}{S}) \end{array} \right), \tag{43}$$

3 We will use equivalently the mathematical, i. e. *absorbing state*, or the ecological, i. e. *monodominance*, terminology.

that is, the matrix is still diagonal and the condition for the linear stability of the symmetric state is simply

$$K'(\frac{1}{S}) < 0 \text{ (stability of the symmetric state).} \tag{44}$$

Slightly more complicated is the case of the broken symmetry state: For simplicity we consider the case of only one species with density $\varphi > 1/S$ and all the others with density ζ, that corresponds to a state $\rho^{\nu}_{bs} = \varphi\delta^{\nu,1} + \zeta(1-\delta^{\nu,1})$, $\nu = 1,\dots,S$ (more general conditions are substantially equivalent and will be given in Appendix A.5). The Jacobian turns out to be of the form

$$J|_{\rho_{bs}} = \left(\begin{array}{c|ccc} a & 0 & \cdots & 0 \\ \hline b & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b & 0 & \cdots & c \end{array}\right), \tag{45}$$

that is still diagonalizable. The eigenvalues are given by

$$\begin{aligned} &\lambda_1 = \varphi(1-\varphi)\left[K'(\varphi) + \tfrac{1}{S-1}K'(\zeta)\right] \\ &\lambda_{2,\dots,S} = \zeta K'(\zeta), \ (S-1)\text{-fold degenerate.} \end{aligned} \tag{46}$$

The conditions for the stability of the broken symmetry state are thus

$$\begin{cases} K'(\zeta) < 0 \\ K'(\varphi) < -\frac{1}{S-1}K'(\zeta) \end{cases} \text{ (stability of the bs state).} \tag{47}$$

3.3.4 *Choice of* K *and numerical simulations*

In the previous section we found that the conditions for the broken-symmetry state to be stationary and stable are $K(\varphi) = K(\zeta)$, $K'(\zeta) < 0$ and $K'(\varphi) < -K'(\zeta)/(S-1)$, while the only conditions in the case of the symmetric active phase is $K'(1/S) < 0$. In this section we will define different functions K satisfying the conditions for different stationary states, and by simulations of the complete master equation by means of the Gillespie algorithm [27] we will derive the scaling laws for some quantities of interest, like the mean time needed to reach one of the absorbing states, τ, or the stationary probability distribution for

the ν-th species population, $P^{\nu}(n)$. For completeness, we report here the Langevin equation, in the Ito prescription, corresponding to the Fokker-Planck equation of the model: It reads, in the rescaled time variable $t \to t/N$,

$$\begin{aligned}\dot{\rho}^{\nu} =& \rho^{\nu}\left[(1-\rho^{\nu})K(\rho^{\nu}) - \sum_{\mu\neq\nu}\rho^{\mu}K(\rho^{\mu})\right] \\ &+ \frac{1}{N}\left\{\rho^{\nu}[(1-\rho^{\nu})K(\rho^{\nu}) + \sum_{\mu\neq\nu}\rho^{\mu}K(\rho^{\mu})]\right\}^{\frac{1}{2}}\xi,\end{aligned} \tag{48}$$

where $\xi(t)$ is a gaussian white noie δ-correlated in time. The first term in the r.h.s. of the equation is the deterministic drift force that we analyzed in the previous section, while the second term takes into account the stochastic fluctuations of the system, and goes to zero in the limit $N \to \infty$.

The VM limit $K(x) \equiv const.$ is trivial since there are no stationary states apart from the absorbing ones (precisely, in the deterministic approximation all the states are stationary). Ecologically more interesting is the case of a linear function $K(z) = a(b-z)$, $a, b > 0$. This case, that is equivalent to the well-known logistic model in ecology, mimic the relative colonization advantage of the rare species with respect to those with abundant populations. The only stable state is the symmetric one, and after an initial transient the full stochastic dynamics is given by gaussian fluctuations around the stable state (Figure 1,a). The extinction of a species is due to a rare large fluctuation in the densities space and is characterized by an exponential timescale of the form $\tau(N) \sim \exp\{kN\}$ –opposed to the typical power-law time of the VM behavior–, where the constant k depends on the specific choice of the parameter a and goes to zero when $a = 0$.

Finally, we want to maintain the decreasing trend of K, that has been proven to be important for the stability of ecosystems, but now we define a function with an "S" shape –Figure 2–, that plays the role of cubic or higher order terms in the ability of colonization. These kind of nonlinear terms are often collectively denoted as *nagumo* terms [28] in mathematical ecology. A cubic K is sufficient to allow for the stability conditions (47), and in this case the states of broken symmetry coexistence are the stationary states of the deterministic dynamics. Depending on the initial conditions, the system will fall in one of the (equivalent) states where a species has a density φ and the others have den-

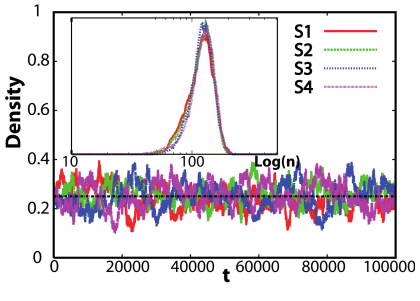
sity ζ. The phase space has S local attractors, and the stochastic dynamics is similar to that of the linear case, except for the fact that large fluctuations make the system jump from an attractor to the other in a typical time $\tau_{switch}(N) \sim \exp\{k_{sw}N\}$. If we look at a history of the process for a time $\gg \tau_{switch}(N)$ the global symmetry will be recovered, but for times $\leqslant \tau_{switch}(N)$ it will appear as non-symmetric. It is important to notice that τ_{switch} diverges exponentially fast in the infinite-size limit, and thus for systems of big sizes and the typical time of observation of a real ecosystem –corresponding practically to a snapshot– the dynamics could appear as completely nonneutral (Figure 1). Also in this case the typical time for the monodominant state is exponential: $\tau(N) \sim \exp\{k'N\}$. As for the precedent case the constant k_{sw} and k' depend on the specific choice of the function K. In a more general case of function K with more "minima" the symmetry breaking could involve up to S different densities simultaneously (see Figure 4).

3.3.5 *Relative species abundance and symmetry breaking*

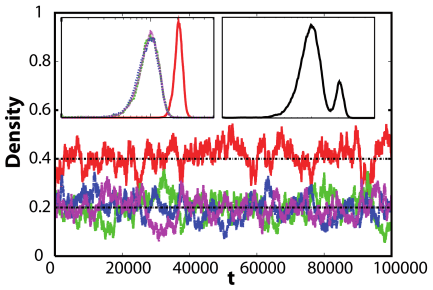
So far we have seen that, from a microscopic[4] level, small nonlinearities in the reproduction rate –that basically summarize all the competing biological and ecological "forces"– can produce big effects in the evolution of the whole system. The problem is that for most macroscopic ecosystems it is very unlikely to be able to follow the dynamical evolution of the species for enough time to answer directly by observations if these reasonings find support in real ecosystems. This is fundamental independently of the answer, either 'yes' or 'no', and we would like to find another way to have some hint on this answer. We need a static quantity that can be measured with field experiments. An appealing quantity is the so-called *Relative Species Abundance* (RSA) distribution, measuring the frequency of species versus their abundance in the considered area.

Precisely, the RSA $P(n)$ is defined as the probability for a species of having exactly n individuals. Historically, the first example of RSA was given in the '40s by the entomologist S. Corbet, that collected data for 620 butterflies in the Malay peninsula. The distribution was monotonically decreasing with long

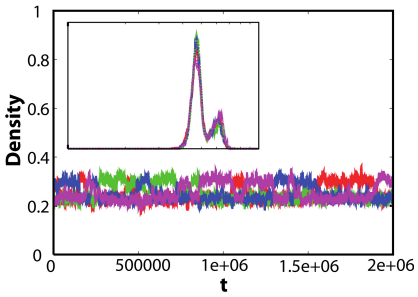
4 By "microscopic" we mean an individual-based description of the total system.



(a) Symmetric



(b) Explicitly broken symmetry



(c) Spontaneously broken symmetry

Figure 1: Example of the evolution of a neutral ecological model with 4 species with global dispersal (see main text) for: (**a**) neutral symmetry. All the species are indistinguishable and fluctuate around the average value $1/4$. In the inset (colors are the same as in the main picture) we show the probabilities $P_i(n)$, and the superposition is perfect within statistical errors, and (**b**) non-symmetric dynamics: species 1 has a different set of birth and death rates with respect to the other three species, and fluctuates around an average density of $2/5$, while the others fluctuate around $1/5$. The probability $P_1(n)$ differs from the others, as shown in the left inset; in the inset on the right, the global probability $P(n)$ is shown. **c**) spontaneously broken neutral symmetry. Here the system behaves differently depending on the observation window of its evolution: for small time scales, the system appear non-symmetric, whereas, for longer time scales, the symmetry is recovered. Unlike case (b), all the species show a bimodal distribution.The probability $P(n)$ in this case superpose virtually exactly on the probabilities $P_i(n)$. The total population is $N = 512$ individuals for the case a and b, and $N = 2048$ individuals for c.

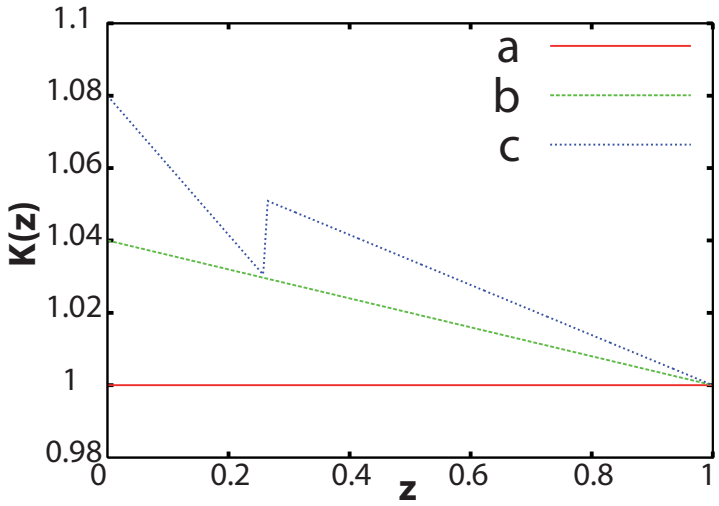


Figure 2: **a)** (Red solid line) $K(z) \equiv 1$, corresponding to the standard voter model with many species. **b)** (Green dashed line) $K(z) = a(b - z)$: This definition of the function $K(z)$ makes the symmetric state stable against perturbations, and the monodominant states unstable, provided $a > 0$. **c)** (Blue dotted line) $K(z)$ allowing S stable stationary states where the neutral symmetry is spontaneously broken by one of the S species.

tails on the rare species. Data were well fitted by the Fisher log-series[5]. After that, it was realized that the Fisher distribution was not suited for most of the dataset collected later: most RSA distributions were not monotonic and showed a peak at intermediate abundances [29] and were better fitter by a Log-Normal distribution. The problem was that these distribution, both the Fisher and the lognormal, were not supported by any theoretical modeling. In this context the neutral theory provided a first conceptual framework, a stochastic and individual-based dynamical model, that predicted a stationary distribution for the RSA that was in good agreement with experiments [5, 23, 24].

The fact that RSA distributions are unimodal is commonly accepted in the ecological community, but the argument is still controversial. The quality of the data sets is usually poor, and often the results change depending on how the data is plotted. It cannot be discarded *a priori* the possibility that real distributions are not unimodal but multimodal, since different peaks

5 The Fisher distribution is given by $P(n) \propto \frac{\alpha^n}{n}$, where $\alpha < 1$ is a system-specific parameter.

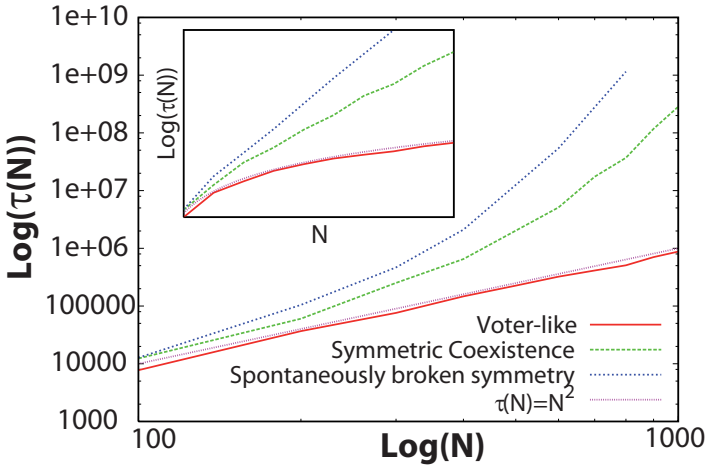


Figure 3: Mean time to extinction $\tau(N)$ for the three different definitions of $K(z)$ in Fig. (2), calculated in the mean field approximation and plotted in Log-Log scale varying N from $N = 100$ to $N = 1000$. For $K = const.$ (red solid line), $\tau(N) \sim N^{\alpha}$ with $\alpha \simeq 2$ (red dotted line) as expected for a Voter-like model, while the two cases of $K = b - az$ (green dashed line), where we chose $a = 0.04$, $b = 1.04$, and $K(z)$ allowing for a spontaneous breaking of the neutral symmetry (blue dotted line) show an exponential behavior $\tau(N) \sim e^{kN}$. In the inset, we show the same plot in a Log-Linear scale, to emphasize the exponential growth.

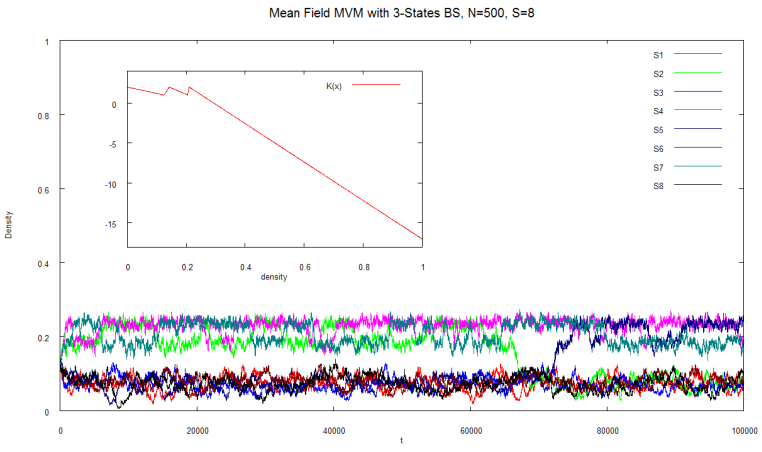


Figure 4: Example of a spontaneously broken neutral symmetry into three different densities.

could be hidden by experimental errors and result in only one mode of the distribution. Recently, a work by R. Vergnon *et al.* [30] discussed this eventuality, and their claim is exactly that "recent analyses of data sampled in communities ranging from corals and fossil brachiopods to birds and phytoplankton suggest that their species abundance distributions have multiple modes". We will now discuss how this case is completely coherent within a spontaneously broken symmetry scenario.

The inset of Figure 1(c) shows the probability $P^{\nu}(n)$ that the ν-th species has n individuals. Due to the global symmetry of the model, as discussed above, the shape of the RSA distribution $P(n) \sim \sum_{\nu} P^{\nu}(n)$ sampled for enough long times is qualitatively identical. It shows two modes, due to the broken symmetry into two stable states, and their relative distance and variance depends on the specific choice of parameters for the function K. The global RSA $P(n)$ would be indistinguishable for the spontaneously or explicitly broken symmetry cases and would be multimodal, at least for a sampling accuracy fine enough to separate small effects of nonneutrality between different species. If the nonlinear effects are too weak or are hidden[6] by other factors the different modes will not be visible and only a unique effective mode will be present. The RSA distribution, then, could not be the right observable to detect the neutral symmetry breaking.

3.4 DISCUSSION

We end this chapter with a brief discussion on the hypotheses we considered and their implications in the dynamical evolution of a complex system, that here we supposed to be an ecological system but due to the very general nature of the model could be intended as a description of several other natural and human processes.

Summarizing, our starting point was an individual-based dynamical modeling of an ecosystem beyond the "linear" approximation, taking into account in an effective way all the nonlinear effects that are ever-present in a real systems and their possible effects on the stability of that ecosystem and the global biodiversity. We considered a symmetric model, generalizing the

6 In the next chapter we will show that this could be the case in presence of spatial heterogeneities.

neutral theory allowing for density dependence. We encoded the nonlinearities in a function representing the relative ability of a species to colonize a free spot given the density of that species in the neighborhood. These kind of models were already considered in the literature and are well established in the theoretical ecology community [26, 31], but none of the previous studies considered the importance of the nonlinear terms in maintaining a rich biodiversity. This was our intent here, and indeed we have shown that a simple non-equilibrium microscopic model for a general S-species ecological community driven by a density-dependent but otherwise completely neutral/symmetric dynamics –*i.e.* the dynamic rules governing the stochastic microscopic process are insensitive to the species' labels– can show a rich and stable heterogeneous biodiversity even at very long times. The striking fact is that species can behave distinctly by spontaneously breaking the neutral symmetry.

4

DISORDER

Chaos is inherent
in all compounded things.
Strive on with diligence.
— ***Buddha***

Nature is certainly not a perfect laboratory where a particular system is prepared under controlled external conditions, known parameters and it is not isolated from the environment. We know that real systems, and many times also experiments, are strongly influenced by factors that are outside our control and too complicated to be clearly understood or modelized as part of the system we are observing. All these factors end up in what we call *disorder*, that can be defined naïvely as an intrinsic component of randomness in the interactions among the microscopic "particles" (the constituents of the system) or in the topological structure over which the dynamics takes place. There are many types and possible definitions of disorder, but the two main classes are those of *annealed* or *quenched* disorder. The former class contains all type of external randomness that evolves at the same timescales of the process under consideration so that it enters actively when we compute average quantities, while the latter applies to systems in which a particular realization of the disorder changes in a timescale much larger than the typical observation times and can thus be considered as "frozen". We will deal with this last class of problems.

4.1 DISORDER IN STATISTICAL PHYSICS

It is well known that quenched disorder can have a dramatic impact in equilibrium systems [32, 33, 34, 35]: consider for example the Ising model hamiltonian where now the couplings J are not constant but are extracted from some probability distribution $P(J)$. Then the new hamiltonian is

$$H(\sigma) = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j \tag{49}$$

and the interaction among spins can be both ferromagnetic and antiferromagnetic. This is the simplest definition of a *spin glass*, and is known as the Edward-Anderson (EA) model. The first obvious consequence of this definition is that, since the matrix J can have both positive and negative entries, the system is *frustrated*, that is, not all couplings can be satisfied simultaneously in the ground-state. This generates a proliferation of metastable states that enriches and modifies the phenomenology of the model and originate a very complicated energy landscape. Although these models are extremely difficult to treat analytically and even by simulations and are still not completely understood, in the last decades many different techniques and tools have been invented to treat these kind of models and many applications, from Biology and Neuroscience to Computer Science and Finance, were found [36, 37, 38].

Nevertheless, as already said above, usually natural systems are not in thermal equilibrium. What do we know then about the consequences of disorder in out-of-equilibrium phenomena? Recently there has been a growing interest in studying the effects of disorder in genuinely nonequilibrium models and in particular in models with absorbing states [39, 40, 41, 42, 43, 44, 45]. In the remainder of this manuscript we will be mainly interested in systems with two symmetric absorbing states; In this specific context Frachebourg, Krapivsky, and Redner [42] studied the influence of quenched disorder in the form of impurities for a model of catalysis with two symmetric absorbing states, showing that a non-trivial steady state emerge. More recently, Masuda et al. [43] showed that a voter model with quenched (random-field like) disorder –creating an intrinsic preference of each individual for a particular state/opinion– hinders the formation of consensus, hence favoring coexistence. Actually, the presence of just a few different "zealots" –not allowed to change their intrinsic state– suffices to prevent consensus [46] or just one frozen spin, acting as a source, constrain the VM to be always in the absorbing phase[1] [47]. Along similar lines, Pigolotti and Cencini [44] analyzed with computer simulations, in the context of neutral ecology, a version of the VM in which at each location there is an intrinsic preference for one particular species, leading to mixed states (no consensus/monodom-

1 In the sense of Directed Percolation, since one absorbing states is removed by construction.

inance) lasting for times that grow exponentially with system size. By studying a similar model, Barghathi and Vojta [45] have very recently stressed that contrarily to what happens in equilibrium systems, where a well known argument first proposed by Imry and Ma[2] (see Appendix A.4) precludes symmetries to be spontaneously broken in low-dimensional systems in the presence of quenched random fields [49], phase transitions that are intrinsically of non-equilibrium, such as those in the GV class, do persist in low-dimensional systems ($D = 1$) in the presence of random fields, even if with a different type of critical behavior [45].

Despite of all these results, a complete and coherent theoretical framework to understand disorder in VM-like systems is missing. In the remainder of this chapter we will consider a VM where each voter experiences an intrinsic tendency to align with a particular opinion, i.e. a voter model in a quenched random external field. This is motivated by the need of at least a first approach to a quantitative study on the effects of environmental heterogeneities in an ecological system, that despite its importance is completely lacking so far in ecology and conservation ecology [50, 51]. In particular we will be interested in the different possible steady states of the model and the typical times to reach one of the absorbing states, as in ecological terms these quantities can be related to the global biodiversity of a system and the persistence of the species in that ecosystem.

4.2 A DISORDERED VOTER MODEL

4.2.1 *Definition of the model and notations*

We consider a Voter Model (see 2.3.2) defined on a D-dimensional lattice ($\Lambda \subset \mathbb{Z}^D$). The lattice has $N = L^D$ sites, denoted by $i, j, \ldots$; at each site i resides a binary or spin variable $\sigma_i \in \{+1, -1\}$ and a random binary field τ_i to which σ_i is locally coupled, favoring its alignment with the field. The values of τ are quenched, that is, a particular realization of the disorder is extracted and does not change during the dynamics. To ensure the global up-down (plus/minus) symmetry, the lattice is bipartite, namely $\Lambda = \Lambda^+ \sqcup \Lambda^-$, with $\Lambda^\pm = \{i \in \Lambda \ : \ \tau_i = \pm 1\}$ and $|\Lambda^+| = |\Lambda^-|$, where $|\bullet|$ denotes the cardinality of a set. This

2 Later this argument was formally proved by Aizenman and Wehr [48].

last constraint can be relaxed for sufficiently large systems to be satisfied just on average over the lattice, and we take τ_i as *i.i.d.* random variables taking values in $\{+1, -1\}$ with uniform probability. At each site the coupling-strength between the spin and the random field is controlled by the free parameter $\epsilon \in [0, 1]$, where $\epsilon = 0$ stands for the uncoupled (pure VM limit) case and $\epsilon = 1$ implies that each spin remains frozen in the direction of its random field. The model is completely defined –in its continuous-time version– by specifying the transition rates W for a generic spin i, namely:

$$W(\sigma_i \to -\sigma_i) = \frac{1 - \epsilon\tau_i\sigma_i}{2z} \sum_{j \in \partial i} (1 - \sigma_i\sigma_j), \tag{50}$$

where ∂i is the set of nearest neighbors of i and z is the lattice coordination number. For a regular lattice one has $z = 2D$. The first (linear) term describes the standard VM dynamics [7], while the second term is proportional to ϵ and the flipping probability is enhanced or reduced depending on whether the spin is aligned or not with its random field.

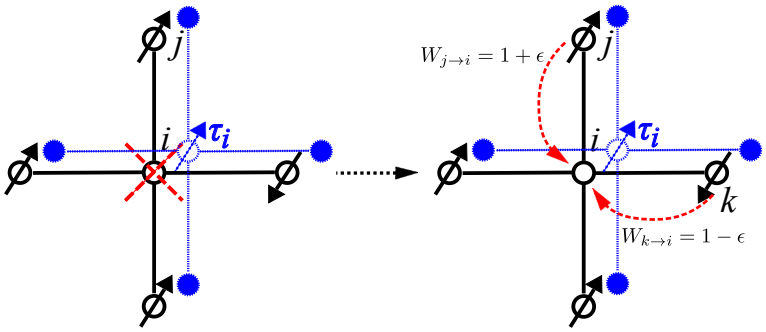


Figure 5: Cartoon of the microscopic dynamical rules of the model

As outlined in the introduction of this section, we are interested in the (quasi-)stationary solution of this model and in the reconstruction of a phase diagram as a function of the two parameters of the models, N and ϵ. Since the complete discrete and spatial model cannot be directly solved, we first consider the mean field (deterministic) approximation and then we extend the analysis to a the model defined on a complete graph but taking into account the stochastic terms for finite N.

4.2.2 *Mapping onto a birth-death Fokker-Planck equation*

As a first step to construct a mean-field solution, let us consider the dynamics on a complete graph, that is, each spin connected to every other spin. The macroscopic state of the system is univocally determined by the value of two variables, x and y, which represent the fraction of up and down spins aligned with their corresponding random fields, respectively:

$$\begin{cases} x \equiv \frac{1}{N}\sum_{i=1}^{N} \delta_{\tau_i,\sigma_i}\delta_{\sigma_i,+1} \\ y \equiv \frac{1}{N}\sum_{i=1}^{N} \delta_{\tau_i,\sigma_i}\delta_{\sigma_i,-1}. \end{cases} \tag{51}$$

These two variables are defined in the interval $[0, 1/2]$ and, since the total number N of spins is constant, the total fraction of up and down spins, call them X and Y respectively, are readily obtained as $X = 1 - Y = 1/2 + x - y \in [0, 1]$. The global magnetization, our order parameter, is given by $\phi = 2(x - y)$. We can now map this spin model onto a birth-death process considering the Master Equation (ME) for $P(x, y, t)$, the probability of having at time t a fraction x and y of up and down spins aligned with their local field, respectively. In this mean-field version the system evolves through the steps $x \to x' = x \pm 1/N$ and $y \to y' = y \pm 1/N$, with transition rates given, up to a global normalization constant, by (see [43, 44])

$$\begin{aligned} W_x^b &= W(x \to x + 1/N) = (1+\epsilon)(\tfrac{1}{2} - x)(\tfrac{1}{2} + x - y) \\ W_x^d &= W(x \to x - 1/N) = (1-\epsilon)x(\tfrac{1}{2} - x + y) \\ W_y^b &= W(y \to y + 1/N) = (1+\epsilon)(\tfrac{1}{2} - y)(\tfrac{1}{2} + y - x) \\ W_y^d &= W(y \to y - 1/N) = (1-\epsilon)y(\tfrac{1}{2} - y + x), \end{aligned} \tag{52}$$

from which one could in principle build the full equations of motion for the model. A standard Kramers-Moyal expansion [6] leads to the Fokker-Planck approximation of the original ME for the evolution in time of $P(x, y, t)$, that we write as

$$\partial_t P(x, y, t) = \partial_x \left[-A_x P + \frac{1}{2N}\partial_x (B_x P) \right] + \partial_y \left[-A_y P + \frac{1}{2N}\partial_y (B_y P) \right], \tag{53}$$

where $A_{x,y} = W_{x,y}^d - W_{x,y}^b$ represent the drift terms and $B_{x,y} = W_{x,y}^d + W_{x,y}^b$ the diffusion terms for the x, y variables, respectively, and time has been rescaled in units of $1/N$.

4.3 STEADY STATE ANALYSIS

4.3.1 *Deterministic limit*

For the time being we focus only on the limit $N \to \infty$, when the diffusion terms can be safely set to zero and the dynamics becomes deterministic, namely $\dot{x} = A_x$ and $\dot{y} = A_y$. Performing a change of variables, proposed first in [43], that will be useful for the later analysis: define $\Sigma \equiv x + y \in [0, 1]$ and $\Delta \equiv x - y \in [-1/2, 1/2]$, from which the global magnetization can be written as $\phi = 2\Delta$. In this notation, the deterministic equations of motion for the couple (Σ, Δ) become

$$\begin{cases} \dot{\Delta} = \epsilon\Delta(1 - 2\Sigma) \\ \\ \dot{\Sigma} = \frac{1}{2}(1 + \epsilon) - \Sigma - 2\epsilon\Delta^2. \end{cases} \tag{54}$$

The analysis of the dynamical system described in Eq. (54) gives already some interesting results: For $\epsilon = 0$ one obtains a line of stable fixed points at $\Sigma = 1/2$ (and arbitrary Δ), i.e. the deterministic dynamics is trivial and the system reaches an absorbing state purely by fluctuations when the noise term is considered, hence, recovering the VM results. Instead, for any $\epsilon > 0$, the phase portrait changes dramatically and the line of fixed points breaks into three fixed points: two of them are unstable corresponding to the absorbing states of the VM dynamics, at $\Sigma = 1/2$ and $\Delta = \pm 1/2$, while the third, at $\Sigma = (1 + \epsilon)/2$ and $\Delta = 0$, is stable and corresponds to an active state with zero magnetization, i.e. a state of symmetric coexistence of the two states.

4.3.2 *Role of the stochastic noise*

For finite-size systems fluctuations cannot be neglected and the system is expected to fluctuate around the deterministic stable fixed point. We expect a priori that only a large collective deviation can bring the system to one of the absorbing states which is, nevertheless, ineluctably reached. To study this we consider the Fokker-Planck eq. (53). It is easy to verify –by computing crossed-derivatives– that this equation does not admit a potential solution for the stationary probability distribution [6]. The lack of a stationary potential reflects the intrinsically non-

equilibrium nature of the problem. A possible strategy would be to construct non-differentiable non-equilibrium potentials following the strategy in [52]. Instead, here, we follow a simpler solution by seeking for a suitable (adiabatic) approximation allowing us to reduce the problem to a one-variable one.

Let us consider $\epsilon \ll 1$, then the equations (54) have two different characteristic relaxation times: Σ relaxes in a time $\mathcal{O}(1)$ whereas Δ in a much longer time-scale $\mathcal{O}(\epsilon^{-1})$. Thus, one can safely assume that the system first relaxes to the nullcline orbit $\dot{\Sigma} = 0$ and then the dynamics is constrained to take place uniquely on such a one-dimensional manifold. Consequently, we substitute the variable Σ by its value in the nullcline orbit, namely

$$\Sigma \to \bar{\Sigma} = \frac{1}{2}(1+\epsilon) - 2\epsilon\Delta^2 \tag{55}$$

and it is treated like a deterministic quantity, that is, fluctuations in its direction are discarded.

Within this approximation the Fokker-Planck equation for the probability distribution of Δ, or equivalently ϕ, $\mathcal{P}(\phi, t)$, is obtained with a change of variables in Eq.(53) from (x, y) to (Δ, Σ) and substituting the variable Σ in the remaining equation with $\bar{\Sigma}$ of Eq. (55). The diffusion term for the variable Σ is neglected and we are left with the following 1-dimensional FP equation

$$\dot{\mathcal{P}}(\phi, t) = -\partial_\phi \left[\mathcal{A}(\phi)\mathcal{P}(\phi, t)\right] + \frac{1}{2}\partial_\phi^2 \left[\mathcal{B}(\phi)\mathcal{P}(\phi, t)\right], \tag{56}$$

with

$$\begin{aligned} \mathcal{A}(\phi) &= -\tfrac{\epsilon^2}{2}\phi(1-\phi^2) \\ \mathcal{B}(\phi) &= \tfrac{1}{N}(1-\epsilon^2)(1-\phi^2). \end{aligned} \tag{57}$$

Eq. (56) is equivalent to the following Langevin equation in the Ito prescription [6]

$$\begin{aligned} \dot{\phi} &= \mathcal{A}(\phi) + \sqrt{\mathcal{B}(\phi)}\eta(t) \\ &= -\tfrac{\epsilon^2}{2}\phi(1-\phi^2) + \sqrt{\tfrac{1}{N}(1-\epsilon^2)(1-\phi^2)}\eta(t), \end{aligned} \tag{58}$$

where η is a δ-correlated in time gaussian white noise with zero mean. Let us emphasize that the main effect of the quenched disorder is to generate a deterministic force which stabilizes the opinion-coexistence state, $\phi = 0$. In the limit $\epsilon \to 0$ we recover

the pure fluctuations-driven VM dynamics [14], while in the opposite limit $\epsilon \to 1$, the dynamics is purely deterministic and the spins align with their corresponding random fields. For $0 < \epsilon < 1$ the stationary solution $\mathcal{P}_s(\phi;\epsilon)$ is formally given by the zero-current condition [6]

$$J = -\mathcal{A}(\phi)\mathcal{P}(\phi,t) + \frac{1}{2}\partial_\phi[\mathcal{B}(\phi)\mathcal{P}(\phi,t)] = 0 \tag{59}$$

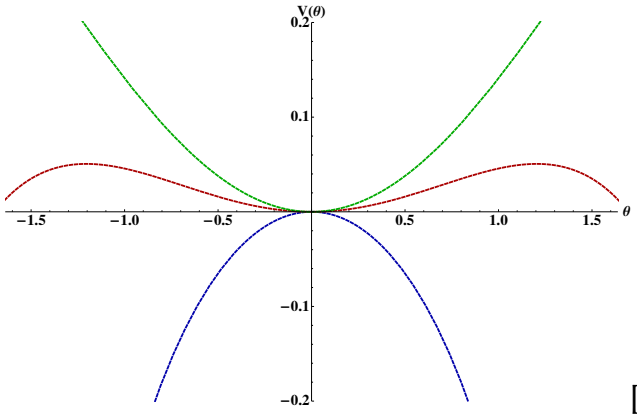
from which

$$\mathcal{P}_s(\phi;\epsilon) = \frac{1}{\mathcal{Z}\mathcal{B}(\phi)} e^{2\int^\phi dx \frac{\mathcal{A}(x)}{\mathcal{B}(x)}}. \tag{60}$$

$\mathcal{Z}$ is supposed to be the normalization constant of $\mathcal{P}_s(\phi;\epsilon)$, but since the diffusion term $\mathcal{B}(\phi) \to 0$ when $\phi \to \pm 1$ and the exponential stays finite, the probability distribution is *not normalizable*, since we know that strictly speaking we are considering only *quasi*-stationary distributions and the real stationary distribution is the generalized function given by

$$\mathcal{P}_\infty = \frac{1}{2}\Big[\delta(\phi-1) + \delta(\phi+1)\Big]. \tag{61}$$

Therefore, as expected, for any finite value of N the only steady state is an absorbing/consensus one: coexistence is always killed on the large time limit.



[h]

Figure 6: Potential $V(\theta)$ of Eq. (64) corresponding to the three observed phases: absorbing ($\epsilon = 0.09$), intermediate ($\epsilon = 0.15$) and coexistence ($\epsilon = 0.4$), blue, red and green curves respectively.

The shed some more light on this problem we perform a change of variables on the Fokker-Planck equation (56)such that its corresponding Langevin equation (58), characterized by

a state-dependent (multiplicative) noise, becomes a new state-independent equation, i.e. the noise becomes additive rather than multiplicative. A suitable transformation is [53, 54]

$$\theta \equiv \frac{1}{\alpha} \arcsin \phi \tag{62}$$

with $\alpha = \sqrt{\frac{1}{N}(1-\epsilon^2)}$, which leads to the following Langevin equation

$$\dot{\theta} = -\frac{dV(\theta)}{d\theta} + \xi, \tag{63}$$

where $V(\theta)$,

$$V(\theta) = \frac{1}{2}\log(\cos(\alpha\theta)) - \frac{\epsilon^2}{4\alpha^2}\sin^2(\alpha\theta), \tag{64}$$

is a potential and ξ is a standard δ-correlated in time Gaussian white noise. $V(\theta)$ is shown in Figure 6 for some particular values of the parameters ϵ and N. We can see that there exists a critical value of ϵ, ϵ_c, to be defined later, below which the potential effectively pushes the system toward the absorbing boundaries. Going above this critical value a local minimum appears in the configuration of zero magnetization $\phi = 0$ (observe that the second term in eq. (64) can be expanded around $\theta = 0$, leading to a parabolic potential around the origin; i.e. disorder creates an effective potential whose minimum corresponds to the opinions-coexistence state). We expect then that for $\epsilon > \epsilon_c$ the time needed to reach the absorbing state will be exponential in the height of the potential barrier, due to the Arrhenius law [6]. Increasing further ϵ the minimum at $\phi = 0$ becomes the absolute minimum, so that the time needed to escape the barrier takes even longer times, long enough to make difficult to compute it by simulations. We define these three regimes as the *absorbing*, *intermediate* and *active* phase respectively: in the absorbing phase symmetry is broken and one of the two possible states of consensus is reached; the active phase is characterized by a coexistence of both states, while the intermediate state is a sort of mixture of the two previous ones: both the consensus states and the one of coexistence are locally stable, so the steady state depends on initial conditions. This view provides a nice illustration of how noise can effectively change the shape of the deterministic potential, allowing for noise-induced phase transitions to occur. However, as $\alpha^2 \propto 1/N$ the only steady states reachable in the thermodynamic limit are the absorbing ones,

hindering the possibility of having a true phase transition in the thermodynamic limit.

4.3.3 *Removing the singularities*

In order to regularize the singularities reported above and explore the possibility of true phase transitions in the large-N limit, we introduce a small "mutation" term. Mutation is defined as the process by which any randomly selected spin can invert its state –at some rate $\nu \geqslant 0$– regardless of its associated random field and the state of its neighbors. With this new mechanism, the transition rates W' become

$$\begin{aligned}
W'^{b}_{x} &= (1+\epsilon)\left[(1-\nu)(\tfrac{1}{2}-x)(\tfrac{1}{2}+x-y)\right] + \tfrac{\nu}{2}(\tfrac{1}{2}-x+y)\\
W'^{d}_{x} &= (1-\epsilon)\left[(1-\nu)x(\tfrac{1}{2}-x+y)\right] + \tfrac{\nu}{2}(\tfrac{1}{2}+x-y)\\
W'^{b}_{y} &= (1+\epsilon)\left[(1-\nu)(\tfrac{1}{2}-y)(\tfrac{1}{2}+y-x)\right] + \tfrac{\nu}{2}(\tfrac{1}{2}+x-y)\\
W'^{d}_{y} &= (1-\epsilon)\left[(1-\nu)y(\tfrac{1}{2}-y+x)\right] + \tfrac{\nu}{2}(\tfrac{1}{2}-x+y).
\end{aligned} \tag{65}$$

Assuming $\nu \ll \epsilon \ll 1$ and discarding all the terms of order ν, $\nu\epsilon$ and higher the Fokker-Planck equation for the global magnetization in presence of speciation becomes

$$\dot{\mathcal{P}}_\nu(\phi,t) = -\partial_\phi\left[\mathcal{A}_\nu(\phi)\mathcal{P}_\nu(\phi,t)\right] + \frac{1}{2}\partial^2_\phi\left[\mathcal{B}_\nu(\phi)\mathcal{P}_\nu(\phi,t)\right] \tag{66}$$

with

$$\begin{aligned}
\mathcal{A}_\nu(\phi) &= -\tfrac{\epsilon^2}{2}\phi(1-\phi^2+2\nu)\\
\mathcal{B}_\nu(\phi) &= [(1-\epsilon^2)(1-\phi^2)+2\nu]/N,
\end{aligned} \tag{67}$$

respectively. Therefore, the stationary probability distribution function $\mathcal{P}^\nu_s(\phi;\epsilon)$ is given up to the leading order in ν by

$$\mathcal{P}^\nu_s(\phi;\epsilon) \propto \frac{1}{(1-\epsilon^2)(1-\phi^2)+2\nu}\exp\left(-\frac{N}{2}\frac{\epsilon^2}{1-\epsilon^2}\phi^2\right), \tag{68}$$

which, owing to ν, does not have any singularity. It is important to notice that if ν is small enough, namely $\nu \ll 2/(2+N)$ (see [44] for a simple derivation in the case of the pure VM), it does not affect significantly the dynamics, apart from removing the absorbing boundaries. We can thus make use of equation (68) to compare directly our approximate solution with the numerical simulation of the complete dynamics. Simulation results are obtained for a complete graph of N spins by means of the

Gillespie algorithm [27]. Results for different values of ϵ, are reported in Figure 7 which shows a good agreement with the theoretical predictions.

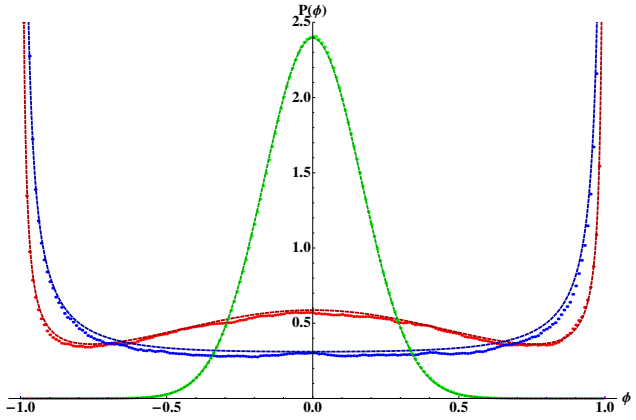


Figure 7: *Dashed Lines:* Stationary probability distribution $\mathcal{P}_s(\phi;\epsilon)$, Eq. (68), for $N = 200$, $\epsilon = 0.09 < \epsilon_c \simeq 0.1$ (blue curve), for $\epsilon = 0.15 > \epsilon_c$ (red curve) and for $\epsilon = 0.4$ (green curve). The curves are computed analytically from Eq. (68) with $\nu = 10^{-4}$. *Dots:* Stationary probability distribution obtained by simulations of the model with the parameters N, ϵ, ν as before.

Finally, from equation (68) it is easy to compute the value of ϵ at which the second derivative of $\mathcal{P}_s^\nu$ computed in $\phi = 0$ changes sign:

$$\epsilon_c \simeq \sqrt{\frac{2}{2+N}}. \tag{69}$$

As expected, $\epsilon_c \to 0$ when $N \to \infty$ since the dynamics becomes deterministic and thus the results of section 2 apply. It is interesting to notice that this transition from the absorbing to the active phase is completely 'noise-driven': any arbitrarily small amount of quenched-noise –i.e. any value of $\epsilon > 0$– leads to a stable active phase. Observe that here, we are in a situation somehow opposite to the one in the previous subsection: once absorbing states are perturbed with the possibility of mutation the only remaining stable-state in the thermodynamic limit is the active one.

4.4 CONNECTION TO THE GENERALIZED-VOTER CLASS

In section 2.3.3 we introduced a coarse-grained field theory for the GV universality class. We recall that the corresponding

Langevin equation (24) is aimed at capturing all the possible features of systems with two-symmetric absorbing states, and reads

$$\begin{aligned}\dot{\phi}(x,t) &= \nabla^2\phi(x,t) + (a\phi - b\phi^3)(1-\phi^2) + \\ &\quad + \sigma\sqrt{1-\phi^2}\,\xi(x,t),\end{aligned} \tag{70}$$

where a and b are constant parameters of the model, $\phi(x,t)$ is a field whose dynamics is frozen if $1-\phi^2 = 0$, and $\xi(x,t)$ is a Gaussian white noise of zero mean and variance σ. With the only requirement that $b \geqslant 0$ this equation reproduces the critical behavior of the GV universality class.

If now we consider equation (56) for the description of the model considered in this paper, we can write it in the equivalent form of the Langevin equation in the Ito prescription [6]

$$\dot{\phi}(t) = -\frac{\epsilon^2}{2}\phi(1-\phi^2) + \sqrt{(1-\epsilon^2)(1-\phi^2)}\,\eta(t) \tag{71}$$

where $\eta(t)$ is a Gaussian white noise with $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = \frac{1}{N}\delta(t-t')$. We can see that the equation coincides with the 1-variable version of Eq.(70) (i.e. Eq.(70) without spatial dependence) once the identifications $b = 0$, $a = -\epsilon^2/2$ and $\sigma = \sqrt{\frac{1}{N}(1-\epsilon^2)}$ are made. Therefore, at least in the case without explicit spatial structure the VM with quenched randonm field resembles a lot the GV dynamics (without quenched disorder!). The main effect of random fields is to create a force which converts the state of coexistence ($\phi = 0$) into a stable one. Such a state is the only possible stationary state for infinitely large systems, while for finite-sizes there is a transition very similar to that of the GV class, without quenched disorder. Then, it is somewhat surprising that also this model is effectively well described by the same equations. It is also noteworthy that $\epsilon = 0$ corresponds to the critical point in the thermodynamic limit and when ϵ is positive, only the active phase exists: the absorbing phase of the AlHammal's equation is not accessible to the present model with quenched random fields.

4.5 DISORDER AND SPONTANEOUS SYMMETRY BREAKING

So far we have seen that quenched disorder pushes a linear VM out of the criticality introducing an effective potential term that

forces the system in an active symmetric phase. In this section we extend the analysis to the larger class of *nonlinear voter models* (NV) by mean field analysis and simulations of the model introduced in [55] in presence of a disordered environment. It is well known [16, 55, 15, 17] that a VM with nonlinear transition rates can undergo a split into two separate phase transition: one in the Ising universality class (spontaneous breaking of the $\mathbb{Z}_2$ symmetry) and one in the Directed Percolation (DP) universality class from active to absorbing, opposed to the Voter universality class characterized by a concomitant $\mathbb{Z}_2$+DP phase transition. On the other hand, a general argument by Imry and Ma [49] (see Appendix A.4) predicts that, at equilibrium, quenched disorder prevents the spontaneous symmetry breaking of a discrete symmetry in $D \leqslant 2$ and of a continuous symmetry in $D \leqslant 4$ (a rigorous proof has been given later in [48]). Thus, we want to study the effect of the disorder, that can be seen as an quenched magnetic field acting on the spins, when the model is tuned to be in the DP active phase but with the broken Ising symmetry, so that the stationary probability distribution for the global magnetization will be characterized by two symmetric maxima at $\phi = \pm\phi_*$ (or equivalently $\rho_* \equiv (\phi_* + 1)/2$), with $0 < \phi_* < 1$ that will depend on the specific choice of the parameters regulating the nonlinearity of the transition rates. We consider the Mean Field limit and the $D = 2$ case, expecting two different qualitative behavior. In fact, Barghathi and Vojta [45] recently studied the validity of the Imry-Ma argument. They studied a purely out-of-equilibrium spin system with two symmetric absorbing state, finding that this argument is violated and the symmetry breaking occurs also in $D = 1$. Nevertheless their model has a peculiar symmetry breaking, since it brings to one absorbing states where the dynamics stop. Our analysis shows that in the case of a nonequilibrium model in a DP-active state but with a spontaneous Ising symmetry breaking, the Imry-Ma argument is still satisfied.

4.5.1 *Mean Field*

From Eq. (71) we can see that in an effective Langevin description for the magnetization field in a disordered VM a random external field act as a ϕ^4 potential term with minimum at zero magnetization. Beside this, it is known [16] that the Generalized

Voter universality class is described by the Langevin equation of the VM plus a term derived from a ϕ^6 potential, of the general form of Eq. (70). We could conjecture that for a nonlinear VM mean field in a random field, with parameters of nonlinearity a and b and disorder strength ϵ, the disorder will still produce an equation of the form of Eq. (70), but with a new effective parameter $a' = a - \epsilon^2/2$. Namely, we expect the system to be described by

$$\dot{\phi}(t) = [(a - \frac{\epsilon^2}{2})\phi - b\phi^3](1 - \phi^2) + \eta(\phi, t) \tag{72}$$

From this equation we can compute the critical value of the external field intensity, ϵ_c^{sb}, at which the disorder destroys the possibility of a spontaneous symmetry breaking, that is, $\epsilon_c^{sb} = \sqrt{2a}$. Analogously, the position of the minimum of the potential (and therefore the maximum in the stationary probability distribution of the magnetization) is expected, when $a, b > 0$, at

$$\phi_* \simeq \pm\sqrt{\frac{a - \epsilon^2/2}{b}}. \tag{73}$$

We simulated the full ME of a microscopic model with nonlinear interaction [55] in the DP active-Ising broken symmetry phase, and in Figure 8 we show the dependence of the peaks position, ϕ_*, on the intensity of the disorder and that it is independent of the size of the system. Our elementary discussion turns out to be in good agreement with simulations.

4.5.2 *Simulations in* $D = 2$

When $D \leqslant 2$, as discussed above, we expect the phenomenology of the system to be radically different, since now the Ising symmetry breaking should not be present. We consider here a nonlinear Voter Model where, if x is the density of spins not aligned with σ_i in its neighborhood, the flipping probability of a randomly selected spin is given by

$$f_i(x) = P(\sigma_i \to -\sigma_i|x) \propto xK(x), \tag{74}$$

where $K(x)$ is an arbitrary nonlinear function of x. Here we will consider a cubic function of the form

$$K(x) = \tilde{a}x^2 + \tilde{b}(\frac{3}{4}x - x^3) \tag{75}$$

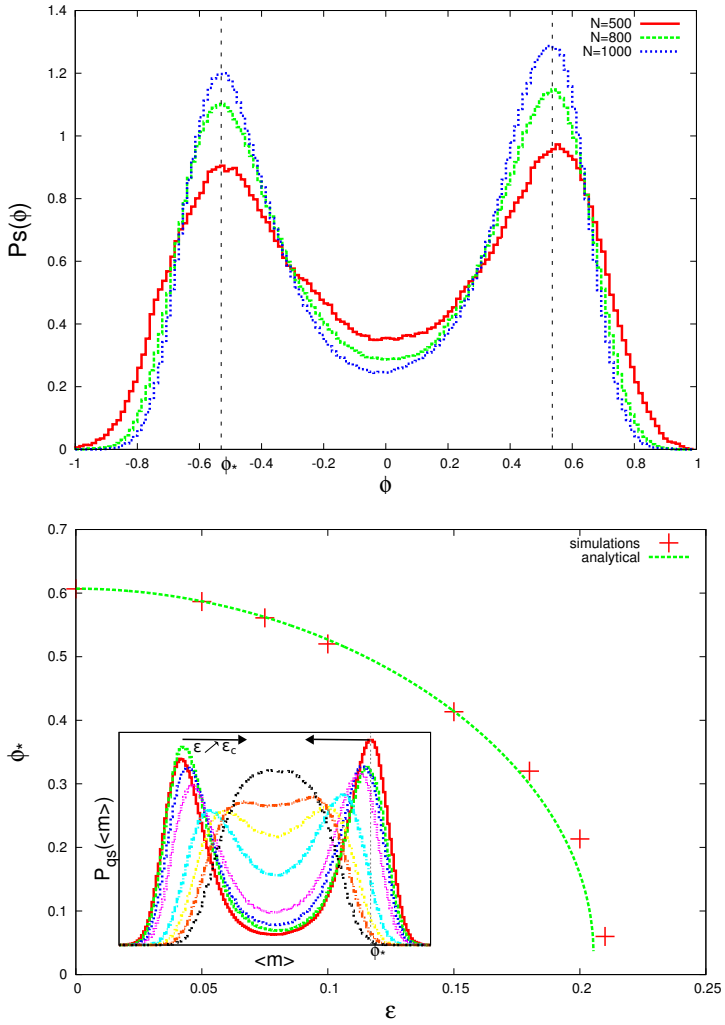


Figure 8: *top:* Stationary probability distributions $P_s(\rho)$ in the mean field NV plotted for $\epsilon = 0.1 < \epsilon_c^{mf}$ and various N. The position of the peak ρ_* does not depend on the size of the system, and the Ising symmetry is still broken. *bottom:* Position of the peak ϕ_* for $a = 0.0175$ and $b = 0.053$ and varying ϵ. The green dashed line represents the prediction Eq. (73). In the inset are shown the probability distribution for some values of ϵ.

that yields a stochastic equation of motion of the form of equation (70) with effective parameters $a = \tilde{a}/4$ and $b = \tilde{b}/16$. If we choose $b > 0, a > 0$, his model has two (meta)stable fixed points at global magnetization $\phi_* = \pm\sqrt{a/b}$ [16, 19]. Next, we include disorder by adding a quenched site-dependent term $\epsilon_i = -\epsilon\tau_i\sigma_i$ that locally breaks the up down symmetry favoring or disfavoring the single spin-flip. The quenched variables τ_i are defined as in Section 2. We simulate this model on regular lattices in $D = 2$ for various linear sizes L and disorder strength ϵ.

Due to finite size effects, the density peak position $\rho_* = (\phi_* + 1)/2$ depends on the size of the system independently of the disorder, but we can extrapolate the value of ρ_* when $N \to \infty$ plotting $\rho_* = \rho_*(x)$ with $x = 1/N$ and look at the trend for small values of x, as shown in Figure 9 for two specific choice of ϵ. We can see that while the symmetry breaking is maintained in the system without quenched disorder and the peak position tend to a value strictly less than $1/2$, the presence of disorder makes the symmetry breaking asymptotically disappear in agreement with the argument of Imry and Ma for the equilibrium Ising simmetry breaking.

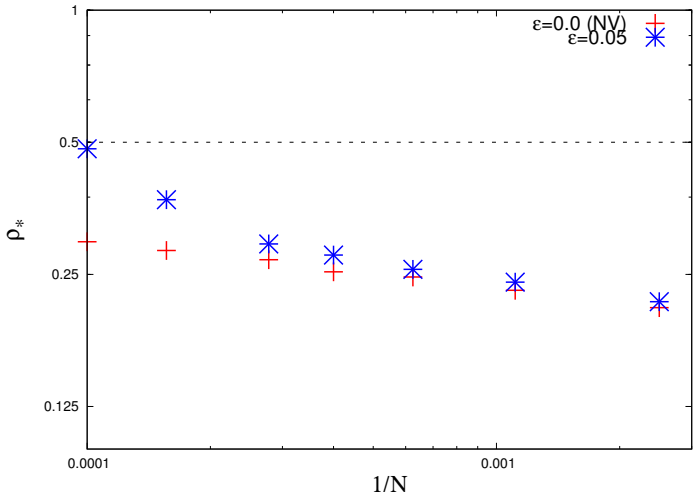


Figure 9: Position of the peaks in the NV plotted as function of $1/N$ for $D = 2$ for the model with the same parameters as in figure 8. The dashed line at $\rho_* = 0.5$ indicates when the symmetric active state is restored.

We stress that clearly this is very different from what we saw for the MF case: in that case we could "tune" the disorder strength in order to shift the magnetization of the stable state in

the broken Ising symmetry phase, while here switching on the interaction automatically destroys the symmetry breaking. We then have access only to a GV absorbing-to-active (symmetric) transition.

In ecological terms this suggests that a neutral-symmetric coexistence is effectively induced by the environment that hides fitness effects or density-dependencies. It could be tempting to conclude that this is a reason of why the neutral assumption revealed so powerful, even if at the present time this is only a speculation.

We also note that to actually see the symmetry breaking in the active phase in the simulations we had to define the neighborhood of each site as a set containing more than the first nearest neighbors –we chose $z = 12$–, in agreement with previous studies of these kind of systems [15, 17].

4.6 SOME CONSIDERATIONS

Summarizing the content of this chapter so far, we have: *i)* investigated analytically the effects of a quenched random 'preference field' in the standard voter model dynamics, finding an effective stochastic description of the dynamics of the global magnetization on a complete graph, function of the total number of spins N and the disorder strength ϵ. *ii)* Given a simple explanation of the non-trivial crossover between the absorbing and the active phases when the size of the system is finite in terms of competing interplay between noise and deterministic drift, which has eluded a clear motivation so far in studies of equivalent models. We stress that, as reported in [44], a qualitatively identical behavior is expected in finite dimensions. *iii)* Pointed out the possibility of an *a priori* unexpected relation in terms of stochastic equations of motion between a linear VM with disorder and the larger class of nonlinear voter models without disorder, and *iv)* checked the validity of the Imry-Ma argument in the case of a nonlinear voter model in the Ising broken symmetry - DP active phase for $D = 2$; By simulations of the disordered model in $D = 2$ and by heuristic analytical arguments well supported by numerical simulations in MF we proved that exactly as it would be expected for an Ising symmetry breaking at equilibrium, disorder inhibits the symmetry breaking in low dimensions while this reasoning does not ap-

ply to the mean field case. This is opposed to the claim of Vojta *et al.* in [45], where they find that a purely non-equlibrium (absorbing) symmetry breaking violates the Imry-Ma prescription. This is not completely surprising: when a system with multiple symmetric absorbing states is in its absorbing phase and the noise does not act on the system sufficiently strong to modify the global non-equilibrium phase, than the symmetry breaking is ineluctable. Our case is intrinsically different, since the system is already in the DP active phase, and the $\mathbb{Z}_2$ symmetry breaking is expected to be in the Ising universality class, so, in a way, it is an equlibrium phase transition even if the detailed balance is not satisfied and because of this the Imry-Ma argument should be satisfied.

Models with symmetric absorbing states mimicking neutral –in its generalized sense– evolution have been applied in many different areas of science, from physics and chemistry where they can model the behavior of kinetic reactions, to biology, genetics and ecology where models like the VM –also called Moran process– [56] or its dual representations –as the coalescent– [57] have been successfully applied to understand, at least at a null-level, different empirical observations. Finally we mention that also in social sciences these models are heavily studied and applied [58].

In all these models and applications, nevertheless, external and environmental "forces" were systematically ignored. A qualitative change in the trustworthiness of the mathematical approach to these problems needs a stronger consideration of the fact that the particles or individuals or agents in the system are never acting in the vacuum, but live in an environment that might both modify the interactions amongst individuals and interact itself with the constituents of the system. What we have done here is a first attempt of formalizing the properties of systems in the GV universality class with quenched disorder, that is still missing in the literature. Thus, this model joins previous studies on different disordered nonequilibrium systems.

We finally note that our description was limited to only two species for clearness and convenience, but we do not expect anything to change qualitatively if the total number of species is augmented to a generic number S.

5

ROLE OF BOUNDARY CONDITIONS OUT OF EQUILIBRIUM

Experience without theory is blind,
but theory without experience is mere intellectual play.
— ***I. Kant***

5.1 INTRODUCTION

It is well known that boundary conditions are important in lattice models and statistical mechanics. In particular, for statistical models at equilibrium, imposing specific boundary conditions to the system is an alternative way of studying phase transitions in addition to looking at the divergence of some thermodynamic quantity at the critical point. Consider for example a two-states Ising model, $\sigma_i = \pm 1$, where the site index i runs over a *finite* lattice Λ in sufficiently large[1] spatial dimensions; When the system's size if finite, it is intuitive to think that the '+' boundary conditions, that is, all spins at the boundaries of the system are fixed in the $+$-state, $\sigma_{\partial\Lambda} \equiv +1$, will make the spins in the bulk a little more likely to be in the $+$ state. In this case we expect then $\langle \sigma_{bulk} \rangle_\Lambda > 0$. This is somehow trivial. The interesting question is what happens when Λ, and in particular the linear size L of the lattice, is *infinite*. In this case the situation is very different depending on the inverse temperature of the system β: it can be proved [59] that if $\beta < \beta_c$, with β_c critical point, the limit $\lim_{\Lambda\to\infty} \langle \sigma_{bulk} \rangle_\Lambda$ exists and it is zero, while for $\beta > \beta_c$ this limit is strictly greater than zero. In other words, boundary conditions do not matter above the critical temperature while they explicitly select one of the two pure states[2] of

1 *i.e.*, allowing for a spontaneous breaking of the $\mathbb{Z}_2$ spin reversal symmetry.

2 A *pure state* is defined as the ensemble of microscopic states of a system over which the system is ergodic. When a symmetry is spontaneously broken it generally breaks the ergodicity of the system and, at equilibrium, the Gibbs measure is given by a sum of different sub-components that are the different symmetric states of the broken symmetry phase, $\langle \bullet \rangle = \sum_\alpha w_\alpha \langle \bullet \rangle_\alpha$.

spontaneous $\pm$ magnetization of the system below the critical temperature.

The importance of this approach to phase transitions revealed to be important also in highly non-trivial situations as, for example, for spin glasses where due to disorder the spin glass-to-paramagnet transition is not easily characterized, and it is even difficult to assess whether this is a true thermodynamical transition or a non-equilibrium glass-transition like phenomenon. In that case response to boundary conditions provide a clearer approach than those based on order parameters [60].

For non-equilibrium systems our knowledge is still unripe and there is not a clear theory equivalent to equilibrium. It is still lacking a solid comprehension of non-equilibrium phase transitions from the point of view of order parameters, and for this reason this second possible approach has not been addressed yet.

But the role of boundary conditions in models on networks genuinely out of equilibrium could reveal much more important than simply for studying the critical properties of a given model. Although still controversial [61], it is becoming more and more evident that many completely different natural systems self-organize in a special point of the parameter space that could resemble a critical point in the physics jargon [62, 63]. Experimental observations in this direction seems to confirm this hypothesis [64, 65, 66, 67, 68] and theoretically, even if there is still not a precise idea of how a complex system could dynamically settle at the critical point, it is clear that a system at criticality has the widest range of possible responses to external stimulations and is more flexible [69]. If this were true, since critical systems are characterized by diverging correlation length, any information coming from the boundaries of the system, that exist for sure since all natural systems are obviously finite, should be able to propagate indefinitely deep into the bulk of the system, and thus external perturbations could reveal to be crucial in the future evolution of the considered system.

Still related to this problem, as already mentioned in the previous chapter, it is a major issue in conservation ecology to determine what are the environmental conditions in macro-ecological systems that allows for a stable and rich in time global biodiversity. On the other hand, it is known that many micro- and macro- ecological systems are well described by neu-

tral theories [23, 66, 70] and show scale invariance. Since the most basic model of neutral dynamics is the voter model, that is critical, we analyze now the effects of *mild boundary conditions*, or in general a small fraction of sites that favor one or the other state, maintaining the global symmetry of the model and both the absorbing states.

Summarizing, what we will present in this chapter is twofold: First, more theoretically minded, is to give a first description of how a non-equilibrium lattice model reacts to a perturbation acting on a lower dimensional manifold of the system when the system size diverges. Second, we apply these results to the case of an ideal two species ecosystem where one border favors one species and the opposite border favors the other and give some quantitative prediction on how this modifies the typical timescales for the extinction of a species.

In the previous chapter we saw how an external random 'preference' field on the voter model induces a stable active phase with zero mean magnetization (symmetric coexistence), and recent works observed that the presence of a single spin fixed on one opinion [47] brings the model to the absorbing phase, while the presence of Z_+, Z_- spins fixed in the $+$ or $-$ opinion respectively make the system persist in the active phase [46]. It is clear however that these last two models are no more in the voter universality class since they have removed one and two absorbing states respectively. But what does happen if we consider a similar –in spirit– model where the absorbing states are preserved? That is, how much strong must this 'preference field' be and on how many spins must it act to induce the active phase?

In ecological terms, alternatively, we want to assess whether the diversification of the habitat can boost the maintenance of the global biodiversity of an ecosystem over large times and spatial scales. We will prove the possibility of maintaining a rich biodiversity for exponentially long times and over all spatial scales by imposing an intrinsic but small habitat preference in a little fraction of sites, while letting the rest of the system evolve through a completely neutral dynamics. We will also give an exact ansatz for the functional form of the mean time to absorption.

5.1.1 *General mathematical settings*

We start from a model that is the paradigm of neutral dynamics, namely the *Voter Model* (VM) –or Moran Model in genetics– on the lattice with two species. The main (and only) difference is that we let a fraction $0 \leqslant \eta N \leqslant N/2$ of spins for each species 'prefer' species A and B –respectively the $+1$ and -1 states of a spin– to fix, where N is the total number of sites. To prefer means that when a site of type, let's say, A is chosen to be replaced by a neighbor individual, it will weight more the presence of individuals of type A than B. This weight is quantified by a parameter $0 \leqslant \epsilon \leqslant 1$, where for $\epsilon = 0$ we recover the standard VM and for $\epsilon = 1$ the non-neutral spins are frozen and the absorbing states are removed as they act as a regular source of new individuals of both species.

In other words, we introduce a local effective fitness on some sites that is coupled to the density of the species in the neighborhood by the coupling constant ϵ. It is important to stress that this model preserves the two symmetric absorbing states typical of the VM, representing the extinction of one of the species. The extension of this model to the case of S symmetric species is straightforward. To be completely clear, the model studied in the previous chapter is a particular case of this model obtained by choosing $\eta = 1/2$.

We want now to formalize mathematically an ecological situation as that described above. This can be mimicked by an interacting particle system of spin variables with, as usual, state space $\Omega = \{+1, -1\}^{\Lambda}$ –the orientation denotes the species– defined by the rates

$$\begin{aligned} W_{\sigma_i \to -\sigma_i} &= W^{VM}_{\sigma_i \to -\sigma_i} + W^{\epsilon}_{\sigma_i \to -\sigma_i} \\ &= \frac{1 - \epsilon \tau_i \sigma_i}{2z} \sum_{j \in \partial i} (1 - \sigma_i \sigma_j). \end{aligned} \tag{76}$$

What changes with respect to the model introduced in chapter (4) with equation (50) is that now the graph is tripartite[3], namely $\Lambda = \Lambda_+ \sqcup \Lambda_- \sqcup \Lambda_\emptyset$ with $|\Lambda_+| = |\Lambda_-| = \eta N = N_\pm$ and $|\Lambda_\emptyset| = (1 - 2\eta)N = N_\emptyset$, and the external field has now three possible values: $\tau_i = \pm 1, 0$ for $i \in \Lambda_+, \Lambda_-, \Lambda_\emptyset$ respectively.

3 The symbol $\sqcup$ denotes the operation of disjoint union.

As a first step, in the next section we will consider the deterministic dynamics of this system, and look at the stable fixed points of the systems depending on the parameters ϵ, η.

5.1.2 $N \to \infty$ *limit, deterministic dynamics*

Following the procedure of section 4.2 we map the model (76) on a complete graph, dropping space, to a birth-death process for three –not two as the previous model– macroscopic variables defining univocally the state of the system. These variables are given by

$$\begin{cases} x \equiv \frac{1}{N} \sum_{i=1}^{N} \delta_{\tau_i,\sigma_i} \delta_{\sigma_i,+1} \\ y \equiv \frac{1}{N} \sum_{i=1}^{N} \delta_{\tau_i,\sigma_i} \delta_{\sigma_i,-1} \\ z \equiv \frac{1}{N} \sum_{i=1}^{N} \delta_{\tau_i,\emptyset} \delta_{\sigma_i,+1} \end{cases} \tag{77}$$

that represent respectively the fraction of spins i with $\sigma_i = \tau_i = 1$, $\sigma_i = \tau_i = -1$ and $\sigma_i = 1, \tau_i = \emptyset$. Note carefully that the three variables are not symmetric. The two absorbing states in these variables are the states with triplets $(x, y, z) = (0, \eta, 0)$ and $(\eta, 0, 1 - 2\eta)$.

Similarly to what we have done in section 4.2, we can easily write down the birth and death rates for the variables x, y, z, $W^{b,d}_{x,y,z}$, the deterministic equations of motion are readily obtained. They read

$$\begin{cases} \dot{x} = (1+\epsilon)(\eta - x)(\eta + x - y + z) - (1-\epsilon)x(1 - \eta - x + y - z) \\ \dot{y} = (1+\epsilon)(\eta - y)(1 - \eta - x + y - z) - (1-\epsilon)y(\eta + x - y + z) \\ \dot{z} = (1 - 2\eta - z)(\eta + x - y + z) - z(1 - \eta - x + y - z). \end{cases} \tag{78}$$

Note that in the above equation the deterministic dynamics for the variable z is not the common trivial VM dynamics – $\dot{z} = 0$– even if the birth-death dynamics of z is completely neutral. To better understand what is going on, we refer to the illustrative cartoon in figure (10) and think about the limit $\epsilon \to 1$. In this case it is clear that the dynamics of z cannot be that of the pure VM since the neutral community is linked to two inexhaustible sources of species A and B respectively. This limit

model, that does not have any absorbing state, has already be studied and solved by Mobilia *et al.* in [46]. Suppose now that $\epsilon < 1$; the absorbing states are restored and the ultimate dynamics has changed, but still the two 'patches' with preference for species A or B act as a, weaker than before, source of individuals of the two species for the neutral patch. This process will end up in an effective continuous flux of individuals in and out the neutral patch, that causes a non-equilibrium steady state of global symmetric coexistence of the two species. As already said, this picture is strictly valid in the deterministic approximation, since rare large fluctuations will always take the system to one of the two absorbing states.

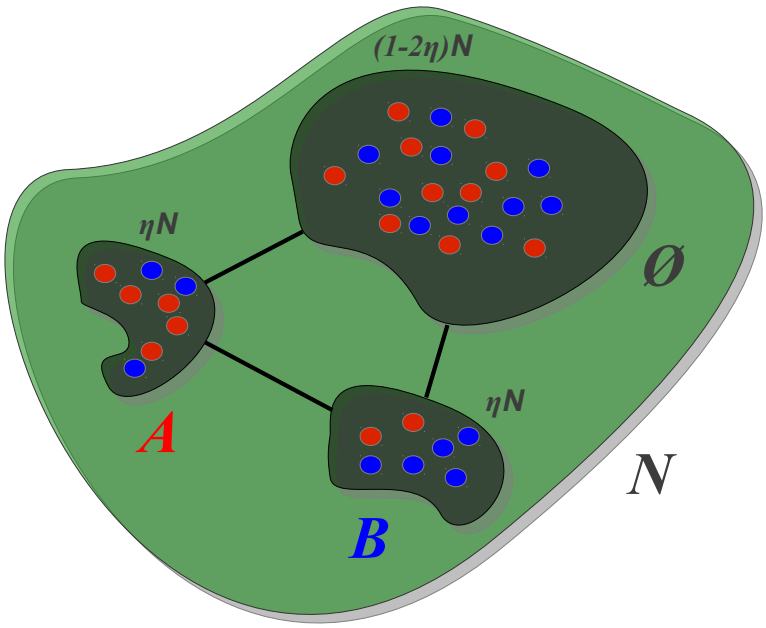


Figure 10: Cartoon of the mean field version of the model: Three communicating patches form our ecosystem of N individuals. The patches A and B, of population ηN each, are more fitted for species A and B respectively, while the patch ∅ contains the remaining individuals and has no fitness.

Indeed, the phase portrait analysis of the fixed points and their stability for the system in equation (78) gives

$$(x^*, y^*, z^*) = \begin{cases} (0, \eta, 0) & \text{(unstable)} \\ (\eta, 0, 1-2\eta) & \text{(unstable)} \\ (\frac{(1+\epsilon)}{2\eta}, \frac{(1+\epsilon)}{2\eta}, \frac{1}{2}-\eta) & \text{(stable).} \end{cases} \tag{79}$$

so, we can see that we still have an active symmetric stable fixed point of the deterministic mean field dynamics that is a global attractor.

In analogy to what we have done in chapter 4 we can expect that, at least for ϵ, η big enough, the (quasi-)stationary probability distribution for the magnetization $P(\phi)$ will be a gaussian centered in 0 and with a variance function of ϵ and η when the finite N full stochastic dynamics is considered, and the mean time to absorption will be an exponential function of the two parameters of the model. We will confirm these suppositions in the next sections by means of different heuristic arguments and numerical simulations.

5.1.3 *Steady state distribution on the complete graph*

In chapter 4 we discussed the limiting case of $\eta = 1/2$, and found that –adding a negligible mutation rate to eliminate the absorbing barriers– the stationary probability distribution has to a very good approximation the gaussian scaling form $P_s(\phi) \sim \exp(-\epsilon^2 N\phi^2/2(1-\epsilon^2))$ as shown in figure (7) (green curve). $\phi \equiv < m >$ is the global mean magnetization of the system. On the other hand we know that for $\eta = 0$ (VM limit) the gaussian term should go to zero since the effective potential term is a constant and the stationary distribution is a δ-function in $\phi = \pm 1$.

Collecting all these considerations, the simplest ansatz for the (quasi-)stationary[4] distribution of the magnetization in our model is then

$$P_s(\phi; \epsilon, \eta) \sim e^{-\eta^\alpha \frac{\epsilon^2}{1-\epsilon^2} N\phi^2}, \tag{80}$$

with $\alpha \simeq 1$ scaling exponent to be determined. For comparison, we simulated numerically the finite N mean-field version of

4 Recall that it is a quasi-stationary distribution since for finite Ns the system will always eventually fall in one absorbing state.

the model for various couples of parameters (ϵ, η) by means of the Gillespie algorithm [27] and computed the steady state distribution for the magnetization. In figure (11) we show the results of these simulations.

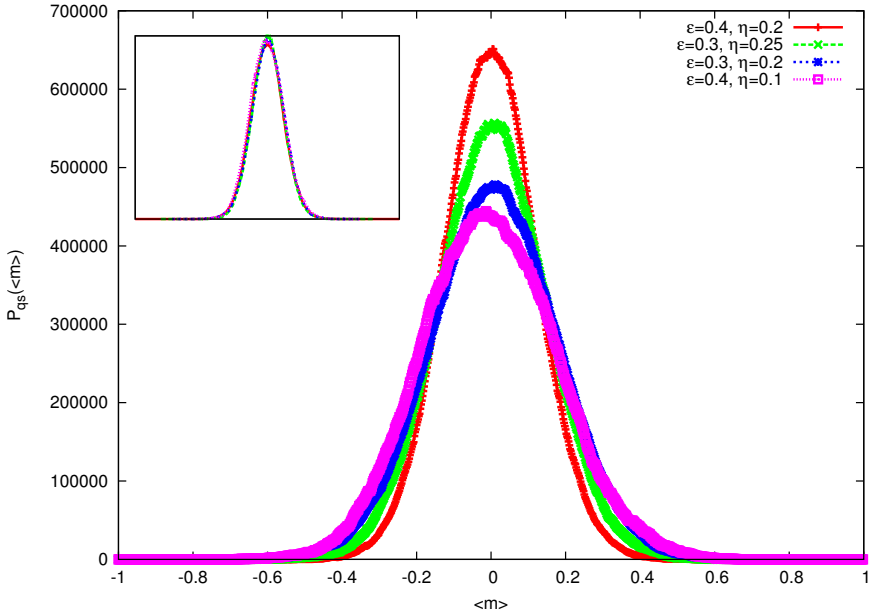


Figure 11: Quasi-stationary Probability distribution for the global magnetization varying the two parameters η and ϵ for a complete graph of $N = 1000$ nodes. In the inset it is shown the good (up to statistical errors and approximations) scaling collapse of the curves following the gaussian ansatz of equation (80) with $\alpha = 1.15$.

The first important question one could ask is whether there is a lower-bound in η below which the preference field is not sufficient to keep the system in the active phase, that is, if there exists a crossover or transition point, $\eta_l(N)$, such that the infinite system is in the active phase if $\eta > \eta_l$ and in the absorbing phase if $\eta < \eta_l$. Rigorous results are not easy to be obtained, so for the time being we will concentrate on the two extreme cases of *extensive* and *intensive* fractions η of spins in the 'preference field'. In the former case we know that in the $N \to \infty$ limit the dynamics becomes deterministic following equation (79) and by definition we will have $\eta_{ext} \sim$const. so that the system will be in the active symmetric phase. Conversely, in the latter case we will have $\lim_{N\to\infty} \eta_{int}(N) = 0$ and thus we expect that it will become an irrelevant perturbation when $N \gg 1$ and in this case the standard VM behavior will be eventually recov-

ered. This is plausible but it might seem in contradiction with the results of [47] and [46], where a single fixed (zealot) spin changes completely the voter behavior. This is easily reconciled recalling the fact the in those works the absorbing states were eliminated by the definition of the models, while in our model they are always present. An indirect confirmation of these reasoning is in figure (12), where we can see that for η extensive, even if very small (in the figure it corresponds to a fraction of $0,01$ with respect to the total number of spins), the mean time to absorption scales exponentially with the size of the system –that is a strong indication of an active phase in the infinite-size limit– where for η intensive it scales, after an initial transient, as a power law with exponent close to that of the pure mean field VM and thus the effect of the external field is washed away and becomes negligible.

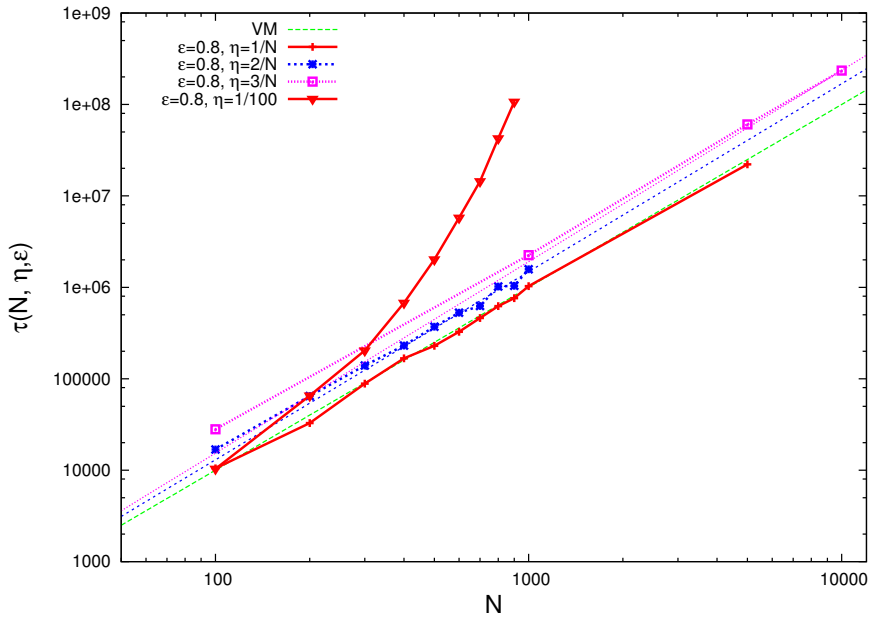


Figure 12: Mean time to absorption for extensive and non-extensive values of ηN at fixed ϵ versus the system's size N. The exponential behavior of the extensive case is a strong signal of active phase in the infinite size limit, as expected. For *intensive* values of $\eta N = const$ the perturbation is strong for small values of N but is likely to have no effect on the infinite system.

The mean-field-like version of our model gives already the possibility of some nice speculations about the large scale biodiversity of a neutral ecosystem that is some way 'connected'

to other areas where the environment separates the species in different niches[5], using the ecological jargon, and it is interesting *per se*. Nevertheless, it is clear that space, in particular the distribution of these fitness patches, and limited connectivity might play a dramatic role modifying substantially the global dynamics. In the next section we will explicitly include space and nearest-neighbors interactions to the model. Finally, a particular case will be discussed that will answer partially the question we arose in the beginning of this chapter: What happens if the fraction of sites more fitted for the two species are at the boundaries of the system and thus are slightly sub-extensive?

5.2 STOCHASTIC DYNAMICS IN TWO DIMENSIONS

5.2.1 *Small patches of environment-induced fitness affect the global biodiversity*

In this section we report numerical simulations of the spatially-extended version of the model (76) on a regular lattice in $D = 2$. It is important to determine what is the dynamics of the system far from the "sources" of particles of type A and B, *i.e.* the areas where the external random field is not zero. Due to the peculiar property of the voter-like models, there is no surface tension at the borders between different domains, and as mentioned in section 2.3.2 the coarsening in only diffusion-driven. When a total fraction $2\eta N$ of the total number of spins feels the external quenched random field, half in one direction and half in the other one, apart from a small region near those sites whose size will depend on the value of ϵ, the dynamics will still be voter-like. This means that there will be no clear separation between domains, the borders will fractal and will percolate through the whole system, resulting in a well mixed population of the two species with only diffusion-driven clustering of domains of single species. The magnitude of this mixing and the fluctuations of the global magnetization will depend on the strength of the external field, ϵ. In figure (13) there are shown

5 An ecological *nice* is defined as a point in some generalized metric space that indicates how an organism or population responds to the distribution of resources and competitors. Each species is thought to have a separate, unique niche and when two species are in the same niche there is not a prevalent species so that the competition if effectively neutral.

snapshots of three realizations of the process for increasing values of ϵ, constant $\eta = 1/100$ and $N = 100 \times 100 = 10^4$ spins. It can be clearly seen the behavior described above. All the same, qualitatively the analysis of the mean field approximation for the steady state is valid also in two dimensions.

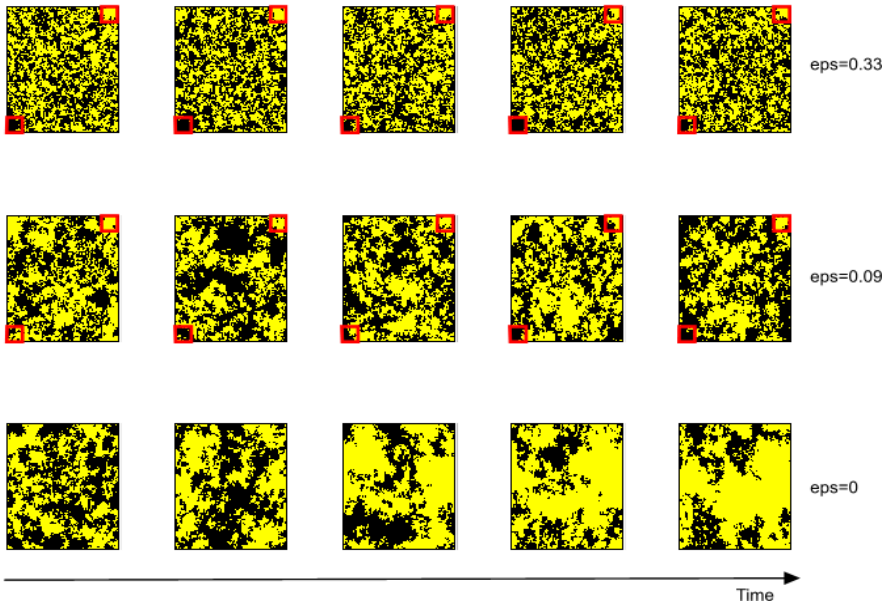


Figure 13: Example of evolution of the model in $D = 2$ for a lattice of size $N = L \times L$, $L = 100$ with fixed $\eta = 0.01$. The sites i with $\tau_i = -1$ are concentrated in a small square in the bottom-left corner, while those with $\tau_i = 1$ are in the top-right corner. All the other sites are neutral $\tau_i = 0$. Top row: $\epsilon = 0.33$, Middle row: $\epsilon = 0.09$, Bottom row: $\epsilon = 0$, pure VM. Random I. C.

5.2.2 *Habitat preference at the boundaries: correlation functions and mean time to absorption*

We now arrive at the topic discussed in the introduction of this chapter, that is, the role of boundary conditions in the voter model, taken as a simple example of a genuinely out-of-equilibrium critical model. For this purpose, we consider a voter model on a cylinder lattice in $D = 2$, $\mathbb{C}_2$, with a sandwich configuration at the borders. *Sandwich configuration* means that the spin on one border of the cylinder, $\partial^+\mathbb{C}_2$, experience the external preference field in the 'up' direction, while the spins of the opposite border, $\partial^-\mathbb{C}_2$, have the external field in the 'down'

direction, both of strength ϵ. In this case $\eta = L^{-(D-1)} = 1/\sqrt{N}$ and is sub-extensive because it fills a manifold of dimension $\dim(\partial C_D) = D - 1$. Consequently, the analysis in the previous section is not of help and this intermediate case in not trivial.

First of all we can compare with simulations the dynamics of this model with that of the pure VM, and in the same spirit of what is shown in figure (13) we compare the voter model dynamics with the sandwich configuration with $\epsilon = 0.5$ in figure (14). The dynamics is still characterized by large fluctuations in the bulk due to the lack of surface tension, but still the coarsening process seems to be missing. We will see later, equation (85), that also in the case of a sub-extensive fraction η of sites with preference the system is maintained in the active phase.

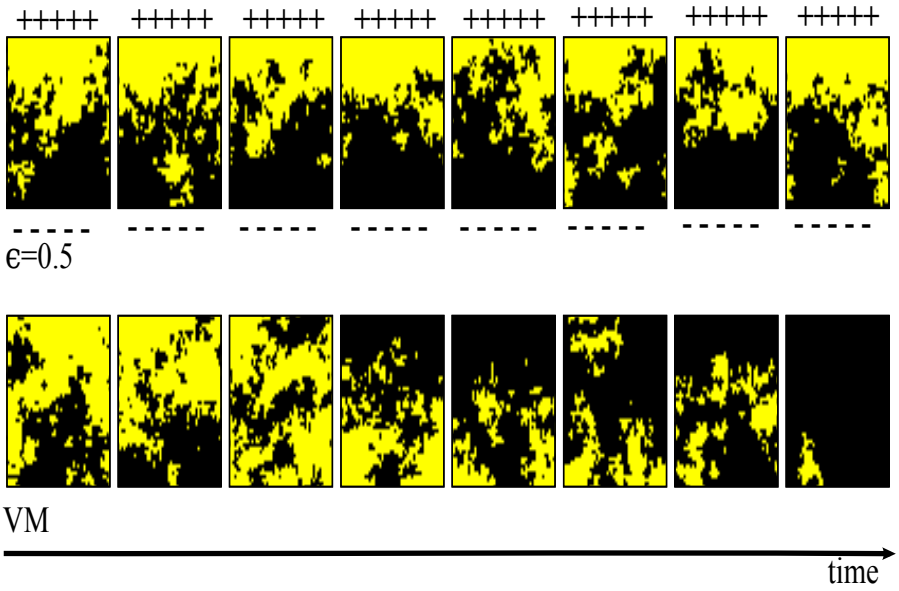


Figure 14: Example of evolution of the model in $D = 2$ for a cylindrical lattice of size $N = L \times L$, $L = 50$ in the sandwich configuration with boundary field strength $\epsilon = 0.5$ (top) and the pure voter model (bottom). Random I. C.

In figure (15) we plot the mean magnetization per site along the direction x parallel to the boundaries[6], $m(x) = \langle \sigma(x, y) \rangle_y$, together with the first neighbors correlation function defined by $G(x, x+1) \equiv \langle \sigma(x, y)\sigma(x+1, y) \rangle_y$. We can see that the mean magnetization, very close to $+1$ and -1 at the respective bound-

6 In this notation a site i on the lattice is determined by the two coordinates x, y, $i = i(x, y)$ and the x direction is parallel to the boundaries while y is moving orthogonally. To simplify the notation we will write for the spin at site i $\sigma_{i(x,y)} \equiv \sigma(x, y)$.

aries as expected, varies linearly along x and is globally zero, thus we expect to have long-range correlations. In particular we expect the dynamics in the bulk to be effectively critical.

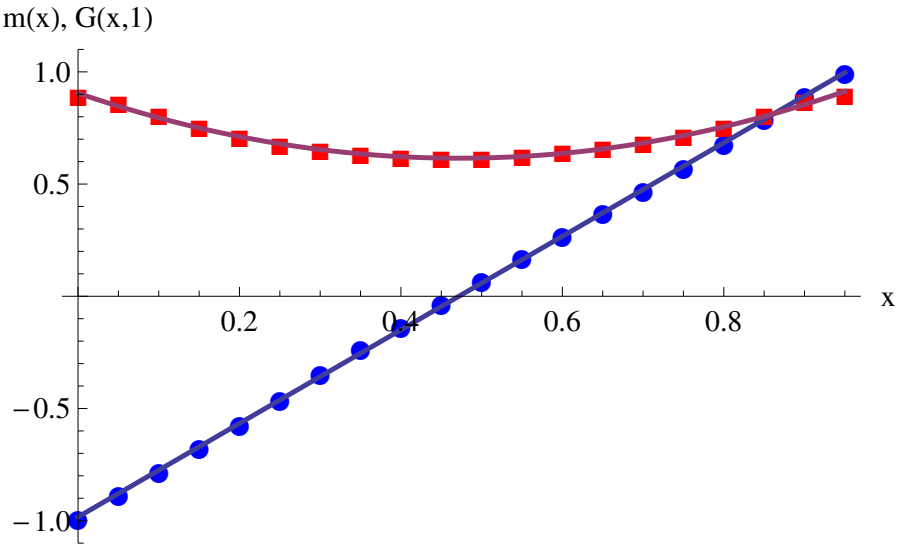


Figure 15: 1−point (dots) and 2−points first neighbors (squares) correlation function for the 'sandwich' configuration. The continuous curves are respectively a linear and quadratic fit.

It is interesting to note that the situation presented here is different to what happens at equilibrium for an Ising model with $(\pm)$-boundary conditions [71, 72]. A general N-spins ising model on a lattice Ω with borders defined by the hamiltonian

$$H = -1/2 \sum_{\langle i,j \rangle} \sigma_i \sigma_j - 1/2 \sum_{i \in \partial^+ \Omega} \sigma_i + 1/2 \sum_{i \in \partial^- \Omega} \sigma_i \qquad (81)$$

in two dimensions has a line λ that separates the two pure phases in the low temperature regime, the ordered phase, that is almost straight and fluctuates locally with an amplitude of order $\mathcal{O}(\sqrt{L})$. In our case, a separation line *strictu sensu* does not exists in any phase since the model has not a curvature driven dynamics in the bulk.

Lastly, the final point we want to address is to deduce by general considerations and heuristic arguments, an ansatz for the mean time to absorption in the sandwich configuration for arbitrary values of N, ϵ.

In section 4.3.2 we derived an effective statistical field theory for the global magnetization's dynamics in the limit $\eta = 1/2$, and from the Langevin equation (58) we expect, qualitatively,

the presence of an effective potential term in the stochastic dynamics that is disorder-induced and depends solely on the disorder strength ϵ. This potential has a minimum at zero magnetization. Furthermore, we know that for $\epsilon = 1$ the well becomes infinite and the dynamics trivial while for $\epsilon < 1$ but N sufficiently large the system is in the symmetric active phase. From general considerations applicable to stochastic processes in presence of a potential, *i.e.* the Arrhenius law, we can argue that, starting from arbitrary (but active) initial conditions the system will rapidly –exponentially fast– relax to the minimum of the potential, while the time needed to escape from the stable state and reach one of the absorbing points is determined by the noise intensity but it will be *exponential* in the depth of the potential 'well', difference of the value of the potential at its maximum height and the minimum in $\phi = 0$. This depth will of course depend in general by ϵ,η and N. From this general reasoning argument we can say that the mean time to absorption $\tau = \tau_N(\epsilon, \eta) \equiv \langle T_{abs}(N)\rangle$, where T_{abs} is the time the system took to reach an absorbing state in a single realization, will have the general form

$$\tau_N(\epsilon, \eta) = f_{VM}(N) e^{g(\epsilon,\eta,N)}. \tag{82}$$

In the above equation (82) the prefactor $f_{VM}(N)$ comes from the typical time of absorption of the pure VM in D dimensions, that is power-law distributed in all dimensions apart from logarithmic corrections. For $D = 2$, in particular, we have $f_{VM}(N) = N \log N$. The function g in the exponential has to be defined, and interesting results will be obtained only if it is non-vanishing in the large N limit. Again from section 4.3.2 we know that the characteristic dependence of the system on ϵ is given by $\epsilon^2/(1-\epsilon^2)$, while from equation (80) we can imagine that the function g will depend on η^β for some exponent $\beta > 0$ and will be linear in N. Hence, we can make the educated guess

$$\tau_N \equiv \langle T_{abs}(N)\rangle = f_{VM}(N) e^{\eta^\beta \frac{\epsilon^2}{1-\epsilon^2} N}. \tag{83}$$

Therefore, we gain an exponential factor in the typical time needed for a species to go extinct. In Fig. (16) we see that this ansatz is almost perfectly supported by simulations results for

$$\beta = \frac{1}{2} \tag{84}$$

on several order of magnitude of the system size, so that at the end of the day we can write, in $D = 2$ and for the sandwich configuration,

$$\begin{aligned} \tau_N \equiv \langle T_{abs}(N) \rangle &= N \log N e^{\sqrt{\eta} \frac{\epsilon^2}{1-\epsilon^2} N} \\ &= N \log N e^{\frac{\epsilon^2}{1-\epsilon^2} N^{3/4}}. \end{aligned} \tag{85}$$

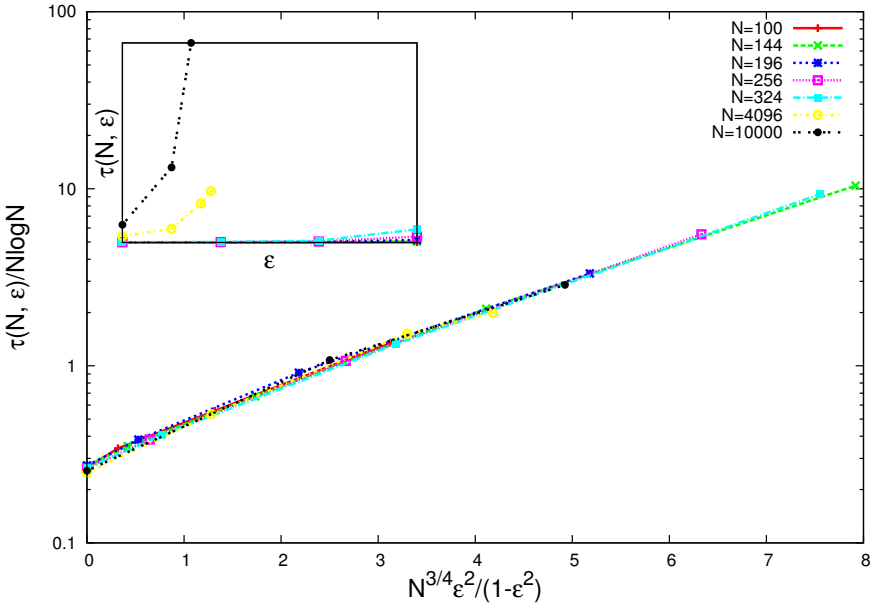


Figure 16: Collapse of the mean time to absorption based on the exponential scaling in Eq. 83 for various ϵ at fixed N. Different curves are for different values of N from $N = 10^2$ to $N = 10^4$. In the inset the same curves are shown without the collapse.

This exponential enhancement of the mean time to absorption is in general a sufficient, even if not necessary, signature of an active phase in the infinite system.

5.3 DISCUSSION

We end this last chapter with some considerations on the possible consequences of the analysis above. First of all, from the pure theoretical-physics-minded aspects, we saw that the voter model is so much sensible to external perturbations that an arbitrarly small, but not infinitesimal, localized effective weak

source of both types of particles in equal measure that does not remove the absorbing states keeps the system far from the absorbing states for an exponentially long time in the system's size N, with a quasi-stationary distribution given by a gaussian centered at zero magnetization with variance $\sigma = \mathcal{O}(1/\sqrt{N})$, both for a complete graph and in finite dimensions. The infinite system's dynamics is deterministic and it is in the active phase. In the case of a cylinder geometry with $(\pm)-$boundary conditions, furthermore, we could give a –supposedly exact– expression for the mean time to absorption for finite N.

This could be extremely important, for example, for applications in ecology or conservation ecology, where our prediction is that in natural areas where biodiversity is in threat because of habitat modifications, perhaps caused by human activities, a small area best fitted for one particular species could favor its proliferation in the whole area of interest. Thus, if these areas in different niches are "balanced", then this could end up in a richer and stable (up to exponential times) biodiversity in all the connected patches, even if they are neutral.

Part III

CONCLUSIONS AND PERSPECTIVES

6

CONCLUSIONS AND PERSPECTIVES

> For every complex problem there is an answer
> that is clear, simple, and wrong.
> — ***H. L. Mencken***

6.1 CONCLUDING REMARKS

The mathematical theory of nonlinear dynamical systems and chaos is one of the scientific revolutions of the twentieth century, together with quantum mechanics and general relativity. Simple equations of motion can give raise to complex structures of extraordinary richness and beauty, and when these concepts were integrated in the dynamics of many interacting particles in a statistical mechanics approach, we obtained the first glance of how the ordinary world we live in can provide phenomena at the same time so simple and so complex.

We are still very far from an even marginal understanding of the mechanisms at the basis of life's phenomena, but we are making theoretical and experimental progresses like never in the past. In this context, the aim of this thesis was to give a –small– original contribution to the theory of interacting particle systems for systems with absorbing states with global symmetries. This was motivated mainly because of the myriad of possible applications that these systems have found in practically all the scientific disciplines. The striking fact is that we are starting to collect different, very distant and in principle totally unrelated, problems under the same light, that is, the emergence of complex behaviors from a microscopic simplicity, where the details are integrated out when going from the micro to the macro and are not important for the global properties of the system. All these hypotheses are corroborated by the increasing number of observational evidences. Almost all "active-matter-phenomena" show scale invariance, power-law distributions, long-range correlations, that are a strong hint of universal emergent behaviors.

In particular, in our case, the prototype of system with emergent properties was a generic ecological systems, where individuals of many species interact and compete in the constant struggle for life. We started from the most basic hypothesis for individuals in the same trophic level, *i.e.* the neutral theory, and developed models of competing symmetric species analyzing first the possible net effects of the sum of several different interactions among individuals, ending up in effective nonlinearities, and then the role of a disordered habitat in the form of a quenched random field that locally breaks the neutral dynamics introducing fitness, both globally and locally. All these problems have been approached mixing analytical and computational techniques of stochastic processes, stochastic field theory, probability theory and dynamical systems.

Altough this is a mostly purely theoretical and speculative physics work, we are confident that practical applications to other pure sciences are about to come.

6.2 SOME FUTURE PERSPECTIVES

> Experience without theory is blind,
> but theory without experience is mere intellectual play.
> — ***I. Kant***

As we were saying, we took the cue for this study from the dynamics of ecological systems. The first prosecution of this work should be a comparison with experimental data. We are willing to set up some experiments, probably with microecological systems, to verify our predictions in a possibly controlled and clear way. Data resulting from these experiments will for sure give strong indications on the future research directions along this line.

From a more theoretical point of view, instead, a important issue is to understand how information propagates through the system in order to generate the emergent collective behaviors that we observe. Up to now, little work has been done on this topic and usually it focus on static distributions of entropy and information content of sent basic messages from an information-theoretic perspective. It is coming out now the need for a dynamical description of the information transmission that can give insights on how the system organizes to its final steady state capable of generating complex outputs. In this

perspective, the integration of the dynamics of interacting particle systems with tools from information theory could produce a breakthrough in our understanding of living matter, as it was the case when stochasticity was added to the dynamical systems' description of population dynamics. Obviously this is neither an easy task, nor a short-term research proposal, but "the aims of pure basic science, unlike those of applied science, are neither fast-flowing nor pragmatic. The quick harvest of applied science is the usable process, the medicine, the machine. The shy fruit of pure science is understanding." (cit. Lincoln Kinnear Barnett, american writer)

Part IV

APPENDICES

A

SOME BASIC RESULTS

A.1 KRAMERS-MOYAL EXPANSION

Consider the generic D−dimensional Master Equation (ME)

$$\dot{P}(\varphi,t)=\sum_{\varphi'}\left[W(\varphi|\varphi',t)P(\varphi',t)-W(\varphi'|\varphi,t)P(\varphi,t)\right] \tag{86}$$

where $\varphi\equiv\{\varphi^{\nu}\}_{\nu=1,\dots,D}$ and $W(\varphi|\varphi',t)\neq 0$ only if $\varphi-\varphi'\in E$ with $|E|<\infty$. Then, under general assumptions [6], equation (5) can be rewritten as

$$\dot{P}(\varphi,t)=\sum_{k=1}^{\infty}\frac{(-)^k}{k!}\partial_{\nu_1}\cdots\partial_{\nu_k}\left[A^{\nu_1\cdots\nu_k}(\varphi,t)P(\varphi,t)\right] \tag{87}$$

where $\partial_{\nu_k}=\partial/\partial(\varphi^{\nu_k})$, the sum over repeated indices is understood, and

$$A^{\nu_1\cdots\nu_k}(\varphi,t)=\sum_{\varphi'}W(\varphi'|\varphi,t)(\varphi'^{\nu_1}-\varphi^{\nu_1})\cdots(\varphi'^{\nu_k}-\varphi^{\nu_k}). \tag{88}$$

The term for $k=1$ corresponds to a deterministic motion of φ:

$$P(\varphi,t)=\delta^{D}(\varphi(t)-\varphi) \tag{89}$$

with $\varphi(t)$ determined by the differential equation

$$\dot{\varphi}^{\nu}(t)=A^{\nu}(\varphi(t),t),\quad \nu=1,\dots,S. \tag{90}$$

The first stochastic contribution to the equation of motions of φ comes from the terms with $k=2$, defining the following Langevin equations for the φ^{ν}s:

$$\dot{\varphi}^{\nu}(t)=A^{\nu}(\varphi(t),t)+B^{\nu\mu}(\varphi(t),t)\xi^{\mu}(t),\quad \nu=1,\dots,D. \tag{91}$$

where $B^{\nu\mu}$ is such that

$$B^{\nu\mu}(\varphi(t),t)B^{\sigma\mu}(\varphi(t),t)=A^{\nu\sigma}(\varphi(t),t) \tag{92}$$

and ξ^{μ} is a gaussian r.v. with correlations:

$$\begin{cases}\langle\xi^{\mu}(t)\rangle=0\\ \langle\xi^{\mu}(t)\xi^{\nu}(t')\rangle=\delta(t-t')\delta^{\mu\nu}\end{cases}\qquad\forall\ \mu,\nu=1,\dots,S. \tag{93}$$

Remark From the definition of $A^{\mu\nu}(\varphi(t), t)$, it can be seen that it is symmetric in μ, ν and it's semipositive defined; this assures that $B^{\mu\nu}$ is well defined, apart from an orthogonal matrix, S ($S \cdot S^T = \mathbb{I}$), $B' = B \cdot S$ implies that $B' \cdot B'^T = B \cdot B^T = A$.

A.2 DUALITY OF THE VOTER MODEL

Consider the voter model on $\Omega = \{0,1\}^\Lambda$ with transition rates

$$c(i,\eta) = \sum_{j:\eta(j)\neq\eta(i)} q(i,j) \tag{94}$$

We will now briefly show that this voter model is dual to a collection of coalescing random walks with transition matrix $Q = [q(i,j)]$. Let

$$H(\eta, A) = \prod_{x\in A} \eta(x) = \mathbb{I}(\eta(x) \equiv 1 \text{ on } A) \tag{95}$$

where $\eta \in \Omega$ is a state of the voter model and $A \subset \Lambda$, $|A| < \infty$. Let $\mathcal{A}_t$ be the Markov chain on the state space $S = \{A \subset \Omega : |A| < \infty\}$ with transition matrix $Q = [Q(A,B)]$ defined as

$$\begin{cases} Q(A, (A\backslash\{i\}) \cup \{j\}) = q(i,j) & \text{for } i \in A, j \notin A, \\ Q(A, A\backslash\{i\}) = \sum_{j\in A,\ j\neq i} q(i,j) & \text{for } i \in A. \end{cases} \tag{96}$$

Now let $\mathcal{L}_\eta$ be the generator of the voter model, then

$$\begin{aligned} \mathcal{L}_\eta H(\cdot, A)(\eta) &= \sum_{\substack{i\in A, j\in\Omega \\ \eta(i)\neq\eta(j)}} q(i,j)\,[H(\eta_i, A) - H(\eta, A)] \\ &= \sum_{\substack{i\in A, j\in\Omega \\ \eta(i)\neq\eta(j)}} q(i,j)\,[1 - 2\eta(i)]\,H(\eta, A\backslash\{i\}) \\ &= \sum_{i\in A, j\in\Omega} q(i,j)\,[\eta(j) - \eta(i)]\,H(\eta, A\backslash\{i\}) \\ &= \sum_{i\in A, j\in\Omega} q(i,j)\,[H(\eta, (A\backslash\{i\}) \cup \{j\}) - H(\eta, A)] \\ &= \sum_{B} Q(A,B)\,[H(\eta, B) - H(\eta, A)] \\ &= \mathcal{L}_{\mathcal{A}} H(\eta, \cdot)(A). \end{aligned} \tag{97}$$

So, $\mathcal{A}_t$ and the voter model η_t are dual with respect to $H(\eta, A)$. An useful property of $\mathcal{A}$ is that $|A_t|$ is non-increasing.

This means that the voter model can be equivalently described by random walks X_t that are independent until they met and coalesce. This random walks reconstruct the 'history' of the spin from time t backward.

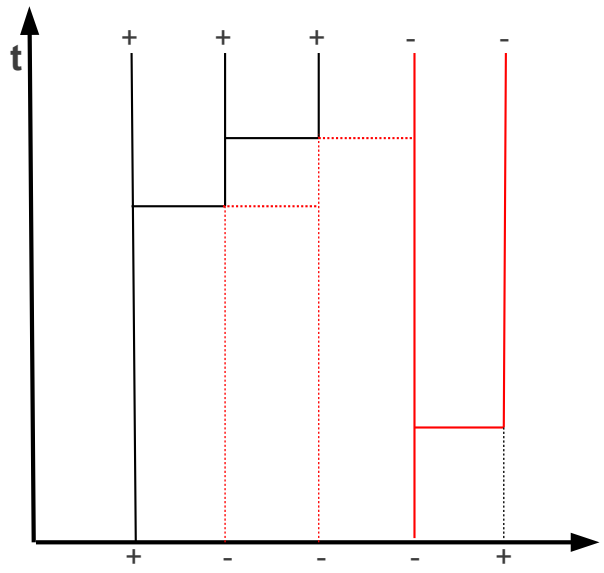


Figure 17: Cartoon of the coalescing random walks dual of the voter model.

A.3 PÓLYA'S THEOREM

A random walk S_n is *recurrent* if the probability of hitting the origin infinitely often is one. If it is zero the walk is said to be *transient*. We will now prove the following theorem [73]:

Theorem (Pólya): A random walk S_n on $\mathbb{Z}^d$ is recurrent if $d \leqslant 2$. If $d \geqslant 3$, the walk is transient and $P(S_n \neq S_0 \; \forall \, n > 0) > 0$.

Proof: Let $N = \sum_n \delta(S_n, S_0)$, then

$$\mathbb{E}[N] = \sum_n \mathbb{E}\delta(S_n, S_0) = \sum_n \mathbb{P}(S_{2n} = 0). \tag{98}$$

In $d = 1$, by Stirling's formula we have

$$\mathbb{P}(S_{2n} = 0) = 2^{-2n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}} \tag{99}$$

so we are done since $\mathbb{E}[N] \sim \sum_n n^{-\frac{1}{2}} = \infty$.

In $d = 2$ the situation is similar since by rotating $\mathbb{Z}^2$ of 45° we get that each step is like moving one step in each of two independent, one dimensional, simple random walks. Hence

$$\mathbb{P}(S_{2n} = 0) \sim \frac{1}{\pi n} \tag{100}$$

and the sum in equation 98 is still divergent.

In $d = 3$ it can be found an upper-bound to the probability $P(S_{2n} = 0)$ that is

$$\mathbb{P}(S_{2n} = 0) \leqslant \frac{c}{n^{\frac{3}{2}}} \tag{101}$$

and thus $\mathbb{E}[N] < \infty$ and the random walk in three dimensions is transient. In $d \geqslant 4$ the random walk is still transient since the first three dimensions are transient.

To prove that $P(S_n \neq S_0 \; \forall \; n > 0) > 0$, let q be the probability that S_n ever returns to its starting point, and assume $q < 1$. Then

$$\mathbb{P}(N = k) = q^{k-1}(1 - q), \; k = 1, 2, \ldots \tag{102}$$

and

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} k\mathbb{P}(N = k) = \sum_{k=1}^{\infty} kq^{k-1}(1 - q) = \frac{1}{1 - q}. \tag{103}$$

The expectation value above is infinite in $d = 1, 2$ so $q = 1$ and finite in $d = 3$, so $q < 1$.

□

From this theorem and the duality of the voter model with coalescing random walks showed above it comes the proof of the absorbing and active phases in the voter model.

A.4 THE IMRY-MA ARGUMENT FOR THE RANDOM FIELD ISING MODEL

Consider an Ising model (with discrete symmetry $\mathbb{Z}_2$) in a quenched random field

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \sum_i h_i \sigma_i, \tag{104}$$

and consider an uniform domain of linear size L in D space dimensions. The free energy that the systems would gain in

the alignment with the external field h is $\mathcal{O}(L^{\frac{D}{2}})$, while due to surface tension the domain wall energy is $\mathcal{O}(L^{D-1})$. Therefore,in $D \leqslant 2$ the energetic balance is favorable if the spins are aligned with the external field and do not form ordered domains so that the symmetry breaking is suppressed. The situation is the opposite when $D \geqslant 3$.

For systems with continuous symmetries, the domain wall energy is of order L^{D-2} so that in this case the marginal dimension is 4.

A.5 GENERALIZATIONS OF THE SPONTANEOUS NEUTRAL SYMMETRY BREAKING STABILITY CONDITION

Suppose that in the model described by equation (34), instead of only one species with a different density, we want to allow for a spontaneous symmetry breaking with S_1 species to have density φ' and the remaining species have density $\frac{1-S_1\varphi}{S-S_1} \equiv \zeta'$. Then the stationary density has the form

$$\rho_{bs}^{\prime\nu} = \varphi'(\delta^{\nu,1} + \cdots + \delta^{\nu,S_1}) + \zeta'(1 - \delta^{\nu,S_1+1} - \cdots - \delta^{\nu,S}). \quad (105)$$

For $S_1 \geqslant \nu \geqslant 1$, $\dot{\rho}_{bs}^{\prime\nu}$ is given by

$$A^{\nu}(\rho_{bs}^{\prime\nu}) = \varphi'(1-\varphi)\Big[K(\varphi') - K(\zeta')\Big] \quad (106)$$

and for $S \geqslant \nu > S_1$

$$A^{\nu}(\rho_{bs}^{\prime\nu}) = \zeta'\varphi\Big[K(\zeta') - K(\varphi')\Big] \quad (107)$$

Thus the condition on K for the stationarity of this density is

$$K(\varphi') = K(\zeta'). \quad (108)$$

The Jacobian matrix is, in this case, of the form:

$$(JA)_{|\rho=\rho_{bs}} = \left(\begin{array}{ccc|ccc} a & \cdots & b & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b & \cdots & a & 0 & \cdots & 0 \\ \hline d & \cdots & d & c & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ d & \cdots & d & 0 & \cdots & c \end{array}\right) \quad (109)$$

with the identifications

$$\begin{aligned} a &\equiv \varphi'\Big[(1-\varphi')K'(\varphi') + \zeta' K'(\zeta')\Big] \\ b &\equiv \varphi'\Big[\zeta' K'(\zeta') - \varphi' K'(\varphi')\Big] \\ c &\equiv \zeta' K'(\zeta') \\ d &\equiv \zeta'\Big[\zeta' K'(\zeta') - \varphi' K'(\varphi')\Big]. \end{aligned} \tag{110}$$

This matrix is still diagonizable, with eigenvalues given by the eigenvalues of the matrix in the upper-left block, that is always diagonizable being symmetric, and the eigenvalue c, that is $(S - S_1)$-fold degenerate.

Example: $S_1 = 2$**.** If we ask $S_1 = 2$, the two eigenvalues of the upper-left block of the matrix Eq. (109) are given by

$$\begin{aligned} \lambda_1 &= a + b = \varphi'\Big[(1-2\varphi')K'(\varphi') + 2\zeta' K'(\zeta')\Big] \\ \lambda_2 &= a - b = \varphi' K'(\varphi') \end{aligned} \tag{111}$$

and then the conditions for the stability of the stationary density are

$$\begin{cases} (1-2\varphi')K'(\varphi') < -2\zeta' K'(\zeta') \\ K'(\varphi') < 0 \\ K'(\zeta') < 0, \end{cases} \tag{112}$$

very similar to those in equation (47). Similar reasoning applies also to the case of $\varphi_1, \varphi_2, \ldots, \varphi_m$, $m \leqslant S$, different densities, and in this case we will have the m variables generalization of equation (112) with $\phi' = \varphi_l$ and $\zeta' = \varphi_{l-1}$, see the inset in figure (4).

B

DYNAMICAL SYSTEMS

B.1 PHASE PORTRAIT, ATTRACTORS AND ORBITS

Since the thesis contains many references to the theory of dynamical systems, we will present here a summary of definitions and the most basic results (see for example [74] for a complete and basic treatment). We start defining rigorously what a (classical) dynamical system is: Let M be a compact differentiable manifold and let μ be a normalized measure on M. Let $\Phi = \{\Phi_t\},\ t \in T = \mathbb{R}, \mathbb{Z}$ a 1-parameter group of measure-preserving diffeomorfisms on M, that is, such that

$$\begin{aligned} &\Phi_t \circ \Phi_s = \Phi_{t+s}, \quad \Phi_0 = \mathbb{I}, \\ &\mu(\Phi_{-t}(A)) = \mu(A),\ A \subset M, \end{aligned} \tag{113}$$

$\forall t, s$. Then the triplet (M, μ, Φ) is a *classical dynamical system*. This means, intuitively, that a dynamical system is an application $T \times M \to M$ that maps the state of the system at time t to the state of the system at time $t + s$. The manifold M is called the *phase space*, and each point on M represents a specific state of the system. We are interested in the case in which $\Phi(x),\ x \in M$ is the *flux* of a field $X \in T_x M$, that is, we can write

$$\dot{x} = X(x),\ x \in M \tag{114}$$

and $\Phi_t(x)$ is the solution of the differential equation above. For concreteness, in all our cases we have $M = \mathbb{R}^D$.

The *orbit* of a point $x \in M$ is defined by the set

$$O_x = \{\Phi_t(x)\ :\ t \in T\}, \tag{115}$$

and the set of all the orbits defines the *phase portrait* of the dynamical system. If X is a vector field $X = (X_1, \ldots, X_n)$, then an orbit is *nullcline* if $X_j(x) = 0$ for any j. A *fixed point* of the dynamics is where all nullclines intersect, that is, one or more points x_0 such that

$$\Phi_t(x) = x\ \forall\ t \in T. \tag{116}$$

A subset $A \subset M$ is *invariant* if it contains all the orbits of its points, that is, if

$$\Phi_t(A) \subset A \ \forall \ t \in T. \tag{117}$$

A fixed point x_0 is said to be an *attractor* if there exists a neighborhood U such that

$$\lim_{t\to\infty} \Phi_t(x) = x_0 \ \forall \ x \in U. \tag{118}$$

If $U = M$ than the attractor is a *global attractor*. Similarly a point x_0 is *stable* if for any neighborhood U of x_0 there exists a neighborhood U_0 such that $\Phi_t(U_0) \subset U \ \forall \ t \geqslant 0$. A fixed point is *unstable* if it is not stable.

B.2 LINEAR STABILITY OF A FIXED POINT

Near a fixed point, we can approximate the dynamical system with its linearized version. In fact, call $\delta x = x - x_0$ a small deviation from the fixed point x_0 caused by a perturbation of the system's state. We want to see if the perturbation is amplified in time or disappears, and how fast. We can write

$$\dot{\delta x} = \dot{x} = X(x) = X(x_0 + \delta x), \tag{119}$$

then, since δx is small, we can write

$$\dot{\delta x} = X(x_0) + JX(x_0)\delta x + o(\delta x) \tag{120}$$

where $JX(x_0) = \partial X/\partial x(x_0)$ is the jacobian matrix of the field X, and it is a constant matrix. This system is linear in δx and it is called the *linear approximation* of the dynamical system. Thus, it is now easy to see that a fixed point x_0 is *locally stable* (*unstable*) if $JX(x_0)$ has all the eigenvalues with negative (positive) real parts.

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