Inauguraldissertation zur Erlangung des akademischen Grades
eines Doktors der Wirtschaftswissenschaften (Dr. rer. pol.)

Ellsberg Games and the Strategic Use of Ambiguity in Normal and Extensive Form Games

vorgelegt von

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an der Fakultät für Wirtschaftswissenschaften
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Bielefeld, 4. Oktober 2013

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Ellsberg Games and the Strategic Use of Ambiguity in Normal and Extensive Form Games

Thèse pour obtenir le grade de docteur ès Sciences Economiques

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Acknowledgments

In the first place I would like to express my gratitude to my supervisor Prof. Frank Riedel. His continuous support and his enthusiasm were essential for the development and completion of my thesis. I want to thank him for pointing out the topic to me, and for giving me a picture what economic research is about. I learned a great deal during the last few years, and I hope I could adapt some of his inspiration and intuition.

I also want to express my gratitude to my supervisor Prof. Jean-Marc Tallon whose support and interest were of great help, especially during my stay in Paris. Likewise, I wish to thank Prof. Pierpaolo Battigalli and Prof. Wolfgang Leininger for agreeing to be external referees for my cotutelle.

I want to thank the members of the Institute for Mathematical Economics at Bielefeld University for their support, interest and inspiring discussions. In particular, I would like to thank my fellow doctoral students in EBIM and BiGSEM for their friendly company in the last years. We spent many nice moments together, and their interest and our fruitful discussions were of great help for completing my Ph.D.

I am also grateful to Paris School of Economics and its members, and to the participants of the Paris Game Theory seminar. In particular, I want to thank Frédéric Koessler whose interest in my research greatly motivated me during my stay in Paris.

I am grateful to the DFG about financial support in the first three years of my Ph.D. via the International Graduate School EBIM. I also thank the IMW for financial support during the fourth year of my Ph.D. Furthermore, I would like to thank the French-German University (UFA-DFH) for offering me a scholarship to finance my cotutelle between Bielefeld University and Paris 1. The scholarship gave me the possibility to visit interesting conferences and workshops, travel between Bielefeld and Paris, and it helps me finance my defense at Bielefeld University. Finally, I wish to thank the rectorate of Bielefeld University for offering me a six-months scholarship for financing the completion of my Ph.D.

At last, I wish to thank Johannes Höwing. I am deeply grateful, not only for his excellent proofreading, but also for his never-ceasing interest in discussing the technical details and difficulties that arouse while I was writing my thesis. For her final proofreading I also wish to thank Friederike Terbeck. Not least, I thank my parents Wilma Sass and Hermann Taube for their love and encouragement.

Introduction

Common sense suggests that a certain strategic ambiguity can be useful in conflicts.[1] "Many different strategies are used to orient toward conflicting interactional goals; some examples include avoiding interaction altogether, remaining silent, or changing the topic", says Eric Eisenberg in his famous article "Ambiguity as strategy in organizational communication" (Eisenberg (1984)), and he points out that applying one's resources of ambiguity is key in successful communication when conflicts of interest are present.

My doctoral thesis introduces such strategic use of ambiguity into games. Although game theory was invented to model conflicts of interest, so far the theory does not allow players to intentionally choose ambiguity as a strategy. To this end, I am going back to the beginnings of game theory. Von Neumann and Morgenstern (1953) introduced mixed strategies as random devices that are used to conceal one's behavior. I take up this interpretation and propose a generalization. In short, I allow players to use Ellsberg urns in addition to probabilistic devices like a roulette wheel or a die. For example, a player can base his action on the draw from an urn that contains hundred red and blue balls and it is only known that the number of red balls is between thirty and fifty. Such urns are objectively ambiguous, by design; players can thus create ambiguity. The recent advances in decision theory, which were motivated by Ellsberg's famous experiments and Knight's distinction between risk and uncertainty, allow to model such strategic behavior formally.

Risk and Ambiguity

For every decision we take, we consider possible future implications. We do so by inferring how our environment will react, and what consequences our decision could have in different states of the world. Knight (1921) observed that not all these possibly influencing future events can be classified in the same way. Whereas it is easy to derive a probability distribution for "red" at a roulette table, it is close to impossible to predict more complicated things such as stock prices or if Germany wins the next world cup, with a precise

[1] Parts of this introduction were published in the IMW working paper Riedel and Sass (2011). A revised version of the working paper is published in Riedel and Sass (2013).

probability distribution.[2] While Knight's point was mainly to draw attention to the role of experience in making good predictions in complex circumstances[3], decision theorists incorporated his distinction between risk and ambiguity into decision models, and this with great success. One major achievement of ambiguity theory is to provide decision models which resolve the famous Ellsberg paradox (Ellsberg (1961)). Most prominent models are by now the multiple-prior expected utility by Gilboa and Schmeidler (1989), Choquet expected utility by Schmeidler (1989) and incomplete preferences by Bewley (2002). Many other authors have added to these models by refining and extending them, others by applying them to economic and behavioral problems. Interesting applications include, e.g., explanation of incomplete contracts (Mukerji (1998)), ambiguity in financial markets (Rigotti and Shannon (2005)), asset pricing under ambiguity (Epstein and Schneider (2008)), optimal stopping with ambiguity (Riedel (2009)), strategies in insurance fraud detection (Lang and Wambach (2010)), and strategic voting (Ellis (2011)). See Luo and Ma (1999) and Mukerji and Tallon (2004) for an overview of applications and Etner, Jeleva, and Tallon (2012) for a comprehensive recent survey on decision theory under ambiguity. A thorough review of philosophical and mathematical aspects of probability and its influence on decision theory is presented in Binmore (2008).

Ambiguity in Games

Ambiguity is naturally present in strategic contexts. A human is a complex opponent and hence strategic interaction can be fruitfully modeled using ambiguity in the Knightian sense. The existing literature on ambiguity in games can be devided into two branches, based on the co-existing interpretations of mixed strategies in the literature. The Bayesian view of mixed strategies has been generalized to ambiguity starting with Dow and Werlang (1994), Lo (1996), and Marinacci (2000), and followed by many other authors. There, the interpretation of mixed strategies as beliefs about others' actions is generalized to uncertain beliefs in the sense of the decision-theoretic literature on Knightian uncertainty (Schmeidler (1989), Gilboa and Schmeidler (1989)). The other interpretation, where a mixed strategy is seen as an objective randomizing device, has to my knowledge not been generalized to ambiguity until now, with one exception. Bade (2011b) allows for ambiguous Anscombe-Aumann acts, and is closer to my approach in that sense. However, she

[2] "The instance in question is so entirely unique that there are no others or not a sufficient number to make it possible to tabulate enough like it to form a basis for any inference of value about any real probability in the case we are interested in", Knight (1921) p. 226.

[3] Knight (1921) writes (p. 228): "Men do form, on the basis of experience, more or less valid opinions as to their own capacity to form correct judgements, and even of the capacities of other men in this regard", and further, p. 229 therein, this is "the most important endowment for which wages are received".

assumes that ambiguity is a subjective part of players' preferences, and thus generalizes Aumann's subjective equilibria (Aumann (1974)). In opposition to her concept using subjective preferences, my approach takes purely the Von Neumann and Morgenstern point of view. Players can use ambiguity-creating devices modeled as sets of probabilities over the pure strategies, and players have ambiguity-averse preferences over these sets. Thus, I generalize the classical model to incorporate Knightian uncertainty in an objective sense, as it was recently axiomatized from a decision-theoretic point of view by Gajdos, Hayashi, Tallon, and Vergnaud (2008).

How does the possibility to use ambiguity as a strategy change the game? I propose in this thesis a model of *Ellsberg games* that differs from the classical Von Neumann and Morgenstern model merely in the one aspect that players may use (objectively ambiguous) Ellsberg urns in addition to mixed strategies; these Ellsberg urns are modeled as convex and compact sets of mixed strategies. A remarkable consequence is that players, once offered the possibility to use these Ellsberg urns, actually use them, even though they are ambiguity-averse.[4] New equilibria emerge, which are not Nash equilibria in the original game, with outcomes that are not in the support of the original Nash equilibrium. I explain this in Section 1.4 with an example of a peace negotiation taken from Greenberg (2000). This game has a unique Nash equilibrium in which war is the outcome; there is another, as I call it, *Ellsberg equilibrium*, in which peace is the outcome. Other examples show that, as in the classical case, my approach will have the most fruitful applications in games of conflict where it is in players' own interest to conceal their behavior. The approach might seem less plausible in common interest games. However, the theory in its abstract form applies there as well, of course, and I think it is useful to study the consequences in such games, too. Interestingly, the concealment of Nash equilibrium and maximin behavior that I observe in two-player Ellsberg games in Section 2.6, has its strongest realization in symmetric coordination games (Section 3.2).

Objective Ambiguity as a Strategy

An important modeling aspect of Ellsberg games is that ambiguity is present in an objective sense, and not in the subjective sense mostly encountered in applications of ambiguity. I do not use ambiguity merely in the beliefs of the players, but in the actual strategies. This has an important implication: using objective ambiguity as a strategy means that players cannot have a look into their own urn. They intentionally choose a device that

[4] In this sense, the assumption of ambiguity-aversion is parsimonious as it makes it harder for players to introduce Knightian uncertainty. They do not use Ellsberg urns for love of uncertainty.

leaves themselves uncertain about the pure strategy eventually played, once uncertainty is resolved. Players choose to know less about their own play than they could know. This may sound surprising or implausible at first. It is, however, the natural generalization of the usual mixed strategy when it is interpreted as an objective device to conceal one's behavior, as in the classical justification by Von Neumann and Morgenstern. The founders of game theory justify the use of random devices in zero-sum games by a thought experiment. If your opponent might find out your strategy, then it is optimal to conceal your behavior by using a random device. I just go one step further and allow players to use Ellsberg urns to conceal their strategy in the objective sense of Von Neumann and Morgenstern.

Notably, this modeling approach also specifies the type of strategic ambiguity we deal with in Ellsberg games. One could imagine to model strategic ambiguity differently, namely in a way that players pick a pure strategy which they then conceal and pretend not to play, only to eventually stick to it anyhow. This kind of behavior is also a type of strategic ambiguity. However, in Ellsberg games, their plan is to use a strategy that gives no clue to the opponent in case their strategy is found out; what also implies that the player himself knows as little (or as much) about his own strategy as his opponent. The former kind of strategic ambiguity is also highly interesting, but is not content of my analysis here.

Creation of Objective Ambiguity

How can players create such objective ambiguity? Indeed, some people think that it is impossible for a player to randomize by throwing a “mental” coin, how should a player then be able to use “mental” Ellsberg urns, which seem even more complex constructs than simple randomizing devices. But Ellsberg urns are not necessarily more difficult to play, on the contrary, they can make the game easier. Ellsberg urns leave room for mistakes: in classical games with only mixed strategies, not playing the exact correct mixture leads automatically to a suboptimal payoff. Not so in Ellsberg games. Ellsberg equilibrium strategies can be different sets of probability distributions which all give the same equilibrium payoff, hence, making small mistakes does not change the players' payoff.

Another interpretation of ambiguous actions in game theory is closer to the behavioral and psychological literature. Gigerenzer (2007), e.g., claims that not assessing all possibilities and information about a choice is often better, more efficient or more satisfying for a human decision maker. Using an Ellsberg strategy where the exact probability of choosing an action is not specified might be viewed as one way to model such mental efficiency.

Many times during my presentations of the working paper Riedel and Sass (2011), the question came up what Ellsberg strategies could actually be in real life. This question always led to vivid discussions, I want to present two of the suggestions made[5] as examples of Ellsberg strategies.

For the first example imagine a tax authority that is interested in minimizing tax fraud. To this end, it uses a mechanism of selective inspection of tax payers, which is too complicated to completely understand and to form a precise prior about the probability of being inspected. Even the tax authority has designed it in a way that does not allow it to form a precise prior, in order to protect its employees from bribery. This too complex mechanism is an example for an Ellsberg urn.

As a second example, think of a state minister of, e.g., environment who promotes a new law of environmental protection. He is head of a large machinery of civil servants and if he suggests a new law, the passing of this law depends (in Germany) on the decisions of Bundesrat and Bundespräsident. When the minister makes a public announcement about the characteristics of the new law, the citizens as well as himself do not know what the law will actually be like when it has gone through the institutional process of being implemented. In that sense the minister is playing with an Ellsberg strategy. This interpretation of Ellsberg strategies is close to the notion of imprecise information, see for example Giraud and Tallon (2011), which is a reason why I adapted the axiomatization of attitude towards imprecise information from Gajdos, Hayashi, Tallon, and Vergnaud (2008).

Having the above interpretations and examples in mind, I suggest the following, for the intent and purpose of this work practical, way how to think of Ellsberg urns while reading this thesis: the players have the possibility to delegate the creation of ambiguity to an independent laboratory. Players can order any kind of Ellsberg urn that they might want to use in the Ellsberg game they are facing. When it comes to playing, the players draw a colored ball from their ambiguous urn and choose their strategy according to the color drawn. Of course, in some games the same kind of commitment problems as with classical mixed strategies can arise.

Why Ellsberg Games?

The important contribution of Ellsberg games to the existing game theory literature lies, to my mind, mainly in providing a model in which people can deliberately choose how

[5] I am grateful to a seminar participant at WZB in October 2012 for suggesting the first example and to a seminar participant at Exeter University in April 2012 for suggesting the second example.

ambiguous they want their strategy to be. In fact, in other models ambiguity has been found, already some time ago, to be a most natural strategy in human interaction. For example, in strategic information transmission models such as suggested by Crawford and Sobel (1982), ambiguity (or vagueness, or concealment) was found to be the prevalent strategy when conflict of interest is present. The partition equilibria in their model, i.e., sending the same message for an interval of types, are in interpretation closely connected to Ellsberg equilibrium. I want to draw attention to the fact that also in games ambiguity is a most natural strategy.

With my thesis I seek to set up such a model which allows the strategic use of ambiguity in games, and explore its basic properties and implications. Of course, there are many aspects which I do not cover in the present text. Furthermore, certain facets of strategic ambiguity in games cannot be explained with the way I define Ellsberg games. However, the modeling choices I make seem to me a good starting point in oder to explore further how ambiguity can be used as a strategy in interactive contexts.

Topics and Main Results of this Work

My doctoral thesis consists of six chapters, Chapter 1 - 5 on strategic use of ambiguity in normal form games and Chapter 6 on dynamic Ellsberg games. I summarize the main results of each chapter in the following.

In Chapter 1, I first discuss the conceptual foundations of my approach. To this end, I compare the two interpretations of mixed strategies as objective random devices (Von Neumann and Morgenstern) or as beliefs about other players' pure actions (the Bayesian view). I discuss in Section 1.2 how the ambiguity-averse players in Ellsberg games evaluate the objective ambiguity inherent in the Ellsberg strategies. Section 1.3 deals with the definition and first characterization of Ellsberg games in normal form. I define the theoretical framework for Ellsberg games in Section 1.3.2, followed by the introduction of Ellsberg equilibrium. Subsequently in Section 1.3.3, I show the equivalence of Ellsberg equilibrium to a reduced form Ellsberg equilibrium, where the state space is represented by the set of pure strategies. This leads to a simplification of Ellsberg equilibrium which facilitates the calculation of equilibria. The proposition on equivalence reads

Proposition 1.3. *Ellsberg equilibrium and reduced form Ellsberg equilibrium are equivalent in the sense that every Ellsberg equilibrium* $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ *induces a payoff-equivalent reduced form Ellsberg equilibrium on* $\Omega^* = S$*; and every reduced form Ellsberg* *equilibrium $\mathcal{Q}^*$ is an Ellsberg equilibrium $((S,\mathcal{F},\mathcal{Q}^*),f^*)$ with f^* the embedding of S in ΔS.*

Most importantly, Ellsberg equilibria generalize Nash equilibria, hence Ellsberg equilibria exist. This is formulated in the following theorem in Section 1.3.4.

Theorem 1.4. *Let $G = \langle N,(S_i),(u_i)\rangle$ be a normal form game. Then a mixed strategy profile $(P_1^*,\ldots,P_n^*)$ of G is a Nash equilibrium of G if and only if the corresponding profile of singletons $(\mathcal{P}_1,\ldots,\mathcal{P}_n)$ with $\mathcal{P}_i = \{\delta_{P_i^*}\}$ is an Ellsberg equilibrium. In particular, Ellsberg equilibria exist when the strategy sets S_i are finite.*

I prove a Principle of Indifference in Distributions in Section 1.3.5. It shows that a player is indifferent between all the mixed strategies which are part of his Ellsberg equilibrium strategy.

Theorem 1.14 (Principle of Indifference in Distributions)**.** *Let $(\mathcal{P}_1^*,\ldots,\mathcal{P}_n^*)$ be an Ellsberg equilibrium of a normal form game $G = \langle N,(S_i),(u_i)\rangle$. Then for all $P_i \in \mathcal{P}_i^*$,*

$$\min_{P_{-i}\in\mathcal{P}_{-i}^*} u_i(P_i,P_{-i}) = c \quad \text{for some } c \in \mathbb{R}\,.$$

In Section 1.3.6, I show that strictly dominated strategies are never used in Ellsberg equilibrium. Precisely,

Proposition 1.16. *Any Ellsberg equilibrium strategy profile $(\mathcal{P}_i^*)_{i\in N}$ must put weight only on strategies that are not strictly dominated.*

Section 1.4 provides the leading example why Ellsberg games are an interesting object to study, namely the peace negotiation example from Greenberg (2000) which I mentioned earlier. The section closes with a discussion of the existing literature and the relation of the different approaches to my model in Section 1.5.

Chapter 2 provides insight into *immunization against strategic ambiguity.* In normal form Ellsberg games players often have a strategy available that immunizes them against any ambiguity played by their opponents. The hedging effect of ambiguity aversion leads to this phenomenon. In this chapter I look at these immunization strategies more closely. I start by giving an example in Section 2.1 and then, after defining immunization strategies, calculate the immunization strategies of a large class of 2×2 games in Section 2.3. I then provide some examples which show that in certain games players immunize themselves against ambiguity by playing their maximin strategy. To generalize this observation, I prove in Section 2.5 under which circumstances in general two-player games the immunization strategy is the maximin strategy of the players. The result is formulated in the following theorem

Theorem 2.8. *Let G be a square two-person normal form game with a completely mixed Nash equilibrium (P^*, Q^*). If player 1 (2) has an immunization strategy $\bar{P}$ $(\bar{Q})$ in G, then $\bar{P}$ $(\bar{Q})$ is a maximin strategy of player 1 (2).*

The last section, Section 2.6, uses the results of the chapter to characterize the role of immunization strategies in Ellsberg equilibria. I explain that in many 2×2 games, Ellsberg equilibria are bounded by the immunization and the Nash equilibrium strategy.

In Chapter 3, I present a number of examples and properties of Ellsberg equilibria of normal form games with two players. I start with an explanation on how to compute Ellsberg equilibria of two-person 2×2 games. Subsequently, in Sections 3.2-3.5, I look at different classes of games: coordination games, conflict games and zero-sum games. I analyze examples and derive general results on the Ellsberg equilibria of these classes of normal form games, supplemented by tables (summarized in Table 3.1) for the Ellsberg equilibria of general coordination and conflict games. Following this, I prove in Section 3.5 that two-person zero-sum games are value preserving.

Theorem 3.18. *Let G be a two-person zero-sum game. Then for all $i \in \{1, 2\}$ and every Ellsberg equilibrium $(\mathcal{P}^*, \mathcal{Q}^*)$ we have*

$$\max_{\mathcal{P}_i \subseteq \Delta S_i} \min_{\mathcal{P}_{-i} \subseteq \Delta S_{-i}} U_i(\mathcal{P}, \mathcal{Q}) = \min_{\mathcal{P}_{-i} \subseteq \Delta S_{-i}} \max_{\mathcal{P}_i \subseteq \Delta S_i} U_i(\mathcal{P}, \mathcal{Q}) = U_i(\mathcal{P}^*, \mathcal{Q}^*) .$$

In Section 3.6, I calculate the Ellsberg equilibria of some classic 2×2 games such as, e.g., Hawk and Dove and the Prisoners' Dilemma. I explain with an example by Myerson, that also games with linear (as opposed to piecewise linear) payoff functions can have proper Ellsberg equilibria. I close the chapter in Section 3.7 with an analysis of a special class of 3×3 games. I consider a slightly modified version of Rock Scissors Paper, and with the help of the experience with 2×2 games I succeed in calculating the Ellsberg equilibria of these games. For a better understanding of how the equilibria of modified circulant games can be determined, I provide a geometric analysis of the equilibrium strategies in the two-dimensional probability simplex.

In Chapter 4, I present some thoughts on a general interpretation of Ellsberg equilibria, and I provide a classification of when ambiguity is an option in two-player Ellsberg equilibria. I am on the one hand interested in human behavior in two-person games and its connection to Ellsberg equilibria. In an experiment with a modified version of Matching Pennies similar to mine, Goeree and Holt (2001) observe that Nash equilibrium prediction was frequently violated; interestingly, Ellsberg equilibria can explain this behavior. In Section 4.2 on the other hand, I classify Ellsberg equilibria that are supported by strategies which are not in the support of any Nash equilibrium of the game. These Ellsberg equilibria arise frequently in games with weakly dominated strategies and have an interesting behavioral interpretation. Finally, in Section 4.3, I characterize in which circumstances and how ambiguity is used in Ellsberg equilibria in two-person games. This analysis provides an answer to the question in which 2×2 games proper Ellsberg equilibria exist. The result is summarized by the following theorem.

Theorem 4.2. *Let G be a two-person normal form game with a unique completely mixed Nash equilibrium (P^*, Q^*). There exist Ellsberg equilibria with unilateral full ambiguity if and only if either P^* or Q^* is maximin. The Ellsberg equilibria are of the following form:*

$$(P^*, [Q_0, Q_1]), \text{ where } Q_0 < Q^* < Q_1, \text{ if } P^* = M_1 ,$$
$$\text{and } ([P_0, P_1], Q^*), \text{ where } P_0 < P^* < P_1, \text{ if } Q^* = M_2 .$$

If P^* is also an immunization strategy for player 1, then the profiles $(P^*, \mathcal{Q})$ with $Q^* \in \mathcal{Q}$ form an Ellsberg equilibrium (and similar for player 2 if Q^* is an immunization strategy).

Chapter 5 analyzes the relation of Ellsberg equilibria with subjective equilibria defined by Aumann (1974). In the preceding chapters I presented the basic properties of Ellsberg equilibria and analyzed how players behave when one allows them to use objective ambiguity in two-player games. Naturally, I want to characterize what can happen in games with more than two players. To tackle this problem, I first investigate the relation of Ellsberg equilibrium to the concept of subjective equilibrium. Since there exists no other solution concept that allows players to use sets of probabilities as their strategy, the question of comparison has to be answered with regard to the attainability of certain outcomes. The objective is to find out under which conditions subjective equilibria can (or cannot) have the same support as Ellsberg equilibria.

I start in Section 5.1 with an example of a three-player game, a mediated Prisoners' Dilemma, where an Ellsberg equilibrium exists that attains an outcome that is not in the support of any Nash equilibria of the game, namely cooperation of the two prisoners. This suggests that in very simple games with more than two players the prediction of Ellsberg equilibrium can be quite different from the Nash equilibrium prediction. Subsequently, in

Section 5.2.1, I define subjective equilibrium in the spirit of Aumann (1974) and Hallin (1976) and subjective beliefs equilibrium like Lo (1996). I show in Section 5.2.2 that every Ellsberg equilibrium contains a subjective equilibrium and a subjective beliefs equilibrium. In two-player games, subjective beliefs equilibria have the same support as Nash equilibria, but this changes for three players. This explains that in Ellsberg games with more than two players outcomes outside the Nash equilibrium support can be attained; this is treated in Section 5.2.3. To this end I prove the following theorem.

Theorem 5.8. *If* $(\mathcal{P}_1^*, \ldots, \mathcal{P}_n^*)$ *is an Ellsberg equilibrium, then there exist* $\mu_{-i}^i \in \mathcal{P}_{-i}^*$ *such that* $(\mu_{-1}^1, \ldots, \mu_{-n}^n)$ *is a subjective beliefs equilibrium and*

$$\arg\max_{P_i \in \Delta S_i} U_i(P_i, \mathcal{P}_{-i}^*) \subseteq \arg\max_{P_i \in \Delta S_i} u_i(P_i, \mu_{-i}^i)\,.$$

Finally, in Chapter 6, the notion of Ellsberg game and Ellsberg equilibrium is extended to dynamic games. It is straightforward that Ellsberg urns can also be used as strategies in dynamic games. In this chapter I develop the theoretical framework to analyze such extensive form Ellsberg games. As in classic extensive form theory, players can use an urn to create ambiguity over their set of pure strategies, or they can place an Ellsberg urn at every decision point in the process of the game. In the first section of the chapter, Section 6.1, I thus extend the notions of mixed strategy and behavioral strategy to extensive form Ellsberg games.

Considering dynamic games leads directly to the question of dynamic consistency of the players' preferences. I discuss this in Section 6.2 and develop a formalism to translate the notion of rectangularity by Epstein and Schneider (2003b) to extensive form Ellsberg games. This property of Ellsberg strategies is then used to prove a version of Kuhn's Theorem (Kuhn (1953)) for extensive form Ellsberg games. My theorem reads

Theorem 6.7. *In an extensive form Ellsberg game* (F, U) *with* $F = (\mathcal{T}, C)$ *satisfying perfect recall, every rectangular Ellsberg strategy profile* $\mathcal{P}$ *induces an Ellsberg behavior strategy profile* $\Theta^{\mathcal{P}}$ *via prior-by-prior updating; every Ellsberg behavior strategy profile* Θ *induces a rectangular Ellsberg strategy profile* $\mathcal{P}^{\Theta}$ *such that prior-by-prior updating of* $\mathcal{P}^{\Theta}$ *yields* Θ*. The induced strategy profiles are payoff-equivalent, i.e.,*

$$U_i(\mathcal{P}) = U_i(\Theta^{\mathcal{P}}) \;\; \textit{and} \;\; U_i(\Theta) = U_i(\mathcal{P}^{\Theta})\,.$$

I present a simple two-player example for intuition, and I provide a picture of the rectangular Ellsberg strategy derived in the example in the two-dimensional probability simplex.

Thereafter in Section 6.4, I define the notion of Ellsberg equilibrium in extensive form Ellsberg games. With the equivalence established in Theorem 6.7, extensive form Ellsberg games can be analyzed by considering Ellsberg behavior strategies only. However, a thorough analysis of Ellsberg equilibria in extensive form Ellsberg games lies beyond the scope of this chapter. Finally, in Section 6.5, I briefly comment on Ellsberg equilibria in extensive form Ellsberg games and compare Ellsberg equilibria to some other extensive form solution concepts.

Introduction en français

Il n'est pas difficile d'imaginer des situations de conflit dans lesquelles on peut bénéficier à être ambiguë. "Many different strategies are used to orient toward conflicting interactional goals; some examples include avoiding interaction altogether, remaining silent, or changing the topic", explique Eric Eisenberg dans son fameux article "Ambiguity as strategy in organizational communication" (Eisenberg (1984)). Il remarque que l'emploi de l'ambiguïté est la clé d'une communication réussie dans la présence des conflits d'intérêts.

Dans ma thèse, j'introduis l'emploi stratégique de l'ambiguïté dans les jeux. Même si la théorie des jeux s'était développée afin de modéliser des conflits d'intérêts, la possibilité d'utiliser, avec intention, l'ambiguïté comme stratégie n'est pas prise en compte. Pour réaliser cela, je me reviens aux fondements de la théorie des jeux. Von Neumann and Morgenstern (1953) ont introduit des stratégies mixtes comme *random devices*[6], utilisées pour dissimuler le comportement du joueur. Je reprends cette interprétation et propose une généralisation. En résumé, je permets aux joueurs d'utiliser des urnes d'Ellsberg en plus des instruments probabilistes comme la roulette ou un dé. Précisément, un joueur peut prendre pour base de son action la couleur d'une boule prise d'une urne avec cent boules rouge et bleu, sur lesquelles il a comme seule information qu'il y en a entre trente et cinquante boules rouges. Telles urnes sont objectivement ambiguës par construction; les joueurs peuvent donc créer de l'ambiguïté. Le progrès récent de la théorie de la décision, motivé par les expériences d'Ellsberg et la distinction du risque et de l'incertitude de Knight, permet de représenter mathématiquement ce comportement stratégique.

Risque et Ambiguïté

Pour toute décision que nous prenons, nous considérons ses implications possibles dans l'avenir. Nous inférons comment notre entourage réagira, et quelles conséquences notre décision pourrait avoir dans différents états du monde. Knight (1921) a observé que tous ces événements ultérieurs ne peuvent pas être traités de la même manière. Tandis qu'il est facile de calculer une distribution de probabilité pour l'événement "rouge" dans les jeux de roulette, il est presque impossible d'estimer des choses plus compliquées, comme

[6] Une sorte de *dispositif aléatoire*. Dans le texte j'utilise l'expression anglaise.

le cours d'une action ou encore si la France gagnera la prochaine coupe du monde, avec une distribution de probabilité précise.[7] Même si le but de Knight était principalement d'attirer l'attention sur le rôle de l'expérience pour pouvoir prendre des bonnes décisions dans des situations complexes[8], sa distinction entre risque et ambiguïté a été ingérée dans des modèles de décision, et avec grand succès. Une conquête principale de la théorie de l'ambiguïté sont des modèles de décision qui dénouent le paradoxe d'Ellsberg (Ellsberg (1961)). Les modèles les plus connus sont jusqu'à présent le "multiple-prior expected utility" de Gilboa and Schmeidler (1989), et "Choquet expected utility" de Schmeidler (1989) et "incomplete preferences" de Bewley (2002). Beaucoup d'autres auteurs ont contribué à ces modèles, ils les ont affinés et étendus, ou ils les ont appliqués à des problèmes économiques ou comportementaux. Par exemple, des applications comprennent l'explication des contrats incomplets (Mukerji (1998)), ambiguïté dans les marchés de capitaux (Rigotti and Shannon (2005)), asset pricing sous ambiguïté (Epstein and Schneider (2008)), optimal stopping avec ambiguïté (Riedel (2009)), stratégies contre l'escroquerie à l'assurance (Lang and Wambach (2010)), et élection stratégique (Ellis (2011)). Dans Luo and Ma (1999) et Mukerji and Tallon (2004) se trouve un aperçu des applications, et dans Etner, Jeleva, and Tallon (2012) un survey récent sur la théorie de la décision sous ambiguïté. Binmore (2008) propose un historique approfondi des aspects philosophes et mathématiques de la probabilité et leur influence sur la théorie de la décision.

Ambiguïté dans les Jeux

Ambiguïté émerge naturellement dans des situations stratégiques. Un être humain est un adversaire complexe et en conséquence l'interaction stratégique peut être modélisé de manière productive avec l'usage d'incertitude Knightienne. La littérature qui existe sur l'ambiguïté dans les jeux se divise en deux parties, s'appuyant sur les deux interprétations des stratégies mixtes qui sont présentes dans le domaine. La vision Bayésienne des stratégies mixtes a été généralisé commençant avec Dow and Werlang (1994), Lo (1996), et Marinacci (2000), et poursuivi par beaucoup d'autres auteurs. Là, l'interprétation des stratégies mixtes comme croyance sur les stratégies pures des adversaire est généralisée aux croyances incertaines, modélisées sur la base des modèles de décision avec incertitude

[7] "The instance in question is so entirely unique that there are no others or not a sufficient number to make it possible to tabulate enough like it to form a basis for any inference of value about any real probability in the case we are interested in", Knight (1921) p. 226.

[8] Knight (1921) écrit (p. 228): "Men do form, on the basis of experience, more or less valid opinions as to their own capacity to form correct judgements, and even of the capacities of other men in this regard", et continue, p. 229 du même ouvrage, cela est "the most important endowment for which wages are received".

Knightienne (Schmeidler (1989), Gilboa and Schmeidler (1989)). L'autre interprétation, où une stratégie mixte est un random device objective, n'a pas été généralisée à l'ambiguïté, autant que je sache, avec une exception. Bade (2011b) propose des Anscombe-Aumann acts ambigus, et dans ce sens est plus près de mon approche. Cependant, elle suppose que l'ambiguïté fait partie des préférences subjectives du joueur, et alors généralise les équilibres subjective de Aumann (Aumann (1974)). Contrairement à elle utilisant des préférences subjectives, mon approche généralise purement l'idée de Von Neumann et Morgenstern. Les joueurs peuvent utiliser des instruments qui créent de l'ambiguïté, modélisés comme ensemble de distributions de probabilité sur des stratégies pures, et les joueurs ont des préférences adverses à l'ambiguïté par rapport à ces stratégies. Ainsi je généralise le modèle classique pour intégrer l'incertitude Knightienne dans un sens objective, comme c'était axiomatisé dans un travail récent par Gajdos, Hayashi, Tallon, and Vergnaud (2008).

Comment l'utilisation stratégique de l'ambiguïté influence-t-elle le jeu? Dans cette thèse, je propose un modèle de *jeux d'Ellsberg*, qui se distingue du modèle classique de Von Neumann et Morgenstern sur un unique aspect: les joueurs peuvent utiliser des urnes d'Ellsberg (objectivement ambiguës) en plus des stratégies mixtes; ces urnes d'Ellsberg sont modélisées comme ensembles convexes et compacts de stratégies mixtes. Curieusement, les joueurs utilisent ces urnes d'Ellsberg à l'équilibre, bien qu'ils soient averses à l'ambiguïté.[9] De nouveaux équilibres, qui ne sont pas des équilibres de Nash dans le jeu original, émergent. En plus, les réalisations ne font pas forcément parties du support des équilibres de Nash. J'explique ce phénomène dans la Section 1.4 avec un exemple sur les négociations de paix de Greenberg (2000). Ce jeu a un unique équilibre de Nash dans lequel la seule réalisation est la guerre; il existe un autre, que j'appelle *l'équilibre d'Ellsberg*, dans lequel la paix est réalisée. Autres exemples indiquent que, comme dans le cadre classique, mon approche a des applications très intéressantes dans des jeux de conflits, quand les joueurs sont intéressés à dissimuler leur comportement. L'approche peut à première vue sembler moins convaincant dans des jeux à intérêt commun. Cependant les conséquences sont parfois remarquables. Par exemple, le raccordement d'équilibre de Nash et de comportement maximin que j'observe dans des jeux d'Ellsberg à deux joueurs dans la Section 2.6 s'accentue plus fortement dans des jeux de coordination symétrique.

[9] Dans ce sens, l'hypothèse d'aversion à l'ambiguïté est parcimonieuse, car elle rend l'introduction d'incertitude Knightienne plus difficile pour les joueurs. Ils n'utilisent pas des urnes d'Ellsberg pour l'amour de l'incertitude.

Ambiguïté Objective Comme Stratégie

Un aspect important des jeux d'Ellsberg est la présence de l'ambiguïté dans un sens objectif, et pas dans le sens subjectif souvent trouvé dans des application d'ambiguïté. Je n'utilise pas l'ambiguïté seulement dans les croyances des joueurs, mais directement dans leurs stratégies. Cela a une implication importante: utilisant l'ambiguïté objective comme stratégie veut dire que les joueur ne peuvent pas voir la composition précise de leur propre urne. Avec intention, ils choisissent un instrument qui les laissent dans le flou de la stratégie finalement réalisée. Les joueurs choisissent de savoir moins de leur propre stratégie qu'il le pourrait. Cela semble surprenant ou peu probable. Cependant, c'est la généralisation naturelle de la stratégie mixte interprétée comme instrument objective de dissimulation, comme elle était introduite par Von Neumann et Morgenstern. Les fondateurs de la théorie des jeux justifie l'usage de random devices dans des jeux à somme nulle avec une expérience de pensée: si ton adversaire peut dépister ta stratégie, il est optimal de dissimuler ton comportement en utilisant un random device. Je surenchéri et donne aux joueurs la possibilité d'utiliser des urnes d'Ellsberg pour dissimuler leur stratégie dans le sens objective de Von Neumann et Morgenstern.

Remarquablement, cette approche de modélisation spécifie aussi le genre d'ambiguïté stratégique rencontré dans les jeux d'Ellsberg. On peut imaginer de modéliser l'ambiguïté stratégique autrement, soit que les joueurs choisissent une stratégie pure, puis font semblant de ne pas vouloir la jouer, avec pour seule fin de la jouer de toute façon. Cependant, dans des jeux d'Ellsberg, les joueurs planifient d'utiliser une stratégie qui ne peut pas être dépistée; ce qui implique qu'eux-mêmes savent aussi peu sur leur stratégie que leurs adversaires. Le premier genre d'ambiguïté stratégique est aussi très intéressant, mais ne fait pas partie de mon analyse.

Création d'Ambiguïté Objective

Comment les joueurs peuvent créer une telle ambiguïté objective? Certes, il y a des gens qui estiment impossible pour un joueur de randomiser en jetant une pièce "imaginaire", comment pourrait-il lui être possible d'utiliser des urnes d'Ellsberg "imaginaire", qui semblent être plus complexes que des random devices. Mais les urnes d'Ellsberg ne sont pas plus difficile à utiliser, en contraire, ils peuvent faciliter le jeu. Les urnes d'Ellsberg donnent de l'espace à des fautes: dans les jeux classiques avec seulement les stratégies mixtes, ne pas jouer la distribution de probabilité exactement correcte mène immédiatement a une réalisation pas optimale. Ce n'est pas le cas dans des jeux d'Ellsberg. Les stratégies d'équilibres d'Ellsberg peuvent être des ensembles différents des distributions de probabilité qui tous donne le même paiement d'équilibre, et alors des petites fautes des joueurs ne changent pas le résultat.

Une autre interprétation des actions ambiguës dans la théorie des jeux est plus près de la littérature comportementale et psychologique. Gigerenzer (2007), par exemple, affirme que ne pas comprendre toutes les possibilités et toute information d'un choix est souvent mieux, plus efficace ou plus satisfaisant pour un décideur humain. Utiliser une stratégie d'Ellsberg dans laquelle la probabilité exacte de choisir une action n'est pas spécifiée peut être regardé comme un moyen de modéliser un telle efficacité mentale.

Souvent pendant mes présentations du document de travail Riedel and Sass (2011), on m'a demandé un exemple d'une stratégie d'Ellsberg dans la vie réelle. Cette question a toujours menée a des vives discussions, et je voudrais présenter deux suggestions proposées[10] comme exemples des stratégies d'Ellsberg.

Le premier exemple se passe dans un bureaux des contributions qui s'intéresse à minimiser la fraude fiscale. A ce fin on utilise un mécanisme d'inspection sélective des contribuables, qui est trop compliqué pour le comprendre complètement et pour estimer une probabilité exacte d'être contrôlé. Même le bureaux des contributions a construit le mécanisme tel qu'il est impossible d'estimer une probabilité exacte pour les employés, pour les protéger de corruption. Ce mécanisme complexe est un exemple d'une stratégie d'Ellsberg.

Pour le seconde exemple, imaginez un ministre d'état de l'environnement, qui fait la publicité pour une nouvelle loi de protection environnementale. Il est le chef d'une machinerie énorme de fonctionnaires et, quand il propose une nouvelle loi, l'adoption dépend (en Allemagne) de la décision de Bundesrat et du président. Quand le ministre annonce publiquement les détails de la loi, ni les citoyens ni lui-même savent exactement, comment la loi sera modifiée après avoir passée tout le processus des institutions. Dans ce sens le ministre joue une stratégie d'Ellsberg. Cette interprétation de stratégie d'Ellsberg est connexe à l'idée d'information imprécise, comme dans Giraud and Tallon (2011), pour cela j'utilise l'axiomatisation d'attitude envers l'information imprécise de Gajdos, Hayashi, Tallon, and Vergnaud (2008).

Avec les interprétations et exemples à disposition, je propose la façon suivante d'imaginer des stratégies d'Ellsberg à la lecture de cette thèse: les joueurs ont la possibilité de déléguer la création de l'ambiguïté à un laboratoire indépendant. Les joueurs peuvent commander toute sorte d'urne d'Ellsberg qu'ils voudraient utiliser dans le jeu. Quand le jeu est

[10] Je remercie un intervenant de séminaire à WZB en octobre 2012 d'avoir proposé le premier exemple, et un intervenant de séminaire à Exeter University en avril 2012 d'avoir proposé le deuxième.

réalisé, les joueurs prennent une boule colorée de leur urne ambiguë et choisissent leur stratégie conforme à la couleur réalisée. Bien sûr, dans quelques jeux le même problème d'engagement peut émerger qu'avec les stratégies mixtes classiques.

Pourquoi des Jeux d'Ellsberg?

La contribution importante des jeux d'Ellsberg à la théorie des jeux déjà existante se trouve, à mon avis, principalement dans la provision d'un modèle dans lequel les joueurs peuvent choisir, avec intention, le dégrée d'ambiguïté de leur stratégie. En effet, dans d'autres modèles classiques, l'ambiguïté a été découverte être une stratégie totalement naturelle dans l'interaction humaine. Par exemple, dans les modèles de transmission stratégique d'information proposés par Crawford and Sobel (1982), l'ambiguïté (ou le vague, ou la dissimulation) s'était trouvée être une stratégie dominante en présence des conflits d'intérêts. Les équilibres partitionels envoient le même message pour un ensemble de types et ont donc une interprétation connexe aux équilibres d'Ellsberg. De même manière, dans les jeux, l'ambiguïté est une stratégie naturelle.

Dans ma thèse, je propose un tel cadre d'analyse permettant d'utiliser l'incertitude Knightienne de manière stratégique, et j'explore les propriétés et implications de ce changement. Bien sûr, il y a beaucoup d'aspects que je ne traite pas. De plus, certaines facettes de l'ambiguïté stratégique ne peux pas être compris avec ma façon de définir les jeux d'Ellsberg. Cependant, mes choix de modélisation semblent un bon point de départ pour explorer plus profondément comment l'ambiguïté est utilisée stratégiquement dans des contextes interactives.

Résultats Principaux

Ma thèse contient six chapitres, les chapitres 1 à 5 traitent de l'utilisation stratégique de l'ambiguïté dans des jeux sous forme normale et le chapitre 6 étudie les jeux sous forme extensive. Je résume les résultats principaux de chaque chapitre comme suit.

Dans le Chapitre 1, je commence avec la discussion des fondations conceptuelles de mon approche. Pour cela je fais la comparaison des deux interprétations des stratégies mixtes comme random devices objectives (Von Neumann et Morgenstern) et comme croyances sur les stratégies pures des autres joueurs (vision Bayésienne). Dans la Section 1.2 j'analyse comment les joueurs averses à l'ambiguïté évaluent l'ambiguité objective dans les stratégies d'Ellsberg. La Section 1.3 traite la définition et une première caractérisation des jeux d'Ellsberg sous forme normale. Je définis le cadre d'analyse des jeux d'Ellsberg dans la Section 1.3.2, suivi par l'introduction des équilibres d'Ellsberg. Ensuite, dans la Section 1.3.3, je présente l'équivalence de l'équilibre d'Ellsberg et sa forme réduite, où l'espace des états est représenté par l'ensemble des stratégies pures. Cela mène à une simplification des équilibres d'Ellsberg qui facilite le calcule des équilibres. La proposition sur l'équivalence est la suivante.

Proposition 1.3. *Les équilibres d'Ellsberg (de la forme normale) et ceux de la forme réduite sont équivalents dans le sens que chaque équilibre d'Ellsberg* $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ *induit un équilibre d'Ellsberg en forme réduite avec le même paiement sur* $\Omega^* = S$*; et chaque équilibre d'Ellsberg en forme réduite* $\mathcal{Q}^*$ *est un équilibre d'Ellsberg* $((S, \mathcal{F}, \mathcal{Q}^*), f^*)$ *où* f^* *est l'encadrement de* S *dans* ΔS*.*

Particulièrement, les équilibres d'Ellsberg généralisent les équilibres de Nash et alors, ils existent. Cela est formulé dans le théorème suivant dans la Section 1.3.4.

Theorem 1.4. *Soit* $G = \langle N, (S_i), (u_i) \rangle$ *un jeu sous forme normale. Un profil de stratégies mixtes* $(P_1^*, \ldots, P_n^*)$ *de* G *est un équilibre de Nash de* G *si et seulement si le profil correspondant de singletons* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ *avec* $\mathcal{P}_i = \{\delta_{P_i^*}\}$ *est un équilibre d'Ellsberg. En conséquence, les équilibres d'Ellsberg existent quand les ensembles de stratégies* S_i *sont finis.*

Je prouve un Principe d'Indifférence en Distributions dans la Section 1.3.5. Cela montre qu'un joueur est indifférent entre toutes les stratégies mixtes qui font partie de sa stratégie d'équilibre d'Ellsberg.

Theorem 1.14 (Principe d'Indifférence en Distributions)**.** *Soit* $(\mathcal{P}_1^*, \ldots, \mathcal{P}_n^*)$ *un équilibre d'Ellsberg d'un jeu sous forme normale* $G = \langle N, (S_i), (u_i) \rangle$*. Pour tout* $P_i \in \mathcal{P}_i^*$*,*

$$\min_{P_{-i} \in \mathcal{P}_{-i}^*} u_i(P_i, P_{-i}) = c \quad \text{pour un } c \in \mathbb{R}\,.$$

Dans la Section 1.3.6, je prouve que les stratégies strictement dominées ne sont jamais utilisées dans un équilibre d'Ellsberg. Précisément,

Proposition 1.16. *Tout profil de stratégies d'équilibre d'Ellsberg* $(\mathcal{P}_i^*)_{i \in N}$ *ne doit mettre un pois positive que sur des stratégies qui ne sont pas strictement dominées.*

La Section 1.4 présente l'exemple principale d'intérêt à étudier les jeux d'Ellsberg, c'est l'exemple des négociations de paix de Greenberg (2000) que j'ai mentionné ci-dessus. La section finit avec la discussion de littérature proche et le rapport entre les différentes approches et mon modèle dans la Section 1.5.

Le Chapitre 2 analyse le concept d'*immunisation contre l'ambiguïté stratégique.* Dans les jeux d'Ellsberg sous forme normale, les joueurs ont souvent une stratégie disponible qui les immunise contre toute l'ambiguïté utilisée par leurs adversaires. L'effet "hedging" de l'aversion contre l'ambiguïté mène à ce phénomène. Dans ce chapitre, j'analyse ces stratégies d'immunisation de plus près. Je commence avec un exemple dans la Section 2.1 et puis, après avoir défini les stratégies d'immunisation, je calcule les stratégies d'immunisation d'une classe large de jeux 2×2 (Section 2.3). Ensuite, je présente des exemples qui montrent que dans certains jeux, les joueurs s'immunisent contre l'ambiguïté en jouant leur stratégie maximin. Pour généraliser cette observation, je prouve dans la Section 2.5 sous quelles circonstances dans des jeux à deux joueurs, la stratégie d'immunisation est exactement la stratégie maximin des joueurs. Le résultat se trouve dans le théorème suivant.

Theorem 2.8. *Soit G un jeu sous forme normale avec deux joueurs et un équilibre de Nash complètement mixte (P^*, Q^*). Si joueur 1 (resp. 2) a une stratégie d'immunisation $\bar{P}$ $(\bar{Q})$ dans G, $\bar{P}$ $(\bar{Q})$ est une stratégie maximin du joueur 1 (resp. 2).*

Dans la dernière partie de ce chapitre, Section 2.6, j'utilise les résultats obtenus pour caractériser le rôle des stratégies d'immunisation dans les équilibres d'Ellsberg. J'explique que dans beaucoup de jeux 2×2, les équilibres d'Ellsberg sont restreints par la stratégie d'immunisation et la stratégie d'équilibre de Nash.

Dans Chapitre 3, je présente des exemples et propriétés d'équilibres d'Ellsberg dans des jeux sous forme normale avec deux joueurs. Je commence par expliquer comment calculer les équilibre d'Ellsberg des jeux 2×2. Dans ce qui suit, dans les Sections 3.2-3.5, j'analyse des classes différentes de jeux: les jeux de coordination, de conflit ou à somme nulle. Je calcule des exemples et je déduis aussi des résultats généraux sur les équilibres d'Ellsberg dans certains jeux sous forme normale (voir le résumé dans Table 3.1 pour les équilibres d'Ellsberg dans des jeux de coordination et de conflit). Après cela, je prouve dans la Section 3.5 que les jeux à somme nulle avec deux joueurs préservent la valeur.

Theorem 3.18. *Soit G un jeu à somme nulle avec deux joueurs. Pour toute $i \in \{1, 2\}$ et tout équilibre d'Ellsberg $(\mathcal{P}^*, \mathcal{Q}^*)$,*

$$\max_{\mathcal{P}_i \subseteq \Delta S_i} \min_{\mathcal{P}_{-i} \subseteq \Delta S_{-i}} U_i(\mathcal{P}, \mathcal{Q}) = \min_{\mathcal{P}_{-i} \subseteq \Delta S_{-i}} \max_{\mathcal{P}_i \subseteq \Delta S_i} U_i(\mathcal{P}, \mathcal{Q}) = U_i(\mathcal{P}^*, \mathcal{Q}^*) .$$

Dans la Section 3.6, je calcule les équilibres d'Ellsberg d'un nombre de jeux classiques 2×2 comme, par exemple, Hawk and Dove, et Prisoners' Dilemma. J'explique avec un exemple de Myerson, que même des jeux ayant des fonctions de paiement linéaires (à l'inverse de linéarité par morceaux) peuvent avoir des propres équilibres d'Ellsberg. Je finis le chapitre dans la Section 3.7 avec une classe spéciale de jeux 3×3. J'analyse une version modifiée de Pierre Papier Ciseaux, et avec la connaissance des résultats pour les jeux 2×2 j'arrive à calculer les équilibre d'Ellsberg dans ces jeux. Pour une meilleure compréhension de comment calculer les équilibres d'Ellsberg dans de tels jeux circulants, je montre une analyse géométrique des stratégies d'équilibres dans le simplexe de dimension deux.

Dans Chapitre 4, je discute l'interprétation générale d'équilibres d'Ellsberg, et je classifie les jeux à deux joueurs dans lesquels l'ambiguïté est une option en équilibre d'Ellsberg. D'une côté je m'intéresse au comportement humain dans des jeux à deux joueurs et sa connexion avec des équilibres d'Ellsberg. Goeree and Holt (2001) observent dans une expérience avec une version modifiée de Matching Pennies similaire que celle que je considère, que la prédiction de Nash est fréquemment violée; remarquablement, les équilibres d'Ellsberg peuvent expliquer ce comportement. Dans la Section 4.2, de l'autre côté, je présente une classification des équilibres d'Ellsberg qui ont un support qui ne fait pas partie du support de l'ensemble des stratégies d'équilibres de Nash du jeu. Tels équilibres d'Ellsberg émergent fréquemment dans des jeux avec des stratégies faiblement dominées et ont une interprétation comportementale intéressante. Finalement, dans la Section 4.3, je caractérise les circonstances dans lesquels l'ambiguïté est utilisée dans les équilibres d'Ellsberg dans des jeux à deux joueurs. Cet analyse présente une réponse à la question dans quels jeux 2×2 existent des propres équilibres d'Ellsberg. Le résultat est résumé par le théorème suivant.

Theorem 4.2. *Soit G un jeu sous forme normale à deux joueurs avec unique équilibre de Nash en stratégies complètement mixtes (P^*, Q^*). Il existe un équilibre d'Ellsberg avec* unilateral full ambiguity[11]*, si et seulement si ou P^* ou Q^* est maximin. Les équilibres d'Ellsberg ont dans ce cas la forme suivante:*

$$(P^*, [Q_0, Q_1]), \ \textit{où}\ Q_0 < Q^* < Q_1, \ \textit{si}\ P^* = M_1 \,,$$

$$\textit{et}\ \ ([P_0, P_1]\,, Q^*), \ \textit{où}\ P_0 < P^* < P_1, \ \textit{si}\ Q^* = M_2 \,.$$

Si P^* est aussi une stratégie d'immunisation pour joueur 1, le profil $(P^*, \mathcal{Q})$ avec $Q^* \in \mathcal{Q}$

[11] Cela définit une sorte d'équilibre d'Ellsberg dans lequel seulement *un* joueur emploie ambiguïté, mais autant qu'il veut; l'autre joue une stratégie mixte.

est un équilibre d'Ellsberg (et pareil pour joueur 2 si Q^* est un stratégie d'immunisation).

Chapitre 5 analyse le rapport entre les équilibres d'Ellsberg et les équilibres subjectifs définis par Aumann (1974). Dans les chapitres précédents, j'ai présenté les propriétés élémentaires des équilibres d'Ellsberg et j'ai analysé comment les joueurs se comportent quand ils ont la possibilité d'utiliser de l'ambiguïté stratégique dans des jeux à deux joueurs. Naturellement, je m'intéresse à comprendre ce qu'il peut arriver dans des jeux avec plus de deux joueurs. Pour commencer à comprendre le changement, je commence à étudier le rapport entre les équilibres d'Ellsberg et le concept des équilibres subjectifs. Comme il n'existe aucun autre concept de solution des jeux où les joueurs utilisent des ensembles des stratégies mixtes comme leur stratégie, la comparaison se fait par rapport à l'obtention de certaines réalisations du jeu. L'objective est de comprendre sous quelles conditions les équilibres subjectifs ont (ou n'ont pas) le même support que les équilibres d'Ellsberg.

Je commence dans la Section 5.1 avec un exemple d'un jeu à trois joueurs, une sorte de Prisoners' Dilemma avec conciliateur, qui a un équilibre d'Ellsberg avec réalisation non-Nash: la coopération des deux prisonniers. Cela suggère que dans des jeux très simples avec plus de deux joueurs, la prédiction d'équilibre d'Ellsberg peut être extrêmement différente de la prédiction classique. Dans ce qui suit, dans la Section 5.2.1, je définis l'équilibre subjectif suivant Aumann (1974) et Hallin (1976) et l'équilibre en croyances subjectives suivant Lo (1996). Je prouve dans la Section 5.2.2 que tout équilibre d'Ellsberg contient un équilibre subjectif et un équilibre en croyances subjectives. Dans des jeux à deux joueurs, des équilibres en croyances subjectives ont le même support comme les équilibres de Nash, mais cela change pour trois joueurs. C'est pour cela que dans des jeux d'Ellsberg avec plus de deux joueurs nous avons des réalisations au-delà des équilibres de Nash; je traite ce point dans la Section 5.2.3. Dans cette partie je prouve le théorème suivant.

Theorem 5.8. *Si $(\mathcal{P}_1^*, \ldots, \mathcal{P}_n^*)$ est un équilibre d'Ellsberg, il existe $\mu_{-i}^i \in \mathcal{P}_{-i}^*$ tel que $(\mu_{-1}^1, \ldots, \mu_{-n}^n)$ est un équilibre en croyances subjectives et*

$$\arg\max_{P_i \in \Delta S_i} U_i(P_i, \mathcal{P}_{-i}^*) \subseteq \arg\max_{P_i \in \Delta S_i} u_i(P_i, \mu_{-i}^i)\,.$$

Finalement, dans le Chapitre 6, la notion des jeux d'Ellsberg et d'équilibre d'Ellsberg est étendue aux jeux sous forme extensive. Il est naturelle que des urnes d'Ellsberg peuvent être utilisées dans des jeux dynamiques de la même manière que dans des jeux sous forme normale. Dans ce chapitre je développe le cadre d'analyse pour de tels jeux d'Ellsberg sous forme extensive. Comme dans la théorie classique des jeux sous forme extensive, les joueurs peuvent utiliser une urne pour créer de l'ambiguïté sur leur ensemble des stratégies pures, ou ils peuvent placer une urne d'Ellsberg à chaque ensemble d'information au courant du jeu. Dans la première partie du chapitre, Section 6.1, j'étends la notion de stratégie mixte et stratégie comportementale pour des jeux d'Ellsberg sous forme extensive.

Travaillant avec jeux dynamiques mène directement à la question de cohérence dynamique des préférences des joueurs. Je discute cette question dans la Section 6.2 et je développe le formalisme pour transmettre la notion de rectangularité de Epstein and Schneider (2003b) aux jeux d'Ellsberg sous forme extensive. Cette propriété est ensuite utilisée pour prouver une version de Théorème de Kuhn (Kuhn (1953)) pour jeux d'Ellsberg sous forme extensive. Mon théorème est le suivant.

Theorem 6.7. *Dans un jeu d'Ellsberg sous forme extensive* (F, U) *avec* $F = (\mathcal{T}, C)$ *satisfaisant mémoire parfaite, chaque profil de stratégies d'Ellsberg rectangulaire* $\mathcal{P}$ *induit un profil de stratégies de comportement d'Ellsberg* $\Theta^{\mathcal{P}}$ *via prior-by-prior updating; chaque profil de stratégies de comportement d'Ellsberg* Θ *induit un profil de stratégies d'Ellsberg rectangulaire* $\mathcal{P}^{\Theta}$ *tel que prior-by-prior updating de* $\mathcal{P}^{\Theta}$ *donne* Θ*. Les profils de stratégies sont équivalents en paiement,*

$$U_i(\mathcal{P}) = U_i(\Theta^{\mathcal{P}}) \;\; et \;\; U_i(\Theta) = U_i(\mathcal{P}^{\Theta})\,.$$

Je présente un exemple simple d'un jeu à deux joueurs pour mieux comprendre l'intention du théorème, et j'explique la rectangularité de la stratégie d'Ellsberg dans l'exemple avec le simplexe de dimension deux. Ensuite, dans la Section 6.4, je définis la notion d'équilibre d'Ellsberg dans un jeu d'Ellsberg sous forme extensive. Avec l'équivalence établie dans Theorem 6.7, les jeux d'Ellsberg sous forme extensive peuvent être analysés en considérant seulement des stratégies de comportement d'Ellsberg, ce qui simplifie largement l'analyse. Cependant, une analyse profonde des équilibres d'Ellsberg dans des jeux d'Ellsberg sous forme extensive ne fait pas partie de ce travail. Finalement, dans la Section 6.5, je présente un bref commentaire et compare les équilibres d'Ellsberg avec quelques autres concepts de solution sous forme extensive.

1 Strategic Use of Ambiguity in Normal Form Games

In the first chapter we present the fundamentals of Ellsberg games.[1] We start by discussing the major aspect in which Ellsberg games differ from other models of ambiguity in games: the difference between objective and subjective ambiguity and the historic development of these notions. In Ellsberg games, players can use additional devices which create ambiguity and these are treated differently than classic randomizing devices. Hence, in Section 1.2 we present the decision-theoretic framework of how to evaluate the utility of the strategies used in Ellsberg games. In Section 1.3 we define the model and show some basic properties of Ellsberg games. These include the possibility to reduce Ellsberg equilibria to the state space of pure strategies, we show the existence of Ellsberg equilibria, and explain and prove the Principle of Indifference in Distributions. Section 1.4 analyzes an important example of an Ellsberg game, where the Ellsberg equilibrium predicts an outcome that is not in the support of any Nash equilibrium of the game. Finally we discuss related literature.

1.1 Ellsberg Urns and Mixed Strategies: Concealment Device versus Beliefs about Opponents' Behavior

Let us go back to the very foundations of game theory. A game consists of a finite set N of players, a finite set of (pure) strategies $S_i, i \in N$ for each player, as well as a collection of payoff functions $u_i : S \to \mathbb{R}$ defined over strategy profiles $S = \times_{i \in N} S_i$. Von Neumann and Morgenstern (1953) introduce mixed strategies as probability vectors P_i over pure strategies S_i. The question then emerges how players evaluate profiles of such mixed strategies $P = (P_1, \dots, P_n)$; as the reader knows, Von Neumann and Morgenstern adopt expected utility (and axiomatize their choice). Here we are going back to these foundations and propose a generalization. We allow players to use Ellsberg urns in addition to probabilistic devices like a roulette wheel or a die. So we imagine that a player can

[1] Parts of this chapter, namely Sections 1.1, 1.3.1-1.3.3, 1.3.5, 1.4 and 1.5, were published in the IMW working paper Riedel and Sass (2011).

credibly commit his behavior on the outcome of an Ellsberg urn whose parameters he has chosen. To give an example, in a Matching Pennies game he would play $HEAD$ if the draw from an urn with 100 red and blue balls yields a red ball, while he himself and the other players only know that the proportion of red balls lies between 30 and 50 percent. The Ellsberg urn thus displays objective, common knowledge of ambiguity. All players know the possible probability distributions of outcomes, but no player has an informational advantage over others. We want to find out what the consequences for game theory are if we change the foundations in such a way.

Before we justify our new approach conceptually, let us go back again to classical game theory and ask how mixed strategies are justified and interpreted there. Our discussion follows closely the excellent account delivered in Reny and Robson (2004).

From a mathematical point of view, mixed strategies lead to convex strategy sets, and if one wants to assign a unique value to zero-sum games, e.g., such convexity is needed. Convexity and linearity of the payoff functions are useful in many other respects as well, of course. Just think about the indifference principle by which we usually find Nash equilibria.

The purely mathematical aspect would not be very compelling, of course, had it not a plausible interpretation. In the words of Von Neumann and Morgenstern (1953), p. 144: "In playing Matching Pennies against an at least moderately intelligent opponent, the player will not attempt to find out the opponent's intentions, but will concentrate on avoiding having his own intentions found out, by playing irregularly 'heads' and 'tails' in successive games. Since we wish to describe the strategy in one play – indeed we must discuss the course in one play and not that of a sequence of plays – it is preferable to express this as follows: The player's strategy consists neither of playing 'tails' and 'heads', but of playing 'tails' with the probability of 1/2 and 'heads' with the probability of 1/2." They then point out that this strategy protects the player against losses as his expected gain is always zero regardless how the opponent plays.

Von Neumann and Morgenstern always interpret these strategies as *objective* random devices like a fair coin or die. In particular, all players assign the same probabilities to the device's outcomes. In zero-sum games, the use of such mixing can be justified as an attempt to conceal your behavior from your opponents. Indeed, Von Neumann and Morgenstern also offer a Stackelberg game-like argument. Suppose that you have to write down your strategy on a sheet of paper before you play. If your opponent sends a spy able to find out what you have written down, then, in a zero-sum game, it is strictly better for you to have concealed your behavior by writing "I will use a fair coin to determine my behavior." A mixed strategy, in Von Neumann and Morgenstern's interpretation, is thus deliberate, objective randomization.

These arguments run into problems in common interest games as one would prefer one's own strategy to be found out by the opponent in simple coordination games (as has been pointed out by Schelling (1960) and Lewis (1969) already). Harsanyi (1967)'s construction allows to resolve this plausibility problem. He shows that mixed strategy equilibria can be interpreted as pure strategy equilibria in nearby incomplete information games where the payoffs are suitably perturbed, and players have private information. A common interpretation of mixed strategies nowadays goes even a step further. Several authors, including Aumann (1987), Armbruster and Boege (1979), Tan and Werlang (1988), Aumann and Brandenburger (1995), propose to forgo Harsanyis's construction and to interpret mixed strategies directly as the *belief* about which pure strategies the other players are going to use. Players are assumed to choose a definite action, and as other players do not know exactly which one, the mixed strategy represents their uncertainty. For more than two players, this requires some consistency among beliefs in equilibrium, of course. We will come back to this below when we discuss beliefs equilibria.

Summing up, we have here two opposing interpretations of mixed strategies: on the one hand, the "objective" interpretation by Von Neumann and Morgenstern where players deliberately use random devices with known probabilities to conceal their behavior, on the other hand, the "subjective" beliefs interpretation where the probability distributions represent players' uncertainty about other players' pure strategy choice.[2] The literature on ambiguity in games has mainly focused on the beliefs interpretation of mixed strategies. So far, it has usually been assumed that the players choose pure strategies and that the opponents are uncertain in the Knightian sense about their choice. Our approach differs from that in the literature as we take up the interpretation of mixed strategies by Von Neumann and Morgenstern and allow players to use objective devices that create Knightian uncertainty.

[2] Both approaches are merged in the framework of Reny and Robson (2004). Reny and Robson unify both views of mixed equilibria with the help of another construction. In their perturbed game, every player i is characterized by a privately known subjective probability $t_i \in [a, b]$, $0 < a < b < 1$ according to which he believes that a spy finds out his strategy. The payoff of the perturbed game is then

$$(1 - t_i)u_i(m_i, m_{-i}) + t_i u_i(m_i, \mathrm{bestreplyto}(m_i))\,.$$

So with probability $1 - t_i$ the normal static payoff is obtained, whereas with probability t_i, the other player finds out one's strategy and is allowed to play a best reply to it. In such a situation, for a generic class of games, one can approximate all Nash equilibria by suitable pure strategy equilibria of the perturbed game, with a and b close to zero. In contrast to Harsanyi (1967), players sometimes do use mixed strategies in the perturbed game, e.g., in zero-sum games. We refer to their insightful paper for a more detailed discussion which is beyond the scope of our aims here.

1.2 Evaluating Objective Ambiguity

Before we come to formalize the possibility of using Ellsberg urns to create ambiguity in games, we show how to represent players' preferences over these objectively ambiguous devices. Knight (1921) is one of the earliest references for the distinction between risk and (Knightian) uncertainty, see Arrow (1951) p. 417. In fact, Knight made the threefold distinction between different types of uncertain outcomes into those predicted by

(1) a priori probability (deductive, "mathematical law")

(2) statistical probability (relative frequency)

(3) estimates ("The instance in question is so entirely unique that there are no others or not a sufficient number to make it possible to tabulate enough like it to form a basis for any inference of value about any real probability in the case we are interested in", Knight (1921) p. 226).

(1) and (2) are, in the subsequent literature, summarized as "risk" and (3) is what we refer to as uncertainty or ambiguity. There has been an ongoing debate about what implications Knight's observation has for preferences and decision making under uncertainty, which is at the heart of microeconomic theory. The discussion has been fueled by the famous Ellsberg experiment published in Ellsberg (1961) (to which the Ellsberg games treated in this thesis owe their name) and has then gained new momentum after the publication of Schmeidler (1989) and Gilboa and Schmeidler (1989), two articles which axiomatize and represent ambiguity-averse preferences in a way that "solves" the Ellsberg paradox. Maxmin expected utility by Gilboa and Schmeidler (1989) says, that an act f is preferred to an act g if and only if there exists a set of priors $\mathcal{C}$ and a utility function u such that

$$\min_{p \in \mathcal{C}} \int f(u)\, dp \geq \min_{p \in \mathcal{C}} \int g(u)\, dp\,.$$

Since this representation is derived from subjective ambiguity-averse preferences, the set of priors $\mathcal{C}$ can be interpreted as the subjective beliefs of the decision maker. In effect, this approach to ambiguity-averse preferences is entirely subjective. It generalizes the expected utility approach with subjective probabilities by Savage (1954) that was turned into the language of horse-races and lotteries and thus made more tractable by Anscombe and Aumann (1963). A number of articles provide other subjective representations of ambiguity-averse preferences, e.g., Hansen and Sargent (2001), Klibanoff, Marinacci, and Mukerji (2005), Maccheroni, Marinacci, and Rustichini (2006), Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011).

After our explanations in Section 1.1 it is clear, that the maxmin expected utility representation derived from subjective preferences is not suitable to represent the ambiguity-averse preferences over the objectively ambiguous devices used in Ellsberg urns. What we need to specify for Ellsberg games is rather an attitude towards imprecise risk, a term coined by Jaffray (1989). With the same motivation as the subjective representations mentioned above, he seeks to distinguish attitude towards risk and uncertainty, where he understands uncertainty objectively: due to unavailable or demonstrably bad information the decision maker is not facing a precise probability distribution. In the same spirit Stinchcombe (2003), Olszewski (2007) and Ahn (2008) model imprecise risk directly by sets of lotteries. We use a generalization of this approach by Gajdos, Hayashi, Tallon, and Vergnaud (2008), pioneered by Wang (2003), that compares pairs $(\mathcal{P}, f)$ where f is an act and $\mathcal{P}$ is a set of probability distributions over the state space. In their representation, $(\mathcal{P}, f)$ is preferred to $(\mathcal{Q}, g)$ if and only if

$$\min_{p \in \phi(\mathcal{P})} \int f(u)\, dp \geq \min_{p \in \phi(\mathcal{Q})} \int g(u)\, dp\,. \tag{1.1}$$

For ϕ the identity ($\phi : \mathcal{P} \to \mathcal{P}$ selects the probability-possibility set and captures the subjective part of the preferences) and a fixed set of probability distributions $\mathcal{P}$, that is, when we evaluate a pair $(\mathcal{P}, f)$ through the induced set of distributions on outcomes, this yields maxmin expected utility by Gilboa and Schmeidler (1989), see Section 3.3 in Gajdos, Hayashi, Tallon, and Vergnaud (2008) for details. For a more detailed review on literature on objective ambiguity aversion see Giraud and Tallon (2011) and Gajdos, Hayashi, Tallon, and Vergnaud (2008).

We apply the utility representation (1.1) to Ellsberg strategies $((\Omega, \mathcal{F}, \mathcal{P}), f)$ which we define in the following section. An Ellsberg strategy consists of a convex and compact set of probability distributions $\mathcal{P}$ on a state space Ω, and an act f. The utility of an Ellsberg strategy $((\Omega, \mathcal{F}, \mathcal{P}), f)$ is defined using (1.1) with $\phi = \text{id}$. The reduced form of an Ellsberg strategy, which will just be a convex and compact set of probability distributions $\mathcal{Q}$, is evaluated with maxmin expected utility derived from (1.1). This representation is mathematically identical to the representation by Gilboa and Schmeidler (1989): An Ellsberg strategy $\mathcal{P}$ is preferred to an Ellsberg strategy $\mathcal{Q}$ if and only if

$$\min_{P \in \mathcal{P}} \int u\, dP \geq \min_{Q \in \mathcal{Q}} \int u\, dQ\,.$$

We thus use the terms maxmin expected utility and maxmin rule, having in mind that we derived this representation from attitude towards objective ambiguity.

1.3 Ellsberg Games

1.3.1 Creating Ambiguity: Objective Ellsberg Urns

Let us formalize the intuitive idea that players can create ambiguity with the help of Ellsberg urns. An Ellsberg urn is, for us, a triple $(\Omega, \mathcal{F}, \mathcal{P})$ of a non-empty set Ω of states of the world, a σ-field $\mathcal{F}$ on Ω (where one can take the power set in case of a finite Ω), and a set of probability measures $\mathcal{P}$ on the measurable space $(\Omega, \mathcal{F})$. This *set* of probability measures represents the Knightian uncertainty of the strategy. A typical example is the classical Ellsberg urn that contains 30 red balls, and 60 balls that are either black or yellow. One ball is drawn from that urn. The state space consists of three elements $\{R, B, Y\}$, $\mathcal{F}$ is the power set, and $\mathcal{P}$ the set of probability vectors (P_1, P_2, P_3) such[3] that $P_1 = 1/3$, $P_2 = k/90, P_3 = (60-k)/90$ for any $k = 0, \ldots, 60$.

We assume that the players of our game have access to and can design the parameters of such Ellsberg urns: imagine that there is an independent, trustworthy laboratory that sets up such urns and reports the outcome truthfully. Note that we allow the player to choose the degree of ambiguity of his urn. He tells the experimentalists of his laboratory to set up such and such an Ellsberg experiment that generates exactly the set of distributions $\mathcal{P}_i$. In this sense, the ambiguity in our formulation of the game is "objective": it is not a matter of agents' *beliefs* about the actions of other players, but rather a property of the device used to determine his action.

1.3.2 Definition of Ellsberg Games

We come now to the game where players can use such urns in addition to the usual mixed strategies (that correspond to roulette wheels or dice). Let $N = \{1, \ldots, n\}$ be the set of players. Each player i has a finite strategy set S_i. Let $S = \times_{i=1}^{n} S_i$ be the set of pure strategy profiles. Players' payoffs are given by functions

$$u_i : S \to \mathbb{R} \qquad (i \in N)\,.$$

The normal form game is denoted $G = \langle N, (S_i), (u_i) \rangle$.

Players can now use different devices. On the one hand, we assume that they have "roulette wheels" or "dice" at their disposal, i.e., randomizing devices with objectively known probabilities. The set of these probabilities over S_i is denoted ΔS_i. The players

[3] We are always going to work with convex sets of probability measures. In this case, this means that we would allow for any $P_2, P_3 \geq 0$ with $P_2 + P_3 = 2/3$ here. In our framework, this is without loss of generality, of course.

evaluate such devices according to expected utility, as in Von Neumann and Morgenstern's formulation of game theory. Moreover, and this is the new part, players can use Ellsberg urns. As we said above, we imagine that the players can credibly commit to base their actions on ambiguous outcomes. Technically, we model the Ellsberg urn of player i as a triple $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$ as explained above. Player i acts in the game by choosing a measurable function (or Anscombe-Aumann act)[4]

$$f_i : (\Omega_i, \mathcal{F}_i) \to \Delta S_i$$

which specifies the classical mixed strategy played once the outcome of the Ellsberg urn is revealed. An *Ellsberg strategy* for player i is then a pair

$$((\Omega_i, \mathcal{F}_i, \mathcal{P}_i), f_i)$$

of an Ellsberg urn and an act. A profile of Ellsberg strategies is then as usual

$$((\Omega, \mathcal{F}, \mathcal{P}), f) = (((\Omega_1, \mathcal{F}_1, \mathcal{P}_1), f_1), \ldots, ((\Omega_n, \mathcal{F}_n, \mathcal{P}_n), f_n)) .$$

To finish the description of our Ellsberg game, we have to determine players' payoffs. We suppose that all players are ambiguity-averse: in the face of ambiguous events (as opposed to simply random events) they evaluate their utility in a cautious and pessimistic way. This behavior in response to ambiguity has been observed in the famous experiments of Ellsberg (1961) and confirmed in further experiments, for example of Pulford (2009) and Camerer and Weber (1992), see also Etner, Jeleva, and Tallon (2012) for references. For our purpose we follow the axiomatization of attitude towards objective but imprecise information in Gajdos, Hayashi, Tallon, and Vergnaud (2008) which we introduced and discussed in Section 1.2. Utility is evaluated as maxmin expected utility like in the axiomatization of Gilboa and Schmeidler (1989), but with the difference of the decision maker facing objective instead of subjective ambiguity.

The payoff of player $i \in N$ at an Ellsberg strategy profile $((\Omega, \mathcal{F}, \mathcal{P}), f)$ is thus the minimal expected utility with respect to all different probability distributions in the convex and compact set $\mathcal{P}$,

$$U_i(((\Omega, \mathcal{F}, \mathcal{P}), f)) := \min_{P_1 \in \mathcal{P}_1, \ldots, P_n \in \mathcal{P}_n} \int_{\Omega_1} \cdots \int_{\Omega_n} u_i(f(\omega)) \, dP_n \ldots dP_1 .$$

We call the described larger game an *Ellsberg game*. An *Ellsberg equilibrium* is, in the

[4] We assume that all such acts in this thesis are measurable, also if not explicitly stated at each appearance.

same spirit as Nash equilibrium, a profile of Ellsberg strategies

$$(((\Omega_1^*, \mathcal{F}_1^*, \mathcal{P}_1^*), f_1^*), \ldots, ((\Omega_n^*, \mathcal{F}_n^*, \mathcal{P}_n^*), f_n^*)),$$

where no player has an incentive to deviate, i.e., for all players $i \in N$, all Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$, and all acts f_i for player i we have

$$U_i\left(((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)\right) \geq U_i\left(((\Omega_i, \mathcal{F}_i, \mathcal{P}_i), f_i), ((\Omega_{-i}^*, \mathcal{F}_{-i}^*, \mathcal{P}_{-i}^*), f_{-i}^*)\right).$$[5]

Definition 1.1. *Let $G = \langle N, (S_i), (u_i)\rangle$ be a normal form game. A profile*

$$(((\Omega_1^*, \mathcal{F}_1^*, \mathcal{P}_1^*), f_1^*), \ldots, ((\Omega_n^*, \mathcal{F}_n^*, \mathcal{P}_n^*), f_n^*))$$

of Ellsberg strategies is an Ellsberg equilibrium *of G if no player has an incentive to deviate from $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$, i.e., for all players $i \in N$, all Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$ and all acts f_i for player i we have*

$$U_i(((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)) \geq U_i(((\Omega_i, \mathcal{F}_i, \mathcal{P}_i), f_i), ((\Omega_{-i}^*, \mathcal{F}_{-i}^*, \mathcal{P}_{-i}^*), f_{-i}^*)), \text{ that is}$$

$$\begin{aligned} &\min_{P_i \in \mathcal{P}_i^*, P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i^*} \int_{\Omega_{-i}^*} u_i(f_i^*(\omega_i), f_{-i}^*(\omega_{-i}))\, dP_{-i}\, dP_i \\ \geq &\min_{P_i \in \mathcal{P}_i, P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i} \int_{\Omega_{-i}^*} u_i(f_i(\omega_i), f_{-i}^*(\omega_{-i}))\, dP_{-i}\, dP_i\,. \end{aligned}$$

1.3.3 Reduced Form Ellsberg Games

This definition of an Ellsberg game depends on the particular Ellsberg urn used by each player i. As there are arbitrarily many possible state spaces[6], the definition of Ellsberg equilibrium might not seem very tractable. Fortunately, there is a more concise way to define Ellsberg equilibrium. The procedure is similar to the reduced form of a correlated equilibrium, see Aumann (1974) or Fudenberg and Tirole (1991). Instead of working with arbitrary Ellsberg urns, we note that the players' payoff depends, in the end, on the set of distributions that the Ellsberg urns and the associated acts induce on the set of strategies. One can then work with that set of distributions directly.

[5] Throughout the text, we follow the notational convention that $(f_i, f_{-i}^*) := (f_1^*, \ldots, f_{i-1}^*, f_i, f_{i+1}^*, \ldots, f_n^*)$. The same convention is used for profiles of pure strategies (s_i, s_{-i}) and probability distributions (P_i, P_{-i}).

[6] In fact, the class of all state spaces is too large to be a well-defined set according to set theory.

Definition 1.2. *Let $G = \langle N, (S_i), (u_i)\rangle$ be a normal form game. A* reduced form Ellsberg equilibrium *of the game G is a profile of sets of probability measures $\mathcal{Q}_i^* \subseteq \Delta S_i$, such that for all players $i \in N$ and all sets of probability measures $\mathcal{Q}_i \subseteq \Delta S_i$ we have*

$$\begin{aligned} &\min_{P_i \in \mathcal{Q}_i^*, P_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i})\, dP_{-i}\, dP_i \\ \geq &\min_{P_i \in \mathcal{Q}_i, P_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i})\, dP_{-i}\, dP_i\,. \end{aligned}$$

The two definitions of Ellsberg equilibrium are equivalent in the following sense.

Proposition 1.3. *Ellsberg equilibrium and reduced form Ellsberg equilibrium are equivalent in the sense that every Ellsberg equilibrium $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ induces a payoff-equivalent reduced form Ellsberg equilibrium on $\Omega^* = S$; and every reduced form Ellsberg equilibrium $\mathcal{Q}^*$ is an Ellsberg equilibrium $((S, \mathcal{F}, \mathcal{Q}^*), f^*)$ with f^* the embedding of S in ΔS.*

Proof. " $\Leftarrow$ " Let $\mathcal{Q}^*$ be a reduced form Ellsberg equilibrium according to Definition 1.2. We choose the states of the world $\Omega = S$ to be the set of pure strategy profiles, thereby we see that player i uses the Ellsberg urn $(S_i, \mathcal{F}_i, \mathcal{Q}_i^*)$, where $\mathcal{F}_i$ is the power set of S_i. We define the act $f_i^* : (S_i, \mathcal{F}_i) \to \Delta S_i$ to be the embedding $f_i^*(s_i) = \delta_{s_i}$ of S_i into ΔS_i, where $\delta_{s_i} \in \Delta S_i$ is the degenerate mixed strategy which puts all weight on the pure strategy s_i. f_i^* induces an image measure $Q_i^{f_i^*}$ of $Q_i \in \mathcal{Q}_i^*$ on ΔS_i,

$$Q_i^{f_i^*} : \delta_{s_i} \mapsto Q_i({f_i^*}^{-1}(\delta_{s_i}))\,.$$

The image measure $Q_i^{f_i^*}$ can be identified with $Q_i \in \mathcal{Q}_i^*$. Thus, the reduced form Ellsberg equilibrium $\mathcal{Q}^*$ can be written as $((S, \mathcal{F}, \mathcal{Q}^*), f^*)$. This strategy is an Ellsberg equilibrium according to Definition 1.1.

" $\Rightarrow$ " Let now $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ be an Ellsberg equilibrium according to Definition 1.1. Every $P_i \in \mathcal{P}_i^*$ induces an image measure $P_i^{f_i^*}$ on ΔS_i that assigns a probability to a distribution $f_i^*(\omega_i) \in \Delta S_i$ to occur. To describe the probability that a pure strategy s_i is played, given a distribution P_i and an Ellsberg strategy $((\Omega_i^*, \mathcal{F}_i^*, \mathcal{P}_i^*), f_i^*)$, we integrate $(f_i^*(\omega_i))(s_i)$ over all states $\omega_i \in \Omega_i$. Thus, we can define Q_i to be

$$Q_i(s_i) := \int_{\Omega_i^*} f_i^*(\omega_i)(s_i)\, dP_i\,. \tag{1.2}$$

Recall that $\mathcal{P}_i$ is a closed and convex set of probability distributions. We call the resulting set of probability measures $\mathcal{Q}_i^*$. $\mathcal{Q}_i^*$ is closed and convex, since $\mathcal{P}_i^*$ is.

A straightforward, if tedious, reasoning shows that the profile $(\mathcal{Q}_1^*, \ldots, \mathcal{Q}_n^*)$ yields the same payoff as $((\Omega_i^*, \mathcal{F}_i^*, \mathcal{P}_i^*), f_i^*)_{i=1,\ldots,n}$.

Now suppose $\mathcal{Q}^*$ is not a reduced form Ellsberg equilibrium. Then for some player $i \in N$ there exists a set $\mathcal{Q}_i$ of probability measures on S_i that yields a higher minimal expected utility. We can then define the Ellsberg strategy $\tilde{\Omega} = S_i$, $\tilde{\mathcal{F}}_i = \mathcal{P}S_i$ (the power set of S_i), and $f_i(s) = \delta_s$ to obtain a profitable deviation in the original Ellsberg game, a contradiction. □

We henceforth call a set $\mathcal{Q}_i \subseteq \Delta S_i$ an Ellsberg strategy whenever it is clear that we are in the reduced form context. The utility of a (reduced) Ellsberg strategy profile is evaluated as

$$U_i(\mathcal{P}_i, \mathcal{P}_{-i}) = \min_{P_i \in \mathcal{P}_i, P_{-i} \in \mathcal{P}_{-i}} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i})\, dP_{-i}\, dP_i\,.$$

In the definition of an Ellsberg game we assume that the Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$ of all players $i \in N$ are stochastically independent. This is done by using product spaces as first suggested by Gilboa and Schmeidler (1989).[7] We define the product $(\Omega, \mathcal{F}, \mathcal{P})$ of n Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$, $i = 1, \ldots, n$, as follows.

$$\Omega := \Omega_1 \times \ldots \times \Omega_n,$$

$$\mathcal{F} := \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n,$$

and $\mathcal{P}$ is the closed convex hull of the set of product measures,

$$\mathcal{P} := \bar{co}\,\{P_1 \otimes \ldots \otimes P_n \mid P_1 \in \mathcal{P}_1, \ldots, P_n \in \mathcal{P}_n\}\,. \tag{1.3}$$

This way the Ellsberg urns are stochastically independent. We usually denote by ΔS_i the set of lotteries on the set of pure strategies S_i of player i. If not otherwise specified, by $\Delta S := \times_{i=1}^n \Delta S_i$, or $\Delta S_{-i} := \times_{j=1, j\neq i}^n \Delta S_j$ we mean the profiles of mixed strategies where each player mixes independently.

Different notions of stochastic independence in the context of ambiguity aversion have been discussed in the literature, see for example Klibanoff (2001), Bade (2011b) and Bade (2011a). In the present context of objective ambiguity in the form of Ellsberg urns the above notion seems the most natural.

[7] Instead of Ellsberg urns they speak of "non-unique probability spaces" $(\Omega, \mathcal{F}, \mathcal{P})$, Gilboa and Schmeidler (1989), p. 150.

1.3.4 Ellsberg Equilibria Generalize Nash Equilibria

Note that the classical game is contained in our formulation: players just choose a singleton $\mathcal{P}_i = \{\delta_{\pi_i}\}$ that puts all weight on a particular (classical) mixed strategy π_i. Now let $(\pi_1, \ldots, \pi_n)$ be a Nash equilibrium of the game G. Can any player unilaterally gain by creating ambiguity in such a situation? The answer is no. Take the game in Figure 1.1 and look at the pure strategy Nash equilibrium (D, R) with equilibrium payoff 1 for both players.

		Player 2 L	Player 2 R
Player 1	U	$3, 3$	$0, 0$
Player 1	D	$0, 0$	$1, 1$

Figure 1.1: Strategic ambiguity does not unilaterally make a player better off.

If player 1 introduces ambiguity, he will play U in some states of the world (without knowing the exact probability of those states). But this does not help here because player 2 sticks to his strategy R, so playing U just leads to a payoff of zero. Unilateral introduction of ambiguity does not increase one's own payoff. We think that this is an important property of our formulation.

Theorem 1.4. *Let $G = \langle N, (S_i), (u_i)\rangle$ be a normal form game. Then a mixed strategy profile $(P_1^*, \ldots, P_n^*)$ of G is a Nash equilibrium of G if and only if the corresponding profile of singletons $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ with $\mathcal{P}_i = \{\delta_{P_i^*}\}$ is an Ellsberg equilibrium. In particular, Ellsberg equilibria exist when the strategy sets S_i are finite.*

We show in Theorem 1.4 that a mixed strategy profile is a Nash equilibrium of a normal form game if and only if it is an Ellsberg equilibrium of that game. The main part of the proof is that every player can find for every profile of Ellsberg strategies a mixed strategy that gives him at least the same utility as his Ellsberg strategy. We first prove the latter result in the following lemma. We let π_i^* abbreviate the constant act that maps every state of the world to the mixed strategy $\pi_i^* \in \Delta S_i$.

Lemma 1.5. *Let $G = \langle N, (S_i), (u_i)\rangle$ be a normal form game. Then for any profile of Ellsberg strategies $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ and every $i \in N$ there exists a mixed strategy $P_i^* \in \Delta S_i$ such that $U_i(\{\delta_{P_i^*}\}, \mathcal{P}_{-i}) \geq U_i(\mathcal{P}_i, \mathcal{P}_{-i})$.*

Proof. As the set of pure strategies is finite, the set of mixed strategies ΔS_i is compact. The linear utility function is continuous, and thus a minimizer in the compact set $\mathcal{P}_i$ exists. □

Proof of Theorem 1.4. Let $\mathcal{P}^*$ with $\mathcal{P}_i^* = \{P_i^*\}$ be an Ellsberg equilibrium of G; then $U_i(\{P_i^*\},\{P_{-i}^*\}) \geq U_i(\mathcal{P}_i,\{P_{-i}^*\})$ holds for all $i \in N$ for all Ellsberg strategies $\mathcal{P}_i \subseteq \Delta S_i$. In particular this holds for all singletons $\{P_i\}$, so $U_i(\{P_i^*\},\{P_{-i}^*\}) \geq U_i(\{P_i\},\{P_{-i}^*\})$ and thus $u_i(P_i^*, P_{-i}^*) \geq u_i(P_i, P_{-i}^*)$ for all $P_i \in \Delta S_i$. Therefore the profile $(P_1^*, \dots, P_n^*)$ is a Nash equilibrium of G.
Next, assume that $(P_1^*, \dots, P_n^*)$ is a Nash equilibrium of G. Suppose it was not an Ellsberg equilibrium of G, that is there exists an Ellsberg strategy $\mathcal{P}_i$ for some player i such that $U_i(\mathcal{P}_i,\{P_{-i}^*\}) > U_i(\{P_i^*\},\{P_{-i}^*\})$. By Lemma 1.5, there exists a P_i' such that $U_i(\{P_i'\},\{P_{-i}^*\}) = U_i(\mathcal{P}_i,\{P_{-i}^*\}) > U_i(\{P_i^*\},\{P_{-i}^*\})$. This contradicts the assumption that $(P_1^*, \dots, P_n^*)$ is a Nash equilibrium of the game G. □

Theorem 1.4 proves that Ellsberg equilibria generalize Nash equilibria. Hence our formulation avoids the existence pitfalls that one encounters when players are assumed to play pure strategies and beliefs are uncertain about those pure actions.

We show that the support of an Ellsberg equilibrium, i.e., the strategies played with probability greater than zero, is identical to the support of the corresponding reduced form Ellsberg equilibrium. First we define what is the support of a (reduced) Ellsberg strategy. We understand null-events in the sense of Savage (1954).[8]

Definition 1.6. *Let* $((\Omega_i, \mathcal{F}_i, \mathcal{P}_i), f_i)$ *be an Ellsberg strategy. The* support *of* $((\Omega_i, \mathcal{F}_i, \mathcal{P}_i), f_i)$ *is defined as the support of* f_i*, that is*

$$\operatorname{supp} f_i := \{s_i \in S_i \mid (f_i(\omega_i))(s_i) > 0 \ \text{ for some non-null } \ \omega_i \in \Omega_i\}\,.$$

Let $\mathcal{Q}_i \subseteq \Delta S_i$ *be a reduced Ellsberg strategy. The support of* $\mathcal{Q}_i$ *is then defined as*

$$\operatorname{supp} \mathcal{Q}_i := \bigcup_{Q_i \in \mathcal{Q}_i} \{s_i \in S_i \mid Q_i(s_i) > 0\}\,.$$

Proposition 1.7. *Let* $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ *be an Ellsberg equilibrium of a normal form game* G, $\mathcal{Q}^* \subseteq \Delta S$ *the equivalent reduced form Ellsberg equilibrium. Then*

$$\operatorname{supp} f^* = \operatorname{supp} \mathcal{Q}^*\,.$$

Proof. It suffices to show the equality for $\operatorname{supp} f_i^*$ and $\operatorname{supp} \mathcal{Q}_i^*$ for some player $i \in N$.

[8] Given an event $E \subseteq \Omega$ and acts f and h, let $f_E h$ be the act such that $(f_E h)(s) = f(s)$ if $s \in E$, and $(f_E h)(s) = h(s)$ otherwise. An event E is *null* if $f_E h \sim f'_E h$ for all acts f and f', otherwise it is *non-null*.

Recall that $\mathcal{Q}_i^*$ is the set of measures on S_i induced by $P_i \in \mathcal{P}_i^*$ under f_i^*, that is

$$\mathcal{Q}_i^*(s_i) := \left\{ Q_i(s_i) = \int_{\Omega_i^*} f_i^*(\omega_i)(s_i)\, dP_i \;\middle|\; P_i \in \mathcal{P}_i^* \right\}.$$

We first show that $\operatorname{supp} f_i^* \subseteq \operatorname{supp} \mathcal{Q}_i^*$. Let $s_i \in \operatorname{supp} f_i^*$, then $f_i^*(\omega_i)(s_i) > 0$ for some $\omega_i \in \Omega_i$. Suppose, $s_i \notin \operatorname{supp} \mathcal{Q}_i^*$. Then

$$\begin{aligned} & Q_i(s_i) = 0 \text{ for all } Q_i \in \mathcal{Q}_i^* \\ \Rightarrow\ & Q_i(s_i) = \int_{\Omega_i} f_i^*(\omega_i)(s_i)\, dP_i = 0 \text{ for all } P_i \in \mathcal{P}_i^* \\ \Rightarrow\ & f_i^*(\omega_i)(s_i) = 0 \text{ for all } \omega_i \in \Omega_i \end{aligned}$$

This contradicts the fact that s_i was in the support of f_i^*.

Let now $s_i \in \operatorname{supp} \mathcal{Q}_i^*$, then $Q_i(s_i) > 0$ for some $Q_i \in \mathcal{Q}_i^*$. Suppose $s_i \notin \operatorname{supp} f_i^*$. Then

$$\begin{aligned} & f_i^*(\omega_i)(s_i) = 0 \text{ for all } \omega_i \in \Omega_i \\ \Rightarrow\ & Q_i(s_i) = \int_{\Omega_i} f_i^*(\omega_i)(s_i)\, dP_i = 0 \text{ for all } \omega_i \in \Omega_i \end{aligned}$$

This contradicts the fact that s_i was in the support of $\mathcal{Q}_i^*$. □

We have seen that Nash equilibria are special cases of Ellsberg equilibria. To simplify reference to different types of Ellsberg equilibria we introduce *proper Ellsberg equilibria*. The notion serves to address exclusively those Ellsberg equilibria which are not Nash equilibria, i.e., those in which ambiguity prevails. This is the result of Corollary 1.9 which follows directly from Theorem 1.4.

Definition 1.8. *Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game with $i \in N$. G has* proper Ellsberg equilibria *when the set of Ellsberg equilibria that are not Nash equilibria is non-empty.*

Corollary 1.9 (of Theorem 1.4)**.** *Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game with $i \in N$. The set of proper Ellsberg equilibria of G coincides with the set of Ellsberg equilibria of G in which at least one player $i \in N$ uses an Ellsberg strategy $\mathcal{P}_i \subseteq \Delta S_i$ which is not a singleton.*

In the same spirit we sometimes speak of *proper Ellsberg strategies* when they are not identical to a pure or mixed strategy.

1.3.5 Principle of Indifference in Distributions

An important characteristic of Nash equilibrium in mixed strategies is what we call here the *Principle of Indifference*: in a Nash equilibrium in mixed strategies players are indifferent between playing the randomized strategy or playing any pure strategy that is in its support. We restate here the corresponding lemma from Osborne and Rubinstein (1994), Lemma 33.2 therein.

Lemma. *Let G be a finite normal form game. Then $P^* \in \times_{i=1}^{N} \Delta S_i$ is a mixed strategy Nash equilibrium of G if and only if for every player $i \in N$ every pure strategy in the support of P_i^* is a best response to P_{-i}^*.*

Osborne and Rubinstein (1994) conclude: "It follows that *every action in the support of any player's equilibrium mixed strategy yields that player the same payoff.*"[9] In the same section the authors point out, that this may no longer be true when players are not Von Neumann and Morgenstern expected utility maximizers. We confirm this observation with an example and show that an alternative version of the Principle of Indifference holds for Ellsberg games.

In Ellsberg games players are no longer expected utility maximizers but ambiguity-averse. Ambiguity-averse preferences have the property that the utility functions often have kinks: in many cases the decision maker switches at some point from one boundary of the set of probability distributions to the other and thus changes the slope of the utility function. In the analysis of Ellsberg games this causes that the Principle of Indifference no longer prevails. In an Ellsberg equilibrium players are in general *not* indifferent between playing the Ellsberg strategy or playing the pure strategies in its support. We illustrate this with an example.

Example 1.10. *We consider a similar coordination game as before, the payoff matrix is as in Figure 1.2. The game has an Ellsberg equilibrium in which player 1 plays strategy U with probability $P \in [3/4, 1]$ and player 2 plays strategy L with probability $Q \in [3/4, 1]$. The maxmin expected utility of player 1 in this equilibrium is $3/4$, because the worst that can happen is that (U, L) is played with the Nash equilibrium mixture $((3/4, 1/4), (3/4, 1/4))$. Now suppose player 1 would change to a pure strategy in the support of the Ellsberg equilibrium, that is to U or D. When he changes to U, his minimal expected utility does not*

[9] Italic in the original text.

		Player 2	
		L	R
Player 1	U	$1,1$	$0,0$
	D	$0,0$	$3,3$

Figure 1.2: No indifference in pure strategies in Ellsberg equilibrium.

change, so he is indifferent in that case. On the contrary, when he plays D, his minimal expected utility is 0*, because he now uses for his maxmin expected utility evaluation that player 2 plays R with probability* 0*. Player 1 is therefore not indifferent between his Ellsberg equilibrium strategy and his pure strategy D.*

To make this argument more precise we present the utility analysis. We first fix a distribution $P \in [P_0, P_1]$ for player 1 and consider his payoff function $U_1(P, [Q_0, Q_1])$. Player 1 *evaluates his minimal expected payoff given player* 2 *plays $Q \in [Q_0, Q_1]$:*

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \leq Q \leq Q_1} PQ + 3(1-P)(1-Q) \\ &= \min_{Q_0 \leq Q \leq Q_1} Q(4P-3) - 3P + 3 \\ &= \begin{cases} Q_0(4P-3) - 3P + 3 & \text{if } P > 3/4\,, \\ 3/4 & \text{if } P = 3/4\,, \\ Q_1(4P-3) - 3P + 3 & \text{if } P < 3/4\,. \end{cases} \end{aligned}$$

In the given Ellsberg equilibrium player 1 thus evaluates

$$\begin{aligned} U_1([3/4,1]\,, [3/4,1]) &= \begin{cases} 3/4(4P-3) - 3P + 3 & \text{if } P > 3/4\,, \\ 3/4 & \text{if } P = 3/4 \end{cases} \\ &= 3/4 \quad \text{for all } P \in [3/4,1]\,. \end{aligned}$$

Now consider the two cases where player 1 plays U, that is $P_0 = P_1 = 1$, or D, that is $P_0 = P_1 = 0$. Then

$$\begin{aligned} U_1(1, [3/4,1]) &= 3/4(4-3) = 3/4\,, \\ \text{and} \quad U_1(0, [3/4,1]) &= 1(-3) + 3 = 0\,. \end{aligned}$$

The last two equations show that the indifference in pure strategies does in general not hold.

Example 1.10 shows violation of the classical Principle of Indifference. However, we can show that in Ellsberg equilibrium players are indifferent between all distributions that are part of their Ellsberg equilibrium strategy. We come to this next.

Recall that a profile $(\mathcal{P}_1^*, \ldots, \mathcal{P}_n^*)$ is a reduced form Ellsberg equilibrium if for all $i \in N$ and all $\mathcal{P}_i \subseteq \Delta S_i$,

$$U_i(\mathcal{P}_i^*, \mathcal{P}_{-i}^*) \geq U_i(\mathcal{P}_i, \mathcal{P}_{-i}^*)\,. \tag{1.4}$$

This formulation states that in equilibrium it is not profitable for any player to unilaterally deviate. Alternatively and in analogy with Nash equilibrium, an Ellsberg equilibrium can be defined as a profile of strategies, where each player only chooses strategies which maximize his (in Ellsberg games) minimal expected utility. That is, $(\mathcal{P}_1^*, \ldots, \mathcal{P}_n^*)$ is a reduced form Ellsberg equilibrium if for all $i \in N$ and all $P_i \in \mathcal{P}_i^*$

$$P_i \in \arg\max_{\mathcal{P}_i \subseteq \Delta S_i} U_i(\mathcal{P}_i, \mathcal{P}_{-i}^*)\,. \tag{1.5}$$

Conditions (1.4) and (1.5) are obviously equivalent. This is formulated in the following proposition.

Proposition 1.11. *Let $G = \langle N, (S_i), (u_i)\rangle$ be a normal form game. A profile of Ellsberg strategies $(\mathcal{P}_1^*, \ldots, \mathcal{P}_n^*)$ satisfies* (1.4) *if an only if it satisfies* (1.5).

For the following theorems we need a lemma concerning the maximization and minimization of a linear function u on some convex set ΔS (for the moment, S does not denote a set of strategy profiles, but simply a set with m elements).

Lemma 1.12. *P is a probability distribution on the set $S = \{s_1, \ldots, s_m\}$. Let $u(P)$ be a linear function that maps the m-dimensional vector P into the real numbers. We denote by ΔS the set of probability distributions P, and by $\mathcal{P} \subset \Delta S$ a closed and convex subset of ΔS. Then*

$$\max_{\mathcal{P} \subseteq \Delta S} \min_{P \in \mathcal{P}} u(P) = \max_{P \in \Delta S} u(P)\,, \tag{1.6}$$

$$\text{and} \quad \min_{\mathcal{P} \subseteq \Delta S} \min_{P \in \mathcal{P}} u(P) = \min_{P \in \Delta S} u(P)\,. \tag{1.7}$$

Proof. We start by showing equation (1.6). It is evident that

$$\max_{P\in\Delta S} u(P) \geq \max_{\mathcal{P}\subseteq\Delta S}\min_{P\in\mathcal{P}} u(P)\,,$$

so it remains to show that the left hand side is always greater than or equal to the right hand side of (1.6). Since the function $u(P)$ is linear, there exists a $P' \in \Delta S$ with

$$\max_{P\in\Delta S} u(P) = u(P')\,.$$

Now we have

$$\max_{\mathcal{P}\subseteq\Delta S}\min_{P\in\mathcal{P}} u(P) \geq \max_{\mathcal{P}\subseteq\Delta S, P'\in\mathcal{P}}\min_{P\in\mathcal{P}} u(P)\,,$$

since maximizing over a smaller number of subsets of ΔS is necessarily less than or equal to the original maximization. Making the set over which we maximize even smaller, we obtain

$$\max_{\mathcal{P}\subseteq\Delta S, P'\in\mathcal{P}}\min_{P\in\mathcal{P}} u(P) \geq \max_{P'}\min_{P=P'} u(P) = u(P')$$

and the equality (1.6) is shown. The argument for equation (1.7) is analog:

$$\min_{\mathcal{P}\subseteq\Delta S}\min_{P\in\mathcal{P}} u(P) \geq \min_{P\in\Delta S} u(P)$$

is evident. Furthermore, there exists $P' \in \Delta S$ such that $\min_{P\in\Delta S} u(P) = u(P')$, and

$$\begin{aligned}\min_{\mathcal{P}\subseteq\Delta S}\min_{P\in\mathcal{P}} u(P) &\leq \min_{\mathcal{P}\subseteq\Delta S, P'\in\mathcal{P}}\min_{P\in\mathcal{P}} u(P)\\ &\leq \min_{P'}\min_{P=P'} u(P) = u(P')\end{aligned}$$

and the equality holds. □

Since the payoff functions in Ellsberg games are multilinear, the preceding lemma gives us the following corollary of which we will make use also in later chapters.

Corollary 1.13. *Let $G = \langle N, (S_i), (u_i)\rangle$ be a normal form game. Then for all $i \in N$,*

$$\begin{aligned}&\max_{\mathcal{P}_i\subseteq\Delta S_i} U_i(\mathcal{P}_i,\mathcal{P}_{-i}) = \max_{P_i\in\Delta S_i}\min_{P_{-i}\in\mathcal{P}_{-i}} u_i(P_i,P_{-i})\,,\\ \textit{and}\quad &\min_{\mathcal{P}_i\subseteq\Delta S_i} U_i(\mathcal{P}_i,\mathcal{P}_{-i}) = \min_{P_i\in\Delta S_i}\min_{P_{-i}\in\mathcal{P}_{-i}} u_i(P_i,P_{-i})\,.\end{aligned}$$

With Corollary 1.13 we thus derive from Proposition 1.11 and Lemma 1.12 that in an

Ellsberg equilibrium $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$, for all $i \in N$,

$$P_i \in \arg\max_{P_i' \in \Delta S_i} U_i(P_i', \mathcal{P}_{-i}^*) \text{ for all } P_i \in \mathcal{P}_i^*\,,$$

or, put differently,

$$U_i(\mathcal{P}_i^*, \mathcal{P}_{-i}^*) = \max_{P_i \in \Delta S_i} \min_{P_{-i} \in \mathcal{P}_{-i}^*} u_i(P_i, P_{-i})\,. \tag{1.8}$$

Suppose the left hand side of (1.8) was strictly smaller than the right hand side. Then there exists a $P_i' \in \Delta S_i$ which maximizes the right hand side, but is not the minimizer of the left hand side. That means player i could deviate from $\mathcal{P}_i^*$ to a set of probability distributions $\mathcal{P}_i'$ with $P_i' \in \mathcal{P}_i'$ for which the minimizer of the left hand side of (1.8) was P_i' and obtain a minimal expected payoff that was higher than $U_i(\mathcal{P}_i^*, \mathcal{P}_{-i}^*)$. This contradicts the assumption that $(\mathcal{P}_i^*, \mathcal{P}_{-i}^*)$ was an Ellsberg equilibrium.

Thence, although U_i is defined as the minimal expected utility over all $\mathcal{P}_1, \dots, \mathcal{P}_n$, de facto player i uses only maximizers in his own Ellsberg equilibrium strategy $\mathcal{P}_i^*$, any other distribution would reduce his utility. In other words, all distributions P_i which player i uses in his Ellsberg equilibrium strategy yield the same utility, that is, player i is indifferent between any $P_i \in \mathcal{P}_i^*$. This is formulated in the following theorem.

Theorem 1.14 (Principle of Indifference in Distributions)**.** *Let $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$ be an Ellsberg equilibrium of a normal form game $G = \langle N, (S_i), (u_i)\rangle$. Then for all $P_i \in \mathcal{P}_i^*$,*

$$\min_{P_{-i} \in \mathcal{P}_{-i}^*} u_i(P_i, P_{-i}) = c \quad \textit{for some } c \in \mathbb{R}\,.$$

Furthermore, a necessary condition for his equilibrium strategy not to be a singleton is therefore that U_i is constant for some strategy $\mathcal{P}_{-i}$. However, this is not a sufficient condition for a strategy to be an Ellsberg equilibrium strategy. When $\mathcal{P}_i$ is a best response to some $\mathcal{P}_{-i}$ then all $P_i \in \mathcal{P}_i$ yield the same minimal expected utility, but $\mathcal{P}_i$ is only an equilibrium strategy, if $\mathcal{P}_{-i}$ is one, too.

An important property of the payoff functions is that they are linear in probabilities. In particular, with the equality established in Proposition 1.11, $U_i(\mathcal{P}_i^*, \mathcal{P}_{-i}^*)$ is linear on ΔS_i for each P_i and linear on $\mathcal{P}_j^*$ for each $P_j \in \mathcal{P}_j^*$, $j \neq i$. This observation suffices to fulfill the assumptions (multilinear functions are convex in each variable) to Fan's Minimax Theorem in Fan (1952). Thus, we have the following minimax theorem for Ellsberg games.

Theorem 1.15 (Minimax Theorem 1)**.** *In an Ellsberg equilibrium $(\mathcal{P}_i^*, \mathcal{P}_{-i}^*)$, for all $i \in N$,*

$$\max_{P_i \in \Delta S_i} \min_{P_{-i} \in \mathcal{P}_{-i}^*} u_i(P_i, P_{-i}) = \min_{P_{-i} \in \mathcal{P}_{-i}^*} \max_{P_i \in \Delta S_i} u_i(P_i, P_{-i})\,.$$

We present more minimax results in Section 3.5 on zero-sum games.

1.3.6 Strict Domination in Ellsberg Games

In this section we show that strictly dominated strategies are never used in Ellsberg equilibria. The question under which conditions *weakly* dominated strategies can be in the support of an Ellsberg equilibrium is postponed to Section 4.2.2. In a normal form game $G = \langle N, (S_i), (u_i)\rangle$ a pure strategy $s_i \in S_i$ is *strictly dominated* if there exists a mixed strategy $P_i \in \Delta S_i$ that yields a higher utility for player i for any strategy profile $s_{-i} \in S_{-i}$.

Proposition 1.16. *Any Ellsberg equilibrium strategy profile $(\mathcal{P}_i^*)_{i\in N}$ must put weight only on strategies that are not strictly dominated.*

Proof. Let $\tilde{s}_i$ be strictly dominated. Then there exists a mixed strategy $P_i \in \Delta S_i$ with $U_i(P_i, \mathcal{P}_{-i}) > U_i(\tilde{s}_i, \mathcal{P}_{-i})$ for all $\mathcal{P}_{-i} \subseteq \Delta S_{-i}$. Let $(\mathcal{P}_i^*, \mathcal{P}_{-i}^*)$ be an Ellsberg equilibrium profile. Suppose for some player $i \in N$, $P_i^*(\tilde{s}_i) > 0$ for some $P_i^* \in \mathcal{P}_i^*$. Then we have:

$$
\begin{aligned}
U_i(\mathcal{P}_i^*, \mathcal{P}_{-i}^*) &= \min_{P_{-i}^*\in\mathcal{P}_{-i}^*} U_i(\mathcal{P}_i^*, P_{-i}^*) = \min_{P_{-i}^*\in\mathcal{P}_{-i}^*} u_i(P_i^*, P_{-i}^*) \\
&= \min_{P_{-i}^*\in\mathcal{P}_{-i}^*} \sum_{s_i\in S_i} u_i(s_i, P_{-i}^*)\, P_i^*(s_i) \\
&= \min_{P_{-i}^*\in\mathcal{P}_{-i}^*} \sum_{s_i\neq\tilde{s}_i} u_i(s_i, P_{-i}^*) P_i^*(s_i) + u_i(\tilde{s}_i, P_{-i}^*) P_i^*(\tilde{s}_i) \\
&< \min_{P_{-i}^*\in\mathcal{P}_{-i}^*} \sum_{s_i\neq\tilde{s}_i} u_i(s_i, P_{-i}^*) P_i^*(s_i) + u_i(P_i, P_{-i}^*) \\
&= \min_{P_{-i}^*\in\mathcal{P}_{-i}^*} \sum_{s_i\neq\tilde{s}_i} u_i(s_i, P_{-i}^*) P_i^*(s_i) + \sum_{s_i\in S_i} u_i(s_i, P_{-i}^*) P_i(s_i)\,. \qquad (1.9)
\end{aligned}
$$

We construct a distribution $P_i' \in \Delta S_i$:

$$
P_i'(s_i) := \begin{cases} P_i^*(s_i) P_i(s_i) & \text{for } s_i \neq \tilde{s}_i\,, \\ P_i(s_i) & \text{for } s_i = \tilde{s}_i\,. \end{cases}
$$

With this distribution we can write

$$
(1.9) \;=\; \min_{P_{-i}^*\in\mathcal{P}_{-i}^*} \sum_{s_i\in S_i} u_i(s_i, P_{-i}^*) P_i'(s_i) = U_i(P_i', \mathcal{P}_{-i^*})\,.
$$

This is a contradiction to the assumption that $(\mathcal{P}_i^*, \mathcal{P}_{-i}^*)$ was an Ellsberg equilibrium. □

From Proposition 1.16 we get an immediate corollary.

Corollary 1.17. *In two-person games, when one round of elimination of dominated strategies yields a unique strategy profile, this is the unique Ellsberg equilibrium.*

The iterated elimination of dominated strategies raises the question of rationalizability in Ellsberg games. It is subject to further research to determine the exact characterization and interdependence of rationalizability and dominance in preference models that account for aversion to objective ambiguity, but we give a brief overview over the main existing contributions in the area of ambiguity-averse preferences.

In classic game theory, the concept of Nash equilibrium has often been criticized in that it assumes correct beliefs in equilibrium. A weaker solution concept that circumvents this criticism is rationalizability. If one allows for correlation in the definition of rationalizability, then in finite normal form games the set of correlated rationalizable strategies coincides with the set of strategies that survive iterated elimination of strictly dominated strategies. Brandenburger and Dekel (1987) show that these solution concepts are equivalent, in some suitable sense, to *a posteriori equilibrium* introduced by Aumann (1974) as a strengthening of subjective correlated equilibrium.

Naturally, it is interesting how these solution concepts relate to each other when we depart from the Bayesian framework and allow for non-expected utility preferences. Notably, four papers provide results on this relation. Epstein (1997) analyzes finite normal form games with two players, where he excludes explicit randomization over pure strategies. He finds that in this case, iterated deletion of strictly dominated strategies and multiple-prior-rationalizability do in general not lead to the same set of strategies. The equivalence is obtained when one allows for explicit randomization, as shown by Klibanoff (1996). This equivalence is generalized by Lo (2000), who shows that rationalizable behavior is indistinguishable for any model of preference satisfying Savage's axiom P3 (eventwise monotonicity). Finally, Chen and Luo (2012) show indistinguishability for another class of payoff functions and ambiguity-averse preferences.

1.4 Non-Nash Outcomes: Strategic Use of Ambiguity in Negotiation Games

Strategic ambiguity can lead to new phenomena that lie outside the scope of classical game theory. To give the reader a first impression of interesting Ellsberg equilibria, we consider the following peace negotiation game taken from Greenberg (2000). There are two small countries who can either opt for peace, or war. If both countries opt for peace, all three players obtain a payoff of 4. If one of the countries does not opt for peace, war breaks out, but the superpower cannot decide whose action started the war. The superpower can punish one country and support the other. The game tree is in Figure 1.3 below.[10]

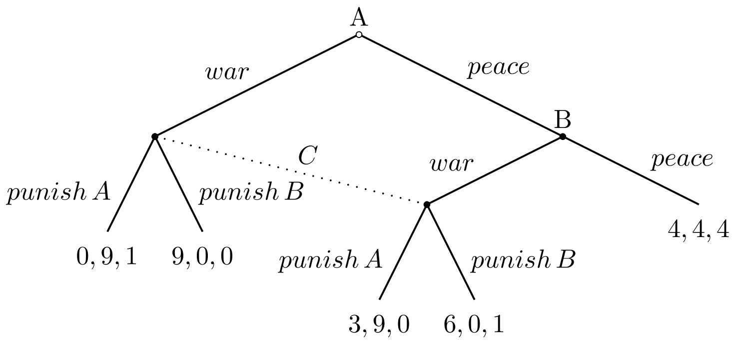

Figure 1.3: Peace negotiation.

As we deal only with static equilibrium concepts in the first chapters, we also present the normal form in Figure 1.4. In the normal form representation of the game country A chooses rows, country B columns, and the superpower chooses the matrix.

	war	*peace*
war	0, 9, 1	0, 9, 1
peace	3, 9, 0	4, 4, 4

punish A

	war	*peace*
war	9, 0, 0	9, 0, 0
peace	6, 0, 1	4, 4, 4

punish B

Figure 1.4: Peace negotiation in normal form.

This game possesses a unique Nash equilibrium where country A mixes with equal probabilities, and country B opts for war. The superpower has no clue who started the war given these strategies. It is thus indifferent about whom to punish and mixes with equal probabilities, as well. War occurs with probability 1. The resulting equilibrium payoff vector is $(4.5, 4.5, 0.5)$.

[10] We take the payoffs as in Greenberg's paper. In case the reader is puzzled by the slight asymmetry between country A and B in payoffs: it does not play a role for our argument. One could replace the payoffs 3 and 6 for country A by 0 and 9.

If the superpower can create ambiguity (and if the countries A and B are ambiguity-averse), the picture changes. Suppose for simplicity, that the superpower creates maximal ambiguity by using a device that allows for any probability between 0 and 1 for its strategy *punish A*. The pessimistic players A and B are ambiguity-averse and thus maximize against the worst case. For both of them, the worst case is to be punished by the superpower, with a payoff of 0. Hence, both prefer to opt for peace given that the superpower creates ambiguity. As this leads to a very desirable outcome for the superpower, it has no incentive to deviate from this strategy. We have thus found an equilibrium where the strategic use of ambiguity leads to an equilibrium outcome outside the support of the Nash equilibrium outcome.

Let us formalize the above considerations. We claim that there is the following type of Ellsberg equilibria. The superpower creates ambiguity about its decision. If this ambiguity is sufficiently large, both players fear to be punished by the superpower in case of war. As a consequence, they opt for peace. In our game with just two actions for the superpower, we can identify an Ellsberg strategy with an interval $[P_0, P_1]$ where $P \in [P_0, P_1]$ is the probability that the superpower punishes country A. Suppose the superpower plays so with $P_0 < 4/9$ and $P_1 > 5/9$. Assume also that country B opts for *peace*. If A goes for war, it uses that prior in $[P_0, P_1]$ which minimizes its expected payoff, which is P_1. This yields $U_A(war, war, [P_0, P_1]) = P_1 \cdot 0 + (1 - P_1) \cdot 9 < 4$. Hence, opting for peace is country A's best reply. The reasoning for country B is similar, but with the opposite probability P_0. If both countries A and B go for peace, the superpower gets 4 regardless of what it does; in particular, the ambiguous strategy described above is optimal. We conclude that $(peace, peace, [P_0, P_1])$ is a (reduced form) Ellsberg equilibrium.

Proposition 1.18. *In Greenberg's game, the strategies* $(peace, peace, [P_0, P_1])$ *with* $P_0 < 4/9$ *and* $P_1 > 5/9$ *form an Ellsberg equilibrium.*

Note that this Ellsberg equilibrium is very different from the game's unique Nash equilibrium. In Nash equilibrium, war occurs in every play of the game. However, in our Ellsberg equilibrium, peace is the unique outcome. By using the strategy $[P_0, P_1]$ which is a set of probability distributions, the superpower *creates* ambiguity. This supports an Ellsberg equilibrium where players' strategies do not lie in the support of the unique Nash equilibrium. We also point out that the countries A and B use different *worst-case priors* in equilibrium, this is a typical phenomenon in Ellsberg equilibria that are supported by strategies which are not in the support of any Nash equilibrium of the game.

Other equilibrium concepts for extensive form games (without Knightian uncertainty) such as conjectural equilibrium Battigalli and Guaitoli (1988), self-confirming equilibrium Fudenberg and Levine (1993), subjective equilibrium Kalai and Lehrer (1995) and mutually acceptable courses of action Greenberg, Gupta, and Luo (2009) can also assure the peace equilibrium outcome in the example by Greenberg. Other equilibrium concepts for extensive form games with Knightian uncertainty are, e.g., Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2012) and Lo (1999). Postponing the analysis of the relation of these equilibrium concepts to Ellsberg equilibrium to Chapter 6 on dynamic Ellsberg games, we only want to stress here that in difference to the existing concepts the driving factor in Ellsberg equilibrium is that ambiguity is employed strategically and objectively.

Greenberg refers to historic peace negotiations between Israel and Egypt (countries A and B in the negotiation example) mediated by the USA (superpower C) after the 1973 war. As explained by Kissinger (1982)[11], the fact that both Egypt and Israel were too afraid to be punished if negotiations broke down partly contributed to the success of the peace negotiations. This story is supported by our Ellsberg equilibrium, a first evidence that Ellsberg equilibria might capture some real world phenomena better than Nash equilibria.

1.5 Related Literature

Several authors introduce Knightian uncertainty into complete-information normal form games. We discuss their concepts and compare them to our approach.

Dow and Werlang (1994), Lo (1996), Marinacci (2000), Eichberger and Kelsey (2000) and Eichberger, Kelsey, and Schipper (2009) all extend the interpretation of Nash equilibrium as an equilibrium in beliefs. For example, Dow and Werlang (1994) interpret their non-additive (Choquet) probabilities as uncertain beliefs about the other player's action. A pair (P_1, P_2) of non-additive probabilities is then a Nash equilibrium under Knightian uncertainty if each action in a support of player 1's belief P_1 is optimal given that he uses P_2 to evaluate his expected payoff, and similarly for player 2. We thus have here a first version of an equilibrium in beliefs. This approach is refined by Marinacci (2000) and extended to n-person games by Eichberger and Kelsey (2000).

Lo (1996) introduces the concept of equilibrium in beliefs under uncertainty where the beliefs are represented by multiple priors over other players' mixed strategies. Each player i has a set of beliefs B_i over what the other players do, so over ΔS_{-i}. The profile (B_i) then forms a beliefs equilibrium if player j puts positive weight only on strategies of player i that maximize i's minimal expected payoff given the belief set B_i. This concept allows for disagreement of players' beliefs, and for correlation. Lo therefore introduces the refinement of a beliefs equilibrium with agreement in which player j and k agree about

[11] See p. 802 therein, in particular.

player i's actions and the beliefs of i over j and k are independent. Lo proves the nice result that every beliefs equilibrium contains a Bayesian beliefs equilibrium (where the belief sets are singletons). As a corollary, he obtains a precursor of Bade (2011b)'s main theorem (which we discuss in a later paragraph): in two player games, every beliefs equilibrium contains a Nash equilibrium.

Note that all the equilibrium concepts discussed above do not specify which action will actually be played in equilibrium. In Lo (1996) players can play any pure or mixed strategy that is a best response to their belief set, in the other equilibrium notions mentioned, players only have access to pure strategies in the support of the capacities. This stands in contrast to Ellsberg equilibrium, where the equilibrium strategy is fixed by the Ellsberg urn chosen. The strategy is a best response to the belief, and the belief coincides with the strategy played.

Klibanoff (1996), Lehrer (2008) and Lo (2009) propose an approach similar to beliefs equilibrium. Uncertainty is present in players' beliefs that are represented by sets of distributions. Equilibrium is defined as a profile of beliefs and an objectively mixed (or pure) strategy for each player, which is the strategy that he plays in equilibrium. These strategies need to be contained in the belief sets. Accordingly, players have to anticipate their opponents' strategy correctly in the sense that the truth is part of their belief. This consistency requirement is weaker than in Nash equilibrium (and weaker than in Ellsberg equilibrium!) and typically the strategies in equilibrium are not best responses to the actual strategies played. Klibanoff (1996) proposes a refinement where only correlated rationalizable beliefs are allowed.[12] Lehrer (2008) develops a model of decision making under uncertainty with partially-specified probabilities, these are used to represent the players' uncertain beliefs about their opponents. Lo (2009) establishes formal epistemic foundations for an equilibrium concept with ambiguity-averse preferences. He finds that epistemically stochastic independence is not necessary for a generalized Nash equilibrium concept. A correlated Nash equilibrium is a pair $\langle \sigma, \Phi \rangle$ consisting of a profile of beliefs Φ_i and a profile of mixed strategies σ_i where, for consistency, each strategy a_i in the support of σ_i is a best response to the belief Φ_i.

Bade (2011b) goes a first step in another direction, away from the beliefs interpretation of Nash equilibrium. She allows players to use acts in the sense of Anscombe-Aumann and players are uncertainty-averse over such acts. In an ambiguous act equilibrium, players play best responses as in Nash equilibrium, but under the generalized framework. A large

[12] Lo (1996) requires every probability distribution in the belief sets to be a best response, therefore every beliefs equilibrium with agreement is a refinement of equilibrium with uncertainty aversion and rationalizable beliefs (this is shown in Lo (1996), Proposition 9).

class of ambiguity-averse preferences are covered. The possible priors for an ambiguous act are part of the players' preferences in her setup. Bade then adds some appropriate consistency properties (agreement on null events) to exclude unreasonably divergent beliefs, and she imposes the rather strong assumption that preferences are *strictly* monotone, following Klibanoff (1996) here. This excludes beliefs on the boundary of strategy sets; such degenerated beliefs are sometimes important, though. For example, it excludes Ellsberg urns with full ambiguity where it is only known that the probability for a red ball is between 0 and 1. Bade's main theorem establishes that under her assumptions, in two-person games the support of ambiguous act equilibria and the support of Nash equilibria coincide.

Note that Aumann (1974), Epstein (1997) and Azrieli and Teper (2011) (amongst others) have also defined games that have Anscombe-Aumann acts as strategies, but to different ends. Aumann (1974) defines such a general game, then imposes Savage expected utility and analyses properties of correlated and subjective equilibrium. Epstein (1997) analyses games very similar to Bade's, but is mainly interested in rationalizability and iterated deletion of strictly dominated strategies in the generalized framework. Azrieli and Teper (2011) define an extension of an incomplete-information game.

In difference to Bade (2011b)'s setup, we let ambiguity be an objective instrument that is not derived from subjective preferences. Players can credibly commit to play an Ellsberg urn with a given and known degree of ambiguity. In Ellsberg games players use devices that create ambiguity, thus we extend the objective random devices interpretation of Nash equilibrium. The articles cited above impose non-expected utility representations derived from subjective preferences, like maxmin expected utility by Gilboa and Schmeidler (1989), Choquet expected utility by Schmeidler (1989), or they fix only certain axioms to allow for a large class of ambiguity-averse preferences. To model the preferences in Ellsberg games we use the representation results by Gajdos, Hayashi, Tallon, and Vergnaud (2008) on attitude towards imprecise information which capture the objective ambiguity we have in mind.

We want to mention some other approaches that introduce Knightian uncertainty into games which are not as closely related to the concept of Ellsberg games. Crawford (1990) was one of the earliest to define an equilibrium concept which incorporated the relaxations of Von Neumann and Morgenstern axioms which were introduced to explain phenomena like the Ellsberg paradox. To this end, Crawford (1990) analyzes an equilibrium concept without the assumption of independence on the underlying preference relation. Stauber (2011) considers incomplete information games with ambiguous beliefs and analyzes the robustness of equilibria of these games. Finally, Perchet (2012) works in the context of repeated games with the possibility of monitoring. The information about the opponents' play is “ambiguously” disturbed and hence the notion of Nash equilibrium generalized. He also generalizes the Lemke-Howson algorithm to compute equilibria of two-player games. In an unpublished working paper, Di Tillio, Kos, and Messner (2012) develop the concept of ‘objective ambiguous strategies’ which leads to equilibrium concepts similar to Ellsberg equilibrium; however, the authors focus on the implications in mechanism design. Of course, the literature reviewed here is not exhaustive, but we hope to have mentioned all closely related concepts.

2 Immunization Against Strategic Ambiguity

The hedging effect of ambiguity aversion leads to what we call *immunization against strategic ambiguity*.[1] In normal form Ellsberg games players often have a strategy available that immunizes them against any ambiguity played by their opponents. In this chapter we look at these immunization strategies more closely. We start by giving an example and then, after defining immunization strategies, calculate the immunization strategies of a large class of 2×2 games. We then provide some examples which show that in certain games players immunize themselves against ambiguity by playing their maximin strategy. To generalize this observation, we prove under which circumstances in general two-player games the immunization strategy is the maximin strategy of the players. The last section uses the results in the chapter to characterize the role of immunization strategies in Ellsberg equilibria. We see that in many 2×2 games, Ellsberg equilibria are bounded by the immunization and the Nash equilibrium strategy.

2.1 Example of an Immunization Strategy

To get an intuition for immunization we take a slightly modified version of Matching Pennies as our example. The results generalize to a large class of two-person 2×2 games as we show below in Proposition 2.3. The payoff matrix for the modified Matching Pennies game is in Figure 2.1.

		Player 2	
		$HEAD$	$TAIL$
Player 1	$HEAD$	$3, -1$	$-1, 1$
	$TAIL$	$-1, 1$	$1, -1$

Figure 2.1: Modified Matching Pennies I.

In our modified version of Matching Pennies, the unique Nash equilibrium is that player 1 mixes uniformly over his strategies, and player 2 mixes with $(1/3, 2/3)$. This yields the

[1] Section 2.1 was published in the IMW working paper Riedel and Sass (2011).

equilibrium payoffs 1/3 and 0. One might guess that one can get an Ellsberg equilibrium where both players use a set of probability measures around the Nash equilibrium distribution as their strategy. This is, somewhat surprisingly, at least to us, not true.

The crucial point to understand here is the following. Players can immunize themselves against ambiguity; in the modified Matching Pennies example, player 1 can use the mixed strategy $(1/3, 2/3)$ to make himself independent of any ambiguity used by the opponent. Indeed, with this strategy, his expected payoff is 1/3 against any mixed strategy of the opponent, and a fortiori against Ellsberg strategies as well. This strategy is also the unique best reply of player 1 to Ellsberg strategies with ambiguity around the Nash equilibrium, in particular, such strategic ambiguity is not part of an Ellsberg equilibrium.

Let us explain this more formally. An Ellsberg strategy for player 2 can be identified with an interval $[Q_0, Q_1] \subseteq [0,1]$ where $Q \in [Q_0, Q_1]$ is the probability to play $HEAD$. Suppose player 2 uses many probabilities around 1/3, so $Q_0 < 1/3 < Q_1$. The minimal expected payoff for player 1 when he uses the mixed strategy with probability P for $HEAD$ is then

$$
\begin{aligned}
U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \le Q \le Q_1} 3PQ - P(1-Q) - (1-P)Q + (1-P)(1-Q) \\
&= \min_{Q_0 \le Q \le Q_1} 6PQ - 2P - 2Q + 1 \\
&= \min \{Q_0(6P-2), Q_1(6P-2)\} + 1 - 2P \\
&= \begin{cases} Q_1(6P-2) + 1 - 2P & \text{if } P < 1/3\,, \\ 1/3 & \text{if } P = 1/3\,, \\ Q_0(6P-2) + 1 - 2P & \text{if } P > 1/3\,. \end{cases}
\end{aligned} \tag{2.1}
$$

We plot the payoff function in Figure 2.2 for $Q_0 = 1/4$ and $Q_1 = 1/2$. By choosing the mixed strategy $P = 1/3$, player 1 becomes immune against any ambiguity and ensures the (Nash) equilibrium payoff of 1/3. If there was an Ellsberg equilibrium with $P_0 < 1/2 < P_1$ and $Q_0 < 1/3 < Q_1$, then the minimal expected payoff would be below 1/3. Hence, such Ellsberg equilibria do not exist.

2.2 Immunization Strategies

We call the strategy that renders the player immune against any ambiguity used by the opponents *immunization strategy.*

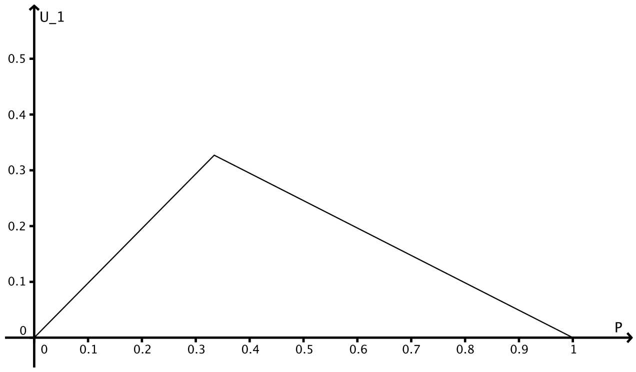

Figure 2.2: Player 1's (minimal expected) payoff (2.1) as a function of the probability P of playing $HEAD$ when player 2 uses the Ellsberg strategy $[1/4, 1/2]$.

Definition 2.1. *Let $G = \langle N, (S_i), (u_i)\rangle$ be a normal form game. A mixed strategy $\bar{P}_i \in \Delta S_i$ for player i is called an* immunization strategy*, if there exists a $v_i \in \mathbb{R}$ such that $U_i(\bar{P}_i, \mathcal{P}_{-i}) = v_i$ for all sets $\mathcal{P}_{-i} \subseteq \Delta S_{-i}$.*

Such immunization plays frequently a role in two-person games, and it need not always be the Nash equilibrium strategy that is used to render oneself immune. In fact, Nash equilibrium and immunization are in some sense opposite concepts: with a Nash equilibrium strategy the player wants to make his opponent indifferent between all his strategies, and with an immunization strategy the player wants to make *himself* indifferent. Consider, e.g., the slightly changed payoff matrix in Figure 2.3.

		Player 2	
		$HEAD$	$TAIL$
Player 1	$HEAD$	$1, -1$	$-1, 1$
	$TAIL$	$-2, 1$	$1, -1$

Figure 2.3: Modified Matching Pennies II.

In the unique Nash equilibrium, player 1 still plays both strategies with probability 1/2 (to render player 2 indifferent); however, in order to be immune against Ellsberg strategies, he has to play $HEAD$ with probability 3/5. Then his payoff is $-1/5$ regardless of what player 2 does. This strategy does not play any role in Nash equilibrium, but note that the payoff to the strategy is the same as in the unique Nash equilibrium in which player

1 plays $HEAD$ with probability 1/2 and player 2 with probability 2/5. In fact, every strategy of player 1 in the interval $[1/2, 3/5]$ yields the same maxmin payoff, which makes this strategy a candidate for an Ellsberg equilibrium strategy.

For solving an Ellsberg game it is helpful to plot the (minimal expected) payoff function as in Figure 2.2. Nevertheless, to understand the general properties of Ellsberg games and their relation to the standard analysis of normal form games, we also have a look at the expected payoff function of the players.

In Figure 2.4 we show player 1's expected payoff function for Modified Matching Pennies I in two different perspectives. Studying the plots, one can see that the payoff is indeed independent of Q at $P = 1/3$, and, furthermore, $P = 1/3$ yields the maximal minimal expected payoff for player 1. So $P = 1/3$ is player 1's maximin strategy. We say more about the general relation between immunization strategies and maximin strategies in Section 2.4.

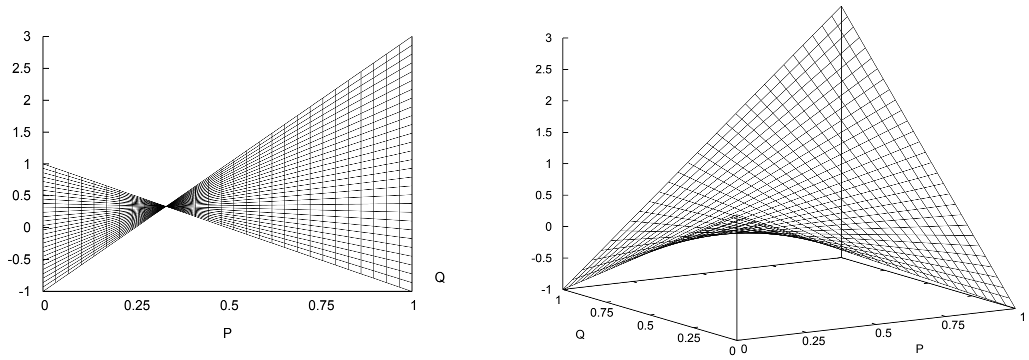

Figure 2.4: Player 1's expected payoff function of the modified Matching Pennies I game, $u_1(P, Q) = 6PQ - 2P - 2Q + 1$, in two different perspectives.

Because of the, in some sense, inverse relation to Nash equilibrium strategies, the necessary conditions for the existence of completely mixed immunization strategies are similar to the existence of completely mixed Nash equilibrium. However, for immunization strategies these are also sufficient. We call a strategy *completely mixed* if it puts positive probability on every pure strategy. Denote with A (B) the payoff matrix of player 1 (2). Then player 1(2) has an immunization strategy if and only if no convex combination of columns (rows) of A (B) dominates convex combinations of other columns (rows). However, the following example shows that existence of completely mixed Nash equilibrium strategies does not imply existence of completely mixed immunization strategies.

Example 2.2. *Consider the game with the following payoff matrix for player 1:*

Player 1	U	2	−1
	D	1	1

As Nash equilibrium strategy player 2 would choose a convex combination of columns such that player 1 is indifferent between U *and* D*, that is,* $(Q^*, 1 - Q^*) = (2/3, 1/3)$*. But to immunize himself, player 1 plays* U *with zero probability and* D *with probability one, which is not a completely mixed strategy.*

2.3 Immunization Strategies of General 2×2 Games

We calculate the immunization strategies of a general two-person 2×2 game. Immunization strategies $(\bar{P}, 1-\bar{P})$ and $(\bar{Q}, 1-\bar{Q})$ for players 1 and 2 are fully described by the probability $\bar{P}$ to play U and $\bar{Q}$ to play L. We use this abbreviated notation also for Nash equilibrium $(P^*, Q^*) := ((P^*, 1 - P^*), (Q^*, 1 - Q^*))$. We have the following proposition.

Proposition 2.3. *Let* $G = \langle \{1, 2\}, (S_i), (u_i) \rangle$ *be a normal form game with payoff matrix*

		Player 2 L	R
Player 1	U	a, e	b, f
	D	c, g	d, h

Then the immunization strategy $\bar{P}$ *of player 1 is*

$$\bar{P} = \frac{d - c}{a - b - c + d},$$

and the immunization strategy $\bar{Q}$ *of player 2 is*

$$\bar{Q} = \frac{h - f}{e - f - g + h},$$

if $0 \leq \bar{P} \leq 1$ *and* $0 \leq \bar{Q} \leq 1$*. That* $\bar{P}$ *and* $\bar{Q}$ *lie between* 0 *and* 1 *is fulfilled when* $d - c \geq 0 > b - a$ *or* $d - c \leq 0 < b - a$ *and, respectively,* $h - f \geq 0 > g - e$ *or* $h - f \leq 0 < g - e$*.*

Proof. We show this only for player 1, the calculation for player 2 is analog. Let $P \in [P_0, P_1]$ denote the probability with which player 1 plays U, and $Q \in [Q_0, Q_1]$ the probability with which player 2 plays L. Assume first that $\bar{P} = \frac{d-c}{a-b-c+d}$. To assure that

$0 \leq \frac{d-c}{a-b-c-d} \leq 1$ we have either

$$\begin{aligned} &d-c \geq 0 \text{ and } a-b-c+d > 0 \text{ and } a-b-c+d \geq d-c\,, \\ \text{or } &d-c \leq 0 \text{ and } a-b-c+d < 0 \text{ and } a-b-c+d \leq d-c\,. \end{aligned}$$

From the first case follows $d-c \geq 0 > b-a$, from the latter case $d-c \leq 0 < b-a$. Now assume without loss of generality that $d-c \geq 0 > b-a$. We derive the minimal expected utility of player 1.

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q\in[Q_0,Q_1]} aPQ + bP(1-Q) + c(1-P)Q + d(1-P)(1-Q) \\ &= \min_{Q\in[Q_0,Q_1]} (a-b-c+d)PQ + (b-d)P + (c-d)Q + d \\ &= \min_{Q\in[Q_0,Q_1]} Q((a-b-c+d)P + c - d) + (b-d)P + d \\ &= \begin{cases} Q_0((a-b-c+d)P + c - d) + (b-d)P + d & \text{if } P > \frac{d-c}{a-b-c+d}\,, \\ \frac{(b-d)(d-c)}{a-b-c+d} + d & \text{if } P = \frac{d-c}{a-b-c+d}\,, \\ Q_1((a-b-c+d)P + c - d) + (b-d)P + d & \text{if } P < \frac{d-c}{a-b-c+d}\,. \end{cases} \end{aligned}$$

Under the assumption that $0 \leq P \leq 1$, the utility function is constant for all $[Q_0, Q_1]$ if and only if $P = \frac{d-c}{a-b-c+d}$. □

The completely mixed Nash equilibrium of the above general game, if it exists, is

$$(P^*, Q^*) = \left(\frac{h-g}{e-f-g+h}\,, \frac{d-b}{a-b-c+d} \right).$$

It is noticeable that the immunization strategies are similar to the opponent's Nash equilibrium strategy. In fact, some of the standard two-person 2×2 games have the property that the immunization strategies are *exactly* the Nash equilibrium strategies of the opponent. This is the case when $c = b$ for player 1 and $f = g$ for player 2. This is what we will assume in Proposition 3.7 on the Ellsberg equilibria of general competitive games in order to make the exposition of the result more comprehensive.

2.4 Immunization as Maximin Strategy

An immunization strategy does not only make a player i indifferent about any ambiguity used by the opponents, it makes him also indifferent to any pure or mixed strategies the opponents might play. Playing the immunization strategy a player can assure himself some payoff v_i that is independent of the opponents' behavior. If this payoff v_i is not less than the maximal Nash equilibrium payoff v_i^*, it can be argued that playing the immunization strategy is smarter than playing the Nash equilibrium strategy. The Nash equilibrium strategy bears the risk of yielding a smaller payoff, if, e.g., some opponent makes a mistake.

A similar phenomenon had been discussed in Aumann and Maschler (1972). The authors notice that in some (and not only in zero-sum!) games the maximin payoff is the same as the Nash equilibrium payoff. In a very instructive paper, Pruzhansky (2011) analyses this observation further and finds, that this is the case if and only if there exists a completely mixed Nash equilibrium and the maximin strategy is an *equalizer*. What the author calls an equalizer is the same as an immunization strategy, only for pure and mixed strategies and not for Ellsberg strategies, that is a strategy $\bar{P}_i$ for which the expected payoff is constant for all strategies $P_{-i} \in \Delta S_{-i}$. It is immediate that every equalizer strategy is an immunization strategy. If the expected payoff $u_i(\bar{P}_i, P_{-i})$ is constant for all $P_{-i} \in \Delta S_{-i}$, then $\min_{P_{-i} \in \mathcal{P}_{-i}} u_i(\bar{P}_i, P_{-i})$ is also constant for all $\mathcal{P}_{-i} \subset \Delta S_{-i}$. The converse is also true, since if $\min_{P_{-i} \in \mathcal{P}_{-i}} u_i(\bar{P}_i, P_{-i})$ is constant for some fixed $\bar{P}_i$ and all $\mathcal{P}_{-i} \subset \Delta S_{-i}$, then it is also constant for all $P_{-i} \in \Delta S_{-i}$.

We now state the definition of maximin strategies and discuss existence and uniqueness.

Definition 2.4. *Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game. A mixed strategy $P_i \in \Delta S_i$ is a* maximin strategy *for player $i \in N$ if it maximizes his minimal expected payoff, that is*

$$P_i \in \arg \max_{P_i' \in \Delta S_i} \min_{P_{-i} \in \Delta S_{-i}} u_i(P_i', P_{-i}) \,.$$

We know that maximin strategies always exist, since u_i is continuous and ΔS_i compact and convex for all $i \in N$. But maximin strategies are in general not unique as the following example shows. We look at two different 2×2 games, one with many maximin strategies, the other with a unique one. The non-uniqueness of the maximin strategy of u_1 in the following example is due to the fact that the Nash equilibrium strategy Q^* of player 2 lies on the boundary, that is, Q^* is not completely mixed.

Whenever we speak about 2×2 games, mixed strategies are represented by one variable per player, typically in our notation $(P, 1-P)$ for player 1 and $(Q, 1-Q)$ for player 2. The general payoff function $u_1(P, Q)$ of player 1 with payoff matrix

$A =$

a	b
c	d

is then

$$u_1(P,Q) = aPQ + bP(1-Q) + c(1-P)Q + d(1-P)(1-Q)$$
$$= (a-b-c+d)PQ + (b-d)P + (c-d)Q\,.$$

Example 2.5 (Unique and non-unique maximin strategies)**.** *We look at two payoff functions for player 1, $u_1(P,Q) = 2PQ - Q$ and $u_1'(P,Q) = 2PQ - Q - P$, which are for example the payoff functions of the following payoff matrices.*

u_1 :

	L	R
U	1	0
D	-1	0

u_1' :

	L	R
U	0	-1
D	-1	0

Figure 2.5: Two games with unique and non-unique maximin strategies.

The functions are plotted in Figure 2.6 and 2.7, respectively, each time shown in the two perspectives P and Q. We see that in Figure 2.6 the indifference strategy of player 2 is

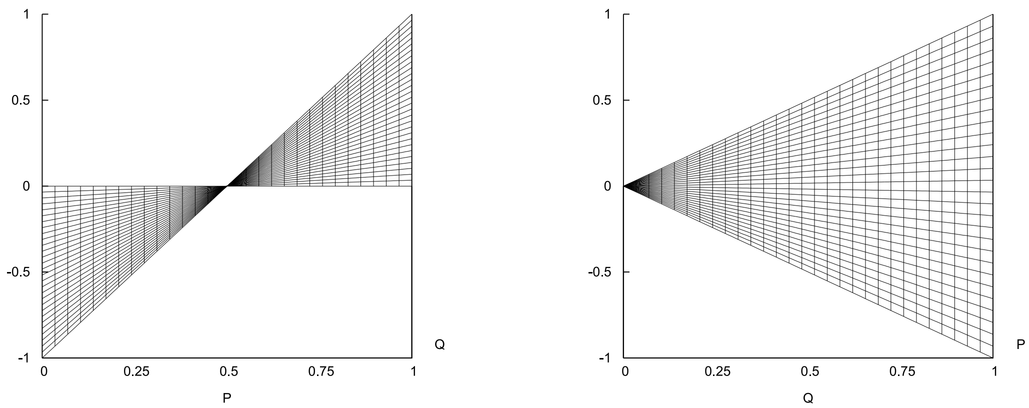

Figure 2.6: $u_1(P,Q) = 2PQ - Q$.

zero, thus not completely mixed. The minimum over Q is 0 whenever $P \geq 1/2$ and else $2P-1$, thus the maximin strategy is not unique. In Figure 2.7 the indifference strategy is $Q^* = 1/2$. The minimum over Q is $-P$ when $P \geq 1/2$ and $P-1$ else. This function is maximized over P when $P = 1/2$. This is the unique maximin strategy of player 1.

Finally, note that immunization strategies are not always maximin strategies as one can

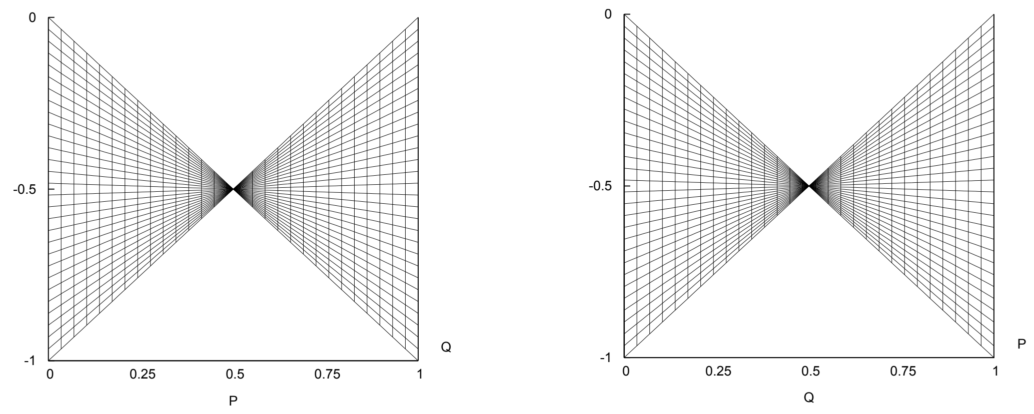

Figure 2.7: $u_1'(P,Q) = 2PQ - Q - P$.

see immediately in the following example. The game does not have a completely mixed Nash equilibrium. We prove in the next section that this is a necessary condition for a game to have immunization strategies that are maximin.

Example 2.6. *In the game in Figure 2.8 below, player 1's maximin strategy is U, and his only immunization strategy is D.*

		Player 2	
		L	R
Player 1	U	$2,1$	$1,0$
	D	$0,0$	$0,1$

Figure 2.8: Game in which the immunization strategy is not the maximin strategy.

2.5 Immunizing is Playing Against Your Zero-Sum-Self

Theorem 2.8 in this section relates maximin strategies and immunization strategies for two-player games with n strategies for each player. We show that if a two-player game with such square payoff matrices has a completely mixed Nash equilibrium, then an immunization strategy is a maximin strategy.

We use the following notation. A two-person normal form game G is described by the payoff matrices A and B of player 1 and player 2, we write $G = (A, B)$. We call G square when A and B are square matrices; in this section we only consider square games. The row vectors of A (B) are denoted by subscripts, a_i (b_i) $\in \mathbb{R}^n$, and the column vectors by superscripts, a^j (b^j) $\in \mathbb{R}^n$. In the following it will be convenient to write the expected utility of a mixed strategy P of player 1 with payoff matrix A as PAQ when player 2 plays mixed strategy Q. Transpose signs are suppressed.

A necessary condition for G to have a completely mixed Nash equilibrium is that no player has weakly or strictly dominated strategies. That is, no row (column) of A (B) is dominated by another row (column) or a convex combination of rows (columns). This condition can be expressed as follows. Let $\tilde{A}$ be the $(n+1) \times n$-matrix consisting of the matrix A with an additional last column $(1, \ldots, 1)$, and $\tilde{B}$ the $n \times (n+1)$-matrix with an additional last row $(1, \ldots, 1)$. Furthermore let $\tilde{k} = (k, \ldots, k, 1)$ and $\tilde{l} = (l, \ldots, l, 1)$ with $k, l \in \mathbb{R}$. Then G has a completely mixed Nash equilibrium $(P, Q) = (P^*, Q^*)$ when it is a nonnegative solution to the two systems of linear equations

$$P\tilde{B} = \tilde{k}\,,$$
$$\tilde{A}Q = \tilde{l}\,.$$

The existence of immunization strategies can be expressed analogously: no column (row) of matrix A (B) is dominated by another column (row) or a convex combination of columns (rows). Let $\tilde{u} = (u, \ldots, u, 1)$ and $\tilde{v} = (v, \ldots, v, 1)$ with $u, v \in \mathbb{R}$. Player 1 has an immunization strategy $P = \bar{P}$ in G when the vector is a nonnegative solution to the system

$$P\tilde{A} = \tilde{u}\,,$$

and player 2 has an immunization strategy $Q = \bar{Q}$ in G when the vector is a nonnegative solution to the system

$$\tilde{B}Q = \tilde{v}\,.$$

Note that for the existence of a completely mixed Nash equilibrium both solutions P^* and Q^* have to exist, whereas the immunization strategy is defined for a single player.

To a square game G we define *associated zero-sum games* G^1 and G^2. G^1 is the game with payoff matrices $(A, -A)$, G^2 the game with payoff matrices $(-B, B)$. We first prove the following lemma.

Lemma 2.7. *Let G be a square two-person normal form game, G^1 and G^2 its associated zero-sum games. If G has a completely mixed Nash equilibrium (P^*, Q^*) and player 1 (2) has an immunization strategy $\bar{P}$ ($\bar{Q}$), then G^1 (G^2) has a Nash equilibrium (P_1^*, Q_1^*) ((P_2^*, Q_2^*)) where $P_1^* = \bar{P}$ and $Q_1^* = Q^*$ ($P_2^* = P^*$ and $Q_2^* = \bar{Q}$).*

Proof. We show this only for player 1, the case for player 2 follows analogously. If (P^*, Q^*) is a completely mixed Nash equilibrium of G, then Q^* solves the system $\tilde{A}Q = \tilde{l}$ for some $l \in \mathbb{R}$. Furthermore, if $\bar{P}$ is an immunization strategy of player 1, then $\bar{P}$ solves the system $P\tilde{A} = \tilde{u}$ for some $u \in \mathbb{R}$. Then $\bar{P}$ also solves $P(\widetilde{-A}) = \widetilde{-u}$ and is therefore a Nash equilibrium strategy for player 1 of the game G^1. □

Now, we present and prove the main result of the section which relates immunization and maximin strategies of a square two-person normal form game.

Theorem 2.8. *Let G be a square two-person normal form game with a completely mixed Nash equilibrium (P^*, Q^*). If player 1 (2) has an immunization strategy $\bar{P}$ $(\bar{Q})$ in G, then $\bar{P}$ $(\bar{Q})$ is a maximin strategy of player 1 (2).*

Proof. We prove the result only for player 1. $\bar{P}$ is an immunization strategy of player 1 in the game G if and only if it is a nonnegative solution to the system $P\tilde{A} = \tilde{u}$. Now by assumption there exists a completely mixed Nash equilbrium (P^*, Q^*) of G, thus with Lemma 2.7 there exists a completely mixed Nash equilibrium (P_1^*, Q_1^*) in G^1. P_1^* is therefore a nonnegative solution to the system $P(\widetilde{-A}) = \tilde{k}$ for some $k \in \mathbb{R}$. Then P_1^* also solves the system $P\tilde{A} = (\widetilde{-k})$ and we see that P_1^* must be the immunization strategy $\bar{P}$ of player 1.

Now, since G^1 is a zero-sum game, $\bar{P}$ is by the classic Minimax Theorem for zero-sum games (which we state in Theorem 3.15) also the maximin strategy in G^1, that means

$$\bar{P} \in \arg\max_P \min_Q PAQ\,.$$

This condition only depends on the payoff matrix A of player 1, thus $\bar{P}(= P_1^*)$ is the maximin strategy in any game where player 1 has the payoff matrix A, in particular in the game G.

To show the analog statement for player 2 we use the associated zero-sum game G^2. □

Lemma 1 in Pruzhansky (2011) proves a result very close to our Theorem 2.8, but with different means by making use of geometric properties of the various types of strategies. The interpretation of the result gets an interesting twist when we use the associated zero-sum games as in our proof. A player who is immunizing himself against strategic ambiguity blindfolds himself and drops the strategic element of the game. Instead of playing the original game, he plays the associated zero-sum game which results from his own payoff matrix. Thus, he plays his optimal strategy $\bar{P}$, the immunization strategy, against an imaginary self that receives the negative of his own payoff.

The assumption of a completely mixed Nash equilibrium in Theorem 2.8 is necessary. In the following example we construct a game with a unique Nash equilibrium in pure strategies (which is also the unique Ellsberg equilibrium) that has completely mixed immunization strategies which are not maximin.

Example 2.9. *Consider the game with payoff matrix*

		Player 2: L	R
Player 1	U	$1, 1$	$0, 2$
	D	$2, 0$	$3, 3$

It has the unique Nash equilibrium in pure strategies (D, R) *which is also the unique Ellsberg equilibrium, and the completely mixed immunization strategy* $(1/2, 1/2)$ *for each player. The unique maximin strategies are the pure strategies D for player 1 and R for player 2.*

2.6 Concealing Nash Equilibrium and Maximin Behavior: Immunization as Ellsberg Equilibrium Strategy

When do players use immunization strategies in Ellsberg equilibrium? We saw in Example 2.9 that immunization strategies are not always used in Ellsberg equilibria, on the other hand, we will see in the next chapter that in many games immunization strategies play a decisive role in Ellsberg equilibrium. We explain why, in 2×2 games, this is the case.

It is well-known that only in two-person zero-sum games the Nash equilibrium strategies are always the maximin strategies. When an immunization strategy (that is, the maximin strategy in games with a completely mixed Nash equilibrium) is *not* the Nash equilibrium strategy, the game has proper Ellsberg equilibria and the immunization strategy plays a special role in Ellsberg equilibrium. If and only if the immunization strategy is the Nash equilibrium strategy (this happens only rarely in general two-person games) immunization strategies are used only in Ellsberg equilibria with unilateral full ambiguity (that is, one player plays his immunization strategy, the other anything in the whole set of probability distributions that includes his Nash equilibrium strategy). This we prove in Theorem 4.2. This type of Ellsberg equilibrium occurs, for example, in symmetric coordination games or of course in zero-sum games. As a result of Theorem 2.8, in two-person zero-sum games, immunization strategies are thus always used in equilibrium, but only in Ellsberg equilibria with unilateral full ambiguity.

We now look at the former case in detail, when the immunization strategy is not the Nash equilibrium strategy.

As long as the Nash equilibrium strategy of the opponent is the distribution used by the player to evaluate his maxmin expected utility, the player is indifferent between any mixed strategy, as the classical Principle of Indifference states. The Nash equilibrium strategy is only used when it is at the boundary of the Ellsberg strategy of the opponent, and the player is only indifferent between the strategies that lie between his immunization strategy and the boundary. Any strategy outside that set makes him change his utility evaluation to the other boundary of the opponents' Ellsberg strategy. Therefore the Ellsberg equilibrium strategy of the player cannot properly contain the immunization strategy in a similar way as it cannot properly contain the Nash equilibrium strategy. The only candidates are thus always between the immunization and the Nash equilibrium strategy, or between the Nash equilibrium strategy and the boundary of the unit interval, if the immunization strategy does not lie in between. In particular, in 2×2 coordination games of the type Battle of the Sexes, there is always an Ellsberg equilibrium in which both players use the set of probabilities between the Nash equilibrium and the maximin strategy, that is $([P^*, M_1], [Q^*, M_2])$ (compare Proposition 3.2). In these games, the strategic use of ambiguity therefore implies concealment of Nash equilibrium and maximin behavior.

3 Strategic Use of Ambiguity in Two-Player Normal Form Games

In this chapter we present a number of examples and properties of Ellsberg equilibria in normal form games with two players.[1] We start with an explanation on how Ellsberg equilibria of two-person 2×2 games are computed. Subsequently, we look at different classes of games: coordination games (Section 3.2), conflict games (Section 3.3) and zero-sum games (Section 3.5). We analyze examples and compute the Ellsberg equilibria of these classes of normal form games. The Ellsberg equilibria of general coordination and conflict games are summarized in Table 3.1 in Section 3.4. For zero-sum games we also show that Ellsberg equilibria are value preserving. In Section 3.6 we calculate the Ellsberg equilibria of some classic 2×2 games such as, e.g., Hawk and Dove and the Prisoners' Dilemma. We explain with an example by Myerson, that also games with linear (as opposed to in Ellsberg games often observed piecewise linear) payoff functions can have proper Ellsberg equilibria.

We close the chapter with an analysis of a special class of 3×3 games. We consider a slightly modified version of Rock Scissors Paper, and with the help of our experience with 2×2 games we succeed in calculating the Ellsberg equilibria of these games. For a better understanding of how the equilibria are determined, we provide a geometric analysis of the equilibrium strategies in the two-dimensional probability simplex.

With this chapter I wish to provide a sort of compendium of the Ellsberg equilibria in the most important normal form games and highlight some structure of Ellsberg equilibria in 2×2 games. The reader should be able to only read certain sections and fully understand how the Ellsberg equilibria are derived. For this reason the impression of repetition of certain steps in the calculations is unavoidable.

[1] Some parts of this chapter, namely Section 3.3.1 and the first half of Section 3.3.2, were published in the IMW working paper Riedel and Sass (2011).

3.1 Computing Ellsberg Equilibria in Two-Player 2×2 Games

We calculate the Ellsberg equilibria of two-player 2×2 games. Here Ellsberg strategies are completely described by the interval of probabilities $[P_0, P_1]$ with which one of the strategies is played. The other strategy is then played with some probability in the interval $[1 - P_1, 1 - P_0]$. In this section, we thus use the notation $[P_0, P_1]$ to denote the Ellsberg strategy of player 1, and $[Q_0, Q_1]$ to denote the Ellsberg strategy of player 2. When it is useful, we also use the notation $\mathcal{P}$ and $\mathcal{Q}$ for the Ellsberg strategies of player 1 and 2, sometimes $\mathcal{P}^*$ and $\mathcal{Q}^*$ is used to indicate Ellsberg equilibrium strategies. In all the 2×2 games we analyze in this chapter, we write $(P^*, Q^*) := ((P^*, 1 - P^*), (Q^*, 1 - Q^*))$ as a short hand for the mixed Nash equilibrium of the game (if it exists). P is the probability that player 1 plays U (or his upper strategy with a sometimes different name), and Q is the probability that player 2 plays L (or his left strategy wit a sometimes different name), throughout the chapter.

Following Proposition 1.11, in order to determine the Ellsberg equilibria of a two-person 2×2 game, we have to find the arg max over $P' \in \Delta S_1$ of player 1's minimal expected utility when P' is played against the equilibrium strategy $\mathcal{Q}$ of the opponent, and likewise for player 2. Thus, we start by calculating the minimal expected utility for both players, where we fix their respective mixtures $P \in \Delta S_1$, $Q \in \Delta S_2$ and minimize over the possible Ellsberg strategies of their opponent,

$$U_1(P, [Q_0, Q_1]) = \min\{u_1(P, Q_0), u_1(P, Q_1)\} ,$$

$$U_2([P_0, P_1], Q) = \min\{u_2(P_0, Q), u_2(P_1, Q)\} .$$

Typically this yields a piecewise linear function for each of the players, depending on the values of P and Q, respectively. Subsequently, we find the best response correspondences $B_1([Q_0, Q_1])$ and $B_2([P_0, P_1])$ by maximizing U_1 and U_2 over P and Q, respectively. Following the Principle of Indifference in Distributions (Theorem 1.14), to find proper Ellsberg equilibria, we analyze where the minimal utility function is flat, because these are best responses in form of intervals of probability distributions. As a final step to determine the Ellsberg equilibria, one finds the intersections of the best response correspondences.

Finally a remark on notation. We often express relations of the value of several variables at once. In oder to keep things as short as possible we introduce some abbreviated notation. For example, $0 \leq P \leq 1/2$ and $0 \leq Q \leq 1/2$ is abbreviated by $0 \leq P, Q \leq 1/2$. In the same way, when $a \geq b$ and $a \geq c$ and $d \geq b$ and $d \geq c$, we simply write $a, d \geq b, c$.

3.2 Strategic Use of Ambiguity in Two-Person Coordination

How can ambiguity be used in standard coordination games? In this section we analyze 2×2 games in which both players want to match each other. We start with an example of a coordination game where both players get the same payoff when they coordinate successfully. In what follows, we present a proposition which characterizes the Ellsberg equilibria of general coordination games.

The section closes with further interesting examples for two-person coordination. The first example is a non-symmetric coordination game; we see that this change to asymmetric payoffs has important implications for the Ellsberg equilibria of the game. The second example is of the type where the players would like to coordinate on different equilibria, that is, games of the type Battle of the Sexes. Thirdly, we discuss the electronic mail game by Rubinstein (1989) and see that coordination on the good equilibrium is more likely when players can use Ellsberg strategies.

3.2.1 Example of a Symmetric Coordination Game

To start, we calculate the Ellsberg equilibria of a symmetric coordination game with the payoffs given by the matrix in Figure 3.1.

		Player 2	
		L	R
Player 1	U	$1,1$	$0,0$
	D	$0,0$	$3,3$

Figure 3.1: Symmetric coordination game.

Note that the Nash equilibria of the game are two pure strategy equilibria (U, L), and (D, R), and a mixed strategy equilibrium

$$(P^*, Q^*) = ((3/4, 1/4), (3/4, 1/4)).$$

Let $[P_0, P_1]$ denote the set of probabilities with which player 1 plays U, and $[Q_0, Q_1]$ the set of probabilities with which player 2 plays L. $[P_0, P_1]$ and $[Q_0, Q_1]$ are thus Ellsberg strategies of player 1 and player 2, respectively. We first fix a distribution $P \in [P_0, P_1]$ for player 1 and consider his payoff function $U_1(P, [Q_0, Q_1])$. Player 1 evaluates his minimal expected payoff given player 2 plays $Q \in [Q_0, Q_1]$:

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \le Q \le Q_1} PQ + 3(1-P)(1-Q) \\ &= \min_{Q_0 \le Q \le Q_1} Q(4P-3) + 3(1-P) \\ &= \begin{cases} Q_0(4P-3) + 3(1-P) & \text{if } P > 3/4\,, \\ 3/4 & \text{if } P = 3/4\,, \\ Q_1(4P-3) + 3(1-P) & \text{if } P < 3/4\,. \end{cases} \end{aligned} \tag{3.1}$$

The maximum of (3.1) over $P \in [P_0, P_1]$ depends on the interval $[Q_0, Q_1]$. Observe that at the boundaries the function reduces to

$$U_1(0\,, [Q_0, Q_1]) = 3(1 - Q_1)\,,$$
$$\text{and} \quad U_1(1\,, [Q_0, Q_1]) = Q_0\,.$$

We find in particular that $U_1(P, [Q_0, Q_1])$ is constant for all P, when $Q_0 = 3/4 = Q_1$. We distinguish six cases where Q_0 and Q_1 can be situated within the interval $[0, 1]$, all these yield different arg max of $U_1(P, [Q_0, Q_1])$. Figure 3.2 pictures these cases. The distinction depends on the mixed strategy $Q_0 = 3/4 = Q_1$ that makes player 1 indifferent between any choice of $[P_0, P_1] \subseteq \Delta S_1$. This is the Nash equilibrium strategy of player 2.

We plot the payoff function in Figure 3.2. The numbering (1)-(6) of the functions refer to (one possible) payoff function when Q_0 and Q_1 take different values. The values are listed below. In the same list we present player 1's best response analysis to the different strategies $[Q_0, Q_1]$:

$$Q_0 > 3/4: \qquad B_1([Q_0, Q_1]) = 1 \tag{1}$$
$$Q_0 = 3/4 < Q_1: \qquad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [3/4, 1]\} \tag{2}$$
$$Q_0 < 3/4 < Q_1: \qquad B_1([Q_0, Q_1]) = 3/4 \tag{3}$$
$$Q_0 < 3/4 = Q_1: \qquad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 3/4]\} \tag{4}$$
$$Q_1 < 3/4: \qquad B_1([Q_0, Q_1]) = 0 \tag{5}$$
$$Q_0 = 3/4 = Q_1: \qquad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 1]\} \tag{6}$$

As this is a symmetric game, we get the same behavior for player 2. We fix the probability $Q \in \Delta S_2$ and derive the payoff functions $U_2([P_0, P_1]\,, Q)$ of player 2. Player 2 evaluates

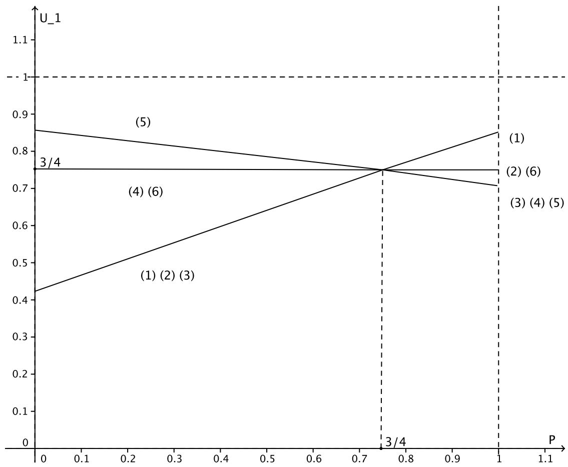

Figure 3.2: Coordination game: payoff function (3.1) of player 1.

his minimal expected payoff given player 1 plays $P \in [P_0, P_1]$.

$$\begin{aligned} U_2([P_0, P_1], Q) &= \min_{P_0 \le P \le P_1} PQ + 3(1-P)(1-Q) \\ &= \min_{P_0 \le P \le P_1} P(4Q-3) + 3(1-Q) \\ &= \begin{cases} P_0(4Q-3) + 3(1-Q) & \text{if } Q > 3/4\,, \\ 3/4 & \text{if } Q = 3/4\,, \\ P_1(4Q-3) + 3(1-Q) & \text{if } Q < 3/4\,. \end{cases} \end{aligned} \tag{3.2}$$

Analogue to the case considered before, the maximum of (3.2) over $Q \in [0,1]$ depends on the interval $[P_0, P_1]$. The plot of $U_2([P_0, P_1], Q)$ is identical to the plot in Figure 3.2. We thus get the following best responses of player 2:

$$\begin{aligned} &P_0 > 3/4 : && B_2([P_0, P_1]) = 1 \\ &P_0 = 3/4 < P_1 : && B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [3/4, 1]\} \\ &P_0 < 3/4 < P_1 : && B_2([P_0, P_1]) = 3/4 \\ &P_0 < 3/4 = P_1 : && B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 3/4]\} \\ &P_1 < 3/4 : && B_2([P_0, P_1]) = 0 \\ &P_0 = 3/4 = P_1 : && B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1]\} \end{aligned}$$

To find the Ellsberg equilibria, we check which of the six cases lead to intersections of the best response correspondences. We explain each of the six cases in turn.

(1) If player 2 plays $Q_0 > 3/4$, player 1 maximizes his payoff function $U_1([P_0, P_1], [Q_0, Q_1])$ by playing $P_0 = P_1 = 1$ and then again the best response of player 2 is $Q_0 = Q_1 = 1$. This is an Ellsberg equilibrium. It coincides with the Nash equilibrium in pure strategies (U, L).

(2) If player 2 plays $Q_0 = 3/4 < Q_1$, player 1's best response is $[P_0, P_1] \subseteq [3/4, 1]$. If he chooses the left border $P_0 > 3/4$, player 2 can improve by playing $Q_0 = Q_1 = 1$, so this would not be an equilibrium. But when player 1 chooses $[P_0, P_1] = [3/4, P_1]$ with $3/4 \leq P_1 \leq 1$ and player 2 chooses $[Q_0, Q_1] = [3/4, Q_1]$ with $3/4 \leq Q_1 \leq 1$ out of his best response set, either of the players can improve by deviating. Thus, $([3/4, P_1], [3/4, Q_1])$ where $3/4 \leq P_1 \leq 1$ and $3/4 \leq Q \leq 1$ is an Ellsberg equilibrium.

(3) If player 2 plays $Q_0 < 3/4 < Q_1$, player 1's best response is the standard mixed strategy $P_0 = P_1 = 3/4$. Player 2 stays with his strategy because he is indifferent between all strategies. This is an Ellsberg equilibrium: $(3/4, [Q_0, Q_1])$, where $Q_0 \leq 3/4 \leq Q_1$.

(4) If player 2 plays $Q_0 < 3/4 = Q_1$, player 1 maximizes his payoff function by playing any interval $[P_0, P_1] \subseteq [0, 3/4]$. This leads to an analog Ellsberg equilibrium to case (2), $([P_0, 3/4], [Q_0, 3/4])$, where $0 \leq P_0 \leq 3/4$ and $0 \leq Q_0 \leq 3/4$.

(5) If player 2 plays $Q_1 < 3/4$, player 1's best response is $P_0 = P_1 = 0$. Thus, player 2 chooses $Q_0 = Q_1 = 0$, this is an Ellsberg equilibrium. It coincides with the Nash equilibrium in pure strategies (D, R).

(6) Lastly, if player 2 plays $Q_0 = 3/4 = Q_1$, player 1's payoff function is constant at $3/4$, thus he is indifferent between all subsets $[P_0, P_1] \subseteq [0, 1]$. If player 1 plays $P_0 \leq 3/4 \leq P_1$, this is an Ellsberg equilibrium, $([P_0, P_1], 3/4)$, where $P_0 \leq 3/4 \leq P_1$. This equilibrium includes the possibility of player 1 choosing the Ellsberg strategy $P_0 = P_1 = 3/4 \in [0, 1]$. This play corresponds to the mixed strategy Nash equilibrium P^*.

Summarizing, we get the following proposition.

Proposition 3.1. *In the symmetric coordination game the Ellsberg equilibria are of the form*

$$(U, L) \text{ and } (D, R)\,,$$

$$([3/4, P_1]\,, [3/4, Q_1])\,, \text{ where } 3/4 \le P_1, Q_1 \le 1\,,$$

$$([P_0, 3/4]\,, [Q_0, 3/4])\,, \text{ where } 0 \le P_0, Q_0 \le 3/4\,,$$

$$(3/4, [Q_0, Q_1])\,, \text{ where } Q_0 \le 3/4 \le Q_1\,,$$

$$\text{and } \; ([P_0, P_1]\,, 3/4)\,, \text{ where } P_0 \le 3/4 \le P_1\,.$$

Looking at the Ellsberg equilibria, we see that player 1 and player 2 can coordinate to use ambiguity. They use the Nash equilibrium to bound their Ellsberg strategies from below or from above. Another equilibrium behavior is that one player uses ambiguity and the other uses his immunization strategy by playing U (or R, respectively) with probability 3/4. In the symmetric coordination game the mixed Nash equilibrium strategies coincide with the immunization strategies, we therefore get Ellsberg equilibria with full ambiguity. This does not happen in general coordination games, as we will see later.

3.2.2 Ellsberg Equilibria in General Coordination Games

We generalize the observations of the preceding section to general 2×2 coordination games. We observe two different types of Ellsberg equilibria in these games: in general, when the Nash equilibrium strategies P^* and Q^* of player 1 and 2 are different probability distributions, there exists an Ellsberg equilibrium in which both player use ambiguity. The sets of probability distributions used in equilibrium are bounded by the Nash equilibrium distribution and the distribution that assures immunization against any ambiguity used by the other player.

In the special case where $P^* = Q^*$ the Ellsberg equilibria are of a different kind. The Ellsberg equilibria in which both players create ambiguity are now bounded by the symmetric Nash equilibrium strategy and 0 or 1. Furthermore, since the immunization strategy coincides with the Nash equilibrium strategy we get Ellsberg equilibria in which one of the players plays his immunization strategy and the other one creates as much ambiguity as he likes.

These observations are formalized in the following proposition. Consider the 2×2 coordination game with payoff matrix in Figure 3.3. We assume that $a, d \ge b, c$ and $e, h \ge f, g$ with $a - b - c + d \ne 0$ and $e - f - g + h \ne 0$.

		Player 2	
		L	R
Player 1	U	a, e	b, f
	D	c, g	d, h

Figure 3.3: General coordination game.

The conditions on the payoffs allow for different types of symmetric and non-symmetric coordination games. Let P^* and Q^* denote the Nash equilibrium strategies for player 1 and 2, respectively. The Nash equilibrium strategies can be computed using standard methods, they are

$$P^* = \frac{h-g}{e-f-g+h} \quad \text{and} \quad Q^* = \frac{d-b}{a-b-c+d}.$$

The immunization strategies of each player are denoted by M_1 and M_2. Furthermore, player 1's Ellsberg strategy is denoted $[P_0, P_1]$, player 2's $[Q_0, Q_1]$. Then we have the following proposition.

Proposition 3.2. *Let P^*, Q^* denote the mixed Nash equilibrium strategies, and M_1, M_2 the immunization strategies of player 1 and 2, respectively. Then the Ellsberg equilibria of the general coordination game are of the following form.*

	$M_1 \leq P^*$	$M_1 \geq P^*$
$M_2 \leq Q^*$	$([P^*, P_1], [Q^*, Q_1])$	$([P^*, P_1], [Q_0, Q^*])$ $P_1 \leq M_1$ and $M_2 \leq Q_0$
$M_2 \geq Q^*$	$([P_0, P^*], [Q^*, Q_1])$ $M_1 \leq P_0$ and $Q_1 \leq M_2$	$([P_0, P^*], [Q_0, Q^*])$

If $M_1 = P^*$ or $M_2 = Q^*$, then an additional type of Ellsberg equilibria arises,

$$(P^*, [Q_0, Q_1]), \ \textit{where } Q_0 \leq Q^* \leq Q_1 \qquad \textit{when } M_1 = P^*;$$
$$([P_0, P_1], Q^*), \ \textit{where } P_0 \leq P^* \leq P_1 \qquad \textit{when } M_2 = Q^*.$$

In any case, the pure and mixed Nash equilibria, that is (U, L), (D, R) and (P^*, Q^*), are also Ellsberg equilibria.

Proof. The Nash equilibrium strategies follow from the usual analysis. To calculate the Ellsberg equilibria of the general coordination game (Figure 3.3), we first derive the utility functions of player 1 and player 2. Due to the assumption that $a, d \geq b, c$ and $e, h \geq f, g$, with $a-b-c+d \neq 0$ and $e-f-g+h \neq 0$, the denominators $a-b-c+d$ and $e-f-g+h$ are positive. This reflects in the payoff functions that the game is a coordination game; player 1 uses Q_0 as a minimizer when $P > \frac{d-c}{a-b-c+d}$, and, the same way, player 2 uses P_0 as a minimizer when $Q > \frac{h-f}{e-f-g+h}$.

$$\begin{aligned}
U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \leq Q \leq Q_1} aPQ + bP(1-Q) + c(1-P)Q + d(1-P)(1-Q) \\
&= \min_{Q_0 \leq Q \leq Q_1} Q((a-b-c+d)P + c - d) + (b-d)P + d \\
&= \begin{cases} Q_0((a-b-c+d)P + c - d) + (b-d)P + d & \text{if } P > \frac{d-c}{a-b-c+d}\,, \\ \frac{(b-d)(c-d)}{a-b-c+d} + d & \text{if } P = \frac{d-c}{a-b-c+d}\,, \\ Q_1((a-b-c+d)P + c - d) + (b-d)P + d & \text{if } P < \frac{d-c}{a-b-c+d}\,, \end{cases}
\end{aligned}$$

$$\begin{aligned}
U_2([P_0, P_1], Q) &= \min_{P_0 \leq P \leq P_1} ePQ + g(1-P)Q + fP(1-Q) + h(1-P)(1-Q) \\
&= \min_{P_0 \leq P \leq P_1} P((e-f-g+h)Q + f - h) + (g-h)Q + h \\
&= \begin{cases} P_0((e-f-g+h)Q + f - h) + (g-h)Q + h & \text{if } Q > \frac{h-f}{e-f-g+h}\,, \\ \frac{(g-h)(h-f)}{e-f-g+h} + h & \text{if } Q = \frac{h-f}{e-f-g+h}\,, \\ P_1((e-f-g+h)Q + f - h) + (g-h)Q + h & \text{if } Q < \frac{h-f}{e-f-g+h}\,. \end{cases}
\end{aligned}$$

We see that

$$M_1 = \frac{d-c}{a-b-c+d}\,, \quad \text{respectively} \quad M_2 = \frac{h-f}{e-f-g+h}$$

are the immunization strategies of player 1 and 2, respectively. At the boundaries the utility functions become

$$\begin{aligned}
U_1(0\,, [Q_0, Q_1]) &= (c-d)Q_1 + d\,, & U_2([P_0, P_1]\,, 0) &= (f-h)P_1 + h\,, \\
U_1(1\,, [Q_0, Q_1]) &= (a-b)Q_0 + b\,, & U_2([P_0, P_1]\,, 1) &= (e-g)P_0 + g\,.
\end{aligned}$$

The payoff function of player 1 is constant for all P, when $Q_1 = Q_0 = Q^*$, and the payoff function of player 2 is constant for all Q, when $P_1 = P_0 = P^*$. We calculate the best response correspondences:

$$Q_0 > Q^* : \qquad B_1([Q_0, Q_1]) = 1 \tag{1}$$
$$Q_0 = Q^* < Q_1 : \qquad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [M_1, 1]\} \tag{2}$$
$$Q_0 < Q^* < Q_1 : \qquad B_1([Q_0, Q_1]) = M_1 \tag{3}$$
$$Q_0 < Q^* = Q_1 : \qquad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, M_1]\} \tag{4}$$
$$Q_1 < Q^* : \qquad B_1([Q_0, Q_1]) = 0 \tag{5}$$
$$Q_0 = Q^* = Q_1 : \qquad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 1]\} \tag{6}$$

$$P_0 > P^* : \qquad B_2([P_0, P_1]) = 1$$
$$P_0 = P^* < P_1 : \qquad B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [M_2, 1]\}$$
$$P_0 < P^* < P_1 : \qquad B_2([P_0, P_1]) = M_2$$
$$P_0 < P^* = P_1 : \qquad B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, M_2]\}$$
$$P_1 < P^* : \qquad B_2([P_0, P_1]) = 0$$
$$P_0 = P^* = P_1 : \qquad B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1]\}$$

To find the Ellsberg equilibria, we look at the different cases in turn.

(1) $Q_0 > Q^*$: player 1 responds $P_0 = P_1 = 1$. If also player 2 chooses $Q_0 = Q_1 = 1$, this is the Ellsberg equilibrium that is identical to the pure Nash equilibrium (U, L).

(2) $\left[\frac{d-b}{a-b-c+d}, Q_1\right] \Rightarrow [P_0, P_1] \subseteq \left[\frac{d-c}{a-b-c+d}, 1\right]$, then, depending on the size of M_1 in relation to P^*, player 1 has the following choices:

- $\frac{d-c}{a-b-c+d} \leq \frac{h-g}{e-f-g+h}$: player 1 can play either
  - $\left[\frac{h-g}{e-f-g+h}, P_1\right]$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[\frac{h-f}{e-f-g+h}, 1\right]$. Now, depending on the size of M_2 in relation to Q^*, player 2 has the following choices:
    * $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: $\left[\frac{d-b}{a-b-c+d}, Q_1\right]$, we thus have here the Ellsberg equilibrium

$$\left(\left[\frac{h-g}{e-f-g+h}, P_1\right], \left[\frac{d-b}{a-b-c+d}, Q_1\right]\right),$$

that is $([P^*, P_1], [Q^*, Q_1])$.

  * $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 1$, thus no Ellsberg equilibrium arises.

- or $\left[P_0, \frac{h-g}{e-f-g+h}\right]$ with $\frac{d-c}{a-b-c+d} \leq P_0 \leq \frac{h-g}{e-f-g+h}$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[0, \frac{h-f}{e-f-g+h}\right]$. Now, depending on the size of M_2 in relation to Q^*, player 2 has the following choices:

  * $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: $\left[\frac{d-b}{a-b-c+d}, Q_1\right]$ with $Q_1 \leq \frac{h-f}{e-f-g+h}$, then the best response of player 1 is $[P_0, P_1] \subseteq \left[\frac{d-c}{a-b-c+d}, 1\right]$. We thus have here the Ellsberg equilibrium

$$\left(\left[P_0, \frac{h-g}{e-f-g+h}\right], \left[\frac{d-b}{a-b-c+d}, Q_1\right]\right),$$

$$\text{where } \frac{d-c}{a-b-c+d} \leq P_0 \text{ and } Q_1 \leq \frac{h-f}{e-f-g+h},$$

$$\text{that is } ([P_0, P^*], [Q^*, Q_1]), \text{ where } M_1 \leq P_0 \text{ and } Q_1 \leq M_2 .$$

  * $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 0$, thus no Ellsberg equilibrium arises.

- $\frac{d-c}{a-b-c+d} \leq \frac{h-g}{e-f-g+h}$: to any $[P_0, P_1]$ the best response is $Q_0 = Q_1 = 1$, thus no Ellsberg equilibrium arises.

(3) $Q_0 < Q^* < Q_1$: player 1 responds with $P_0 = P_1 = M_1$. Only when $M_1 = P^*$, player 1 sticks to his strategy and we get the equilibrium $(P^*, [Q_0, Q_1])$, where $Q_0 < Q^* < Q_1$.

(4) $\left[Q_0, \frac{d-b}{a-b-c+d}\right] \Rightarrow [P_0, P_1] \subseteq \left[0, \frac{d-c}{a-b-c+d}\right]$, then, depending on the size of M_1 in relation to P^*, player 1 has the following choices:

- $\frac{d-c}{a-b-c+d} \geq \frac{h-g}{e-f-g+h}$: player 1 can play either

  - $\left[P_0, \frac{h-g}{e-f-g+h}\right]$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[0, \frac{h-f}{e-f-g+h}\right]$. Now, depending on the size of M_2 in relation to Q^*, player 2 has the following choices:

    * $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: $\left[Q_0, \frac{d-b}{a-b-c+d}\right]$, we thus have here the Ellsberg equilibrium

$$\left(\left[P_0, \frac{h-g}{e-f-g+h}\right], \left[Q_0, \frac{d-b}{a-b-c+d}\right]\right),$$

$$\text{that is } ([P_0, P^*], [Q_0, Q^*]) .$$

    * $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 0$, thus no Ellsberg equilibrium arises.

  - or $\left[\frac{h-g}{e-f-g+h}, P_1\right]$ with $\frac{h-g}{e-f-g+h} \leq P_1 \leq \frac{d-c}{a-b-c+d}$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[\frac{h-f}{e-f-g+h}, 1\right]$. Now, depending on the size of M_2 in relation to Q^*, player 2 has the following choices:

    * $\frac{d-b}{a-b-c+d} \geq \frac{h-f}{e-f-g+h}$: $\left[Q_0, \frac{d-b}{a-b-c+d}\right]$ with $\frac{h-f}{e-f-g+h} \leq Q_0$, then the best response of player 1 is $[P_0, P_1] \subseteq \left[0, \frac{d-c}{a-b-c+d}\right]$. We thus have here the Ellsberg equilibrium

$$\left(\left[\frac{h-g}{e-f-g+h}, P_1\right], \left[Q_0, \frac{d-b}{a-b-c+d}\right]\right),$$

$$\text{where } P_1 \leq \frac{d-c}{a-b-c+d} \text{ and } \frac{h-f}{e-f-g+h} \leq Q_0\,,$$

$$\text{that is } ([P^*, P_1], [Q_0, Q^*]),\ \text{where } P_1 \leq M_1 \text{ and } M_2 \leq Q_0\,.$$

    * $\frac{d-b}{a-b-c+d} \leq \frac{h-f}{e-f-g+h}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 1$, thus no Ellsberg equilibrium arises.

- $\frac{d-c}{a-b-c+d} \leq \frac{h-g}{e-f-g+h}$: to any $[P_0, P_1]$ the best response is $Q_0 = Q_1 = 1$, thus no Ellsberg equilibrium arises.

(5) $Q_1 < Q^*$: player 1 responds $P_0 = P_1 = 0$, if player 2 chooses $Q_0 = Q_1 = 0$ this is the Ellsberg equilibrium that is identical to the pure Nash equilibrium (D, R).

(6) $Q_0 = Q^* = Q_1$: player 1 responds with $[P_0, P_1] \subseteq [0, 1]$. Only when $Q^* = M_2$, player 2 sticks to his strategy and we get the equilibrium $([P_0, P_1], Q^*)$, where $P_0 < P^* < P_1$.

□

Notably, the largest Ellsberg equilibria in the lower left and the upper right cell of the table in Proposition 3.2 are $([M_1, P^*], [Q^*, M_2])$ and $([P^*, M_1], [M_2, Q^*])$, respectively. The associated coordination games are those of the type Battle of the Sexes. In these games, where despite the wish to coordinate the players are in some conflict of interest, we see that the use of imprecise probabilistic devices consolidates Nash behavior and maximin behavior in an even clearer way than in the conflict games discussed above: *both* players use their Nash equilibrium and maximin strategies as boundaries of their largest Ellsberg equilibrium strategies. Moreover, we observe again that knowledge of the Nash equilibrium and immunization strategies suffices to compute all Ellsberg equilibria of the game.

3.2.3 Further Examples of Two-Person Coordination

We now calculate the Ellsberg equilibria of three other classes of coordination games. All these games are covered by Proposition 3.2 and can thus be simply read off the table, but to better understand the derivation of the Ellsberg equilibria and their mathematical structure, we present a detailed analysis. We calculate the Ellsberg equilibria of two non-symmetric coordination games, of the game of Battle of the Sexes, and of Rubinstein's electronic mail game.

Two Non-Symmetric Coordination Games

We consider two examples of coordination games which are less symmetric than the game in Figure 3.1. The first non-symmetric example of a coordination game has the payoff matrix given in Figure 3.4. This analysis is interesting, because the game does not allow for the same use of ambiguity in equilibrium as the very symmetric example which we considered in the beginning. We see that as soon as the players are in some kind of opposition to each other, the use of ambiguity gets more interesting. The Nash equilibria

		Player 2	
		L	R
Player 1	U	$4,5$	$1,1$
	D	$0,0$	$3,4$

Figure 3.4: Non-symmetric coordination game I.

of the game are the two pure equilibria (U, L) and (D, R) and the mixed Nash equilibrium $(P^*, Q^*) = ((3/7, 4/7), (1/5, 4/5))$. We use the same notation as before. The payoff function $U_1(P, [Q_0, Q_1])$ for a fixed $P \in \Delta S_1$ is

$$U_1(P, [Q_0, Q_1]) = \begin{cases} Q_1(5P-2) + 2 - P & \text{if } P < 2/5\,, \\ 8/5 & \text{if } P = 2/5\,, \\ Q_0(5P-2) + 2 - P & \text{if } P > 2/5\,. \end{cases} \tag{3.3}$$

$U_1(P, [Q_0, Q_1])$ is constant at $8/5$ when

$$U_1(0\,, [Q_0, Q_1]) = 2 - 2Q_1 = 8/5 = 1 + 3Q_0 = U_1(1\,, [Q_0, Q_1])\,,$$

this is the case when $Q_1 = Q_0 = Q^* = 1/5$. Thereby we get the following best response correspondence for player 1:

$$Q_0 > 1/5: \qquad B_1([Q_0, Q_1]) = 1 \tag{1}$$
$$Q_0 = 1/5 < Q_1: \qquad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [2/5, 1]\} \tag{2}$$
$$Q_0 < 1/5 < Q_1: \qquad B_1([Q_0, Q_1]) = 2/5 \tag{3}$$
$$Q_0 < 1/5 = Q_1: \qquad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 2/5]\} \tag{4}$$
$$Q_1 < 1/5: \qquad B_1([Q_0, Q_1]) = 0 \tag{5}$$
$$Q_0 = 1/5 = Q_1: \qquad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 1]\} \tag{6}$$

The same analysis for player 2 yields the best response correspondence

$$\begin{aligned}
&P_0 > 3/7: && B_2([P_0, P_1]) = 1\\
&P_0 = 3/7 < P_1: && B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [2/7, 1]\}\\
&P_0 < 3/7 < P_1: && B_2([P_0, P_1]) = 2/7\\
&P_0 < 3/7 = P_1: && B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 2/7]\}\\
&P_1 < 3/7: && B_2([P_0, P_1]) = 0\\
&P_0 = 3/7 = P_1: && B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1]\}\,.
\end{aligned}$$

We look at the intersections of the best response correspondences to find the Ellsberg equilibria:

(1) $Q_0 > 1/5 \Rightarrow P_0 = P_1 = 1$, when player 2 chooses $Q_0 = Q_1 = 1$. This is the Ellsberg equilibrium that coincides with the pure Nash equilibrium (U, L).

(2) $Q_0 = 1/5 < Q_1 \Rightarrow [P_0, P_1] \subseteq [2/5, 1]$, if player 1 chooses $[P_0, 3/7]$ with $2/5 \leq P_0 < 3/7$ then player 2's best response is $[Q_0, Q_1] \subseteq [0, 2/7]$. Thus, we obtain the Ellsberg equilibrium

$$([P_0, 3/7], [1/5, Q_1])\,, \text{ where } 2/5 \leq P_0 < 3/7 \text{ and } 1/5 < Q_1 \leq 2/7\,.$$

(3) $Q_0 < 1/5 < Q_1 \Rightarrow P_0 = P_1 = 2/5 \Rightarrow Q_0 = Q_1 = 0$. This contradicts that $Q_0 < 1/5 < Q_1$, thus this is not an Ellsberg equilibrium.

(4) $Q_0 < 1/5 = Q_1 \Rightarrow [P_0, P_1] \subseteq [0, 2/5]$. We observe that $2/5 < 3/7$, thus player 2 responds optimally with $Q_0 = Q_1 = 0$. Thus, this does not lead to an Ellsberg equilibrium.

(5) The case $Q_1 < 1/5$ results, analogously to (1), in the Ellsberg equilibrium that coincides with the other pure Nash equilibrium (D, R).

(6) In the last case, players find it optimal to deviate from the immunization strategy, since it does not coincide with their Nash equilibrium strategy: $Q_1 = 1/5 = Q_0 \Rightarrow [P_0, P_1] \subseteq [0, 1]$, then player 2 responds optimally with $Q_0 = Q_1 = 2/7$ and player 1 would, since $2/7 > 1/5$, respond with his pure strategy U. But what player 1 can do is respond with his respective Nash equilibrium strategy $3/7$, then we have the equilibrium $(3/7, 1/5)$.

Proposition 3.3. *The Ellsberg equilibria in the non-symmetric coordination game I are of the form*

$$(U, L) \text{ and } (D, R)\,,$$

$$([P_0, 3/7]\,, [1/5, Q_1])\,, \text{ where } 2/5 \leq P_0 \leq 3/7 \text{ and } 1/5 \leq Q_1 \leq 2/7\,.$$

This example in contrast to the symmetric coordination game, Figure 3.1, shows that when the coordination game is not symmetric in the sense that $P^* = Q^*$, the ambiguity intervals in equilibrium are not bounded by 0 and 1, but by the mixed strategy Nash equilibrium or the immunization strategies (or both). Furthermore, there are no Ellsberg equilibria with one player creating as much ambiguity as he wants and the other player playing his Nash equilibrium strategy.

To see an application of Proposition 3.2, we quickly derive the Ellsberg equilibria of the game in Figure 3.5 using the proposition.

		Player 2	
		L	R
Player 1	U	$4, 5$	$3, 1$
	D	$0, 2$	$4, 5$

Figure 3.5: Non-symmetric coordination game II.

The mixed strategy Nash equilibria are $P^* = 3/7$ and $Q^* = 1/5$, and the immunization strategies are $M_1 = 4/5$ and $M_2 = 4/7$. We see that $P^* < M_1$ and $Q^* < M_2$. Thus, all Ellsberg equilibria are of the form $([P_0, 3/7]\,, [Q_0, 1/5])$, in addition to the Ellsberg equilibria that are identical to the pure Nash equilibria (U, L) and (D, R).

Battle of the Sexes

The next example is the game of Battle of the Sexes. In this classic game, players wish to coordinate, but they have different preferences on which action they want to coordinate. We choose the payoffs as in the matrix in Figure 3.6. A popular interpretation of this setting is a man and a woman who want to spend the evening together; the man (player 2) wants to take the woman (player 1) to watch a football match (L and U), whereas the woman wants to take him to the opera (D and R). Both prefer spending the evening together instead of spending it alone.

		Player 2: L	Player 2: R
Player 1	U	$1, 2$	$0, 0$
	D	$0, 0$	$2, 1$

Figure 3.6: Battle of the Sexes.

Note that the Nash equilibria of the game are two pure equilibria (U, L) and (D, R) and a mixed strategy Nash equilibrium $(P^*, Q^*) = ((1/3, 2/3), (2/3, 1/3))$. We use the same notation as before. To find the Ellsberg equilibria of the Battle of the Sexes game, we first derive the payoff function $U_1(P, [Q_0, Q_1])$ of player 1.

$$
\begin{aligned}
U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \leq Q \leq Q_1} PQ + 2(1-P)(1-Q) \\
&= \min_{Q_0 \leq Q \leq Q_1} Q(3P-2) + 2(1-P) \\
&= \begin{cases} Q_0(3P-2) + 2(1-P) & \text{if } P > 2/3\,, \\ 2/3 & \text{if } P = 2/3\,, \\ Q_1(3P-2) + 2(1-P) & \text{if } P < 2/3\,. \end{cases}
\end{aligned}
$$

Observe that at the boundaries the function reduces to

$$
\begin{aligned}
&U_1(0\,, [Q_0, Q_1]) = 2(1-Q_1)\,, \\
\text{and} \quad &U_1(1\,, [Q_0, Q_1]) = Q_0\,.
\end{aligned}
$$

We see that $U_1(P, [Q_0, Q_1])$ is constant at $2/3$ for all $P \in \Delta S_1$ when $Q_0 = 2/3 = Q_1$.

Therefore we get the following best responses for player 1:

$$
\begin{aligned}
&Q_0 > 2/3 : && B_1([Q_0, Q_1]) = 1 \\
&Q_0 = 2/3 < Q_1 : && B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [2/3, 1]\} \\
&Q_0 < 2/3 < Q_1 : && B_1([Q_0, Q_1]) = 2/3 \\
&Q_0 < 2/3 = Q_1 : && B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 2/3]\} \\
&Q_1 < 2/3 : && B_1([Q_0, Q_1]) = 0 \\
&Q_0 = 2/3 = Q_1 : && B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 1]\}
\end{aligned}
$$

Now we fix the probability $Q \in [Q_0, Q_1]$ and consider the payoff function $U_2([P_0, P_1], Q)$ of player 2. Player 2 evaluates his minimal expected payoff given player 1 plays $P \in [P_0, P_1]$:

$$
\begin{aligned}
U_2([P_0, P_1], Q) &= \min_{P_0 \le P \le P_1} 2PQ + (1-P)(1-Q) \\
&= \min_{P_0 \le P \le P_1} P(3Q-1) + 1 - Q \\
&= \begin{cases} P_0(3Q-1) + 1 - Q & \text{if } Q > 1/3\,, \\ 2/3 & \text{if } Q = 1/3\,, \\ P_1(3Q-1) + 1 - Q & \text{if } Q < 1/3\,. \end{cases}
\end{aligned}
$$

Observe that at the boundaries the function reduces to

$$
\begin{aligned}
&U_2([P_0, P_1], 0) = 1 - P_1\,, \\
\text{and} \quad &U_2([P_0, P_1], 1) = 2P_0\,.
\end{aligned}
$$

We see that $U_2([P_0, P_1], Q)$ is constant at $2/3$ when $P_0 = 1/3 = P_1$. Therefore we get the following best responses for player 2:

$$
\begin{aligned}
&P_0 > 1/3 : && B_2([P_0, P_1]) = 1 && (1) \\
&P_0 = 1/3 < P_1 : && B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [1/3, 1]\} && (2) \\
&P_0 < 1/3 < P_1 : && B_2([P_0, P_1]) = 1/3 && (3) \\
&P_0 < 1/3 = P_1 : && B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1/3]\} && (4) \\
&P_1 < 1/3 : && B_2([P_0, P_1]) = 0 && (5) \\
&P_0 = 1/3 = P_1 : && B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1]\} && (6)
\end{aligned}
$$

Now we can calculate the Ellsberg equilibria:

(1) $P_0 > 1/3 \Rightarrow Q_0 = Q_1 = 1 \Rightarrow P_0 = P_1 = 1$. This is the Ellsberg equilibrium that coincides with the pure strategy Nash equilibrium (U, L).

(2) $P_0 = 1/3 < P_1 \Rightarrow [Q_0, Q_1] \subseteq [1/3, 1]$. Player 2 has the possibility to play different subsets of $[1/3, 1]$, but only one leads to a best response that is consistent with the original play of player 1,

$$[Q_0, Q_1] = [Q_0, 2/3]\,, \text{ where } 1/3 \leq Q_0 \leq 2/3 \Rightarrow [P_0, P_1] \subseteq [0, 2/3]\ .$$

If in the last case player 1 chooses to play $[P_0, P_1] = [1/3, P_1]$ with $1/3 \leq P_1 \leq 2/3$, then this is an Ellsberg equilibrium, i.e.,

$$([1/3, P_1]\,, [Q_0, 2/3])\,, \text{ where } 1/3 \leq P_1 \leq 2/3 \text{ and } 1/3 \leq Q_0 \leq 2/3\,.$$

(3) $P_0 < 1/3 < P_1 \Rightarrow Q_0 = Q_1 = 1/3 \Rightarrow P_0 = P_1 = 0$, this is not an equilibrium.

(4) $P_0 < 1/3 = P_1 \Rightarrow [Q_0, Q_1] \subseteq [0, 1/3] \Rightarrow P_0 = P_1 = 0$, this is not an equilibrium.

(5) $P_1 < 1/3 \Rightarrow Q_0 = Q_1 = 0 \Rightarrow P_0 = P_1 = 0$, this is the analog Ellsberg equilibrium to (1), which now coincides with the pure strategy Nash equilibrium (D, R).

(6) $P_0 = 1/3 = P_1 \Rightarrow [Q_0, Q_1] \subseteq [0, 1]$. We consider two cases: if player 2 plays $Q_0 < 2/3 < Q_1$, player 1's best response is $P_0 = P_1 = 2/3$, so this is not an equilibrium. But if player 2 chooses $Q_0 = P_0 = 2/3$ and player 1 $P_0 = P_1 = 1/3$, this is an Ellsberg equilibrium. It coincides with the mixed strategy Nash equilibrium P^*.

Proposition 3.4. *The Ellsberg equilibria in the Battle of the Sexes game are of the following form:*

$$(U, L) \textit{ and } (D, R)\,,$$

$$([1/3, P_1]\,, [Q_0, 2/3])\,, \textit{ where } 1/3 \leq P_1 \leq 2/3 \textit{ and } 1/3 \leq Q_0 \leq 2/3\,.$$

In the Ellsberg equilibrium the players may use Ellsberg strategies that include the whole interval $[1/3, 2/3]$ on U and L. Again, as in the preceding non-symmetric coordination example, the equilibrium interval is bounded by the mixed Nash equilibrium strategy and the immunization strategy.

This can be interpreted as an approximation of the two interests present in the Battle of the Sexes game. Whereas in Nash equilibrium the players stick to the different distributions over their pure strategies which reflect the conflict of interest, in Ellsberg equilibrium they play identical strategies, each of them incorporating the Nash equilibrium strategy of the opponent: $([1/3, 2/3], [1/3, 2/3])$.

Rubinstein's Electronic Mail Game

As a final example we look at the following game, taken from Rubinstein (1989). Originally, Rubinstein considered two games, one with a payoff matrix as in Figure 3.7, and a second one, where the payoffs for (U, L) and (D, R) are interchanged. He observes that the introduction of an imperfect communication mechanism (electronic mail which fails to reach the opponent with some small probability) leads to selection of the "better" equilibrium (D, R) (as opposed to (U, L)). We only look at one of these games, namely the one in Figure 3.7, and analyze the outcome of the game when players are allowed to use Ellsberg strategies. We find that the possibility to use Ellsberg strategies also leads to equilibrium selection, if so in a weak sense. In the only type of proper Ellsberg equilibrium of Rubinstein's electronic mail game, (U, L) can still be the outcome with some probability, but chances are small compared to occurrence of the outcome (D, R).

		Player 2	
		L	R
Player 1	U	$0, 0$	$0, -2$
	D	$-2, 0$	$1, 1$

Figure 3.7: Rubinstein's electronic mail game.

We calculate the minimal expected utility function for both players,

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \le Q \le Q_1} -2(1-P)Q + (1-P)(1-Q) \\ &= \min_{Q_0 \le Q \le Q_1} Q(3P-3) - P + 1 \\ &= \begin{cases} 0 & \text{if } P = 1\,, \\ Q_1(3P-3) - P + 1 & \text{if } P < 1\,, \end{cases} \end{aligned}$$

$$U_2([P_0, P_1], Q) = \begin{cases} 0 & \text{if } Q = 1\,, \\ P_1(3Q-3) - Q + 1 & \text{if } Q < 1\,. \end{cases}$$

At the boundaries the functions reduce to

$$U_1(0\,,[Q_0,Q_1]) = -3Q_1 + 1\,, \qquad U_2([P_0,P_1]\,,0) = -3P_1 + 1\,,$$

$$U_1(1\,,[Q_0,Q_1]) = 0\,, \qquad U_2([P_0,P_1]\,,1) = 0\,.$$

The utility functions are constant when $Q_1 = 1/3$ and $P_1 = 1/3$, respectively. Thus, this is the important conclusion, in Ellsberg equilibrium, players must use $1/3$ as *upper* bound of their equilibrium strategy, the lower bound is variable. To have $1/3$ as the lower bound will not lead to an Ellsberg equilibrium. With this reasoning we have, for example, the equilibrium

$$([0,1/3]\,,[0,1/3])\,,$$

in which both players play (U,L) only with a probability between 0 and $1/3$. Playing this Ellsberg equilibrium strategy thus helps to coordinate on the good equilibrium (D,R). Note that there is no Ellsberg equilibrium that facilitates coordination on (U,L) in the same way, because an Ellsberg strategy with lower bound $1/3$ (and upper bound not equal to $1/3$) can never be a best response. These Ellsberg equilibria can of course also be read off the table in Proposition 3.2. We have the following proposition.

Proposition 3.5. *The Ellsberg equilibria of Rubinstein's electronic mail game have the following form:*

(U,L) *and* $(D,R)\,,$

and the proper Ellsberg equilibria $([P_0,1/3]\,,[Q_0,1/3])\,,$

where $0 \le P_0, Q_0 \le 1/3\,.$

3.3 Strategic Use of Ambiguity in Two-Person Conflicts

Our approach to games has its most natural and fruitful applications to conflicts where players are at least to some degree in opposition to each other. We start this section by discussing a modified version of Matching Pennies. We then provide a general analysis of 2×2 conflict games, where for reasons of clarity we first calculate the equilibria of a more restricted case, before we prove the more general result. The general analysis will be followed by the derivation of Ellsberg equilibria of a standard symmetric Matching Pennies game and a non-symmetric conflict game.

While our predictions are broader than the classical Nash equilibrium, they remain restrictive, and, at least in principle, testable. Our results allow to explain the experimental findings of Goeree and Holt (2001), who show that humans tend to deviate from Matching Pennies in asymmetric modified Matching Pennies games, but tend to play Nash equilibrium in symmetric Matching Pennies. This corresponds and is consistent with our Ellsberg equilibria. We present and explain the experiment in Section 4.1 in the next chapter.

3.3.1 Example of a Conflict Game: Modified Matching Pennies

In the following game the standard Matching Pennies game is modified in an asymmetric way such that one player gets higher payoff in one strategy profile. All other payoffs remain the same. The game has the payoff matrix given in Figure 3.8. Observe that the

		Player 2 L	Player 2 R
Player 1	U	$3, -1$	$-1, 1$
	D	$-1, 1$	$1, -1$

Figure 3.8: Modified Matching Pennies.

modification is solely an increase in player 1's payoff in case (U, L) is played. Note that there is a unique mixed strategy Nash equilibrium

$$(P^*, Q^*) = ((1/2, 1/2), (1/3, 2/3))$$

that yields expected utility $u_1(P^*, Q^*) = 1/3$ and $u_2(P^*, Q^*) = 0$ for player 1 and player 2.

Suppose player 1 chooses to play U with some probability $P \in [P_0, P_1]$, whereas player 2 plays U with some probability $Q \in [Q_0, Q_1]$. To find the Ellsberg equilibria, we first derive the payoff function $U_1(P, [Q_0, Q_1])$ of player 1.

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \le Q \le Q_1} 3PQ - P(1-Q) - (1-P)Q + (1-P)(1-Q) \\ &= \min_{Q_0 \le Q \le Q_1} Q(6P-2) - 2P + 1 \\ &= \begin{cases} Q_0(6P-2) - 2P + 1, & \text{if } P > 1/3\,, \\ 1/3, & \text{if } P = 1/3\,, \\ Q_1(6P-2) - 2P + 1, & \text{if } P < 1/3\,. \end{cases} \end{aligned} \tag{3.4}$$

Observe that at the boundaries the utility function reduces to

$$U_1(0\,,[Q_0,Q_1]) = 1-2Q_1\,,$$
$$\text{and}\quad U_1(1\,,[Q_0,Q_1]) = 4Q_0-1\,.$$

We see that $U_1(P,[Q_0,Q_1])$ is constant at 1/3 for all $P \in [0,1]$ when $Q_0 = 1/3 = Q_1$. We distinguish six cases where Q_0 and Q_1 can be situated within the interval $[0,1]$, all these give different arg max of $U_1(P,[Q_0,Q_1])$. Figure 3.9 pictures these cases. The distinction depends on the mixed strategy $Q_0 = 1/3 = Q_1$ that makes player 1 indifferent between any choice of $[P_0,P_1] \subseteq \Delta S_1$. This is the Nash equilibrium strategy of player 2.

We plot the payoff function in Figure 3.9. The numbering (1)-(6) of the functions refer to (one possible) payoff function when Q_0 and Q_1 take different values. The values are listed below Figure 3.9. In the same list we present player 1's best response analysis to the different strategies $[Q_0,Q_1]$. We see that the payoff function is constant when $Q_0 = 1/3 = Q_1$ (case (6)), then player 1 has any set of distributions in the whole interval $[0,1]$ as a best response. In cases (2) and (4), only one boundary of player 2 is fixed at 1/3, hence only subintervals of $[1/3,1]$ or $[0,1/3]$ are possible best responses for player 1. When player 2 plays an interval around his Nash equilibrium strategy (case (3)), player 1's best response is the classic mixed strategy 1/3.

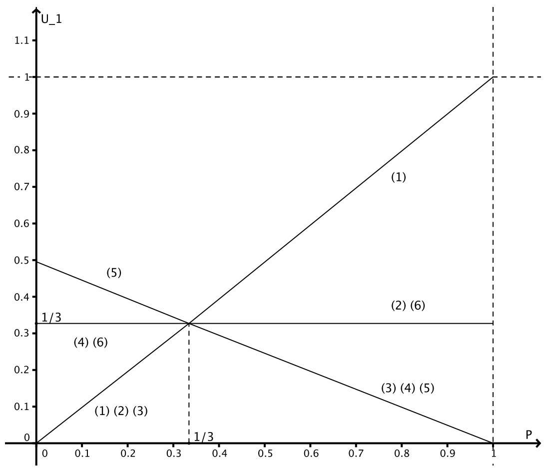

Figure 3.9: Modified Matching Pennies: payoff function (3.4) of player 1 for different strategies $[Q_0,Q_1] \subseteq \Delta S_2$.

$Q_0 > 1/3:$	$B_1([Q_0, Q_1]) = 1$	(1)
$Q_0 = 1/3 < Q_1:$	$B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [1/3, 1]\}$	(2)
$Q_0 < 1/3 < Q_1:$	$B_1([Q_0, Q_1]) = 1/3$	(3)
$Q_0 < 1/3 = Q_1:$	$B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 1/3]\}$	(4)
$Q_1 < 1/3:$	$B_1([Q_0, Q_1]) = 0$	(5)
$Q_0 = 1/3 = Q_1:$	$B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 1]\}$	(6)

Next we look at the behavior of player 2. We fix the probability $Q \in \Delta S_2$ and derive the payoff function $U_2([P_0, P_1], Q)$ of player 2.

$$
\begin{aligned}
U_2([P_0, P_1], Q) &= \min_{Q_0 \leq Q \leq Q_1} (1-P)Q + (1-Q)P \\
&= \min_{Q_0 \leq Q \leq Q_1} P(1-2Q) + Q \\
&= \begin{cases} P_0(1-2Q) + Q, & \text{if } Q < 1/2\,, \\ 1/2, & \text{if } Q = 1/2\,, \\ P_1(1-2Q) + Q, & \text{if } Q > 1/2\,. \end{cases}
\end{aligned} \tag{3.5}
$$

Observe that at the boundaries the function reduces to

$$
\begin{aligned}
&U_2([P_0, P_1], 0) = P_0\,, \\
\text{and} \quad &U_2([P_0, P_1], 1) = 1 - P_1\,.
\end{aligned}
$$

The plot of $U_2([P_0, P_1], Q)$ is given in Figure 3.10, and the best responses of player 2 are listed below the figure.

$P_0 > 1/2:$	$B_2([P_0, P_1]) = 0$	(1)
$P_0 = 1/2 < P_1:$	$B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1/2]\}$	(2)
$P_0 < 1/2 < P_1:$	$B_2([P_0, P_1]) = 1/2$	(3)
$P_0 < 1/2 = P_1:$	$B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [1/2, 1]\}$	(4)
$P_1 < 1/2:$	$B_2([P_0, P_1]) = 1$	(5)
$P_0 = 1/2 = P_1:$	$B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1]\}$	(6)

Now we can calculate the Ellsberg equilibria by finding the intersections of the best re-

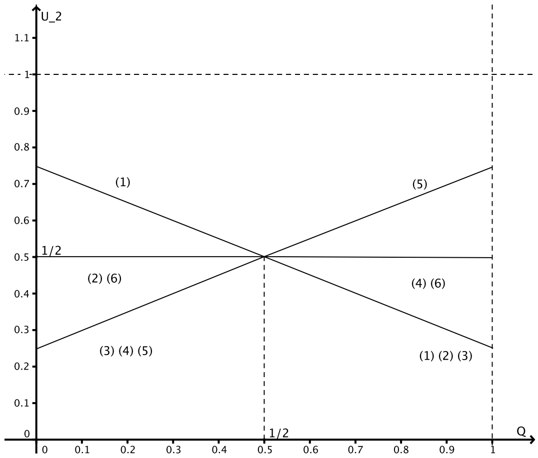

Figure 3.10: Modified Matching Pennies: payoff function (3.5) of player 2 for different strategies $[P_0, P_1] \subseteq \Delta S_2$.

sponse correspondences.

(1) $P_0 > 1/2 \Rightarrow Q_0 = Q_1 = 0 \Rightarrow P_0 = P_1 = 0$, thus this is not an equilibrium.

(2) $P_0 = 1/2 < P_1 \Rightarrow [Q_0, Q_1] \subseteq [0, 1/2]$. Player 2 can choose different subsets of the interval $[0, 1/2]$, but only one leads to a best response of player 1 that is consistent with his original play:

$$[Q_0, Q_1] \subseteq [1/3, 1/2] \Rightarrow [P_0, P_1] \subseteq [1/3, 1] \ .$$

If player 1 chooses to play $[P_0, P_1] = [1/2, P_1]$ with $1/2 \leq P_1 \leq 1$, then this is an Ellsberg equilibrium,

$$([1/2, P_1], [1/3, Q_1]), \text{ where } 1/2 \leq P_1 \leq 1 \text{ and } 1/3 \leq Q_1 \leq 1/2 \,.$$

(3) $P_0 < 1/2 < P_1 \Rightarrow [Q_0, Q_1] = 1/2 \Rightarrow P_0 = P_1 = 1$, thus this is not an equilibrium.

(4) $P_0 < 1/2 = P_1 \Rightarrow [Q_0, Q_1] \subseteq [1/2, 1] \Rightarrow P_0 = P_1 = 1$, thus this is not an equilibrium.

(5) $P_1 < 1/2 \Rightarrow Q_0 = Q_1 = 1 \Rightarrow P_0 = P_1 = 1$, thus this is not an equilibrium.

(6) $P_0 = 1/2 = P_1 \Rightarrow [Q_0, Q_1] \subseteq [0, 1]$. If player 2 chooses $Q_0 < 1/3 < Q_1$, then player 1's best response is $P_0 = P_1 = 1/3$, thus this is not an equilibrium. But if player 2

decides to play $Q_0 = Q_1 = 1/3$ and player 1 plays $P_0 = P_1 = 1/2$, no player can gain by deviating from this strategy. This is the mixed strategy Nash equilibrium (P^*, Q^*).

Proposition 3.6. *In modified Matching Pennies the Ellsberg equilibria are of the form* $([1/2, P_1], [1/3, Q_1])$, *where* $1/2 \leq P_1 \leq 1$ *and* $1/3 \leq Q_1 \leq 1/2$.

In addition to the Nash equilibrium in mixed strategies, in asymmetric modified Matching Pennies are Ellsberg equilibria in which both players create ambiguity. In the game analyzed above, in particular, player 2 uses a interval between his Nash equilibrium strategy and his immunization strategy. We generalize this observation in the next section.

3.3.2 Ellsberg Equilibria in General Conflict Games

In this section we determine the Ellsberg equilibria for competitive games with more general payoffs. We will see that in the general case when $P^* \neq Q^*$ there exists an Ellsberg equilibrium where both players create ambiguity. The set of probability distributions which each player uses in equilibrium is bounded by his Nash equilibrium strategy and his immunization strategy. We provide two propositions. In Proposition 3.7 below we restrict to the case where the immunization strategies are equal to the opponents' Nash equilibrium strategies. It will become clear that this simplifies the notation and the comprehensiveness of the result. In Proposition 3.9 we derive a further generalization.

First consider the competitive two-person 2×2 game with payoff matrix in Figure 3.11. We assume that $a, c > b$ and $d, f < e$.

		Player 2	
		L	R
Player 1	U	a, d	b, e
	D	b, e	c, f

Figure 3.11: General conflict game I.

Due to the assumptions on the payoff, our game has conflicting interests and no pure strategy Nash equilibria. In the unique mixed strategy Nash equilibrium, player 1 plays U with probability

$$P^* = \frac{f - e}{d - 2e + f},$$

and player 2 plays L with probability

$$Q^* = \frac{c - b}{a - 2b + c}.$$

Proposition 3.7. *The Ellsberg equilibria of the above game are the following:*
For $P^* > Q^*$ all Ellsberg equilibria are of the form

$$([P^*, P_1], [Q^*, Q_1]) \text{ for } P^* \leq P_1 \leq 1, Q^* \leq Q_1 \leq P^* ;$$

for $P^* < Q^*$ all Ellsberg equilibria are of the form

$$([P_0, P^*], [Q_0, Q^*]) \text{ for } 0 \leq P_0 \leq P^*, P^* \leq Q_0 \leq Q^* ;$$

and for $P^* = Q^*$ all Ellsberg equilibria are of the form

$$\begin{aligned} &(Q^*, [Q_0, Q_1]), \text{ where } Q_0 \leq Q^* \leq Q_1 , \\ \text{and } &([P_0, P_1], P^*), \text{ where } P_0 \leq P^* \leq P_1 . \end{aligned}$$

Proof. The Nash equilibrium strategies follow from the usual analysis. The conditions on the payoffs assure that the Nash equilibrium is completely mixed, i.e., $0 < P^* < 1$ and $0 < Q^* < 1$. Let now $[P_0, P_1]$ and $[Q_0, Q_1]$ be Ellsberg strategies of player 1 and 2, where $P \in [P_0, P_1]$ is the probability of player 1 to play U, and $Q \in [Q_0, Q_1]$ is the probability of player 2 to play L. Let us compute the minimal expected payoff. The minimal expected payoff of player 1 when he plays the mixed strategy P is

$$\begin{aligned} \min_{Q_0 \leq Q \leq Q_1} u_1(P, Q) &= \min_{Q_0 \leq Q \leq Q_1} aPQ + bP(1-Q) + b(1-P)Q + c(1-P)(1-Q) \\ &= \min_{Q_0 \leq Q \leq Q_1} Q(b - c + P(a - 2b + c)) + bP + c - cP \\ &= \begin{cases} Q_1(b - c + P(a - 2b + c)) + bP + c - cP & \text{if } P < \frac{c-b}{a-2b+c} , \\ \frac{ac-b^2}{a-2b+c} & \text{if } P = \frac{c-b}{a-2b+c} , \\ Q_0(b - c + P(a - 2b + c)) + bP + c - cP & \text{if } P > \frac{c-b}{a-2b+c} . \end{cases} \end{aligned}$$

Note that the payoff function is constant at $P = \frac{c-b}{a-2b+c}$, which is player 2's Nash equilibrium strategy. Depending on Q_0 and Q_1 the minimal payoff function can have six different forms. It can be strictly increasing, strictly decreasing, have flat parts or be completely constant. To determine how player 1 maximizes his minimal payoff for different Q_0 and Q_1, we look at the borders of the minimal payoff function, where $P = 0$ and $P = 1$. This gives us two functions

$$\min_{Q_0 \le Q \le Q_1} u_1(0\,,Q) = Q_1(b-c)+c\,,$$

$$\text{and} \quad \min_{Q_0 \le Q \le Q_1} u_1(1\,,Q) = Q_0(a-b)+b\,.$$

Note that $b-c<0$ and $a-b>0$, that is, the minimal payoff function is decreasing with Q_1 at $P=0$ and increasing with Q_0 at $P=1$. When

$$Q_1 = \frac{c-b}{a-2b+c} = Q^*\,, \quad \text{then} \quad \min_{Q_0 \le Q \le Q_1} u_1(0\,,Q) = \frac{ac-b^2}{a-2b+c}\,,$$

that is, the minimal payoff function is constant for $0 \le P \le Q^*$. The same is true for the other boundary. When

$$Q_0 = \frac{c-b}{a-2b+c} = Q^*\,, \quad \text{then} \quad \min_{Q_0 \le Q \le Q_1} u_1(1\,,Q) = \frac{ac-b^2}{a-2b+c}\,,$$

that is, the minimal payoff function is constant for $Q^* \le P \le 1$.
With this analysis one can see immediately that when $Q_0 = Q^* = Q_1$, the minimal payoff function is constant for all $P \in [0,1]$ thus any Ellsberg strategy $[P_0,P_1] \subseteq [0,1]$ is a best response for player 1.
Assume that $Q_0 > Q^*$, then (since $a-b>0$) the minimal payoff function is strictly increasing and the best response of player 1 is $P_0 = P_1 = 1$. The opposite is true for $Q_1 < Q^*$ and thus the best response is $P_0 = P_1 = 0$.
Observe that when $Q_0 < Q^* < Q_1$, the values of both boundary functions drop below Q^* and the function takes its maximum at the kink $P = Q^*$. Therefore player 1's best response in this case is $P_0 = P_1 = Q^*$.

Two cases are still missing. The minimal expected payoff function can be flat exclusively to the left or to the right of Q^*. For all $P \in \left[0, \frac{c-b}{a-2b+c}\right] = [0,Q^*]$, Player 1's utility is constant at $\frac{ac-b^2}{a-2b+c}$ when $Q_1 = \frac{c-b}{a-2b+c} = Q^*$, and it is strictly decreasing for $P > Q^*$. Hence, all $P \le Q^*$ are optimal for player 1. He can thus use any Ellsberg strategy $[P_0,P_1]$ with $P_1 \le Q^*$ as a best reply. Similarly, the payoff is constant for all $P \ge Q^*$ when $Q_0 = Q^*$ (and strictly increasing for $P < Q^*$). This means that player 1's best response to a strategy $[Q_0,Q^*]$ is any strategy $[P_0,P_1] \subseteq [0,Q^*]$, and player 1's best response to a strategy $[Q^*,Q_1]$ with $Q^* \le Q_1 \le 1$ is any strategy $[P_0,P_1] \subseteq [Q^*,1]$.

We repeat the same analysis for player 2. His minimal expected utility when he plays the mixed strategy Q is

$$\begin{aligned}\min_{P_0\leq P\leq P_1} u_2(P,Q) &= \min_{P_0\leq P\leq P_1} dPQ + eP(1-Q) + e(1-P)Q + f(1-P)(1-Q)\\ &= \min_{P_0\leq P\leq P_1} P(e-f+Q(d-2e+f)) + eQ + f - fQ\\ &= \begin{cases} P_0(e-f+Q(d-2e+f)) + eQ + f - fQ & \text{if } Q < \frac{f-e}{d-2e+f}\,,\\ \frac{df-e^2}{d-2e+f} & \text{if } Q = \frac{f-e}{d-2e+f}\,,\\ P_1(e-f+Q(d-2e+f)) + eQ + f - fQ & \text{if } Q > \frac{f-e}{d-2e+f}\,. \end{cases}\end{aligned}$$

Note that the payoff function has a fixed value at $Q = \frac{f-e}{d-2e+f}$, which is player 1's Nash equilibrium strategy. Again, as for player 1, depending on P_0 and P_1 the minimal payoff function can have six different forms. We note the two functions that describe the minimal payoff function at the borders $Q = 0$ and $Q = 1$:

$$\min_{P_0\leq P\leq P_1} u_1(P,0) = P_0(e-f) + f\,,$$

$$\text{and} \qquad \min_{P_0\leq P\leq P_1} u_1(P,1) = P_1(d-e) + e\,.$$

Note that $e - f > 0$ and $d - e < 0$, that is, the minimal payoff function is increasing with P_0 in $P = 0$ and decreasing with P_1 in $P = 1$. When

$$P_0 = \frac{f-e}{d-2e+f} = P^*\,, \quad \text{then} \quad \min_{P_0\leq P\leq P_1} u_2(P,0) = \frac{df-e^2}{d-2e+f}\,,$$

that is, the minimal payoff function is constant for $0 \leq Q \leq P^*$. The same is true for the other boundary: When

$$P_1 = \frac{f-e}{d-2e+f} = P^*\,, \quad \text{then} \quad \min_{P_0\leq P\leq P_1} u_2(P,1) = \frac{df-e^2}{d-2e+f}\,,$$

that is, the minimal payoff function is constant for $P^* \leq Q \leq 1$.

Similar to the analysis of player 1 we now get the following best responses of player 2. When $P_0 = P^* = P_1$ player 2 can use any strategy $[Q_0, Q_1] \subseteq [0, 1]$, when $P_0 > P^*$ the best response is $Q_0 = Q_1 = 0$ and when $P_1 < P^*$ then $Q_0 = Q_1 = 1$. When $P_0 < P^* < P_1$ the minimal payoff function takes its maximum at the kink $Q = P^*$ and accordingly player 2's best response is $Q_0 = Q_1 = P^*$.

Finally note that player 2's utility is constant at $\frac{df-e^2}{d-2e+f}$ for all $Q \in \left[0, \frac{f-e}{d-2e+f}\right] = [0, P^*]$ when $P_0 = \frac{f-e}{d-2e+f} = P^*$, and it is strictly decreasing for $Q > P^*$. Hence, all $Q \leq P^*$ are optimal for player 2. He can thus use any Ellsberg strategy $[Q_0, Q_1]$ where $Q_1 \leq P^*$ as a best reply. Similarly, the payoff is constant for all $Q \geq P^*$ when $P_0 = P^*$ (and strictly increasing for $Q < P^*$). This means that player 2's best response to a strategy $[P^*, P_1]$ is any strategy $[Q_0, Q_1] \subseteq [0, P^*]$, and player 2's best response to a strategy $[P^*, P_1]$ with $P^* \leq P_1 \leq 1$ is any strategy $[Q_0, Q_1] \subseteq [P^*, 1]$.

In Ellsberg equilibrium no player wants to unilaterally deviate from his equilibrium strategy. We analyze in the following which Ellsberg strategies have best responses such that no player wants to deviate.

We assume first that $Q^* < P^*$. Three Ellsberg strategies can quickly be excluded to be part of an Ellsberg equilibrium. Suppose player 2 plays $[Q_0, Q_1]$ with $Q_0 > Q^*$, then player 1's best response is $P_0 = P_1 = 1$ and since we are looking at a competitive game, player 2 would want to deviate from his original strategy to $Q_0 = Q_1 = 0$. A similar reasoning leads to the result that an Ellsberg strategy $[Q_0, Q_1]$ with $Q_1 < Q^*$ cannot be an equilibrium strategy. Thirdly, suppose player 2 plays $[Q_0, Q_1]$ with $Q_0 < Q < Q_1$, then player 1 would respond with $P_0 = P_1 = Q^*$. Since $Q^* < P^*$ player 2 would deviate from his original strategy to $Q_0 = Q_1 = 1$.

Now suppose player 2 plays $Q_0 = Q_1 = Q^*$, then player 1 can respond with any $[P_0, P_1] \subseteq [0, 1]$. Any choice with $P_0 \geq P^*$, $P_1 < P^*$ or $P_0 < P^* < P_1$ leads to contradictions similar to the cases above. The possibilities that P_0 and P_1 are such that $P_0 < P_1 = P^*$ or $P_0 = P_1 = P^*$ (which are Ellsberg equilibria), are contained in the Ellsberg equilibria that arise in the two remaining cases below.

Suppose player 2 plays $[Q^*, Q_1]$ with $Q^* \leq Q_1 \leq 1$, then if player 1 responds with $[P_0, P_1] = [P^*, P_1]$ with $P^* \leq P_1 \leq 1$ player 2 would play any strategy $[Q_0, Q_1] \subseteq [0, P^*]$ as a best response. Because $Q^* < P^*$, player 2 can choose $[Q_0, Q_1] = [Q^*, Q_1]$ with $Q^* \leq Q_1 \leq P^*$. These strategies are Ellsberg equilibria

$$([P^*, P_1], [Q^*, Q_1]), \text{ where } P^* \leq P_1 \leq 1 \text{ and } Q^* \leq Q_1 \leq P^*.$$

In the case $Q^* < P^*$ this is the only type of Ellsberg equilibrium. Note that the Nash equilibrium is contained in these equilibrium strategies.

When we assume that $P^* < Q^*$, the analysis is very similar. We skip the first four cases and only look at the cases where the minimal payoff function has flat parts. Suppose player 2 plays $[Q_0, Q^*]$ with $0 \leq Q_0 \leq Q^*$, then if we let player 1 pick $[P_0, P_1] \subseteq [P_0, P^*]$ with $0 \leq P_0 \leq P^*$, player 2's best response is any subset $[Q_0, Q_1] \subseteq [P^*, 1]$. Again, because $P^* < Q^*$, he can choose $[Q_0, Q_1] = [Q_0, Q^*]$ with $P^* \leq Q_0 \leq Q^*$ as a best response. Player 1 would not want to deviate and thus these strategies are Ellsberg equilibria

$$([P_0, P^*], [Q_0, Q^*]), \text{ where } 0 \leq P_0 \leq P^* \text{ and } P^* \leq Q_0 \leq Q^*.$$

As before, this is the only type of Ellsberg equilibrium in the case $P^* < Q^*$.

Finally let $P^* = Q^*$. Repeat the considerations above having in mind the equality of the Nash equilibrium strategies. Since it was precisely the difference between P^* and Q^* that led to the Ellsberg equilibria in the above cases, we see that no Ellsberg equilibria exist where both players create ambiguity. But, in difference to the above analysis, two types of Ellsberg equilibria with unilateral ambiguity arise that could not be sustained above.

Remember that when player 2 plays $[Q_0, Q_1]$ with $Q_0 < Q^* < Q_1$ then it is optimal for player 1 to respond with $P_0 = P_1 = Q^*$. Since $P^* = Q^*$, these strategies are in equilibrium, even for $Q_0 \leq Q^* \leq Q_1$. One observes that, as long as player 2 makes sure that the mixed Nash equilibrium strategy Q^* is strictly contained in his Ellsberg strategy, player 1 responds with Q^* and we have Ellsberg equilibria, in which player 1 immunizes against the ambiguity of player 2. An analogous type of Ellsberg equilibrium exists for player 2 immunizing against the ambiguity of player 1 by playing $Q_0 = Q_1 = P^*$. Thus, we have the following Ellsberg equilibria

$$(Q^*, [Q_0, Q_1]), \text{ where } Q_0 \leq Q^* \leq Q_1,$$

$$\text{and} \quad ([P_0, P_1], P^*), \text{ where } P_0 \leq P^* \leq P_1.$$

These equilibria do not exist in the non-symmetric case $P^* \neq Q^*$ since the immunization strategies are in general not equilibrium strategies. Due to the assumptions on the payoffs in this proposition, the immunization strategy equals the Nash equilibrium strategy of the opponent. Thereby the immunization strategy is an equilibrium strategy only when $P^* = Q^*$. □

Remarks 3.8. *1. In Proposition 3.7 we restrict to the case with (U, D) and (L, R) giving the same payoffs (b, e) for both players. The more general competitive game* *yields two more types of Ellsberg equilibria as we find in Proposition 3.9. The nice feature of the above restriction is that players use the mixed Nash equilibrium strategy of their respective opponent as their immunization strategy.*

2. *Observe the asymmetry in the Ellsberg equilibria in the preceding proposition: no matter if $P^* < Q^*$ or $Q^* < P^*$, it is always player 2 who creates ambiguity between the Nash equilibrium strategies, player 1 never does so. This is due to the assumptions on the payoffs. If we assume that $a, c < b$ and $d, f > e$, player 1 plays between P^* and Q^*.*

For a further generalization consider now the 2×2 conflict game with payoff matrix in Figure 3.12. We assume that $a, d > b, c$ and $e, h < f, g$.

		Player 2 L	Player 2 R
Player 1	U	a, e	b, f
	D	c, g	d, h

Figure 3.12: General conflict game II.

As before, P^* and Q^* are the probabilities with which U respectively L are played. Let

$$P^* = \frac{h - g}{e - f - g + h}, \quad \text{respectively} \quad Q^* = \frac{d - b}{a - b - c + d}$$

denote the Nash equilibrium strategies for player 1 and 2, respectively. The immunization strategies of each player are denoted by M_1, respectively M_2.

Proposition 3.9. *Let P^*, Q^* denote the mixed strategy Nash equilibria and M_1, M_2 the immunization strategies of player 1 and 2, respectively. Then the Ellsberg equilibria of the general conflict game above are of the following form.*

	$M_1 \leq P^*$	$M_1 \geq P^*$
$M_2 \leq Q^*$	$([P_0, P^*], [Q^*, Q_1])$ $M_1 \leq P_0$	$([P_0, P^*], [Q_0, Q^*])$ $M_2 \leq Q_0$
$M_2 \geq Q^*$	$([P^*, P_1], [Q^*, Q_1])$ $Q_1 \leq M_2$	$([P^*, P_1], [Q_0, Q^*])$ $P_1 \leq M_1$

If $M_1 = P^*$ or $M_2 = Q^*$, then an additional type of Ellsberg equilibria arises,

$$(P^*, [Q_0, Q_1]), \text{ where } Q_0 \leq Q^* \leq Q_1 \qquad \text{when } M_1 = P^*;$$
$$([P_0, P_1], Q^*), \text{ where } P_0 \leq P^* \leq P_1 \qquad \text{when } M_2 = Q^*.$$

In any case, we have the Ellsberg equilibrium which is identical to the Nash equilibrium in mixed strategies, (P^*, Q^*).

Proof. The Nash equilibrium strategies follow from the usual analysis. To calculate the Ellsberg equilibria of the general conflict game II (Figure 3.12), we first derive the utility functions of player 1 and player 2. Due to the assumption that $a, d > b, c$ and $e, h < f, g$, the denominator $a - b - c + d$ is positive, and the denominator $e - f - g + h$ is negative. This reflects the competitiveness of the game in the payoff functions; player 1 uses Q_0 as a minimizer when $P > \frac{d-c}{a-b-c+d}$, and on the contrary, player 2 uses P_0 as a minimizer when $Q < \frac{h-f}{e-f-g+h}$.

$$\begin{aligned}
U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \leq Q \leq Q_1} aPQ + bP(1-Q) + c(1-P)Q + d(1-P)(1-Q) \\
&= \min_{Q_0 \leq Q \leq Q_1} Q((a-b-c+d)P + c - d) + (b-d)P + d \\
&= \begin{cases} Q_0((a-b-c+d)P + c - d) + (b-d)P + d & \text{if } P > \frac{d-c}{a-b-c+d}, \\ \frac{(b-d)(c-d)}{a-b-c+d} + d & \text{if } P = \frac{d-c}{a-b-c+d}, \\ Q_1((a-b-c+d)P + c - d) + (b-d)P + d & \text{if } P < \frac{d-c}{a-b-c+d}, \end{cases}
\end{aligned}$$

$$\begin{aligned}
U_2([P_0, P_1], Q) &= \min_{P_0 \leq P \leq P_1} ePQ + g(1-P)Q + fP(1-Q) + h(1-P)(1-Q) \\
&= \min_{P_0 \leq P \leq P_1} P((e-f-g+h)Q + f - h) + (g-h)Q + h \\
&= \begin{cases} P_0((e-f-g+h)Q + f - h) + (g-h)Q + h & \text{if } Q < \frac{h-f}{e-f-g+h}, \\ \frac{(g-h)(h-f)}{e-f-g+h} + h & \text{if } Q = \frac{h-f}{e-f-g+h}, \\ P_1((e-f-g+h)Q + f - h) + (g-h)Q + h & \text{if } Q > \frac{h-f}{e-f-g+h}. \end{cases}
\end{aligned}$$

We see that

$$M_1 = \frac{d-c}{a-b-c+d}, \quad \text{respectively} \quad M_2 = \frac{h-f}{e-f-g+h}$$

are indeed the immunization strategies of player 1 and 2 respectively. At the boundaries the utility functions become

$$U_1(0\,,[Q_0,Q_1]) = (c-d)Q_1 + d\,, \qquad U_2([P_0,P_1]\,,0) = (f-h)P_0 + h\,,$$
$$U_1(1\,,[Q_0,Q_1]) = (a-b)Q_0 + b\,, \qquad U_2([P_0,P_1]\,,1) = (e-g)P_1 + g\,.$$

The payoff function of player 1 is constant when $Q_0 = Q_1 = Q^*$, and the payoff function of player 2 is constant when $P_0 = P_1 = P^*$. The best response correspondences for both players are listed below. We see immediately that, of course, there cannot be an Ellsberg equilibrium in pure strategies: when player 2 plays L, player 1 best responds U, and when player 1 plays U, player 2 best responds R. The intersections of the best response correspondences are discussed in detail below.

$$Q_0 > Q^*: \qquad B_1([Q_0,Q_1]) = 1 \tag{1}$$
$$Q_0 = Q^* < Q_1: \qquad B_1([Q_0,Q_1]) = \{[P_0,P_1] \subseteq [M_1,1]\} \tag{2}$$
$$Q_0 < Q^* < Q_1: \qquad B_1([Q_0,Q_1]) = M_1 \tag{3}$$
$$Q_0 < Q^* = Q_1: \qquad B_1([Q_0,Q_1]) = \{[P_0,P_1] \subseteq [0,M_1]\} \tag{4}$$
$$Q_1 < Q^*: \qquad B_1([Q_0,Q_1]) = 0 \tag{5}$$
$$Q_0 = Q^* = Q_1: \qquad B_1([Q_0,Q_1]) = \{[P_0,P_1] \subseteq [0,1]\} \tag{6}$$

$$P_0 > P^*: \qquad B_2([P_0,P_1]) = 0$$
$$P_0 = P^* < P_1: \qquad B_2([P_0,P_1]) = \{[Q_0,Q_1] \subseteq [0,M_2]\}$$
$$P_0 < P^* < P_1: \qquad B_2([P_0,P_1]) = M_2$$
$$P_0 < P^* = P_1: \qquad B_2([P_0,P_1]) = \{[Q_0,Q_1] \subseteq [M_2,1]\}$$
$$P_1 < P^*: \qquad B_2([P_0,P_1]) = 1$$
$$P_0 = P^* = P_1: \qquad B_2([P_0,P_1]) = \{[Q_0,Q_1] \subseteq [0,1]\}$$

To find the Ellsberg equilibria, we look at the different cases in turn.

(1) $Q_0 > Q^*$: player 1 responds $P_0 = P_1 = 1$ and player 2 chooses $Q_0 = Q_1 = 0$, thus this is not an Ellsberg equilibrium.

(2) $\left[\frac{d-b}{a-b-c+d}, Q_1\right] \Rightarrow [P_0,P_1] \subseteq \left[\frac{d-c}{a-b-c+d}, 1\right]$, then, depending on the size of M_1 in relation to P^*, player 1 has the following choices:

- $\frac{d-c}{a-b-c+d} \leq \frac{h-g}{e-f-g+h}$: player 1 can play either

– $\left[\frac{h-g}{e-f-g+h}, P_1\right]$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[0, \frac{h-f}{e-f-g+h}\right]$. Now, depending on the size of M_2 in relation to Q^*, player 2 has the following choices:

  * $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: $\left[\frac{d-b}{a-b-c+d}, Q_1\right]$ with $Q_1 \leq \frac{h-f}{e-f-g+h}$, we thus have here the Ellsberg equilibrium

$$\left(\left[\frac{h-g}{e-f-g+h}, P_1\right], \left[\frac{d-b}{a-b-c+d}, Q_1\right]\right),$$

$$\text{where } Q_1 \leq \frac{h-f}{e-f-g+h},$$

$$\text{that is } ([P^*, P_1], [Q^*, Q_1]), \text{ where } Q_1 \leq M_2 .$$

  * $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 0$, thus no Ellsberg equilibrium arises.

– or $\left[P_0, \frac{h-g}{e-f-g+h}\right]$ with $\frac{d-c}{a-b-c+d} \leq P_0 \leq \frac{h-g}{e-f-g+h}$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[\frac{h-f}{e-f-g+h}, 1\right]$. Now, depending on the size of M_2 in relation to Q^*, player 2 has the following choices:

  * $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: $\left[\frac{d-b}{a-b-c+d}, Q_1\right]$, then the best response of player 1 is $[P_0, P_1] \subseteq \left[\frac{d-c}{a-b-c+d}, 1\right]$. We thus have here the Ellsberg equilibrium

$$\left(\left[P_0, \frac{h-g}{e-f-g+h}\right], \left[\frac{d-b}{a-b-c+d}, Q_1\right]\right),$$

$$\text{where } \frac{d-c}{a-b-c+d} \leq P_0 ,$$

$$\text{that is } ([P_0, P^*], [Q^*, Q_1]), \text{ where } M_1 \leq P_0 .$$

  * $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 1$, thus no Ellsberg equilibrium arises.

- $\frac{d-c}{a-b-c+d} \geq \frac{h-g}{e-f-g+h}$: to any $[P_0, P_1]$ the best response is $Q_0 = Q_1 = 0$, thus no Ellsberg equilibrium arises.

(3) $Q_0 < Q^* < Q_1$: player 1 responds with $P_0 = P_1 = M_1$. Only when $M_1 = P^*$, player 1 sticks to his strategy and we get the equilibrium $(P^*, [Q_0, Q_1])$, where $Q_0 < Q^* < Q_1$.

(4) $\left[Q_0, \frac{d-b}{a-b-c+d}\right] \Rightarrow [P_0, P_1] \subseteq \left[0, \frac{d-c}{a-b-c+d}\right]$, then, depending on the size of M_1 in relation to P^*, player 1 has the following choices:

- $\frac{d-c}{a-b-c+d} \geq \frac{h-g}{e-f-g+h}$: player 1 can play either
  - $\left[P_0, \frac{h-g}{e-f-g+h}\right]$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[\frac{h-f}{e-f-g+h}, 1\right]$. Now, depending on the size of M_2 in relation to Q^*, player 2 has the following choices:
    * $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: $\left[Q_0, \frac{d-b}{a-b-c+d}\right]$ with $\frac{h-f}{e-f-g+h} \leq Q_0$, we thus have here the Ellsberg equilibrium

$$\left(\left[P_0, \frac{h-g}{e-f-g+h}\right], \left[Q_0, \frac{d-b}{a-b-c+d}\right]\right),$$

$$\text{where } \frac{h-f}{e-f-g+h} \leq Q_0 \,,$$

$$\text{that is } \left(\left[P_0, P^*\right], \left[Q_0, Q^*\right]\right), \text{ where } M_2 \leq Q_0 \,.$$

    * $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 1$, thus no Ellsberg equilibrium arises.
  - or $\left[\frac{h-g}{e-f-g+h}, P_1\right]$ with $\frac{h-g}{e-f-g+h} \leq P_1 \leq \frac{d-c}{a-b-c+d}$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[0, \frac{h-f}{e-f-g+h}\right]$. Now, depending on the size of M_2 in relation to Q^*, player 2 has the following choices:
    * $\frac{d-b}{a-b-c+d} \leq \frac{h-f}{e-f-g+h}$: $\left[Q_0, \frac{d-b}{a-b-c+d}\right]$, then the best response of player 1 is $[P_0, P_1] \subseteq \left[0, \frac{d-c}{a-b-c+d}\right]$. We thus have here the Ellsberg equilibrium

$$\left(\left[\frac{h-g}{e-f-g+h}, P_1\right], \left[Q_0, \frac{d-b}{a-b-c+d}\right]\right),$$

$$\text{where } P_1 \leq \frac{d-c}{a-b-c+d} \,,$$

$$\text{that is } \left(\left[P^*, P_1\right], \left[Q_0, Q^*\right]\right), \text{ where } P_1 \leq M_1 \,.$$

    * $\frac{d-b}{a-b-c+d} \geq \frac{h-f}{e-f-g+h}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 0$, thus no Ellsberg equilibrium arises.
- $\frac{d-c}{a-b-c+d} \leq \frac{h-g}{e-f-g+h}$: to any $[P_0, P_1]$ the best response is $Q_0 = Q_1 = 0$, thus no Ellsberg equilibrium arises.

(5) $Q_1 < Q^*$: player 1 responds $P_0 = P_1 = 0$ and player 2 chooses $Q_0 = Q_1 = 1$ thereafter, thus this is not an Ellsberg equilibrium.

(6) $Q_0 = Q^* = Q_1$: player 1 responds with $[P_0, P_1] \subseteq [0, 1]$. Only when $M_2 = Q^*$, player 2 sticks to his strategy and we get the equilibrium $([P_0, P_1], Q^*)$, where $P_0 < P^* < P_1$.

□

3.3.3 Further Examples of Two-Person Conflicts

Now we calculate the Ellsberg equilibria of some more examples of conflict games. We start with the classic symmetric Matching Pennies game to observe how the Ellsberg equilibria change compared to the modified case above. Then we apply Proposition 3.9 to a non-symmetric conflict game.

Matching Pennies

We look at a symmetric Matching Pennies game with the payoff matrix given in Figure 3.13. Note that the game has a unique mixed strategy Nash equilibrium in which both players mix with equal probabilities,

$$(P^*, Q^*) = ((1/2, 1/2), (1/2, 1/2)).$$

		Player 2: L	Player 2: R
Player 1	U	$1, -1$	$-1, 1$
	D	$-1, 1$	$1, -1$

Figure 3.13: Symmetric Matching Pennies.

Suppose player 1 chooses to play U with probability $P \in [P_0, P_1]$, whereas player 2 plays U with probability $Q \in [Q_0, Q_1]$. To find the Ellsberg equilibria, we first derive the payoff function $U_1(P, [Q_0, Q_1])$ of player 1,

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \le Q \le Q_1} PQ - (1-P)Q - P(1-Q) + (1-P)(1-Q) \\ &= \min_{Q_0 \le Q \le Q_1} Q(4P-2) - 2P + 1 \\ &= \begin{cases} Q_0(4P-2) - 2P + 1 & \text{if } P > 1/2\,, \\ 0 & \text{if } P = 1/2\,, \\ Q_1(4P-2) - 2P + 1 & \text{if } P < 1/2\,. \end{cases} \end{aligned}$$

Observe that at the boundaries the function reduces to

$$U_1(0\,,[Q_0,Q_1]) = 1 - 2Q_1\,,$$
$$\text{and}\quad U_1(1\,,[Q_0,Q_1]) = Q_0 - 1\,.$$

We see that $U_1(P,[Q_0,Q_1])$ is constant at 0 when $Q_0 = Q_1 = 1/2$. Therefore the best responses of player 1 are as follows:

$$\begin{array}{ll}
Q_0 > 1/2: & B_1([Q_0,Q_1]) = 1 \\
Q_0 = 1/2 < Q_1: & B_1([Q_0,Q_1]) = \{[P_0,P_1] \subseteq [1/2,1]\} \\
Q_0 < 1/2 < Q_1: & B_1([Q_0,Q_1]) = 1/2 \\
Q_0 < 1/2 = Q_1: & B_1([Q_0,Q_1]) = \{[P_0,P_1] \subseteq [0,1/2]\} \\
Q_1 < 1/2: & B_1([Q_0,Q_1]) = 0 \\
Q_0 = 1/2 = Q_1: & B_1([Q_0,Q_1]) = \{[P_0,P_1] \subseteq [0,1]\}
\end{array}$$

We get an analog behavior for player 2, just for the opposite strategies. We fix the probability $Q \in \Delta S_2$ and derive the payoff function $U_2([P_0,P_1]\,,Q)$ of player 2. Player 2 evaluates his minimal expected payoff given player 1 plays $P \in [P_0,P_1]$.

$$\begin{aligned}
U_2([P_0,P_1]\,,Q) &= \min_{P_0 \le P \le P_1} -PQ + (1-P)Q + P(1-Q) - (1-P)(1-Q) \\
&= \min_{P_0 \le P \le P_1} P(-4Q+2) + 2Q - 1 \\
&= \begin{cases} P_0(-4Q+2) + 2Q - 1 & \text{if } Q < 1/2\,, \\ 0 & \text{if } Q = 1/2\,, \\ P_1(-4Q+2) + 2Q - 1 & \text{if } Q > 1/2\,. \end{cases}
\end{aligned}$$

Observe that at the boundaries the function reduces to

$$U_2([P_0,P_1]\,,0) = 2P_0 - 1\,,$$
$$\text{and}\quad U_2([P_0,P_1]\,,1) = 1 - 2P_1\,.$$

The best response correspondence of player 2 is thus the following:

$$P_0 > 1/2: \qquad B_2([P_0, P_1]) = 0 \tag{1}$$
$$P_0 = 1/2 < P_1: \qquad B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1/2]\} \tag{2}$$
$$P_0 < 1/2 < P_1: \qquad B_2([P_0, P_1]) = 1/2 \tag{3}$$
$$P_0 < 1/2 = P_1: \qquad B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [1/2, 1]\} \tag{4}$$
$$P_1 < 1/2: \qquad B_2([P_0, P_1]) = 1 \tag{5}$$
$$P_0 = 1/2 = P_1: \qquad B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1]\} \tag{6}$$

Now we can solve for the Ellsberg equilibria by finding the intersections of the best response correspondences.

(1) $P_0 > 1/2 \Rightarrow Q_0 = Q_1 = 0 \Rightarrow P_0 = P_1 = 0$, this contradicts the original play by player 1, thus this is not an equilibrium.

(2) $P_0 = 1/2 < P_1 \Rightarrow [Q_0, Q_1] \subseteq [0, 1/2] \Rightarrow [P_0, P_1] \subseteq [0, 1/2]$, this again contradicts the original play by player 1, thus this is not an equilibrium.

(3) $P_0 < 1/2 < P_1 \Rightarrow Q_0 = Q_1 = 1/2 \Rightarrow [P_0, P_1] \subseteq [0, 1]$, this is an Ellsberg equilibrium, $([P_0, P_1], 1/2)$, where $P_0 < 1/2 < P_1$.

(4) $P_0 < 1/2 = P_1 \Rightarrow [Q_0, Q_1] \subseteq [1/2, 1] \Rightarrow P_0 = P_1 = 1$, thus this is not an equilibrium.

(5) $P_1 < 1/2 \Rightarrow Q_0 = Q_1 = 1 \Rightarrow P_0 = P_1 = 1$, thus this is not an equilibrium.

(6) $P_0 = 1/2 = P_1 \Rightarrow [Q_0, Q_1] \subseteq [0, 1]$. We consider two cases: if player 2 chooses $Q_0 < 1/2 < Q_1$, then player 1's best response is $P_0 = P_1 = 1/2$, thus $(1/2, [Q_0, Q_1])$, where $Q_0 < 1/2 < Q_1$, is an Ellsberg equilibrium. If player 2 decides to play $Q_0 = Q_1 = 1/2$, no player can gain by deviating from this strategy. This is the mixed strategy Nash equilibrium (P^*, Q^*).

Proposition 3.10. *In Matching Pennies the Ellsberg equilibria are* $([P_0, P_1], 1/2)$*, where* $P_0 < 1/2 < P_1$*, and* $(1/2, [Q_0, Q_1])$*, where* $Q_0 < 1/2 < Q_1$*.*

The Ellsberg equilibria can also be derived by applying Proposition 3.9. We see in this symmetric example that essentially no other behavior than in the mixed Nash equilibrium arises. The players may create complete ambiguity about their randomization, but this does not affect the opponent's best response.

Non-symmetric Conflict Game

To demonstrate the application of Proposition 3.9, we derive the Ellsberg equilibria of the game in Figure 3.14.

		Player 2 L	Player 2 R
Player 1	U	$3, -1$	$-2, 4$
	D	$0, 5$	$4, 2$

Figure 3.14: Non-symmetric conflict game.

We find that $P^* = 3/8$, $Q^* = 2/3$ and $M_1 = 4/9$, $M_2 = 1/4$. That means, since $M_1 > P^*$ and $M_2 < Q^*$, the Ellsberg equilibria of the above game are of the form $([P_0, 3/8], [Q_0, 2/3])$, where $1/4 \leq Q_0 \leq 2/3$.

3.4 Overview of Ellsberg Equilibria in Coordination and Conflict Games

In the preceding sections we derived the Ellsberg equilibria of different two-person 2×2 games. We summarize the results of Propositions 3.2 and 3.9 in Table 3.1. It gives an overview over the Ellsberg equilibria of general coordination and conflict games with payoff matrices as in Figure 3.15. For coordination games we assume that $a, d > c, b$

		Player 2 L	Player 2 R
Player 1	U	a, e	b, f
	D	c, g	d, h

Figure 3.15: General game.

and $e, h > g, f$, and for competitive games we assume that $a, d < c, b$ and $e, h < g, f$. We denote by P^* (Q^*) the probability that player 1 (2) play U (L) in the mixed Nash equilibrium, and by M_1 (M_2) the probability of U (L) in the immunization strategies of the respective players. Then the Ellsberg equilibria are as in Table 3.1.

Remark 3.11. *When any of the weak inequalities between Nash equilibrium strategies and immunization strategies are in fact equalities, some of the Ellsberg equilibria in Table 3.1 become redundant but not wrong.*

Table 3.1: Ellsberg equilibria in general coordination (blue) and conflict games.

	$M_1 \leq P^*$	$M_1 \geq P^*$
$M_2 \leq Q^*$	$([P^*, P_1], [Q^*, Q_1])$	$([P^*, P_1], [Q_0, Q^*])$ $P_1 \leq M_1$ and $M_2 \leq Q_0$
	$([P_0, P^*], [Q^*, Q_1])$ $M_1 \leq P_0$	$([P_0, P^*], [Q^*, Q_1])$ $M_2 \leq Q_0$
$M_2 \geq Q^*$	$([P_0, P^*], [Q^*, Q_1])$ $M_1 \leq P_0$ and $Q_1 \leq M_2$	$([P_0, P^*], [Q_0, Q^*])$
	$([P^*, P_1], [Q^*, Q_1])$ $Q_1 \leq M_2$	$([P^*, P_1], [Q_0, Q^*])$ $P_1 \leq M_1$

If $M_1 = P^*$ or $M_2 = Q^*$, then in both types of games an additional type of Ellsberg equilibria arises,

$$(P^*, [Q_0, Q_1]), \text{ where } Q_0 \leq Q^* \leq Q_1 \qquad \text{when } M_1 = P^*;$$

$$([P_0, P_1], Q^*), \text{ where } P_0 \leq P^* \leq P_1 \qquad \text{when } M_2 = Q^*.$$

In both types of games and all the above cases we have the Ellsberg equilibria which are identical to the Nash equilibria in pure (if they exist) and mixed strategies.

The results assembled in the above table cover all generic games, except when there is exactly one pure Nash equilibrium. In that case the pure Nash equilibrium is the only Ellsberg equilibrium. We showed this in Section 1.3.6.

3.5 Zero-Sum Games

Historically, much interest was paid to two-person zero-sum games which describe situations in which players are in pure opposition to each other. One player maximizing his own payoff is equivalent to him trying to minimize the payoff of his opponent: this makes these games mathematically very tractable.

Naturally, it is interesting to see what the Ellsberg equilibria of these zero-sum games are. A *two-person zero-sum game* is a game $G = \langle \{1, 2\}, (S_i), (u_i) \rangle$ such that

$$u_2(s_1, s_2) = -u_1(s_1, s_2) \quad \text{for all} \quad s_1 \in S_1 \quad \text{and} \quad s_2 \in S_2.$$

We showed in Theorem 2.8 that in two-person games with a completely mixed Nash equilibrium immunization strategies are maximin strategies. In the context of two-person zero-sum games this implies that immunizing against ambiguity is an equilibrium strategy. This observation is not true for general games as we have seen in the preceding sections, and it is crucial for the type of Ellsberg equilibria that arise in two-person zero-sum games. This is the content of the first part of this section. One player always finds it optimal to play his immunization strategy, and thus no Ellsberg equilibria in which both players create ambiguity arise. We prove this result in Proposition 3.12 and Corollary 3.13 for 2×2 games, a more general case is treated in Theorem 4.2 in the next chapter.

The second part of this section treats Minimax Theorems in Ellsberg games which assure that Ellsberg equilibria of zero-sum games are value preserving.

3.5.1 Ellsberg Equilibria in Two-Person 2×2 Zero-Sum Games

Note that Proposition 3.7 on the Ellsberg equilibria of general competitive games holds likewise for zero-sum games, but due to the assumptions on the payoffs the proposition restricts to zero-sum games where $P^* = Q^*$. This can be seen easily by setting $d := -a$, $-c := f$ and $e := -b$, then $P^* = \frac{f-e}{d-2e+f} = \frac{c-b}{a-2b+c} = Q^*$. Therefore, in the Ellsberg equilibria of two-person 2×2 zero-sum games under the assumptions of Proposition 3.7 only one player creates ambiguity. This also holds for general two-person 2×2 zero-sum games, although in those games in general $P^* \neq Q^*$. The result then hinges on the fact that the immunization strategy of player i is always equal to his own Nash equilibrium strategy.

We now calculate the Ellsberg equilibria of a general two-person 2×2 zero-sum game with the payoff matrix given in Figure 3.16. We assume without loss of generality that $a, b, c, d \geq 0$. Player 1's Ellsberg strategy is as usual denoted $[P_0, P_1]$, player 2's $[Q_0, Q_1]$.

		Player 2	
		L	R
Player 1	U	$a, -a$	$b, -b$
	D	$c, -c$	$d, -d$

Figure 3.16: General zero-sum game.

Proposition 3.12. *(a) In the general two-person zero-sum game in Figure 3.16 the players' immunization strategies are their own Nash equilibrium strategies. By immunizing themselves against ambiguity they can assure themselves the value of the game,* $\frac{ad-bc}{a-b-c+d}$ *and* $-\frac{ad-bc}{a-b-c+d}$*, respectively.*

(b) *Assume that $a, d > b, c$ or $b, c > a, d$. Then the Ellsberg equilibria of the game are of the form*

$$([P_0, P_1], Q^*), \quad \textit{where } 0 \le P_0 \le P^* \le P_1 \le 1,$$
$$\textit{and } (P^*, [Q_0, Q_1]), \quad \textit{where } 0 \le Q_0 \le Q^* \le Q_1 \le 1.$$

(c) *Assume that $a = c$ and $b > d$, i.e., U is a weakly dominant strategy for player 1. We further assume without loss of generality $d < c < b$. The Ellsberg equilibria in this case are*

$$([P_0, P_1], 1), \quad \textit{where } P_1 > \frac{d-c}{d-b} \textit{ and } 0 \le P_0 \le P_1,$$
$$\textit{and } \left(\frac{d-c}{d-b}, [Q_0, 1]\right), \quad \textit{where } 0 \le Q_0 \le 1.$$

Proof. (a) The Nash equilibrium of the game in Figure 3.16 is (P^*, Q^*), where $P^* = \frac{d-c}{a-b-c+d}$ and $Q^* = \frac{d-b}{a-b-c+d}$. We first calculate the minimal expected utility of player 1, when he plays $P \in [P_0, P_1]$.

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \le Q \le Q_1} aPQ + bP(1-Q) + c(1-P)Q + d(1-P)(1-Q) \\ &= \min_{Q_0 \le Q \le Q_1} Q(P(a-b-c+d) + c - d) + (b-d)P + d \\ &= \begin{cases} Q_0(P(a-b-c+d) + c - d) + (b-d)P + d & \text{if } P > P^*, \\ \frac{ad-bc}{a-b-c+d} & \text{if } P = P^*, \\ Q_1(P(a-b-c+d) + c - d) + (b-d)P + d & \text{if } P < P^*. \end{cases} \end{aligned}$$

The same analysis for player 2 yields

$$U_2([P_0, P_1], Q) = \begin{cases} P_0(Q(-a+b+c-d) - b + d) + (-c+d)Q - d & \text{if } Q < Q^*, \\ \frac{bc-ad}{a-b-c+d} & \text{if } Q = Q^*, \\ P_1(Q(-a+b+c-d) - b + d) + (-c+d)Q - d & \text{if } Q > Q^*. \end{cases}$$

As usual we look at the utility function on the borders, this gives

$$U_1(0, [Q_0, Q_1]) = Q_1(c-d) + d, \qquad U_2([P_0, P_1], 0) = P_0(d-b) - d,$$
$$U_1(1, [Q_0, Q_1]) = Q_0(a-b) + b, \qquad U_2([P_0, P_1], 1) = P_1(b-a) - c.$$

The utility function of player 1 is constant when player 2 plays his Nash equilibrium strategy Q^*, that is at the same time his immunization strategy. Player 2's utility function is constant when player 1 plays his Nash equilibrium (and immunization) strategy P^*.

(b) Because the Nash and the immunization strategy coincide, we see immediately that the two types of Ellsberg equilibria stated in the proposition are indeed equilibria. Due to the competitiveness of the game we conclude that no other Ellsberg equilibria exist.

(c) Since $a = c$, we find the following utility functions for player 1 and player 2:

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \le Q \le Q_1} Q(P(d-b) + c - d) + (b-d)P + d \\ &= \begin{cases} Q_0(P(d-b) + c - d) + (b-d)P + d & \text{if } P < \frac{d-c}{d-b}\,, \\ c & \text{if } P = \frac{d-c}{d-b}\,, \\ Q_1(P(d-b) + c - d) + (b-d)P + d & \text{if } P > \frac{d-c}{d-b}\,. \end{cases} \end{aligned}$$

$$\begin{aligned} U_2([P_0, P_1], Q) &= \min_{P_0 \le P \le P_1} P(Q(b-d) - b + d) + (d-c)Q - d \\ &= \begin{cases} -c & \text{if } Q = 1\,, \\ P_1(Q(b-d) + d - b) + (d-c)Q - d & \text{if } Q < 1\,. \end{cases} \end{aligned}$$

As usual we look at the utility function on the borders, this gives

$$\begin{aligned} U_1(0\,, [Q_0, Q_1]) &= Q_0(c-d) + d\,, & U_2([P_0, P_1]\,, 0) &= P_1(d-b) - d\,, \\ U_1(1\,, [Q_0, Q_1]) &= Q_1(c-b) + b\,, & U_2([P_0, P_1]\,, 1) &= -c\,. \end{aligned}$$

We see in the best response correspondences that player 2 finds it optimal to play $Q_0 = Q_1 = 0$ when player 1 plays $P_1 > \frac{d-c}{d-b}$ and $Q_0 = Q_1 = 1$ when $P_1 < \frac{d-c}{d-b}$. When player 1 plays exactly $P_1 = \frac{d-c}{d-b}$, player 2 is indifferent between all distributions in $[0, 1]$.

When $Q_0 = Q_1 = 1$ player 1 is indifferent between all distributions in $[0, 1]$. For all $Q_0 < 1 = Q_1$ player 1 responds optimally with $[P_0, P_1] \subseteq \left[\frac{d-c}{d-b}\,, 1\right]$ and finally when $Q_1 < 1$ player 1 wants to play $P_0 = P_1 = 1$.

From this analysis we find that the Ellsberg equilibria are

$$([P_0, P_1], 1), \quad \text{where } P_1 > \frac{d-c}{d-b},$$

$$\text{and } \left(\frac{d-c}{d-b}, [Q_0, 1]\right), \quad \text{where } 0 \leq Q_0 \leq 1.$$

□

The cases treated in parts *(b)* and *(c)* of Proposition 3.12 are exhaustive in the sense that any other constellation of payoffs would have strictly dominant strategies and we know from Proposition 1.16 that those games do not have proper Ellsberg equilibria. Thus, we can state the following corollary.

Corollary 3.13. *In two-person 2×2 zero-sum games, no proper Ellsberg equilibria exist in which both players create ambiguity.*

To apply Proposition 3.12 we consider the following example.

Example 3.14. *We derive the Ellsberg equilibria of the zero-sum game in Figure 3.17 by applying Proposition 3.12 (c).*

		Player 2	
		L	R
Player 1	U	$2, -2$	$3, -3$
	D	$2, -2$	$1, -1$

Figure 3.17: Example of a zero-sum game.

The Ellsberg equilibria of the game are of the form

$$([P_0, P_1], 1), \; \textit{where } P_1 > 1/2 \textit{ and } (1/2, [Q_0, 1]).$$

3.5.2 Minimax Theorems in Ellsberg Games

We let $G = \langle \{1, 2\}, (S_i), (u_i) \rangle$ be a zero-sum game throughout the whole section, that is $u_2(s_1, s_2) = -u_1(s_1, s_2)$ for all $s_1 \in S_1$ and $s_2 \in S_2$. The following proposition collects the classic Minimax results for zero-sum games, see, e.g., Myerson (1997) p. 123, for a proof.

Proposition 3.15. *Let G be a zero-sum game with value v.*

(i) (P^*, Q^*) *is a Nash equilibrium of* G *if and only if*

$$P^* \in \arg \max_{P \in \Delta S_1} \min_{Q \in \Delta S_2} u_1(P, Q)$$

$$\text{and} \quad Q^* \in \arg \min_{Q \in \Delta S_2} \max_{P \in \Delta S_1} u_1(P, Q)\,. \tag{3.6}$$

(ii) If (P^*, Q^*) *is a Nash equilibrium of* G*, then*

$$u_1(P^*, Q^*) = \max_{P \in \Delta S_1} \min_{Q \in \Delta S_2} u_1(P, Q) = \min_{Q \in \Delta S_2} \max_{P \in \Delta S_1} u_1(P, Q) = v\,. \tag{3.7}$$

The proposition says that player 1 wants to maximize his minimal expected payoff, and player 2 likewise. This interpretation of the result is possible, since for any function $f(x)$, $\max_x(-f(x)) = -\min_x(f(x))$ and $\arg\min_x(-f(x)) = \arg\max_x f(x)$, and therefore expression (3.6) reads

$$Q^* \in \arg \max_{Q \in \Delta S_2} \min_{P \in \Delta S_1} u_2(P, Q)\,. \tag{3.8}$$

If we want to state a similar relation for Ellsberg games, we must be careful, since it is not true in general that $U_2(\mathcal{P}, \mathcal{Q}) = -U_1(\mathcal{P}, \mathcal{Q})$. Therefore the equivalence between (3.6) and (3.8) does not carry over to Ellsberg games. Anyhow, we can show an equivalent of the classical minimax theorem (3.7) and thus find that in two-person zero-sum games the Ellsberg equilibria all yield the same payoff, and this payoff is the value v of the game for player 1, and $-v$ for player 2. We prove two lemmas that yield the result, which is, thereafter, presented in Theorem 3.18.

Lemma 3.16. *Let* G *be a zero-sum game with two players. Then the following holds.*

$$\min_{\mathcal{Q} \subseteq \Delta S_2} \max_{\mathcal{P} \subseteq \Delta S_1} U_1(\mathcal{P}, \mathcal{Q}) \overset{(1)}{=} \max_{\mathcal{P} \subseteq \Delta S_1} \min_{\mathcal{Q} \subseteq \Delta S_2} U_1(\mathcal{P}, \mathcal{Q}) \overset{(2)}{=} u_1(P^*, Q^*) = v\,, \tag{3.9}$$

$$\text{and} \quad \min_{\mathcal{P} \subseteq \Delta S_1} \max_{\mathcal{Q} \subseteq \Delta S_2} U_2(\mathcal{P}, \mathcal{Q}) \overset{(1)}{=} \max_{\mathcal{Q} \subseteq \Delta S_2} \min_{\mathcal{P} \subseteq \Delta S_1} U_2(\mathcal{P}, \mathcal{Q}) \overset{(2)}{=} -u_1(P^*, Q^*) = -v\,. \tag{3.10}$$

Proof. We start by showing equality (2) of equation (3.9), followed by equality (1). (3.10) is shown below. For all equalities we use the fact that for all linear functions $f(x, y)$,

$\min_x \min_y f(x,y) = \min_y \min_x f(x,y)$, and Lemma 1.12.

$$\max_{\mathcal{P}\subseteq\Delta S_1} \min_{\mathcal{Q}\subseteq\Delta S_2} U_1(\mathcal{P},\mathcal{Q}) = \max_{\mathcal{P}\subseteq\Delta S_1} \min_{\mathcal{Q}\subseteq\Delta S_2} \min_{P\in\mathcal{P}} \min_{Q\in\mathcal{Q}} u_1(P,Q)$$
$$= \max_{\mathcal{P}\subseteq\Delta S_1} \min_{\mathcal{Q}\subseteq\Delta S_2} \min_{Q\in\mathcal{Q}} \min_{P\in\mathcal{P}} u_1(P,Q) \overset{\text{Lemma 1.12}}{=} \max_{\mathcal{P}\subseteq\Delta S_1} \min_{Q\in\Delta S_2} \min_{P\in\mathcal{P}} u_1(P,Q)$$
$$= \max_{\mathcal{P}\subseteq\Delta S_1} \min_{P\in\mathcal{P}} \min_{Q\in\Delta S_2} u_1(P,Q) \overset{\text{Lemma 1.12}}{=} \max_{P\in\Delta S_1} \min_{Q\in\Delta S_2} u_1(P,Q) = u_1(P^*,Q^*) = v\,.$$

To proof equality (1) of equation (3.9) we need the Minimax Theorem 1 which we presented in Theorem 1.15. Then we have

$$\min_{\mathcal{Q}\subseteq\Delta S_2} \max_{\mathcal{P}\subseteq\Delta S_1} U_1(\mathcal{P},\mathcal{Q}) = \min_{\mathcal{Q}\subseteq\Delta S_2} \max_{\mathcal{P}\subseteq\Delta S_1} \min_{P\in\mathcal{P}} \min_{Q\in\mathcal{Q}} u_1(P,Q)$$
$$= \min_{\mathcal{Q}\subseteq\Delta S_2} \max_{P\in\Delta S_1} \min_{Q\in\mathcal{Q}} u_1(P,Q) \overset{\text{Thm. 1.15}}{=} \min_{\mathcal{Q}\subseteq\Delta S_2} \min_{Q\in\mathcal{Q}} \max_{P\in\Delta S_1} u_1(P,Q)$$
$$= \min_{Q\in\Delta S_2} \max_{P\in\Delta S_1} u_1(P,Q) = u_1(P^*,Q^*) = \max_{\mathcal{P}\subseteq\Delta S_1} \min_{\mathcal{Q}\subseteq\Delta S_2} U_1(\mathcal{P},\mathcal{Q})\,.$$

We now come to equality (2) of equation (3.10).

$$\max_{\mathcal{Q}\subseteq\Delta S_2} \min_{\mathcal{P}\subseteq\Delta S_1} U_2(\mathcal{P},\mathcal{Q}) = \max_{\mathcal{Q}\subseteq\Delta S_2} \min_{\mathcal{P}\subseteq\Delta S_1} \min_{Q\in\mathcal{Q}} \min_{P\in\mathcal{P}} u_2(P,Q)$$
$$= \max_{\mathcal{Q}\subseteq\Delta S_2} \min_{\mathcal{P}\subseteq\Delta S_1} \min_{P\in\mathcal{P}} \min_{Q\in\mathcal{Q}} u_2(P,Q) \overset{\text{Lemma 1.12}}{=} \max_{\mathcal{Q}\subseteq\Delta S_2} \min_{P\in\Delta S_1} \min_{Q\in\mathcal{Q}} u_2(P,Q)$$
$$= \max_{\mathcal{Q}\subseteq\Delta S_2} \min_{Q\in\mathcal{Q}} \min_{P\in\Delta S_1} u_2(P,Q) \overset{\text{Lemma 1.12}}{=} \max_{P\in\Delta S_1} \min_{Q\in\Delta S_2} u_2(P,Q)$$
$$= \max_{P\in\Delta S_1} \min_{Q\in\Delta S_2} -u_1(P,Q) = -u_1(P^*,Q^*) = v\,.$$

Lastly, we show equality (1) of equation (3.10).

$$\min_{\mathcal{P}\subseteq\Delta S_1} \max_{\mathcal{Q}\subseteq\Delta S_2} U_2(\mathcal{P},\mathcal{Q}) = \min_{\mathcal{P}\subseteq\Delta S_1} \max_{\mathcal{Q}\subseteq\Delta S_2} \min_{Q\in\mathcal{Q}} \min_{P\in\mathcal{P}} u_2(P,Q)$$
$$= \min_{\mathcal{P}\subseteq\Delta S_1} \max_{Q\in\Delta S_2} \min_{P\in\mathcal{P}} u_2(P,Q) \overset{\text{Thm. 1.15}}{=} \min_{\mathcal{P}\subseteq\Delta S_1} \min_{P\in\mathcal{P}} \max_{Q\in\Delta S_2} -u_1(P,Q)$$
$$= \min_{P\in\Delta S_1} \max_{Q\in\Delta S_2} -u_1(P,Q) = -u_1(P^*,Q^*) = \max_{\mathcal{Q}\subseteq\Delta S_2} \min_{\mathcal{P}\subseteq\Delta S_1} U_2(\mathcal{P},\mathcal{Q})\,.$$

□

Lemma 3.17. *Let $(\mathcal{P}^*,\mathcal{Q}^*)$ be an Ellsberg equilibrium of the two-person zero-sum game*

G. Then

$$U_1(\mathcal{P}^*, \mathcal{Q}^*) = u_1(P^*, Q^*) = v\,, \tag{3.11}$$

$$\textit{and}\;\; U_2(\mathcal{P}^*, \mathcal{Q}^*) = -u_1(P^*, Q^*) = -v\,. \tag{3.12}$$

Proof. We start by showing equation (3.11). Equation (3.12) follows analogously.

$$\begin{aligned}
& U_2(\mathcal{P}^*, \mathcal{Q}^*) \geq U_2(\mathcal{P}^*, \mathcal{Q}) \text{ for all } \mathcal{Q} \subseteq \Delta S_2 \\
\Rightarrow\; & U_2(\mathcal{P}^*, \mathcal{Q}^*) = \max_{\mathcal{Q} \subseteq \Delta S_2} U_2(\mathcal{P}^*, \mathcal{Q}) \\
\Rightarrow\; & \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_2(P, Q) = \max_{\mathcal{Q} \subseteq \Delta S_2} \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_2(P, Q) \\
\Rightarrow\; & \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_2(P, Q) = \max_{Q \in \Delta S_2} \min_{P \in \mathcal{P}^*} u_2(P, Q) \\
\Rightarrow\; & \max_{P \in \mathcal{P}^*} \max_{Q \in \mathcal{Q}^*} u_1(P, Q) = \min_{Q \in \Delta S_2} \max_{P \in \mathcal{P}^*} u_1(P, Q) \\
\Rightarrow\; & \max_{P \in \mathcal{P}^*} \max_{Q \in \mathcal{Q}^*} u_1(P, Q) \leq \min_{Q \in \Delta S_2} \max_{P \in \Delta S_1} u_1(P, Q)\,.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& U_1(\mathcal{P}^*, \mathcal{Q}^*) \geq U_1(\mathcal{P}, \mathcal{Q}^*) \text{ for all } \mathcal{P} \subseteq \Delta S_1 \\
\Rightarrow\; & U_1(\mathcal{P}^*, \mathcal{Q}^*) = \max_{\mathcal{P} \subseteq \Delta S_1} U_1(\mathcal{P}, \mathcal{Q}^*) \\
\Rightarrow\; & \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_1(P, Q) = \max_{\mathcal{P} \subseteq \Delta S_1} \min_{P \in \mathcal{P}} \min_{Q \in \mathcal{Q}^*} u_1(P, Q) \\
\Rightarrow\; & \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_1(P, Q) = \max_{P \in \Delta S_1} \min_{Q \in \mathcal{Q}^*} u_1(P, Q) \\
\Rightarrow\; & \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_1(P, Q) \geq \max_{P \in \Delta S_1} \min_{Q \in \Delta S_2} u_1(P, Q)\,.
\end{aligned}$$

From the above relations and Proposition 3.15 follows that

$$\max_{P \in \mathcal{P}^*} \max_{Q \in \mathcal{Q}^*} u_1(P, Q) \leq \min_{Q \in \Delta S_2} \max_{P \in \Delta S_1} u_1(P, Q) = \max_{P \in \Delta S_1} \min_{Q \in \Delta S_2} u_1(P, Q) \leq \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_1(P, Q)$$

and we finally have $U_1(\mathcal{P}^*, \mathcal{Q}^*) = u_1(P^*, Q^*) = v\,.$ □

From the two preceding lemmas we get the following minimax theorem for Ellsberg games. Since, as explained above, in general $U_2(\mathcal{P}, \mathcal{Q}) \neq -U_1(\mathcal{P}, \mathcal{Q})$, the equation is shown and stated for both players 1 and 2. In the classic mixed strategy case it suffices to state the minimax relation for one of the players.

Theorem 3.18 (Minimax Theorem 2)**.** *Let G be a two-person zero-sum game. Then for all $i \in \{1,2\}$ and every Ellsberg equilibrium $(\mathcal{P}^*, \mathcal{Q}^*)$ we have*

$$\max_{\mathcal{P}_i \subseteq \Delta S_i} \min_{\mathcal{P}_{-i} \subseteq \Delta S_{-i}} U_i(\mathcal{P}, \mathcal{Q}) = \min_{\mathcal{P}_{-i} \subseteq \Delta S_{-i}} \max_{\mathcal{P}_i \subseteq \Delta S_i} U_i(\mathcal{P}, \mathcal{Q}) = U_i(\mathcal{P}^*, \mathcal{Q}^*) \,.$$

3.6 Ellsberg Equilibria of Some Classic 2×2 Games

We have already seen the Ellsberg equilibria of some important classes of games: of symmetric and asymmetric coordination games, competitive games and zero-sum games. In an undergraduate game-theory course one analyzes usually three more types of games that became sort of classic, the Hawk and Dove game, the Prisoners' Dilemma and the Stag Hunt. We derive the Ellsberg equilibria of these games and find some special properties which we present in this section. These games have in common that the payoff functions only depend on the lower bound of the probability interval of the opponent. Thus they are linear, opposed to the piecewise linearity observed in the preceding examples. We then present a fourth game with linear payoff functions which is an example by Myerson (1997). We demonstrate that despite the well-behaved (i.e., linear) payoff functions, in some of these games proper Ellsberg equilibria arise.

3.6.1 Hawk and Dove

We start with the two-player game which is known as Hawk and Dove with the payoff matrix in Figure 3.18.

		Player 2 D	Player 2 H
Player 1	D	$3,3$	$1,4$
	H	$4,1$	$0,0$

Figure 3.18: Hawk and Dove.

The game has two pure Nash equilibria, (H,D) and (D,H) and a Nash equilibrium in mixed strategies, $(P^*,Q^*) = ((1/2,1/2),(1/2,1/2))$. We stay with the notation used in the preceding sections and denote by $[P_0,P_1]$ the set of probabilities with which player 1 plays D, whereas player 2 plays D with the set of probabilities $[Q_0,Q_1]$. To find the

Ellsberg equilibria, we first derive the payoff function $U_1(P,[Q_0,Q_1])$ of player 1.

$$\begin{aligned} U_1(P,[Q_0,Q_1]) &= \min_{Q_0 \le Q \le Q_1} 3(PQ + P(1-Q)) + 4(1-P)Q \\ &= \min_{Q_0 \le Q \le Q_1} Q(4-2P) + P \\ &= Q_0(4-2P) + P \text{ for all } P \in [P_0, P_1]\,. \end{aligned}$$

The payoff function is linear and depends only on the lower bound of $[Q_0, Q_1]$. The best response of player 1 is $P_0 = P_1 = 0$ when $Q_0 < 1/2$, and $P_0 = P_1 = 1$ when $Q_0 > 1/2$, these lead to the pure Nash equilibria (H, D) and (D, H). When player 2 chooses $Q_0 = 1/2$ the payoff function is constant at 2, thus the best response of player 1 is $[P_0, P_1] \subseteq [0, 1]$. This is an Ellsberg equilibrium, as long as $P_0 < 1/2 < P_1$. Furthermore, player 1 may choose $P_0 = P_1 = 1/2$ which results in the mixed strategy Nash equilibrium P^*. Since the game is symmetric, the analysis for player 2 yields the same results. Hence we get the following proposition.

Proposition 3.19. *The Ellsberg equilibria in the Hawk and Dove game are*

$$\begin{aligned} &([P_0,P_1]\,,1/2)\,,\ \textit{where } P_0 < 1/2 < P_1, \\ \textit{and}\quad &(1/2,[Q_0,Q_1])\,,\ \textit{where } Q_0 < 1/2 < Q_1. \end{aligned}$$

It is not surprising that in this completely symmetric game only Ellsberg equilibria exist which do not add interesting behavior to the analysis. When a player creates ambiguity about his Nash equilibrium strategy, the optimal response of his opponent does not change compared to the usual analysis.

3.6.2 Prisoners' Dilemma

Let us now look at the Prisoners' Dilemma with the payoff matrix in Figure 3.19.

		Player 2	
		C	D
Player 1	C	3, 3	0, 4
	D	4, 0	1, 1

Figure 3.19: Prisoners' Dilemma.

This game has a unique Nash equilibrium in pure strategies (D, D). To find the Ellsberg equilibrium, we first derive the payoff function $U_1(P, [Q_0, Q_1])$ of player 1.

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q \in [Q_0, Q_1]} 3PQ + 4(1-P)Q + (1-Q)(1-P) \\ &= \min_{Q \in [Q_0, Q_1]} 3Q - P + 1 \\ &= 3Q_0 - P + 1 \text{ for all } P \in [P_0, P_1]\,. \end{aligned}$$

The payoff function is again linear and depends only on the lower bound of $[Q_0, Q_1]$. The best response of player 1 is always $P_0 = P_1 = 0$, no matter where Q_0 lies in the interval $[0, 1]$. The analysis for player 2 yields the same result, since the game is symmetric. This results in the Nash equilibrium in pure strategies (D, D).

Proposition 3.20. *The Ellsberg equilibrium of the Prisoners' Dilemma is* (D, D).

Because the only Nash equilibrium of the game is in strictly dominant strategies, the possibility of using ambiguity is not seized. However, when we add a mediator to the Prisoners' Dilemma, this mediator causes the prisoners to cooperate. We analyze the three-player mediated Prisoners' Dilemma with non-Nash outcome in Section 5.1.

3.6.3 Stag Hunt

Next we calculate the Ellsberg equilibria of the Stag Hunt game. We choose the following payoff matrix.

		Player 2	
		S	H
Player 1	S	$2, 2$	$0, 1$
	H	$1, 0$	$1, 1$

Figure 3.20: Stag Hunt.

The game has two pure Nash equilibria (S, S) and (H, H), and one Nash equilibrium in mixed strategies $(P^*, Q^*) = ((1/2, 1/2), (1/2, 1/2))$. We calculate the minimal payoff functions of the two players.

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \le Q \le Q_1} 2PQ + (1-P)Q + (1-P)(1-Q) \\ &= \min_{Q_0 \le Q \le Q_1} 2PQ - P + 1 \\ &= 2PQ_0 - P + 1 \text{ for all } P \in [0, 1]\,, \end{aligned}$$

$$\text{and} \quad U_2([P_0, P_1], Q) = 2P_0Q - P + 1 \text{ for all } Q \in [0,1] .$$

The payoff function of player 1 is constant at 1 for $Q_0 = 1/2$, and the payoff function of player 2 likewise for $P_0 = 1/2$. They are increasing for $Q_0 \geq 1/2$ ($P_0 \geq 1/2$) and decreasing for $Q_0 \leq 1/2$ ($P_0 \leq 1/2$). Thus, we can immediately see that ambiguity is only used when $P_0 = Q_0 = 1/2$, and then the best responses are the whole interval $[0,1]$ for both players. Therefore, we have the following proposition.

Proposition 3.21. *The Ellsberg equilibria of the Stag Hunt game are* (S,S)*,* (H,H) *and the proper Ellsberg equilibria*

$$([1/2, P_1], [1/2, Q_1]), \; \textit{where } 1/2 \leq P_1 \leq 1 \textit{ and } 1/2 \leq Q_1 \leq 1 .$$

This proposition shows that in the Stag Hunt game the payoff dominant equilibrium (S,S) is preferred over the risk dominant equilibrium (H,H), when ambiguity as a strategy is allowed.

3.6.4 Example by Myerson

The last example of this section is taken from Myerson (1997). He chose this example to show the power and limits in the interpretation of Nash equilibrium. We present it here, because it has linear payoff functions also for ambiguity-averse players, and nevertheless proper Ellsberg equilibria. The game has the payoff matrix depicted in Figure 3.21. It has the unique Nash equilibrium in mixed strategies $(P^*, Q^*) = ((3/4, 1/4), (1/2, 1/2))$ with expected payoff $(0, 0)$, but it can be argued that it would be more reasonable for players to coordinate on (U, L) and thus *guarantee* an equilibrium payoff of 0. Of course, this is not a Nash equilibrium since player 1 would want to deviate.

		Player 2	
		L	R
Player 1	U	$0,0$	$0,-1$
	D	$1,0$	$-1,3$

Figure 3.21: Myerson's game.

We calculate the payoff functions of the two players.

$$\begin{aligned} U_1(P,[Q_0,Q_1]) &= \min_{Q_0 \le Q \le Q_1} (1-P)Q - (1-P)(1-Q) \\ &= \min_{Q_0 \le Q \le Q_1} Q(2-2P) + P - 1 \\ &= \begin{cases} Q_0(2-2P)+P-1, & \text{if } P<1\,, \\ 0, & \text{if } P=1\,. \end{cases} \end{aligned} \tag{3.13}$$

Note that for P equal to 1, the utility function of player 1 is 0, and for $P=0$ it is equal to $2Q_0-1$. In Figure 3.22 one can easily derive the best responses of player 1 which are listed below.

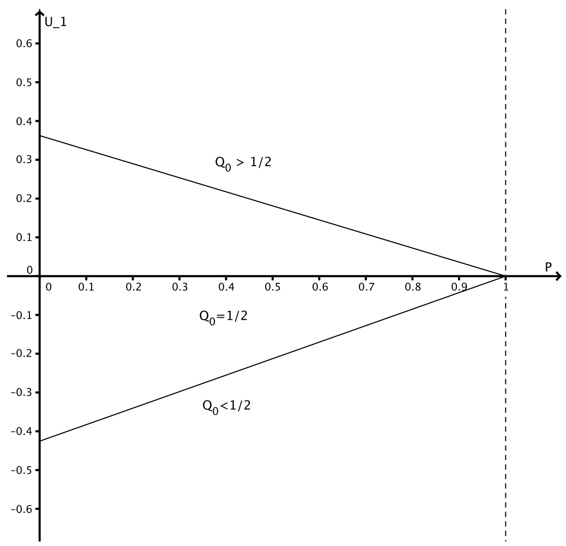

Figure 3.22: Payoff function (3.13) of player 1 in Myerson's game.

$$\begin{aligned} Q_0 > 1/2: &\qquad B_1([Q_0,Q_1]) = \{[P_0,P_1] | P_0 = P_1 = 0\} \\ Q_0 = 1/2: &\qquad B_1([Q_0,Q_1]) = \{[P_0,P_1] \subseteq [0,1]\} \\ Q_0 < 1/2: &\qquad B_1([Q_0,Q_1]) = \{[P_0,P_1] | P_0 = P_1 = 1\} \end{aligned}$$

We do the same analysis for player 2.

$$\begin{aligned}
U_2([P_0, P_1], Q) &= \min_{P_0 \le P \le P_1} -P(1-Q) + 3(1-P)(1-Q) \\
&= \min_{P_0 \le P \le P_1} P(4Q-4) - 3Q + 3 \\
&= \begin{cases} 0, & \text{if } Q = 1\,, \\ P_1(4Q-4) - 3Q + 3, & \text{if } Q < 1\,. \end{cases}
\end{aligned}$$

Similarly to case U_1, utility function of player 2 is 0 when Q is equal to 1, and for $Q = 0$ it is $3 - 4P_1$. The plot of U_2 looks very much like the payoff function for player 1. The best responses of player 2 are

$$\begin{aligned}
P_1 < 3/4: &\qquad B_2([P_0, P_1]) = \{[Q_0, Q_1] | Q_0 = Q_1 = 0\} \\
P_1 = 3/4: &\qquad B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1]\} \\
P_1 > 3/4: &\qquad B_2([P_0, P_1]) = \{[Q_0, Q_1] | Q_0 = Q_1 = 1\}\,.
\end{aligned}$$

Hence, we obtain

Proposition 3.22. *The Ellsberg equilibria in Myerson's game are*

$$([P_0, 3/4], [1/2, Q_1])\,, \quad \textit{where } 0 \le P_0 \le 3/4 \ \textit{ and } \ 1/2 \le Q_1 \le 1\,.$$

Thus, both players may use ambiguity in equilibrium, although the payoff functions of both players are linear. In a sense, one of the Ellsberg equilibria, that is $(3/4, [1/2, 1])$, makes coordination on the outcome (U, L) more likely.

We make an interesting observation studying these examples: even when both payoff functions are linear (and not, as often in Ellsberg games, piecewise linear) there exist Ellsberg equilibria in which ambiguity is used. But the existence of a unique strict Nash equilibrium in pure strategies as in the Prisoners' Dilemma causes, of course, that no ambiguity is used even when we allow for it. We proved this in Section 1.3.6.

3.7 Ellsberg Equilibria of 3×3 Games and Their Geometry

We can apply the insights on the Ellsberg equilibria of 2×2 games in the preceding sections to investigate the Ellsberg equilibria of larger games. We discuss the Ellsberg equilibria of a special class of modified 3×3 zero-sum games, that is, modified *circulant* games. Circulant games have square payoff matrices in which the payoffs are circularly permuted in every row (column). A general circulant 3×3 zero-sum game has the following payoff matrix for some $a, b, c \in \mathbb{R}$.

Player 2

Player 1		L	M	R
	T	$a, -a$	$b, -b$	$c, -c$
	C	$c, -c$	$a, -a$	$b, -b$
	B	$b, -b$	$c, -c$	$a, -a$

Figure 3.23: General circulant game.

For example, the classic game Rock Scissors Paper (RSP) is a circulant game, this is the game we analyze in this section. Our RSP has the payoff matrix given in Figure 3.24, the strategies are denoted R for Rock, S for Scissors, P for Paper.

Player 2

Player 1		R	S	P
	R	$0, 0$	$1, -1$	$-1, 1$
	S	$-1, 1$	$0, 0$	$1, -1$
	P	$1, -1$	$-1, 1$	$0, 0$

Figure 3.24: Rock Scissors Paper.

We fix some new notation for Ellsberg strategies when we deal with games with more than two pure strategies. An Ellsberg strategy for player 1 is, as before, denoted by $\mathcal{P}$, for player 2 by $\mathcal{Q}$. They are convex sets of probability distributions (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) in ΔS_1 and ΔS_2, respectively, that is $\mathcal{P} \subseteq \Delta S_1$, $\mathcal{Q} \subseteq \Delta S_2$. An Ellsberg strategy $\mathcal{P}$ is described as follows. Fix two vectors $x, y \in [0,1]^3$ in the unit cube with components (x_1, x_2, x_3) and (y_1, y_2, y_3). Then

$$\mathcal{P} = \left\{ (P_1, P_2, P_3) \in \mathbb{R}^3 \;\middle|\; \sum_{i=1}^{3} P_i = 1, P_i \geq 0, x_i \leq P_i \leq y_i, i = 1, 2, 3 \right\}. \tag{3.14}$$

Note that $\mathcal{P}$ is completely described by giving the possible range of probabilities P_1 and P_2. The range of P_3 (given by x_3 and y_3) then follows directly from the assumption that every triple $(P_1, P_2, P_3) \in \mathcal{P}$ is a probability distribution and some careful calculations.

We use the following abbreviation for (3.14):

$$\mathcal{P} = (\{x_i \le P_i \le y_i\})_{i=1,2,3} = (\{x_1 \le P_1 \le y_1\}, \{x_2 \le P_2 \le y_2\}, \{x_3 \le P_3 \le y_3\}) .$$

We write $\{x_i \le P_i\}$ when $y_i = 1$, and $\{P_i \le y_i\}$ when $x_i = 0$. We drop the subscript $i = 1, 2, 3$ when no confusion can arise. To describe an Ellsberg strategy $\mathcal{Q}$ for player 2, we use the vectors $w, z \in [0, 1]^3$. An Ellsberg strategy for player 1 is then for example

$$(\{1/3 \le P_1 \le 1/2\}, \{P_2 \le 2/3\}, \{P_3 \le 2/3\}) .$$

When we calculate Ellsberg equilibria, we typically find a large number of equilibria which are very similar but differ in that one or more boundaries of the set of probability distributions are variable. Then we use the following notation. Fix $\hat{x}, \hat{y} \in [0, 1]^3$, then

$$\begin{aligned} &(\{\hat{x}_i \le x_i \le P_i \le y_i \le \hat{y}_i\}) \\ &:= \{(\{x_i \le P_i \le y_i\}) \mid \text{for all } x, y \in [0, 1]^3 \text{ such that } \hat{x}_i \le x_i \le y_i \le \hat{y}_i\} . \end{aligned}$$

As before, we suppress $\hat{x}_i$ and $\hat{y}_i$ when they are 0 or 1, respectively. Such a set of Ellsberg strategies is then for example

$$(\{1/3 \le P_1 \le y_1 \le 1/2\}, \{x_2 \le P_2 \le 2/3\}, \{P_3 \le y_3 \le 2/3\}) .$$

Note that this describes the set of Ellsberg strategies which necessarily include the probability distribution $(1/3, 2/3, 0)$ at the boundary of *each* Ellsberg strategy contained in the set. This is the type of set of Ellsberg strategies which we encounter frequently in Ellsberg equilibrium analysis.

We use the following notation. We denote by $P_1(Q_1)$ the probability with which player 1(2) plays R, $P_2(Q_2)$ the probability with which player 1(2) plays S, $P_3(Q_3)$ the probability with which player 1(2) plays P. The only Nash equilibrium of the game is $(P^*, Q^*) = ((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$. As RSP's equilibrium strategies are Nash, maximin, and immunization strategies according to Theorem 4.2, RSP has Ellsberg equilibria with unilateral ambiguity, and, according to Theorem 3.18, the value of all Ellsberg equilibria is equal to the minimax value.

Proposition 3.23. *All Ellsberg equilibria of RSP have the (minimal) expected payoff zero.*

Ellsberg equilibria with unilateral ambiguity of RSP are, e.g.,

$$((\{P_i = 1/3\}), (\{w_i \leq Q_i \leq z_i\}))\,,$$

$$\textit{where } 0 \leq w_i \leq 1/3 \leq z_i \leq 1 \textit{ for all } i = 1, 2, 3\,,$$

$$\textit{and} \quad ((\{x_i \leq P_i \leq y_i\}), (\{Q_i = 1/3\}))\,,$$

$$\textit{where } 0 \leq x_i \leq 1/3 \leq y_i \leq 1 \textit{ for all } i = 1, 2, 3\,.$$

As in the Matching Pennies game, the situation changes when we slightly modify the original zero-sum game. We consider the following modification (Figure 3.25), where player 1 gets a payoff of 2, instead of 1, when (R, S) is played.

		Player 2		
		R	S	P
	R	$0, 0$	$2, -1$	$-1, 1$
Player 1	S	$-1, 1$	$0, 0$	$1, -1$
	P	$1, -1$	$-1, 1$	$0, 0$

Figure 3.25: Modified Rock Scissors Paper.

Now the Nash equilibrium is

$$(P^*, Q^*) = ((1/3, 1/3, 1/3), (1/3, 1/4, 5/12))\,.$$

As in 2×2 games, the immunization strategies play an important role in the Ellsberg equilibria of the modified RSP. In our modified game, they are

$$(M_1, M_2) = ((1/4, 1/3, 5/12), (1/3, 1/3, 1/3))\,.$$

We have the following proposition.

Proposition 3.24. *The profiles*

$$((\{1/3 \leq P_1 \leq y_1 \leq 2/3\}, \{P_2 = 1/3\}, \{x_3 \leq P_3 \leq 1/3\}),$$
$$(\{w_1 \leq Q_1 \leq 1/3\}, \{1/4 \leq Q_2 \leq z_2 \leq 1/3\}, \{5/12 \leq Q_3 \leq z_3 \leq 3/4\}))$$

are nontrivial Ellsberg equilibria of the modified RSP.

This Ellsberg equilibrium can be read as player 1 using the strategy: "I will play Rock at least with probability 1/3 but not with probability higher than 2/3, I will play Paper with probability less than 1/3 and Scissors with exactly probability 1/3."

Before we prove this, we discuss the intuition of the result. We know from the analysis of 2×2 games that ambiguity can only be a best response if the opponent uses his Nash equilibrium probability at the boundary of his Ellsberg strategy, and additionally this is the worst case measure which the player uses in his utility evaluation. In this case, the utility function has flat parts and the player best responds with ambiguity. When the opponent plays exactly his Nash equilibrium distribution (without using ambiguity himself), the whole utility function of the player is flat and he can play any Ellsberg strategy he likes, it will always be a best response. However, recall that this can only be an Ellsberg equilibrium if the opponent's Nash equilibrium strategy is exactly his immunization strategy.

Consider the Ellsberg equilibrium in Proposition 3.24. We can draw the Ellsberg strategies as a projection into the 2-simplex (see, e.g., Ritzberger (2002) p. 36 for an introduction) to understand how the characteristics we observe in 2×2 games are apparent in the modified RSP.

To this end we use the 'largest' Ellsberg equilibrium, that is

$$((\{1/3 \leq P_1 \leq 2/3\}, \{P_2 = 1/3\}, \{P_3 \leq 1/3\}),$$
$$(\{Q_1 \leq 1/3\}, \{1/4 \leq Q_2 \leq 1/3\}, \{5/12 \leq Q_3 \leq 3/4\}))\,.$$

In the equilateral triangle in Figures 3.26 and 3.27, for every point on the edge opposite the vertex R, the probability that R is played is zero. On the other hand, at the vertex R, R is played with probability one and S and P with probability zero. Sets of probability distributions, as we encounter in Ellsberg equilibria, are drawn as gray areas. The gray areas are the possible probabilities for each component of $(P_1, P_2, P_3) \in \mathcal{P}$ (in Figure 3.26) and $(Q_1, Q_2, Q_3) \in \mathcal{Q}$ (in Figure 3.27), the intersection is then the set of probability distributions which satisfy all three conditions of the Ellsberg strategy. The intersection is framed by a thick black line.

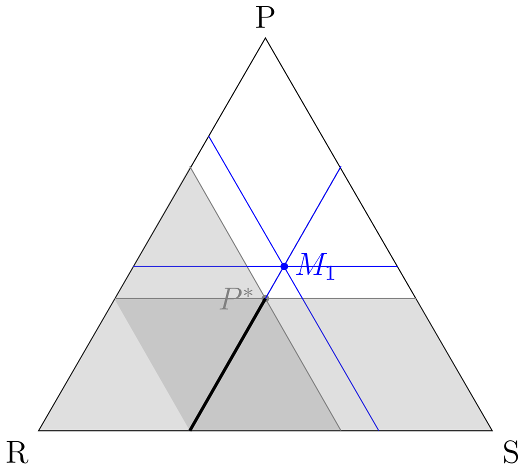

Figure 3.26: Ellsberg equilibrium strategy of player 1 in the modified RSP.

As one can see in Figure 3.26, the Ellsberg equilibrium strategy of player 1 is a line, because $\{P_2 = 1/3\}$ contains only a single element. His Nash equilibrium strategy $P^* = (1/3, 1/3, 1/3)$ lies at the boundary of the Ellsberg equilibrium strategy. Player 2 plays a set of probability distributions depicted in Figure 3.27. Again, $Q^* = (1/3, 1/4, 5/12)$ lies at the boundary of the Ellsberg equilibrium strategy. As we have seen in 2×2 games, the 'largest' Ellsberg equilibrium is bounded by the immunization strategy M_2.

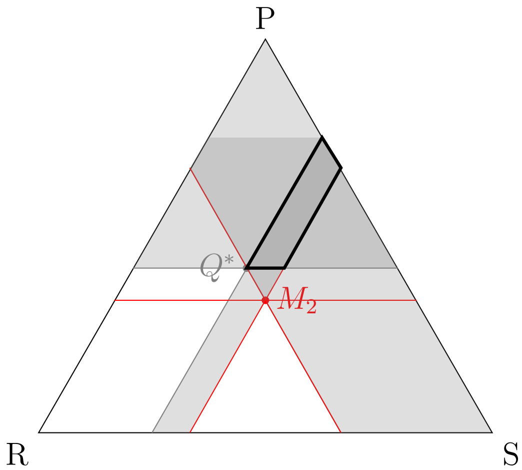

Figure 3.27: Ellsberg equilibrium strategy of player 2 in the modified RSP.

Proof of Proposition 3.24. We start by calculating the minimal expected utility functions of player 1 and 2. Player 1 minimizes over Q_1 and Q_2. Q_3 is expressed as $1 - Q_1 - Q_2$. The lower and upper bounds of Q_1 and Q_2 are w_1, z_1 and w_2, z_2, respectively.

$$
\begin{aligned}
&U_1(P_1, P_2, Q_1, Q_2) \\
&= \min_{Q_1,Q_2} 2P_1Q_2 - P_1(1 - Q_1 - Q_2) - P_2Q_1 \\
&\qquad + P_2(1 - Q_1 - Q_2) + (1 - P_1 - P_2)Q_1 - (1 - P_1 - P_2)Q_2 \\
&= \min_{Q_2} \begin{cases} w_1(1 - 3P_2) + 4P_1Q_2 - P_1 + P_2 - Q_2 & \text{if } P_2 < 1/3\,, \\ 4P_1Q_2 - P_1 - Q_2 + 1/3 & \text{if } P_2 = 1/3\,, \\ z_1(1 - 3P_2) + 4P_1Q_2 - P_1 + P_2 - Q_2 & \text{if } P_2 > 1/3 \end{cases} \\
&= \begin{cases} \left.\begin{matrix} w_2(4P_1 - 1) - 3P_2w_1 + w_1 - P_1 + P_2 & \text{if } P_1 > 1/4\,, \\ w_1 - 3P_2w_1 - 1/4 + P_2 & \text{if } P_1 = 1/4\,, \\ z_2(4P_1 - 1) - 3P_2w_1 + w_1 - P_1 + P_2 & \text{if } P_1 < 1/4\,, \end{matrix}\right\} P_2 < 1/3\,, \\ \\ \left.\begin{matrix} w_2(4P_1 - 1) + 1/3 - P_1 & \text{if } P_1 > 1/4\,, \\ 1/12 & \text{if } P_1 = 1/4\,, \\ z_2(4P_1 - 1) + 1/3 - P_1 & \text{if } P_1 < 1/4\,, \end{matrix}\right\} P_2 = 1/3\,, \\ \\ \left.\begin{matrix} w_2(4P_1 - 1) - 3P_2z_1 + z_1 - P_1 + P_2 & \text{if } P_1 > 1/4\,, \\ z_1 - 3P_2z_1 - 1/4 + P_2 & \text{if } P_1 = 1/4\,, \\ z_2(4P_1 - 1) - 3P_2z_1 + z_1 - P_1 + P_2 & \text{if } P_1 < 1/4\,, \end{matrix}\right\} P_2 > 1/3\,. \end{cases}
\end{aligned}
$$

Player 2 minimizes over P_1 and P_2, P_3 is expressed as $1 - P_1 - P_2$. The lower and upper bounds of P_1 and P_2 are x_1, y_1 and x_2, y_2, respectively.

$$
\begin{aligned}
&U_2(P_1, P_2, Q_1, Q_2) \\
&= \min_{P_1,P_2} -P_1Q_2 + P_1(1 - Q_1 - Q_2) + P_2Q_1 \\
&\qquad - P_2(1 - Q_1 - Q_2) - (1 - P_1 - P_2)Q_1 + (1 - P_1 - P_2)Q_2 \\
&= \min_{P_2} \begin{cases} x_1(1 - 3Q_2) + 3P_2Q_1 - P_2 - Q_1 + Q_2 & \text{if } Q_2 < 1/3\,, \\ 3P_2Q_1 - P_2 - Q_1 + 1/3 & \text{if } Q_2 = 1/3\,, \\ y_1(1 - 3Q_2) + 3P_2Q_1 - P_2 - Q_1 + Q_2 & \text{if } Q_2 > 1/3 \end{cases}
\end{aligned}
$$

$$= \begin{cases} \left.\begin{array}{ll} x_2(3Q_1-1) - 3Q_2x_1 + x_1 - Q_1 + Q_2 & \text{if } Q_1 > 1/3\,, \\ x_1 - 3Q_2x_1 - 1/3 + Q_2 & \text{if } Q_1 = 1/3\,, \\ y_2(3Q_1-1) - 3Q_2x_1 + x_1 - Q_1 + Q_2 & \text{if } Q_1 < 1/3\,, \end{array}\right\} Q_2 < 1/3\,, \\ \\ \left.\begin{array}{ll} x_2(3Q_1-1) + 1/3 - Q_1 & \text{if } Q_1 > 1/3\,, \\ 0 & \text{if } Q_1 = 1/3\,, \\ y_2(3Q_1-1) + 1/3 - Q_1 & \text{if } Q_1 < 1/3\,, \end{array}\right\} Q_2 = 1/3\,, \\ \\ \left.\begin{array}{ll} x_2(3Q_1-1) - 3Q_2y_1 + y_1 - Q_1 + Q_2 & \text{if } Q_1 > 1/3\,, \\ y_1 - 3Q_2y_1 - 1/3 + Q_2 & \text{if } Q_1 = 1/3\,, \\ y_2(3Q_1-1) - 3Q_2y_1 + y_1 - Q_1 + Q_2 & \text{if } Q_1 < 1/3\,, \end{array}\right\} Q_2 > 1/3\,. \end{cases}$$

Now we proceed as follows to derive the Ellsberg equilibria. Recall the Nash equilibria and immunization strategies of the game:

$$(P^*, Q^*) = ((1/3, 1/3, 1/3), (1/3, 1/4, 5/12))\,,$$
$$\text{and} \quad (M_1, M_2) = ((1/4, 1/3, 5/12), (1/3, 1/3, 1/3))\,.$$

Player 1 has to use his Nash equilibrium strategies at the boundary of his Ellsberg equilibrium strategy, except for the component P_2, where the Nash equilibrium probability is the same as the immunization strategy. Furthermore, the set of probability distributions in the Ellsberg equilibrium may not extend across the immunization strategy.

Hence, for the first component for player 1, we consider either $\{1/4 \leq x_1 \leq P_1 \leq 1/3\}$, or $\{1/3 \leq P_1\}$. In Ellsberg equilibrium, player 2 uses 1/3 as his worst case measure, thus in the first case we only consider those elements of U_2 which use y_1 as worst case measure, in the second case only those which use x_1. y_1 is only used when $Q_2 \geq 1/3$. Since the range of Q_2 must contain 1/4 at the boundary, which is impossible when $Q_2 \geq 1/3$, $\{1/4 \leq x_1 \leq P_1 \leq 1/3\}$ cannot be an Ellsberg equilibrium strategy. We proceed with the second case: x_1 is only used for $Q_2 \leq 1/3$, this is compatible with the Nash equilibrium strategy and yields $\{1/4 \leq Q_2 \leq z_2 \leq 1/3\}$. Finally we check if $\{1/3 \leq P_1\}$ is a best response to $\{1/4 \leq Q_2 \leq z_2 \leq 1/3\}$. Player 1 must use w_2 in his utility evaluation U_1, this is true whenever $P_1 \geq 1/4$ which is compatible with the strategy $\{1/3 \leq P_1\}$. Thence we have found a best response pair.

In the second component, player 1 can play $\{P_2 = 1/3\}$, which is his Nash equilibrium probability and therefore makes player 2 indifferent between all $\{w_1 \leq Q_1 \leq z_1\}$, and at the same time his best response to player 2 playing any probability $\{w_1 \leq Q_1 \leq z_1\}$ with $0 \leq w_1 \leq 1/3 \leq z_1 \leq 1$. Or, conversely, player 2 can play $\{Q_1 = 1/3\}$ and player 1 $\{x_2 \leq P_2 \leq y_2\}$ with $0 \leq x_2 \leq 1/3 \leq y_2 \leq 1$. The latter case collapses to Nash equilibrium. The restrictions on w_1, z_1, which are $0 \leq w_1 \leq 1/3 = z_1$, follow from the third component and the fact that each element of the Ellsberg equilibrium strategy must be a probability distribution.

We cannot determine the range for P_3 and Q_3 from the two utility functions, since P_3 and Q_3 are only implicitly given. Thus, we derive the utility functions again, now using $(P_1, 1 - P_1 - P_3, P_3)$ and $(1 - Q_2 - Q_3, Q_2, Q_3)$ as probabilities. Since we already know that only the boundaries x_1 and w_2 are relevant for the equilibrium, we get

$$\begin{aligned}
&U_1(P_1, P_3, Q_2, Q_3)\\
&= \min_{Q_2, Q_3} 2P_1Q_2 - P_1Q_3 - (1 - P_1 - P_3)(1 - Q_2 - Q_3)\\
&\qquad + (1 - P_1 - P_3)Q_3 + P_3(1 - Q_2 - Q_3) - P_3Q_2\\
&= \begin{cases} w_3(2 - 3P_1 - 3P_3) + w_2(P_1 - 3P_3 + 1) + 2P_3 + P_1 - 1 & \text{if } P_3 < 2/3 - P_1\,, \\ w_2(5/3 - 4P_3) + P_3 - 1/3 & \text{if } P_3 = 2/3 - P_1\,, \\ z_3(2 - 3P_1 - 3P_3) + w_2(P_1 - 3P_3 + 1) + 2P_3 + P_1 - 1 & \text{if } P_3 > 2/3 - P_1\,, \end{cases}
\end{aligned}$$

when $P_1 > 3P_3 - 1$.
On the other hand,

$$\begin{aligned}
&U_2(P_1, P_3, Q_2, Q_3)\\
&= \min_{P_1, P_3} -P_1Q_2 + P_1Q_3 + (1 - P_1 - P_3)(1 - Q_2 - Q_3)\\
&\qquad - (1 - P_1 - P_3)Q_3 - P_3(1 - Q_2 - Q_3) + P_3Q_2\\
&= \begin{cases} x_3(3Q_2 + 3Q_3 - 2) + x_1(3Q_3 - 1) - Q_2 - 2Q_3 + 1 & \text{if } Q_2 > 2/3 - Q_3\,, \\ x_1(3Q_3 - 1) - Q_3 + 1/3 & \text{if } Q_2 = 2/3 - Q_3\,, \\ y_3(3Q_2 + 3Q_3 - 2) + x_1(3Q_3 - 1) - Q_2 - 2Q_3 + 1 & \text{if } Q_2 < 2/3 - Q_3\,, \end{cases}
\end{aligned}$$

when $Q_3 > 1/3$.

Now we use the following reasoning: we see from U_2 that Q_3 must be greater than (or in fact equal to, this is suppressed in the shortened statement of U_2) 1/3. Since 5/12 is the Nash equilibrium probability of Q_3, we get the two following possible probability sets for Q_3:

$$\text{either } \{1/3 \leq w_3 \leq Q_3 \leq 5/12\}, \quad \text{or } \{5/12 \leq Q_3 \leq z_3\}.$$

In the first case, player 1 must use z_3 in his utility evaluation, according to U_1 this is the case only when $P_3 \geq 2/3 - P_1$. We know that $P_1 \geq 1/3$, therefore $P_3 \geq 1/3$. This leads to the set of probabilities $\{1/3 \leq P_3 \leq y_3 \leq 5/12\}$ for P_3, since 1/3 as the Nash equilibrium probability must be part of the set. Here x_3 is used by player 2, hence from U_2 we see this is only the case if $Q_2 \geq 2/3 - Q_3$. $Q_3 \geq 1/3$ and therefore Q_2 must be greater than 1/3. This is not possible, since we have already found Q_2 to be played with the set of probabilities $\{1/4 \leq Q_2 \leq z_2 \leq 1/3\}$.

Consider the second possible set for Q_2. Here w_3 is used by player 1, and hence $P_2 \leq 1/3$. This leads to $\{x_3 \leq P_3 \leq 1/3\}$ for P_3. Player 2 uses y_3 and we conclude that Q_2 must be less than or equal to 1/3. This is exactly what we found to be true for Q_2 before. Therefore we have found that the profile

$$\begin{aligned}&((\{1/3 \leq P_1 \leq y_1 \leq 2/3\}, \{P_2 = 1/3\}, \{x_3 \leq P_3 \leq 1/3\}),\\ &\qquad (\{w_1 \leq Q_1 \leq 1/3\}, \{1/4 \leq Q_2 \leq z_2 \leq 1/3\}, \{5/12 \leq Q_3 \leq z_3 \leq 3/4\}))\end{aligned}$$

is an Ellsberg equilibrium of the modified RSP game. □

4 Ellsberg Games, Human Behavior and Observational Implications

In this chapter we present some thoughts on a general interpretation of Ellsberg equilibria and a classification of when ambiguity is an option in two-player Ellsberg equilibria.[1]

We are on the one hand interested in human behavior in two-person games and its connection to Ellsberg equilibria. In an experiment with a modified version of Matching Pennies, Goeree and Holt (2001) observed that Nash equilibrium prediction was frequently violated; interestingly, our Ellsberg equilibria can explain this behavior. In Section 4.2 on the other hand, we classify Ellsberg equilibria that are supported by strategies which are not in the support of any Nash equilibrium of the game. These Ellsberg equilibria arise frequently in games with weakly dominated strategies and have an interesting behavioral interpretation. Finally in Section 4.3, we characterize in which circumstances and how ambiguity is used in equilibrium in two-person games. This analysis provides an answer to the question in which two-person games proper Ellsberg equilibria exist.

4.1 Human Behavior in Matching Pennies Games and Ellsberg Equilibria

We have now seen a number of different two-player games and have calculated their Ellsberg equilibria. Recall the modified Matching Pennies example in Section 3.3.1. Whereas the support of the Ellsberg and Nash equilibria of the game is obviously the same, we think that the Ellsberg equilibria reveal a new class of behavior not encountered in game theory before. It might be very difficult for humans to play exactly a randomizing strategy with equal probabilities. Indeed, the ability to do so has been a source of debate since the early days of game theory, and some claim that humans cannot randomize, see Dang (2009) for a recent account and references therein. Our result shows that it is not necessary to randomize exactly to support a similar equilibrium outcome (with the same expected payoff). It is just enough that your opponent knows that you are randomizing with some

[1] Some parts of this chapter, namely Section 4.1 and the first part of Section 4.2, were published in the IMW working paper Riedel and Sass (2011).

probability, and that it could be that this probability is one half, but not less. It is thus sufficient that the player is able to control the lower bound of his device. This might be easier to implement than the perfectly random behavior required in classical game theory.

In fact, there are experimental findings which suggest that the Ellsberg equilibrium strategy in the modified Matching Pennies game is closer to real behavior than the Nash equilibrium prediction. To illustrate this, let us consider the interesting results by Goeree and Holt (2001) who ran experiments on three different versions of Matching Pennies; the three payoff matrices can be seen in Table 4.1.

In the first game, we have a typical symmetric conflict game with a unique mixed Nash equilibrium in which both players randomize uniformly over both pure strategies. The aggregate play of humans in the experiment is closely consistent with the Nash equilibrium prediction, 48% of players choosing “Top” or “Left”, respectively. Remember that the probabilities in a mixed strategy equilibrium are chosen in such a way as to render the opponent indifferent between his two pure strategies. As a consequence, if we change the payoffs of player 1 only (while keeping the ordering of payoffs), his Nash equilibrium strategy does not change because he has to make player 2 indifferent between his two pure actions, and player 2’s payoffs have not been modified.

In the second game, called the asymmetric Matching Pennies game, player 1 gets 320 instead of 80 in the upper left outcome. All other payoffs remain the same. Many humans now deviate from Nash, as is reported in brackets, 96% of the players taking the action “Top”. Interestingly, also the humans playing the role of player 2 change their behavior, and most of them play “Right”, the best reply to “Top”.

In the third case, player 1’s payoff in the upper left outcome is decreased to a lowly 44. Then only 8% of players choose “Top”; 80% of humans in the role of player 2 choose “Left”.

While aggregate behavior by humans is certainly inconsistent with the predictions of Nash equilibrium, it is consistent with Ellsberg equilibria. We summarize the results in Table 4.2.

In the symmetric game, our Proposition 3.7 essentially predicts only Nash equilibrium behavior, and this is what we observe in the experiment as well. In the asymmetric Matching Pennies game, the Nash equilibrium strategies are $P^* = 1/2$ for player 1 and $Q^* = 1/8$ for player 2. According to our proposition, the Ellsberg equilibria allow for probabilities in the interval $[1/2, 1]$ for player 1 choosing “Top”, and for the interval $[1/8, 1/2]$ for player 2 choosing “Left”. The observed percentages of 96% and 16% do lie in these intervals. And in the “reversed” version of the game, the Nash equilibrium strategies are $P^* = 1/2$

Table 4.1: The Goeree-Holt results on three different versions of Matching Pennies.

		Left (48)	Right (52)
Symmetric	Top (48)	80,40	40,80
	Bottom (52)	40,80	80,40
		Left (16)	Right(84)
Asymmetric	Top (96)	**320**,40	40,80
	Bottom (4)	40,80	80,40
		Left (80)	Right (20)
Reversed	Top (8)	**44**,40	40,80
	Bottom (92)	40,80	80,40

Table 4.2: Comparison of Nash and Ellsberg predictions with the experimental observations. We record the probabilities (or intervals of probabilities) for each player to play the first pure strategy ("Top", respectively "Left") and the observed aggregate frequency of these actions in the Goeree-Holt experiments.

Game	Nash Equilibrium		Ellsberg Equilibrium		Observations	
	Player 1	Player 2	Player 1	Player 2	Player 1	Player 2
symmetric	0.5	0.5	0.5	0.5	0.48	0.48
asymmetric	0.5	0.125	[0.5,1]	[0.125,0.5]	0.96	0.16
reversed	0.5	$0.\overline{90}$	[0,0.5]	[0.5,$0.\overline{90}$]	0.08	0.8

and $Q^* = 10/11$. So we have the reversed relation $Q^* > P^*$. The Ellsberg equilibria allow for probabilities for "Top" in the interval $[0, 1/2]$ for player 1, and for probabilities in $[1/2, 10/11]$ for player 2. The aggregate observed quantities of 8% and 80% do lie in these intervals.

4.2 Observational Implications of Ellsberg Equilibria

Game Theory studies equilibrium outcomes of social conflicts when rational agents interact. Human beings are quite different from rational agents in general, so one can only expect to see a consistency with Nash equilibrium predictions and human behavior when the situation is controlled in such a way as to bring out the rational part of humans.

Nevertheless, it does make sense to ask what the observational implications of our theory are. For three player games, this is quite clear, as our theory predicts new equilibria outside the support of Nash equilibria. This is a testable implication, and we shall proceed one day to carry out such a test.

For two player games, the situation is more subtle. Both the Nash equilibrium and the Ellsberg equilibria have full support, so the only thing that we can learn from our theory seems to be that either action is fine in a one shot game. This is indeed the stance of Bade (2011b), in line with a number of predecessors.

There is, however, a way to distinguish the predictions of Ellsberg equilibria and Nash equilibria even in two player games. To understand this, we first need to explain what the law of large numbers looks like under ambiguity. The classical law states that the frequency of $HEAD$ in an infinite sequence of independent coin tosses will converge to the probability of $HEAD$. Now let us look at a typical Ellsberg urn that contains 100 balls, red and black, and we only know that the number of red balls is between 30 and 60. What can we say about the average frequency drawn from independent repetitions of the Ellsberg experiment? The natural guess would be that the average lies in the interval between 30% and 60% in the long run. This is indeed correct, and mathematical versions of that theorem have recently been proven, see Maccheroni and Marinacci (2005) and Epstein and Schneider (2003a), e.g., Peng (2007) has obtained the result that the average frequency will indeed fluctuate between both bounds, and *every* point in the interval $[0.3, 0.6]$ is an accumulation point of the sequence.

What is then the empirical content of such laws of large numbers? If we adhere to the point of view that our observed humans play independently one shot games, and that they should play equilibrium strategies, then the average frequency will converge to the Nash equilibrium strategy according to the classical theory, and will fluctuate between two bounds according to the new Ellsberg theory.

We thus do get observational differences between the two theories, and we interpret the Goeree-Holt results as a first evidence that our theory can accommodate deviations from Nash equilibrium observed in laboratories.

4.2.1 Two-Player Games: Strategies Outside the Nash Support

We now present a two-player example in which a strategy is supported in Ellsberg equilibrium which is not in the support of a Nash equilibrium of the game. When this happens, an Ellsberg equilibrium is observationally different to any Nash equilibrium of the game: if an outside observer watched two players play a one-shot strategic game, he would be able to distinguish if the players used ambiguity in equilibrium or not.

Bade (2011b) introduces the notion of observational equivalence to strategic two-person games with subjective randomization devices. She shows that in two-person games with fairly general preference structure observational equivalence is always satisfied. To make the theorem hold she uses one assumption on the preferences which is not common to the mostly used preference representations, the property of strict monotonicity.

The following example is taken from Bade (2011b). She chooses it to show that without the condition of strict monotonicity the conditions of the theorem are not fulfilled and thus we have observational differences. In opposition to her, we think that the introduction of ambiguity offers an interesting outcome of the game. In addition, our analysis allows to give a full description of all possible Ellsberg equilibria of this game. The payoff matrix of the game is given in Figure 4.1.

		Player 2	
		L	R
Player 1	U	$10, 1$	$0, 0$
	D	$11, 0$	$0, 1$

Figure 4.1: Example of a 2×2 game which has Ellsberg equilibria with support outside the Nash equilibrium support.

We first explain why strict monotonicity is not fulfilled in this game. Strict monotonicity means that if there is one non-null state[2] in which an Ellsberg strategy is strictly preferred to another, this should be so in all non-null states. Player 1 strictly prefers D to U when in some state player 2 plays L with positive probability. Whenever player 2 plays L with some Ellsberg strategy $[0, P_1]$, $P_1 > 0$, player 1 discounts in his utility evaluation completely the possibility of L to occur, although it is not a null-state. Thus, D is not for all states strictly preferred to U.

The game has a pure Nash equilibrium (D, R). Notice that the game does not have a Nash equilibrium with full support, since L is never played in equilibrium. One must admit that although U is a weakly dominated strategy, it would be appealing for player 1 if he could get player 2 to play L. Player 1 could assure himself a payoff of 10 as opposed to a payoff of 0. Nash equilibrium does not allow this "coordination" on (U, L). We will now see that this is different when we look at the Ellsberg equilibria of the game.

We calculate the Ellsberg equilibria. Suppose player 1 chooses to play U with the set of probabilities $[P_0, P_1]$, whereas player 2 plays L with the set of probabilities $[Q_0, Q_1]$. To

[2] Bade (2011b) also defines non-null events in the sense of Savage (1954).

find the Ellsberg equilibria, we first derive the payoff function $U_1(P, [Q_0, Q_1])$ of player 1.

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \le Q \le Q_1} 10PQ + 11(1-P)Q \\ &= \min_{Q_0 \le Q \le Q_1} Q(11-P) \\ &= Q_0(11-P) \text{ for all } P \in [P_0, P_1]\,. \end{aligned} \tag{4.1}$$

The payoff function is linear and depends only on the lower bound of $[Q_0, Q_1]$. On the boundaries we have

$$\begin{aligned} U_1(0, [Q_0, Q_1]) &= 11Q_0\,, \\ U_1(1, [Q_0, Q_1]) &= 10Q_0\,. \end{aligned}$$

The best response of player 1 is $P_0 = P_1 = 0$ when $Q_0 > 0$, only when $Q_0 = 0$ the payoff function is constant equal to zero and thus player 1 is indifferent between any subsets of $[0, 1]$. Thus, his best response in the latter case is $[P_0, P_1] \subseteq [0, 1]$. This can be easily observed in the plot of (4.1) in Figure 4.2. We plot $U_1(P, [Q_0, Q_1])$ for (1) $Q_0 = 0$, (2) $Q_0 = 1/2$ and (3) $Q_0 = 1$.

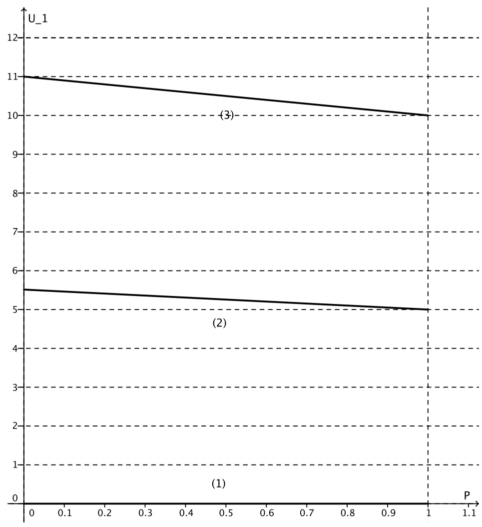

Figure 4.2: Payoff function (4.1) of player 1 for the cases (1) $Q_0 = 0$, (2) $Q_0 = 1/2$ and (3) $Q_0 = 1$.

The case is different for player 2, here we get a dependence on the lower and upper bound of $[P_0, P_1]$, and thus kinks in the payoff function.

$$
\begin{aligned}
U_2([P_0, P_1], Q) &= \min_{P_0 \le P \le P_1} QP + (1-Q)(1-P) \\
&= \min_{P_0 \le P \le P_1} P(2Q-1) + 1 - Q \\
&= \begin{cases} P_0(2Q-1)+1-Q, & \text{if } Q > 1/2\,, \\ 1/2, & \text{if } Q = 1/2\,, \\ P_1(2Q-1)+1-Q, & \text{if } Q < 1/2\,. \end{cases}
\end{aligned} \tag{4.2}
$$

The maximum of (4.2) over $Q \in [Q_0, Q_1]$ depends on the interval $[P_0, P_1]$. Observe that at the boundaries player 2's payoff function is the following.

$$
\begin{aligned}
U_2([P_0, P_1], 0) &= 1 - P_1\,, \\
U_2([P_0, P_1], 1) &= P_0\,.
\end{aligned}
$$

Player 2 maximizes (4.2) over $[Q_0, Q_1]$, his best responses are

$$
\begin{aligned}
&P_0 > 1/2: & B_2([P_0, P_1]) &= 1 & (1)\\
&P_0 = 1/2 < P_1: & B_2([P_0, P_1]) &= \{[Q_0, Q_1] \subseteq [1/2, 1]\} & (2)\\
&P_0 < 1/2 < P_1: & B_2([P_0, P_1]) &= 1/2 & (3)\\
&P_0 < 1/2 = P_1: & B_2([P_0, P_1]) &= \{[Q_0, Q_1] \subseteq [0, 1/2]\} & (4)\\
&P_1 < 1/2: & B_2([P_0, P_1]) &= 0 & (5)\\
&P_0 = 1/2 = P_1: & B_2([P_0, P_1]) &= \{[Q_0, Q_1] \subseteq [0, 1]\} & (6)
\end{aligned}
$$

Now we can solve for the Ellsberg equilibria by finding the intersections of the best response correspondences.

(1) $P_0 > 1/2 \Rightarrow Q_0 = Q_1 = 1 \Rightarrow P_0 = P_1 = 0$, thus this is not an equilibrium.

(2) $P_0 = 1/2 < P_1 \Rightarrow [Q_0, Q_1] \subseteq [1/2, 0] \Rightarrow P_0 = P_1 = 0$, thus this is not an equilibrium.

(3) $P_0 < 1/2 < P_1 \Rightarrow Q_0 = Q_1 = 1/2 \Rightarrow P_0 = P_1 = 0$, thus this is not an equilibrium.

(4) $P_0 < 1/2 = P_1 \Rightarrow [Q_0, Q_1] \subseteq [0, 1/2]$. We look at different subsets of the interval $[0, 1/2]$. When player 2 chooses $Q_0 > 0$, then player 1 always finds it optimal to play

$P_0 = P_1 = 0$. But when player 2 chooses $[Q_0, Q_1] \subseteq [0, Q_1]$ with $0 \leq Q_1 \leq 1/2$, then player 1 is indifferent on the whole interval of P and may choose $[P_0, P_1] \subseteq [0, 1]$. If in the latter case player 1 chooses to play $[P_0, P_1] = [P_0, 1/2]$ with $0 \leq P_0 \leq 1/2$, then this is an Ellsberg equilibrium, i.e.,

$$([P_0, 1/2], [0, Q_1]), \text{ where } 0 \leq P_0 \leq 1/2 \text{ and } 0 \leq Q_1 \leq 1/2 .$$

(5) $P_1 < 1/2 \Rightarrow Q_0 = Q_1 = 0 \Rightarrow [P_0, P_1] \subseteq [0, 1]$, thus if player 1 chooses $[P_0, P_1]$ with $0 \leq P_0 \leq P_1 \leq 1/2$ this is an Ellsberg equilibrium. Note that the pure strategy Nash equilibrium (D, R) is contained in this Ellsberg equilibrium.

(6) $P_0 = 1/2 = P_1 \Rightarrow [Q_0, Q_1] \subseteq [0, 1]$. We again consider two cases: if player 2 chooses $[Q_0, Q_1] = [Q_0, 1]$ with $0 \leq Q_0 \leq 1$, then player 1's best response is $P_0 = P_1 = 0$, thus this is not an equilibrium. But if player 2 decides to play $[Q_0, Q_1] = [0, Q_1]$ with $0 \leq Q_1 \leq 1$ and player 1 plays $P_0 = P_1 = 1/2 \in [0, 1]$, no player can gain by deviating from this strategy. We get the Ellsberg equilibrium

$$(1/2, [0, Q_1]), \text{ where } 0 \leq Q_1 \leq 1 .$$

Proposition 4.1. *The Ellsberg equilibria of the game in Figure 4.1 are of the form*

$$([P_0, 1/2], [0, Q_1]), \textit{ where } 0 \leq P_0 \leq 1/2 \textit{ and } 0 \leq Q_1 \leq 1/2 ,$$
$$([P_0, P_1], 0), \textit{ where } 0 \leq P_0 \leq P_1 \leq 1/2 ,$$
$$\textit{and } (1/2, [0, Q_1]), \textit{ where } 0 \leq Q_1 \leq 1 .$$

In the first and third of the Ellsberg equilibria there is some chance that (U, L) is played. Player 2 creates ambiguity and makes sure that the probability that L is not played at all is contained in his Ellsberg strategy. Then player 1 finds it optimal to play U with positive probability, either by playing a random strategy as in the third equilibrium, or by creating ambiguity himself.

Of course, in all these equilibria the minimal expected utility of player 1 is 0. Therefore player 1, being ambiguity-averse, in his own estimation does not gain any utility by using an Ellsberg strategy instead of the pure Nash equilibrium strategy. Even more, player 2 can assure himself at most a utility of 1/2 which is worse than his (strict) Nash equilibrium payoff. Nevertheless, these Ellsberg equilibria are in some sense pareto dominating, because with some positive probability the outcome of the game is (U, L).

The preceding example (Figure 4.1) suggests that if we allow for not strictly monotonous payoff functions in two-person games (which is the case in most ambiguity averse preference representations) we can obtain Ellsberg equilibria which are supported by pure strategies that do not appear in the support of any Nash equilibrium. These equilibria pareto dominate the Nash equilibria of these games.

4.2.2 Weakly Dominated Strategies and Observational Differences

Different papers have found that the existence of weakly dominated strategies plays an important role in the classification of games with ambiguity averse players. We comment on some of the existing results.

Theorem 5 in Klibanoff (1996) shows that Weak Admissibility (WA) of a preference relation $\succcurlyeq$:[3]

$$\text{for all acts } f, g, \text{ if for all } s \in S \; f(s) \succcurlyeq g(s)\,, \text{ then } f \succcurlyeq g \text{ and}$$
$$[f \succ g \text{ if and only if for some non-null event } E\,, f(s) \succ g(s) \text{ for all } s \in E]$$

implies agreement on null events (AGR),

$$\text{let } \mathcal{P}_i \text{ be a closed and convex subset of } \Delta S_i$$
$$P(E) = 0 \text{ if and only if for all } P \in \mathcal{P}_i\,, P(E) = 0$$

in the maxmin expected utility representation by Gilboa and Schmeidler (1989). The same holds for the objective ambiguity representation we use in Ellsberg games. Agreement on null events has the following interpretation in the context of games. Each event (that is, e.g., each strategy of the opponent) is given either zero probability by all distributions in $\mathcal{P}_i$, or positive probability by all distributions in $\mathcal{P}_i$ (i.e., the distributions in $\mathcal{P}_i$ are mutually absolutely continuous, see, e.g., Epstein and Marinacci (2007)).

Practically, this excludes proper Ellsberg strategies which touch the boundary of the simplex. In 2×2 games that would be strategies of the type $[0, P_1]$ and $[P_0, 1]$. When we exclude these types of strategies, those Ellsberg strategies in Proposition 4.1 that have a support outside the Nash support, collapse.

And really, two-player games with weakly dominated strategies (where the indifference is at the minimal payoff) have to be excluded if we want to assure *no* observational differences between Ellsberg equilibria and Nash equilibria in two-person normal form games. This is proved in Theorem 1 on observational equivalence in Bade (2011b) for a large class of

[3] This is the same property as strict monotonicity of preferences, in Schmeidler (1989) (axiom vi) and Bade (2011b) (MON).

ambiguity averse preferences which also includes Ellsberg games. She finds that (WA) is a necessary condition for the theorem to hold.

In a similar spirit, Eichberger and Kelsey (2000) prove in Proposition 5.1, that in n-person normal form games where the minimal payoffs for each strategy are different, in an "equilibrium under uncertainty" no weakly dominated strategies are used. But note that in their concept of equilibrium under uncertainty players are not allowed to use mixed strategies, thus weak domination in the proposition is only weak domination in pure strategies.

4.3 Existence of Proper Ellsberg Equilibria: When is Ambiguity an Option?

In the preceding examples one could observe that in some games ambiguity is played in equilibrium, either by both or at least by one of the players, and in some games no proper Ellsberg equilibria exist. In, e.g., the Prisoners' Dilemma which has a unique Nash equilibrium in pure strategies, no ambiguity is observed in Ellsberg equilibrium: in fact, the only Ellsberg equilibrium is the unique Nash equilibrium. In the modified Matching Pennies game exist Ellsberg equilibria where both players create ambiguity, and in zero-sum games no more than one player uses ambiguity in Ellsberg equilibrium. We now have a closer look at the conditions under which proper Ellsberg equilibria (see Definition 1.8) are possible.

Intuitively, when thinking about ambiguity in normal form games, we imagine players using some kind of ambiguity "around" their mixed Nash equilibrium strategies. If we take P^* to be the mixed Nash equilibrium strategy, this would be some Ellsberg strategy $[P^* - \epsilon, P^* + \epsilon]$ with $\epsilon > 0$. Surprisingly, these kind of equilibria, where *both* players create ambiguity around their Nash strategy, never exist. When one of the players uses this type of strategy in Ellsberg equilibrium we speak of *Ellsberg equilibrium with unilateral full ambiguity*, i.e., equilibrium profiles of the types $(P^*, \Delta S_2)$ or $(\Delta S_1, Q^*)$. "Full", because the player can then use as much ambiguity as he wants (also the whole simplex), as long as his mixed Nash equilibrium strategy is contained in the set.

There is only one type of generic 2×2 games in which such Ellsberg equilibria arise: when the immunization strategy of one player is exactly his Nash equilibrium strategy. Then the opponent can use any set of probability measures, as long as his Nash equilibrium is an element of that Ellsberg strategy. We generalize this observation to games with more strategies.

Theorem 4.2. *Let G be a two-person normal form game with a unique completely mixed Nash equilibrium (P^*, Q^*). There exist Ellsberg equilibria with unilateral full ambiguity if and only if either P^* or Q^* is maximin. The Ellsberg equilibria are of the following form:*

$$(P^*, [Q_0, Q_1]), \text{ where } Q_0 < Q^* < Q_1, \text{ if } P^* = M_1\,,$$

$$\text{and } ([P_0, P_1]\,, Q^*), \text{ where } P_0 < P^* < P_1, \text{ if } Q^* = M_2\,.$$

If P^* is also an immunization strategy for player 1, then the profiles $(P^*, \mathcal{Q})$ with $Q^* \in \mathcal{Q}$ form an Ellsberg equilibrium (and similar for player 2 if Q^* is an immunization strategy).

Proof. Let $\left(\bar{P}, \Delta S_2\right)$ be an equilibrium with unilateral full ambiguity. Then $\bar{P}$ is a best reply to ΔS_2, or in other words, $\bar{P}$ is a maximin strategy.

By the indifference principle (Theorem 1.14), player 2 is indifferent between all $Q \in \Delta S_2$ when player 1 plays $\bar{P}$. It follows that $(\bar{P}, Q^*)$ is a Nash equilibrium. As we have assumed uniqueness of Nash equilibrium, $\bar{P} = P^*$, and P^* is therefore maximin.

Similarly, if P^* is maximin, then the singleton P^* is a best reply in the Ellsberg game to ΔS_2. Also, player 2 is indifferent against P^*, so ΔS_2 is a best reply to P^* for player 2.

Now suppose that P^* is also an immunization strategy. Let $v^* = u_1(P^*, Q^*)$ be the Nash equilibrium payoff, and $\bar{v} = \min_{Q \in \Delta S_2} u_1(P^*, Q)$ be the maximin payoff. As P^* immunizes player 1, we have $v^* = \bar{v}$. As P^* is part of a completely mixed Nash equilibrium, any $\mathcal{Q} \subset \Delta S_2$ is a best reply in the Ellsberg game for player 2. □

In an unpublished working paper, Ryan (1999) proves a result for 2×2 normal form games using the Beliefs Equilibrium concept by Lo (1996). Beliefs Equilibrium is a wider (it allows correlation of beliefs) and entirely subjective concept (see Section 1.5 on the related literature for details), however, the result applies to 2×2 Ellsberg games. He shows that if $\mathcal{Q}$ is a closed and convex subset of ΔS_2 and player 1's set of best responses $B_1(\mathcal{Q})$ does not contain pure strategies, then player 1 has a unique maximin strategy M_1 and $B_1(\mathcal{Q}) = M_1$. The property of no pure strategies in the set of best responses has the geometric representation of a minimal expected utility function with a unique optimal point in the interior of $[0, 1]$ (see, e.g., graph (3) in Figure 3.2). Theorem 4.2 adds to this result. The Ellsberg strategy $\mathcal{Q}$ with the property above is part of an Ellsberg equilibrium, if and only if $B_1(\mathcal{Q}) = M_1$ and $M_1 = P^*$ is the Nash equilibrium strategy of player 1.

The result of Theorem 4.2 also provokes reflection on the robustness of Ellsberg equilibria. From what we have seen, when player 1's Nash equilibrium strategy is equal to his immunization strategy, Ellsberg equilibrium has the particular form that player 1 plays his Nash equilibrium strategy and player 2 can play an arbitrarily large set of probability distributions (provided this set contains his Nash equilibrium strategy). When we change the game just a little by adding ϵ to some payoff of player 1, this Ellsberg equilibrium type collapses and player 2 is now much more constrained in his equilibrium play (compare table 3.1). Thus, the equilibrium type in Theorem 4.2 is not robust to small changes in the payoff matrix. On the other hand, starting from a game where the Nash equilibrium strategies are not identical to the immunization strategies, the Ellsberg equilibria change continuously with changes in the payoff matrix.

5 A Third Player Can Cause Non-Nash Behavior

In the preceding chapters we presented the basic properties of Ellsberg equilibria and how players behave when we allow them to use objective ambiguity in two-player games. Naturally, we want to characterize what can happen in games with more than two players. To tackle this problem, we first investigate the relation of Ellsberg equilibrium to the concept of subjective equilibrium. Since there exists no other solution concept that lets players use sets of probabilities as their strategy, the question of comparison has to be answered with regard to the attainability of certain outcomes. The objective is to find out under which conditions subjective equilibria can (or cannot) have the same support as Ellsberg equilibria.

We start with an example of a three-player game, where an Ellsberg equilibrium exists that attains an outcome that is not in the support of any Nash equilibria of the game. This suggests that in very simple games with more than two players, the prediction of Ellsberg equilibrium can be quite different from the Nash equilibrium prediction. Subsequently, we define subjective equilibrium in the spirit of Aumann (1974) and Hallin (1976) and subjective beliefs equilibrium like Lo (1996). We show that every Ellsberg equilibrium contains a subjective equilibrium and a subjective beliefs equilibrium. In two player games, subjective beliefs equilibria have the same support as Nash equilibria, but this changes for three players. This explains that in Ellsberg games with more than two players outcomes outside the Nash equilibrium support can be attained.

5.1 Three-Person Games with Non-Nash Outcomes

We want to give a simple example of a game with more than two players, where an Ellsberg equilibrium predicts very different behavior than the Nash equilibrium. To this end, we consider a classic example from Aumann (1974). He presents a three-person game where one player has some mediation power to influence his opponents' choice. The original game is given by the payoff matrix in Figure 5.1, where we let player 1 choose rows, player 2 choose columns and player 3 choose matrices.

	L	R
U	$0, 8, 0$	$3, 3, 3$
D	$1, 1, 1$	$0, 0, 0$

l

	L	R
U	$0, 0, 0$	$3, 3, 3$
D	$1, 1, 1$	$8, 0, 0$

r

Figure 5.1: Aumann's example.

Player 3 is indifferent between his strategies l and r, since he gets the same payoffs for both. As long as player 3 chooses l with a probability higher than 3/8, L is an optimal strategy for player 2 regardless what player 1 does; and player 1 would subsequently play D. By the same reasoning, as long as player 3 plays r with a probability higher than 3/8, D is optimal for player 1, and player 2 plays L then. Thus, the Nash equilibria of this game are all of the form (D, L, P^*), where P^* is any classical mixed strategy.

This example has also been analyzed in other literature on ambiguity in games, see Eichberger, Kelsey, and Schipper (2009), Lo (2009) and Bade (2011b). Bade chooses one ambiguous act equilibrium with maxmin expected utility preferences to show that non-Nash outcomes can be sustained in games with more than two players. We characterize all Ellsberg equilibria[1] and provide an interpretation of the example which highlights the strategic use of ambiguity as a mediation tool.

Let us now explain how Aumann's example can be interpreted to illustrate the strategic use of ambiguity. Suppose players 1 and 2 are prisoners, and player 3 the police officer. Let us rearrange the matrix game and put it in the form displayed in Figure 5.2. We swap the strategies of player 2 and rename the strategies of player 1 and 2 to C = "cooperate" and D = "defect" as in the classical Prisoners' Dilemma. We can merge the two matrices into one, because the strategy choice of the police officer is simply the choice of a probability that influences the payoffs of prisoners 1 and 2 in case of unilateral defection from (C, C). If he chooses $P = 1$, this corresponds to strategy l in the original game (i.e., prisoner 2 gets all the reward), $P = 0$ would be strategy r (i.e., prisoner 1 gets all the reward). The choice of the objectively mixed strategy $P = 1/2$ leads to the classical symmetric Prisoners' Dilemma with a payoff of 4 in case of unilateral defection.

In this interpretation, players 1 and 2 are facing a sort of Prisoners' Dilemma situation mediated by a player 3, the police officer. Given the payoffs, the police officer is most interested in cooperation between the prisoners. The police officer can influence how high

[1] In this special game and the preference representation chosen by Bade, the ambiguous act equilibrium coincides with one of the Ellsberg equilibria. Recall that in difference to ambiguous act equilibria, Ellsberg equilibria use ambiguity objectively.

the reward would be for unilateral defection by using an objective randomizing device. Nevertheless, in every Nash equilibrium of the game, the players obtain the inefficient outcome of 1.

		Prisoner 2	
		C	D
Prisoner 1	C	$3, 3, 3$	$0, 8P, 0$
	D	$8(1-P), 0, 0$	$1, 1, 1$

Figure 5.2: Mediated Prisoners' Dilemma.

Now suppose we let the players use Ellsberg strategies. The police officer could create ambiguity by announcing: "I'm not sure about who of you I will want to punish and who I will want to reward for reporting on your partner. I might also reward you both equally... I simply don't tell you what mechanism I will use to decide about this."

Let us exhibit Ellsberg strategies that support this behavior. If prisoner 2 expects P to be lower than $3/8$ and prisoner 1 expects P to be higher than $5/8$, they would prefer to cooperate. This behavior corresponds to the police officer playing an Ellsberg strategy $[P_0, P_1]$ with $0 \leq P_0 < 3/8$ and $5/8 < P_1 \leq 1$. The ambiguity averse prisoners 1 and 2 evaluate their utility with $P = P_0$ and $P = P_1$, respectively. Consequently they would prefer to play (C, C). This gives an Ellsberg equilibrium in which the prisoners cooperate.

Proposition 5.1. *In the mediated Prisoners' Dilemma, the Ellsberg strategy profiles*

$$(C, C, [P_0, P_1]) \text{ with } P_0 < 3/8 \text{ and } P_1 > 5/8$$

are Ellsberg equilibria that achieve the efficient outcome $(3, 3, 3)$.

Again, as in the example by Greenberg (see Figure 1.3 in Section 1.4), it is important to see that both prisoners use different worst-case probabilities to compute their expected payoffs. Aumann (1974) has already commented on this behavior. He observes that (C, C) can be a "subjective equilibrium point" if players 1 and 2 have non-common beliefs about the objectively mixed strategy player 3 is going to use. In his analysis player 1 believes $P = 3/4$ and player 2 believes $P = 1/4$. Note that in Ellsberg equilibrium the players have the *common* belief $P \in [P_0, P_1]$.

5.2 Subjective Equilibria and Ellsberg Equilibria

In a subjective equilibrium the players may base their actions on events of which the probability of realization is not commonly agreed on. These events are thus not the outcome of the roll of a die, but something more subjective such as the outcome of a horse race, certain stock exchange fluctuations or the outcome of a soccer match. For a game this implies that a player may best respond to a belief he has about his opponents' play, but this belief does not have to coincide with the actual strategy played.

The notion of subjective equilibrium as a static concept has been first introduced by Aumann (1974). The only authors that to our knowledge subsequently worked with this static notion were Hallin (1976) and Brandenburger and Dekel (1987). Hallin (1976) shows under which conditions subjective equilibrium payoffs can be attained by an objective equilibrium, and when, on the other hand, subjective equilibrium payoffs can strictly dominate objective ones. In both results he allows for correlated beliefs and correlated strategies. Brandenburger and Dekel (1987) show that correlated rationalizability is equivalent to a special case of subjective equilibria, that is subjective correlated equilibria that are also optimal after the state of the world is revealed (Aumann (1974) calls these "a posteriori equilibria"). Subjectivity in extensive form or repeated games has been defined and examined in another branch of literature not treated in this section.

We give an example of a subjective equilibrium before stating the definition. Consider the standard matching pennies game in Figure 5.3.

		Player 2: L	Player 2: R
Player 1	U	$1, -1$	$-1, 1$
	D	$-1, 1$	$1, -1$

Figure 5.3: Matching Pennies game.

This game has only one Nash equilibrium (in Aumann (1974)'s language: equilibrium in objective mixed strategies, objective, because all players have the same probability distribution on the state space, and mixed, because the strategies cannot be correlated), that is when both players mix with probabilities $(1/2, 1/2)$. The expected payoff is then zero for both players. Now we relax the assumption that the beliefs in equilibrium have to be correct. Then (U, L) can be a subjective equilibrium when player 1 beliefs player 2 will play L (which is true), but player 2 thinks player 1 will play D and thus correctly best responds to his belief by playing L. Obviously this equilibrium seems unsatisfying, since one of the players beliefs something wrong. Nevertheless, this is a subjective equilibrium of the matching pennies game, with subjective expected utility $(1, 1)$.

Aumann (1974) considers a similar equilibrium of the matching pennies game in example (2.2) of that article. He supposes that there is an event to which players 1 and 2 ascribe subjective probabilities 1 and 0, respectively. Player 2 plays left in any case (he does not observe D), player 1 plays U if D occurs, else he plays D. Aumann claims, p. 76, that these strategies form an equilibrium in subjective mixed strategies, but I believe it only works when the strategies are correlated. At least player 1 has to believe that player 2 plays a strategy that is correlated to his. Else player 1's strategy cannot be a best response to any belief player 1 might have.

5.2.1 Definition of Subjective Equilibrium

As we said above, a subjective equilibrium relaxes the common belief assumption in Nash equilibrium by letting players base their actions on events that players may assign different probabilities to.

Let $N = 1, \ldots, n$ denote the set of players, S_i the set of pure strategies of each player. A normal form game $G = \langle N, (S_i), (u_i) \rangle$ is defined by the players' strategies and their utility functions. We equip each player i with a subjective randomizing device Ω_i that is the set of future states of the world. Each player i has a subjective probability distribution $\pi^i = (\pi^i_1, \ldots, \pi^i_n)$ on the profile $\Omega = (\Omega_1, \ldots, \Omega_n)$. We assume that all the state spaces are stochastically independent. A subjectively mixed strategy for player i is a measurable function $f_i : \Omega \to \Delta S_i$, where ΔS_i is the set of lotteries over pure strategies $s_i \in S_i$. The set of measurable functions f_i is denoted $(\Delta S_i)^\Omega$, lotteries over S_i, i.e., mixed strategies, are denoted by p_i. The utility for player i of a profile f of subjectively mixed strategies is evaluated as subjective payoff expectation,

$$\bar{U}_i(f_1, \ldots, f_n) := \int_\Omega u_i(f(\omega))\, d\pi^i \,.$$

Definition 5.2. *A* subjective equilibrium *of the strategic game $G = \langle N, (S_i), (u_i) \rangle$ is a profile $(f^*_1, \ldots, f^*_n)$ of subjectively mixed strategies, such that for all $i \in N$ there exists a profile of probability distributions π^i on Ω, such that*

$$f^*_i \in \arg \max_{f_i \in (\Delta S_i)^\Omega} \int_{\Omega_i} \int_{\Omega_{-i}} u_i(f_i(\omega_i), f^*_{-i}(\omega_{-i}))\, d\pi^i_{-i}\, d\pi^i_i \,.$$

Or alternatively, if for all $i \in N$ there exists a profile of probability distributions π^i on Ω, such that for all $f_i : \Omega \to \Delta S_i$,

$$\bar{U}_i(f^*_i, f^*_{-i}) \geq \bar{U}_i(f_i, f^*_{-i}) \,.$$

Note that we gave a definition for the private event case. Every player bases his action on a private event, and the state spaces of all the players are stochastically independent. Hallin (1976) gave the analog definition for the public event case and thus leaves the possibility for players to correlate their actions and have correlated beliefs.

The above definition can, the same as correlated and Ellsberg equilibrium, be reduced to the case where the states of the world are the pure strategies of each player. To this end, let $\mu^i_{-i} = (\mu^i_1, \ldots, \mu^i_{i-1}, \mu^i_{i+1}, \ldots, \mu^i_n)$ denote a profile of probability distributions over the sets $\Delta S_1, \ldots, \Delta S_{i-1}, \Delta S_{i+1}, \ldots, \Delta S_n$, respectively, this means $\mu^i_{-i} \in \Delta(\Delta S_{-i})$. μ^i_{-i} is the belief of player i about which mixed strategy players $N \setminus \{i\}$ will play, therefore it has to be a probability distribution over the set of mixed strategies. We state the definition below. The subjective expected payoff $\bar{U}_i$ of the profile of acts f now reduces to the subjective expected payoff of the beliefs which are represented by mixed strategies. Therefore we denote the subjective expected payoff in the reduced case by u_i.

Definition 5.3. *A* reduced form subjective equilibrium *of a strategic game $G = \langle N, (S_i), (u_i)\rangle$ is a profile $(p^*_1, \ldots, p^*_n)$ of mixed strategies, if for each $i \in N$ there exists a profile of probability distributions μ^i_{-i} on the set of lotteries ΔS_{-i}, such that*

$$p^*_i \in \arg\max_{p_i \in \Delta S_i} u_i(p_i, \mu^i_{-i}) = \arg\max_{p_i \in \Delta S_i} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i})\, d\mu^i_{-i}\, dp_i\,.$$

Or alternatively, if for all $i \in N$ there exists a profile of probability distributions μ^i_{-i} on the set of lotteries ΔS_{-i}, such that for all $p_i \in \Delta S_i$,

$$u_i(p^*_i, \mu^i_{-i}) \geq u_i(p_i, \mu^i_{-i})\,.$$

Proposition 5.4. *The definitions 5.2 and 5.3 are equivalent.*

Proof. The proof is analog to the proof of Proposition 1.3 on the reduced form Ellsberg equilibrium. □

In the reduced form subjective equilibria players formulate their beliefs directly about which strategy their opponents will play, in the non-reduced form the belief is formed about the probability distribution governing the state space that each opponent uses to subjectively mix their strategy.

Whichever definition the reader prefers, since the beliefs are inconsistent (that is, they can differ between the players), Aumann (1974) argues that in subjective equilibria someone might want to renegotiate after the state of the world is revealed. This can be observed very well in the example of a subjective equilibrium of the matching pennies game presented at the beginning of the chapter. In that equilibrium player 2 best responds to a completely false belief and thus is happy with his choice only before the state D is revealed. A posteriori he would obviously like to change his action.

These problems can be avoided by making further assumptions on the equilibrium. Aumann (1974) calls them "a posteriori equilibria", when he assumes that no player wants to unilaterally deviate even *after* the state of the world is revealed. In his definition of a posteriori equilibria he allows for correlated randomizing. He shows that the set of payoffs to a posteriori equilibrium points coincides with the set of payoffs to subjective correlated equilibrium points, when the π^i are mutually continuous with respect to each other. That is, if $\pi^i(\omega) = 0$ for one i, then $\pi^i(\omega) = 0$ for all $i \in N$. This property is also called *agreement on null-events*.

Brandenburger and Dekel (1987) go even further and show that the set of payoffs to a posteriori equilibria coincides with the set of correlated rationalizable payoffs.

5.2.2 Relation with Ellsberg Equilibrium

We now analyze how Ellsberg equilibria are connected with subjective equilibria. With his definition of a subjectively mixed strategy, Aumann (1974) was the first to model a strategy that is based on an uncertain event in the Knightian sense. Of course, at the time, no ambiguity averse preference representations were available and Aumann assumes that an uncertain event is characterized by the fact that players may have different priors on its realization. Thus, he assumes that players have a probability distribution on the set of states of the world Ω according to Savage (1954) subjective expected utility. With this assumption the use of subjectively mixed strategies boils down to the fact that the beliefs of the players may differ in equilibrium. When we get rid of the subjective expected utility assumption and instead assume ambiguity averse preferences, these lead to ambiguous acts as defined by Bade (2011b). In contrast, by playing Ellsberg strategies players may pick any kind of ambiguous randomizing device that they find reasonable, that is, which is utility maximizing. Preferences are imposed upon the existence of objective ambiguity.

In spite of the different preference representations in Aumann (1974) and our setup, both concepts rely on the fact that each player may base his action on a set of states of the world for which different probability distributions exist. In Aumann (1974) each player may have a different one, but every player has only one distribution, in Ellsberg games all players have the same prior, but the prior is a *set* of probability distributions on each of the state spaces. How do the sets of equilibria in subjective games and Ellsberg games relate to each other? We start by proving that every Ellsberg equilibrium contains a subjective equilibrium.

Proposition 5.5. *For every Ellsberg equilibrium* $(\mathcal{P}_1^*, \ldots, \mathcal{P}_n^*)$ *of a normal form game*

$G = \langle N, (S_i), u_i \rangle$, $i \in N$, there exists a subjective equilibrium $(p_1^*, \dots, p_n^*)$ with beliefs $\mu_{-i}^i \in \Delta S_{-i}$ of G, such that

$$p_i^* \in \mathcal{P}_i^* \,, \\ \textit{and} \quad \mu_{-i}^i \in \Delta \mathcal{P}_{-i}^* \,.$$

Proof. When a Nash equilibrium $(q_1, \dots, q_n)$ is element of the Ellsberg equilibrium $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$, that is $q_i \in \mathcal{P}_i^*$ for all $i \in N$, then we define

$$p_i^* := q_i \,, \\ \text{and} \quad \mu_{-i}^i := q_{-i} \,,$$

and the proposition holds since every Nash equilibrium is a subjective equilibrium in which the subjective probability distributions coincide.

Now let there not be a Nash equilibrium like above. Take $p_i^* \in \mathcal{P}_i^*$ to be any element of $\mathcal{P}_i^*$ and

$$\mu_{-i}^i \in \left\{ P'_{-i} \,\middle|\, P'_{-i} = \arg \min_{P_{-i} \in \mathcal{P}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) \, dP_{-i} \, dp_i^* \right\} \tag{5.1}$$

to be any minimizer given $p_i^* \in \mathcal{P}_i^*$. We need to show that $(p_1^*, \dots, p_n^*)$ with beliefs μ_{-i}^i is indeed a subjective equilibrium. But this is obvious, since $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$ is an Ellsberg equilibrium and

$$p_i^* \in \arg \max_{p_i \in \Delta S_i} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) \, d\mu_{-i}^i \, dp_i \quad \text{for all} \quad i \in N \,.$$

Thus, $(p_1^*, \dots, p_n^*)$ with the beliefs μ_{-i}^i as in (5.1) for each $i \in N$ is a subjective equilibrium. □

Remarks 5.6. *1. To find a subjective equilibrium contained in an Ellsberg equilibrium, not* every *candidate for the beliefs μ_{-i}^i can be chosen within $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$. To see this, consider the Modified Matching Pennies example, Figure 3.8: if player 1 played his Nash equilibrium strategy $1/2$ and his belief about player 2's strategy was not the minimizer $1/4$, he would not stay with his mixed strategy.*

2. On the other hand, for the beliefs μ_{-i}^i of every player $i \in N$, we need not necessarily choose the minimizer as constructed in (5.1) in the proof. To see this, consider the peace negotiation example in Figure 1.3: every belief of countries A and B about the play of the superpower C that fulfills $\mu_C^A(\textit{punish}\, A) > 5/9$ and $\mu_C^B(\textit{punish}\, A) < 4/9$

is a candidate for a subjective equilibrium.

This result is of course not very surprising, because due to the inconsistency of beliefs one can obtain a large set of outcomes with the subjective equilibrium notion.

The converse of the statement does not hold, that is, it is not possible to find for every subjective equilibrium of a normal form game G an Ellsberg equilibrium that contains the subjective equilibrium in the way prescribed by the proposition. We give an example to show this. Consider the matching pennies game in Figure 5.3. We explained that the pair (U, L) with beliefs $\mu_2^1(D) = 1$ and $\mu_1^2(L) = 1$ is a subjective equilibrium of the game. In contrast, the only Ellsberg equilibria are $((1/2, 1/2), [Q_0, Q_1])$ with $Q_0 < 1/2 < Q_1$ and $([P_0, P_1], (1/2, 1/2))$ with $P_0 < 1/2 < P_1$, where $P \in [P_0, P_1], Q \in [Q_0, Q_1]$ are the probabilities of player 1 playing U and player 2 playing L, respectively. It is obvious that the subjective equilibrium (U, L) cannot be contained in any Ellsberg equilibrium.

5.2.3 Subjective Equilibria Explain Non-Nash Behavior in Three-Person Ellsberg Games

We now give an intuitive explanation for the observation of strategies with non-Nash support in Ellsberg games with more than two players. To this end, we define one more subjective equilibrium notion. The *subjective beliefs equilibrium* is a profile of possible non-common beliefs about the opponents' play, where the belief μ_i^j of player j about the play of player i must be a best response of player i to his belief μ_{-i}^i about the others' play.

Definition 5.7. *A* subjective beliefs equilibrium *is a profile* $(\mu_{-1}^1, \ldots, \mu_{-n}^n)$*, where* $\mu_{-i}^i \in \Delta S_{-i}$*, such that for all* $i \in N$*, for every* $j \neq i$*,*

$$\mu_i^j \in \arg\max_{P_i \in \Delta S_i} u_i(P_i, \mu_{-i}^i)\,.$$

This is what Lo (1996) calls a *Bayesian Beliefs Equilibrium*, with the difference that we do not allow for correlated beliefs. The difference of subjective beliefs equilibrium to subjective equilibrium lies in the fact that players do no longer best respond to some subjective belief, but the (possibly non-common) subjective beliefs of the other players must be a best response to the subjective beliefs. One can see clearly now that Klibanoff (1996)'s equilibrium under uncertainty is a generalization of subjective equilibrium, whereas Lo (1996) generalizes the subjective beliefs equilibrium.

We show in Theorem 5.8 that every Ellsberg equilibrium contains a subjective beliefs equilibrium. In two-player games, every subjective beliefs equilibrium is a Nash equilibrium. When there is only one opponent, there is no room for disagreeing beliefs. This changes when we allow for more players, as Lo (1996) points out, because now players j and k may disagree about what player i will play. The consequence of this is immediate: we have an intuitive explanation, why in games with more than two players Ellsberg equilibria arise that are not obtainable by a Nash equilibrium.

Theorem 5.8. *If $(\mathcal{P}_1^*, \ldots, \mathcal{P}_n^*)$ is an Ellsberg equilibrium, then there exist $\mu_{-i}^i \in \mathcal{P}_{-i}^*$ such that $(\mu_{-1}^1, \ldots, \mu_{-n}^n)$ is a subjective beliefs equilibrium and*

$$\arg\max_{P_i \in \Delta S_i} U_i(P_i, \mathcal{P}_{-i}^*) \subseteq \arg\max_{P_i \in \Delta S_i} u_i(P_i, \mu_{-i}^i)\,.$$

Proof. It is sufficient to show that there exists $\mu_{-i}^i \in \mathcal{P}_{-i}^*$ such that

$$\arg\max_{P_i \in \Delta S_i} U_i(P_i, \mathcal{P}_{-i}^*) \subseteq \arg\max_{P_i \in \Delta S_i} u_i(P_i, \mu_{-i}^i)\,.$$

The latter inclusion means that the set of best responses to $\mathcal{P}_{-i}^*$ is contained in the set of best responses to μ_{-i}^i. This and the assumption that $(\mathcal{P}_1^*, \ldots, \mathcal{P}_n^*)$ is an Ellsberg equilibrium imply

$$\mu_i^j \in \mathcal{P}_i^* \subseteq \arg\max_{P_i \in \Delta S_i} U_i(P_i, \mathcal{P}_{-i}^*) \subseteq \arg\max_{P_i \in \Delta S_i} u_i(P_i, \mu_{-i}^i)\,,$$

and therefore $(\mu_{-1}^1, \ldots, \mu_{-n}^n)$ is a subjective beliefs equilibrium. We know from the Minimax Theorem 1 (Theorem 1.15) and the Principle of Indifference in Distributions (Theorem 1.14) that

$$\max_{P_i \in \Delta S_i} \min_{P_{-i} \in \mathcal{P}_{-i}^*} u_i(P_i, P_{-i}) = \min_{P_{-i} \in \mathcal{P}_{-i}^*} \max_{P_i \in \Delta S_i} u_i(P_i, P_{-i}) = c^*\,.$$

Now, $P_i^* \in \arg\max_{P_i \in \Delta S_i} U_i(P_i, \mathcal{P}_{-i}^*)$ if and only if (by the Principle of Indifference in Distributions) $\min_{P_{-i} \in \mathcal{P}_{-i}^*} u_i(P_i^*, P_{-i}) = c^*$. From this follows that

$$\begin{aligned} &u_i(P_i^*, P_{-i}) \geq c^* \\ &\text{for all } P_{-i} \in \mathcal{P}_{-i}^*, \text{ for all } P_i^* \in \arg\max_{P_i \in \Delta S_i} U_i(P_i, \mathcal{P}_{-i})^*\,. \end{aligned} \tag{5.2}$$

Take $\mu_{-i}^i \in \arg\min_{P_{-i} \in \mathcal{P}_{-i}^*} \max_{P_i^* \in \Delta S_i} u_i(P_i^*, P_{-i})$. Then we have

$$u_i(P_i^*, \mu_{-i}^i) \leq c^* = \max_{P_i^* \in \Delta S_i} u_i(P_i^*, \mu_{-i}^i) \text{ for all } P_i^* \in \Delta S_i\,. \tag{5.3}$$

When we combine equations (5.2) and (5.3), we have

$$u_i(P_i^*, \mu_{-i}^i) = c^* \text{ for all } P_i^* \in \arg\max_{P_i \in \Delta S_i} U_i(P_i, \mathcal{P}_{-i}^*)\,.$$

That is,

$$\arg\max_{P_i \in \Delta S_i} U_i(P_i, \mathcal{P}^*_{-i}) \subseteq \arg\max_{P_i \in \Delta S_i} u_i(P_i, \mu^i_{-i})\,.$$

□

Since subjective beliefs equilibria and Nash equilibria are equivalent in two-person games, Theorem 5.8 has the following corollary. The reasoning of Theorem 5.8 and the corollary are based on Lo (1996), Proposition 3. The corollary gives an intuitive explanation, why in two-player games the Ellsberg equilibrium behavior cannot be “too far” from the Nash equilibrium behavior. We discuss this in Remark 5.10.

Corollary 5.9. *When G is a two-player game, then for every Ellsberg equilibrium $(\mathcal{P}^*, \mathcal{Q}^*)$ there exists $P^* \in \mathcal{P}^*$ and $Q^* \in \mathcal{Q}^*$ such that (P^*, Q^*) is a Nash equilibrium and*

$$\arg\max_{P \in \Delta S_i} U_i(P, \mathcal{Q}^*) \subseteq \arg\max_{P \in \Delta S_i} u_i(P, Q^*)\,.$$

Remark 5.10. *Note that the statement of Corollary 5.9 has an interesting consequence for the interpretation of Bade (2011b)'s theorem on observational indifference. When we relax Bade's assumption on strictly monotone payoff functions and allow for weakly dominated strategies (like in the example on observational differences in two-player games, Figure 4.1), we are likely to observe Ellsberg equilibria with a support greater than the Nash equilibrium support. But, and this follows from Corollary 5.9 and the proof of Theorem 5.8, the minimizers used by the players to derive the utility in Ellsberg equilibrium remain the Nash equilibrium strategies.*

6 Strategic Use of Ambiguity in Dynamic Games

It is straightforward that Ellsberg urns can also be used as strategies in dynamic games.[1] In this chapter we develop the theoretical framework to analyze such extensive form Ellsberg games. As in classic extensive form theory, players can use an urn to create ambiguity over their set of pure strategies, or they can place an Ellsberg urn at every decision point in the process of the game. In the first section of the chapter we thus extend the notions of mixed strategy and behavioral strategy to extensive form Ellsberg games.

Considering dynamic games leads directly to the question of dynamic consistency of the players' preferences. We discuss this in Section 6.2 and develop a formalism to translate rectangularity of Epstein and Schneider (2003b) to extensive form Ellsberg games. This property of Ellsberg strategies is then used to prove a version of Kuhn's Theorem (Kuhn (1953)) for extensive form Ellsberg games. We present an example for intuition. Thereafter we define Ellsberg equilibrium in extensive form Ellsberg games and finally, in Section 6.5, we compare Ellsberg equilibria to other extensive form solution concepts.

6.1 Extensive Form Ellsberg Games

We focus on dynamic games that can be represented in finite game trees. This implies a finite number n of players and a finite number of moves for each player. For simplicity, we do not allow any chance moves. We use the model of Osborne and Rubinstein (1994) for extensive form games with imperfect information, and extend it to allow for imprecise probabilistic devices.

Definition 6.1. *A finite* extensive form game *is a tuple* $(N, H, W, l, (\mathcal{I}_i), (u_i))$ *whose components are defined by:*

- *A finite set N of players i;*
- *A finite set H of sequences of actions $(a^k)_{k=1,\ldots,K}$ with $\emptyset \in H$, which represent the histories;*

[1] This chapter was published as IMW working paper Sass (2013).

- *A set $W \subset H$ of terminal histories; for each $h \in H \setminus W$, $A(h) := \{a \mid (h,a) \in H\}$ defines the set of actions available after history h;*

- *A player function $l : H \setminus W \to N$;*

- *For each player $i \in N$ an information partition $\mathcal{I}_i$ of $\{h \in H \mid l(h) = i\}$ with the property that $A(h) = A(h')$ whenever $h, h' \in I_i \in \mathcal{I}_i$. For $I_i \in \mathcal{I}_i$ we denote by $A(I_i)$ the set $A(h)$ and by $l(I_i)$ the player $l(h)$ for any $h \in I_i$.*

- *For each player $i \in N$ the preferences on lotteries over W can be represented by a Von Neumann and Morgenstern expected utility function u_i.*

The set of pure strategies of a player i is denoted S_i. The utility of a pure strategy profile $s \in S_1 \times \ldots \times S_n$ is simply $u_i(w)$, where w is the terminal history that results when the profile s is played. The set of mixed strategies P_i is denoted ΔS_i. We assume a product structure on $\Delta S := \Delta S_1 \times \ldots \times \Delta S_n$ and thus have stochastic independence of mixed strategies. Thus, if $P = (P_1, \ldots, P_n) \in \Delta S$ and $s \in S$, then

$$P(s) = P(s_1, \ldots, s_n) = \prod_{i=1}^{n} P_i(s_i) \text{ for all } s \in S\,.$$

The expected utility $u = (u_1, \ldots, u_n) : S \to \mathbb{R}^n$ of a mixed strategy profile P (we use the same notation as for the utility of a pure strategy) is $u_i(P) = \sum_{s \in S} P(s) u_i(s)$. A behavioral strategy of player i is a function $\theta_i = (\theta_i(I_i))_{I_i \in \mathcal{I}_i}$ with $\theta_i(I_i) \in \Delta A(I_i)$ that assigns to each information set I_i of player i a probability distribution over the set of actions available at I_i. The set of behavioral strategies of player i is denoted $\mathcal{O}_i$, the set of profiles $\theta = (\theta_1, ..., \theta_n)$ of behavioral strategies by $\mathcal{O} = \mathcal{O}_1 \times \ldots \times \mathcal{O}_n$.

We assume that in addition to classic randomizing devices, players can use imprecise probabilistic devices to choose among their pure strategies or among their available actions at each information set. Therefore, in addition to classical pure, mixed and behavioral strategies, we define Ellsberg strategies and Ellsberg behavioral strategies as convex and compact sets of mixed and behavioral strategies.

Definition 6.2. *An* Ellsberg strategy *of player i is a convex and compact set $\mathcal{P}_i$ of probability distributions $P_i : S_i \to \mathbb{R}_+$, such that $\sum_{s_i \in S_i} P_i(s_i) = 1$.*

Definition 6.3. *An* Ellsberg behavior strategy Θ_i *of player i is a function that assigns to each information set I_i of player i a convex and compact set of probability distributions θ_i over the set of actions available at I_i.*

A profile of Ellsberg behavior strategies is denoted $\Theta = (\Theta_1, \ldots, \Theta_n)$, and Θ is constituted of profiles of behavior strategies $\theta = (\theta_1, \ldots, \theta_n)$. For any history $h \in I_i \in \mathcal{I}_i$ and action $a \in A(h)$ we denote by $\theta_i(h)(a)$ the probability $\theta_i(I_i)(a)$ assigned by $\theta_i(I_i)$ to the action a.

For a profile $P = (P_1, ..., P_n)$ of mixed strategies, we define the outcome Π_P of P to be the probability distribution over the terminal histories that results when each player i follows the precepts of P_i. The probability that Π_P assigns to a terminal history w is $\prod_{i \in N} \pi_i(w)$ where $\pi_i(w)$ is the sum of the probabilities according to P_i of all the pure strategies of player i that are *consistent* (i.e. that result in the terminal history w) with w. The same way we define Π_θ for a profile of behavioral strategies. The probability that Π_θ assigns to $w = (a^1, ..., a^K)$ is $\prod_{k=0}^{K-1} \theta_{l(a^1,...,a^k)}(a^1, ..., a^k)(a^{k+1})$, where for $k = 0$ the history $(a^1, ..., a^k)$ is the initial history.

For a profile of Ellsberg strategies $\mathcal{P}$ or Ellsberg behavior strategies Θ, the outcomes $\Pi_\mathcal{P}$ and Π_Θ are sets of probabilities defined as

$$\Pi_\mathcal{P}(w) := \left\{ \Pi_P(w) \;\middle|\; P \in \mathcal{P} \right\}, \tag{6.1}$$

$$\Pi_\Theta(w) := \left\{ \Pi_\theta(w) \;\middle|\; \theta \in \Theta \right\}.$$

Finally we specify players' preferences over the new strategic devices. We assume that the players are ambiguity-averse according to Gajdos, Hayashi, Tallon, and Vergnaud (2008) in the special case $\phi = \text{id}$. The utility of both an Ellsberg and Ellsberg behavior strategy profile $\mathcal{P}$, Θ is then evaluated with a maxmin rule using the worst case probability distribution. We have stochastic independence of the Ellsberg strategies by assuming a product structure for every mixed strategy profile P contained in $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_n)$.[2] Thus, the utility of an Ellsberg strategy profile $\mathcal{P}$ for player i is

$$U_i(\mathcal{P}) = \min_{P \in \mathcal{P}} \sum_{w \in W} \Pi_P(w) u_i(w). \tag{6.2}$$

[2] In the definition of a normal form Ellsberg game we assume that the Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$ of all players $i \in N$ are stochastically independent. This is done by using product spaces as first suggested by Gilboa and Schmeidler (1989) (instead of Ellsberg urns they speak of "non-unique probability spaces" $(\Omega, \mathcal{F}, \mathcal{P})$, p. 150 therein). We define the product $(\Omega, \mathcal{F}, \mathcal{P})$ of n Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$, $i = 1, \ldots, n$, as follows: $\Omega := \Omega_1 \times \ldots \times \Omega_n$, $\mathcal{F} := \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, and $\mathcal{P}$ is the closed convex hull of the set of product measures, $\mathcal{P} := \bar{co}\{P_1 \otimes \ldots \otimes P_n \mid P_1 \in \mathcal{P}_1, \ldots, P_n \in \mathcal{P}_n\}$. This way the Ellsberg urns are stochastically independent. Different notions of stochastic independence in the context of ambiguity aversion have been discussed in the literature, see for example Klibanoff (2001), Bade (2011b) and Bade (2011a). In the present context of objective ambiguity in the form of Ellsberg urns the above notion seems the most natural.

The utility of an Ellsberg behavior strategy profile Θ for a player i is then

$$U_i(\Theta) = \min_{\theta \in \Theta} \sum_{w \in W} \Pi_\theta(w) u_i(w)\,. \tag{6.3}$$

With the ambiguity-averse preference representation at hand, we can now give the definition of an extensive form Ellsberg game.

Definition 6.4. *An n-player* extensive form Ellsberg game *is a tuple* $(N, H, W, l, (\mathcal{I}_i)(U_i))$ *of which the components satisfy the conditions in Definition 6.1, and* $U = (U_1, \dots, U_n)$ *is the ambiguity-averse preference representation in* (6.2) *and* (6.3).

By specifying in the definition how players evaluate Ellsberg strategies we allow players to use Ellsberg urns as imprecise probabilistic devices. Of course, it is still possible to play classic mixed and behavioral strategies. Then U_i reduces to the Von Neumann and Morgenstern expected utility u_i.

6.2 Rectangular Ellsberg Strategies and Dynamic Consistency

Dynamic consistency requires that preferences over outcomes at some point in the game will not be reversed or contradicted at a later point. A compelling feature of classic extensive form games lies in the fact that for a rational player it does not matter whether he plans his strategy in advance or executes it whenever one of his information sets is reached; this means that additional information arriving at some point in the game is irrelevant for the choice of the strategy. This equivalence is captured by Kuhn's Theorem which we discuss and extend to Ellsberg games in the next section. When players are expected utility maximizers, Kuhn's Theorem establishes a connection between conditional preferences at different points in the game tree via Bayes' updating rule: when players update in that way, their choices following the expected utility model are dynamically consistent.

Updating of ambiguity-averse preference representations and its relation to dynamic consistency has been extensively analyzed. Gilboa and Schmeidler (1993) propose two updating rules for multiple-prior expected utilities, full Bayesian updating and maximum likelihood updating, which are in general not dynamically consistent. Epstein and Schneider (2003b) characterize a particular set of priors which they call *rectangular*, for which full Bayesian updating is dynamically consistent. They show, more precisely, that multiple-prior utilities like those used in Ellsberg games are dynamically consistent if and only if each set of priors is rectangular and it is updated by Bayes' Rule applied prior by prior. Riedel (2004) uses rectangularity to prove dynamic consistency of dynamic coherent risk measures. Later Hanany and Klibanoff (2007) use a weaker notion of dynamic consistency to obtain a larger set of dynamically consistent updating rules for maxmin expected utilities.

We translate the property of rectangularity in Epstein and Schneider (2003b) to Ellsberg games. First we describe the information structure of an extensive form Ellsberg game (F, U) with a filtration $\{\mathcal{W}_t\}_0^T$ on a state space W. This is only possible, when the extensive form F satisfies perfect recall, otherwise for some t, $\mathcal{W}_t \nsubseteq \mathcal{W}_{t+1}$. Thus, we assume that F satisfies perfect recall.[3] Furthermore, we assume finite trees, $T < \infty$, throughout. The information structure of (F, U) is captured in the extensive form F in form of a rooted tree $\mathcal{T} = (W, N)$. The root W corresponds to time 0, and the sequence of decisions taken by the players defines the information available at time t. Let the set W of plays be the state space. Then, staying with Epstein and Schneider (2003b), we assume that $\mathcal{W}_0$ is trivial (that is, consists only of W and $\varnothing$) and that for each t, $\mathcal{W}_t$ is generated by the finite information partition at time t. Then $\mathcal{W}_t(w)$ denotes the partition component containing the play w. $\mathcal{W}_t(w)$ corresponds to some information set h, at which w is possible, along with a number of other plays. At $t+1$, the number of possible plays is narrowed down again. At time T, $\mathcal{W}_T(w)$ is the singleton set of play $\{w\}$ that has materialized.

Epstein and Schneider (2003b) consider lotteries over adapted consumption processes, and preferences over $\mathcal{W}_t$-measurable acts from W into such lotteries. Then, for simplicity, they assume full support (Axiom 5 therein): every non-empty event in $\mathcal{W}_T$ is considered possible at time 0. To apply the result by Epstein and Schneider (2003b) we assume the same condition. For the game (F, U) this implies that we only consider strategies with full support. This is without loss of generality. To allow for strategies without full support, we apply the construction to a suitable subset of plays. It is up to further research to determine whether this procedure has implications for the strategic analysis of an extensive form Ellsberg game.

In the description of the information structure $\{\mathcal{W}_t\}_0^T$, it is irrelevant, *who* takes the decision at some time t. The filtration only represents what is known to all players at some time t, therefore it does not define a game, but only an event tree. When we define a rectangular Ellsberg strategy profile $\mathcal{P}$, we therefore only impose restrictions on the set of induced distributions over plays: as long as the set of induced realization probabilities $\Pi_{\mathcal{P}}$ (see (6.1)) is rectangular, any Ellsberg strategy profile $\mathcal{P}$ which induces $\Pi_{\mathcal{P}}$ is admissible.

[3] For a precise definition of this notion in the notation introduced here see Ritzberger (2002), p. 124.

We specify this in Definition 6.5. For the extension of Kuhn's theorem to extensive form Ellsberg games this leaves some degrees of freedom for the choice of an Ellsberg strategy profile $\mathcal{P}$ given some set of realization probabilities $\Pi_{\mathcal{P}}$.

We have already found a way to describe Ellsberg strategy profiles, that is, "the big urn" over all pure strategies, in the setting of Epstein and Schneider (2003b): they are represented by the set of realization probabilities $\Pi_{\mathcal{P}}$. Now we see how Ellsberg behavior strategies are described in this setting. Define the set of $\mathcal{W}_t$-conditionals of measures π on $(W, \mathcal{W}_T)$ as

$$\Pi^t_{\mathcal{P}}(w)(\cdot) = \left\{\pi^t(w)(\cdot) := \pi(\cdot|\mathcal{W}_t)(w) \;\middle|\; \pi \in \Pi_{\mathcal{P}}\right\},$$

this is the set of Bayesian updates at a time t. The set of conditional one-step-ahead measures is defined by

$$\Pi^t_{\mathcal{P},+1}(w) = \left\{\pi^t_{+1}(w) \;\middle|\; \pi \in \Pi_{\mathcal{P}}\right\},$$

where π^t_{+1} is the restriction of π^t to $\mathcal{W}_{t+1}$. The sets $\Pi^t_{\mathcal{P}}$ and $\Pi^t_{\mathcal{P},+1}$ can be viewed as realizations of $\mathcal{W}_t$-measurable correspondences into $\Delta(W, \mathcal{W}_T)$ and $\Delta(W, \mathcal{W}_{t+1})$, respectively. The name "conditional one-step-ahead measures" has the following intuition. Each measure in $\Pi^t_{\mathcal{P},+1}(w)$ is a measure on $\mathcal{W}_{t+1}$, thus one can think of $\Pi^t_{\mathcal{P},+1}(w)$ as the set of measures describing beliefs about the "next step": the belief at time t about what will happen at time $t+1$. Think of it as cutting the tree at time t and time $t+1$, then one is only left with distributions over the choices at the information sets at time t, and the distributions are induced by $\Pi_{\mathcal{P}}$. From this explanation it is clear that the conditional one-step-ahead measures are the induced Ellsberg behavior strategies. Epstein and Schneider (2003b) explain further that from any set of one-step-ahead conditionals, e.g. some set Π^t_{+1}, a rectangular set Π can be constructed with backward construction via

$$\Pi = \{\pi \in \Delta(W, \mathcal{W}_t) \mid \pi^t_{+1} \in \Pi^t_{+1}(w) \text{ for all } t \text{ and } w\}.$$

We see that Π is the set of all measures π whose one-step ahead conditionals conform with the Π^t_{+1}.

When is an Ellsberg strategy dynamically consistent (rectangular)? The Bayesian theory says, a probability distribution π on $(W, \mathcal{W}_T)$ can be for every t decomposed into its conditionals and marginals in the form

$$\pi^t(w) = \sum_{w' \in W} \pi^{t+1}(w') \cdot \pi^t_{+1}(w). \tag{6.4}$$

The set $\Pi_{\mathcal{P}}$ of probability distributions (and thus $\mathcal{P}$) on $(W, \mathcal{W}_T)$ is rectangular if it admits a corresponding decomposition. Details are explained in the proof of Theorem 6.7 and in Epstein and Schneider (2003b). First, we define rectangularity precisely.

Definition 6.5. *An Ellsberg strategy profile $\mathcal{P}$ in an extensive form Ellsberg game (F, U) is* rectangular*, if the set of realization probabilities $\Pi_{\mathcal{P}}$ is rectangular in the following sense. Let the information structure of the game (F, U) be described by the filtration $\{\mathcal{W}_t\}_0^T$. Then $\Pi_{\mathcal{P}}$ is $\{\mathcal{W}_t\}$-rectangular, if for all w and all t*

$$\Pi_{\mathcal{P}}^t(w) = \left\{ \sum_{w' \in W} \pi^{t+1}(w') \cdot \pi_{+1}^t(w) \;\middle|\; \pi^{t+1}(w') \in \Pi_{\mathcal{P}}^{t+1}(w') \text{ for all } w' \in W, \pi_{+1}^t(w) \in \Pi_{\mathcal{P},+1}^t(w) \right\}. \tag{6.5}$$

An Ellsberg strategy $\mathcal{P}_i$ for player i is rectangular if it is part of a rectangular Ellsberg strategy profile $\mathcal{P}$. When necessary, we denote the restriction of a set X to its rectangular subset with X^R.

Remark 6.6. *Definition 6.5 also applies to the set of realization probabilities Π_Θ. Π_Θ is rectangular, if it has an analog decomposition as in* (6.5)*. In difference to Definition 6.5 this is not a property of the Ellsberg behavior strategy profile Θ.*

Observe that the inclusion $\subset$ is always satisfied by applying (6.4) for every $\pi \in \Pi_{\mathcal{P}}$. In order to assure the inclusion in the other direction, $\Pi_{\mathcal{P}}$ is required to have the special "rectangular" form. Combinations of conditionals and marginals that arise from *different* $\pi, \pi' \in \Pi_{\mathcal{P}}$ have to lie in $\Pi_{\mathcal{P}}^t$. In Example 6.9 in the following section, we explain the geometric representation of a rectangular Ellsberg strategy. Epstein and Schneider (2003b) point out some features of rectangularity. One of these is particularly important in the setting of extensive form Ellsberg games: rectangularity imposes no restrictions on one-step-ahead conditionals. This means that players can use any Ellsberg behavior strategy they wish. Only when they want to represent this Ellsberg behavior strategy by an outcome-equivalent Ellsberg strategy they have to use the induced rectangular Ellsberg strategy. We show this in detail in the following section.

6.3 Kuhn's Theorem for Extensive Form Ellsberg Games

Kuhn (1953) proved that an extensive form satisfies perfect recall if and only if, for every probability distribution on plays that is induced by some mixed strategy profile, there is a behavior strategy profile that induces the same distribution, and for every probability distribution on plays that is induced by some behavior strategy profile, there is a mixed strategy profile that induces the same distribution. Under expected utility this implies that the strategies yield the same utility.

In this section we assume, as it is assumed in most applications, that the extensive form $F = (\mathcal{T}, C)$ satisfies perfect recall. We then show that under this assumption and the condition that the Ellsberg strategies are rectangular we get an equivalent result of Kuhn's theorem for extensive form Ellsberg games, namely:

Theorem 6.7. *In an extensive form Ellsberg game (F, U) with $F = (\mathcal{T}, C)$ satisfying perfect recall, every rectangular Ellsberg strategy profile $\mathcal{P}$ induces an Ellsberg behavior strategy profile $\Theta^{\mathcal{P}}$ via prior-by-prior updating; every Ellsberg behavior strategy profile Θ induces a rectangular Ellsberg strategy profile $\mathcal{P}^{\Theta}$ such that prior-by-prior updating of $\mathcal{P}^{\Theta}$ yields Θ. The induced strategy profiles are payoff-equivalent, i.e.,*

$$U_i(\mathcal{P}) = U_i(\Theta^{\mathcal{P}}) \text{ and } U_i(\Theta) = U_i(\mathcal{P}^{\Theta}).$$

Proof. We construct the induced strategies as follows. Note that we only consider strategies with full support according to Axiom 5 in Epstein and Schneider (2003b). This is without loss of generality. To allow for strategies without full support, we apply the construction to a suitable subset of plays.

A rectangular Ellsberg strategy profile $\mathcal{P}$ has by definition a rectangular set of realization probabilities $\Pi_{\mathcal{P}}$. Due to rectangularity, $\Pi_{\mathcal{P}}$ possesses at any point in time t a decomposition into marginals and one-step-ahead conditionals according to (6.5). The set of conditional one-step-ahead measures at t, $\Pi^t_{\mathcal{P},+1}$, defines the induced Ellsberg behavior strategy profile $\Theta^{\mathcal{P}}$ by setting

$$\Theta_i^{\mathcal{P},\alpha} := \Pi^t_{\mathcal{P},+1}, \tag{6.6}$$

when player i is at information set α at time t. The construction is graphically captured in the diagram in Figure 6.1.

The construction of an induced rectangular Ellsberg strategy profile $\mathcal{P}^{\Theta}$ is similar. Any Ellsberg behavior strategy profile Θ has a set of realization probabilities Π_{Θ}. This set can be restricted to its rectangular subset Π^R_{Θ} by requiring that the conditional one-step-ahead measures are again the Ellsberg behavior strategy profile Θ in the sense that

$$\Pi^{R,t}_{\Theta,+1} = \Theta_i^{\alpha} \tag{6.7}$$

when player i is at information set α at time t (dashed line in Figure 6.2). Then we

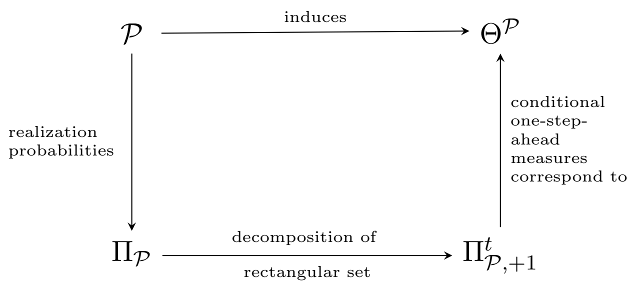

Figure 6.1: Rectangular $\mathcal{P}$ induces Ellsberg behavior strategy $\Theta^{\mathcal{P}}$.

define $\Pi^R_{\mathcal{P}} := \Pi^R_{\Theta}$. $\Pi^R_{\mathcal{P}}$ is uniquely determined by the process of conditional one-step-ahead correspondences $\Pi^{R,t}_{\Theta,+1}$. The rectangular set is the set of realization probabilities of the induced rectangular Ellsberg strategy profile $\mathcal{P}^{\Theta}$. The construction is graphically captured in the diagram in Figure 6.2.

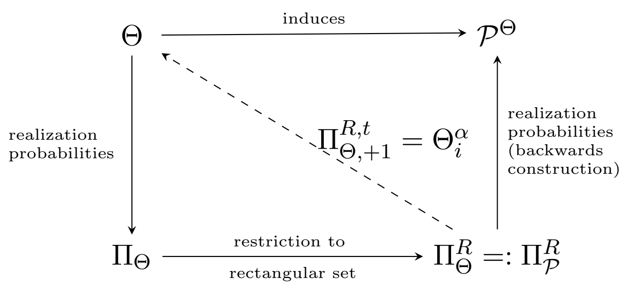

Figure 6.2: Θ induces rectangular Ellsberg strategy $\mathcal{P}^{\Theta}$.

It remains to show that $\mathcal{P}$ and $\Theta^{\mathcal{P}}$, as well as Θ and $\mathcal{P}^{\Theta}$, yield the same minimal expected utility.

Recall that

$$U_i(\mathcal{P}) = \min_{P \in \mathcal{P}} P(s)u_i(s)\,.$$

This is equal to the minimal expected utility of the realization probabilities of $\mathcal{P}$,

$$U_i(\mathcal{P}) = U_i(\Pi_{\mathcal{P}}) = \min_{\pi_P \in \Pi_{\mathcal{P}}} \sum_{w \in W} \pi_P(w) u_i(w)\,,$$

where $\pi_P(w) = \sum_{s \in S, w \in s} P(s)$. Now consider the minimal expected utility of $\Theta^{\mathcal{P}}$, with $\pi_\theta(w) = \prod_{i=1}^n \theta_i(c_i)$, where $(c_i)_{i=1}^n$ is the unique sequence of choices that yield w.

$$\begin{aligned} U_i(\Theta^{\mathcal{P}}) = U_i(\Pi_{\Theta^{\mathcal{P}}}) &= \min_{\pi_\theta \in \Pi_{\Theta^{\mathcal{P}}}} \sum_{w \in W} \pi_\theta(w) u_i(w) \\ &= \min_{\theta \in \Theta^{\mathcal{P}}} \sum_{w \in W} \left(\prod_{i=1}^n \theta_i(c_i) \right) u_i(w) && (6.8) \\ &= \min_{\pi_{+1}^t \in \Pi_{\mathcal{P},+1}^t} \sum_{w \in W} \left(\prod_{t=0}^T \pi_{+1}^t(w) \right) u_i(w) && (6.9) \\ &= \min_{\pi_P \in \Pi_{\mathcal{P}}} \sum_{w \in W} \pi_P(w) u_i(w) && (6.10) \\ &= U_i(\Pi_{\mathcal{P}}) = U_i(\mathcal{P})\,. \end{aligned}$$

From (6.8) to (6.9) we get by definition of θ_i in the construction of the proof, see (6.6). We can replace $\prod_{i=1}^n \theta_i(c_i)$ by $\prod_{t=0}^T \pi_{+1}^t(w)$, where $(c_i)_{i=1}^n$ is the unique sequence of choices which yields w. The equality of (6.9) and (6.10) results from the rectangularity of $\mathcal{P}$. Because of rectangularity, $\prod_{t=1}^T \pi_{+1}^t(w)$ is an element of $\Pi_{\mathcal{P}}$, that is, $\prod_{t=0}^T \pi_{+1}^t(w) = \pi_P(w)$.

Likewise, we show the equality of $U_i(\Theta)$ and $U_i(\mathcal{P}^\Theta)$. We use (6.7) for the equality of (6.11) and (6.12), and rectangularity in the equality of (6.12) and (6.13).

$$\begin{aligned} U_i(\Theta) = U_i(\Pi_\Theta) &= \min_{\pi_\theta \in \Pi_\Theta} \sum_{w \in W} \pi_\theta(w) u_i(w) \\ &= \min_{\theta \in \Theta} \sum_{w \in W} \left(\prod_{i=1}^n \theta_i(c_i) \right) u_i(w) && (6.11) \\ &= \min_{\pi_{+1}^t \in \pi_{\Theta,+1}^{R,t}} \sum_{w \in W} \left(\prod_{t=0}^T \pi_{+1}^t(w) \right) u_i(w) && (6.12) \\ &= \min_{\pi_P \in \Pi_{\mathcal{P}^\Theta}} \sum_{w \in W} \pi_P(w) u_i(w) && (6.13) \\ &= U_i(\Pi_{\mathcal{P}^\Theta}) = U_i(\mathcal{P}^\Theta)\,. \end{aligned}$$

□

Remark 6.8. *Note that the induced Ellsberg behavior strategy profile $\Theta^{\mathcal{P}}$ is unique, but the induced Ellsberg strategy profile $\mathcal{P}^{\Theta}$ is not. This is because the construction of $\mathcal{P}^{\Theta}$ from the set of realization probabilities $\Pi^R_{\mathcal{P}}$ is in general not unique.*

Theorem 6.7 has the following interpretation. For every Ellsberg behavior strategy profile there exists an Ellsberg strategy profile that yields the same utility for all players. This induced Ellsberg strategy profile has to be chosen rectangular if we want it to be dynamically consistent and equivalent to the original Ellsberg behavior strategy. Note that this does not impose any restriction on the choice of the Ellsberg behavior strategy profile (this is also pointed out in Epstein and Schneider (2003b), p. 10), but only allows a restricted set of Ellsberg strategy profiles. Conversely, for every rectangular Ellsberg strategy, there exists an equivalent Ellsberg behavior strategy.

To gain some intuition about the nature of rectangular Ellsberg strategies and the inductions explained in the proof of Theorem 6.7, we now discuss a most simple example.

Example 6.9. *Consider the following example taken from Osborne and Rubinstein (1994), p. 93. We believe this to be the simplest two-player information structure with which we can illustrate the equivalence of rectangular Ellsberg strategies and Ellsberg behavior strategies. The game is presented in Figure 6.3.*

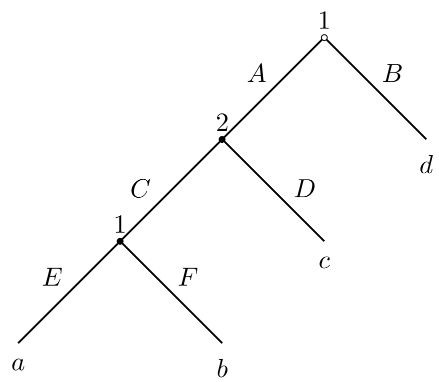

Figure 6.3: Game where player 1 moves before and after player 2.

We assume that $a, b, c, d \in \mathbb{R}$ are payoffs for player 1 with $d < a = b$. To keep things as simple as possible, we let player 2 choose C at all times, thus c will never be reached. Player 1 has four pure strategies: (A, E), (A, F), (B, E) and (B, F), we denote the set of pure strategies of player 1 by S_1. Player 2 only has C and D as pure strategies.

First, we start with Ellsberg behavior strategies $\Theta_1 = (\Theta_1^0, \Theta_1^1)$ *and* Θ_2 *that specify a set of probability distributions at every information set of player 1 and 2, respectively. Suppose player 1 plays*

$$\Theta_1^0 = \left\{(\theta_1^0, 1-\theta_1^0) \mid \theta_1^0 \in [0,1],\ 1/4 \leq \theta_1^0 \leq 3/4\right\},$$
$$\Theta_1^1 = \left\{(\theta_1^1, 1-\theta_1^1) \mid \theta_1^1 \in [0,1],\ 1/2 \leq \theta_1^1 \leq 2/3\right\},$$

with Θ_1^0 *the set of distributions over* A *and* B*, and* Θ_1^1 *the set of distributions over* E *and* F*. For the purpose of the example, let player 2 have an Ellsberg behavior strategy* Θ_2 *that chooses* C *with probability* 1*, without creating any ambiguity.*
The set of plays W *is then given by* $W = \{E, F, B\}$ *with respect to the outcomes* a, b *and* d*. We derive the set of realization probabilities* Π_Θ *corresponding to the Ellsberg behavior strategy* Θ*,*

$$\Pi_\Theta(E) = \{\theta_1^0 \cdot \theta_1^1 \mid \theta_1^0 \in \Theta_1^0, \theta_1^1 \in \Theta_1^1\},$$
$$\Pi_\Theta(F) = \{\theta_1^0 \cdot (1-\theta_1^1) \mid \theta_1^0 \in \Theta_1^0, \theta_1^1 \in \Theta_1^1\},$$
$$\Pi_\Theta(B) = \{1-\theta_1^0 \mid \theta_1^0 \in \Theta_1^0\}.$$

Now we can calculate the maxmin expected utility for player 1 of the Ellsberg behavior strategy Θ*.*

$$\begin{aligned} U_1(\Theta) = U_1(\Pi_\Theta) &= \min_{\pi_\theta \in \Pi_\Theta} \sum_{w \in W} \pi_\theta(w) u_i(w) \\ &= \min_{\pi_\theta \in \Pi_\Theta} \pi_\theta(E) \cdot a + \pi_\theta(F) \cdot b + \pi_\theta(B) \cdot d \\ &= \min_{\theta \in \Theta} \theta_1^0 \cdot \theta_1^1 \cdot a + \theta_1^0 \cdot (1-\theta_1^1) \cdot b + (1-\theta_1^0) \cdot d \\ &= 1/4 \cdot 1/2 \cdot a + 1/4 \cdot 1/2 \cdot b + 3/4 \cdot d \\ &= 1/8 \cdot a + 1/8 \cdot b + 3/4 \cdot d\,. \end{aligned} \tag{6.14}$$

What is the rectangular Ellsberg strategy profile $\mathcal{P}^\Theta$ *induced by the Ellsberg behavior strategy profile* Θ*? We follow the construction in Figure 6.2.*
The set of induced realization probabilities Π_Θ *is a hexagon (colored green in Figure 6.4), given by the surface that lies in the intersection of* $\Pi_\Theta(E) = [1/8, 1/2]$*,* $\Pi_\Theta(F) = [1/12, 3/8]$ *and* $\Pi_\Theta(B) = [1/4, 3/4]$*. It can be easily seen that* Π_Θ *is too large in the sense that it induces conditional probabilities over* E *and* F *which lie outside the set* Θ_1^1 *of original Ellsberg behavior strategies. Take for example the distribution* $(1/8, 3/8, 1/2)$ *(probability*

distribution over E, F, B) at the lower right corner of the green hexagon. This distribution yields a conditional one-step-ahead probability for E which is equal to $\frac{1/8}{1/2} = 1/4$, and $1/4$ does not lie in the set $\Theta_1^1(E)$. Hence we have to restrict Π_Θ to its rectangular subset Π_Θ^R, this is exactly the set that yields the correct conditionals.

To apply the construction of Epstein and Schneider (2003b) we represent the game in Figure 6.9 by a state space $W = \{E, F, B\}$ (the set of plays) and a filtration $\{\mathcal{W}_t\}_0^2$ where

$$\begin{aligned}\mathcal{W}_0 &= \{\{E, F, B\}\}\,,\\ \mathcal{W}_1 &= \{\{E, F\}, \{B\}\}\,,\\ \mathcal{W}_2 &= \{\{E\}, \{F\}, \{B\}\}\,.\end{aligned}$$

$\Pi_\Theta^R \subset \Pi_\Theta$ is the set of distributions on $(W, \mathcal{W}_2)$ that for all $t \in \{0, 1, 2\}$ and all $w \in W$ admits a decomposition

$$\begin{aligned}\Pi_\Theta^{R,t}(w) = \Bigg\{ \sum_{w' \in W} \pi^{t+1}(w') \cdot \pi_{+1}^t(w) \;\Bigg|\\ \pi^{t+1}(w') \in \Pi_\Theta^{t+1}(w') \text{ for all } w' \in W, \pi_{+1}^t(w) \in \Pi_{\Theta,+1}^t(w) \Bigg\}\,.\end{aligned} \tag{6.15}$$

The set Π_Θ^R excludes exactly all those distributions in Π_Θ that yield conditionals outside Θ_1^1. The rectangular set Π_Θ^R is depicted as the blue rectangle in Figure 6.4.

From the construction of $\Pi_\Theta^{R,t}$, it follows that

$$\begin{aligned}\Pi_{\Theta,+1}^{R,0} &= \Theta_1^0\,,\\ \Pi_{\Theta,+1}^{R,1} &= \Theta_1^1\,,\end{aligned}$$

that is, the induced conditional one-step-ahead measures are exactly the components of the Ellsberg behavior strategy which we started with. The set Π_Θ^R can also be constructed recursively as the set of all distributions $\pi \in \Delta(W, \mathcal{W}_T)$ for which $\pi_{+1}^t \in \Pi_{\Theta,+1}^{R,t}(w)$ for all t and all w, see Epstein and Schneider (2003b) p. 8 for details. Hence, to find the induced Ellsberg strategy $\mathcal{P}^\Theta$ we first define

$$\Pi_{\mathcal{P}}^R := \Pi_\Theta^R$$

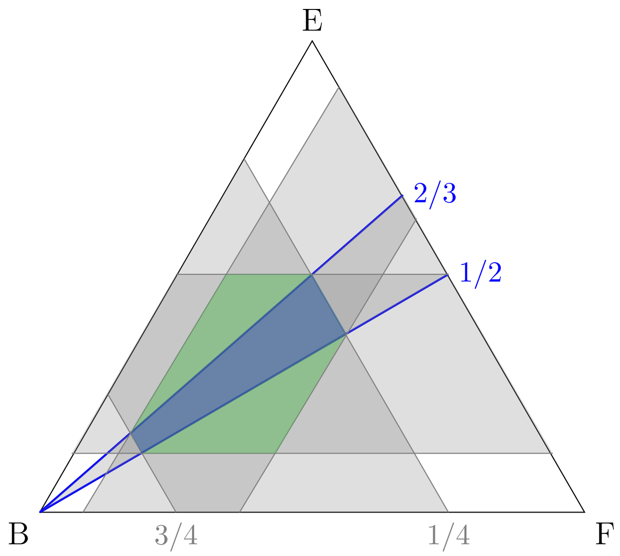

Figure 6.4: Set of realization probabilities Π_Θ (green hexagon), rectangular set of probabilities Π^R_Θ (blue rectangle). The grey numbers indicate the set of probabilities for B, the blue numbers the set of conditional one-step-ahead probabilities for E.

as in Figure 6.2. Then the rectangular set of realization probabilities $\Pi^R_{\mathcal{P}}$ *is*

$$\Pi^R_{\mathcal{P}} = \Big\{\pi = (\pi_1, \pi_2, \pi_3) \in \Delta(W, \mathcal{W}_T) \,\Big|\, \pi_1/(1-\pi_3) \in [1/2, 2/3] = \Pi^{R,1}_{\mathcal{P},+1}(E)\,,$$

$$\pi_2/(1-\pi_3) \in [1/3, 1/2] = \Pi^{R,1}_{\mathcal{P},+1}(F), \pi_3 \in [1/4, 3/4] = \Pi^{R,0}_{\mathcal{P},+1}(B)\Big\}$$

$$= \big\{\pi \in \Delta(W, \mathcal{W}_T) \,\big|\, \pi_3 \in [1/4, 3/4], \pi_1 \in [-\pi_3/2 + 1/2, -2\pi_3/3 + 2/3]\big\}\,. \qquad (6.16)$$

The notation in (6.16) *is derived as follows. Every point in the simplex is fully described by probabilities for* B *and* E. *The two lines passing through* $B = 1$ *and* $E = 2/3$, $B = 1$

and $E = 1/2$ are functions of B and E with equations

$$\pi_1 = \pi_3/2 + 1/2\,,$$

$$\pi_1 = 2\pi_3/3 + 2/3\,.$$

Now, finally, we construct $\mathcal{P}^{\Theta}$ from the rectangular set of realization probabilities $\Pi^R_{\mathcal{P}}$. We find

$$\mathcal{P}^{\Theta}_1 = \Big\{ P \in \Delta S_1 \mid P(A,E) = \pi_1, P(A,F) = \pi_2, P(B,E) = \frac{\pi_1}{\pi_1+\pi_2}\pi_3, \\ P(B,F) = \frac{\pi_2}{\pi_1+\pi_2}\pi_3, \text{ for all } (\pi_1,\pi_2,\pi_3) \in \Pi^R_{\mathcal{P}} \Big\}\,.$$

We calculate the maxmin expected utility of player 1 of the Ellsberg strategy profile $\mathcal{P}^{\Theta}$.

$$\begin{aligned} U_1(\mathcal{P}^{\Theta}) &= \min_{P\in\mathcal{P}} \sum_{s\in S} P(s)u_i(s) \\ &= \min_{P\in\mathcal{P}} P(A,E)\cdot a + P(A,F)\cdot b + (P(B,E)+P(B,F))\cdot d \\ &= 1/8\cdot a + 1/8\cdot b + (3/8+3/8)\cdot d \\ &= 1/8\cdot a + 1/8\cdot b + 3/4\cdot d\,. \end{aligned} \tag{6.17}$$

Player 1 uses $P = (1/8, 1/8, 3/8, 3/8)$ as his worst case probability distribution, because it puts the greatest available probability to the worst outcome d. We thus find that the Ellsberg strategy profile $\mathcal{P}^{\Theta}$ induced by Θ yields the same maxmin expected utility as Θ:

$$U_1(\mathcal{P}^{\Theta}) \overset{(6.17)}{=} 1/8\cdot a + 1/8\cdot b + 3/4\cdot d \overset{(6.14)}{=} U_1(\Theta)\,.$$

Furthermore, doing the reasoning backwards, one sees directly that with the rectangular construction of $\Pi^R_{\mathcal{P}} = \Pi^R_{\Theta}$ in (6.15), the Ellsberg strategy yields the correct one-step-ahead conditionals for A, B and E, F.

Now, we start out from the rectangular Ellsberg strategy profile $\mathcal{P}$ (the blue rectangle in Figure 6.4) and construct the induced Ellsberg behavior strategy profile $\Theta^{\mathcal{P}}$. We proceed as sketched in the diagram in Figure 6.1. The Ellsberg strategy profile has a set of realization probabilities $\Pi_{\mathcal{P}}$ which are by definition also rectangular. Then from Definition 6.5, $\Pi_{\mathcal{P}}$ possesses a decomposition of $\Pi^t_{\mathcal{P}}$ for all t into marginals and one-step-ahead conditionals. The rectangular set $\Pi_{\mathcal{P}}$ is constructed in the way to get exactly $\Pi^0_{\mathcal{P},+1} = \Theta^{\mathcal{P},0}_1$ and $\Pi^1_{\mathcal{P},+1} =$

$\Theta_1^{\mathcal{P},1}$ *as one-step-ahead conditionals, and hence*

$$\Theta_1^{\mathcal{P},0} = \left\{(\theta_1^0, 1-\theta_1^0) \mid \theta_1^0 \in [0,1], 1/4 \le \theta_1^0 \le 3/4\right\},$$
$$\Theta_1^{\mathcal{P},1} = \left\{(\theta_1^1, 1-\theta_1^1) \mid \theta_1^1 \in [0,1], 1/2 \le \theta_1^1 \le 2/3\right\}.$$

Obviously we have

$$U_1(\mathcal{P}) = U_1(\mathcal{P}^\Theta) \overset{(6.17)}{=} 1/8 \cdot a + 1/8 \cdot b + 3/4 \cdot d \overset{(6.14)}{=} U_1(\Theta) = U_1(\Theta^\mathcal{P}).$$

6.4 Ellsberg Equilibrium in Extensive Form Ellsberg Games

An Ellsberg equilibrium in an extensive form Ellsberg game is defined straightforwardly as in the static case. A profile of Ellsberg strategies is an Ellsberg equilibrium profile, if no player finds it profitable to deviate unilaterally.

Definition 6.10. *An* Ellsberg equilibrium *of an extensive form Ellsberg game* (F, U) *is a profile* $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$ *of Ellsberg strategies such that for all* $i = 1, \dots, n$ *and every* $\mathcal{P}_i \subseteq \Delta S_i$

$$U_i(\mathcal{P}_i^*, \mathcal{P}_{-i}^*) \ge U_i(\mathcal{P}_i, \mathcal{P}_{-i}^*).$$

An *Ellsberg equilibrium in Ellsberg behavior strategies* is defined analogously. Given Theorem 6.7, the two definitions are equivalent for extensive form Ellsberg games with perfect recall.

Corollary 6.11 (of Theorem 6.7)**.** *If a rectangular Ellsberg strategy profile* $\mathcal{P}$ *is an Ellsberg equilibrium of an extensive form Ellsberg game* $(\mathcal{F}, U)$*, then the Ellsberg behavior strategy profile* $\Theta^\mathcal{P}$ *is an Ellsberg equilibrium in Ellsberg behavior strategies. If an Ellsberg behavior strategy profile* Θ *is an Ellsberg equilibrium in Ellsberg behavior strategies, then the rectangular Ellsberg strategy profile* $\Theta^\mathcal{P}$ *is an Ellsberg equilibrium.*

Every Nash equilibrium is an Ellsberg equilibrium. Conversely, every Ellsberg equilibrium in which $\mathcal{P}$ is a single probability distribution is a Nash equilibrium.

6.5 Relation to Other Extensive Form Solution Concepts

We discuss the relation of Ellsberg equilibrium to other extensive form solution concepts with an example which Fudenberg and Kreps (1988) used to show that mistakes about play off the equilibrium path can lead to non-Nash outcomes. Subsequently this "horse"-like game[4] has been used as an example in a number of papers (Battigalli, Gilli, and Molinari (1992), Fudenberg and Levine (1993), Rubinstein and Wolinsky (1994), Groes, Jacobsen, Sloth, and Tranaes (1998), Lo (1999)). The information structure is the same as in Greenberg (2000)'s peace negotiation example, and also the incentive structure is very similar. We are in a situation, where ambiguity can be used as a threat against deviation.[5]

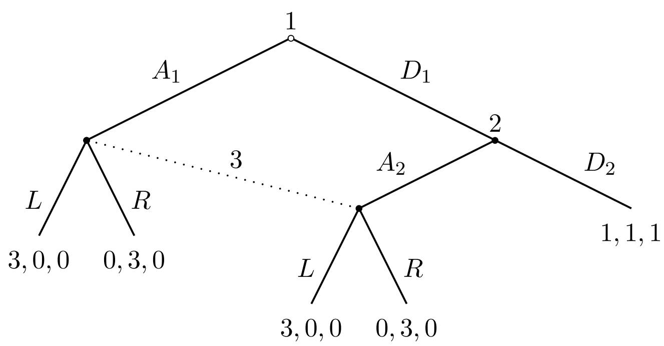

Figure 6.5: Fudenberg and Kreps' three-player game.

Proposition 6.12. *The proper Ellsberg equilibria of Fudenberg and Kreps' Three-Player game are of the form* $(A_1, A_2, [R_0, R_1])$ *where* $R \in [R_0, R_1]$ *is the probability that player 3 plays L, and* $R_0 < 1/3$ *and* $R_1 > 2/3$.

Proof. Let P, Q denote the probability that player 1 plays A_1, player 2 plays A_2, respectively. Then we can calculate the minimal expected utility for each player.

$$\begin{aligned} & U_1(P, Q, [R_0, R_1]) \\ & \quad = \min_{R_0 \leq R \leq R_1} PQR + 3P(1-Q)R + 3(1-P)QR + 3(1-P)(1-Q)R + PQ(1-R) \\ & \quad = \min_{R_0 \leq R \leq R_1} R(-3PQ + 3) + PQ \\ & \quad = R_0(-3PQ + 3) + PQ \text{ for all } P, Q \in [0,1]\,. \end{aligned}$$

[4] Selten (1975) is first to analyze this "horse"-like information structure. In his numerical example, however, the possibility to use Ellsberg strategies does not lead to Ellsberg equilibria outside the Nash equilibrium support, because the incentive structure is such that players 1 and 2 are not in opposition to each other.

[5] Kelsey and Spanjers (2004) observe that "Ambiguity can make threats more effective." In the incentive structure of the games considered here, this observation is confirmed.

$$U_2(P, Q, [R_0, R_1]) = \min_{R_0 \leq R \leq R_1} R(3PQ - 3) - 2PQ + 3$$
$$= R_1(3PQ - 3) - 2PQ + 3 \text{ for all } P, Q \in [0, 1]\,.$$

$$U_3(P, Q, [R_0, R_1]) = \max_{R_0 \leq R \leq R_1} PQR + PQ(1 - R)$$
$$= PQ\,.$$

Player 1 gets a payoff less than 1 if player 1 or 2 defect, when

$$R_0(-3PQ + 3) + PQ < 1$$
$$\Leftrightarrow \; R_0 < 1/3\,.$$

Player 2 gets a payoff less than 1 if player 1 or 2 defect, when

$$R_1(3PQ - 3) - 2PQ + 3 < 1$$
$$\Leftrightarrow \; R_1 > 2/3\,.$$

This yields the equilibrium $(A_1, A_2, [R_0, R_1])$ where $R_0 < 1/3$ and $R_1 > 2/3$. □

The information structure of the game has been characterized by Fudenberg and Levine (1993) in the analysis of their concept of Self-confirming equilibrium. They define a property of *unobserved deviators* which captures the fact that a player does not observe who of his opponents leaves the equilibrium path at some point in the game. These games have a natural property to allow for "wrong" beliefs on plays *off* the equilibrium path. Therefore games with unobserved deviators can have Self-confirming or Conjectural (Battigalli (1987), Battigalli and Guaitoli (1988), Battigalli and Guaitoli (1997)) equilibria which are not Nash equilibria. Fudenberg and Levine (1993) show that games that have only *observed* deviators do not have non-Nash equilibria. We suppose that a similar characterization holds for Ellsberg equilibria as well. This is an interesting question for further research.

We briefly present other extensive form solution concepts with Knightian uncertainty. Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2012) define Self-confirming equilibrium with model uncertainty. Their concept also incorporates Knightian uncertainty into extensive form games, but is quite different from our extensive form Ellsberg games. The differences lie especially in the fact that players may only play pure strategies in Self-confirming equilibrium with model uncertainty, and that uncertainty is present in the environment and not in the strategies. In Aryal and Stauber (2013) players, anticipating possible small mistakes (trembles), can have ambiguous beliefs about their opponents' strategies. A further extensive form solution concept with Knightian uncertainty is Multiple Priors Nash equilibrium by Lo (1999). Lo allows beliefs to be represented by multiple priors and demands every distribution in the sets of beliefs to be a best response to the beliefs on the other players. Players only have pure strategies at their disposition. Lo (1999) also looks at the example in Figure 6.5 and shows that the path (D_1, D_2) can be supported by a Multiple Priors Nash equilibrium, that is, that there exist sets of beliefs such that playing (D_1, D_2) is optimal. However, this equilibrium in beliefs is entirely subjective and in that aspect differs fundamentally from Ellsberg equilibrium. In the same way, Nash equilibrium with lower probabilities in Groes, Jacobsen, Sloth, and Tranaes (1998) is distinguished from Ellsberg equilibrium. The Ellsberg equilibrium outcomes in Fudenberg and Kreps' game can also be achieved with subjective equilibrium by Kalai and Lehrer (1995), but, again, the fundamentals of the two concepts differ.

σ^*- equilibrium by Ma (2000) is closest to our approach. In his model, players use ambiguous plans modeled by sets of probability distributions over acts. The author also mentions the possibility to *create* Knightian uncertainty. In difference to our approach he relies entirely on subjective preference representations and focuses on belief systems and thus on equilibrium in beliefs. An interesting aspect of his paper is that it models the possibility to create ambiguity with pre-play communication.

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Summary

In this thesis I propose a framework for normal and extensive form games where players can use Knightian uncertainty strategically. In such *Ellsberg games*, ambiguity-averse players may render their actions objectively ambiguous by using devices such as Ellsberg urns, in addition to the standard mixed strategies. This simple change in the foundations leads to a number of interesting phenomena.

While Nash equilibria remain equilibria in the extended game, there arise new *Ellsberg equilibria* with distinct outcomes. This happens especially in games with an information structure in which a player has the possibility to threaten his opponents. I illustrate this with the example of a negotiation game with three players. This mediated peace negotiation does not have a Nash equilibrium with peace outcome, but does have a peace equilibrium when ambiguity is a possible strategy. That a game with more than two players can have interesting non-Nash Ellsberg equilibria is traced back to results on subjective equilibria.

Ellsberg equilibria are mathematically characterized by the Principle of Indifference in Distributions. In an Ellsberg equilibrium, players are indifferent between all mixed strategies contained in the Ellsberg equilibrium strategy. Furthermore, I observe that in two-player games players can immunize against strategic ambiguity by playing their maximin strategy (if a completely mixed Nash equilibrium exists).

I analyze Ellsberg equilibria in two-person games with common and conflicting interests. I provide a number of examples and general results how to determine the Ellsberg equilibria of these games. The equilibria of conflicting interest games (modified Matching Pennies) turn out to be consistent with experimental deviations from Nash equilibrium play.

Finally, I define extensive form Ellsberg games. Under the assumption of dynamically consistent (rectangular) Ellsberg strategies, I prove a result analog to Kuhn's theorem: rectangular Ellsberg strategies and Ellsberg behavior strategies are equivalent.

Keywords Knightian Uncertainty in Games, Strategic Ambiguity, Ellsberg Games, Extensive Form Ellsberg Games, Kuhn's Theorem

Résumé

Dans cette thèse, je propose un cadre d'analyse permettant d'étudier les jeux sous forme normale et les jeux sous forme extensive dans lesquels les joueurs peuvent utiliser l'incertitude Knightienne de manière stratégique. Dans ces jeux, appelés *jeux d'Ellsberg*, les joueurs adverses à l'ambiguïté ont la possibilité de rendre leurs actions objectivement ambiguës en utilisant comme instrument stratégique des urnes d'Ellsberg, en plus des stratégies mixtes usuelles. Ce changement simple mène à de nombreux phénomènes intéressants.

Bien que les équilibres de Nash restent des équilibres dans le jeu étendu, il peut exister de nouveaux *équilibres d'Ellsberg* avec des résultats distincts des résultats d'équilibre de Nash. Ceci se produit notamment dans des jeux dont la structure d'information permet à un joueur de menacer ses adversaires. J'illustre ce phénomène à l'aide d'un exemple de négociation de paix à trois joueurs. Dans ce jeu, la paix n'est jamais une issue d'équilibre de Nash, mais le jeu possède un équilibre d'Ellsberg menant à la paix lorsque les joueurs peuvent utiliser des stratégies ambiguës. Dans les jeux à plus de deux joueurs, l'existence d'équilibres d'Ellsberg qui ne sont pas des équilibres de Nash est expliquée à l'aide de résultats sur les équilibres subjectifs.

Les équilibres d'Ellsberg se caractérisent mathématiquement par le Principe d'Indifférence dans les Distributions. Les joueurs sont indifférents entre toutes les stratégies mixtes contenues dans la stratégie d'équilibre d'Ellsberg. De plus, on observe que dans les jeux à deux joueurs, les joueurs peuvent s'immuniser contre l'ambiguïté stratégique en jouant leurs stratégies maximin (lorsqu'il existe un équilibre de Nash complètement mixte).

J'analyse des équilibres d'Ellsberg dans des jeux à deux joueurs à intérêts communs et opposés. Je présente des exemples et des résultats généraux permettant de déterminer les équilibres d'Ellsberg dans ces jeux. Les équilibres d'Ellsberg des jeux à intérêts opposés (des jeux "cache bouton" modifiés) sont en adéquation avec les comportements observés dans des études expérimentales.

Finalement, je définis les jeux d'Ellsberg sous forme extensive. Sous l'hypothèse de cohérence dynamique (rectangularité) des stratégies d'Ellsberg, je démontre un résultat analogue au théorème de Kuhn: des stratégies d'Ellsberg rectangulaires sont équivalentes à des stratégies d'Ellsberg comportementales.

Mots clés Incertitude Knightienne dans les jeux, ambiguïté stratégique, jeux d'Ellsberg, jeux d'Ellsberg sous forme extensive, Théorème de Kuhn

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Gedruckt auf alterungsbeständigem Papier nach DIN ISO 9706.