CONTRIBUTIONS TO AUTOMORPHISMS OF AFFINE SPACES

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To Anna and Johannes

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OVERVIEW

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2. Introduction

In 1872, Felix Klein wrote in his accession to the University of Erlangen a scientific program for the classification of geometric subjects, which later became famous as "Erlanger Programm". He formulated the following general problem: "*Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben. Man entwickle die auf die Gruppe bezügliche Invariantentheorie.*" (see [Kle93]). This can be freely translated into modern mathematical language as

"*Study mathematical objects via their automorphisms.*"

This approach has been successively implemented in many areas of mathematics, for example the study of manifolds via the mapping class group, Riemannian manifolds via isometries, field extensions via the Galois group, algebraic varieties via automorphisms.

In this thesis, we focus on the study of the affine space $\mathbb{A}^n$ via its automorphisms. Shafarevich introduced on the automorphism group $\mathcal{G}_n := \mathrm{Aut}(\mathbb{A}^n)$ the structure of an "infinite dimensional variety", a so-called ind-variety (see Section 3.2). The slogan of this thesis is

"Study the automorphism group of the affine space within the framework of ind-varieties."

The thesis is organized as follows. In Section 3 we introduce the basic concepts and notions that we will need. In Section 4 we give an overview of the results in the articles of this thesis. Thereafter we list all these articles.

We work over an uncountable algebraically closed field k of characteristic zero, if not explicitely stated otherwise.

3. Fundamentals

3.1. **The automorphism group $\mathcal{G}_n = \mathrm{Aut}(\mathbb{A}^n)$.** An *endomorphism* of $\mathbb{A}^n = k^n$ is a map of the form

$$\mathbf{g} \colon \mathbb{A}^n \to \mathbb{A}^n \,,\ (a_1, \ldots, a_n) \mapsto (g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n))$$

where $g_1, \ldots, g_n \in k[x_1, \ldots, x_n]$ are polynomials and we use the notation $\mathbf{g} = (g_1, \ldots, g_n)$. An *automorphism* of $\mathbb{A}^n$ is an endomorphism that admits an inverse which is also an endomorphism. We denote by $\mathcal{E}_n$ the monoid of endomorphisms and by $\mathcal{G}_n$ the group of automorphisms. Moreover we define the *degree* of $\mathbf{g} = (g_1, \ldots, g_n) \in \mathcal{E}_n$ as $\deg \mathbf{g} := \max_i \deg g_i$.

There are two prominent subgroups in $\mathcal{G}_n$: The group Aff_n of *affine linear automorphisms* (i.e. the automorphisms $\mathbf{g}$ with $\deg \mathbf{g} \leq 1$) and the group $\mathcal{J}_n$ of *triangular automorphisms* (i.e. the automorphisms $(g_1, \ldots, g_n)$ where $g_i = g_i(x_i, \ldots, x_n)$ depends only on $x_i, \ldots, x_n$ for each i). The group $T\mathcal{G}_n$ of *tame automorphisms* is the subgroup of $\mathcal{G}_n$ generated by Aff_n and $\mathcal{J}_n$.

Whereas the group $\mathcal{G}_1 = \mathrm{Aff}_1$ is an algebraic group, for $n > 1$, the group $\mathcal{G}_n$ is not an algebraic group anymore.

In the case $n = 2$, it is known that $\mathcal{G}_2$ has a decomposition as an amalgamated product of the subgroups Aff_2 and $\mathcal{J}_2$ over their intersection and consists thus only of tame automorphisms (see [Jun42] and [vdK53]). Due to this decomposition a lot is known, for example: every algebraic group is conjugate to a subgroup of Aff_2 or to a subgroup of $\mathcal{J}_2$ (see [Kam79]); every action of the additive group $\mathbb{G}_a$ on $\mathbb{A}^2$ is a modified translation, i.e. for suitable coordinates (x, y) of $\mathbb{A}^2$ and for some polynomial $p \in k[y]$ the action has the form $(t, (x, y)) \mapsto (x + tp(y), y)$ (see [Ren68]).

In the case $n = 3$, it was long time conjectured by Nagata that a certain automorphism is non-tame (see [Nag72]). This automorphism is now called Nagata-automorphism (see Section 3.3 for a definition). In 2003, Shestakov and Umirbaev proved that the Nagata-automorphism is non-tame (see [SU04]). This gives an indication, that $\mathcal{G}_3$ is rather difficult to understand. But never the less, a good strategy to get a better insight to the group $\mathcal{G}_3$ is to try to reduce a given problem to the 2-dimensional case, if possible.

It is still an open problem, whether $T\mathcal{G}_n = \mathcal{G}_n$ for $n \geq 4$, but many specialists believe that $T\mathcal{G}_n \neq \mathcal{G}_n$ for $n \geq 4$.

3.2. **Ind-varieties and ind-groups.** Shafarevich introduced on the group $\mathcal{G}_n$ the structure of an “infinite dimensional variety”, a so-called ind-variety (see [Sha66] and [Sha81]). Actually, it turned out that the automorphism group of any affine variety has such a structure. First ideas leading to this notion go back to Ramanujam in [Ram64] where he studies algebraic families in automorphism groups. For a general reference we refer to [Kum02].

Definition 3.1. An *ind-variety* is a set X together with a *filtration*, i.e. a chain of varieties $X_1 \subseteq X_2 \subseteq \ldots$ such that $X = \bigcup_{i=1}^{\infty} X_i$ and for all i the subset $X_i \subseteq X_{i+1}$ is closed. In this case we denote $X = \varinjlim X_i$. If every X_i is affine we call $X = \varinjlim X_i$ *affine*. We endow every ind-variety $X = \varinjlim X_i$ with the *ind-topology*, i.e. a subset $A \subseteq X$ is closed if and only if $A \cap X_i$ is closed in X_i for all i.

A basic example of an affine ind-variety is the *infinite dimensional affine space* $\mathbb{A}^\infty = \varinjlim \mathbb{A}^i$ where $\mathbb{A}^i$ is linearly embedded in $\mathbb{A}^{i+1}$. Another basic example is the set of the endomorphisms of $\mathbb{A}^n$: $\mathcal{E}_n = \varinjlim \mathcal{E}_{n,i}$ where $\mathcal{E}_{n,i}$ denotes the endomorphisms of degree $\leq i$.

Definition 3.2. A *morphism* of ind-varities $f \colon \varinjlim X_i \to \varinjlim Y_j$ is a map such that for all i there exists $j = j(i)$ such that $f(X_i) \subseteq Y_j$ and the restriction $f|_{X_i} \colon X_i \to Y_j$ is a morphism of (ordinary) varieties. Two filtrations $X_1 \subseteq X_2 \subseteq \ldots$ and $X_1' \subseteq X_2' \subseteq \ldots$ on a set X are called *equivalent*, if the identity map $\mathrm{id} \colon \varinjlim X_i \to \varinjlim X_i'$ is an isomorphism of ind-varieties. One doesn't distinguish between equivalent filtrations on a set X.

Example 3.3. If V is a countable dimensional k-vector space, then V has the structure of an ind-variety by choosing a filtration $V_1 \subseteq V_2 \subseteq \ldots$ of finite dimensional subspaces. Two filtrations of V by finite dimensional subspaces are always equivalent.

Clearly, every morphism of ind-varieties is continuous. A basic example of a morphism is the map $\mathrm{Jac}_0 \colon \mathcal{E}_n \to \mathbb{A}^1$, $\mathbf{g} \mapsto \det D_0(\mathbf{g})$, where $D_0(\mathbf{g})$ denotes the differential in the origin.

Definition 3.4. For any affine ind-variety $X = \varinjlim X_i$, the morphisms $X \to \mathbb{A}^1$ are exactly the elements of the projective limit $\varprojlim \mathcal{O}(X_i)$. We call these morphisms the *regular functions* on X and we define $\mathcal{O}(X) := \varprojlim \mathcal{O}(X_i)$.

Definition 3.5. An *ind-group* is an ind-variety such that the product and the inverse are morphisms of ind-varieties.

Let H be a subgroup of an ind-group $G = \varinjlim G_i$. We say that H is an *ind-subgroup* of G if H can be turned into an ind-group $H = \varinjlim H_k$ such that to every k there exists $i = i(k)$ such that $H_k \subseteq G_i$ is closed. If the ground field is uncountable, one easily verifies that the ind-structure of H is then unique.

A subgroup $H \subseteq G = \varinjlim G_i$ is called *algebraic*, if it is a closed subset of some G_i. An element $g \in G$ is called *algebraic*, if the closure of the cyclic group generated by g is an algebraic subgroup of G.

Likewise one defines *ind-monoid*, *ind-submonoid*, *algebraic submonoid* and *algebraic element* of an ind-monoid.

For every algebraic element g of an *affine* ind-group G there exists a (unique) Chevalley-Jordan decomposition $g = g_s g_u = g_u g_s$ into a semisimple part g_s and a unipotent part g_u. This enables us to speak of unipotent and semisimple elements in an arbitrary ind-group.

Example 3.6. The endomorphisms $\mathcal{E}_n = \varinjlim \mathcal{E}_{n,i}$ form an affine ind-monoid. The automorphism group $\mathcal{G}_n$ is then a locally closed subset of $\mathcal{E}_n$ (see [BCW82]) and thus $\mathcal{G}_n$ has the structure of an affine ind-variety via

$$\mathcal{G}_n = \varinjlim \mathcal{G}_{n,i}\,, \quad \text{where } \mathcal{G}_{n,i} \text{ is the set of } \mathbf{g} \in \mathcal{G}_n \text{ with } \deg \mathbf{g} \leq i\,.$$

As $\mathcal{G}_n \to \mathcal{G}_n$, $\mathbf{g} \mapsto \mathbf{g}^{-1}$ is a morphism of ind-varieties (see [BCW82]), it follows that $\mathcal{G}_n$ is an affine ind-group. The subgroup $\mathrm{Aff}_n \subseteq \mathcal{G}_n$ is algebraic and $\mathcal{J}_n \subseteq \mathcal{G}_n$ is a closed subgroup consisting of algebraic elements but it is not an algebraic subgroup. $\mathcal{J}_n$ has the structure of and ind-group through $\mathcal{J}_n = \varinjlim \mathcal{J}_{n,i}$, where $\mathcal{J}_{n,i} := \mathcal{J}_n \cap \mathcal{G}_{n,i}$.

More generally, for every *affine* variety X one can define on $\mathrm{Aut}(X)$ the structure of an affine ind-group (in a natural way).

Proposition 3.7. *Let X be an affine variety. Then $\mathrm{Aut}(X)$ has the structure of an ind-group, such that for all algebraic groups G, the G-actions $G \times X \to X$ correspond to the ind-group homomorphisms $G \to \mathrm{Aut}(X)$.*

Lemma 3.8. *Let X and Y be affine varieties. Then the set of morphisms $\mathrm{Mor}(X, Y)$ has a canonical structure of an ind-variety.*

Proof. Take a closed embedding $Y \subseteq \mathbb{A}^n$ and denote by $I \subseteq \mathcal{O}(\mathbb{A}^n)$ the vanishing ideal of Y. The countable dimensional k-vector space $\mathrm{Mor}(X, \mathbb{A}^n)$ has the structure of an ind-variety by Example 3.3. It follows, that

$$\mathrm{Mor}(X, Y) = \{\ f \in \mathrm{Mor}(X, \mathbb{A}^n) \mid \varphi \circ f = 0 \text{ for all } \varphi \in I\ \}$$

is closed in $\mathrm{Mor}(X, \mathbb{A}^n)$ and it has thus the structure of an ind-variety.

One can check that the ind-structure on $\mathrm{Mor}(X, Y)$ does not depend on the choice of the embedding $Y \subseteq \mathbb{A}^n$. □

The next (easy) Lemma we state without proof.

Lemma 3.9. *Let X, Y and Z be affine varieties. Then we have a bijection*

$$\begin{array}{ccc} \mathrm{Mor}(X \times Y, Z) & \overset{1:1}{\longleftrightarrow} & \mathrm{Mor}(X, \mathrm{Mor}(Y, Z)) \\ f & \longmapsto & (x \mapsto (y \mapsto f(x, y))) \end{array}$$

In fact, the bijection is an isomorphism of ind-varieties.

Proof of Proposition 3.7. Take any closed embedding $X \subseteq \mathbb{A}^n$ and let $p\colon \mathcal{E}_n \to \mathrm{Mor}(X, \mathbb{A}^n)$ be the canonical k-linear projection. Thus $\mathrm{Mor}(X, \mathbb{A}^n) = \varinjlim p(\mathcal{E}_{n,i})$ is filtrated by finite dimensional subspaces. and $\mathrm{End}(X) = \varinjlim \mathrm{End}(X)_i$ is an ind-variety, where $\mathrm{End}(X)_i = \mathrm{End}(X) \cap p(\mathcal{E}_{n,i})$. From the construction it follows that $\mathrm{End}(X) \times \mathrm{End}(X) \to \mathrm{End}(X)$, $(f, g) \mapsto f \circ g$ is a morphism and hence $\mathrm{End}(X)$ is an affine ind-monoid.

The set

$$\mathrm{Aut}(X) = \{\, (f, h) \in \mathrm{End}(X) \times \mathrm{End}(X) \mid f \circ h = h \circ f = \mathrm{id}\,\}$$

is closed in $\mathrm{End}(X) \times \mathrm{End}(X)$ and it has thus the structure of an ind-variety. As $\mathrm{End}(X)$ is an ind-monoid, the composition

$$\mathrm{Aut}(X) \times \mathrm{Aut}(X) \to \mathrm{Aut}(X)\,,\ ((f_1, h_1), (f_2, h_2)) \mapsto (f_1 \circ f_2, h_2 \circ h_1)$$

is a morphism and taking inverses

$$\mathrm{Aut}(X) \to \mathrm{Aut}(X)\,,\ (f, h) \mapsto (h, f)\,.$$

is a morphism as well. Thus, $\mathrm{Aut}(X)$ is an affine ind-group.

Let G be an algebraic group. If $\rho\colon G \times X \to X$ is a G-action, then $G \to \mathrm{End}(X)$, $g \mapsto \rho_g$ is a morphism by Lemma 3.9, where $\rho_g\colon X \to X$ is defined by $\rho_g(x) := \rho(g, x)$. Hence $G \to \mathrm{End}(X) \times \mathrm{End}(X)$, $g \mapsto (\rho_g, \rho_{g^{-1}})$ is a morphism and thus induces a homomorphism of ind-groups $G \to \mathrm{Aut}(X)$. Conversely, if $G \to \mathrm{Aut}(X)$ is a homomorphism of ind-groups, then

$$G \to \mathrm{Aut}(X) \subseteq \mathrm{End}(X) \times \mathrm{End}(X) \xrightarrow{\mathrm{pr}_1} \mathrm{End}(X)$$

is a morphism and thus $G \times X \to X$ is a G-action by Lemma 3.9. □

Remark 3.10. i) In fact, if $\mathcal{E}$ is any ind-monoid, then the group of units of $\mathcal{E}$ has the structure of an ind-group, exactly in the same way as in the proof of Proposition 3.7.

ii) The ind-structure introduced on $\mathrm{Aut}(\mathbb{A}^n) = \mathcal{G}_n$ in Proposition 3.7 coincides with the ind-structure introduced in Example 3.6. This follows from the fact that the projection on the first factor $\mathcal{E}_n \times \mathcal{E}_n \to \mathcal{E}_n$ induces an isomorphism of ind-groups $\mathrm{Aut}(\mathbb{A}^n) \to \mathcal{G}_n$, since $\mathcal{G}_n \subseteq \mathcal{E}_n$ is locally closed and since $\mathcal{G}_n \to \mathcal{G}_n \subseteq \mathcal{E}_n$, $\mathbf{g} \mapsto \mathbf{g}^{-1}$ is a morphism.

Surprisingly, the situation is completely different in the case of the group of birational maps $\mathrm{Bir}(\mathbb{P}^n)$. Blanc and Furter showed recently that for $n \geq 2$ there exists no filtration on $\mathrm{Bir}(\mathbb{P}^n)$ that turns it into an ind-group, such that "families" of birational maps parametrized by a variety A correspond to morphisms of ind-varieties $A \to \mathrm{Bir}(\mathbb{P}^n)$ (see [BF12]).

3.3. **Unipotent elements.**

Definition 3.11. Let G be an ind-group. An algebraic element $g \in G$ is called *unipotent*, if g is equal to its unipotent part g_u in the Chevalley-Jordan decomposition. This condition is equivalent to saying, that the closure of $\langle g \rangle$ is isomorphic to the additive group $\mathbb{G}_a = (k, +)$ or that it is trivial. The subset of all unipotent elements of G is denoted by G_u.

Let X be an irreducible affine variety. Note that we have a bijective correspondence

$$\mathrm{Aut}(\mathbb{A}^n)_u = \{\, \text{unipotent elements in } \mathrm{Aut}(X) \,\} \xleftrightarrow{1:1} \{\, \mathbb{G}_a\text{-actions on } X \,\}$$

given in the following manner: If $\mathbf{u} \in \mathrm{Aut}(X)$ is unipotent, then $\mathbb{G}_a \simeq \overline{\langle \mathbf{u} \rangle} \subseteq \mathrm{Aut}(X)$ and thus we get a $\mathbb{G}_a$-action on X by the homomorphism $\mathbb{G}_a \to \mathrm{Aut}(X)$ that sends 1 to $\mathbf{u}$. Conversely, if $\rho\colon \mathbb{G}_a \to \mathrm{Aut}(X)$ is a homomorphism, then $\mathbf{u} := \rho(1) \in \mathrm{Aut}(X)$ is unipotent. We have also a bijective correspondence

$$\{\, \mathbb{G}_a\text{-actions on on } X \,\} \xleftrightarrow{1:1} \{\, \text{locally nilpotent derivations on } \mathcal{O}(X) \,\}$$

that is given as follows: If $\rho\colon \mathbb{G}_a \times X \to X$ is a $\mathbb{G}_a$-action, then the comorphism $\rho^*\colon \mathcal{O}(X) \to \mathcal{O}(X)[t]$ induces a locally nilpotent derivation $D\colon \mathcal{O}(X) \to \mathcal{O}(X)$, $D(f) := \frac{\mathrm{d}}{\mathrm{d}t}\rho^*(f)|_{t=0}$. If $D\colon \mathcal{O}(X) \to \mathcal{O}(X)$ is a locally nilpotent derivation, then $\mathbb{G}_a \to \mathrm{Aut}(X)$, $t \mapsto \mathrm{Exp}(tD)$ defines a $\mathbb{G}_a$-action on X where the comorphism of $\mathrm{Exp}(tD)$ is

$$\mathcal{O}(X) \to \mathcal{O}(X)\,,\ f \mapsto \sum_{i=0}^{\infty} \frac{t^i}{i!} D^i(f)\,.$$

For a general reference on the theory of locally nilpotent derivations see [Fre06].

Let $\mathbf{u} \in \mathrm{Aut}(X)$ be unipotent. We denote by $\mathcal{O}(X)^{\mathbf{u}}$ the invariant ring of $\mathbf{u}$. If D is the locally nilpotent derivation that corresponds to $\mathbf{u}$, then $\mathcal{O}(X)^{\mathbf{u}} = \ker D$ and moreover it is the invariant ring of the $\mathbb{G}_a$-action on X, corresponding to $\mathbf{u}$. The algebraic quotient $X \to X /\!\!/ \mathbb{G}_a$ is given by the inclusion $\mathcal{O}(X)^{\mathbf{u}} \subseteq \mathcal{O}(X)$.

Definition 3.12. Let $\mathbf{u} \in \operatorname{Aut}(X)$ be unipotent and let D be the corresponding locally nilpotent derivation. For every $f \in \mathcal{O}(X)^{\mathbf{u}}$ we denote by $f \cdot \mathbf{u}$ the unipotent automorphism corresponding to the locally nilpotent derivation fD and we call $f \cdot \mathbf{u}$ a *modification* of $\mathbf{u}$. A unipotent automorphism $\mathbf{u} \neq \mathbf{id}$ is called *irreducible*, if $\mathbf{u} = f \cdot \mathbf{v}$ implies that f is a unit in $\mathcal{O}(X)^{\mathbf{u}}$. If $\mathcal{O}(X)$ is a unique factorization domain and $\mathbf{u} \neq \mathbf{id}$, then there exists an irreducible $\mathbf{u}'$ such that $\mathbf{u} = f \cdot \mathbf{u}'$ and we call such a decomposition a *standard decomposition*. Moreover, $\mathbf{u}'$ is unique up to a modification by a unit of $\mathcal{O}(X)^{\mathbf{u}}$.

The most basic unipotent elements in $\mathcal{G}_n = \operatorname{Aut}(\mathbb{A}^n)$ are the *translations*, i.e. automorphisms of the form $(x_1 + 1, x_2, \ldots, x_n)$ for a suitable coordinate system $(x_1, \ldots, x_n)$ of $\mathbb{A}^n$. A modification of such a translation is and automorphism of the form $(x_1 + f(x_2, \ldots, x_n), x_2, \ldots, x_n)$ for a polynomial $f(x_2, \ldots, x_n)$ depending only on $x_2, \ldots, x_n$. By abuse of language, we call such an automorphism a *modified translation*.

More general examples of unipotent automorphisms in $\mathcal{G}_n$ are the triangular automorphisms $(g_1, g_2, \ldots, g_n)$ that satisfy $g_i = x_i + p_i(x_{i+1}, \ldots, x_n)$ and p_i depends only on the variables $x_{i+1}, \ldots, x_n$.

A very famous unipotent automorphism in $\mathcal{G}_3$ is the Nagata-automorphism. It is defined in the following way

$$\mathbf{u}_N := (x + py + \frac{1}{2}p^2 z, y + pz, z) \quad \text{where} \quad p := xz - \frac{1}{2}y^2 \,.$$

In fact, $\mathbf{u}_N$ is a modification of $\mathbf{u} := (x + y + \frac{1}{2}z, y + z, z)$, namely $\mathbf{u}_N = p \cdot \mathbf{u}$.

An important invariant of a unipotent automorphism (for the action by conjugation) is its plinth ideal (scheme), that we introduce now.

Definition 3.13. Let $\mathbf{u} \in \operatorname{Aut}(X)$ be unipotent and let D be its corresponding locally nilpotent derivation of $\mathcal{O}(X)$. The *plinth ideal of* $\mathbf{u}$ is the intersection $\operatorname{im} D \cap \ker D \subseteq \ker D$. We call the corresponding closed subscheme $\Gamma \subseteq X /\!\!/ \mathbb{G}_a = \operatorname{Spec}(\ker D)$ the *plinth scheme* of $\mathbf{u}$.

For example, the plinth scheme of a unipotent $\mathbf{u} \in \mathcal{G}_n$ is empty if and only if $\mathbf{u}$ is a translation. The plinth ideal of a modified translation $(x_1 + f(x_2, \ldots, x_n), x_2, \ldots, x_n)$ is the principal ideal $(f) \subseteq k[x_2, \ldots, x_n]$.

The reduced plinth scheme Γ_{red} has a nice geometric interpretation: its complement in $X /\!\!/ \mathbb{G}_a$ is the biggest open subset, such that the algebraic quotient $X \to X /\!\!/ \mathbb{G}_a$ is a locally trivial principal $\mathbb{G}_a$-bundle over it.

Definition 3.14. An algebraic group U is called *unipotent* if every element is unipotent. An ind-group U is called *unipotent* if $U = \varinjlim U_i$ where U_i is a unipotent algebraic group for all i.

So far, we do not know, if an ind-group G is unipotent (in the sense of Definition 3.14), if all elements of G are unipotent.

4. Outline of the articles

4.1. **Topologies on ind-varieties and irreducibility questions.** There is another natural way to endow an affine ind-variety $X = \varinjlim X_i$ with a topology beside the ind-topology (see Section 3.2). Namely, Kambayashi introduced in [Kam96] and [Kam03] the *Zariski-topology* with respect to the regular functions of the ind-variety: a subset $A \subseteq X = \varinjlim X_i$ is closed if and only if there exists a subset $E \subseteq \mathcal{O}(X)$, such that A is the zero set of E, i.e.

$$A = V(E) := \{ x \in X \mid f(x) = 0 \text{ for all } f \in E \} .$$

The ind-topology is in general finer than the Zariski-topology. One part of [Sta12b] is devoted to the comparison of the ind-topology and the Zariski-topology on an affine ind-variety. Already on the most basic affine ind-variety, these topologies are different:

Example 4.1 (Example 1 in [Sta12b]). Let $f_n \in k[x_1, \ldots, x_n] = \mathcal{O}(\mathbb{A}^n)$ be recursively defined as

$$f_1 := x_1 , \quad f_{n+1} := f_n^2 + x_{n+1} .$$

Then $A := \bigcup_n V_{\mathbb{A}^n}(f_n)$ is a proper closed subset of the infinite-dimensional affine space $\mathbb{A}^\infty = \varinjlim \mathbb{A}^n$ with respect to the ind-topology, but it is dense in $\mathbb{A}^\infty$ with respect to the Zariski topology.

It turned out that for a big class of ind-varieties these topologies are different.

Theorem 4.2 (Theorem A in [Sta12b]). *Let $X = \varinjlim X_n$ be an affine ind-variety. If there exists $x \in X$ such that X_n is normal or Cohen-Macaulay in x for infinitely many n, and the local dimension of X_n at x tends to infinity, then the ind-topology and the Zariski topology are different.*

The idea of the proof is to pullback a certain modification of the subset $A \subseteq \mathbb{A}^\infty$ in Example 4.1 to the affine ind-variety X, via a well chosen morphism of ind-varieties $X \to \mathbb{A}^\infty$.

If we allow only affine ind-varieties that admit a filtration by normal varieties, it is even possible to characterize the affine varieties, such that these topologies are different.

Theorem 4.3 (Corollary B in [Sta12b]). *Let $X = \varinjlim X_i$ be an affine ind-variety such that X_i is normal for infinitely many i. Then the ind-topology and the Zariski-topology on X coincide if and only if for all $x \in X$ the local dimension of X_i at x is bounded for all i.*

The other part of [Sta12b] is devoted to the study of the irreducibility of an affine ind-variety $X = \varinjlim X_i$. It turned out that in general the irreducibility depends on the topology:

Example 4.4 (Example 4 in [Sta12b]). Let $g_n, f_n \in k[x_1, \ldots, x_n]$ be defined as

$$g_n := x_1 + \ldots + x_n , \qquad f_1 := x_1 , \quad f_{n+1} = f_n^2 + x_{n+1} .$$

The affine ind-variety $X := \varinjlim (V_{\mathbb{A}^n}(f_n \cdot g_n))$ decomposes into the proper closed subsets $\bigcup_n V_{\mathbb{A}^n}(f_n)$ and $\bigcup_n V_{\mathbb{A}^n}(g_n))$ (with respect to the ind-topology) and thus X is reducible in the ind-topology. On the other hand one can see, that $\mathcal{O}(X)$ is an integral domain, which is equivalent to the irreducibility of X in the Zariski-topology.

Another interesting example is the following

Example 4.5 (Example 5 in [Sta12b]). Let $g_n \in k[x_1, \ldots, x_n]$ be recursively defined as

$$g_1 := x_1 - 1 , \quad g_{n+1} := (x_1 - (n+1)) \cdot g_n - x_{n+1} .$$

Let $X_n := V_{\mathbb{A}^n}(g_n) \cup V_{\mathbb{A}^n}(x_2, \ldots, x_n) \subseteq \mathbb{A}^n$. Then X_n consists of two irreducible components for all $n > 1$, but one can see that the limit $X = \varinjlim X_n$ is irreducible (with respect to the ind-topology and thus also with respect to the Zariski-topology).

For the property of connectedness, the situation is different from the property of irreducibility.

Remark 4.6. An ind-variety $X = \varinjlim X_i$ is connected in the ind-topology if and only if it is connected in the Zariski-topology.

Proof. As the ind-topology is finer than the Zariski-topology, we have only to show the following: if X is non-connected in the ind-topology, then it is non-connected in the Zariski-topology. Let $A, B \subseteq X$ be non-empty, disjoint, subsets, that are closed with respect to the ind-topology and such that $A \cup B = X$. Then X_i is the disjoint union of the closed subsets $A \cap X_i$ and $B \cap X_i$. Let $f_i \colon X_i \to \mathbb{A}^1$ be defined by $f_i|_{A \cap X_i} \equiv 1$ and $f_i|_{B \cap X_i} \equiv 0$. Hence, $(f_i)_i, (1 - f_i)_i \in \varprojlim \mathcal{O}(X_i) = \mathcal{O}(X)$. Therefore, $A = V((f_i)_i)$ and $B = V((1 - f_i)_i)$ are closed subsets of X with respect to the Zariski-topology and thus X is non-connected with respect to the Zariski-topology. □

4.2. **Automorphisms of the affine Cremona group.** A natural problem in the study of a group G is to determine its automorphisms. There are always the inner automorphisms $G \to G$, $h \mapsto ghg^{-1}$ where $g \in G$. In case $G = \mathcal{G}_n$, there are beside the inner automorphisms another natural class of group automorphisms: every field automorphism $\tau \colon k \to k$ induces a group automorphism

$$\tau \colon \mathcal{G}_n \to \mathcal{G}_n\,, \quad \mathbf{g} \mapsto \tau_n \circ \mathbf{g} \circ {\tau_n}^{-1}$$

where $\tau_n \colon \mathbb{A}^n \to \mathbb{A}^n$ is defined by $\tau_n(x_1, \ldots, x_n) = (\tau(x_1), \ldots, \tau(x_n))$.

Déserti proved in [Dés06] that all group automorphisms $\mathcal{G}_2 \to \mathcal{G}_2$ are inner automorphisms up to field automorphisms. Together with Kraft we generalized in [KS12] the result of Déserti in the following way (recall that $\mathcal{G}_2 = T\mathcal{G}_2$).

Theorem 4.7 (Main Theorem in [KS12])**.** *Let $\theta \colon \mathcal{G}_n \to \mathcal{G}_n$ be a group automorphism. Then there exists a $\mathbf{g} \in \mathcal{G}_n$ and a field automorphism $\tau \colon k \to k$ such that*

$$\theta(\mathbf{f}) = \tau(\mathbf{g} \circ \mathbf{f} \circ \mathbf{g}^{-1}) \quad \textit{for all } \mathbf{f} \in T\mathcal{G}_n\,.$$

Remark 4.8. Recently, Urech generalized our result to an algebraically closed field of any characteristic.

We describe now the strategy of the proof of Theorem 4.7. Let $\theta \colon \mathcal{G}_n \to \mathcal{G}_n$ be a group automorphism.

The first and hardest step is to prove that for the standard torus $D_n \subseteq \mathcal{G}_n$ the image $\theta(D_n)$ is an algebraic group that is isomorphic to D_n (see Lemma 3.3 and Proposition 3.4 in [KS12]). Białynicki-Birula proved in [BB66] that every faithful action of D_n on $\mathbb{A}^n$ is linearizable and thus $\theta(D_n)$ is conjugate to D_n.

Now, we prove that for a one-dimensional unipotent D_n-stable subgroup $U \subseteq \mathcal{G}_n$, the image $\theta(U)$ is again unipotent and one-dimensional. If $U \subseteq \mathcal{G}_n$ is such a group, then for a fixed $\mathbf{u}_0 \in U$ with $\mathbf{u}_0 \neq \mathbf{id}$ the map

$$\theta(D_n) \twoheadrightarrow \theta(U) \setminus \{\mathbf{id}\} \subseteq \mathcal{G}_n\,, \quad \theta(\mathbf{d}) \mapsto \theta(\mathbf{d}) \circ \theta(\mathbf{u}_0) \circ \theta(\mathbf{d})^{-1}$$

is a morphism of ind-varieties. Thus $\theta(U) \subseteq \mathcal{G}_n$ is a constructible subgroup contained in some $\mathcal{G}_{n,i}$ and therefore $\theta(U)$ is an algebraic subgroup of $\mathcal{G}_n$ (see Lemma 2.1

in [KS12]). As $\theta(U)$ has no element of finite order $\neq \mathbf{id}$, $\theta(U)$ is unipotent. Moreover, $\theta(D_n)$ normalizes $\theta(U)$ and since $\theta(U)$ consists only of two $\theta(D_n)$-orbits its dimension is one. This result can then be generalized to unipotent D_n-stable subgroups of arbitrary dimension (see Proposition 6.1 in [KS12]).

The next step is to prove that two different one-dimensional unipotent D_n-stable subgroups have different associated characters (see Lemma 4.1 and Remark 4.2 in [KS12]).

Using the above facts, we show now, that for any algebraic group $G \subseteq \mathcal{G}_n$ containing D_n, the image $\theta(G)$ is again an algebraic group of the same dimension. Let $U_1, \ldots, U_r$ be the different one-dimensional unipotent D_n-stable subgroups of G. Then $X := D_n \circ U_1 \circ \ldots \circ U_r$ is dense in G, which implies that $G = X \circ X$ (see Lemma 2.1 in [KS12]). Thus $\theta(G) = \theta(X) \circ \theta(X)$ is an algebraic subgroup. As $\theta(U_1), \ldots, \theta(U_r)$ are different one-dimensional unipotent $\theta(D_n)$-stable subgroups of $\theta(G)$ and $\theta(D_n)$ is an n-dimensional subtorus, we have $\dim \theta(G) \geq \dim G$. The same arguments applied to θ^{-1} yields equality.

Now, we prove that $\theta(\mathrm{GL}_n)$ is linearizable. As GL_n contains no non-trivial normal unipotent subgroup, the same is true for $\theta(\mathrm{GL}_n)$, and thus it is reductive. As $\theta(\mathrm{GL}_n)$ acts faithfully on $\mathbb{A}^n$ and since it contains a torus of dimension n, there is no non-constant $\theta(\mathrm{GL}_n)$-invariant function, hence $\theta(\mathrm{GL}_n)$ is linearizable by Proposition 5.1 in [KP85]). By composing θ with an inner automorphism, we can assume that $\theta(\mathrm{GL}_n) = \mathrm{GL}_n$.

Using the fact that the subgroup of translations $T_n \subseteq \mathrm{Aff}_n \subseteq \mathcal{G}_n$ is the only commutative unipotent subgroup normalized by GL_n (see Lemma 4.4 in [KS12])), it follows that $\theta(T_n) = T_n$. In summary, we get $\theta(\mathrm{Aff}_n) = \mathrm{Aff}_n$.

The last step is to prove the theorem for the restriction $\theta|_{\mathrm{Aff}_n} \colon \mathrm{Aff}_n \to \mathrm{Aff}_n$. After this done, we can assume, that $\theta|_{\mathrm{Aff}_n} = \mathrm{id}_{\mathrm{Aff}_n}$. From this fact one can then deduce that $\theta|_{\mathcal{J}_n} = \mathrm{id}_{\mathcal{J}_n}$, which proves the theorem (see Proposition 7.1 in [KS12])).

In the article [Sta12a] we generalize the techniques used in the proof of Theorem 4.7. Our main result is the following.

Theorem 4.9 (Main Theorem in [Sta12a])**.** *Let $\theta \colon \mathcal{G} \to \mathcal{G}$ be a group automorphism of an ind-group $\mathcal{G}$ that is the identity on a closed torus $T \subseteq \mathcal{G}$. If $\mathcal{U} \subseteq \mathcal{G}$ is a unipotent ind-subgroup that is normalized by T and if the neutral element of $\mathcal{U}$ is the only element that commutes with T, then $\theta(\mathcal{U})$ is a unipotent ind-subgroup of $\mathcal{G}$ and $\theta|_{\mathcal{U}} \colon \mathcal{U} \to \theta(\mathcal{U})$ is an isomorphism of ind-groups.*

Kuroda gave a characterization of the non-tame modifications of certain unipotent automorphisms (see Theorem 2.3 in [Kur11]). This result implies that for $\mathbf{u} := (x + y + \frac{1}{2}z, y + z, z)$ the modification $f \cdot \mathbf{u}$ is non-tame if and only if $f \in \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} \setminus k[z]$. Recall that the Nagata-automorphism $\mathbf{u}_N$ is also a modification of $\mathbf{u}$ (see Section 3.3). Clearly, all the modifications of $\mathbf{u}$ lie in the centralizer $\mathrm{Cent}(\mathbf{u})$. As a consequence of Theorem 4.9 we proved in [Sta12a] the following generalization of Theorem 4.7.

Theorem 4.10 (Application in [Sta12a])**.** *Let $\theta : \mathcal{G}_3 \to \mathcal{G}_3$ be a group automorphism that is the identity on the tame automorphisms $T\mathcal{G}_3$. Then θ fixes $\mathrm{Cent}(\mathbf{u})$ pointwise, where $\mathbf{u} = (x + y + \frac{1}{2}z, y + z, z)$. In particular θ fixes the non-tame automorphisms $f \cdot \mathbf{u}$ where $f \in \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} \setminus k[z]$ and thus also the Nagata-automorphism $\mathbf{u}_N$.*

We describe now the idea of the proof of Theorem 4.10. The first step is to calculate the centralizer $\mathrm{Cent}(\mathbf{u})$ (see Proposition 1 in [Sta12a]). It turned out that

Cent($\mathbf{u}$) is the following semi-direct product

$$\mathrm{Cent}(\mathbf{u}) = C \circ (\mathcal{H} \circ \mathcal{F}) \subseteq \mathcal{G}_3$$

where

$$\begin{aligned} C &= \{ (ax, ay, az) \mid a \in k^* \} \\ \mathcal{F} &= \{ f \cdot \mathbf{u} \mid f \in \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} \} \\ \mathcal{H} &= \{ h \cdot \mathbf{e} \mid h \in \mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{u}, \mathbf{e} \rangle} \}, \quad \mathbf{e} := (x+1, y, z). \end{aligned}$$

One can check, that $\mathcal{U} := \mathcal{H} \circ \mathcal{F} \subseteq \mathrm{Cent}(\mathbf{u})$ consists only of algebraic elements and every element $\neq \mathbf{id}$ has infinite order. Thus, it follows that $\mathcal{U}$ is the set of unipotent elements of Cent($\mathbf{u}$). It turns out that some two-dimensional torus $T \subseteq \mathcal{G}_3$ normalizes $\mathcal{U}$ and that $\mathbf{id} \in \mathcal{U}$ is the only element that commutes with T. Theorem 4.9 applied to $\mathcal{H}$ and $\mathcal{F}$ yields then that $\theta(\mathcal{U}) = \mathcal{U}$ and that θ preserves $\mathcal{F}$, since $\mathcal{F} = \mathrm{Cent}_{\mathcal{U}}[\mathcal{U}, \mathcal{U}]$.

The next step is now to prove that θ is actually the identity on $\mathcal{F}$. This step done, it follows that θ is the identity on the centralizer Cent($\mathbf{u}$), as $\mathcal{H}$ and C are subgroups of the tame automorphisms $T\mathcal{G}_3$.

If we assume in addition that $\theta \colon \mathcal{G}_n \to \mathcal{G}_n$ is an *automorphism of ind-groups*, then Belov-Kanel and Yu proved recently that θ is an inner automorphism of $\mathcal{G}_n$ (see [BKY13]). It is still an open problem, if every (abstract) group automorphism $\mathcal{G}_n \to \mathcal{G}_n$, $n \geq 3$ is inner up to a field automorphism.

4.3. **Automorphisms of $\mathbb{A}^2$ preserving a curve.** In this section the ground field k is arbitary, if not explicitely stated otherwise. Together with Blanc we investigated the group of automorphisms in $\mathcal{G}_2$ that preserve a closed curve $\Gamma \subseteq \mathbb{A}^2$, i.e. a closed equidimensional subvariety, that is reduced and one-dimensional. We denote this group by $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$. If $\Gamma \subseteq \mathbb{A}^2 = \mathrm{Spec}\, k[x,y]$ is defined by some polynomial in $k[x]$, then one can easily see that $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is not an algebraic group. Our main result says that this is the only case, where $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is not algebraic.

Theorem 4.11 (Theorem 1 in [BS13])**.** *Let k be any field and let Γ be a closed curve in $\mathbb{A}^2 = \mathrm{Spec}\, k[x,y]$. Applying an automorphism of $\mathbb{A}^2$, one of the following holds:*

i) The curve Γ is the zero-set of a square-free polynomial $F(x) \in k[x]$ and

$$\mathrm{Aut}(\mathbb{A}^2, \Gamma) = \{ (ax, by + P(x)) \mid a, b \in k^*, P \in k[x],\ F(ax)/F(x) \in k^* \}.$$

ii) The group $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is equal to

$$\{ g \in \mathrm{Aff}_2 \mid g(\Gamma) = \Gamma \} \quad \text{or} \quad \{ g \in \mathcal{J}_{2,i} \mid g(\Gamma) = \Gamma \}$$

for some integer i. Moreover, the action of $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ on Γ gives an isomorphism of $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ with a closed subgroup of $\mathrm{Aut}(\Gamma)$ ($\mathcal{J}_{2,i}$ is defined in Example 3.6).

In particular, $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group if and only if there is no automorphism of $\mathbb{A}^2$ which sends Γ onto a union of parallel lines in $\mathbb{A}^2$.

Our proof is based on methods from birational geometry. Namely, we consider *natural completions of* $\mathbb{A}^2$, i.e. pairs (X, B), where X is either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$, $n \geq 1$ and $B = B_X \subseteq X$ is a closed subset, together with an isomorphism that identifies $X \setminus B$ with $\mathbb{A}^2$. A *birational map* $(X, B) \dashrightarrow (X, B')$ is then a birational map $X \dashrightarrow X'$ that induces an isomorphism $X \setminus B \to X' \setminus B'$. If $\Gamma \subseteq X \setminus B = \mathbb{A}^2$ is a closed curve, then we denote by $\mathrm{Bir}((X, B), \Gamma)$ the group of all birational maps $\varphi \colon (X, B) \dashrightarrow (X, B)$, such that $\varphi(\Gamma) = \Gamma$. By definition we have thus $\mathrm{Bir}((X, B), \Gamma) = \mathrm{Aut}(\mathbb{A}^2, \Gamma)$. Every birational map $(X, B) \dashrightarrow (X', B')$ can be written uniquely (up to isomorphisms) as a finite composition of elementary links, i.e. birational maps of the following types:

- a blow-up $(\mathbb{F}_1, B_{\mathbb{F}_1}) \to (\mathbb{P}^2, B_{\mathbb{P}^2})$ of a point in $B_{\mathbb{P}^2}$ or its inverse
- a birational map $(\mathbb{F}_n, B_{\mathbb{F}_n}) \dashrightarrow (\mathbb{F}_{n+1}, B_{\mathbb{F}_{n+1}})$ which is the composition of a blow-up and a contraction $\mathbb{F}_n \leftarrow S \to \mathbb{F}_{n+1}$ or its inverse

(see Proposition 2.9 and Proposition 2.14 in [BS13]). These are the elementary links used in the Sarkisov program (see [Cor95] and [Isk96]), compatible with the boundaries.

The difficulty in the proof of Theorem 4.11 is to show the following statement (see Corollary 3.9 in [BS13]): To every natural completion (X, B) of $\mathbb{A}^2$ and to every closed curve $\Gamma \subseteq X \setminus B = \mathbb{A}^2$, there exists a natural completion (X', B') of $\mathbb{A}^2$ and a birational map $\varphi \colon (X, B) \dashrightarrow (X', B')$, such that either $X' = \mathbb{P}^2$ and $\varphi(\Gamma)$ is a projective line in $\mathbb{P}^2$ (and thus we are in case i)) or the birational map φ induces an embedding

$$\mathrm{Bir}((X, B), \Gamma) \hookrightarrow \mathrm{Aut}(X', B') .$$

The key tool in proving this statement is a suitable analysis of the boundary points of Γ in B_X under birational maps of natural completions $(X, B_X) \dashrightarrow (X', B_{B_{X'}})$ (see Proposition 3.4 in [BS13]).

One can see that the group $\mathrm{Aut}(X', B')$ is an algebraic subgroup of $\mathrm{Aut}(\mathbb{A}^2)$, equal to Aff_2 if $X' = \mathbb{P}^2$, and equal to $\mathcal{J}_{2,i}$ if $X = \mathbb{F}_i$ (see Lemma 2.6 in [BS13]). Now, if $\mathrm{Bir}((X, B), \Gamma) \hookrightarrow \mathrm{Aut}(X', B')$, then it follows that we are in case ii), as $\mathrm{Aut}(\mathbb{A}^2, \Gamma) = \mathrm{Bir}((X, B), \Gamma)$.

Moreover, in [BS13] we describe precisely the group $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ in case Γ is geometrically irreducible, the ground field k is perfect and $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is not 0-dimensional.

Theorem 4.12 (Theorem 2 in [BS13]). *Let Γ be a geometrically irreducible closed curve in $\mathbb{A}^2$, defined over a perfect field k. Applying an automorphism of $\mathbb{A}^2 = \mathrm{Spec}\, k[x, y]$, one of the following holds:*

i) The curve Γ is the line with equation x and

$$\mathrm{Aut}(\mathbb{A}^2, \Gamma) = \{\, (ax, by + P(x)) \mid a, b \in k^*,\ P \in k[x] \,\} .$$

ii) The curve Γ has equation $x^b - \lambda y^a$, where $\lambda \in k^*$ and $a, b > 1$ are coprime integers. Moreover, $\mathrm{Aut}(\mathbb{A}^2, \Gamma) = \{\, (t^a x, t^b y) \mid t \in k^* \,\}$.

iii) The curve Γ has equation $x^b y^a - \lambda$, where $\lambda \in k^*$ and $a, b \geq 1$ are coprime integers. Moreover, $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ contains the group $G := \{\, (t^a x, t^{-b} y) \mid t \in k^* \,\}$, and it is equal to G if $(a, b) \neq (1, 1)$, or is the group $G \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by the exchange (y, x) if $(a, b) = (1, 1)$.

iv) The curve Γ has equation $\lambda x^2 + \nu y^2 - 1$, where $\lambda, \nu \in k$, $-\lambda\nu$ is not a square in k and $\mathrm{char}(k) \neq 2$. Moreover, $\mathrm{Aut}(\mathbb{A}^2, \Gamma) = T \rtimes \langle \sigma \rangle$, where

$$T = \left\{ \begin{pmatrix} a & -\nu b \\ \lambda b & a \end{pmatrix} \;\middle|\; a^2 + \lambda\nu b^2 = 1 \right\} , \ \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and T is an anisotropic torus.

v) The curve Γ has equation $x^2+\mu xy+\nu y^2-1$, the polynomial $x^2+\mu x+1$ has no root in k, and $\mathrm{char}(k)=2$. Moreover, $\mathrm{Aut}(\mathbb{A}^2,\Gamma)=T\rtimes\langle\sigma\rangle$, where

$$T=\left\{\begin{pmatrix}a & b\\ b & a+\mu b\end{pmatrix}\;\middle|\; a^2+\mu ab+b^2=1\right\},\ \sigma=\begin{pmatrix}1&\mu\\0&1\end{pmatrix}$$

and T is an anisotropic torus which is isomorphic to Γ.

vi) The group $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ is a 0-dimensional subgroup of Aff_2 or $\mathcal{J}_2$.

Let us assume that $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ is of positive dimension. The first step is to see that $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ contains a closed algebraic group G (defined over k) which is isomorphic to $\mathbb{G}_m$ (over the algebraic closure $\bar{k}$) or to $\mathbb{G}_a$ (over k) (see Lemma 4.3 in [BS13]). In fact, if $\mathbb{G}_a$ is a closed subgroup, then it turned out that Γ is an affine line in $\mathbb{A}^2$ and we are in case i) (see Lemma 4.7 in [BS13]). If $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ does not contain a closed subgroup isomorphic to $\mathbb{G}_a$, then it turned out that $G\subseteq\mathrm{GL}_2(k)$ in suitable coordinates (see Lemma 4.4 in [BS13]). Hence Γ is invariant under a torus

$$t\mapsto\begin{pmatrix}t^a & 0\\ 0 & t^b\end{pmatrix}$$

up to a coordinate change $\psi\in\mathrm{GL}_2(\bar{k})$, where $a,b>0$ are coprime integers. As Γ is not an affine line, the equation of $\psi(\Gamma)$ is

$$\begin{array}{rcll} x^b-\eta y^a &=& 0 & \text{if } a>0,\\ x^by^{-a}-\eta &=& 0 & \text{if } a<0\end{array}$$

for a suitable $\eta\neq 0$ in the algebraic closure $\bar{k}$. If $a>0$, then $x^b-\eta y^a=0$ has exactly one point at infinity, which is therefore defined over k. One can see then, that we are in case ii). If $a<0$, then $x^by^{-a}-\eta=0$ has exactly two points at infinity. If both of these points are defined over k, then one can see that we are in case iii) and if the two points are not defined over k, then it turned out that we are either in case iv) or v) (see the proof of Proposition 4.5 in [BS13]).

4.4. **Centralizer of a unipotent automorphism in $\mathcal{G}_3$.** Let X be an affine variety. A classification of the unipotent elements up to conjugacy in the automorphism group $\mathrm{Aut}(X)$ is only known for a few varieties X. For example when $X=\mathbb{A}^2$ we have a classification: every unipotent automorphism is a modified translation.

Now, we consider the case, where $X=\mathbb{A}^3$. In contrast to the two-dimensional case, there is no classification known for the unipotent elements. As a first step towards a classification, one can study the centralizer of a unipotent element in $\mathcal{G}_3$. This is the content of the article [Sta13].

Let $\mathbf{id}\neq\mathbf{u}\in\mathcal{G}_3$ be unipotent and let $\mathbf{u}=d\cdot\mathbf{u}'$ be a standard decomposition. It turned out that the plinth scheme Γ of $\mathbf{u}$ is the main object in the study of the centralizer $\mathrm{Cent}(\mathbf{u})$. This is due to the exact sequence (induced by the natural action of $\mathrm{Cent}(\mathbf{u})$ on the algebraic quotient $\mathbb{A}^3/\!\!/\mathbb{G}_a$)

$$(*)\qquad 1\to\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'}\cdot\mathbf{u}'\hookrightarrow\mathrm{Cent}(\mathbf{u})\to\mathrm{Aut}(\mathbb{A}^3/\!\!/\mathbb{G}_a,\Gamma)$$

where $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'}\cdot\mathbf{u}'$ denotes the modifications of $\mathbf{u}'$ and $\mathrm{Aut}(\mathbb{A}^3/\!\!/\mathbb{G}_a,\Gamma)$ denotes the group of automorphisms of $\mathbb{A}^3/\!\!/\mathbb{G}_a$ that preserve Γ (see Proposition 5.1 in [Sta13]).

Theorem 4.13 (see Proposition A in [Sta13])**.** *If $\mathbf{id}\neq\mathbf{u}\in\mathcal{G}_3$ is a modified translation, then the sequence in $(*)$ splits.*

Hence, the challenging case is, when $\mathbf{u}$ is not a modified translation, which we assume from now on. The following notion is crucial to state the results.

Definition 4.14. A scheme that is isomorphic to $\mathbb{A}^1 \times F$ is called a *fence*, where $F \subseteq \mathbb{A}^1$ is a proper closed subscheme.

Theorem 4.15 (Theorem B, Proposition C, Theorem D in [Sta13])**.** *Let* $\mathbf{id} \neq \mathbf{u} \in \mathcal{G}_3$ *be unipotent, let* $\Gamma \subseteq \mathbb{A}^3/\!\!/\mathbb{G}_a$ *be its plinth scheme and let* $\mathbf{u} = d \cdot \mathbf{u}'$ *be a standard decomposition. If* $\mathbf{u}$ *is not a modified translation, then:*

i) *The set of unipotent elements* $\mathrm{Cent}(\mathbf{u})_u$ *is a closed normal subgroup, and there exists a closed algebraic group* R *such that* $\mathrm{Cent}(\mathbf{u}) \simeq \mathrm{Cent}(\mathbf{u})_u \rtimes R$ *as ind-groups.*
ii) *All elements in* $\mathrm{Cent}(\mathbf{u})$ *are algebraic.*
iii) *The subgroup* $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ *is characteristic in* $\mathrm{Cent}(\mathbf{u})$*.*
iv) *If* Γ *is not a fence, then* $\mathrm{Cent}(\mathbf{u})_u = \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$*.*
v) *Assume that* Γ *is a fence. Let* $\mathrm{Iner}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)$ *be the group of automorphisms of* $\mathbb{A}^3/\!\!/\mathbb{G}_a$ *that induce the identity on* Γ*. Then the set of unipotent elements* $\mathrm{Iner}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)_u$ *is a group and*

$$1 \to \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \mathrm{Cent}(\mathbf{u})_u \overset{p}{\twoheadrightarrow} \mathrm{Iner}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)_u \to 1$$

is a split short exact sequence of ind-groups. Moreover, p *induces an isomorphism* $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{u}, \mathbf{e} \rangle} \cdot \mathbf{e} \xrightarrow{\sim} \mathrm{Iner}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)_u$ *for a certain irreducible unipotent* $\mathbf{e} \in \mathcal{G}_3$*.*

Now, we give an idea of the proof of this theorem. By classical results, $\mathbb{A}^3/\!\!/\mathbb{G}_a$ is isomorphic to $\mathbb{A}^2$ (see [Miy80, Miy81]) and the plinth scheme Γ is defined by one equation (see [DK09]), i.e. the plinth ideal is principal. Let $a \in \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}}$ be a generator of the plinth ideal. By using Theorem 4.11, one can see that Γ is a fence if and only if $\mathrm{Aut}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)$ is an algebraic group (see Proposition 3.1 in [Sta13]).

As mentioned already, $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ is a family of unipotent automorphisms in $\mathrm{Cent}(\mathbf{u})$. If Γ is a fence, then we have beside this family another one: In fact, since Γ is a fence, one can see that there exists a coordinate system (x, y) of $\mathbb{A}^2 \simeq \mathbb{A}^3/\!\!/\mathbb{G}_a$ such that the generator of the plinth ideal a lies in $k[y]$ (see Proposition 3.3 in [Sta13]). It turned out that there exists an irreducible unipotent $\mathbf{e} \in \mathrm{Cent}(\mathbf{u})$ that induces the automorphism $(x + a, y)$ on $\mathbb{A}^2 \simeq \mathbb{A}^3/\!\!/\mathbb{G}_a$. Thus, $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{e}, \mathbf{u} \rangle} \cdot \mathbf{e}$ is a family of unipotent automorphisms in $\mathrm{Cent}(\mathbf{u})$.

Let us define

$$N(\mathbf{u}) := \begin{cases} \mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{e}, \mathbf{u} \rangle} \cdot \mathbf{e} \circ \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' & \text{if } \Gamma \text{ is a fence} \\ \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' & \text{otherwise.} \end{cases}$$

Then $N(\mathbf{u}) \subseteq \mathrm{Cent}(\mathbf{u})$ is a closed normal subgroup (see Proposition 5.15 in [Sta13]). If Γ is a fence, then $\mathrm{Iner}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)_u$ are the modifications of $(x + a, y)$ by elements of $k[y]$ and if Γ is not a fence, then $\mathrm{Iner}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)_u = \{\mathbf{id}\}$. It follows that $N(\mathbf{u})$ fits into the following split short exact sequence of ind-groups

$$(**) \qquad 1 \to \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow N(\mathbf{u}) \overset{p}{\twoheadrightarrow} \mathrm{Iner}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)_u \to 1$$

and if Γ is a fence, then p induces an isomorphism $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{u}, \mathbf{e} \rangle} \cdot \mathbf{e} \xrightarrow{\sim} \mathrm{Iner}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)_u$ (see Proposition 5.15 in [Sta13]).

As $a \in \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}}$ is a generator of the plinth ideal, there exists $s \in \mathcal{O}(\mathbb{A}^3)$, such that the comorphism $\mathbf{u}^*$ satisfies $\mathbf{u}^*(s) = s + a$. Hence, the inclusion $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}}[s] \subseteq \mathcal{O}(\mathbb{A}^3)$ induces a $\mathbb{G}_a$-invariant dominant morphism $\varphi\colon \mathbb{A}^3 \to \mathbb{A}^3/\!\!/\mathbb{G}_a \times \mathbb{A}^1$, where the $\mathbb{G}_a$-action on $\mathbb{A}^3/\!\!/\mathbb{G}_a \times \mathbb{A}^1$ is given by the unipotent automorphism $\tilde{\mathbf{u}} : (x, y, s) \mapsto (x, y, s + a)$. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^3 & \xrightarrow{\varphi} & \mathbb{A}^3/\!\!/\mathbb{G}_a \times \mathbb{A}^1 \\ \downarrow & & \downarrow \mathrm{pr}_1 \\ \mathbb{A}^3/\!\!/\mathbb{G}_a & \xrightarrow{\sim} & \mathbb{A}^3/\!\!/\mathbb{G}_a \end{array}$$

and the isomorphism $\mathbb{A}^3/\!\!/\mathbb{G}_a \xrightarrow{\sim} \mathbb{A}^3/\!\!/\mathbb{G}_a$ sends the plinth scheme of $\mathbf{u}$ onto the plinth scheme of $\tilde{\mathbf{u}}$.

Now, if Γ is not a fence, then $\mathrm{Aut}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)$ is an algebraic group and by Theorem 4.13 there exists an algebraic subgroup $\tilde{R} \subseteq \mathrm{Cent}(\tilde{\mathbf{u}})$ such that $\mathrm{Cent}(\tilde{\mathbf{u}})$ is the semi-direct product of $N(\tilde{\mathbf{u}})$ and $\tilde{R}$. If Γ is a fence, it follows from the sequence $(**)$ and from Theorem 4.13, that we have an exact sequence

$$1 \to N(\tilde{\mathbf{u}}) \hookrightarrow \mathrm{Cent}(\tilde{\mathbf{u}}) \twoheadrightarrow \mathrm{Aut}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)/\,\mathrm{Iner}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)_u \to 1$$

and the sequence splits. One can see that $\mathrm{Aut}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)/\,\mathrm{Iner}(\mathbb{A}^3/\!\!/\mathbb{G}_a, \Gamma)_u$ is an algebraic group, and thus $\mathrm{Cent}(\tilde{\mathbf{u}})$ is the semi-direct product of $N(\tilde{\mathbf{u}})$ and some algebraic subgroup $\tilde{R} \subseteq \mathrm{Cent}(\tilde{\mathbf{u}})$ (see Lemma 5.18 in [Sta13]).

The next step is to prove that the preimage $R := \eta^{-1}(\tilde{R})$ is an algebraic subgroup of $\mathrm{Cent}(\mathbf{u})$, where $\eta\colon \mathrm{Cent}(\mathbf{u}) \to \mathrm{Cent}(\tilde{\mathbf{u}})$ is the injective homomorphism induced by $\varphi\colon \mathbb{A}^3 \to \mathbb{A}^3/\!\!/\mathbb{G}_a \times \mathbb{A}^1$. Since η maps $N(\mathbf{u})$ onto $N(\tilde{\mathbf{u}})$, one can see that $\mathrm{Cent}(\mathbf{u})$ is the semi-direct product of $N(\mathbf{u})$ and R (see Theorem 5.17 in [Sta13]). One can then deduce ii).

We claim that $N(\mathbf{u})$ is the set of unipotent elements of $\mathrm{Cent}(\mathbf{u})$. To achieve this goal, we need the fact that $N(\mathbf{u})$ satisfies the property (Sat) (see Proposition 5.15 in [Sta13]).

Definition 4.16. We say that a subset $S \subseteq \mathcal{G}_n$ satisfies the *property (Sat)* if for all unipotent automorphisms $\mathbf{w} \in \mathcal{G}_n$ and for all $0 \neq f \in \mathcal{O}(\mathbb{A}^n)^{\mathbf{w}}$ we have

$$\text{(Sat)} \qquad f \cdot \mathbf{w} \in S \quad \Longrightarrow \quad \mathbf{w} \in S\,.$$

One can check that $N(\mathbf{u}) \subseteq \mathrm{Cent}(\mathbf{u})_u$. If $\mathbf{v} \in \mathrm{Cent}(\mathbf{u}) \setminus N(\mathbf{u})$ is a unipotent element, then $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{v}, \mathbf{u}\rangle} \cdot \mathbf{v} \cap N(\mathbf{u}) = \{\mathbf{id}\}$, as $N(\mathbf{u})$ satisfies (Sat). But since $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{v}, \mathbf{u}\rangle} \cdot \mathbf{v} \subseteq \mathrm{Cent}(\mathbf{u})$, we get a contradiction to the fact, that $\mathrm{Cent}(\mathbf{u})$ is the semi-direct product of $N(\mathbf{u})$ and an algebraic group. One can then deduce i), iv) and v).

To achieve iii), the idea is to prove that $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' = \mathrm{Cent}_G(G^{(i)})$ for a certain characteristic subgroup $G \subseteq \mathrm{Cent}(\mathbf{u})$, where $G^{(i)}$ denotes the i-th derived group of G (see Proposition 5.21 [Sta13]).

From Theorem 4.15 v) we get immediately the following corollary. It gives us a geometric description of the plinth scheme Γ, in case it is a fence.

Corollary 4.17 (Corollary E in [Sta13]). *Let $\mathbf{u} \in \mathcal{G}_3$ be unipotent. Assume that $\mathbf{u}$ is not a modified translation and that the plinth scheme Γ is a fence. Then Γ is the fixed point scheme of the induced action of* $\mathrm{Cent}(\mathbf{u})_u$ *on the algebraic quotient* $\mathbb{A}^3/\!\!/\mathbb{G}_a$.

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ON THE TOPOLOGIES ON IND-VARIETIES AND RELATED IRREDUCIBILITY QUESTIONS

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ABSTRACT. In the literature there are two ways of endowing an affine ind-variety with a topology. One possibility is due to SHAFAREVICH and the other to KAMBAYASHI. In this paper we specify a large class of affine ind-varieties where these two topologies differ. We give an example of an affine ind-variety that is reducible with respect to SHAFAREVICH's topology, but irreducible with respect to KAMBAYASHI's topology. Moreover, we give a counter-example of a supposed irreducibility criterion given in [Sha81] which is different from the counter-example given by HOMMA in [Kam96]. We finish the paper with an irreducibility criterion similar to the one given by SHAFAREVICH.

0. **Introduction.** In the 1960s, in [Sha66], SHAFAREVICH introduced the notion of an infinite-dimensional variety and infinite-dimensional group. In this paper, we call them ind-variety and ind-group, respectively. His motivation was to explore some naturally occurring groups that allow a natural structure of an infinite-dimensional analogue to an algebraic group (such as the group of polynomial automorphisms of the affine space). More precisely, he defined an ind-variety as the successive limit of closed embeddings

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \ldots$$

of ordinary algebraic varieties X_n and an ind-group as a group that carries the structure of an ind-variety compatible with the group structure. We denote the limit of $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ by $\varinjlim X_n$ and call $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ a *filtration*. If all X_n are affine, then $\varinjlim X_n$ is called *affine*. For example, one can define a filtration on the group of polynomial automorphisms of the affine space via the degree of an automorphism. Further examples of ind-groups are $\mathrm{GL}_n(k[t])$, $\mathrm{SL}_n(k[t])$, etc., where the filtrations are given via the degrees of the polynomial entries of the matrices (for properties of these filtrations in case $n = 2$ see [Sha04]). Fifteen years after his first paper [Sha66], SHAFAREVICH wrote another paper with the same title [Sha81], where he gave more detailed explanations of some statements of his first paper. Moreover, he endowed an ind-variety $\varinjlim X_n$ with the weak topology induced by the topological spaces $X_1 \subseteq X_2 \subseteq \ldots$. Later KAMBAYASHI defined (affine) ind-varieties in [Kam96] and [Kam03] via a different approach. Namely, he defined an affine ind-variety as a certain spectrum of a so-called pro-affine algebra (see Section 1 for the definition). This pro-affine algebra is then the ring of regular functions on the affine ind-variety. With this approach KAMBAYASHI introduced a topology in a natural way on an affine ind-variety. Namely, a subset is closed

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if it is the zero-set of some regular functions on the affine ind-variety. In analogy to the Zariski topology defined on an ordinary affine variety, we call this topology again *Zariski topology*. In this paper, we call the weak topology on an affine ind-variety *ind-topology* to prevent confusion, as the weak topology is finer than the Zariski topology. The Zariski topology and the ind-topology differ in general. For example, it follows from Exercise 4.1.E, IV. in [Kum02] that these topologies differ on the infinite-dimensional affine space $\mathbb{A}^\infty = \varinjlim \mathbb{A}^n$ (see Example 1). The aim of this paper is to specify classes of affine ind-varieties where these topologies differ or coincide, and to study questions concerning the irreducibility of an affine ind-variety (with respect to these topologies).

This paper is organized as follows. We give some basic definitions and notations in Section 1. In the next section we describe a large class of ind-varieties where the two topologies differ. The main result of this paper is the following

Theorem A. *Let $X = \varinjlim X_n$ be an affine ind-variety. If there exists $x \in X$ such that X_n is normal or Cohen-Macaulay at x for infinitely many n, and the local dimension of X_n at x tends to infinity, then the ind-topology and the Zariski topology are different.*

This theorem follows from a more general statement given in Proposition 1 (see also Remark 1). As a corollary to this theorem we get

Corollary B. *Let $X = \varinjlim X_n$ be an affine ind-variety such that X_n is normal for infinitely many n. Then the ind-topology and the Zariski topology coincide if and only if for all $x \in X$ the local dimension of X_n at x is bounded for all n.*

This corollary follows from a more general statement given in Corollary 6. As a contrast to Theorem A, we show in Proposition 7 that the two topologies coincide if $X = \varinjlim X_n$ is "locally constant" with respect to the Zariski topology. More precisely we prove

Proposition C. *If $X = \varinjlim X_n$ is an affine ind-variety such that every point has a Zariski open neighbourhood U with $U \cap X_n = U \cap X_{n+1}$ for all sufficiently large n, then the ind-topology and the Zariski topology coincide.*

Section 3 contains an example of an affine ind-variety that is reducible with respect to the ind-topology, but irreducible with respect to the Zariski topology. This is the content of Example 4.

In the last section we give a counter-example to Proposition 1 in [Sha81] (see Example 5). The content of the proposition is: an ind-variety $X = \varinjlim X_n$ is irreducible with respect to the ind-topology if and only if the set of irreducible components of all X_n is directed under inclusion. One can see that the latter condition is equivalent to the existence of a filtration $X'_1 \hookrightarrow X'_2 \hookrightarrow \ldots$ where each X'_n is irreducible and $\varinjlim X'_n = X$. In [Kam96], Homma gave a counter-example to that supposed irreducibility criterion. But in contrast to his counter-example, the number of irreducible components of X_n in our counter-example is bounded for all n. We finish the paper with the following irreducibility criterion. The proposition follows from Proposition 8.

Proposition D. *Let $X = \varinjlim X_n$ be an affine ind-variety where the number of irreducible components of X_n is bounded for all n. Then X is irreducible with respect to the ind-topology (Zariski topology) if and only if there exists a chain of irreducible subvarieties $X'_1 \subseteq X'_2 \subseteq \ldots$ in X (i.e., X'_n is an irreducible subvariety of some X_m) such that $\bigcup_n X'_n$ is dense in X with respect to the ind-topology (Zariski topology).*

1. **Definitions and notation.** Throughout this paper we work over an uncountable algebraically closed field k. We use the definitions and notation of Kambayashi in [Kam03] and Kumar in [Kum02]. Let us recall them briefly. A *pro-affine algebra* is a complete and separated commutative topological k-algebra such that 0 admits a countable base of open neighbourhoods consisting of ideals. Let A be a pro-affine algebra and let $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \ldots$ be a base for $0 \in A$ as mentioned above. Let $A_n = A/\mathfrak{a}_n$ and let $\mathfrak{Spm}(A)$ be the set of closed maximal ideals of A. Then we have

$$A = \varprojlim A_n \quad \text{and} \quad \mathfrak{Spm}(A) = \bigcup_{n=1}^{\infty} \mathfrak{Spm}(A_n)$$

(cf. 1.1 and 1.2 in [Kam03]).

Definition 1. An *affine ind-variety* is a pair $(\mathfrak{Spm}(A), A)$ where A is a pro-affine algebra such that $A/\mathfrak{a}_n$ is reduced and finitely generated for some countable base of ideals $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \ldots$ of $0 \in A$. We call A the *coordinate ring* of the affine ind-variety and the elements of A *regular functions*. Two ind-varieties are called *isomorphic* if the underlying pro-affine algebras are isomorphic as topological k-algebras. Such an isomorphism induces then a bijection of the spectra.

One can construct affine ind-varieties in the following way. Consider a *filtration of affine varieties*, i.e., a countable sequence of closed embeddings of affine varieties

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \ldots .$$

Let $X = \bigcup_{n=1}^{\infty} X_n$ as a set and let $\mathcal{O}(X) := \varprojlim \mathcal{O}(X_n)$. We endow $\mathcal{O}(X)$ with the topology induced by the product topology of $\prod_n \mathcal{O}(X_n)$, where $\mathcal{O}(X_n)$ carries the discrete topology for all n. Then $(\mathfrak{Spm}(\mathcal{O}(X)), \mathcal{O}(X))$ is an affine ind-variety and there is a natural bijection $X \to \mathfrak{Spm}(\mathcal{O}(X))$ induced by the bijections $X_n \to \mathfrak{Spm}(\mathcal{O}(X_n))$. In the following, we denote this ind-variety by $\varinjlim X_n$. In fact, every affine ind-variety can be constructed in this way (up to isomorphy). Two filtrations $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ and $X'_1 \hookrightarrow X'_2 \hookrightarrow \ldots$ induce the same affine ind-variety (up to isomorphy) if and only if there exists a bijection

$$f\colon \bigcup_{n=1}^{\infty} X_n \to \bigcup_{n=1}^{\infty} X'_n$$

with the following property: for every i there exists j_i and for every j there exists i_j, such that $f|_{X_i}\colon X_i \to X_{j_i}$ and $f^{-1}|_{X_j}\colon X_j \to X_{i_j}$ are closed embeddings of affine varieties. Such filtrations are called *equivalent*.

2. **Topologies on affine ind-varieties.** So far we have not established any topology on the set $\mathfrak{Spm}(A)$ of an affine ind-variety $(\mathfrak{Spm}(A), A)$. As mentioned in the introduction there are two ways to introduce a topology on the set $\mathfrak{Spm}(A)$. The first possibility is due to Shafarevich [Sha66], [Sha81] and we call it the *ind-topology*. A subset $Y \subseteq \mathfrak{Spm}(A)$ is closed in this topology if and only if $A \cap \mathfrak{Spm}(A_n)$ is a closed subset of $\mathfrak{Spm}(A_n)$ for all n. One can easily check that this topology does

not depend on the choice of the ideals $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$. The second possibility is due to KAMBAYASHI [Kam96], [Kam03] and we call it the *Zariski topology.* The closed subsets in this topology are the subsets of the form

$$V_{\mathfrak{Spm}(A)}(E) := \{ \mathfrak{m} \in \mathfrak{Spm}(A) \mid \mathfrak{m} \supseteq E \} ,$$

where E is any subset of A. Clearly, the ind-topology is finer than the Zariski topology. But in general these two topologies on $\mathfrak{Spm}(A)$ differ. In the next proposition (which implies Theorem A) we specify a large class of affine ind-varieties where the two topologies differ.

Proposition 1. *We assume that* $\mathrm{char}(k) = 0$*. Let* $X = \varinjlim X_n$ *be an affine ind-variety. Assume that there exists* $x \in X$ *such that* $\mathcal{O}_{X_n,x}$ *satisfies* SERRE*'s condition* (S_2) *for infinitely many* n *and* $\dim_x X_n \to \infty$ *if* $n \to \infty$*. Then there exists a subset* $Y \subseteq X$ *such that*

i) Y *is closed in* X *with respect to the ind-topology,*
ii) Y *is not closed in* X *with respect to the Zariski topology.*

In particular, there exists no isomorphism $X \to X$ *of affine ind-varieties that is a homeomorphism if we endow the first* X *with the ind-topology and the second* X *with the Zariski topology.*

Remark 1. A Noetherian ring A satisfies SERRE's condition (S_2) if $\operatorname{depth} A_{\mathfrak{p}} \geq \min\{\dim A_{\mathfrak{p}}, 2\}$ for all primes $\mathfrak{p} \subseteq A$. For example, this is satisfied if A is normal (and hence also if A is a unique factorization domain) or Cohen-Macaulay (and hence also if A is Gorenstein, locally a complete intersection or regular) (see Theorem 23.8 [Mat86]).

We will use the following lemmata to prove Proposition 1.

Lemma 2. *Let* Z *and* Y *be affine varieties and assume that there exists a closed embedding* $Z \hookrightarrow Y$*. If* $f\colon Z \twoheadrightarrow \mathbb{A}^{\dim Z}$ *is a finite surjective morphism, then there exists a finite surjective morphism* $g\colon Y \twoheadrightarrow \mathbb{A}^{\dim Y}$ *such that* $g|_Z = \iota \circ f$*, where* $\iota : \mathbb{A}^{\dim Z} \hookrightarrow \mathbb{A}^{\dim Y}$ *is given by* $\iota(v) = (v, 0)$*.*

Proof. Let $A := \mathcal{O}(Z)$, $B := \mathcal{O}(Y)$ and let $\psi\colon B \twoheadrightarrow A$ be the surjective homomorphism induced by $Z \hookrightarrow Y$. Further, let $f_1, \dots, f_n$ be the coordinate functions of f. By assumption $k[f_1, \dots, f_n] \subseteq A$ is an integral extension and $f_1, \dots, f_n$ are algebraically independent. Choose generators $b_1, \dots, b_l$ of the k-algebra B such that $\psi(b_i) = f_i$ for $i = 1, \dots, n$. For every $j = n+1, \dots, l$ there exists a monic polynomial $p_j \in k[b_1, \dots, b_n][T]$ such that $h_j := p_j(b_j) \in \ker(\psi)$, since $k[f_1, \dots, f_n] \subseteq A$ is integral. Thus,

$$k[b_1, \dots, b_n, h_{n+1}, \dots, h_l] \subseteq B$$

is an integral extension. If $b_1, \dots, b_n, h_{n+1}, \dots, h_l$ are algebraically independent, then we are done. Otherwise, there exists a non-zero polynomial $f(X_1, \dots, X_l)$ with coefficients in k such that $f(b_1, \dots, b_n, h_{n+1}, \dots, h_l) = 0$. Exactly the same as in the proof of Lemma 2, §33 [Mat86] one can see that there exist $c_1, \dots, c_{l-1} \in k$ such that h_l is integral over $k[b'_1, \dots, b'_n, h'_{n+1}, \dots, h'_{l-1}]$, where $b'_i := b_i - c_i h_l$ and $h'_i := h_i - c_i h_l$. Thus,

$$k[b'_1, \dots, b'_n, h'_{n+1}, \dots, h'_{l-1}] \subseteq B$$

is an integral extension. By induction there exists m with $n \leq m < l$ and algebraically independent elements $b''_1, \dots, b''_n, h''_{n+1}, \dots, h''_m \in B$ such that B is integral over $k[b''_1, \dots, b''_n, h''_{n+1}, \dots, h''_m]$ and $b''_i - b_i$, $h''_i \in \ker(\psi)$. This proves the lemma. □

Remark 2. From an iterative use of the lemma above we can deduce the following. For every affine ind-variety $X = \varinjlim X_n$ there exists a surjective map of the underlying sets $X \twoheadrightarrow \mathbb{A}^\infty$ such that the restriction to every X_n yields a finite surjective morphism $X_n \twoheadrightarrow \mathbb{A}^{\dim X_n}$.

Lemma 3. *We assume that* $\mathrm{char}(k) = 0$*. Let* Y *be an irreducible affine variety and let* X *be an affine scheme of finite type over* k *that is reduced in an open dense subset. If* $f \colon X \to Y$ *is a dominant morphism, then there exists an open dense subset* $U \subseteq Y$ *such that* $f^{-1}(u)$ *is reduced in an open dense subset for all* $u \in U$*.*

Proof. Without loss of generality, one can assume that f is flat and surjective (see Theorem 14.4 (Generic freeness) [Eis95]). Since X is reduced in an open dense subset, there exists an open dense subset $X' \subseteq X$ such that all fibres of $f|_{X'} \colon X' \to Y$ are reduced (see Corollary 10.7, Ch. III (Generic smoothness) and Theorem 10.2, Ch. III [Har77]; here we use $\mathrm{char}(k) = 0$). Let $K := X \setminus X'$ be endowed with the reduced induced closed subscheme structure of X and let $g := f|_K \colon K \to Y$. If g is not dominant, then the fibres of f over an open dense subset are reduced and we are done. Hence we can assume that g is dominant. Again according to Theorem 24.1 [Mat86] there exists an open dense subset $U \subseteq Y$ such that $g|_{g^{-1}(U)} \colon g^{-1}(U) \twoheadrightarrow U$ is flat and surjective. Thus, we have for all $u \in U$ and $x \in g^{-1}(u)$

$$\dim_x g^{-1}(u) = \dim_x g^{-1}(U) - \dim_u U < \dim_x X - \dim_u Y = \dim_x f^{-1}(u)\,.$$

It follows that $f^{-1}(u) \setminus g^{-1}(u)$ is a reduced open dense subscheme of $f^{-1}(u)$ for all $u \in U$. This implies the lemma. □

According to Ex. 11.10 [Eis95] we have the following criterion for reducedness of a Noetherian ring.

Lemma 4. *A Noetherian ring* A *is reduced if and only if*

(R_0) *the localization of* A *at each prime ideal of height* 0 *is regular,*
(S_1) A *has no embedded associated prime ideals.*

One can see that condition (R_0) is satisfied for a Noetherian ring A if $\mathrm{Spec}(A)$ is reduced in an open dense subset. Thus we get the following

Lemma 5. *Let* X *be a Noetherian affine scheme that is reduced in an open dense subset. If* $\mathcal{O}_{X,x}$ *satisfies* (S_1) *for a point* $x \in X$*, then* $\mathcal{O}_{X,x}$ *is reduced.*

Now we have the preliminary results to prove Proposition 1. The strategy is as follows. First we construct $0 \neq f_n \in \mathcal{O}(X_n)$ such that $f_n(x) = 0$, $f_n|_{X_{n-1}} = f_{n-1}^2$ and $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$ is reduced. The main part of the proof is devoted to showing the reducedness and for that matter we use the condition (S_2) of the local ring $\mathcal{O}_{X_n,x}$. Then we define $Y := \bigcup_n V_{X_n}(f_n)$. It follows that Y is closed in X with respect to the ind-topology. Afterwards, we prove that Y is not closed in X with respect to the Zariski topology. For that purpose, we take $\varphi = (\varphi_n) \in \mathcal{O}(X) = \varprojlim \mathcal{O}(X_n)$ that vanishes on Y, and we show that φ_n vanishes also on all irreducible components of X_n passing through x. The latter we deduce from the fact that

$$\varphi_n = \varphi_{n+i}|_{X_n} \in f_{n+i}|_{X_n}\mathcal{O}_{X_n,x} = f_n^{2^i}\mathcal{O}_{X_n,x}$$

for all $i \geq 0$ and KRULL's Intersection Theorem.

Proof of Proposition 1. For the sake of simpler notation, we assume that $\mathcal{O}_{X_n,x}$ satisfies (S_2) and $\dim_x X_n = n$ for all n. Let X'_n be the union of all irreducible components of X_n containing x and let W_n be the union of all irreducible components of all X_i with $i \leq n$, not containing x and of strictly smaller dimension than n. Then, $X'_1 \cup W_1 \hookrightarrow X'_2 \cup W_2 \hookrightarrow \ldots$ is an equivalent filtration of X to $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$, since $\dim_x X_n \to \infty$. Thus, we can further impose that $\dim_x X_n = \dim X'_n = \dim X_n$ and $\dim_p X_n < \dim X_n$ for all $p \notin X'_n$. As $\mathcal{O}_{X_n,x}$ satisfies (S_2), it follows from Corollary 5.10.9 [Gro65] that X'_n is equidimensional.

Now, we construct the $0 \neq f_n \in \mathcal{O}(X_n)$. From Lemma 2 it follows that we can choose algebraically independent elements $x_1, \ldots, x_n \in \mathcal{O}(X_n)$ such that $\mathcal{O}(X_n)$ is finite over $k[x_1, \ldots, x_n]$ and x_n restricted to X_{n-1} is zero. We can assume that the finite morphism $X_n \twoheadrightarrow \mathbb{A}^n$ induced by $k[x_1, \ldots, x_n] \subseteq \mathcal{O}(X_n)$ sends x to $0 \in \mathbb{A}^n$. Since $\dim_p X_n < \dim X_n$ for all $p \notin X'_n$ and X'_n is equidimensional, it follows that

$$(*) \qquad k[x_1, \ldots, x_n] \hookrightarrow \mathcal{O}(X_n) \twoheadrightarrow \mathcal{O}(K) \quad \text{is injective}$$

for all irreducible components K of X'_n. Let us define

$$f_1 := c_1 x_1 \quad \textit{and} \quad f_{n+1} := f_n^2 + c_{n+1} x_{n+1} \, ,$$

where $c_1, c_2, \ldots \in k$, not all equal to zero. It follows that $f_n(x) = 0$ and $f_{n+1}|_{X_n} = f_n^2$. The aim is to prove that $c_1, c_2, \ldots \in k$ can be chosen such that not all are equal to zero and $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$ is reduced for $n > 1$. Consider the morphism

$$\psi_n \colon Z_n \longrightarrow \mathbb{A}^n \, ,$$

where Z_n is the affine scheme with coordinate ring

$$S_n := \mathcal{O}(X'_n)[c_1, \ldots, c_n]/(f_n)$$

and ψ_n is the restriction of the canonical projection $X'_n \times \mathbb{A}^n \twoheadrightarrow \mathbb{A}^n$ to the closed subscheme Z_n. If $(c_1, \ldots, c_n)$ is fixed, then $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$ is the local ring of the fibre $\psi_n^{-1}(c_1, \ldots, c_n)$ in the point $(x, c_1, \ldots, c_n) \in Z_n$. For that reason we will study the fibres of the morphism $\psi_n \colon Z_n \to \mathbb{A}^n$. We claim that Z_n is reduced in an open dense subset for $n > 1$. To prove this claim, we mention first that

$$(S_n)_{x_n} \simeq \mathcal{O}(X'_n)_{x_n}[c_1, \ldots, c_n]/(f_{n-1}^2 + c_n x_n) \simeq \mathcal{O}(X'_n)_{x_n}[c_1, \ldots, c_{n-1}]$$

is reduced. Let $R_n := k[x_1, \ldots, x_n][c_1, \ldots, c_n]/(f_n)$. It follows that the morphisms $\mathrm{Spec}(S_n) \twoheadrightarrow \mathrm{Spec}(R_n)$ and $\mathrm{Spec}(S_n/(x_n)) \twoheadrightarrow \mathrm{Spec}(R_n/(x_n))$ are both finite and surjective. As $\dim R_n/(x_n) < \dim R_n$ for $n > 1$ we get $\dim S_n/(x_n) < \dim S_n$. Since X'_n is equidimensional one can deduce from $(*)$ that Z_n is equidimensional. Hence, $\mathrm{Spec}((S_n)_{x_n}) \subseteq Z_n$ is an open dense reduced subscheme.

Since $\{x\} \times \mathbb{A}^n$ is contained in Z_n, it follows that ψ_n is surjective. For $n > 1$ there exists an open dense subset $U_n \subseteq \mathbb{A}^n$ such that

$$\psi_n|_{\psi_n^{-1}(U_n)} \colon \psi_n^{-1}(U_n) \twoheadrightarrow U_n$$

is surjective and flat, and every fibre is reduced in an open dense subset (see Lemma 3 and Theorem 24.1 [Mat86]). With the aid of $(*)$ it follows that f_n is an $\mathcal{O}_{X_n,x}$-regular sequence for every choice $(c_1, \ldots, c_n) \in U_n$. Since $\mathcal{O}_{X_n,x}$ satisfies (S_2), we get from Corollary 5.7.6 [Gro65] that $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$ satisfies (S_1). But as $\psi_n^{-1}(c_1, \ldots, c_n)$ is reduced in an open dense subset, it follows from Lemma 5 that it is reduced in the point $(x, c_1, \ldots, c_n)$. Hence, for $n > 1$ it follows that $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$

is reduced if we choose $(c_1, \ldots, c_n) \in U_n$. For $i \geq n$ let $\pi_n^i : \mathbb{A}^i \twoheadrightarrow \mathbb{A}^n$ be the projection onto the first n components. As the field k is uncountable, one can choose inductively

$$0 \neq c_1 \in \bigcap_{i \geq 1} \pi_1^i(U_i)\,, \quad (c_1, \ldots, c_n, c_{n+1}) \in \bigcap_{i \geq n+1} \pi_{n+1}^i(U_i) \cap \{(c_1, \ldots, c_n)\} \times \mathbb{A}^1\,.$$

Hence, $(c_1, \ldots, c_n) \in U_n$ for all $n > 1$ and not all $c_1, c_2, \ldots$ are equal to zero. This finishes the construction of the f_n.

Let us define $Y := \bigcup_n V_{X_n}(f_n)$. Since $f_{n+1}|_{X_n} = f_n^2$ for all n, Y satisfies i). Take any $\varphi = (\varphi_n) \in \varprojlim \mathcal{O}(X_n)$ that vanishes on Y. We claim that $\varphi|_{X'} = 0$, where $X' := \bigcup_n X'_n$. It is enough to prove that $\varphi_n = 0$ in $\mathcal{O}_{X_n,x}$. Since $\varphi_m|_{Y_m} = 0$ and $\mathcal{O}_{X_m,x}/f_m\mathcal{O}_{X_m,x}$ is reduced, it follows that $\varphi_m \in f_m\mathcal{O}_{X_m,x}$. Using $f_{m+1}|_{X_m} = f_m^2$ again, we get by induction

$$\varphi_n = \varphi_{n+i}|_{X_n} \in f_n^{2^i}\mathcal{O}_{X_n,x} \quad \text{for all } i \geq 0\,,\ n > 1\,.$$

But according to KRULL's Intersection Theorem (see Theorem 8.10 [Mat86]), we have $\bigcap_{i \geq 0} f_n^i \mathcal{O}_{X_n,x} = 0$, hence $\varphi_n = 0$ in $\mathcal{O}_{X_n,x}$. Since $f_n|_{X'_n} \neq 0$ (cf. ($*$)), we get $X' \cup Y \supsetneq Y$. Thus Y satisfies ii) according to the afore mentioned claim. □

The following example is a special case of the construction in the proof of Proposition 1. We mention it here, since we will use it in future examples.

Example 1 (See Ex. 4.1.E, IV. in [Kum02])**.** Let $f_n \in k[x_1, \ldots, x_n] = \mathcal{O}(\mathbb{A}^n)$ be recursively defined as

$$f_1 := x_1\,, \quad f_{n+1} := f_n^2 + x_{n+1}\,.$$

Then $\bigcup_n V_{\mathbb{A}^n}(f_n)$ is a proper closed subset of the infinite-dimensional affine space $\mathbb{A}^\infty = \varinjlim \mathbb{A}^n$ with respect to the ind-topology, but it is dense in $\mathbb{A}^\infty$ with respect to the Zariski topology.

Let $\mathcal{G}$ be the group of polynomial automorphisms of the affine space $\mathbb{A}^n$, where n is a fixed number ≥ 2. We prove in the next example that the ind-topology and the Zariski topology on $\mathcal{G}$ differ if we consider $\mathcal{G}$ as an affine ind-variety via the filtration given by the degree of an automorphism.

Example 2. First, we define on $\mathcal{G}$ a filtration of affine varieties (via the degree). Let $\mathcal{E}$ be the set of polynomial endomorphisms of the affine space $\mathbb{A}^n$ and let $\mathcal{E}_d$ be the subset of all $\varphi \in \mathcal{E}$ of degree $\leq d$. Denote by $U_d \subseteq \mathcal{E}_d$ the subset of all $\varphi \in \mathcal{E}_d$ such that $\mathrm{Jac}(\varphi) \in k^*$. One can see that $U_d \subseteq \mathcal{E}_d$ is a locally closed subset and it inherits the structure of an affine variety from $\mathcal{E}_d$. With Corollary 0.2 [Kam79] and the estimate of the degree of the inverse of an automorphism due to GABBER (see Corollary 1.4 in [BCW82]) one can deduce that $\mathcal{G}_d \subseteq U_d$ is a closed subset. Thus $\mathcal{G}_d$ is locally closed in $\mathcal{E}_d$ and it inherits the structure of an affine variety from $\mathcal{E}_d$. Moreover, one can see that $\mathcal{G}_d$ is closed in $\mathcal{G}_{d+1}$. In the following, we consider $\mathcal{G}$ as an affine ind-variety via the filtration $\mathcal{G}_1 \hookrightarrow \mathcal{G}_2 \hookrightarrow \ldots$ of affine varieties.

We claim that the ind-topology and the Zariski topology on $\mathcal{G}$ differ. Consider the subset

$$M := \{\, (x_1 + p, x_2, \ldots, x_n) \in \mathcal{G} \mid p \in k[x_n] \,\} \subseteq \mathcal{G}\,.$$

It is closed in $\mathcal{G}$ with respect to the ind-topology. We consider M as an affine ind-variety via $M := \varinjlim M \cap G_d$ and thus $M \simeq \mathbb{A}^\infty$ as affine ind-varieties. According to Example 1 there exists a proper subset $Y \subsetneq M$ that is closed with respect to the ind-topology, but it is dense in M with respect to the Zariski topology. Hence, every regular function on $\mathcal{G}$ vanishing on Y, vanishes also on M. This implies the claim.

Remark 3. A similar argument as in Example 2 shows that the ind-topology and the Zariski topology differ on $\mathrm{GL}_n(k[t])$ and also on $\mathrm{SL}_n(k[t])$.

Next, we give a corollary to Proposition 1 which implies Corollary B. Before we state the corollary, we introduce the following notation. For any ind-variety $X = \varinjlim X_n$ we choose connected components X_n^i of X_n, $i = 1, \ldots, k_n$, such that

$$X_n = \bigcup_{i=1}^{k_n} X_n^i \quad \text{and} \quad X_n^i \subseteq X_{n+1}^i \text{ for all } i = 1, \ldots, k_n$$

(it can be that $X_n^i = X_n^j$ for $i \neq j$). We remark that the decomposition of an ind-variety into connected components is the same for the ind-topology and the Zariski topology.

Corollary 6. *We assume that* $\mathrm{char}(k) = 0$. *Let* $X = \varinjlim X_n$ *be an affine ind-variety such that for i fixed, the number of irreducible components of* X_n^i *is bounded for all* n. *Moreover, assume that* $\mathcal{O}(X_n)$ *satisfies* (S_2) *for infinitely many* n. *Then the following statements are equivalent:*

i) *The ind-topology and the Zariski topology on* X *coincide.*
ii) *For all* $x \in X$ *the local dimension of* X_n *at* x *is bounded for all* n.
iii) *Every connected component of* X *is contained in some* X_n.

Proof. Every connected component of X is equal to some $X^i := \bigcup_n X_n^i$.
i) ⇒ ii): This follows from Proposition 1.
ii) ⇒ iii): As X_n^i satisfies (S_2) and is connected, X_n^i is equidimensional (see Corollary 5.10.9 [Gro65]). Thus, $X_n^i = X_{n+1}^i$ for n large enough, as the number of irreducible components of X_n^i is bounded for all n. Thus, $X^i \subseteq X_n$ for some n.
iii) ⇒ i): This follows from the fact that every connected component of X is closed and open with respect to the Zariski topology. □

As a contrast to Proposition 1, the two topologies coincide if the affine ind-variety is "locally constant" with respect to the Zariski topology. The following proposition coincides with Proposition C.

Proposition 7. *Let* $X = \varinjlim X_n$ *be an affine ind-variety. Assume that every* $x \in X$ *has a Zariski open neighbourhood* $U_x \subseteq X$ *such that* $U_x \cap X_n = U_x \cap X_{n+1}$ *for all sufficiently large* n. *Then the two topologies on* X *coincide.*

Proof. Let $Y \subseteq X$ be a closed subset with respect to the ind-topology. One can see that $Y \cap U_x$ is closed in U_x with respect to the Zariski topology for all $x \in X$. This proves that Y is closed in X with respect to the Zariski topology. □

The following example is an application of the proposition above. We construct a proper ind-variety (i.e., it is not a variety) such that the ind-topology and the Zariski topology coincide and moreover, it is connected.

Example 3. Let L_n be defined as

$$L_n := V_{\mathbb{A}^n}(x_1 - 1, x_2 - 1, \ldots, x_{n-1} - 1) \subseteq \mathbb{A}^n .$$

Remark that $L_n \cap L_{n+1} = \{(1, \dots, 1)\} \subseteq \mathbb{A}^n$ for all n and $L_n \cap L_m = \varnothing$ for all n, m with $|n - m| \geq 2$. Let $X := \varinjlim X_n$ where $X_n := L_1 \cup \ldots \cup L_n \subseteq \mathbb{A}^n$. It follows that $X \subseteq \mathbb{A}^\infty$ is a closed connected subset in the ind-topology. We claim that the ind-topology and the Zariski topology on X coincide. According to the proposition above it is enough to show that X is "locally constant" with respect to the Zariski topology. Let $x \in X$. Then there exists N such that $x \in L_N$, but $x \notin L_{N+1}$. Let $U_x := X \setminus V_{\mathbb{A}^\infty}(f_1, \dots, f_N) \subseteq X$ where $f_i \in \mathcal{O}(\mathbb{A}^\infty)$ is given by

$$f_i|_{\mathbb{A}^n} = x_i - 1 \in k[x_1, \dots, x_n] \quad \text{for all } n \geq N.$$

Thus, $U_x \subseteq X$ is a Zariski open neighbourhood of x. Moreover, for all $n > N$ we have $L_n \subseteq V_{\mathbb{A}^\infty}(f_1, \dots, f_N)$. Hence we have $U_x \cap X_n = U_x \cap X_{n+1}$ for all $n \geq N$.

As remarked before Corollary 6, connectedness of an affine ind-variety is the same for both topologies. But this is no longer true for irreducibility as we will see in the next section (see Example 4).

3. **Irreducibility via the coordinate ring.** It is well known that an affine variety X is irreducible if and only if the coordinate ring $\mathcal{O}(X)$ is an integral domain. This statement remains true for affine ind-varieties endowed with the Zariski topology. The proof is completely analogous to the proof for affine varieties. In the case of the ind-topology it is still true that $\mathcal{O}(X)$ is an integral domain if X is irreducible, as the ind-topology is finer than the Zariski topology. But the converse is in general false. In the following we give an example of an affine ind-variety X, which is reducible in the ind-topology, but its coordinate ring $\mathcal{O}(X)$ is an integral domain and thus it is irreducible in the Zariski topology.

Example 4. Throughout this example we work in the ind-topology. Let $g_n \in k[x_1, \dots, x_n]$ be defined as

$$g_n := x_1 + \ldots + x_n\,,$$

and let f_n be defined as in Example 1. By construction, f_n and g_n are irreducible polynomials. The affine ind-variety $X := \varinjlim (V_{\mathbb{A}^n}(f_n) \cup V_{\mathbb{A}^n}(g_n))$ decomposes into the proper closed subsets $\bigcup_n V_{\mathbb{A}^n}(f_n)$ and $\bigcup_n V_{\mathbb{A}^n}(g_n))$ and thus X is reducible. We claim that $\mathcal{O}(X) = \varprojlim k[x_1, \dots, x_n]/(f_n g_n)$ is an integral domain. Assume towards a contradiction that there exist $(\varphi_n), (\psi_n) \in \prod_{n=1}^\infty k[x_1, \dots, x_n]$ such that (φ_n) and (ψ_n) define non-zero elements in $\mathcal{O}(X)$, but $(\varphi_n \psi_n)$ defines zero in $\mathcal{O}(X)$. By definition, there exists $\alpha_n \in k[x_1, \dots, x_n]$ such that

$$(*) \qquad \varphi_{n+1}(x_1, \dots, x_n, 0) = \varphi_n + f_n g_n \alpha_n \quad \text{for all } n\,.$$

Since $(\varphi_n \psi_n)$ defines zero in $\mathcal{O}(X)$, it follows that $f_n g_n$ divides $\varphi_n \psi_n$ for $n > 0$. Hence we can assume without loss of generality that f_n divides φ_n for infinitely many n. Eq. ($*$) and the definition of f_{n+1} show that f_n divides φ_n for all n. Since $(\varphi_n) \neq 0$ in $\mathcal{O}(X)$ there exists $N > 1$ such that g_N does not divide φ_N. Let $\rho_n \in k[x_1, \dots, x_n]$ such that $\varphi_n = f_n \rho_n$. It follows that g_N does not divide ρ_N, in particular $\rho_N \neq 0$. According to ($*$) and the definition of f_{n+1} we have

$$(**) \qquad \rho_n = f_n \cdot \rho_{n+1}(x_1, \dots, x_n, 0) - g_n \cdot \alpha_n \quad \text{for all } n\,.$$

Since g_N does not divide ρ_N it follows that there exists $p \in \mathbb{A}^N$ with $g_N(p) = 0$ and $\rho_N(p) \neq 0$. Let $\gamma_n \colon \mathbb{A}^1 \to \mathbb{A}^n$ be the curve defined by $\gamma_n(t) = (p, 0, \dots, 0) +$

$(t, -t, 0, \ldots, 0)$ for $n \geq N$. Since $g_n(\gamma_n(t)) = 0$ it follows from ($**$) that $\rho_n(\gamma_n(t)) = f_n(\gamma_n(t))\rho_{n+1}(\gamma_{n+1}(t))$. This implies

$$0 \neq \rho_N(\gamma_N(t)) = \left(\prod_{i=N}^{n-1} f_i(\gamma_i(t))\right) \cdot \rho_n(\gamma_n(t)) \quad \text{for all } n \geq N\,.$$

Since $f_i(\gamma_i(t))$ is a polynomial of degree 2^{i-1} for all $i \geq N$, it follows that the polynomial $\rho_N(\gamma_N(t))$ is of unbounded degree, a contradiction.

4. **Irreducibility via the filtration.** One would like to give a criterion for connectedness or irreducibility in terms of the filtration $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ of the affine ind-variety. In the case of connectedness Shafarevich gave a nice description via the filtration (see Proposition 2 [Sha81]) and Kambayashi gave a proof for it (see Proposition 2.4 [Kam96]) (the proof works in both topologies, as connectedness of an affine ind-variety is the same for both topologies). In the case of irreducibility, things look different.

If we start with a filtration $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ of irreducible affine varieties, then one can see that $\varinjlim X_n$ is an irreducible affine ind-variety in both topologies. Likewise one can ask if every irreducible affine ind-variety is obtained from a filtration of irreducible affine varieties. One can see that the latter property is equivalent to the following condition: the set $\mathscr{K}$ of all irreducible components of all X_n is directed under inclusion for some (and hence every) filtration $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$. Shafarevich claims in [Sha81] that the latter condition is equivalent to the irreducibility of X in the ind-topology. But Homma gave in [Kam96] a counter-example X to this statement. For every filtration $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ of Homma's counter-example X the number of irreducible components of X_n tends to infinity if $n \to \infty$. Here we give another counter-example. Namely, we construct an irreducible affine ind-variety $X = \varinjlim X_n$ (irreducible with respect to both topologies) such that $\mathscr{K}$ is not directed, but X_n consists of exactly two irreducible components for $n > 1$.

Example 5. Let us define $g_n \in k[x_1, \ldots, x_n]$ recursively by

$$g_1 := (x_1 - 1)\,, \qquad g_{n+1} := (x_1 - (n+1)) \cdot g_n - x_{n+1}\,.$$

By construction every g_n is an irreducible polynomial. Let $Y_n := V_{\mathbb{A}^n}(g_n) \subseteq \mathbb{A}^n$. It follows that $Y_n \subseteq Y_{n+1}$ for all n. Let further $Z_n := V_{\mathbb{A}^n}(x_2, \ldots, x_n) \subseteq \mathbb{A}^n$ and $X_n := Y_n \cup Z_n$. It follows that $X_n \subseteq X_{n+1}$ is a closed subset for all n. Let $X := \varinjlim X_n$. We get

$$Y_n \cap Z_n = V_{\mathbb{A}^n}(g_n, x_2, \ldots, x_n) = V_{\mathbb{A}^n}(\prod_{i=1}^{n}(x_1 - i), x_2, \ldots, x_n) = \{e_1, 2e_1, \ldots, ne_1\}\,,$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{A}^n$. The set $\mathscr{K}$ defined above is not directed and X_n decomposes in two irreducible components for $n > 1$. It remains to show that X is irreducible with respect to the ind-topology, as in that case X is also irreducible in the Zariski topology. As Y_n is irreducible for all n, it follows that $Y = \bigcup_n Y_n$ is irreducible. Since

$$Z_m \subseteq \overline{\bigcup_{n=1}^{\infty} Y_n \cap Z_n} \subseteq \overline{Y} \subseteq X \qquad \text{for all } m\,,$$

we have $X = \overline{Y}$, where the closure is taken in the ind-topology. Since Y is irreducible, as a consequence X is also irreducible.

We conclude this paper with a criterion for the irreducibility of an affine ind-variety $X = \varinjlim X_n$ where the number of irreducible components of X_n is bounded for all n. Unfortunately we need for this criterion also information about the closure of a subset in the "global" object X and not only about the filtration $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ itself. The following proposition implies Proposition D.

Proposition 8. *Let $X = \varinjlim X_n$ be an affine ind-variety such that the number of irreducible components of X_n is bounded by l for all n. Then X is irreducible in the ind-topology (Zariski topology) if and only if for all n there exists an irreducible component F_n of X_n such that $F_1 \subseteq F_2 \subseteq \ldots$ and $\bigcup_n F_n$ is dense in X with respect to the ind-topology (Zariski topology).*

Proof. One can read the proof either with respect to the ind-topology or with respect to the Zariski topology. Let $X = \varinjlim X_n$ be irreducible. For all n let us write $X_n = X_n^1 \cup \ldots \cup X_n^l$ where X_n^i is an irreducible component of X_n and for all n we have $X_n^i \subseteq X_{n+1}^i$ (it can be that $X_n^i = X_n^j$ for $i \neq j$). Thus, one gets

$$X = \overline{\bigcup_n X_n^1} \cup \ldots \cup \overline{\bigcup_n X_n^l} \, .$$

Since X is irreducible the claim follows. The converse of the statement is clear. □

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ON AUTOMORPHISMS OF THE AFFINE CREMONA GROUP

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Abstract. We show that every automorphism of the group $\mathcal{G}_n := \mathrm{Aut}(\mathbb{A}^n)$ of polynomial automorphisms of complex affine n-space $\mathbb{A}^n = \mathbb{C}^n$ is inner up to field automorphisms when restricted to the subgroup $T\mathcal{G}_n$ of tame automorphisms. This generalizes a result of Julie Deserti who proved this in dimension $n = 2$ where all automorphisms are tame: $T\mathcal{G}_2 = \mathcal{G}_2$. The methods are different, based on arguments from algebraic group actions.

1. **Notation.** Let $\mathcal{G}_n := \mathrm{Aut}(\mathbb{A}^n)$ denote the group of polynomial automorphisms of complex affine n-space $\mathbb{A}^n = \mathbb{C}^n$. For an automorphism $\mathbf{g}$ we use the notation $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ if

$$\mathbf{g}(a) = (g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)) \quad \text{for } a = (a_1, \ldots, a_n) \in \mathbb{A}^n$$

where $g_1, \ldots, g_n \in \mathbb{C}[x_1, \ldots, x_n]$. Moreover, we define the degree of $\mathbf{g}$ by $\deg \mathbf{g} := \max(\deg g_1, \ldots, \deg g_n)$. The product of two automorphisms is denoted by $\mathbf{f} \circ \mathbf{g}$.

The automorphisms of the form $(g_1, \ldots, g_n)$ where $g_i = g_i(x_i, \ldots, x_n)$ depends only on $x_i, \ldots, x_n$, form the *Jonquière subgroup* $\mathcal{J}_n \subset \mathcal{G}_n$. Moreover, we have the inclusions $D_n \subset \mathrm{GL}_n \subset \mathrm{Aff}_n \subset \mathcal{G}_n$ where D_n is the group of *diagonal automorphisms* $(a_1 x_1, \ldots, a_n x_n)$ and Aff_n is the group of *affine transformations* $\mathbf{g} = (g_1, \ldots, g_n)$ where all g_i have degree 1. The group Aff_n is the semidirect product of GL_n with the commutative unipotent subgroup $\mathcal{T}_n$ of translations. The subgroup $T\mathcal{G}_n \subset \mathcal{G}_n$ generated by $\mathcal{J}_n$ and Aff_n is called the group of *tame* automorphisms.

Main Theorem. *Let θ be an automorphism of $\mathcal{G}_n$. Then there is an element $\mathbf{g} \in \mathcal{G}_n$ and a field automorphism $\tau \colon \mathbb{C} \to \mathbb{C}$ such that*

$$\theta(\mathbf{f}) = \tau(\mathbf{g} \circ \mathbf{f} \circ \mathbf{g}^{-1}) \text{ for all tame automorphisms } \mathbf{f} \in T\mathcal{G}_n.$$

After some preparation in the following sections the proof is given in Section 7. For $n = 2$ where $T\mathcal{G}_2 = \mathcal{G}_2$ this result is due to Julie Deserti [Dés06]. In fact, she proved this for any uncountable field K of characteristic zero. Our methods work for any algebraically closed field of characteristic zero.

Recently, Alexei Belov-Kanel and Jie-Tai Yu proved that every ind-group automorphism $\mathcal{G}_n \to \mathcal{G}_n$ is inner [BKY13].

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2. **Ind-group structure and locally finite automorphisms.** The group $\mathcal{G}_n$ has the structure of an ind-group given by $\mathcal{G}_n = \bigcup_{d\geq 1}(\mathcal{G}_n)_d$ where $(\mathcal{G}_n)_d$ are the automorphisms of degree $\leq d$ (see [Kum02]). Each $(\mathcal{G}_n)_d$ is an affine variety and $(\mathcal{G}_n)_d \subset (\mathcal{G}_n)_{d+1}$ is closed for all d. This defines a topology on $\mathcal{G}_n$ where a subset $X \subset \mathcal{G}_n$ is closed (resp. open) if and only if $X \cap (\mathcal{G}_n)_d$ is closed (resp. open) in $(\mathcal{G}_n)_d$ for all d. All subgroups mentioned above are closed subgroups, except possibly $T\mathcal{G}_n$.

In addition, multiplication $\mathcal{G}_n \times \mathcal{G}_n \to \mathcal{G}_n$ and inverse $\mathcal{G}_n \to \mathcal{G}_n$ are morphisms of ind-varieties where for the latter one has to use the fact that $\deg \mathbf{f}^{-1} \leq (\deg \mathbf{f})^{n-1}$. This seems to be a classical result for birational maps of $\mathbb{P}^n$ based on Bézout's Theorem (see [BCW82, Corollary (1.4) and Theorem (1.5)]). It follows that for every subgroup $G \subset \mathcal{G}_n$ the closure $\bar{G}$ in $\mathcal{G}_n$ is also a subgroup.

A closed subgroup G contained in some $(\mathcal{G}_n)_d$ is called an *algebraic subgroup*. In fact, such a G is an affine algebraic group which acts faithfully on $\mathbb{A}^n$, and for every affine algebraic group H acting on $\mathbb{A}^n$ the image of H in $\mathcal{G}_n$ is an algebraic subgroup.

A subset $X \subset \mathcal{G}_n$ is called *bounded constructible*, if X is a constructible subset of some $(\mathcal{G}_n)_d$.

Lemma 2.1. *Let $G \subset \mathcal{G}_n$ be a subgroup and let $X \subset G$ be a subset which is dense in G and bounded constructible. Then G is an algebraic subgroup, and $G = X \circ X$.*

Proof. By assumption $G \subset \bar{X} \subset (\mathcal{G}_n)_d$ for some d and so $\bar{G} = \bar{X}$ is an algebraic subgroup. Moreover, there is a subset $U \subset X$ which is open and dense in $\bar{G}$. Then $U \circ U = \bar{G}$, and so $\bar{G} = G = X \circ X$. □

An element $\mathbf{g} \in \mathcal{G}_n$ is called *locally finite* if it induces a locally finite automorphism of the algebra $\mathbb{C}[x_1, \ldots, x_n]$ of polynomial functions on $\mathbb{A}^n$. This is equivalent to the condition that the linear span of $\{(\mathbf{g}^m)^*(f) \mid m \in \mathbb{Z}\}$ is finite dimensional for all $f \in \mathbb{C}[x_1, \ldots, x_n]$.

More generally, an action of a group G on an affine variety X is called *locally finite* if the induced action on the coordinate ring $\mathcal{O}(X)$ is locally finite, i.e. for all $f \in \mathcal{O}(X)$ the linear span $\langle Gf \rangle$ is finite dimensional. It is easy to see that the image of G in $\mathrm{Aut}(X)$ is dense in an algebraic group $\bar{G}$ which acts algebraically on X. In fact, one first chooses a finite dimensional G-stable subspace $W \subset \mathcal{O}(X)$ which generates $\mathcal{O}(X)$, and then defines $\bar{G} \subset \mathrm{GL}(W)$ to be the closure of the image of G inside $\mathrm{GL}(W)$.

The next result will be used in the following section. We start again with an action of a group G on an affine variety X and assume that $x_0 \in X$ is a fixed point. Then we obtain a representation $\tau\colon G \to \mathrm{GL}(T_{x_0}X)$ on the tangent space at x_0, given by $\tau(g) := d_{x_0}g$.

Lemma 2.2. *Let G act faithfully on an irreducible affine variety X. Assume that $x_0 \in X$ is a fixed point and that there is a G-stable decomposition $\mathfrak{m}_{x_0} = V \oplus \mathfrak{m}_{x_0}^2$. Then the tangent representation $\tau\colon G \to \mathrm{GL}(T_{x_0}X)$ is faithful.*

Proof. Let $g \in \ker \tau$. Then g acts trivially on V, hence on all powers V^j of V. This implies that the action of g on $\mathcal{O}(X)/\mathfrak{m}_{x_0}^k$ is trivial for all $k \geq 1$. Since $\bigcap_k \mathfrak{m}_{x_0}^k = \{0\}$ the claim follows. □

We remark that a G-stable decomposition $\mathfrak{m}_{x_0} = V \oplus \mathfrak{m}_{x_0}^2$ like in the lemma above always exists if G is a reductive algebraic group.

3. **Tori and centralizers.** For the convenience of the reader we recall two important results about fixed point sets of group actions which we will need below. A complex variety X is called *$\mathbb{Z}/p\mathbb{Z}$-acyclic* if $H_j(X, \mathbb{Z}/p\mathbb{Z}) = 0$ for $j > 0$ and $H_0(X, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. The first result goes back to P. A. Smith [Smi34].

Proposition 3.1 (Corollary to Theorem 7.5 in [Ser09])**.** *Let G be a finite p-group and let X be an affine G-variety. If X is $\mathbb{Z}/p\mathbb{Z}$-acyclic, then so is X^G.*

The second result is due to Fogarty and describes the *tangent cone* $C(X^G, x)$ of the fixed point set X^G.

Proposition 3.2 (Theorem 5.2 in [Fog73])**.** *Let G be a reductive group. If X is an affine G-variety, then for each point $x \in X$ we have $C(X^G, x) = C(X, x)^G$.*

Define $\mu_k := \{\mathbf{g} \in D_n \mid \mathbf{g}^k = \mathrm{id}\}$. We have $\mu_k \simeq (\mathbb{Z}/k)^n$, and $\mu_\infty := \bigcup_k \mu_k \subset D_n$ is the subgroup of elements of finite order where $\mu_\infty \simeq (\mathbb{Q}/\mathbb{Z})^n$. The next lemma about the centralizer of μ_k is easy.

Lemma 3.3. *For every $k > 1$ we have $\mathrm{Cent}_{\mathcal{G}_n}(\mu_k) = \mathrm{Cent}_{\mathrm{GL}_n}(\mu_k) = D_n$.*

The following result is crucial for the proof of the main theorem.

Proposition 3.4. *Let $\mu \subset \mathcal{G}_n$ be a finite subgroup isomorphic to μ_2. Then the centralizer $\mathrm{Cent}_{\mathcal{G}_n}(\mu)$ is a diagonalizable algebraic subgroup of $\mathcal{G}_n$, i.e. isomorphic to a closed subgroup of a torus. Moreover $\dim \mathrm{Cent}_{\mathcal{G}_n}(\mu) \le n$.*

Proof. We first remark that $\mathrm{Cent}_{\mathcal{G}_n}(\mu)$ is a closed subgroup of $\mathcal{G}_n$. By Proposition 3.1 the fixed point set $F := (\mathbb{A}^n)^{\mu'}$ of every subgroup $\mu' \subset \mu$ is $\mathbb{Z}/2$-acyclic, in particular non-empty and connected. We also know that F is smooth and that $T_a F = (T_a \mathbb{A}^n)^{\mu'}$ since μ' is linearly reductive (see Proposition 3.2). If $a \in (\mathbb{A}^n)^\mu$, then the tangent representation of μ on $T_a\mathbb{A}^n$ is faithful, by Lemma 2.2 above, and so a is an isolated fixed point. Hence, $(\mathbb{A}^n)^\mu = \{a\}$.

Choose generators $\sigma_1, \dots, \sigma_n$ of μ such that the images in $\mathrm{GL}(T_a\mathbb{A}^n)$ are reflections, i.e. have a single eigenvalue -1, and set $H_i := (\mathbb{A}^n)^{\sigma_i}$. The tangent representation shows that H_i is a hypersurface, hence defined by an irreducible polynomial $f_i \in \mathbb{C}[x_1, \dots, x_n]$. Moreover, $\sigma_i^*(f_i) = -f_i$ and $\sigma_i^*(f_j) = f_j$ for $j \ne i$. It follows that the linear subspace $V := \mathbb{C}f_1 \oplus \cdots \oplus \mathbb{C}f_n \subset \mathbb{C}[x_1, \dots, x_n]$ is μ-stable. In addition, any $\mathbf{g} \in G := \mathrm{Cent}_{\mathcal{G}_n}(\mu)$ fixes a and stabilizes all $\mathbb{C}f_i$ and so, by the following Lemma 3.6 applied to the morphism $\varphi := (f_1, \dots, f_n)\colon \mathbb{A}^n \to \mathbb{A}^n$, the action of G on $\mathbb{A}^n$ is locally finite. Since G is a closed subgroup of $\mathcal{G}_n$, it follows that it is an algebraic subgroup of $\mathcal{G}_n$, and its image in $\mathrm{GL}(V)$ is a closed subgroup contained in a maximal torus, hence a diagonalizable group.

Finally, $\mathfrak{m}_a = V \oplus \mathfrak{m}_a^2$, and thus the homomorphism $G \to \mathrm{GL}(T_a\mathbb{A}^n)$ is injective, by Lemma 2.2. Hence the claim. □

Remark 3.5. It is not difficult to show that the proposition holds for every finite commutative subgroup μ of rank n. In fact, the proof carries over to subgroups isomorphic to μ_p where p is a prime, and every finite commutative subgroup μ of rank n contains such a group.

Lemma 3.6. *Let $G \subset \mathrm{Aut}(\mathbb{A}^n)$ be a subgroup and let $\varphi\colon \mathbb{A}^n \to X$ be a dominant morphism such that $\dim X = n$. Assume that $\varphi^*(\mathcal{O}(X))$ is a G-stable subalgebra and that the induced action of G on X is locally finite. Then the same holds for the action of G on $\mathbb{A}^n$.*

Proof. Put $A := \varphi^*(\mathcal{O}(X)) \subset \mathbb{C}[x_1, \dots, x_n]$ and denote by $R \subset \mathbb{C}[x_1, \dots, x_n]$ the integral closure of A. We first claim that the action of G on R is locally finite. In fact, let $f \in R$ and let $f^m + a_1 f^{m-1} + \cdots + a_m = 0$ be an integral equation of f over A. By assumption, the spaces $\langle Ga_i \rangle$ are all finite dimensional, and so there is a $d \in \mathbb{N}$ such that $\deg ga_i < d$ for all $g \in G$ and all a_i. Since gf satisfies the equation $(gf)^m + (ga_1)(gf)^{m-1} + \cdots + (ga_m) = 0$ we get $\deg(gf) < d$ for all $g \in G$, hence the claim.

Therefore, we can assume that X is normal and that $\varphi\colon \mathbb{A}^n \to X$ is birational. Choose an open set $U \subset \mathbb{A}^n$ such that $\varphi(U) \subset X$ is open and φ induces an isomorphism $U \xrightarrow{\sim} \varphi(U)$. Define $Y := \bigcup_{g \in G} gU \subset \mathbb{A}^n$. Then the induced morphism $\psi := \varphi|_Y \colon Y \to \varphi(Y)$ is G-equivariant and a local isomorphism. This implies that ψ is a G-equivariant isomorphism.

By assumption, the action of G on X is locally finite, and so G is dense in an algebraic group $\bar{G}$ which acts regularly on X. Clearly, the open set $\varphi(Y)$ is $\bar{G}$-stable and thus the action of $\bar{G}$ on $\mathcal{O}(\varphi(Y))$ is locally finite. Now the claim follows, because $\mathbb{C}[x_1, \dots, x_n] \subset \mathcal{O}(Y)$ is a G-stable subalgebra. □

The proposition above has an interesting consequence for the linearization problem for finite group actions on affine 3-space $\mathbb{A}^3$. In this case it is known that every faithful action of a non-finite reductive group on $\mathbb{A}^3$ is linearizable (Kraft-Russell, see [KR12]).

Corollary 3.7. *Let $\mu \subset \mathcal{G}_3$ be a commutative subgroup of rank three. If the centralizer of μ is not finite, then μ is conjugate to a subgroup of D_3.*

4. D_n**-stable unipotent subgroups.** Recall that every commutative unipotent group U has a natural structure of a $\mathbb{C}$-vector space, given by the exponential map $\exp\colon T_eU \xrightarrow{\sim} U$. Thus $\mathrm{Aut}(U) = \mathrm{GL}(U)$ and every action of an algebraic group on U by group automorphisms is given by a linear representation.

A (non-zero) locally nilpotent vector field $\delta = \sum_{i=1}^n h_i \frac{\partial}{\partial x_i}$ defines a (non-trivial) $\mathbb{C}^+$-action on $\mathbb{A}^n$, hence a one-dimensional unipotent subgroup

$$U_\delta = \{\mathrm{Exp}(t\delta) := (\exp(t\delta)(x_1), \dots, \exp(t\delta)(x_n)) \mid t \in \mathbb{C}^+\} \subseteq \mathcal{G}_n,$$

and $U_\delta = U_{\delta'}$ if and only if δ' is a scalar multiple of δ. In the following we denote by $e_1, \dots, e_n$ the standard basis of $\mathbb{Z}^n$, and by $\varepsilon_1, \dots, \varepsilon_n$ the standard basis of the character group of D_n.

Lemma 4.1. *Let $U = U_\delta \subset \mathcal{G}_n$ be a one-dimensional unipotent subgroup. Then U_δ is normalized by D_n if and only if δ is of the form $cx^\gamma \frac{\partial}{\partial x_i}$, where*

$$x^\gamma = x_1^{\gamma_1} \cdots x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}} \cdots x_n^{\gamma_n}$$

and $c \in \mathbb{C}^*$. In particular, $U_\delta = \{(x_1, \dots, x_i + s(cx^\gamma), \dots, x_n) \mid s \in \mathbb{C}\}$, and $\mathbf{d} \circ \mathrm{Exp}(s\delta) \circ \mathbf{d}^{-1} = \mathrm{Exp}(t^{e_i - \gamma} s\delta)$ for $\mathbf{d} = (t_1x_1, \dots, t_nx_n) \in D_n$.

Proof. If U_δ is normalized by D_n, then $\mathbf{d}^* \circ \delta \circ (\mathbf{d}^*)^{-1} \in \mathbb{C}^*\delta$ for all $\mathbf{d} \in D_n$. Writing $\delta = \sum_i h_i \frac{\partial}{\partial x_i}$ it follows that each h_i is a monomial of the form $h_i = a_i x^{\beta + e_i}$ for some $\beta \in \mathbb{Z}^n$. If $\beta_i \geq 0$ an induction on m shows that, for all $m \geq 1$, we have

$$\delta^m(x_i) = b_m^{(i)} x^{m\beta + e_i}, \quad \text{where } b_m^{(i)} = a_i \prod_{l=1}^{m-1} (lb + a_i) \text{ and } b = \sum_{j=1}^n a_j \beta_j.$$

Assume that $\beta_i \geq 0$ for all i. Since δ is locally nilpotent there is a minimal $m_i \geq 0$ such that $b^{(i)}_{m_i+1} = 0$. This implies $a_i = -m_i b$. Since $\delta \neq 0$, we get

$$0 \neq b = \sum_{i=1}^{n} a_i\beta_i = -b\sum_{i=1}^{n} m_i\beta_i,$$

and so $\sum m_i\beta_i = -1$. But this is a contradiction, because $m_i, \beta_i \geq 0$ for all i. Therefore $a_i x^{\beta+e_i} \neq 0$ implies that $\beta_j \geq 0$ for all $j \neq i$, and $\beta_i = -1$. Thus there is only one term in the sum, i.e. $\delta = a_i x^{\gamma}\frac{\partial}{\partial x_i}$ where $\gamma := \beta + e_i$ has the claimed form. □

Remark 4.2. This lemma can also be expressed in the following way: *There is a bijective correspondence between the D_n-stable one-dimensional unipotent subgroups $U \subset \mathcal{G}_n$ and the characters of D_n of the form $\lambda = \sum_j \lambda_j\varepsilon_j$ where one λ_i equals* 1 *and the others are* ≤ 0. We will denote this set of characters by $X_u(D_n)$:

$$X_u(D_n) := \{\lambda = \sum \lambda_j\varepsilon_j \mid \exists\, i \text{ such that } \lambda_i = 1 \text{ and } \lambda_j \leq 0 \text{ for } j \neq i\}.$$

If $\lambda \in X_u(D_n)$, then U_λ denotes the corresponding one-dimensional unipotent subgroup normalized by D_n.

Remark 4.3. In [Lie11, Theorem 1] Alvaro Liendo shows that the locally nilpotent derivations normalized by the torus $D_n' := D_n \cap \mathrm{SL}_n$ have exactly the same form.

Lemma 4.4. *The subgroup $\mathcal{T}_n$ of translations is the only commutative unipotent subgroup normalized by* GL_n.

Proof. If $U \subset \mathcal{G}_n$ is a commutative unipotent subgroup normalized by GL_n, then all the weights of the representation of GL_n on $T_eU \simeq U$ must belong to $X_u(D_n)$. The dominant weights of GL_n are $\sum_i \lambda_i\varepsilon_i$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and only those of the form $\lambda = \varepsilon_1 + \sum_{i>1}\lambda_i\varepsilon_i$ where $0 \geq \lambda_2 \geq \cdots \geq \lambda_n$ occur in $X_u(D_n)$. If $\lambda \neq \varepsilon_1$, i.e. $\lambda = \varepsilon_1 + \lambda_k\varepsilon_k + \lambda_{k+1}\varepsilon_{k+1} + \cdots$ where $\lambda_k < 0$, then the weight $\lambda' := (\lambda_k + 1)\varepsilon_k + \lambda_{k+1}\varepsilon_{k+1} + \cdots$ is dominant and $\lambda' \prec \lambda$. Therefore λ' appears in the irreducible representation of GL_n of highest weight λ, but $\lambda' \notin X_u(D_n)$. Thus U and $\mathcal{T}_n$ are isomorphic as GL_n-modules, hence contain the same D_n-stable one-dimensional unipotent subgroups, and so $U = \mathcal{T}_n$. □

5. **Maximal tori.** It is clear that $D_n \subset \mathcal{G}_n$ is a maximal commutative subgroup of $\mathcal{G}_n$ since it coincides with its centralizer, see Lemma 3.3. Moreover, Białynicki-Birula proved in [BB66] that a faithful action of an n-dimensional torus on $\mathbb{A}^n$ is linearizable (cf. [KS92, Chap. I.2.4, Theorem 5]).Thus we have the following result.

Lemma 5.1. *D_n is a maximal commutative subgroup of $\mathcal{G}_n$. Moreover, every algebraic subgroup of $\mathcal{G}_n$, which is isomorphic to D_n is conjugate to D_n.*

Now let $G \subset \mathcal{G}_n$ be an algebraic subgroup which is normalized by D_n. Then the non-zero weights of the representation of D_n on the Lie algebra $\mathrm{Lie}\, G$ belong to $X_u(D_n)$, and the weight spaces are one-dimensional. It follows that the non-zero weight spaces of $\mathrm{Lie}\, G$ are in bijective correspondence with the D_n-stable one-dimensional unipotent subgroups of G.

Lemma 5.2. *Let $G \subset \mathcal{G}_n$ be an algebraic subgroup normalized by a torus $D \subset \mathcal{G}_n$ of dimension n, let $U_1, \ldots, U_r$ be the D-stable one-dimensional unipotent subgroups of G, and put $X := U_1 \circ \cdots \circ U_r \subset G$.*

(a) *If G is unipotent, then $G = X \circ X$ and $\dim G = r$.*
(b) *If $D \subset G$, then $G^0 = D \circ X \circ D \circ X$ and $\dim G = r + n$.*

Proof. (a) The canonical map $U_1 \times \cdots \times U_r \to G$ is dominant, and so $X \subset G$ is constructible and dense. Thus $X \circ X = G$, by Lemma 2.1, and $\dim G = \dim \mathrm{Lie}\, G = r$.

(b) Similarly, we see that $D \circ X \subset G^0$ is constructible and dense, and therefore $D \circ X \circ D \circ X = G^0$, and $\dim G = \dim \mathrm{Lie}\, G = \dim \mathrm{Lie}\, D + r$. □

6. **Images of algebraic subgroups.** The next two propositions are crucial for the proof of our main theorem.

Proposition 6.1. *Let θ be an automorphism of $\mathcal{G}_n$. Then*
(a) *$D := \theta(D_n)$ is a torus of dimension n, conjugate to D_n.*
(b) *If U is a D_n-stable unipotent subgroup, then $\theta(U)$ is a D-stable unipotent subgroup of the same dimension.*
(c) *$\mathcal{T} := \theta(\mathcal{T}_n)$ is a commutative unipotent subgroup of dimension n, normalized by D, and the representation of D on $\mathcal{T}$ is faithful.*

Proof. (a) We have $D_n = \mathrm{Cent}_{\mathcal{G}_n}(\mu_2)$, by Lemma 3.3, and thus $D = \theta(D_n) = \mathrm{Cent}_{\mathcal{G}_n}(\theta(\mu_2))$. Proposition 3.4 implies that D is a diagonalizable algebraic subgroup with $\dim D \le n$, hence $D = D^0 \times F$ for some finite group F. If k is prime to the order of F, then $\theta(\mu_k) \subset D^0$ and so $\dim D^0 = n$, because $\mu_k \simeq (\mathbb{Z}/k)^n$. Hence $D = D^0$ is an n-dimensional torus which is conjugate to D_n, by Lemma 5.1.

(b) First assume that $\dim U = 1$. Then U consists of two D_n-orbits, $O := U \setminus \{\mathrm{id}\}$ and $\{\mathrm{id}\}$. It follows that $\theta(U)$ consists of the two D-orbits $\theta(O)$ and $\{\mathrm{id}\}$, and so $\theta(U)$ is bounded constructible and thus a commutative algebraic group normalized by D. Since it does not contain elements of finite order it is unipotent, and since it consists of only two D-orbits it is of dimension 1.

Now let U be arbitrary, $\dim U = r$, and let $U_1, \ldots, U_r$ be the different D_n-stable one-dimensional unipotent subgroups of U. Then $X := U_1 \circ U_2 \circ \cdots \circ U_r \subset U$ is dense and constructible and $U = X \circ X$, by Lemma 5.2(a). Applying θ implies that $\theta(X) = \theta(U_1) \circ \cdots \circ \theta(U_r)$ is bounded constructible and connected, as well as $\theta(U) = \theta(X) \circ \theta(X)$, and thus $\theta(U)$ is a connected algebraic subgroup of $\mathcal{G}_n$ normalized by D. Since every element of $\theta(U)$ has infinite order, $\theta(U)$ must be unipotent. Moreover, $\dim \theta(U) \ge r$, since $\theta(U)$ contains the D-stable one-dimensional unipotent subgroups $\theta(U_i)$, $i = 1, \ldots, r$. The same argument applied to θ^{-1} finally gives $\dim \theta(U) = r$.

(c) This statement follows from (b) and the fact that $\mathcal{T}_n$ contains a dense D_n-orbit with trivial stabilizer. □

The same arguments, this time using Lemma 5.2(b), gives the next result.

Proposition 6.2. *Let θ be an automorphism of $\mathcal{G}_n$ and let $G \subset \mathcal{G}_n$ be an algebraic subgroup which contains a torus D of dimension n.*
(a) *The image $\theta(G)$ is an algebraic subgroup of $\mathcal{G}_n$ of the same dimension $\dim G$.*
(b) *We have $\theta(G^0) = \theta(G)^0$. In particular, $\theta(G)$ is connected if G is connected.*
(c) *If G is reductive, then so is $\theta(G)$, and then $\theta(G)$ is conjugate to a closed subgroup of GL_n.*

Proof. As above, let $U_1, \dots, U_r$ be the different D-stable one-dimensional unipotent subgroups of G, and put $X := U_1 \circ \cdots \circ U_r$. Then $D \circ X$ is constructible in G^0, and $D \circ X \circ D \circ X = G^0$, by Lemma 5.2(b). Applying θ we see that $\theta(D \circ X \circ D \circ X) = \theta(D) \circ \theta(X) \circ \theta(D) \circ \theta(X)$ is bounded constructible and connected, and so $\theta(G^0)$ is a connected algebraic subgroup of $\mathcal{G}_n$, of finite index in $\theta(G)$. Since the $\theta(U_i)$ are different $\theta(D)$-stable one-dimensional unipotent subgroups of $\theta(G)$ we have $\dim \theta(G) \geq \dim \theta(D) + r = \dim G$. Using θ^{-1} we get equality. This proves (a) and (b).

For (c) we remark that if G contains a normal unipotent subgroup U, then $\theta(U)$ is a normal unipotent subgroup of $\theta(G)$. Moreover, a reductive subgroup G containing a torus of dimension n has no non-constant invariants, and so G is linearizable (see [KP85, Proposition 5.1]). □

7. **Proof of the Main Theorem.** Let θ be an automorphism of $\mathcal{G}_n$. It follows from Proposition 6.2 that there is a $\mathbf{g} \in \mathcal{G}_n$ such that $\mathbf{g} \circ \theta(\mathrm{GL}_n) \circ \mathbf{g}^{-1} \subset \mathrm{GL}_n$. Therefore we can assume that $\theta(\mathrm{GL}_n) = \mathrm{GL}_n$. The subgroup $\mathcal{T}_n$ of translations is the only commutative unipotent subgroup normalized by GL_n, by Lemma 4.4. Therefore, $\theta(\mathcal{T}_n) = \mathcal{T}_n$ and so $\theta(\mathrm{Aff}_n) = \mathrm{Aff}_n$. Now the theorem follows from the next proposition. □

Proposition 7.1. (a) *Every automorphism θ of Aff_n with $\theta(\mathrm{GL}_n) = \mathrm{GL}_n$ and $\theta(\mathcal{T}_n) = \mathcal{T}_n$ is of the form $\theta(\mathbf{f}) = \tau(\mathbf{g} \circ \mathbf{f} \circ \mathbf{g}^{-1})$ where $\mathbf{g} \in \mathrm{GL}_n$ and τ is an automorphism of the field $\mathbb{C}$.*

(b) *If θ is an automorphism of $\mathcal{G}_n$ such that $\theta|_{\mathrm{Aff}_n} = \mathrm{Id}_{\mathrm{Aff}_n}$, then $\theta|_{\mathcal{J}_n} = \mathrm{Id}_{\mathcal{J}_n}$.*

Proof. (a) It is enough to prove that $\theta(\mathbf{f}) = \mathbf{g} \circ \tau(\mathbf{f}) \circ \mathbf{g}^{-1}$ for some $\mathbf{g} \in \mathrm{GL}_n$ and some automorphism $\tau \colon \mathbb{C} \to \mathbb{C}$ of the field $\mathbb{C}$. Let $Z = \mathbb{C}^* \subseteq \mathrm{GL}_n$ be the center of GL_n and define $\theta_0 := \theta|_Z \colon Z \to Z$, $\theta_1 := \theta|_{\mathcal{T}_n} \colon \mathcal{T}_n \to \mathcal{T}_n$. It follows that θ_0 and θ_1 are abstract group homomorphisms of $\mathbb{C}^*$ and $\mathcal{T}_n$ respectively, and for all $c \in \mathbb{C}^*$, $\mathbf{t} \in \mathcal{T}_n$ we get

$$(*) \qquad \theta_1(c \cdot \mathbf{t}) = \theta_1(c \circ \mathbf{t} \circ c^{-1}) = \theta_0(c) \circ \theta_1(\mathbf{t}) \circ \theta_0(c)^{-1} = \theta_0(c) \cdot \theta_1(\mathbf{t}),$$

where "$\cdot$" denotes scalar multiplication. We claim that $\tau \colon \mathbb{C} \to \mathbb{C}$ defined by $\tau|_{\mathbb{C}^*} = \theta_0$, $\tau(0) = 0$, is an automorphism of the field $\mathbb{C}$. Indeed, using $(*)$ one sees that $\tau(a+b) = \tau(a) + \tau(b)$ for all $a, b \in \mathbb{C}^*$ such that $a + b \neq 0$. As $\theta_0(-1) = -1$ it follows that $\tau(-a) = -\tau(a)$ for all $a \in \mathbb{C}^*$ and so $\tau(a + (-a)) = \tau(a) + \tau(-a)$. This implies the claim.

Thus we can assume that $\theta_0 = \mathrm{id}_{\mathbb{C}^*}$. Using $(*)$ again, it follows that θ_1 is linear. Considering θ_1 as an element of GL_n we have $\theta_1(\mathbf{t}) = \theta_1 \circ \mathbf{t} \circ \theta_1^{-1}$, and thus we can assume that $\theta_1 = \mathrm{id}_{\mathcal{T}_n}$. But this implies that $\theta(\mathbf{g}) = \mathbf{g}$ for all $\mathbf{g} \in \mathrm{GL}_n$, because

$$\mathbf{g} \circ \mathbf{t} \circ \mathbf{g}^{-1} = \theta(\mathbf{g} \circ \mathbf{t} \circ \mathbf{g}^{-1}) = \theta(\mathbf{g}) \circ \mathbf{t} \circ \theta(\mathbf{g})^{-1}$$

for all $\mathbf{t} \in \mathcal{T}_n$.

(b) Let $U \subset \mathcal{G}_n$ be a one-dimensional unipotent D_n-stable subgroup. We first claim that $\theta(U) = U$ and that $\theta|_U$ is linear. In fact, $U' := \theta(U)$ is a one-dimensional unipotent D_n-stable subgroup, by Proposition 6.1(b), and the characters λ and λ' associated to U and U' (see Remark 4.2) have the same kernel, because

$$(**) \quad \theta(\lambda(\mathbf{d}) \cdot u) = \theta(\mathbf{d} \circ u \circ \mathbf{d}^{-1}) = \mathbf{d} \circ \theta(u) \circ \mathbf{d}^{-1} = \lambda'(\mathbf{d}) \cdot \theta(u) \quad \text{for } \mathbf{d} \in D_n,\ u \in U.$$

Hence $\lambda = \pm\lambda'$. If $\lambda = -\lambda'$, then $U \subseteq \mathrm{GL}_n$ and so $U' = U$, since $\theta|_{\mathrm{GL}_n} = \mathrm{Id}_{\mathrm{GL}_n}$, hence a contradiction. Thus $\lambda = \lambda'$, and so $U = U'$ and (∗∗) shows that $\theta|_U$ is linear, proving our claim.

As a consequence, $\theta|_{U_\lambda} = a_\lambda \,\mathrm{Id}_{U_\lambda}$ for all $\lambda \in X_u(D_n)$, with suitable $a_\lambda \in \mathbb{C}^*$. If $\lambda_i = 1$ put $\gamma_i := 0$ and $\gamma_j := -\lambda_j$. Then $\mathbf{f} = (x_1, \ldots, x_i + x^\gamma, \ldots, x_n) \in U_\lambda$, see Lemma 4.1. Conjugation with the translation $\mathbf{t}\colon x \mapsto x - \sum_{j\neq i} e_j$ gives

$$\mathbf{t}\circ\mathbf{f}\circ\mathbf{t}^{-1} = (x_1, \ldots, x_i + h_\gamma, \ldots, x_n) \text{ where } h_\gamma := (x_1+1)^{\gamma_1}(x_2+1)^{\gamma_2}\cdots(x_n+1)^{\gamma_n}.$$

Now we apply θ to get $\theta(\mathbf{t}\circ\mathbf{f}\circ\mathbf{t}^{-1}) = \mathbf{t}\circ\theta(\mathbf{f})\circ\mathbf{t}^{-1}$. Since all the monomials $x^{\gamma'}$ with $\gamma' \leq \gamma$ appear in h_γ it follows that the corresponding coefficients $a_{\lambda'}$ must all be equal. In particular, $a_\lambda = a_{\varepsilon_i} = 1$ since $U_{\varepsilon_i} \subset \mathcal{T}_n$. This shows that $\theta|_{\mathcal{J}_n} = \mathrm{Id}_{\mathcal{J}_n}$. □

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A NOTE ON AUTOMORPHISMS OF THE AFFINE CREMONA GROUP

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ABSTRACT. Let $\mathcal{G}$ be an ind-group and let $\mathcal{U} \subseteq \mathcal{G}$ be a unipotent ind-subgroup. We prove that an abstract automorphism $\theta\colon \mathcal{G} \to \mathcal{G}$ maps $\mathcal{U}$ isomorphically onto a unipotent ind-subgroup of $\mathcal{G}$, provided that a closed torus $T \subseteq \mathcal{G}$ normalizes $\mathcal{U}$ and the action of T on $\mathcal{U}$ by conjugation fixes only the neutral element. As an application we generalize a result by HANSPETER KRAFT and the author in [KS11] as follows: If an abstract automorphism of the affine Cremona group $\mathcal{G}_3$ in dimension 3 fixes the group of tame automorphisms $T\mathcal{G}_3$, then it also fixes a whole family of non-tame automorphisms (including the NAGATA automorphism).

0. **Introduction.** Throughout this note we denote by $\mathcal{G}_n$ the group of polynomial automorphisms $\mathrm{Aut}(\mathbb{A}^n)$ of the complex affine space $\mathbb{A}^n = \mathbb{C}^n$. Such an automorphism has the form $\mathbf{g} = (g_1, \ldots, g_n) \in \mathcal{G}_n$ with polynomials $g_1, \ldots, g_n \in \mathbb{C}[x_1, \ldots, x_n]$. We define $\deg \mathbf{g} := \max_i \deg g_i$. The tame automorphism group $T\mathcal{G}_n$ is the subgroup of $\mathcal{G}_n$ generated by the affine linear automorphisms (i.e. the automorphisms $\mathbf{g}$ with $\deg \mathbf{g} \leq 1$) and the triangular automorphisms (i.e. the automorphisms $(g_1, \ldots, g_n)$ where $g_i = g_i(x_i, \ldots, x_n)$ depends only on $x_i, \ldots, x_n$ for each i). The main result of [KS11] is the following.

Theorem 1. *Let $\theta\colon \mathcal{G}_n \to \mathcal{G}_n$ be an abstract automorphism. Then there exists $\mathbf{g} \in \mathcal{G}_n$ and a field automorphism $\tau\colon \mathbb{C} \to \mathbb{C}$ such that*

$$\theta(\mathbf{f}) = \tau(\mathbf{g} \circ \mathbf{f} \circ \mathbf{g}^{-1}) \quad \textit{for all tame automorphisms } \mathbf{f} \in T\mathcal{G}_n\,.$$

If θ preserves in addition the ind-group structure of $\mathcal{G}_n$ (see below for a definition), then ALEXEI BELOV-KANEL and JIE-TAI YU proved recently that θ is an inner automorphism of $\mathcal{G}_n$ (see [BKY13]).

In dimension $n = 2$ all automorphisms are tame (cf. [Jun42] and [vdK53]). But in dimension $n = 3$, IVAN P. SHESTAKOV and UALBAI U. UMIRBAEV showed that the famous NAGATA automorphism $\mathbf{u}_N \in \mathcal{G}_3$ (see below for a definition) is non-tame (cf. [SU04]). It is an open problem if there exist non-tame automorphisms in dimension $n > 3$. A natural question is whether the theorem above extends to the entire automorphism group $\mathcal{G}_n$, i.e. whether $\theta(\mathbf{f}) = \tau(\mathbf{g} \circ \mathbf{f} \circ \mathbf{g})$ for all $\mathbf{f} \in \mathcal{G}_n$. If this would be true, then every abstract automorphism of $\mathcal{G}_n$ would preserve algebraic subgroups of $\mathcal{G}_n$ (see below for a definition). In fact, a main tool in the proof of Theorem 1 is to show, that certain algebraic subgroups are sent into isomorphic algebraic subgroups under an abstract automorphism of $\mathcal{G}_n$. The main point of this

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note is to refine this techniques. In order to state the main result we introduce the concept of an ind-group and related terms.

A group $\mathcal{G}$ is called an *ind-group* if it is endowed with a filtration by affine varieties $G_1 \subseteq G_2 \subseteq \dots$, each one closed in the next, such that $\mathcal{G} = \bigcup_{i=1}^{\infty} G_i$ and such that the map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(x, y) \mapsto x \cdot y^{-1}$ is a morphism of ind-varieties (see [Kum02, ch. IV] for an introduction to ind-varieties and ind-groups). We then write $\mathcal{G} = \varinjlim G_i$. For example, $\mathcal{G}_n = \varinjlim G_{n,i}$ is an ind-group, where $G_{n,i}$ is the set of all automorphisms $\mathbf{g} \in \mathcal{G}_n$ with $\deg \mathbf{g} \leq i$ (see [BCW82]). We endow an ind-group $\mathcal{G} = \varinjlim G_i$ with the following topology: a subset $X \subseteq \mathcal{G}$ is closed if and only if $X \cap G_i$ is closed in G_i with respect to the Zariski topology. If $\mathcal{H} \subseteq \mathcal{G}$ is a closed subgroup, then $\mathcal{H}$ inherits in a canonical way an ind-structure from $\mathcal{G}$, namely $\mathcal{H} = \varinjlim \mathcal{H} \cap G_i$. But for our purposes we need a more general definition of an ind-subgroup.

Definition 1. Let $\mathcal{H}$ be a subgroup of an ind-group $\mathcal{G} = \varinjlim G_i$. We say that $\mathcal{H}$ is an *ind-subgroup* of $\mathcal{G}$ if $\mathcal{H}$ can be turned into an ind-group $\mathcal{H} = \varinjlim H_k$ such that to every k there exists $i = i(k)$ such that $H_k \subseteq G_i$ is closed. Clearly, the ind-structure of $\mathcal{H}$ is then unique. We say that $\mathcal{H}$ is an *algebraic subgroup* of $\mathcal{G}$, if $\mathcal{H}$ is closed in $\mathcal{G}$ and contained in some G_i.

We say that an ind-group $\mathcal{U}$ is *unipotent* if $\mathcal{U} = \varinjlim U_i$ where U_i is a unipotent algebraic group for all i. Every element in a unipotent ind-group is unipotent. We don't know whether an ind-group consisting only of unipotent elements is always unipotent. If the ind-group is commutative, then we are able to prove this.

Main Theorem. *Let $\theta\colon \mathcal{G} \to \mathcal{G}$ be an abstract automorphism of an ind-group $\mathcal{G}$ that is the identity on a closed torus $T \subseteq \mathcal{G}$. If $\mathcal{U} \subseteq \mathcal{G}$ is a unipotent ind-subgroup that is normalized by T and if the neutral element of $\mathcal{U}$ is the only element that is fixed under conjugation by T, then $\theta(\mathcal{U})$ is a unipotent ind-subgroup of $\mathcal{G}$ and $\theta|_{\mathcal{U}}\colon \mathcal{U} \to \theta(\mathcal{U})$ is an isomorphism of ind-groups.*

Recall that there exists a bijective correspondence between locally nilpotent derivations of $\mathbb{C}[x_1, \dots, x_n]$ and unipotent elements of $\mathcal{G}_n$, given by $D \mapsto \exp(D)$ where

$$\exp(D) = \left(\sum_{i=0}^{\infty} \frac{D^i(x_1)}{i!}, \dots, \sum_{i=0}^{\infty} \frac{D^i(x_n)}{i!} \right).$$

If D is a locally nilpotent derivation and $f \in \ker D$ then fD is again a locally nilpotent derivation and we call $\exp(fD)$ a *modification* of $\exp(D)$. For example, the Nagata automorphism $\mathbf{u}_N$ is a modification of $\mathbf{u} := \exp D$ where

$$D = z\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\,, \quad p := xz - \frac{1}{2}y^2 \in \ker D \quad \text{and} \quad \mathbf{u}_N = \exp(pD)\,.$$

Recently, Shigeru Kuroda gave a characterization of the non-tame modifications of certain unipotent automorphisms (see [Kur11, Theorem 2.3]). This result implies that for all $f \in \ker D \setminus \mathbb{C}[z]$ the modification $\exp(fD)$ of $\mathbf{u}$ is non-tame. Clearly, all the modifications of $\mathbf{u}$ lie in the centralizer $\operatorname{Cent}(\mathbf{u})$. As a consequence of our Main Theorem we get the following result.

Application. *Let $\theta : \mathcal{G}_3 \to \mathcal{G}_3$ be an abstract automorphism that is the identity on the tame automorphisms $T\mathcal{G}_3$. Then θ fixes $\operatorname{Cent}(\mathbf{u})$ where $\mathbf{u} = \exp(D)$ and $D = z \cdot \partial/\partial y + y \cdot \partial/\partial x$. In particular, θ fixes the non-tame automorphisms $\exp(fD)$ where $f \in \ker D \setminus \mathbb{C}[z]$ and thus θ fixes the Nagata automorphism $\mathbf{u}_N$.*

Remark 1. All the results and proves work over any uncountable algebraically closed field of characteristic zero.

I would like to thank Hanspeter Kraft for many fruitful discussions.

1. **Proof of the Main Theorem.** Let V be a commutative unipotent algebraic group. Recall that V has a unique $\mathbb{C}$-vector space structure such that the product in V corresponds to addition. Also recall that a map of commutative unipotent algebraic groups $V \to V'$ is a homomorphism of algebraic groups if and only if it is $\mathbb{C}$-linear.

We start with a lemma that proves the Main Theorem in the case when $\mathcal{U} \subseteq \mathcal{G}$ is an algebraic subgroup isomorphic to $\mathbb{C}^+$.

Lemma 1. *Let $\theta : \mathcal{G} \to \mathcal{G}$ be an abstract automorphism that is the identity on a closed torus $T \subseteq \mathcal{G}$ and let $U \subseteq \mathcal{G}$ be an algebraic subgroup isomorphic to $\mathbb{C}^+$ which is normalized by T with character λ. If λ is non-trivial, then $\theta(U) \subseteq \mathcal{G}$ is an algebraic subgroup isomorphic to $\mathbb{C}^+$ and T normalizes $\theta(U)$ with the same character λ. Moreover, $\theta|_U : U \to \theta(U)$ is an isomorphism of algebraic groups.*

Proof. Let $U' := \theta(U) \subseteq \mathcal{G}$. Choose $u_0 \in U$. Then, $U' \setminus \{e\} = \{\, \mathbf{t} \cdot \theta(\mathbf{u}_0) \cdot \mathbf{t}^{-1} \mid \mathbf{t} \in T \,\}$ and $\{e\}$ are constructible subsets of some filter set of $\mathcal{G}$ and since U' is a group it follows that $U' \subseteq \mathcal{G}$ is an algebraic subgroup (see [Hum75, 7.4 Proposition A]). Since U' has no element $\neq e$ of finite order U' is unipotent. As U' is a toric variety with exactly two orbits, U' is one-dimensional (see also [KS11, Proposition 2]). Let λ' be the character of U'. We have

$$(*) \qquad \theta(\lambda(\mathbf{t})\mathbf{u}_0) = \theta(\mathbf{t} \cdot \mathbf{u}_0 \cdot \mathbf{t}^{-1}) = \mathbf{t} \cdot \theta(\mathbf{u}_0) \cdot \mathbf{t}^{-1} = \lambda'(\mathbf{t})\theta(\mathbf{u}_0) \quad \text{for } \mathbf{t} \in T\,.$$

Hence, it follows that λ and λ' have the same kernel and thus $\lambda = \pm\lambda'$. If we take $\mathbf{t} \in T$ such that $\lambda(\mathbf{t}) = 2$ then eq. $(*)$ implies that $\lambda' \neq -\lambda$. Hence, $\lambda = \lambda'$ and $\theta|_U : U \to U'$ is $\mathbb{C}$-linear by eq. $(*)$. ☐

Proof of the Main Theorem. Let $U \subseteq \mathcal{U}$ be an algebraic subgroup that is normalized by T. Choose closed algebraic subgroups $V_1, \ldots, V_r \subseteq U$ which are isomorphic to $\mathbb{C}^+$ and which are normalized by T, such that $\operatorname{Lie} U = \operatorname{Lie} V_1 \oplus \ldots \oplus \operatorname{Lie} V_r$. Thus, for suitable indices $i_1, \ldots, i_m$ we have $U = V_{i_1} \cdot \ldots \cdot V_{i_m}$ (see [Hum75, 7.5 Proposition]). According to Lemma 1, $\theta(U) = \theta(V_{i_1}) \cdot \ldots \cdot \theta(V_{i_m})$ is a constructible subset of some $G_i \subseteq \mathcal{G}$ and thus $\theta(U)$ is an algebraic subgroup of $\mathcal{G}$. Again, since no element $\neq e$ in U has finite order, $\theta(U)$ is unipotent. Consider the following commutative diagram.

$$\begin{array}{ccc} V_{i_1} \times \ldots \times V_{i_m} & \xrightarrow{\theta \times \ldots \times \theta} & \theta(V_{i_1}) \times \ldots \times \theta(V_{i_m}) \\ \downarrow & & \downarrow \\ U & \xrightarrow{\theta|_U} & \theta(U) \end{array}$$

The vertical maps are induced by the product in $\mathcal{U}$ and hence they are surjective morphisms. The top horizontal map is an isomorphism of varieties by Lemma 1. The lemma below due to Hanspeter Kraft implies that the abstract group homomorphism $\theta|_U$ is an isomorphism of algebraic groups.

As we can replace the filtration of $\mathcal{U}$ by a filtration of unipotent algebraic subgroups, each one normalized by T, it follows that $\theta(\mathcal{U}) \subseteq \mathcal{G}$ is a unipotent ind-subgroup and $\theta|_{\mathcal{U}} : \mathcal{U} \to \theta(\mathcal{U})$ is an isomorphism of ind-groups. ☐

Lemma 2 (Hanspeter Kraft)**.** *Let X and Y be affine varieties and let $f\colon X \to Y$ be an abstract map. If there exists a surjective morphism $g\colon Z \twoheadrightarrow X$ such that the composition $f \circ g\colon Z \to Y$ is a morphism and if X is normal, then f is a morphism.*

For the proof of this lemma, one shows that the graph of f is closed in $X \times Y$.

2. **Proof of the Application.** First, we determine the structure of $\mathrm{Cent}(\mathbf{u})$. Denote by E the partial derivative with respect to x. The ind-subgroups of $\mathcal{G}_3$ listed below are clearly contained in $\mathrm{Cent}(\mathbf{u})$.

$$\begin{aligned} C &:= \{\, (ax, ay, az) \mid a \in \mathbb{C}^* \,\} \\ \mathcal{F} &:= \{\, \exp(fD) \mid f \in \ker D \,\} \\ \mathcal{H} &:= \{\, \exp(hE) \mid h \in \ker E \cap \ker D \,\} \end{aligned}$$

Proposition 1. *We have a semi-direct product decomposition*

$$\mathrm{Cent}(\mathbf{u}) = C \ltimes (\mathcal{H} \ltimes \mathcal{F})\,.$$

Proof. Recall that $p = xz - (1/2)y^2$, $z \in \ker D$. In fact, $R := \ker D$ is the polynomial ring $\mathbb{C}[z,p]$ by [DF98, Proposition 2.3]. We have $R[x] = \mathbb{C}[z,x,y^2]$ and hence a decomposition $\mathbb{C}[x,y,z] = R[x] \oplus yR[x]$. Let $\mathbf{g} = (g_1, g_2, g_3) \in \mathrm{Cent}(\mathbf{u})$. Write $g_1 = v + yq$ with polynomials $v, q \in R[x]$. In $\mathbb{C}[x,y,z,t]$ we have, by definition,

$$v(x + ty + \frac{1}{2}t^2 z) + (y + tz)q(x + ty + \frac{1}{2}t^2 z) \;=\; v(x) + yq(x) + tg_2 + \frac{1}{2}t^2 g_3\,.$$

A comparison of the coefficients with respect to the variable t shows that $v = r + sx$ with $r, s \in R$, and $q \in R$. Hence, we get $g_1 = r + sx + qy$, $g_2 = sy + qz$, $g_3 = sz$, where $s \in \mathbb{C}^*$. Up to post composition with an element of C we can assume that $s = 1$. Thus,

$$\mathbf{g} \circ \exp(qD)^{-1} = (x + r - \frac{1}{2}q^2 z, y, z)\,.$$

One easily sees that this automorphism belongs to $\mathcal{H}$. □

Let $\mathcal{U}$ be the ind-subgroup $\mathcal{H} \circ \mathcal{F} \subseteq \mathcal{G}$. Every element $\mathbf{g} = \exp(hE) \circ \exp(fD) \in \mathcal{H} \circ \mathcal{F}$ satisfies

$$\mathbf{g}^m = \exp(mhE) \circ \exp\left(\sum_{i=0}^{m-1} \exp(ihE)^*(f) D \right)\,.$$

Since $\mathcal{H} \cap \mathcal{F} = \{\mathbf{id}\}$, this shows that every element $\neq \mathbf{id}$ of $\mathcal{U}$ has infinite order and is contained in an algebraic subgroup of $\mathcal{U}$. Thus every element of $\mathcal{U}$ is unipotent. Now, Proposition 1 implies that $\mathcal{U}$ is the set of unipotent elements of $\mathrm{Cent}(\mathbf{u})$. An easy calculation shows

$$(\triangle) \qquad \mathcal{F} = \mathrm{Cent}_{\mathcal{U}}[\mathcal{U}, \mathcal{U}]$$

where $[\mathcal{U}, \mathcal{U}]$ is the commutator subgroup of $\mathcal{U}$. Let $T := \{\, (a^2 b^{-1} x, ay, bz) \mid a, b \in \mathbb{C}^* \,\}$ which is a closed algebraic subgroup of $\mathcal{G}_3$. The torus T normalizes $\mathrm{Cent}(\mathbf{u})$, $\mathcal{H}$ and $\mathcal{F}$ and hence it normalizes also $\mathcal{U}$. In fact, it follows from eq. ($\triangle$) that T is the largest subgroup of the standard torus in $\mathcal{G}$ that normalizes $\mathrm{Cent}(\mathbf{u})$.

Proof of the Application. As $\mathbf{u}$ is a tame automorphism, θ preserves $\mathrm{Cent}(\mathbf{u})$. A straightforward calculations shows, that the neutral element is the only element of $\mathcal{H}$ (of $\mathcal{F}$), that commutes with T. Thus the Main Theorem applied to the unipotent ind-subgroups $\mathcal{H}$ and $\mathcal{F}$ of $\mathcal{G}_3$ implies that $\theta(\mathcal{U}) = \mathcal{U}$. From eq. ($\triangle$) it follows that

θ preserves $\mathcal{F}$. We define $\mathscr{V}_{\mathcal{F}}$ as the set of all algebraic subgroups of $\mathcal{F}$ which are isomorphic to $\mathbb{C}^+$ and which are normalized by T. One can see that the elements of $\mathscr{V}_{\mathcal{F}}$ correspond to the locally nilpotent derivations $z^m p^n D$ for all $m, n \geq 0$ and hence different elements in $\mathscr{V}_{\mathcal{F}}$ have different characters. This implies that for all $V \in \mathscr{V}_{\mathcal{F}}$ there exists $a_V \in \mathbb{C}^*$, such that $\theta|_V = a_V \operatorname{id}_V$. Let $\mathbf{e} := (x-1, y, z) \in \mathcal{H}$. An easy calculation shows

$$\mathbf{e} \circ \exp(z^m p^n D) \circ \mathbf{e}^{-1} = \exp\left(\sum_{i=0}^{n} \binom{n}{i} z^{i+m} p^{n-i} D \right) .$$

Applying θ to the last equation and using the fact that $\exp(z^j D)$ and $\mathbf{e}$ are tame automorphisms yields $a_V = 1$ for all $V \in \mathscr{V}_{\mathcal{F}}$. As all the subgroups $V \in V_{\mathcal{F}}$ generate $\mathcal{F}$, it follows that θ is the identity on $\mathcal{F}$. Since $\mathcal{H}$ and C consist of tame automorphisms this finishes the proof. □

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AUTOMORPHISMS OF THE PLANE PRESERVING A CURVE

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Abstract. We study the group of automorphisms of the affine plane preserving some given curve, over any field. The group is proven to be algebraic, except in the case where the curve is a bunch of parallel lines. Moreover, a classification of the groups of positive dimension occuring is also given in the case where the curve is geometrically irreducible and the field is perfect.

1. Introduction

Let $\mathbf{k}$ be any field. This article studies (closed) curves $\Gamma \subset \mathbb{A}^2 = \mathbb{A}^2_{\mathbf{k}}$ and the group of automorphisms of $\mathbb{A}^2$ (defined over $\mathbf{k}$) which preserve this curve. We will denote this group by $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$. In other words, we study polynomials in $\mathbf{k}[x,y]$ and the $\mathbf{k}$-algebra automorphisms of $\mathbf{k}[x,y]$ that send the polynomial on a multiple of itself. We will always assume that the curve is reduced, i.e. that the polynomial does not contain any multiple factor. For our purpose, it is a natural assumption.

If Γ has equation in $\mathbf{k}[x]$, we will say that Γ is a *fence*. In this case, $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is easy to describe, and it is in fact an infinite union of algebraic groups. Our main result consists of showing that this is the only case where such phenomenon occurs.

Recall that $\mathrm{Aut}(\mathbb{A}^2)$ has the structure of an ind-variety. More precisely, the set $\mathrm{Aut}(\mathbb{A}^2)_d$ of automorphisms of degree $\leqslant d$ is an algebraic variety and $\mathrm{Aut}(\mathbb{A}^2)_d$ is closed in $\mathrm{Aut}(\mathbb{A}^2)_{d+1}$ for any d. This gives to $\mathrm{Aut}(\mathbb{A}^2) = \bigcup_{d=1}^{\infty} \mathrm{Aut}(\mathbb{A}^2)_d$ the structure of an ind-variety, and since composition and taking inverse preserve this structure, $\mathrm{Aut}(\mathbb{A}^2)$ is an ind-group (see [Kum02, Chapter IV] for precise definitions of "ind-variety" and "ind-group").

By definition, an algebraic subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ is a closed subgroup of bounded degree. Let us recall that the group $\mathrm{Aut}(\mathbb{A}^2)$ contains the following natural algebraic subgroups

$$\begin{aligned} \mathrm{Aff}(\mathbb{A}^2) &= \{(x,y) \mapsto (ax+by+e, cx+dy+f) \mid a,b,c,d,e,f \in \mathbf{k}, \\ &\qquad ad-bc \neq 0\}, \\ \mathrm{J}_n &= \{(x,y) \mapsto (ax+P(y), by+c) \mid a,b \in \mathbf{k}^*, c \in \mathbf{k}, P \in \mathbf{k}[y], \\ &\qquad \deg(P) \leqslant n\}. \end{aligned}$$

Moreover, $\mathrm{Aut}(\mathbb{A}^2)$ is generated by the union of these groups (Jung - van der Kulk's Theorem [Jun42], [vdK53]), any algebraic subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ is contained in one of these subgroups [Kam79, Theorem 4.3].

For any curve $\Gamma \subset \mathbb{A}^2$, the group $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is a closed ind-subgroup of $\mathrm{Aut}(\mathbb{A}^2)$. The question is to know if this is an algebraic subgroup, i.e. if it has bounded degree.

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Theorem 1. *Let Γ be a curve in $\mathbb{A}^2$. Applying an automorphism of $\mathbb{A}^2$, one of the following holds:*

i) The curve Γ has equation $F(x) = 0$, where $F(x) \in \mathbf{k}[x]$ is a square-free polynomial and

$$\begin{array}{rcl}\operatorname{Aut}(\mathbb{A}^2, \Gamma) & = & \{(x, y) \mapsto (ax + b, cy + P(x)) \mid a, c \in \mathbf{k}^*, \, b \in \mathbf{k}\,, \\ & & P \in \mathbf{k}[x]\,,\ F(ax + b)/F(x) \in \mathbf{k}^*\}\,.\end{array}$$

ii) The group $\operatorname{Aut}(\mathbb{A}^2, \Gamma)$ is equal to

$$\{g \in \operatorname{Aff}(\mathbb{A}^2) \mid g(\Gamma) = \Gamma\} \quad or \quad \{g \in \mathrm{J}_n \mid g(\Gamma) = \Gamma\}$$

for some integer n. Moreover, the action of $\operatorname{Aut}(\mathbb{A}^2, \Gamma)$ on Γ gives an isomorphism of $\operatorname{Aut}(\mathbb{A}^2, \Gamma)$ with a closed subgroup of $\operatorname{Aut}(\Gamma)$, (this latter being an algebraic group).

In particular, $\operatorname{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group if and only if there is no automorphism of $\mathbb{A}^2$ which sends Γ onto a fence.

Remark that the existence of an automorphism of $\mathbb{A}^2$ which sends Γ onto a line implies that $\Gamma \simeq \mathbb{A}^1$, and that the converse is true in characteristic 0 by the Abhyankar-Moh Theorem [AM75], but false in general. So Theorem 1 implies, in positive characteristic, that non-trivial embeddings of $\mathbb{A}^1$ into $\mathbb{A}^2$ are rigid in the sense that they admit only few compatible automorphisms of $\mathbb{A}^2$.

Another consequence of Theorem 1 is that the fixed locus of an automorphism of $\mathbb{A}^2$ only contains points and curves equivalent to lines, a result already observed by Friedland and Milnor in [FM89], in the case where $\mathbf{k} = \mathbb{C}$ (see [Jel03] for some generalisations to higher dimensions).

Remark that in the case where $\mathbf{k} = \mathbb{C}$, the observation on fixed points of Friedland and Milnor, the Abhyankar-Moh Theorem and the Lin-Zaidenberg Theorem [ZL83] imply Theorem 1 in the case where the curve Γ has an irreducible component non-isomorphic to $\mathbb{C}^*$. The interesting case of Theorem 1 is then, if $\mathbf{k} = \mathbb{C}$, the description of $\operatorname{Aut}(\mathbb{A}^2, \Gamma)$ when Γ is $\mathbb{C}^*$ (or a union of such curves). Note that there exist only partial classifications of the closed embeddings of $\mathbb{C}^*$ into $\mathbb{A}^2$ (see [CNKR09]), and that there exist complicated torsion-abelian closed subgroups of $\operatorname{Aut}(\mathbb{A}^2)$ which are not conjugated to a subgroup of $\operatorname{Aff}(\mathbb{A}^2)$ or to the Jonquières group $\bigcup_{n=1}^{\infty} \mathrm{J}_n$ (see [Wri79]). Theorem 1 implies that such groups do not preserve any curve.

The proof of Theorem 1 is done in Section 3, using tools of birational geometry of surfaces introduced in Section 2.

In Section 4, we refine the theorem by describing more precisely the possibilities for the group $\operatorname{Aut}(\mathbb{A}^2, \Gamma)$, in the case where Γ is geometrically irreducible and the ground field $\mathbf{k}$ is perfect. We obtain then the following result (the tori T of cases iv), v) are precisely described in Proposition 4.5).

Theorem 2. *Let Γ be a geometrically irreducible closed curve in $\mathbb{A}^2$, defined over the perfect field $\mathbf{k}$. Applying an automorphism of $\mathbb{A}^2$, one of the following holds:*

i) The curve Γ is the line with equation $x = 0$ and

$$\operatorname{Aut}(\mathbb{A}^2, \Gamma) = \{(x, y) \mapsto (ax, by + P(x)) \mid a, b \in \mathbf{k}^*, \ P \in \mathbf{k}[x]\}\,.$$

ii) The curve Γ has equation $x^b = \lambda y^a$, where $\lambda \in \mathbf{k}^*$ and $a, b > 1$ are coprime integers. Moreover, $\operatorname{Aut}(\mathbb{A}^2, \Gamma)$ is equal to the group $\mathbf{k}^*$ acting diagonally via $(x, y) \mapsto (t^a x, t^b y)$.

iii) *The curve* Γ *has equation* $x^b y^a = \lambda$, *where* $\lambda \in \mathbf{k}^*$ *and* $a, b \geqslant 1$ *are coprime integers. Moreover,* $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ *contains the group* $\mathbf{k}^*$ *acting diagonally via* $(x, y) \mapsto (t^a x, t^{-b} y)$, *and is equal to this group if* $(a, b) \neq (1, 1)$, *or is the group* $\mathbf{k}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ *generated by* $\mathbf{k}^*$ *and by* $(x, y) \mapsto (y, x)$ *if* $(a, b) = (1, 1)$.

iv) *The curve* Γ *has equation* $\lambda x^2 + \nu y^2 = 1$, *where* $\lambda, \nu \in \mathbf{k}^*$, $-\lambda\nu$ *is not a square in* $\mathbf{k}$ *and char*$(\mathbf{k}) \neq 2$. *The group* $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ *is the subgroup of* $\mathrm{GL}(2, \mathbf{k})$ *preserving the form* $\lambda x^2 + \nu y^2$, *which is isomorphic to* $T \rtimes \mathbb{Z}/2\mathbb{Z}$ *for some non-*$\mathbf{k}$*-split torus* T.

v) *The curve* Γ *has equation* $x^2 + \mu xy + y^2 = 1$, *where* $\mu \in \mathbf{k}^*$ *and* $x^2 + \mu x + 1$ *has no root in* $\mathbf{k}$, *and char*$(\mathbf{k}) = 2$. *The group* $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ *is the subgroup of* $\mathrm{GL}(2, \mathbf{k})$ *preserving the form* $x^2 + \mu xy + y^2$, *which is isomorphic to* $T \rtimes \mathbb{Z}/2\mathbb{Z}$ *for some non-*$\mathbf{k}$*-split torus* T.

vi) *The group* $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ *is a zero-dimensional (hence finite) subgroup of* $\mathrm{Aff}(\mathbb{A}^2)$ *or* J_n *for some* n.

2. Reminders on completions of $\mathbb{A}^2$

In this section, we define natural completions of $\mathbb{A}^2$, and links between them.

2.1. Natural completions of $\mathbb{A}^2$.

Example 2.1. The morphism

$$\begin{array}{ccc} \mathbb{A}^2 & \hookrightarrow & \mathbb{P}^2 \\ (x, y) & \mapsto & (x : y : 1) \end{array}$$

yields an isomorphism $\mathbb{A}^2 \xrightarrow{\sim} \mathbb{P}^2 \backslash L_{\mathbb{P}^2}$, where $L_{\mathbb{P}^2}$ is the line of $\mathbb{P}^2$ with equation $z = 0$.

Example 2.2. For $n \geqslant 1$, the n-th Hirzebruch surface $\mathbb{F}_n$ is

$$\mathbb{F}_n = \{((a : b : c), (u : v)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid bv^n = cu^n\}.$$

Let $E_n, L_{\mathbb{F}_n} \subset \mathbb{F}_n$ be the curves given by $(1 : 0 : 0) \times \mathbb{P}^1$ and $v = 0$, respectively. The morphism

$$\begin{array}{ccc} \mathbb{A}^2 & \hookrightarrow & \mathbb{F}_n \\ (x, y) & \mapsto & ((x : y^n : 1), (y : 1)) \end{array}$$

gives an isomorphism $\mathbb{A}^2 \xrightarrow{\sim} \mathbb{F}_n \backslash (E_n \cup L_{\mathbb{F}_n})$. Note that E_n is the unique section of $\pi \colon \mathbb{F}_n \to \mathbb{P}^1$ of self-intersection $-n$, and $L_{\mathbb{F}_n}$ is a smooth fibre, and has thus self-intersection 0.

Remark 2.3. In Example 2.2, we could also have chosen $n = 0$, which yields the surface $\mathbb{F}_0$, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, via $((x : y : z), (u : v)) \mapsto ((x : y), (u : v))$, but we will not need this one in the sequel.

Definition 2.4. A *natural completion* (of $\mathbb{A}^2$) is a pair (X, B) which is given in Example 2.1 or Example 2.2: either $(X, B) = (\mathbb{P}^2, L_{\mathbb{P}^2})$ or $(X, B) = (\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n})$ for some $n \geqslant 1$. The isomorphism $\mathbb{A}^2 \to X \backslash B$ given in the examples will be called *canonical isomorphism*, or *canonical embedding of* $\mathbb{A}^2$ *into* X.

A birational map (respectively birational morphism) $(X, B) \dashrightarrow (X', B')$ between two natural completions is a birational map (respectively birational morphism) $X \dashrightarrow X'$ inducing an isomorphism $X \backslash B \to X' \backslash B'$.

We denote by $\mathrm{Aut}(X, B)$ the group of automorphisms of (X, B): it is the group of automorphisms of X which leave B (or equivalently $X \backslash B = \mathbb{A}^2$) invariant.

Remark 2.5. Given two natural completions (X, B), (X', B'), any birational map $\varphi\colon (X, B) \dashrightarrow (X', B')$ restricts to an isomorphism $X\backslash B \to X'\backslash B'$, and corresponds thus, via the canonical isomorphisms $\mathbb{A}^2 \simeq X\backslash B$ and $\mathbb{A}^2 \simeq X'\backslash B'$, to a unique automorphism of $\mathbb{A}^2$. Moreover, every automorphism of $\mathbb{A}^2$ is obtained in this way.

Let us recall some easy fact on the automorphisms of natural completions:

Lemma 2.6. *Let (X, B) be a natural completion of $\mathbb{A}^2$, and let $\iota\colon \mathbb{A}^2 \to X\backslash B$ be the canonical isomorphism associated. The group $\mathrm{Aut}(X, B)$ corresponds via ι to the following subgroups of $\mathrm{Aut}(\mathbb{A}^2)$ (see the introduction for the definition of $\mathrm{Aff}(\mathbb{A}^2)$ and J_n):*

1) $X = \mathbb{P}^2$, $\mathrm{Aut}(X, B) = \mathrm{Aff}(\mathbb{A}^2) \simeq \mathrm{GL}(2, \mathbf{k}) \ltimes \mathbf{k}^2$
2) $X = \mathbb{F}_n$, $\mathrm{Aut}(X, B) = \mathrm{J}_n \simeq (\mathbf{k}^*)^2 \ltimes (\mathbf{k} \ltimes \mathbf{k}^{n+1})$.

Proof. The first assertion follows from the fact that $\mathrm{Aut}(\mathbb{P}^2)$ is the group of linear automorphisms, equal to $\mathrm{PGL}(3, \mathbf{k})$.

Recall that the $\mathbb{P}^1$-bundle structure given by the projection on the second factor is unique, and thus that the group of automorphisms of $\mathbb{F}_n$ preserve the $\mathbb{P}^1$-bundle structure (since $n \geqslant 1$). In particular, all automorphisms of $\mathrm{Aut}(\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n})$ restrict on $\mathbb{A}^2$ to automorphisms preserving the $\mathbb{A}^1$-bundle $\mathbb{A}^2 \to \mathbb{A}^1, (x, y) \mapsto y$. The precise description on the degree follows from a straightforward calculation. □

2.2. **Elementary links.** There are two kinds of very simple birational maps between natural completions. The first one are automorphisms, that we described in Lemma 2.6, and the second one are elementary links, that we describe now. The results of this section are classical, we just remind them to the reader for self-containedness and in order to fix notation.

Definition 2.7. A *link* is a birational map $\varphi\colon (X, B) \dashrightarrow (X', B')$ between two natural completions, which is not an isomorphism, such that both φ and φ^{-1} have at most one $\overline{\mathbf{k}}$-base-point (and in particular there is no infinitely near base-point). An *elementary link* is a link which does not decompose into $\varphi = \varphi' \circ \varphi''$, where φ', φ'' are links.

One can easily see that Definition 2.7 is equivalent to the classical definitions of elementary links which appear in the minimal model program (see [Cor95, Theorem of page 225] and [Isk96, Definition 2.2, page 597]):

Lemma 2.8. *Any elementary link is one of the following three types:*

i) *link of type* I*: a map $(\mathbb{P}^2, L_{\mathbb{P}^2}) \dashrightarrow (\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1})$ consisting of blowing-up one $\mathbf{k}$-rational point of $L_{\mathbb{P}^2}$.*

ii) *link of type* II*: maps $(\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n}) \dashrightarrow (\mathbb{F}_m, E_m \cup L_{\mathbb{F}_m})$ given by the blow-up of a $\mathbf{k}$-rational point p of $L_{\mathbb{F}_n}$ followed by the contraction of the strict transform of $L_{\mathbb{F}_n}$. Moreover, $m = n + 1$ if $p \in E_n$ and $m = n - 1$ if $p \notin E_n$.*

iii) *link of type* III*: a morphism $(\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1}) \to (\mathbb{P}^2, L_{\mathbb{P}^2})$ given by the contraction of the curve E_1 onto a $\mathbf{k}$-rational point of $L_{\mathbb{P}^2}$.*

The inverse of a link of type I, II, III *is a link of type* III, II, I *respectively.*

Proof. Let $\varphi\colon (X, B) \dashrightarrow (X', B')$ be a birational map, which is an elementary link.

Suppose first that φ is a morphism, which implies that φ is the contraction of a (-1)-curve onto a $\mathbf{k}$-rational point. This implies that $X = \mathbb{F}_1$ and $X' = \mathbb{P}^2$, and that we are in the third case. If φ^{-1} is a morphism, we get symmetrically the first case.

In the remaining cases, both φ and φ^{-1} have exactly one base-point. These points are thus defined over $\mathbf{k}$, and both maps contract one irreducible curve. The curves have thus either self-intersection -1 or self-intersection 0, depending if they contain the base-points. In the first case, we get a link $(\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1}) \dashrightarrow (\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1})$, which is not an elementary link since it factors through $\mathbb{P}^2$ as the composition of links of type III and I. In the second case, we get a link of type II described above. □

Proposition 2.9. *Let $\varphi\colon (X,B) \dashrightarrow (X',B')$ be a birational map between two natural completions of $\mathbb{A}^2$. If φ is not an isomorphism, then there exist $m \geqslant 1$ and elementary links $\varphi_1,\dots,\varphi_m$ such that $\varphi = \varphi_m \cdots \varphi_1$.*

Remark 2.10. We call $\varphi = \varphi_m \cdots \varphi_1$ a *decomposition into m elementary links*. If φ is an isomorphism, we sometimes say that it decomposes into 0 elementary links. This is coherent with the fact that the composition of an elementary link with an isomorphism of completions is an elementary link.

Proposition 2.9 implicitly follows from the work made in [Lam02] or from [BD11, Theorem 3.0.2]. The proof given here is however direct. Note also that Jung-van der Kulk's Theorem is a direct consequence of this Proposition.

Proof. As all birational maps between projective smooth surfaces, φ admits a minimal resolution, i.e. two birational morphisms $\epsilon\colon Y \to X$, $\eta\colon Y \to X'$ such that $\varphi = \eta\epsilon^{-1}$, and such that no (-1)-curve of Y (not necessarily defined over $\mathbf{k}$) is contracted by both ϵ and η. Moreover, each point blown-up by η and ϵ belongs to the boundaries B, B', as proper or infinitely near points.

We proceed by induction on the number of $\overline{\mathbf{k}}$-points blown-up by η and ϵ, corresponding to the number of base-points of φ^{-1} and φ. If η is an isomorphism, φ is only a sequence of blow-ups in the boundary of B. Because of the nature of B and B', this implies that φ is either an isomorphism or a link of type I, from $X = \mathbb{P}^2$ to $X' = \mathbb{F}_1$. Similarly, φ is an isomorphism or a link of type III if ϵ is an isomorphism.

We can thus assume that η (respectively ϵ) contract at least one (-1)-curve of Y, not contracted by ϵ (respectively η), and which is thus the strict transform of an irreducible curve $E \subset B$ (respectively $E' \subset B'$) of self-intersection $\geqslant -1$. If $E^2 = -1$, we factorise Y with a link of type III from $X = \mathbb{F}_1$ to $\mathbb{P}^2$. We do the same with φ^{-1} if $(E')^2 = -1$.

Looking at the self-intersections of the curves of the boundaries B and B', the remaining case is when $E^2 \geqslant 0$, $(E')^2 \geqslant 0$ and only one (-1)-curve of Y is contracted by η (respectively ϵ). This implies that φ and φ^{-1} have exactly one proper base-point. If $E^2 = 1$, we factorise φ with a link of type I from $X = \mathbb{P}^2$ to $\mathbb{F}_1$. Otherwise, $E^2 = 0$ and we factorise φ with a link of type II from $X = \mathbb{F}_n$ to $\mathbb{F}_m$ with $m = n \pm 1$. It remains to see that $m = 0$ is impossible. Indeed, otherwise the (-1)-curve of X would not pass through the unique proper base-point of φ and would thus be sent by φ onto a curve of self-intersection $\geqslant 0$, not contracted by φ^{-1}. □

Corollary 2.11. *Let $\varphi\colon (X,B) \dashrightarrow (X',B')$ be a birational map between two natural completions of $\mathbb{A}^2$. All $\overline{\mathbf{k}}$-base-points of φ (that belong to X as proper or infinitely near points) are defined over $\mathbf{k}$.*

Proof. Follows from Proposition 2.9 and from the fact that the base-points of any elementary link are defined over $\mathbf{k}$ (Lemma 2.8). □

For our purpose, we will need the decomposition into elementary links given by Proposition 2.9, but we will also need to have more precise information on the composition of links and their base-points, provided by Lemma 2.12 and Proposition 2.14 below.

Lemma 2.12. *Let $\varphi\colon (X_0, B_0) \dashrightarrow (X_1, B_1)$ and $\psi\colon (X_1, B_1) \dashrightarrow (X_2, B_2)$ be two elementary links. The birational map $\psi \circ \varphi$ is an isomorphism if and only if one of the following occurs:*

i) $X_1 = \mathbb{F}_1$, $X_0 = X_2 = \mathbb{P}^2$ (*i.e.* ψ *and* φ^{-1} *are both of type* III)

ii) *The maps ψ and φ^{-1} are both of type* I *or both of type* II*, and share the same base-point.*

Moreover, if $X_1 = \mathbb{F}_n$ and $X_0 = X_2 = \mathbb{F}_{n+1}$, for some $n \geqslant 1$, the map $\psi \circ \varphi$ is always an isomorphism.

Proof. If neither ψ nor φ^{-1} has a base-point, we have $X_1 = \mathbb{F}_1$ and $X_0 = X_2 = \mathbb{P}^2$, and both ψ and φ^{-1} are contractions of the (-1)-curve of $\mathbb{F}_1$. This implies that $\psi \circ \varphi$ is an automorphism of $\mathbb{P}^2$.

If ψ has a base-point, which is not a base-point of φ^{-1}, then $\psi \circ \varphi$ has a base-point, and is thus not an isomorphism. The same holds exchanging the role of ψ and φ.

The last case is when both ψ and φ^{-1} are links of type I or II, having the same base-point. The description of the links (Lemma 2.8) implies that $\psi \circ \varphi$ is an isomorphism. It remains to see that this is always the case when $X_1 = \mathbb{F}_n$ and $X_0 = X_2 = \mathbb{F}_{n+1}$. Indeed, this is true as the base-point of every elementary link from $(\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n})$ to $(\mathbb{F}_{n+1}, E_{n+1} \cup L_{\mathbb{F}_{n+1}})$ is the intersection point of E_n and $L_{\mathbb{F}_n}$. □

Definition 2.13. A decomposition of a birational map $\varphi\colon (X, B) \dashrightarrow (X', B')$ into elementary links is called *reduced*, if the composition of two consecutive elementary links is never an automorphism.

Proposition 2.14. *Let $\varphi\colon (X, B) \dashrightarrow (X', B')$ be a birational map between two natural completions of $\mathbb{A}^2$, and let $\varphi = \varphi_m \cdots \varphi_1$ be a reduced decomposition of φ into elementary links, with $m \geqslant 1$. Then, the following hold:*

i) *The number l of base-points of φ is equal to the number of elementary links φ_i of type* I *or* II*. If $l \geqslant 1$, then φ has a unique proper base-point, which is equal to the base-point of φ_1 if φ_1 is of type* I *or* II*, or to the preimage by φ_1 of the base-point of φ_2 if φ_1 is of type* III.

ii) *If $\varphi = \psi_k \cdots \psi_1$ is another decomposition into elementary links of φ, then $k \geqslant m$. Moreover, if $k = m$ the decomposition is unique, up to isomorphisms of completions.*

Proof. We prove *i)* by induction on m, the case $m = 1$ being obvious. Let $m \geqslant 2$.

Suppose that φ_1 is a link of type III. It contracts $E_1 \subset X$ onto a point which is not a base-point of φ_2, and thus not of $\varphi_m \cdots \varphi_2$, by induction hypothesis. In consequence, the number of base-points of φ is equal to the one of $\varphi_m \cdots \varphi_2$, and φ has a unique proper base-point, being the preimage by φ_1 of the one of φ_2.

Suppose now that φ_2 is a link of type III, which implies that φ_1 is a link $\mathbb{F}_2 \dashrightarrow \mathbb{F}_1$ of type II, with one base-point q. If $m = 2$, the result is clear. If $m \geqslant 3$, we use the induction hypothesis and see that the unique proper base-point of $\varphi_m \cdots \varphi_2$ is a point of $L_{\mathbb{F}_1} \backslash E_1$, which is not a base-point of $(\varphi_1)^{-1}$. This implies that φ has one more base-point as $\varphi_m \cdots \varphi_2$. Moreover, the proper base-point of $\varphi_m \cdots \varphi_2$ being on $L_{\mathbb{F}_1}$, contracted by $(\varphi_1)^{-1}$ onto q, this implies that q is the only proper base-point of φ.

To prove *i*), it remains to study the case where φ_1 and φ_2 are links of type I or II. As in the previous case, $\varphi_m \cdots \varphi_2$ has a unique proper base-point (here equal to the one of φ_2), which is not a base-point of $(\varphi_1)^{-1}$ but belongs to the curve contracted by $(\varphi_1)^{-1}$ onto the base-point of φ_1. This again implies that φ has one more base-point as $\varphi_m \cdots \varphi_2$ and that its unique proper base-point is the one of φ_1.

Now that *i*) is proved, let us show that it implies *ii*). Suppose the existence of another decomposition, that we can assume to be reduced, of length $k \leqslant m$: $\varphi = \varphi_m \cdots \varphi_1 = \psi_k \cdots \psi_1$. If φ has no base-point, both decomposition consist of one link of type III and the result is clear. Otherwise, the unique base-point of φ determines the first link, so we have $\psi_1 = \alpha\varphi_1$ for some isomorphism of natural completions α. Proceeding by induction, we get $k = m$ and the unicity of the decomposition. □

Proposition 2.14 implies that the "length" of a reduced decomposition of φ only depends on φ. The following definition is thus natural.

Definition 2.15. The number of elementary links in a reduced decomposition of φ is called the *length of* φ and we denote it by $\mathrm{len}(\varphi)$.

3. Birational maps preserving a curve and the proof of Theorem 1

This section is devoted to the proof of Theorem 1, which is done using elementary links and by looking at the singularities of the curve obtained in the natural completions.

Definition 3.1. If $\Gamma \subseteq X$ is a (closed) curve with no component in B and $\varphi\colon (X,B) \dashrightarrow (X,B)$ is a birational map, then we denote by $\varphi(\Gamma)$ the closure of $\varphi(\Gamma \cap X\backslash B)$ in X and call it the *image of* Γ *under* φ. Moreover, we denote by $\mathrm{Bir}((X,B),\Gamma)$ the group of birational maps $\varphi\colon (X,B) \dashrightarrow (X,B)$ such that $\varphi(\Gamma) = \Gamma$.

Remark 3.2. If $\Gamma \subset X$ is a curve having no component in B, then $\mathrm{Bir}((X,B),\Gamma) = \{g \in \mathrm{Aut}(X\backslash B) \mid g(\Gamma\backslash B) = (\Gamma\backslash B)\}$. In particular, $\mathrm{Bir}((X,B),\Gamma)$ corresponds to the group $\mathrm{Aut}(\mathbb{A}^2, \Gamma \cap \mathbb{A}^2)$ of automorphisms of $X\backslash B = \mathbb{A}^2$ that preserve the closed curve $\Gamma \cap \mathbb{A}^2$ of $\mathbb{A}^2$.

To study maps preserving a curve, we will study the singularities of the curves on the boundary.

Definition 3.3. Let Γ be a curve and let $p \in \Gamma$ be a $\overline{\mathbf{k}}$-point. We define the *height* $\mathrm{ht}_\Gamma(p)$ *of* Γ *in* p inductively, as follows

i) If Γ is smooth in p, then we define $\mathrm{ht}_\Gamma(p) := 0$.

ii) Otherwise, let $\pi\colon \tilde{\Gamma} \to \Gamma$ be the blow-up of Γ in p and let $p_1, \ldots, p_n$ be the points of $\tilde{\Gamma}$ with $\pi(p_i) = p$. We define $\mathrm{ht}_\Gamma(p) := \max_i\{\, \mathrm{ht}_{\tilde{\Gamma}}(p_i) + 1 \,\}$.

Recall that for any curves Γ_1, Γ_2 having no common component on a smooth projective surface, the intersection number $\Gamma_1 \cdot \Gamma_2$ is non-negative, and corresponds to the sum of the intersection numbers $(\Gamma_1 \cdot \Gamma_2)_p$, where p runs over all $\overline{\mathbf{k}}$-points of $\Gamma_1 \cap \Gamma_2$. The intersection number $(\Gamma_1 \cdot \Gamma_2)_p$ satisfies $(\Gamma_1 \cdot \Gamma_2)_p \geqslant m_p(\Gamma_1) \cdot m_p(\Gamma_2)$, where $m_p(\Gamma_i)$ is the multiplicity of Γ_i at p, and equality holds if and only if the tangent cones of Γ_1 and Γ_2 at p are distinct.

The next proposition is the key ingredient in the proof of Theorem 1.

Proposition 3.4. *Let $\varphi\colon (X,B) \dashrightarrow (X',B')$ be a birational map between natural completions of $\mathbb{A}^2$, admitting a reduced decomposition $\varphi = \varphi_m \cdots \varphi_1$ into elementary links, with $m \geqslant 1$. Let $\Gamma \subset X$ be a curve having no component in B and let $\Gamma' = \varphi(\Gamma) \subset X'$ be its image under φ. Suppose that one of the following holds:*

a) φ_1 is a link of type III *(from $\mathbb{F}_1$ to $\mathbb{P}^2$) and $(E_1 \cdot \Gamma)_{p_0} > 1$, where p_0 is defined by $\{p_0\} = E_1 \cap L_{\mathbb{F}_1} =: A$.*

b) φ_1 is a link of type I *or* II*, and the finite set $A \subset L_X$ of $\overline{\mathbf{k}}$-points lying on Γ which are not (proper) base-points of φ satisfies $\sum_{p\in A}(L_X \cdot \Gamma)_p > 1$.*

Then, there exists a $\overline{\mathbf{k}}$-point $q \in L_{X'}$ such that

$$\operatorname{ht}_{\Gamma'}(q) \geqslant l + \max_{p\in A} \operatorname{ht}_\Gamma(p)\,,$$

where l is the number of base-points of φ^{-1}. If $l \geqslant 1$, then q can be chosen as the unique proper base-point of φ^{-1}, which is a $\mathbf{k}$-point.

Proof. We proceed by induction on the number m of elementary links in the reduced decomposition of φ. We distinguish the following cases, depending on the nature of the first link φ_1.

i) If φ_1 is a link of type III, it contracts the curve $E_1 \subseteq \mathbb{F}_1$ onto a $\mathbf{k}$-point $q \in \mathbb{P}^2$. By assumption, we have $(E_1 \cdot \Gamma)_{p_0} > 1$. The multiplicity of $\Gamma_1 = \varphi_1(\Gamma)$ at q is equal to $E_1 \cdot \Gamma > 1$, which implies that Γ_1 is singular at q and that $\operatorname{ht}_{\Gamma_1}(q) \geqslant \operatorname{ht}_\Gamma(p_0) + 1$.

If $m = 1$, we have $l = 1$ and $\operatorname{ht}_{\Gamma_1}(q) \geqslant \operatorname{ht}_\Gamma(p) + 1 = l + \max_{p\in A} \operatorname{ht}_\Gamma(p)$ (because $A = \{p_0\}$), so we are done. So, assume $m > 1$. The point q is not a base point of φ_2, since otherwise $\varphi_2 \circ \varphi_1$ would be an automorphism of $\mathbb{F}_1$. Moreover, $(L_{\mathbb{P}^2} \cdot \Gamma_1)_q > 1$ because Γ_1 is singular at q. The claim follows by applying the induction hypothesis to $\varphi_m \cdots \varphi_2$, as q is not a base-point of $\varphi_m \cdots \varphi_2$.

ii) Suppose that φ_1 is a link of type II, i.e. $\varphi_1\colon \mathbb{F}_{n'} \dashrightarrow \mathbb{F}_n$ where $n' = n \pm 1$. By definition there exist blow-ups $\varepsilon\colon S \to \mathbb{F}_{n'}$ and $\eta\colon S \to \mathbb{F}_n$ of $\mathbf{k}$-points $q' \in L_{\mathbb{F}_{n'}}$ and $q \in L_{\mathbb{F}_n}$ respectively, such that $\varphi_1 = \eta \circ \varepsilon^{-1}$. Denote by $\tilde{\Gamma} \subseteq S$ the strict transform of Γ under ε^{-1}, and by $E_q \subset S$ the irreducible curve contracted by η onto q, which is the strict transform of $L_{\mathbb{F}_{n'}}$ under ε^{-1}. The map ε^{-1} is an isomorphism in a neighbourhood of every point of $A \subset L_{\mathbb{F}_{n'}}$. By hypothesis, one has $\sum_{p\in A}(L_{\mathbb{F}_{n'}} \cdot \Gamma)_p > 1$, which implies that $\tilde{\Gamma} \cdot E_q > 1$ on S. The curve $\Gamma_1 = \eta(\tilde{\Gamma}) \subset \mathbb{F}_n$ is thus singular at q, and the height of Γ_1 at q is at least equal to $1 + \max_{p\in A} \operatorname{ht}_\Gamma(p)$. If $m = 1$, then we are done. If $m > 1$, q is not a base point of φ_2, since otherwise $\varphi_2 \circ \varphi_1$ would be an isomorphism. Moreover, q is a singular point of Γ_1 which belongs to $L_{\mathbb{F}_n}$, and if φ_2 is of type III, then $n = 1$, $n' = 2$, so q is the intersection point of E_1 and $L_{\mathbb{F}_1}$. In any case, the point q belongs to the set "A" associated to $\varphi_m \cdots \varphi_2$, so the result follows by induction.

iii) The last case is when φ_1 is of type I, i.e. when it is a map $\varphi_1\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$. If $m = 1$, then $l = 0$ and the result is obvious, by choosing q as the image of a point of A with a maximal height. If $m > 1$, the link φ_2 is a map $\mathbb{F}_1 \dashrightarrow \mathbb{F}_2$ centered at the intersection point of E_1 and $L_{\mathbb{F}_1}$. In particular, the map φ_1 induces an isomorphism at a neighbourhood of any point of A, sending $L_{\mathbb{P}^2}$ onto $L_{\mathbb{F}_1}$, and sends the set A onto the set "A" associated to $\varphi_m \cdots \varphi_2$. The result follows by induction hypothesis. □

The following corollary is a direct consequence of Proposition 3.4.

Corollary 3.5. *Let $\varphi \in \mathrm{Bir}((X,B),\Gamma)\backslash \mathrm{Aut}(X,B)$ and let $\varphi = \varphi_m \cdots \varphi_1$ be its reduced decomposition into elementary links. We assume that $(L_X \cdot \Gamma)_p > 1$ for some $\overline{\mathbf{k}}$-point p and if φ_1 is of type* III *(i.e. a link from $\mathbb{F}_1$ to $\mathbb{P}^2$), then we also assume $\{p\} = E_1 \cap L_{\mathbb{F}_1}$.*

Then the height in Γ of the proper base point q of φ is (strictly) bigger than the height in Γ of every other point of $\Gamma \cap L_X$. In particular, $(L_X \cdot \Gamma)_q > 1$, φ and φ^{-1} have the same proper base-point q, and either $\Gamma \cap L_X = \{q\}$ or q is a singular point of Γ.

Proposition 3.6. *Let (X,B) be a natural completion of $\mathbb{A}^2$ and let Γ be a curve in X having no component in B. Then, there exists a natural completion (X',B') of $\mathbb{A}^2$ and a birational map $\varphi\colon (X,B) \dashrightarrow (X',B')$ such that one of the following holds:*

1) $\mathrm{Bir}((X',B'),\varphi(\Gamma)) \subseteq \mathrm{Aut}(X',B')$*;*
2) $\varphi(\Gamma)$ *intersects transversally $L_{X'}$, i.e. $(\varphi(\Gamma)\cdot L_{X'})_p \leqslant 1$ for any $p \in X'(\overline{\mathbf{k}})$.*

Proof. We define $(X_0,B_0) = (X,B)$, define $\varphi_0 = \mathrm{id} \in \mathrm{Aut}(X_0,B_0)$ and construct inductively a sequence of elementary links

$$(*) \qquad (X_0,B_0) \overset{\varphi_1}{\dashrightarrow} (X_1,B_1) \overset{\varphi_2}{\dashrightarrow} (X_2,B_2) \overset{\varphi_3}{\dashrightarrow} \ldots .$$

Let $i \geqslant 0$ and assume that $\varphi_0,\ldots,\varphi_i$ are already constructed. We write $\Gamma_i = \varphi_i \cdots \varphi_1(\Gamma)$, $G_i = (\mathrm{Bir}(X_i,B_i),\Gamma_i)$ and define now φ_{i+1} in the following way:

i) If $G_i \subseteq \mathrm{Aut}(X_i,B_i)$ or Γ_i intersects transversally L_{X_i}, we define $(X_{i+1},B_{i+1}) = (X_i,B_i)$ and $\varphi_{i+1} = \mathrm{id}$.

ii) If *i)* doesn't hold and the decomposition into elementary links of every $g \in G_i\backslash \mathrm{Aut}(X_i,B_i)$ starts with the link $\tau_0\colon \mathbb{F}_1 \to \mathbb{P}^2$ (which contracts the (-1)-curve of $\mathbb{F}_1$), we define $\varphi_{i+1} = \tau_0$, and thus set $(X_{i+1},B_{i+1}) = (\mathbb{P}^2, L_{\mathbb{P}^2})$.

iii) If *i)* and *ii)* do not occur, there exists $g \in G_i\backslash \mathrm{Aut}(X_i,B_i)$ such that the decomposition into elementary links of g starts with a link τ_1 being not from $\mathbb{F}_1$ to $\mathbb{P}^2$. In this case we define $\varphi_{i+1} = \tau_1$, and (X_{i+1},B_{i+1}) as the target of this link.

Now, we claim that the sequence $(*)$ satisfies the following two properties.

a) If φ_{i+1} is not an automorphism, then we have for all $g \in G_i$

$$\mathrm{len}(\varphi_{i+1} g \varphi_{i+1}^{-1}) < \mathrm{len}(g) \quad \text{or} \quad \mathrm{len}(\varphi_{i+1} g \varphi_{i+1}^{-1}) = \mathrm{len}(g) = 0\,.$$

b) The sequence $(*)$ is stationary after finitely many steps, i.e. φ_i is an automorphism for i large enough.

Property *a)* will serve to show Property *b)*, which directly implies the result. It thus remains to prove these two properties.

Proof of Property a):

In case *i)*, there is nothing to prove.

If we are in case *ii)*, then for all $g \in G_i\backslash \mathrm{Aut}(X_i,B_i)$ the decomposition into elementary links starts with $\tau_0\colon \mathbb{F}_1 \to \mathbb{P}^2$ and ends with $(\tau_0)^{-1}$, since G_i is a group. The conjugation of G_i by τ_0 decreases the length of any element of $G_i\backslash \mathrm{Aut}(X_i,B_i)$ by two. Moreover, $(X_i,B_i) = (\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1})$, so $\mathrm{Aut}(X_i,B_i)$ preserves the curve

E_1 contracted by τ_0, which implies that $\tau_0 \operatorname{Aut}(X_i, B_i)(\tau_0)^{-1} \subset \operatorname{Aut}(\mathbb{P}^2, L_{\mathbb{P}^2}) = \operatorname{Aut}(X_{i+1}, B_{i+1})$. It follows that a) is satisfied.

Assume that we are in case *iii*). As we are not in case *i*) there exists $p_0 \in X_i$ such that $(\Gamma_i \cdot L_{X_i})_{p_0} > 1$. Moreover, by definition of case *iii*), there exists $g \in G_i \backslash \operatorname{Aut}(X_i, B_i)$ such that the decomposition into elementary links starts with $\varphi_{i+1} \colon (X_i, B_i) \dashrightarrow (X_{i+1}, B_{i+1})$, and φ_{i+1} has a base-point p since it does not go from $\mathbb{F}_1$ to $\mathbb{P}^2$. Applying Corollary 3.5 to g, we see that $(\Gamma_i \cdot L_{X_i})_p > 1$ and that the height in Γ of p is bigger than the height in Γ of every other point of $\Gamma \cap L_X$. This implies that any element $h \in \operatorname{Aut}(X_i, B_i) \cap G_i$ fixes p, and thus satisfies $\varphi_{i+1} h \varphi_{i+1}^{-1} \in \operatorname{Aut}(X_{i+1}, B_{i+1})$. If $X_i = \mathbb{F}_1$, then φ_{i+1} is a link from $\mathbb{F}_1$ to $\mathbb{F}_2$, so p is the intersection point $E_1 \cap L_{\mathbb{F}_1}$. We can thus apply Corollary 3.5 to any element $h \in G_i \backslash \operatorname{Aut}(X_i, B_i)$, and see that p is the unique proper base-point of h, so the reduced decomposition of h starts with φ_{i+1}. This yields *a*).

Proof of Property b):

By property *a*), for all i, $\varphi_i \cdots \varphi_1$ is a reduced decomposition into elementary links, if we are for no index $< i$ in case *i*). For any $i \geqslant 0$ such that neither φ_{i+1} nor φ_{i+2} is an isomorphism, we define $p_i \in L_{X_i} \subset B_i \subset X_i$ as the unique proper base-point of $\varphi_{i+2} \circ \varphi_{i+1}$.

Under the assumption that neither of the three maps $\varphi_{i+1}, \varphi_{i+2}$ and φ_{i+3} is an isomorphism we show that $(\Gamma_i \cdot L_{X_i})_{p_i} \geqslant (\Gamma_{i+1} \cdot L_{X_{i+1}})_{p_{i+1}}$ and describe the cases when equality holds:

- φ_{i+1} *is a link of type* II: we can write $\varphi_{i+1} = \eta \circ \pi^{-1}$, where $\pi \colon S \to X_i$ is the blow-up of p_i and $\eta \colon S \to X_{i+1}$ is the blow-up of a point $q \neq p_{i+1}$. Denote by $\widetilde{\Gamma}_i, \widetilde{L_{X_i}} \subset S$ the strict transforms of Γ_i and L_{X_i}, and let E_{p_i} be the exceptional divisor of π. It follows that η contracts $\widetilde{L_{X_i}}$. Note that η^{-1} restricts to an isomorphism in a neighbourhood of p_{i+1}, which sends $L_{X_{i+1}}$ onto E_{p_i}. This yields the following estimate
$$\begin{aligned} (\Gamma_i \cdot L_{X_i})_{p_i} &\geqslant m_{p_i}(\Gamma_i) \\ &= \widetilde{\Gamma}_i \cdot E_{p_i} \\ &\geqslant (\widetilde{\Gamma}_i \cdot E_{p_i})_{\eta^{-1}(p_{i+1})} \\ &= (\Gamma_{i+1} \cdot L_{X_{i+1}})_{p_{i+1}} \end{aligned}$$
where $m_{p_i}(\Gamma_i)$ denotes the multiplicty of Γ_i in p_i. Moreover, if $(\Gamma_i \cdot L_{X_i})_{p_i} = m_{p_i}(\Gamma_i) = (\Gamma_{i+1} \cdot L_{X_{i+1}})_{p_{i+1}} > 1$, then $\operatorname{ht}_{\Gamma_i}(p_i) > \operatorname{ht}_{\Gamma_{i+1}}(p_{i+1})$.
- φ_{i+1} *is a link of type* I: Remark that $(\varphi_{i+1})^{-1} \colon \mathbb{F}_1 \to \mathbb{P}^2$ is the blow-up of $p_i \in \mathbb{P}^2$ and that Γ_{i+1} and $L_{\mathbb{F}_1}$ are the strict transforms of Γ_i and $L_{\mathbb{P}^2}$ under $(\varphi_{i+1})^{-1}$ respectively. It follows that $(\Gamma_i \cdot L_{\mathbb{P}^2})_{p_i} \geqslant (\Gamma_{i+1} \cdot L_{\mathbb{F}_1})_{p_{i+1}}$.
- φ_{i+1} *is a link of type* III: Since φ_{i+1} is an isomorphism in a neighbourhood of p_i and $p_{i+1} = \varphi_{i+1}(p_i)$ it follows that $(\Gamma_i \cdot L_{\mathbb{F}_1})_{p_i} = (\Gamma_{i+1} \cdot L_{\mathbb{P}_2})_{p_{i+1}}$ and $\operatorname{ht}_{\Gamma_i}(p_i) = \operatorname{ht}_{\Gamma_{i+1}}(p_{i+1})$.

Now, assume towards a contradiction, that the sequence $(*)$ is never stationary, i.e. φ_i is not an isomorphism for all $i \geqslant 1$. According to this case-by-case-analysis, we see that $(\Gamma_i \cdot L_{X_i})_{p_i}$ is a decreasing sequence in i and for every $r > 1$ there are only finitely many i with $(\Gamma_i \cdot L_{X_i})_{p_i} = r$. Thus, there exists I such that for all $i \geqslant I$ we have $(\Gamma_i \cdot L_{X_i})_{p_i} \leqslant 1$. As the sequence $(*)$ is not stationary, we have a φ_{i+1} with $i \geqslant I$ which is a link of type I or II. The point p_i is thus the base-point of φ_{i+1} and satisfies $(\Gamma_i \cdot L_{X_i})_{p_i} \leqslant 1$. By Corollary 3.5 applied to an automorphism of $G_i\backslash \mathrm{Aut}(X_i, B_i)$ that starts with φ_{i+1}, this implies that $(\Gamma_i \cdot L_{X_i})_q = 1$ for all $q \in L_{X_i} \cap \Gamma_i$. Hence, we are in case i), a contradiction. □

Lemma 3.7. *Let $(X, B) = (\mathbb{P}^2, L_{\mathbb{P}^2})$, and let Γ be a conic (as always, reduced but not necessarily irreducible) in $X = \mathbb{P}^2$, intersecting $L_{\mathbb{P}^2}$ into two distinct points (not necessarily defined over $\mathbf{k}$).*

Then, $\mathrm{Bir}((X, B), \Gamma) \subseteq \mathrm{Aut}(X, B)$.

Proof. Suppose, for contradiction, the existence of $g \in \mathrm{Bir}((X, B), \Gamma)$, which is not an automorphism of $\mathbb{P}^2$.

Applying Proposition 3.4, one of the two points of $\Gamma \cap L_X$ is a base-point of g (and in particular it is defined over $\mathbf{k}$). By Proposition 2.9 and Lemma 2.12, there exists an integer $n \geqslant 2$ such that the reduced decomposition of g into elementary links starts with $\varphi_{2n}\varphi_{2n-1}\cdots\varphi_1$, where $\varphi_1 : \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$, $\varphi_{2n} : \mathbb{F}_1 \to \mathbb{P}^2$, $\varphi_i : \mathbb{F}_{i-1} \dashrightarrow \mathbb{F}_i$ and $\varphi_{i+n-1} : \mathbb{F}_{n-i+2} \dashrightarrow \mathbb{F}_{n-i+1}$ for $i = 2, \ldots, n$. The curve $\varphi_1(\Gamma)$ intersects transversally E_1 into one point and $L_{\mathbb{F}_1}$ into one point, and does not pass through the intersection point of E_1 and $L_{\mathbb{F}_1}$, blown-up by φ_2. This implies that the same holds for $\varphi_2\varphi_1(\Gamma)$, with E_2 and $L_{\mathbb{F}_2}$, and that $\varphi_2\varphi_1(\Gamma)$ passes through the point blown-up by $(\varphi_2)^{-1}$. Proceeding by induction, the curve $\varphi_n\cdots\varphi_1(\Gamma) \subset \mathbb{F}_n$ intersects transversally E_n into one point and $L_{\mathbb{F}_n}$ into one point, and passes through the point blown-up by $(\varphi_n)^{-1}$. In consequence, it does not pass through the point blown-up by φ_{n+1}, which implies that $\varphi_{n+1}\varphi_n\cdots\varphi_1(\Gamma) \subset \mathbb{F}_{n-1}$ intersects $E_{n-1} \cup L_{\mathbb{F}_{n-1}}$ into two distinct points, both being on E_{n-1}. Proceeding by induction, the curve $\varphi_{2n-1}\cdots\varphi_n\cdots\varphi_1(\Gamma) \subset \mathbb{F}_1$ intersects $E_1 \cup L_{\mathbb{F}_1}$ into two distinct points, both on E_1. The curve $\varphi_{2n}\cdots\varphi_1(\Gamma) \subset \mathbb{P}^2$ intersects $L_{\mathbb{P}^2}$ into one point q, and is singular at this point, with two branches. The remaining part of the decomposition of g is equal to $\varphi_m\cdots\varphi_{2n+1}$, and its unique proper base-point is different from q. Proposition 3.4 implies that $g(\Gamma)$ is singular at a point of $B = L_{\mathbb{P}^2}$, which is a contradiction. □

Proposition 3.8. *Let (X, B) be a natural completion of $\mathbb{A}^2$ and let Γ be a curve in X. If Γ intersects transversally L_X, there exists a natural completion (X', B') of $\mathbb{A}^2$ and a birational map $\varphi : (X, B) \dashrightarrow (X', B')$ such that one of the following holds:*

i) $\mathrm{Bir}((X', B'), \varphi(\Gamma)) \subseteq \mathrm{Aut}(X', B')$*;*

ii) $X' = \mathbb{P}^2$ *and* $\varphi(\Gamma) \subset \mathbb{P}^2$ *is defined by a polynomial in* $\mathbf{k}[x]$*.*

Proof. We can assume the existence of $g \in \mathrm{Bir}((X, B), \Gamma)\backslash \mathrm{Aut}(X, B)$, admitting a reduced decomposition $g = \varphi_m\cdots\varphi_1$ into elementary links.

a) Suppose that $X = \mathbb{P}^2$. By Corollary 3.5, the curve Γ intersects L_X into at most 2 points, hence Γ is a line or a conic. If Γ is a line the equation can be chosen to be $x = 0$, and if Γ is a conic, the result follows from Lemma 3.7.

b) Suppose that $X \not\simeq \mathbb{P}^2$ but that there exists $1 \leqslant i \leqslant m$ such that φ_i is a link $\varphi_i : \mathbb{F}_1 \to \mathbb{P}^2$ of type III, contracting E_1 onto a point $q \in L_{\mathbb{P}^2}$. If $\Gamma' = \varphi_i\cdots\varphi_1(\Gamma)$ intersects transversally $L_{\mathbb{P}^2}$, we conclude by applying case a) to Γ'. There exists thus a point $p \in L_{\mathbb{P}^2}$ with $(\Gamma' \cdot L_{\mathbb{P}^2})_p > 1$. We claim that p can be chosen to be equal to q. If $i = 1$, this is true because $\Gamma' = \varphi_1(\Gamma)$ and Γ intersects transversally $L_{\mathbb{F}_1}$, which is the strict transform of $L_{\mathbb{P}^2}$. If $i > 1$, the claim follows from Proposition 3.4, applied to $(\varphi_1)^{-1}\cdots(\varphi_i)^{-1} : (\mathbb{P}^2, L^2_{\mathbb{P}}) \dashrightarrow (X, B)$. Since φ_{i+1} is a link of type I and q is not a base-point of φ_{i+1}, $\varphi_{i+1}(\Gamma')$ does not intersect transversally $L_{\mathbb{F}_1}$. This implies that $m \geqslant i + 2$, and that $(\varphi_m\cdots\varphi_{i+1})^{-1}$ has at least one base-point. Applying

Proposition 3.4 to $\varphi_m \cdots \varphi_{i+1}$, the curve $\Gamma = \varphi_m \cdots \varphi_{i+1}(\Gamma')$ has a singular point at the proper base-point of $(\varphi_m \cdots \varphi_{i+1})^{-1}$. Since this one lies on L_X, we get a contradiction.

c) We can now assume that $X = \mathbb{F}_n$ for some $n \geqslant 1$ and that all φ_i are links of type II. By Corollary 3.5, $L_{\mathbb{F}_n} \cap \Gamma$ contains at most 2 points and if it contains 2, then one of these is a base-point of φ_1. If one of the points of $L_{\mathbb{F}_n} \cap \Gamma$ is the intersection point p_n of $L_{\mathbb{F}_n}$ and E_n, we perform an elementary link $\psi\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$. Because $L_{\mathbb{F}_n} \cap \Gamma$ contains at most 2 points, the curve $\psi(\Gamma)$ intersects transversally $L_{\mathbb{F}_{n+1}}$, into at most 2 points. Moreover, writing $\{p_{n+1}\} = L_{\mathbb{F}_{n+1}} \cap E_{n+1}$, we obtain $(\psi(\Gamma) \cdot E_{n+1})_{p_{n+1}} = (\Gamma \cdot E_n)_{p_n} - 1$. Performing a sequence of elementary links if needed, we reduce to the case where $L_{\mathbb{F}_n} \cap \Gamma$ contains at most 2 points and that none of them belong to E_n.

If $L_{\mathbb{F}_n} \cap \Gamma$ is empty, then Γ is contained in a finite set of fibres of $\mathbb{F}_n \to \mathbb{P}^1$. Going from $\mathbb{F}_n$ to $\mathbb{F}_1$ and then to $\mathbb{P}^2$, we send the fibres onto lines of the form $x = a$ where $a \in \overline{\mathbf{k}}$, and see that $L_{\mathbb{F}_n}$ has equation in $\mathbf{k}[x]$.

It remains to see that it is not possible, in the case where $L_{\mathbb{F}_n} \cap \Gamma$ contains 1 or 2 points that belong to $L_{\mathbb{F}_n} \backslash E_n$, to have $g \in \mathrm{Bir}((X,B),\Gamma) \backslash \mathrm{Aut}(X,B)$ having a reduced decomposition consisting only of links of type II. Such an element has a decomposition $g = \varphi_{2k} \cdots \varphi_1$, where $k \geqslant 1$, φ_i is a link $\mathbb{F}_{n+i-1} \dashrightarrow \mathbb{F}_{n+i}$ for $i = 1, \ldots, k$, φ_i is a link $\mathbb{F}_{n+2k-i+1} \dashrightarrow \mathbb{F}_{n+2k-i}$ for $i = k+1, \ldots, 2k$, and where $(\varphi_k)^{-1}$ and φ_{k+1} do not have the same base-point (see Lemma 2.12). In particular, the base-point of φ_1 is the intersection point of E_n and $L_{\mathbb{F}_n}$, and does not lie on Γ, so $\varphi_1(\Gamma)$ passes through the base-point of $(\varphi_1)^{-1}$, which is not a base-point of φ_2. By induction, we deduce that $(\varphi_i \cdots \varphi_1)(\Gamma)$ passes through the base-point of $(\varphi_i)^{-1}$, for $i = 1, \ldots, 2k$. In the case $i = 2k$, this implies that $g(\Gamma)$ passes through the base-point of $(\varphi_{2k})^{-1}$, i.e. through the intersection point of E_n and $L_{\mathbb{F}_n}$, contradicting the fact that $g(\Gamma) = \Gamma$. □

Corollary 3.9. *Let (X,B) be a natural completion of $\mathbb{A}^2$ and let Γ be a curve in X having no component in B. Then, there exists a natural completion (X',B') of $\mathbb{A}^2$ and a birational map $\varphi\colon (X,B) \dashrightarrow (X',B')$ such that one of the following holds:*

1) $\mathrm{Bir}((X',B'),\varphi(\Gamma)) \subseteq \mathrm{Aut}(X',B')$*;*
2) $X' = \mathbb{P}^2$ *and* $\varphi(\Gamma) \subset \mathbb{P}^2$ *is defined by a polynomial in* $\mathbf{k}[x]$.

Proof. Follows directly from Propositions 3.6 and 3.8. □

Theorem 1 is now a direct consequence of Corollary 3.9:

Proof of Theorem 1. In the case where the equation of Γ is in $\mathbf{k}[x]$, i.e. when Γ is a fence, the explicit description of $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ is an easy calculation. If Γ is not equivalent to a fence by an automorphism of $\mathbb{A}^2$, Corollary 3.9 and Lemma 2.6 imply that we can conjugate $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ to a subgroup of $\mathrm{Aff}(\mathbb{A}^2)$ or J_n for some n. This implies that $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ is an algebraic group. Moreover, we obtain a morphism of algebraic groups $\mathrm{Aut}(\mathbb{A}^2,\Gamma) \to \mathrm{Aut}(\Gamma)$. It remains to observe that curves fixed pointwise by elements of J_n and $\mathrm{Aff}(\mathbb{A}^2)$ are fences in a suitable coordinate system of $\mathbb{A}^2$. □

3.1. **Generalisation to other subsets.**

Definition 3.10. If $\Delta \subset \mathbb{A}^2(\overline{\mathbf{k}})$ is any subset, we denote by $\mathrm{Aut}(\mathbb{A}^2,\Delta)$ the group of automorphisms of $\mathbb{A}^2$ that leave the set Δ invariant, and denote by $\mathrm{Aut}_F(\mathbb{A}^2,\Delta)$ the group of automorphisms of $\mathbb{A}^2$ that fix any element of Δ.

By Definition, $\mathrm{Aut}_F(\mathbb{A}^2,\Delta)$ is always a normal subgroup of $\mathrm{Aut}(\mathbb{A}^2,\Delta)$. Moreover, if $\mathrm{Aut}(\mathbb{A}^2,\Delta)$ is an algebraic group, then $\mathrm{Aut}_F(\mathbb{A}^2,\Delta)$ is a an algebraic subgroup. Theorem 1 implies the following result:

Proposition 3.11. *Let $\Delta \subset \mathbb{A}^2(\overline{\mathbf{k}})$ be a subset. Applying an element $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$, the following hold:*

i) The set Δ is contained in a fence given by $f(x) = 0$ where $f \in \mathbf{k}[x]$, and the group $\mathrm{Aut}_F(\mathbb{A}^2,\Delta)$ is not algebraic: it contains the group

$$\{(x,y) \mapsto (x, y + p(x)f(x)) \mid p \in \mathbf{k}[x]\} \simeq \mathbf{k}[x].$$

ii) The group $\mathrm{Aut}(\mathbb{A}^2,\Delta)$ is equal to $\{g \in \mathrm{Aff}(\mathbb{A}^2) \mid g(\Delta) = \Delta\}$, and then $\mathrm{Aut}(\mathbb{A}^2,\Delta)$ is an algebraic subgroup of $\mathrm{Aff}(\mathbb{A}^2)$. Moreover, $\mathrm{Aut}_F(\mathbb{A}^2,\Delta)$ is trivial.

iii) There exists an integer $n \geqslant 1$ such that the group $\mathrm{Aut}(\mathbb{A}^2,\Delta)$ is equal to $\{g \in \mathrm{J}_n \mid g(\Delta) = \Delta\}$, and then $\mathrm{Aut}(\mathbb{A}^2,\Delta)$ is an algebraic subgroup of J_n. Moreover, $\mathrm{Aut}_F(\mathbb{A}^2,\Delta)$ is trivial.

Proof. Denote by $I(\Delta) \subset \mathbf{k}[x,y]$ the ideal of polynomials vanishing on Δ, and by $\overline{\Delta} \subset \mathbb{A}^2(\overline{\mathbf{k}})$ the closure of Δ, which is the set of points where $I(\Delta)$ vanishes. Then we obtain $\mathrm{Aut}(\mathbb{A}^2,\Delta) = \mathrm{Aut}(\mathbb{A}^2,\overline{\Delta})$ and $\mathrm{Aut}_F(\mathbb{A}^2,\Delta) = \mathrm{Aut}_F(\mathbb{A}^2,\overline{\Delta})$. We can thus replace Δ with $\overline{\Delta}$.

If Δ is a finite union of points, we get case (i). Otherwise, Δ is the union of one curve Γ (reduced but not necessarily irreducible) and of a finite number of points of $\mathbb{A}^2(\overline{\mathbf{k}})$, and $\mathrm{Aut}(\mathbb{A}^2,\Delta) \subset \mathrm{Aut}(\mathbb{A}^2,\Gamma)$. The result follows then from the description of Γ and $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$, given in Theorem 1. □

In the case where Δ is finite, the group $\mathrm{Aut}(\mathbb{A}^2,\Delta)$ is quite big; indeed it is often maximal. This is the case for example when $\mathbf{k} = \mathbb{C}$, as pointed out to us by J.-P. Furter and P.-M. Poloni. This is a consequence of the following observation.

Lemma 3.12. *Let G be a group acting on a set S. Let $\Delta \subset S$ be a finite subset of $r \geqslant 1$ points. Suppose that G acts $2r$-transitively on S, and that $|S| > 2r$. Then,*

$$G_\Delta = \{g \in G \mid g(\Delta) = \Delta\}$$

is a maximal subgroup of G.

Proof. Since G_Δ is not trivial and not equal to G (because of r-transitivity), it suffices to take $a \in G\backslash G_\Delta$ and to show that a and G_Δ generate G. We can write $\Delta = \{x_1,\dots,x_r\}$, with $a(x_1),\dots,a(x_k) \notin \Delta$ and $a(x_{k+1}),\dots,a(x_r) \in \Delta$, where $1 \leqslant k \leqslant r$. The hypotheses yield the existence of $g \in G$ that fixes $x_1,\dots,x_r,a(x_2),\dots,a(x_k)$, and does not fix $a(x_1)$. Then, $g \in G_\Delta$ and $f = a^{-1}ga$ fixes $x_2,\dots,x_r$ but $f(x_1) \notin \Delta$.

It remains to see that any $h \in G\backslash G_\Delta$ is generated by f and G_Δ. We write $\Delta = \{z_1,\dots,z_r\}$, with $h(z_1)$, $\dots$, $h(z_j) \notin \Delta$ and $h(z_{j+1})$, $\dots$, $h(z_r) \in \Delta$, where $1 \leqslant j \leqslant r$. Replacing h with its composition with an element of G_Δ we can assume that $h(z_i) = z_i$ for $i = j+1,\dots,r$. For $i = 1,\dots,j$, we choose $g_i \in G_\Delta$ that sends z_i onto x_1 and sends $h(z_i)$ onto $f(x_1)$. Then, $(g_i)^{-1}fg_i$ sends z_i onto $h(z_i)$ and fixes $\Delta\backslash\{z_i\}$. Composing this element with an element of G_Δ, we find an element f_i, generated by f and G_Δ, which sends z_i onto $h(z_i)$ and fixes $(\Delta \cup h(\Delta))\backslash\{z_i,h(z_i)\}$.

Since $h^{-1}f_1\cdots f_j$ belongs to G_Δ, this achieves the proof. □

Corollary 3.13. *Assume that the ground field* $\mathbf{k}$ *is not a finite field of characteristic* 2 *and let* $\Delta \subset \mathbb{A}^n(\mathbf{k})$ *be a finite proper non-empty set, with* $n \geqslant 2$. *Then,* $\mathrm{Aut}(\mathbb{A}^n, \Delta)$ *is a maximal subgroup of* $\mathrm{Aut}(\mathbb{A}^n)$.

Proof. If $\mathbf{k}$ is infinite, we use Lemma 3.12 and the fact that $\mathrm{Aut}(\mathbb{A}^2)$ acts m-transitively on $\mathbb{A}^2(\mathbf{k})$ for every $m \geqslant 1$, which can be seen using the subgroup

$$\{(x_1, \ldots, x_n) \mapsto (x_1 + p(x_2, \ldots, x_n), x_2, \ldots, x_n) \mid p \in \mathbf{k}[x_2, \ldots, x_n]\},$$

and permutations of coordinates.

If $\mathbf{k}$ is a finite field of characteristic > 2, the group $\mathrm{Aut}(\mathbb{A}^2)$ acts m-transitively on $\mathbb{A}^2(\mathbf{k})$ for each m by [Mau01]; we can then apply Lemma 3.12 to Δ or its complement. □

Corollary 3.14. *Assume that the ground field* $\mathbf{k}$ *is a finite field of characteristic* 2 *and let* $\Delta \subset \mathbb{A}^n(\mathbf{k})$ *be a finite proper non-empty set, with* $n \geqslant 2$.

Then, the group $\mathrm{Aut}(\mathbb{A}^n, \Delta)$ *is a maximal subgroup of* $\mathrm{Aut}(\mathbb{A}^n)$ *if and only if* $|\Delta| \neq \frac{1}{2}|\mathbb{A}^n(\mathbf{k})|$.

Proof. Let us write $|\mathbb{A}^n(\mathbf{k})| = 2m$ for some integer m.

If $|\Delta| < m$, the fact that $\mathrm{Aut}(\mathbb{A}^n, \Delta)$ is a maximal subgroup of $\mathrm{Aut}(\mathbb{A}^n)$ follows from Lemma 3.12 and from the fact that the action of $\mathrm{Aut}(\mathbb{A}^n)$ on the $2m$ points of $\mathbb{A}^n(\mathbf{k})$ give all even permutations (see [Mau01]), and thus acts $(2m-2)$-transitively. If $|\Delta| > m$, we exchange Δ with its complement.

If $|\Delta| = m$, there exists an automorphism φ of $\mathbb{A}^n$ exchanging Δ with its complement (by the result of [Mau01] cited before). Denoting by H the group generated by $\mathrm{Aut}(\mathbb{A}^n, \Delta)$ and φ, we have $\mathrm{Aut}(\mathbb{A}^n, \Delta) \subsetneq H \subsetneq \mathrm{Aut}(\mathbb{A}^n)$. □

Remark 3.15. Corollaries 3.13 and 3.14 raise the question of describing all maximal subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ in general.

3.2. **Generalisation to higher dimension.** Let us show that in higher dimension, the hypersurfaces $X \subset \mathbb{A}^n$ such that $\mathrm{Aut}(\mathbb{A}^n, X)$ is not an algebraic group are not as simple as in dimension 2:

Example 3.16. Let $X \subset \mathbb{A}^3$ be the hypersurface with equation $xy = f(z)$, for some polynomial $f \in \mathbf{k}[z]$. Then, $\mathrm{Aut}(\mathbb{A}^3, X)$ contains the group

$$\left\{(x, y, z) \mapsto \left(x, y + \frac{f(z + xq(x)) - f(z)}{x}, z + xq(x)\right) \mid q \in \mathbf{k}[x]\right\} \simeq \mathbf{k}[x]$$

and thus it is not an algebraic group.

A possible generalisation of Theorem 1 would be to show that every hypersurface $X \subset \mathbb{A}^3$ such that $\mathrm{Aut}(\mathbb{A}^3, X)$ is not an algebraic group admits an $\mathbb{A}^1$-fibration $X \to \mathbb{A}^1$ given by a coordinate projection.

4. Classification of the possible group actions and the proof of Theorem 2

This section is devoted to the proof of Theorem 2, which describes more precisely the curves and groups appearing in Theorem 1, in the case where the ground field $\mathbf{k}$ is perfect, and where the curve is geometrically irreducible.

4.1. The possibilities for Γ and $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$.

Lemma 4.1. *Let Γ be an affine geometrically irreducible curve, defined over a perfect field $\mathbf{k}$. The group $\mathrm{Aut}(\Gamma)$ is an affine algebraic group. If it has positive dimension, one of the following holds:*

i) $\Gamma \simeq \mathbb{A}^1$*;*
ii) Γ *is a unicuspidal curve with normalization* $\mathbb{A}^1$*;*
iii) Γ *is isomorphic, over* $\overline{\mathbf{k}}$*, to* $\mathbb{A}^1\backslash\{0\}$*.*

Proof. Let $\tilde{\Gamma}$ be the normalization of Γ, which can be viewed as an open subset of a smooth projective curve X, defined over $\mathbf{k}$. We write $(X\backslash\tilde{\Gamma})(\overline{\mathbf{k}}) = \{x_1,\ldots,x_r\}$ its complement, which is a finite set of points $x_1,\ldots,x_r$, not necessarily all defined over $\mathbf{k}$ (the union is however invariant by $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$). We also denote by $x_{r+1},\ldots,x_m$ the $\overline{\mathbf{k}}$-points of $\tilde{\Gamma}$ which are send onto the singular $\overline{\mathbf{k}}$-points of Γ. As before, not all are necessarily defined over $\mathbf{k}$.

This yields a natural inclusion

$$\mathrm{Aut}(\Gamma) \subseteq \{g \in \mathrm{Aut}(X) \mid g(\{x_1,\ldots,x_m\}) = \{x_1,\ldots,x_m\}\},$$

and a group homomorphism $\mathrm{Aut}(\Gamma) \to \mathrm{Sym}_m$. The kernel is of the same dimension as $\mathrm{Aut}(\Gamma)$.

Let us recall easy classical facts on automorphisms of smooth projective curves. If the genus of X is at least 2, then $\mathrm{Aut}(X)$ is finite. If the genus is 1, the subgroup of $\mathrm{Aut}(X)$ that fixes a point is also finite. If X is rational, the subgroup of automorphisms fixing three points is trivial.

If the dimension of $\mathrm{Aut}(X)$ is positive, we obtain that X is rational and that $1 \leqslant r \leqslant m \leqslant 2$.

If $r = m = 1$, then $\Gamma = \tilde{\Gamma} \simeq \mathbb{A}^1$ (every form of the affine line over a perfect field is trivial, see [Rus70]).

If $r = 1$ and $m = 2$, then $\tilde{\Gamma} \simeq \mathbb{A}^1$ and Γ is a unicuspidal curve.

If $r = m = 2$, then $\Gamma = \tilde{\Gamma}$ is smooth and isomorphic, over $\overline{\mathbf{k}}$, to $\mathbb{A}^1\backslash\{0\}$. □

Remark 4.2. Lemma 4.1 is false over a non-perfect field, since there are non-trivial forms of the affine line and its additive group; see [Rus70] for a classification of such curves.

Lemma 4.3. *Assume that* $\mathbf{k}$ *is a perfect field. Let* $\Gamma \subseteq \mathbb{A}^2$ *be a closed geometrically irreducible curve and assume that* $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ *is an algebraic group that is of positive dimension (over* $\overline{\mathbf{k}}$*). Then,* $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ *contains a closed subgroup* G *defined over* $\mathbf{k}$*, which is either*

i) a one-dimensional torus, i.e. isomorphic to $\mathbb{G}_m$ *over* $\overline{\mathbf{k}}$*,*
ii) or isomorphic to $\mathbb{G}_a$ *over* $\mathbf{k}$*.*

Proof. The algebraic group $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ is isomorphic to a closed subgroup of $\mathrm{Aut}(\Gamma)$ (Theorem 1). This gives three possibilities for Γ, according to Lemma 4.1.

a) $\Gamma \simeq \mathbb{A}^1$, hence $\mathrm{Aut}(\Gamma) \simeq \mathbb{G}_a \rtimes \mathbb{G}_m$, and contains thus a closed torus $\mathbb{G}_m$.

b) Γ is a unicuspidal curve, in which case $\mathrm{Aut}(\Gamma)$ is a torus.

c) Γ is isomorphic, over $\overline{\mathbf{k}}$, to $\mathbb{A}^1\backslash\{0\}$. The connected component of the identity $\mathrm{Aut}(\mathbb{A}^2,\Gamma)^0$ is a connected algebraic group, defined over $\mathbf{k}$, which is a torus. □

The two possibilities given by Lemma 4.3 are described respectively in §4.2 and §4.3. We will in particular show that the second case does not occur.

4.2. **Torus actions.**

Lemma 4.4. *Assume that* $\mathbf{k}$ *is a perfect field. Let* $\Gamma \subseteq \mathbb{A}^2$ *be a closed geometrically irreducible curve and assume that* $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ *is an algebraic group that contains a closed one-dimensional torus* T*. Then there exists an automorphism* $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ *such that* $\varphi \circ T \circ \varphi^{-1} \subseteq \mathrm{GL}(2, \mathbf{k})$.

Proof. By Corollary 3.9 we can assume that either $\mathrm{Aut}(\mathbb{A}^2, \Gamma) \subseteq \mathrm{Aut}(\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n})$ or $\mathrm{Aut}(\mathbb{A}^2, \Gamma) \subseteq \mathrm{Aut}(\mathbb{P}^2, L_{\mathbb{P}^2})$.

Moreover, we can assume that T is defined over $\mathbf{k}$ and T is isomorphic to $\mathbb{G}_m$ over $\overline{\mathbf{k}}$ (Lemma 4.3).

Assume $\mathrm{Aut}(\mathbb{A}^2, \Gamma) \subseteq \mathrm{Aut}(\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n})$. Thus T acts on $L_{\mathbb{F}_n}$. If this action is trivial, then there exists a fixed point on $L_{\mathbb{F}_n} \backslash E_n$ that is defined over $\mathbf{k}$. Otherwise, T has exactly two fixed points on $L_{\mathbb{F}_n} \simeq \mathbb{P}^1$ defined over $\overline{\mathbf{k}}$. As $L_{\mathbb{F}_n} \cap E_n$ is a fixed point of the T-action that is defined over $\mathbf{k}$, there is a fixed point on $L_{\mathbb{F}_n} \backslash E_n$ that is defined over $\mathbf{k}$. Thus by performing elementary links, we can assume that $T \subseteq \mathrm{Aut}(\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1})$. But T preserves the exceptional divisor E_1 and therefore $\varphi \circ T \circ \varphi^{-1} \subseteq \mathrm{Aut}(\mathbb{P}^2, L_{\mathbb{P}^2})$, where $\varphi\colon \mathbb{F}_1 \to \mathbb{P}_2$ denotes the contraction of E_1 onto a point in $L_{\mathbb{P}^2}$.

Thus we are left over with the case $T \subseteq \mathrm{Aut}(\mathbb{P}^2, L_{\mathbb{P}^2})$. It is enough to show that the induced action of T on $\mathbb{A}^2 = \mathbb{P}^2 \backslash L_{\mathbb{P}^2}$ has a fixed point that is defined over $\mathbf{k}$. The set of points of $\mathbb{A}^2(\overline{\mathbf{k}})$ that are fixed by $T(\overline{\mathbf{k}})$ consists either of one affine line or one point. This set is invariant by the action of the Galois group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$, and is thus defined over $\mathbf{k}$. Using again the fact that every form of the affine line over a perfect field is trivial (see [Rus70]), we find a $\mathbf{k}$-point of $\mathbb{A}^2$ fixed by T. □

Proposition 4.5. *Assume that* $\mathbf{k}$ *is perfect. Let* $\Gamma \subseteq \mathbb{A}^2$ *be a closed geometrically irreducible curve and assume that* $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ *contains a closed one-dimensional torus. After conjugation by an automorphism of* $\mathbb{A}^2$*, the curve* Γ

i) has equation $x = 0$*, or*
ii) has equation $x^b = \lambda y^a$ *where* $a, b > 1$ *are coprime integers and* $\lambda \in \mathbf{k}^*$*, or*
iii) has equation $x^b y^a = \lambda$ *where* $a, b \geqslant 1$ *are coprime integers and* $\lambda \in \mathbf{k}^*$*, or*
iv) has equation $\lambda x^2 + \nu y^2 = 1$*, where* $\lambda, \nu \in \mathbf{k}^*$*,* $-\lambda\nu$ *is not a square in* $\mathbf{k}$ *and* $\mathit{char}(\mathbf{k}) \neq 2$*, or*
v) has equation $x^2 + \mu xy + y^2 = 1$*, where* $\mu \in \mathbf{k}^*$*,* $x^2 + \mu x + 1$ *has no root in* $\mathbf{k}$ *and* $\mathit{char}(\mathbf{k}) = 2$.

Moreover, the group $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ *is respectively equal to*

i) $\{(x, y) \mapsto (ax, by + P(x)) \mid a, b \in \mathbf{k}^*, P \in \mathbf{k}[x]\} \simeq \mathbf{k}[x] \rtimes (\mathbf{k}^*)^2$*;*
ii) $\{(x, y) \mapsto (t^a x, t^b y) \mid t \in \mathbf{k}^*\} \simeq \mathbf{k}^*$*;*
iii) $\{(x, y) \mapsto (t^a x, t^{-b} y) \mid t \in \mathbf{k}^*\} \simeq \mathbf{k}^*$ *if* $(a, b) \neq (1, 1)$*;*
$\{(x, y) \mapsto (tx, t^{-1} y) \mid t \in \mathbf{k}^*\} \cup \{(x, y) \mapsto (ty, t^{-1} x) \mid t \in \mathbf{k}^*\} \simeq \mathbf{k}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ *if* $(a, b) = (1, 1)$*;*
iv) $T \rtimes \langle\sigma\rangle \simeq T \rtimes \mathbb{Z}/2\mathbb{Z}$*, where* $T, \{\sigma\} \subset \mathrm{GL}(2, \mathbf{k})$ *are given by*

$$T = \left\{ \begin{pmatrix} a & -\nu b \\ \lambda b & a \end{pmatrix} \middle| a^2 + \lambda\nu b^2 = 1 \right\}, \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover T *is a torus, which is not* $\mathbf{k}$*-split.*

v) $T \rtimes \langle\sigma\rangle \simeq T \rtimes \mathbb{Z}/2\mathbb{Z}$*, where* $T, \{\sigma\} \subset \mathrm{GL}(2, \mathbf{k})$ *are given by*

$$T = \left\{ \begin{pmatrix} a & b \\ b & a + \mu b \end{pmatrix} \middle| a^2 + \mu ab + b^2 = 1 \right\}, \sigma = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}.$$

Moreover T is a torus which is isomorphic to Γ (and which is not $\mathbf{k}$-split).

Proof. By Theorem 1, we can suppose that $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ is an algebraic group, and using Lemma 4.4, we can moreover assume that Γ is preserved by a torus $T \subset \mathrm{GL}(2,\mathbf{k})$.

There exists an element $\psi \in \mathrm{GL}(2,\overline{\mathbf{k}})$ which conjugates $T(\overline{\mathbf{k}})$ to

$$\lambda \mapsto \begin{pmatrix} \lambda^a & 0 \\ 0 & \lambda^b \end{pmatrix}$$

for integers a, b with $(a,b) \neq (0,0)$ and a, b are coprime. If a or b is equal to zero, then it follows that Γ is a line. Hence we can assume that a and b are non-zero. Now, let $(x_0,y_0) \in \psi(\Gamma(\overline{\mathbf{k}}))$, such that $(x_0,y_0) \neq (0,0)$. If x_0 or y_0 is zero, then Γ is again a line. Hence, we may assume that $x_0 \neq 0 \neq y_0$. By symmetry, we can assume that $b > 0$. Then the equation of $\psi(\Gamma)$ is

$$\begin{array}{ll} y_0^a x^b - x_0^b y^a = 0 & \text{if } a > 0\,, \\ y_0^a x^b y^{-a} - x_0^b = 0 & \text{if } a < 0\,. \end{array}$$

In the first case, we can assume that $a > 1$ or $b > 1$, otherwise the curve is a line. This implies that $a \neq b$, since both are coprime, and we can thus assume that $a > b$. The Galois group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ fixes the unique point at infinity, which is then defined over $\mathbf{k}$. Hence, we can assume that the unique point of the closure of Γ at infinity is the direction $y = 0$, and that the equation of Γ is the polynomial $(\alpha x + \beta y)^b - (\gamma y)^a \in \mathbf{k}[x,y]$, for some $\alpha,\beta,\gamma \in \overline{\mathbf{k}}$, $\alpha\gamma \neq 0$. If $\beta/\alpha \in \mathbf{k}$, we make a change of coordinates $(x,y) \mapsto (x - \frac{\beta}{\alpha}y, y)$ and obtain an equation of the form $x^b - \lambda y^a$ for some $\lambda \in \mathbf{k}$, as desired. It remains to see that β/α always belong to $\mathbf{k}$. If the characteristic of $\mathbf{k}$ is zero, we develop $(\alpha x + \beta y)^b$ and divide the coefficient of $x^{b-1}y$ by the coefficient of x^b. If the characteristic of $\mathbf{k}$ is $p > 0$, we write $b = qm$, where q is a power of p and p does not divide m. Developing, we find

$$(\alpha x + \beta y)^b = (\alpha^q x^q + \beta^q y^q)^m = \alpha^b x^b + m\alpha^{b-q}\beta^q x^{b-q} y^q + \ldots$$

hence $(\beta/\alpha)^q \in \mathbf{k}$, which implies, since $\mathbf{k}$ is a perfect field, that $\beta/\alpha \in \mathbf{k}$.

In the second case ($a < 0$), the closure of the curve $\psi(\Gamma)$ has two points at infinity in $\mathbb{P}^2$. If $a \neq -b$, the two points have different multiplicities. In consequence the Galois group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ has to fix the two points at infinity of Γ, which are thus defined over $\mathbf{k}$. We can assume that these points correspond to the directions $x = 0$ and $y = 0$, and that $\psi \in \mathrm{GL}(2,\overline{\mathbf{k}})$ is diagonal. The equation of Γ is then of the form $x^b y^a - \lambda$ for some $\lambda \in \mathbf{k}^*$, and we get case *iii*). The only remaining case is when $(-a,b) = (1,1)$ and the two points at infinity of Γ are exchanged by $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$. The equation of $\psi(\Gamma)$ being of the form $xy = x_0y_0$, the equation of Γ is of the form

$$\lambda x^2 + \mu xy + \nu y^2 = 1\,,$$

where $\lambda,\mu,\nu \in \mathbf{k}$, and $\lambda x^2 + \mu xy + \nu y^2 \in \mathbf{k}[x,y]$ is irreducible. When the characteristic of $\mathbf{k}$ is not 2, we can make a change of coordinates $(x,y) \mapsto (x - \frac{\mu}{2\lambda}y, y)$ and assume that $\mu = 0$. The two points at infinity are thus given by $\lambda x^2 + \nu y^2 = 0$. Because the two points are not defined over $\mathbf{k}$, we find that $-\nu\lambda$ is not a square in $\mathbf{k}$, and obtain *iv*). If the characteristic of $\mathbf{k}$ is 2, the elements λ,ν are squares in $\mathbf{k}$, since $\mathbf{k}$ is perfect. Making a diagonal change of variables, we can then assume that $\lambda = \nu = 1$. Then $x^2 + \mu x + 1$ has no root in $\mathbf{k}$, and we get *v*).

It remains to prove that $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$ has the desired form. Case *i*) is a direct calculation. In cases *ii*), *iii*), *iv*), *v*), it can be checked that the group given are contained in $\mathrm{Aut}(\mathbb{A}^2,\Gamma)$, so we just need to see that there is no other automorphism. The group homomorphism $\mathrm{Aut}(\mathbb{A}^2,\Gamma)\to\mathrm{Aut}(\Gamma)$ being injective, the only case to consider is the curve $x^by^a-\lambda=0$, with $(a,b)\neq(1,1)$, and to prove that there is no automorphism of $\mathbb{A}^2$ inducing on Γ an"exchange" of the two points at infinity. These two points are $p_1=(1:0:0)\in\mathbb{P}^2$ and $p_2=(0:1:0)\in\mathbb{P}^2$, and have multiplicity a and b respectively on Γ, and $\mathrm{ht}_\Gamma(p_1)=\mathrm{ht}_\Gamma(p_2)>0$ if $b>1$. We can moreover assume $a>b\geqslant 1$. The hypothetic automorphism extends to a birational map $\varphi\colon\mathbb{P}^2\dashrightarrow\mathbb{P}^2$ which is not an automorphism of $\mathbb{P}^2$, and thus decomposes into a sequence of elementary links $\varphi=\varphi_m\cdots\varphi_1$ (Proposition 2.9). By Corollary 3.5, we have $b=1$ and the point blown-up by the first link $\varphi_1\colon(\mathbb{P}^2,L_{\mathbb{P}^2})\dashrightarrow(\mathbb{F}_1,E_1\cup L_{\mathbb{F}_1})$ is p_1. Looking at the equation of Γ in $\mathbb{A}^2$, we can describe the closure of Γ on $\mathbb{F}_1$. This projective curve is smooth, intersects transversally $L_{\mathbb{F}_1}$ in one point away from E_1, corresponding to p_2, and intersects E_1 into one point q_1, with multiplicity a, corresponding to p_1; this latter is moreover not on $L_{\mathbb{F}_1}$. In consequence, the next links of type II do not affect the point q_1, and after the first link of type III, the image of the curve is singular at a point of $\mathbb{P}^2$ with multiplicity $\geqslant a$ and height 1. This point being not the base-point of the next elementary links, the image of Γ by φ has again a singular point, corresponding to the image of p_1. It is thus not possible to "exchange" p_1 and p_2. □

4.3. $\mathbb{G}_a$**-actions.** The classification of all $\mathbb{G}_a$-actions on $\mathbb{A}^2$ is known when the ground field $\mathbf{k}$ is of characteristic 0 [Ren68] or algebraically closed of positive characteristic [Miy71]. The following lemma gives the generalisation of MIYANISHI's result to the case where $\mathbf{k}$ is perfect. The proof is probably known to the specialists, we include it for the sake of completeness and lack of reference.

Proposition 4.6. *Assume that* $\mathbf{k}$ *is a perfect field of characteristic* $p>0$. *Then every* $\mathbb{G}_a$*-action on* $\mathbb{A}^2$ *that is defined over* $\mathbf{k}$ *has the form*

$$(t,x,y)\mapsto\left(x,y+tf_0(x)+t^pf_1(x)+\ldots+t^{p^n}f_n(x)\right).$$

Proof. By [Miy71] it follows that the $\mathbb{G}_a$-action has the claimed form over the algebraic closure $\overline{\mathbf{k}}$. Thus there exists a $\mathbb{G}_a$-invariant polynomial $f\in\overline{\mathbf{k}}[x,y]$ which is a variable of $\overline{\mathbf{k}}[x,y]$, i.e. which admits $g\in\overline{\mathbf{k}}[x,y]$ such that $\overline{\mathbf{k}}[f,g]=\overline{\mathbf{k}}[x,y]$. Let $G:=\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ be the Galois group of the extension $\overline{\mathbf{k}}/\mathbf{k}$, and assume that the $\mathbb{G}_a$-action is non-trivial. Then $\overline{\mathbf{k}}[f]$ is the subring of $\mathbb{G}_a$-invariant polynomials of $\overline{\mathbf{k}}[x,y]$ and since the $\mathbb{G}_a$-action is defined over $\mathbf{k}$, the subspace $\overline{\mathbf{k}}[f]$ is invariant under G. Thus, the action of G on $\overline{\mathbf{k}}[f,g]=\overline{\mathbf{k}}[x,y]$ is given by

$$\sigma(f)=a_\sigma f+c_\sigma\,,\quad \sigma(g)=b_\sigma g+d_\sigma$$

where $a_\sigma,b_\sigma\in\overline{\mathbf{k}}^*$, $c_\sigma\in\overline{\mathbf{k}}$ and $d_\sigma\in\overline{\mathbf{k}}[f]$. It is enough to show that the 1-cocycle

$$G\to\mathrm{J}_n(\overline{\mathbf{k}})\simeq(\overline{\mathbf{k}}^*)^2\ltimes(\overline{\mathbf{k}}\ltimes\overline{\mathbf{k}}[f])\,,\quad \sigma\mapsto(a_\sigma,b_\sigma,c_\sigma,d_\sigma)$$

is a 1-coboundary. The vanishing of $H^1(G,J_n(\overline{\mathbf{k}}))$ follows from the vanishing of $H^1(G,\overline{\mathbf{k}}^*)$ (see [NSW00, (6.2.1) Theorem]) and from the vanishing of $H^1(G,\overline{\mathbf{k}}[f])=\varinjlim_n H^1(G,\overline{\mathbf{k}}[f]_{\leqslant n})$ (see [NSW00, (1.5.1) Proposition] and [NSW00, (6.1.1) Theorem]) by using exact sequences (here $\overline{\mathbf{k}}[f]_{\leqslant n}$ denotes the polynomials in f of degree $\leqslant n$). □

Lemma 4.7. *Assume that* $\mathbf{k}$ *is perfect. Let* $\Gamma \subseteq \mathbb{A}^2$ *be a closed geometrically irreducible curve that is defined over* $\mathbf{k}$ *and assume that it is preserved under a non-trivial* $\mathbb{G}_a$*-action (defined over* $\mathbf{k}$*). Then there exists an automorphism* $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ *such that* $\varphi(\Gamma)$ *is an affine line in* $\mathbb{A}^2$*.*

Proof. By Proposition 4.6 (in case $char(\mathbf{k}) = p > 0$) and by [Ren68] (in case $char(\mathbf{k}) = 0$) we can conjugate the action to an action of the form $(t, x, y) \mapsto (x, y+p(t, x))$ where $p \in \mathbf{k}[t, x]$ is a non-zero polynomial. Hence, every geometrically irreducible invariant curve is a line in $\mathbb{A}^2$. □

Lemma 4.7 implies that the second case of Lemma 4.3 does not occur. The proof of Theorem 2 is now clear:

Proof of Theorem 2. By Theorem 1, either Γ is a line or $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic subgroup of $\mathrm{Aff}(\mathbb{A}^2)$ or J_n for some $n \geqslant 1$. If $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group of positive dimension, it contains a closed one-dimensional torus by Lemmas 4.3 and 4.7. The description of the possible cases follows then from Proposition 4.5. □

4.4. **The case of finite groups.** There are plenty of examples where $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is finite. The simplest way to get such examples is to take a finite subgroup $G \subset \mathrm{Aut}(\mathbb{A}^2)$ and to look for invariant curves. Since G has a finite action on $\mathbf{k}[x, y]$, one can find a lot of invariant polynomials, and in practice most of them are irreducible.

In characteristic zero, the group G is reductive and thus contained, up to conjugation, in GL_2 (see [Kam79]). In positive characteristic, there are however plenty of non-linearisable subgroups of $\mathrm{Aut}(\mathbb{A}^2)$, and, as far as we know, there is for the moment no classification of the conjugacy classes of such subgroups.

Example 4.8. Let $\mathbf{k}$ be of characteristic $p > 0$. For any integer $a > 1$

$$\varphi\colon (x, y) \mapsto (x, y + x^a)$$

is a non-linearisable automorphism of order p of $\mathbb{A}^2$, which preserves the family of curves of the form

$$y^p - yx^{a(p-1)} = q(x)\,,$$

where $q \in \mathbf{k}[x]$.

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CENTRALIZER OF A UNIPOTENT AUTOMORPHISM IN THE AFFINE CREMONA GROUP

IMMANUEL STAMPFLI

ABSTRACT. Let $\mathbf{u}$ be a unipotent polynomial automorphism of the affine 3-space $\mathbb{A}^3$. We describe the centralizer $\mathrm{Cent}(\mathbf{u})$ inside the group of polynomial automorphisms of $\mathbb{A}^3$. First, we treat the case when $\mathbf{u}$ is a modified translation.

In the other case, we describe the subset $\mathrm{Cent}(\mathbf{u})_u$ of unipotent elements of $\mathrm{Cent}(\mathbf{u})$ and prove that it is a closed normal subgroup of $\mathrm{Cent}(\mathbf{u})$. Moreover, we show that $\mathrm{Cent}(\mathbf{u})$ is the semi-direct product of $\mathrm{Cent}(\mathbf{u})_u$ with a closed algebraic subgroup $R \subseteq \mathrm{Cent}(\mathbf{u})$. Finally, we prove that the subgroup of $\mathrm{Cent}(\mathbf{u})$ consisting of those automorphisms that induce the identity on the algebraic quotient $\operatorname{Spec} \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}}$ form a characteristic subgroup of $\mathrm{Cent}(\mathbf{u})$.

1. INTRODUCTION

Throughout this paper, we fix an uncountable algebraically closed field k of characteristic zero. An interesting and important object in affine algebraic geometry is the automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ of the affine n-space $\mathbb{A}^n = k^n$. Let $k^{[n]}$ be the polynomial ring in n variables over the field k. For polynomials $g_1, \ldots, g_n \in k^{[n]}$, we use the notation $\mathbf{g} = (g_1, \ldots, g_n)$ to describe the automorphism

$$\mathbf{g} \colon \mathbb{A}^n \to \mathbb{A}^n\,, \quad (a_1, \ldots, a_n) \mapsto (g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n))\,.$$

We define $\deg \mathbf{g} := \max_i \deg g_i$. Prominent examples of automorphisms are the *affine automorphisms*, i.e. the automorphisms $(g_1, \ldots, g_n)$ where $\deg(\mathbf{g}) \leq 1$ and the *triangular automorphisms*, i.e. the automorphisms $(g_1, \ldots, g_n)$ where $g_i = g_i(x_i, \ldots, x_n)$ depends only on $x_i, \ldots, x_n$ for each i.

A further important class of automorphisms form the unipotent automorphisms which we introduce now. Recall that a derivation D of $k^{[n]}$ is called *locally nilpotent*, if for every $f \in k^{[n]}$ there exists an integer $n = n(f)$ such that $D^n(f) = 0$ (see [Fre06] for a reference on the theory of locally nilpotent derivations).

Definition 1.1. An automorphism of the form $\mathrm{Exp}(D) \in \mathrm{Aut}(\mathbb{A}^n)$ is called *unipotent* where D is a locally nilpotent derivation of $k^{[n]} = k[x_1, \ldots, x_n]$ and

$$\mathrm{Exp}(D) = \left(\sum_{i=0}^{\infty} \frac{D^i(x_1)}{i!}, \ldots, \sum_{i=0}^{\infty} \frac{D^i(x_n)}{i!} \right).$$

Note that we have a bijective correspondence between locally nilpotent derivations of $k^{[n]}$ and unipotent automorphisms of $\mathbb{A}^n$ given by Exp.

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Definition 1.2. Let $\mathbf{u} = \operatorname{Exp}(D)$ be a unipotent automorphism of $\mathbb{A}^n$ and let $f \in \ker D$. Then fD is a locally nilpotent derivation and we call $\operatorname{Exp}(fD)$ a *modification* of $\mathbf{u}$. We then denote

$$\mathbf{f} \cdot \mathbf{u} := \operatorname{Exp}(fD)\,.$$

We call $\mathbf{u}$ irreducible, if $\mathbf{u} \neq \mathbf{id}$ and the following holds: if $\mathbf{u} = f \cdot \mathbf{u}'$ for some unipotent $\mathbf{u}' \in \operatorname{Aut}(\mathbb{A}^n)$ and some $f \in \ker D$, then $f \in k^*$. If $\mathbf{u} \neq \mathbf{id}$, then there exists an irreducible $\mathbf{u}' \in \operatorname{Aut}(\mathbb{A}^n)$ such that $\mathbf{u} = d \cdot \mathbf{u}'$ for some $d \in \ker D$ and $\mathbf{u}'$ is unique up to a modification by some element in k^*. We call then $\mathbf{u} = d \cdot \mathbf{u}'$ a *standard decomposition*.

A natural problem is to describe the centralizer $\operatorname{Cent}(\mathbf{f}) \subseteq \operatorname{Aut}(\mathbb{A}^n)$ of a given automorphism $\mathbf{f} \in \operatorname{Aut}(\mathbb{A}^n)$. We consider in this paper the case when $n = 3$ and $\mathbf{f}$ is unipotent.

In dimension $n = 2$ Shmuel Friedland and John Milnor proved that every automorphism of $\mathbb{A}^2$ is conjugate to a composition of generalized Hénon maps (i.e. a map $(p(x) - ay, x)$ where $p \in k[x]$ and $0 \neq a \in k^*$) or to a triangular automorphism (cf. [FM89, Theorem 2.6]). In the first case, Stéphane Lamy showed that the centralizer of such an automorphism is isomorphic to a semi-direct product of $\mathbb{Z}$ with a finite cyclic group $\mathbb{Z}_q$ (cf. [Lam01, Proposition 4.8]). In the second case, assuming in addition that the automorphism is unipotent, it has the form $\mathbf{u} = (x + d(y), y)$ for some polynomial d, up to conjugation. Thus, $\mathbf{u} = d \cdot \mathbf{u}'$ is a standard decomposition where $\mathbf{u}' = (x+1, y)$. One can check that the centralizer $\operatorname{Cent}(\mathbf{u})$ fits in the following split short exact sequence

$$1 \to \mathcal{O}(\mathbb{A}^2)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \operatorname{Cent}(\mathbf{u}) \twoheadrightarrow \operatorname{Aut}(\mathbb{A}^1, \Gamma) \to 1 \qquad (1)$$

where $\mathcal{O}(\mathbb{A}^2)^{\mathbf{u}'}$ denotes the $\mathbf{u}'$-invariant functions and $\operatorname{Aut}(\mathbb{A}^1, \Gamma)$ denotes the automorphisms of $\mathbb{A}^1$ preserving the principal divisor $\Gamma := \operatorname{div}(d)$ in $\mathbb{A}^1$.

In dimension $n = 3$, Cinzia Bisi proved that any automorphism $\mathbf{g}$ that commutes with a so-called regular automorphism $\mathbf{f}$ satisfies $\mathbf{g}^m = \mathbf{f}^k$ for certain integers k, m (cf. [Bis08, Main Theorem 1.1]). As a counterpart to the regular automorphisms, one can regard the unipotent automorphisms (a regular automorphism is always algebraically stable and thus can not be unipotent). The work of David Finston and Sebastian Walcher [FW97] can be seen as a first step in the study of the centralizer of a unipotent automorphism. They explore the centralizer of a triangulable (locally nilpotent) derivation inside the algebra of all derivations of the polynomial ring $k^{[3]}$.

Before we state our main results, we introduce some notion from the theory of ind-varieties and ind-groups (see [Kum02, ch. IV] for an introduction). An *(affine) ind-variety* is a set X together with a filtration by affine varieties $X_1 \subseteq X_2 \subseteq \dots$, each one closed in the next, such that $X = \bigcup_{i \geq 1} X_i$. We write then $X = \varinjlim X_i$. We endow an ind-variety $X = \varinjlim X_i$ with the following topology: a subset $A \subseteq X$ is closed if and only if $A \cap X_i$ is closed in X_i with respect to the Zariski topology.

An ind-variety $G = \varinjlim G_i$ that is also a group is called an *ind-group*, if the map $G \times G \to G$, $(x, y) \mapsto x \cdot y^{-1}$ is a morphism of ind-varieties. For example, $\operatorname{Aut}(\mathbb{A}^n)$ is an ind-group with the filtration $\operatorname{Aut}(\mathbb{A}^n)_1 \subseteq \operatorname{Aut}(\mathbb{A}^n)_2 \subseteq \dots$ where $\operatorname{Aut}(\mathbb{A}^n)_i$ is the set of all automorphisms $\mathbf{g} \in \operatorname{Aut}(\mathbb{A}^n)$ with $\deg \mathbf{g} \leq i$ (see [BCW82]). In fact, one can check, that $\operatorname{Aut}(\mathbb{A}^n)$ is a locally closed subset of $\operatorname{End}(\mathbb{A}^n) = \varinjlim \operatorname{End}(\mathbb{A}^n)_i$ where $\operatorname{End}(\mathbb{A}^n)_i$ denotes the polynomial endomorphisms of $\mathbb{A}^n$ with degree $\leq i$.

A subgroup H of an ind-group $G = \varinjlim G_i$ is called *algebraic*, if H is closed in G and contained in some G_i. We call an element $g \in G$ *algebraic*, if the closure of the cyclic group $\langle g \rangle$ is an algebraic subgroup of G.

2. Statement of the main results

In order to state our main results we introduce some notation and recall some facts about $\mathbb{G}_a$-actions on $\mathbb{A}^3$. Let $\mathbf{id} \neq \mathbf{u} = \operatorname{Exp}(D) \in \operatorname{Aut}(\mathbb{A}^3)$ be a unipotent automorphism and let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. The automorphism $\mathbf{u}$ induces a $\mathbb{G}_a$-action on $\mathbb{A}^3$ via $(t, v) \mapsto (t \cdot \mathbf{u})(v)$. We denote by $\pi \colon \mathbb{A}^3 \to \mathbb{A}^3 /\!\!/ \mathbb{G}_a$ its algebraic quotient. The invariant ring satisfies $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} = \ker D$ and thus $\mathbb{A}^3 /\!\!/ \mathbb{G}_a = \operatorname{Spec}(\ker D)$.

Definition 2.1. Let $\mathbf{u} = \operatorname{Exp}(D)$ be a unipotent automorphism of $\mathbb{A}^3$. We call the ideal $\operatorname{im} D \cap \ker D$ of $\ker D$ the *plinth ideal* and we denote

$$\operatorname{pl} D := \operatorname{im} D \cap \ker D\,.$$

By [DK09, Theorem 1] the plinth ideal is principal. We fix some generator of the plinth ideal and denote it by $a = a(D)$. We denote further by

$$\Gamma := \operatorname{div}(a) = \operatorname{div}(a(D))$$

the principal divisor in $\mathbb{A}^3 /\!\!/ \mathbb{G}_a$ corresponding to a and call it the *plinth divisor* of D (respectively of $\mathbf{u}$).

Note that $\mathbb{A}^3 /\!\!/ \mathbb{G}_a \simeq \mathbb{A}^2$ by Miyanishi's Theorem (cf. [Fre06, Theorem 5.1]) and that the restriction $\pi|_{\mathbb{A}^3 \setminus \pi^{-1}(\Gamma)} \colon \mathbb{A}^3 \setminus \pi^{-1}(\Gamma) \twoheadrightarrow \mathbb{A}^2 \setminus \Gamma$ is a trivial principal $\mathbb{G}_a$-bundle.

We have an induced action of $\operatorname{Cent}(\mathbf{u})$ on the algebraic quotient $\mathbb{A}^3 /\!\!/ \mathbb{G}_a$ that preserves Γ. This implies that there is an exact sequence of ind-groups (see Proposition 5.1)

$$1 \to \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \operatorname{Cent}(\mathbf{u}) \xrightarrow{p} \operatorname{Aut}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)\,. \tag{2}$$

In contrast to the 2-dimensional case (see (1)), the homomorphism p is in general not surjective (see [Sta12, Proposition 1]). So, it is interesting to ask, what can be said about the image of p in $\operatorname{Aut}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)$.

The description of $\operatorname{Cent}(\mathbf{u})$ special in the case when $\mathbf{u}'$ is a translation (see Section 5.2 for a precise definition of "translation"). By abuse of language we call then $\mathbf{u} = d \cdot \mathbf{u}'$ a *modified translation*.

2.1. The case when u is a modified translation.

The next result is the content of Proposition 5.5.

Proposition A. *Let* $\mathbf{id} \neq \mathbf{u} \in \operatorname{Aut}(\mathbb{A}^3)$ *be unipotent and let* $\mathbf{u} = d \cdot \mathbf{u}'$ *be a standard decomposition. If* $\mathbf{u}$ *is a modified translation, then*

$$1 \to \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \operatorname{Cent}(\mathbf{u}) \to \operatorname{Aut}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma) \to 1$$

is a split short exact sequence of ind-groups where Γ *is the plinth divisor of* $\mathbf{u}$*.*

2.2. **The case when u is not a modified translation.** The next result follows from Theorem 5.17, Theorem 5.19, Corollary 5.20 and Proposition 5.21.

Theorem B. *Let* $\mathbf{id} \neq \mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ *be unipotent and let* $\mathbf{u}' = d \cdot \mathbf{u}$ *be a standard decomposition. If* $\mathbf{u}$ *is not a modified translation, then*

i) The set of unipotent elements $\mathrm{Cent}(\mathbf{u})_u$ *of the centralizer* $\mathrm{Cent}(\mathbf{u})$ *is a closed normal subgroup, and there exists a closed algebraic subgroup* $R \subseteq \mathrm{Cent}(\mathbf{u})$ *consisting only of semi-simple elements such that* $\mathrm{Cent}(\mathbf{u}) \simeq \mathrm{Cent}(\mathbf{u})_u \rtimes R$ *as ind-groups.*
ii) All elements in $\mathrm{Cent}(\mathbf{u})$ *are algebraic.*
iii) The subgroup $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \subseteq \mathrm{Cent}(\mathbf{u})$ *is characteristic.*

In the next two results we describe the group $\mathrm{Cent}(\mathbf{u})_u$ of unipotent elements more precisely. There are two cases for Γ which are completely different. For this distinction we have to introduce the following term.

Definition 2.2. Let $\Gamma = \sum_i n_i \Gamma_i$ be an effective divisor in $\mathbb{A}^2$. We call Γ a *fence*, if $\Gamma_i \simeq \mathbb{A}^1$ for all i and the Γ_i are pairwise disjoint.

2.2.1. *The case when* Γ *is not a fence.* This is the generic case. As Γ is not a fence, the underlying variety cannot be a union of orbits of a non-trivial $\mathbb{G}_a$-action on $\mathbb{A}^2$. Hence, there exists no non-trivial unipotent automorphism of $\mathbb{A}^2$ preserving Γ. Thus all the unipotent automorphisms of $\mathrm{Cent}(\mathbf{u})_u$ induce the identity on the algebraic quotient. Hence we get from (2) the next result.

Proposition C. *Let* $\mathbf{id} \neq \mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ *be unipotent and let* $\mathbf{u} = d \cdot \mathbf{u}'$ *be a standard decomposition. If* $\mathbf{u}$ *is not a modified translation and the plinth divisor* Γ *of* $\mathbf{u}$ *is not a fence, then the subgroup of unipotent automorphisms in* $\mathrm{Cent}(\mathbf{u})$ *is given by*

$$\mathrm{Cent}(\mathbf{u})_u = \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \,.$$

2.2.2. *The case when* Γ *is a fence.* This is the hard case. As Γ is a fence there exists a proper non-empty open subset $C \subseteq \mathbb{A}^1$ such that we have the following commutative diagram

$$\begin{array}{ccccc}
\mathbb{A}^3 & \supseteq & \mathbb{A}^3 \setminus \pi^{-1}(\Gamma) & \simeq & (C \times \mathbb{A}^1) \times \mathbb{A}^1 \\
\downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\
\mathbb{A}^2 & \supseteq & \mathbb{A}^2 \setminus \Gamma & \simeq & C \times \mathbb{A}^1
\end{array}$$

where pr denotes the projection onto the first two factors (see Proposition 3.3). There exists coordinates (u, v, w) of $(C \times \mathbb{A}^1) \times \mathbb{A}^1$ such that the automorphism induced by $\mathbf{u}$ on $(C \times \mathbb{A}^1) \times \mathbb{A}^1$ is given by $(u, v, w) \mapsto (u, v, w+1)$ and the unipotent automorphism $(u, v, w) \mapsto (u, v+1, w)$ of $(C \times \mathbb{A}^1) \times \mathbb{A}^1$ extends to an irreducible unipotent automorphism $\mathbf{e}$ on $\mathbb{A}^3$ that commutes with $\mathbf{u}$ (see Section 5.3). Thus, $\mathrm{Cent}(\mathbf{u})_u$ contains $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{u}, \mathbf{e} \rangle} \cdot \mathbf{e}$ beside $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$. Let $\mathrm{Iner}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)$ be the group of automorphisms of $\mathbb{A}^3 /\!\!/ \mathbb{G}_a$ that induce the identity on Γ. The difficulty in the next theorem lies in proving that the image of $p \colon \mathrm{Cent}(\mathbf{u}) \to \mathrm{Aut}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)$ lies in $\mathrm{Iner}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)$. The result follows from Proposition 5.15 and from Theorem 5.19.

Theorem D. *Let* $\mathbf{id} \neq \mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ *be unipotent and let* $\mathbf{u} = d \cdot \mathbf{u}'$ *be a standard decomposition. Assume that* $\mathbf{u}$ *is not a modified translation and that the plinth divisor* Γ *is a fence. Then the set* $\mathrm{Iner}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)_u$ *of unipotent automorphisms in* $\mathrm{Iner}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)$ *is a group and the sequence induced by* (2)

$$1 \to \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \mathrm{Cent}(\mathbf{u})_u \overset{p}{\twoheadrightarrow} \mathrm{Iner}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)_u \to 1$$

is a split short exact sequence of ind-groups. Moreover, p induces an isomorphism $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{u}, \mathbf{e} \rangle} \cdot \mathbf{e} \xrightarrow{\sim} \mathrm{Iner}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)_u$ *for a certain irreducible unipotent* $\mathbf{e} \in \mathrm{Aut}(\mathbb{A}^3)$.

Let $|\Gamma|$ be the underlying scheme of the plinth divisor Γ. Then the reduced scheme $|\Gamma|_{\mathrm{red}}$ has the following geometric description. The complement of $|\Gamma|_{\mathrm{red}}$ is the maximal open subset of $\mathbb{A}^3 /\!\!/ \mathbb{G}_a$ such that the algebraic quotient $\pi \colon \mathbb{A}^3 \to \mathbb{A}^3 /\!\!/ \mathbb{G}_a$ is a locally trivial principal $\mathbb{G}_a$-bundle over it. So far - to the authors knowledge - there is now geometric description of $|\Gamma|$. At least in the case when Γ is a fence we can give the following geometric description of $|\Gamma|$ (which is an immediate consequence of Theorem D).

Corollary E. *Let* $\mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ *be unipotent. Assume that* $\mathbf{u}$ *is not a modified translation and that the plinth divisor* Γ *is a fence. Then* $|\Gamma|$ *is the fixed point scheme of the induced action of* $\mathrm{Cent}(\mathbf{u})_u$ *on the algebraic quotient* $\mathbb{A}^3 /\!\!/ \mathbb{G}_a$.
(see [Fog73] *for a definition of "fixed point scheme")*

3. Automorphisms of $\mathbb{A}^2$ that preserve a divisor

Recall that for a non-trivial $\mathbb{G}_a$-action on $\mathbb{A}^3$, the algebraic quotient $\mathbb{A}^3 /\!\!/ \mathbb{G}_a$ is always isomorphic to $\mathbb{A}^2$. In view of the exact sequence (2) we study in this section the automorphisms of $\mathbb{A}^2$ that preserve a given effective divisor Γ in $\mathbb{A}^2$. Clearly, Γ is completely determined by its underlying scheme and vice versa. Thus we can and will identify this underlying scheme with Γ. We denote by $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ the subgroup of all $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^2)$ such that the scheme-theoretic image $\mathbf{g}(\Gamma)$ is again Γ and we denote by $\mathrm{Iner}(\mathbb{A}^2, \Gamma)$ the subgroup of all $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^2, \Gamma)$ such that the pullback to Γ is the identity. Thus we have an exact sequence

$$1 \to \mathrm{Iner}(\mathbb{A}^2, \Gamma) \hookrightarrow \mathrm{Aut}(\mathbb{A}^2, \Gamma) \to \mathrm{Aut}(\Gamma)\,.$$

In the following proposition we list some facts about $\mathrm{Iner}(\mathbb{A}^2, \Gamma)$ and $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$.

Proposition 3.1. *Let* Γ *be a non-trivial effective divisor of* $\mathbb{A}^2$.

i) The subgroup $\mathrm{Aut}(\mathbb{A}^2, \Gamma) \subseteq \mathrm{Aut}(\mathbb{A}^2)$ *is closed and all elements of* $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ *are algebraic.*
ii) The following statements are equivalent
  - *a)* Γ *is a fence*
  - *b)* $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ *contains unipotent automorphisms* $\neq \mathbf{id}$.
  - *c)* $\mathrm{Iner}(\mathbb{A}^2, \Gamma) \neq \{\mathbf{id}\}$
  - *d)* $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ *is not an algebraic group*

For the proof of this proposition we recall some facts about $\mathrm{Aut}(\mathbb{A}^2)$. As mentioned in the introduction, an automorphism of $\mathbb{A}^2$ is conjugate to a composition of generalized Hénon maps or to a triangular automorphism (cf. [FM89, Theorem 2.6]). But a composition of generalized Hénon maps never preserves an algebraic curve, by [BS13, Theorem 1]. This implies the following result.

Theorem 3.2. *An automorphism of $\mathbb{A}^2$ that preserves an algebraic curve is conjugate to a triangular automorphism.*

Another result that we will constantly use, is the ABHYANKAR-MOH-SUZUKI-Theorem which says that all closed embeddings $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$ are equivalent (see [AM75]). In fact we will use a slightly more general version of it.

Proposition 3.3. *Let Γ be a fence in $\mathbb{A}^2$. Then there exists a 0-dimensional closed subscheme F of $\mathbb{A}^1$, an automorphism of $\mathbb{A}^2$ and an isomorphism $\Gamma \simeq F \times \mathbb{A}^1$ such that the following diagram commutes*

$$\begin{array}{ccc} \Gamma & \subseteq & \mathbb{A}^2 \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ F \times \mathbb{A}^1 & \subseteq & \mathbb{A}^2 \end{array}$$

Proof. Clearly, we can assume that $\Gamma \neq \varnothing$. Moreover, we can easily reduce to the case, when Γ is a reduced scheme. Let Γ_i, $i \in I$ be the irreducible components of Γ. Let $i_0 \in I$ be fixed. By the ABHYANKAR-MOH-SUZUKI-Theorem, there exists a trivial $\mathbb{A}^1$-bundle $f\colon \mathbb{A}^2 \twoheadrightarrow \mathbb{A}^1$ such that Γ_{i_0} is a fiber of f. Now, if the restriction $f|_{\Gamma_i}\colon \Gamma_i \to \mathbb{A}^1$ is non-constant, then it is surjective, since $\Gamma_i \simeq \mathbb{A}^1$. But this implies that $\Gamma_i \cap \Gamma_{i_0} \neq \varnothing$, a contradiction. Thus every Γ_i is a fiber of f. This implies the proposition. □

Proof of Proposition 3.1.

i) Assume that $\Gamma = \operatorname{div}(a)$ for some non-zero $a \in k^{[2]} = k[x, y]$. For $(f_1, f_2) \in k[x, y]^2$ we denote by $a_{ij}(f_1, f_2)$ the coefficient of the monomial $x^i y^j$ in the polynomial $a(f_1, f_2)$. The subgroup $\operatorname{Aut}(\mathbb{A}^2, \Gamma)$ of $\operatorname{Aut}(\mathbb{A}^2)$ is defined by the equations

$$a_{ij}(f_1, f_2)a_{kl}(x, y) = a_{kl}(f_1, f_2)a_{ij}(x, y) \quad \text{for all pairs } (i, j),\ (k, l)\,.$$

This proves the first statement.

Let $\mathbf{g} \in \operatorname{Aut}(\mathbb{A}^2, \Gamma)$. By Theorem 3.2, $\mathbf{g}$ is conjugate to a triangular automorphism and hence $\mathbf{g}$ is algebraic. This proves the second statement.

ii) *a*) ⇒ *b*): This follows immediately form Proposition 3.3.

b) ⇒ *c*): Let $\mathbf{g} \in \operatorname{Aut}(\mathbb{A}^2, \Gamma)$ be a unipotent automorphism $\neq \mathbf{id}$. Choose some $a \in k^{[2]}$ such that $\Gamma = \operatorname{div}(a)$. As $\mathbf{g}$ preserves Γ, it follows that a is a semi-invariant for the $\mathbb{G}_a$-action on $\mathbb{A}^2$ induced by $\mathbf{g}$. Since $\mathbb{G}_a$ has no non-trivial character, a is an invariant. Hence, $\mathbf{id} \neq a \cdot \mathbf{g} \in \operatorname{Iner}(\mathbb{A}^2, \Gamma)$.

c) ⇒ *d*): Let $\mathbf{g} \in \operatorname{Iner}(\mathbb{A}^2, \Gamma)$ with $\mathbf{g} \neq \mathbf{id}$. By Theorem 3.2, there exists a trivial $\mathbb{A}^1$-bundle $f\colon \mathbb{A}^2 \twoheadrightarrow \mathbb{A}^1$ such that $\mathbf{g}$ preserves the fibration of $\mathbb{A}^2$ induced by f. Let Γ_i, $i \in I$ be the irreducible components of the reduced scheme associated to Γ. If every Γ_i lies in a fiber of f, then Γ is a fence and thus $\operatorname{Aut}(\mathbb{A}^2, \Gamma)$ is not an algebraic group. Therefore we can assume that $f(\Gamma_i) \subseteq \mathbb{A}^1$ is dense for some i. As $\mathbf{g}$ is the identity on Γ_i, it follows that $\mathbf{g}$ maps each fiber on itself. Hence, there exists $\alpha \in k^*$ and a polynomial $b(y)$ such that for each $y \in \mathbb{A}^1$ the restriction of $\mathbf{g}$ to the fiber $f^{-1}(y)$ is given by

$$\mathbf{g}_y\colon \mathbb{A}^1 \to \mathbb{A}^1\,, \quad x \mapsto \alpha x + b(y)\,.$$

As $\mathbf{g}$ is the identity on Γ_i, it follows that $\mathbf{g}_y$ has a fixed point for all $y \in f(\Gamma_i)$. If $\alpha = 1$, then $\mathbf{g}_y$ is the identity map for all $y \in f(\Gamma_i) \subseteq \mathbb{A}^1$. Since $f(\Gamma_i)$ is dense in $\mathbb{A}^1$ we get a contradiction to the fact, that $\mathbf{g} \neq \mathbf{id}$. Thus, $\alpha \neq 1$. But this implies that $\mathbf{g}_y$ has exactly one fixed point for each $y \in \mathbb{A}^1$. Thus, $\Gamma_i \simeq \mathbb{A}^1$ and it is the only irreducible component of Γ. Therefore, Γ is again a fence and $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is not an algebraic group.

d) $\Rightarrow$ *a*): Assume that Γ is not a fence. By [BS13, Theorem 1], $\mathrm{Aut}(\mathbb{A}^2, \Gamma_{\mathrm{red}})$ is an algebraic group, where Γ_{red} is the reduced scheme associated to Γ. Now, $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group as well, since it is a closed subgroup of $\mathrm{Aut}(\mathbb{A}^2, \Gamma_{\mathrm{red}})$.

□

4. Some basic properties of locally nilpotent derivations

Let A be a k-algebra and assume it is a unique factorization domain (UFD). Let B be a locally nilpotent derivation of A. We call B *irreducible*, if $B \neq 0$ and the following holds: if $B = fB'$ for some locally nilpotent derivation B' and some $f \in \ker B$, then $f \in A^*$ where A^* denotes the subgroup of units of A. Remark, if $A = k^{[n]}$, then a unipotent automorphism $\mathbf{u} = \mathrm{Exp}(D)$ is irreducible if and only if D is irreducible. We list some basic facts about locally nilpotent derivations, that we will use later on (see [Fre06] for proofs):

i) The units of A lie in $\ker B$. In particular, $k \subseteq \ker B$.
ii) The kernel $\ker B$ is factorially closed in A, i.e. if $f, g \in A$ such that $fg \in \ker B$, then $f, g \in \ker B$.
iii) If $S \subseteq \ker B$ is a multiplicative system, then B extends uniquely to a locally nilpotent derivation of the localization A_S.
iv) If $B \neq 0$, then there exists $f \in A$ such that $B(f) \in \ker B$ and $B(f) \neq 0$.
v) If $s \in A$ such that $B(s) = 1$, then A is a polynomial ring in s over $\ker B$ and $B = \partial/\partial s$.
vi) For $f \in A$, the derivation fB is locally nilpotent if and only if $f \in \ker B$.
vii) If B is irreducible and E is another locally nilpotent derivation of A such that $E(\ker B) = 0$, then there exists $f \in \ker B$ such that $E = fB$.
viii) If $B \neq 0$, then there exists a unique irreducible locally nilpotent derivation B' (up to multiplication by some element of A^*) such that $\ker B = \ker B'$.
ix) The exponential $\exp(B) = \sum_{i=0}^{\infty} B^i/i!$ is a k-algebra automorphism of A and the map exp defines an injection from the set of locally nilpotent derivations of A to the set of k-algebra automorphisms of A.
x) If $B(f) \in fA$, then $B(f) = 0$.

5. Centralizer of a unipotent automorphism in $\mathrm{Aut}(\mathbb{A}^3)$

5.1. **The first unipotent subgroup in** $\mathrm{Cent}(\mathbf{u})$**.** Let $\mathbf{id} \neq \mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ be unipotent and let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. There exists an obvious subgroup of unipotent automorphisms in $\mathrm{Cent}(\mathbf{u})$: The modifications of $\mathbf{u}'$, i.e. the subgroup $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$. This subgroup has another characterization:

Proposition 5.1. *Let* $\mathbf{id} \neq \mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ *be unipotent and let* $\mathbf{u} = d \cdot \mathbf{u}'$ *be a standard decomposition. The subgroup* $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ *consists of those automorphisms of* $\mathbb{A}^3$ *that commute with* $\mathbf{u}$ *and that induce the identity on* $\mathbb{A}^3 /\!\!/ \mathbb{G}_a$*, i.e. the sequence*

$$1 \to \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \mathrm{Cent}(\mathbf{u}) \xrightarrow{p} \mathrm{Aut}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)$$

is exact, where Γ *denotes the plinth divisor of* $\mathbf{u}$*. Moreover, the homomorphisms in the sequence above are homomorphisms of ind-groups.*

This result is an immediate consequence of the following remark and the next lemma.

Remark 5.2. Choose generators $\tilde{y}, \tilde{z}$ of the polynomial ring $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}}$ and choose a k-linear retraction $r\colon \mathcal{O}(\mathbb{A}^3) \twoheadrightarrow \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}}$. The map $p\colon \mathrm{Cent}(\mathbf{u}) \to \mathrm{Aut}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)$ is a morphism of ind-varieties due to the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{End}(\mathbb{A}^3) & \xrightarrow[\text{morph.}]{\mathbf{g}\mapsto(\mathbf{g}^*(\tilde{y}),\mathbf{g}^*(\tilde{z}))} & \mathcal{O}(\mathbb{A}^3)^2 & \overset{r\times r}{\underset{\text{lin.}}{\twoheadrightarrow}} & (\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}})^2 \\
\uparrow{\scriptstyle \text{loc. closed}} & & & & {\scriptstyle \text{loc. closed}}\uparrow \\
\mathrm{Cent}(\mathbf{u}) & & \xrightarrow{\quad p \quad} & & \mathrm{Aut}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)\,.
\end{array}$$

Lemma 5.3. *Let A be a k-algebra and assume it is a UFD, let B, B' be non-zero locally nilpotent derivations of A such that B' is irreducible and* $\ker B = \ker B'$. *If $\varphi\colon A \to A$ is a k-algebra automorphism, then we have*

$$\varphi|_{\ker B} = \mathrm{id} \ \textit{ and } \varphi \circ B = B \circ \varphi \quad \textit{if and only if} \quad \varphi = \exp(fB')\,, \ f \in \ker B\,.$$

Proof. Assume that $\varphi|_{\ker B}$ is the identity and φ commutes with B. There exists $0 \neq d \in \ker B$, such that $A_d = \ker(B)_d[s]$ is a polynomial ring in an element $s \in A_d$ and $B(s) = 1$, if we extend B to A_d. Since φ commutes with B there exists $g \in \ker(B)_d$ such that the extension $\tilde{\varphi}$ to A_d of φ satisfies $\tilde{\varphi}(s) = s + g$. Now, we have $\varphi = \exp(gB)|_A$. A density argument shows that $\exp(tgB)(A) = A$ for all $t \in k$. Since

$$gB = \frac{(\exp(tgB) - \mathrm{id})}{t}\Big|_{t=0}\,,$$

we have $gB(A) \subseteq A$. Hence gB is a locally nilpotent derivation of A that vanishes on $\ker B = \ker B'$. Thus, $gB = fB'$ for some $f \in \ker B$. The converse is clear. □

5.2. Centralizer of a modified translation in $\mathrm{Aut}(\mathbb{A}^3)$.

Definition 5.4. We call an automorphism $\mathbf{f} \in \mathrm{Aut}(\mathbb{A}^3)$ a *translation*, if there exists a coordinate system (x, y, z) of $\mathbb{A}^3$ such that $\mathbf{f} = (x + 1, y, z)$.

Clearly, a translation is unipotent. Note, that a unipotent $\mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ is a translation if and only if the plinth divisor of $\mathbf{u}$ is empty. By abuse of language, we call a modification of a translation a *modified translation*.

Proposition 5.5. *Let* $\mathbf{id} \neq \mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ *be a modified translation and let* $\mathbf{u} = d\cdot\mathbf{u}'$ *be a standard decomposition. Denote by* Γ *the plinth divisor of* $\mathbf{u}$. *Then*

$$1 \to \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \hookrightarrow \mathrm{Cent}(\mathbf{u}) \xrightarrow{p} \mathrm{Aut}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma) \to 1$$

is a split short exact sequence of ind-groups. Moreover there exists a closed subgroup of $\mathrm{Cent}(\mathbf{u})$ *that is mapped via p isomorphically onto* $\mathrm{Aut}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)$.

Proof. By Proposition 5.1, the sequence above is left exact. Since $\mathbf{u}'$ is a translation, we can identify the algebraic quotient $\pi\colon \mathbb{A}^3 \to \mathbb{A}^2$ with the map $(x, y, z) \mapsto (y, z)$ and $\Gamma = \mathrm{div}(d)$. Let $\mathbf{f} \in \mathrm{Aut}(\mathbb{A}^2, \Gamma)$. Then $\mathbf{f}^*(d) = \lambda d$ for some $\lambda = \lambda(\mathbf{f}) \in k^*$. One can see that $\mathrm{Aut}(\mathbb{A}^2, \Gamma) \to k^*$, $\mathbf{f} \mapsto \lambda(\mathbf{f})$ defines a morphism of ind-varieties. Thus

$$H := \{\, (\lambda x, f_1, f_2) \mid \mathbf{f} = (f_1, f_2) \in \mathrm{Aut}(\mathbb{A}^2, \Gamma) \text{ and } \mathbf{f}^*(d) = \lambda d \,\}$$

is a closed subgroup of $\mathrm{Cent}(\mathbf{u})$ (note that the subgroup $\mathrm{Aut}(\mathbb{A}^2, \Gamma) \subseteq \mathrm{Aut}(\mathbb{A}^2)$ is closed by Proposition 3.1) and $p|_H\colon H \to \mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is an isomorphism of ind-groups. □

5.3. **The second unipotent subgroup in** $\mathrm{Cent}(\mathbf{u})$**.** Let $\mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ be unipotent and let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. Throughout this subsection we assume that the plinth divisor $\Gamma = \mathrm{div}(a)$ of $\mathbf{u}$ is a fence. There exists another subgroup of unipotent automorphisms inside $\mathrm{Cent}(\mathbf{u})$ in addition to $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$, that we describe in this subsection.

Lemma 5.6. *Let* $\mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ *be a unipotent automorphism. If the plinth divisor* $\Gamma = div(a)$ *is a fence, then there exists a variable* z *of* $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} = k^{[2]}$ *such that* $a \in k[z]$ *and any such* z *is a variable of* $\mathcal{O}(\mathbb{A}^3) = k^{[3]}$*.*

Proof. By Proposition 3.3 there exists a coordinate system (z, w) of $\mathbb{A}^2$ such that the embedding $\mathrm{div}(a) = \Gamma \subseteq \mathbb{A}^2$ is given by the standard embedding $F \times \mathbb{A}^1 \subseteq \mathbb{A}^2$ for some 0-dimensional closed subscheme F of $\mathbb{A}^1$. Thus $a \in k[z]$. Since π is a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^2 \setminus \Gamma$, it follows that only finitely many fibers of $z \colon \mathbb{A}^3 \to \mathbb{A}^1$ are non-isomorphic to $\mathbb{A}^2$. Thus z is a variable of $k^{[3]}$, according to KALIMAN's Theorem [DK09, Theorem 3] (cf. also [BEHEK08, Theorem 3.1]). □

Remark 5.7. If $\mathbf{u} = \mathrm{Exp}(D)$ is irreducible, then Γ is a fence if and only if $\mathrm{rank}\, D \leq 2$ (i.e. there exists a variable z of $k^{[3]}$ that lies in $\ker D$). This follows from the lemma above and from [DF98, Theorem 2.4, Proposition 2.3].

Definition 5.8. Let A be a UFD and let $P \in A[x, y]$ be a polynomial in x and y over A. We denote

$$\Delta_P := -P_y \frac{\partial}{\partial x} + P_x \frac{\partial}{\partial y}$$

where P_x and P_y denote the partial derivatives of P with respect to x and y respectively. Obviously, Δ_P is a derivation of $A[x, y]$ and $\Delta_P(P) = 0$.

Let D, D' be locally nilpotent derivations of $\mathcal{O}(\mathbb{A}^3)$ such that $\mathbf{u} = \mathrm{Exp}(D)$ and $\mathbf{u}' = \mathrm{Exp}(D')$. Let $z \in \ker D$ be a variable such that $a \in k[z]$ and let (x, y, z) be a coordinate system of $k^{[3]}$ (see Lemma 5.6). Let $A := k[z]$. It follows now from Remark 5.7 and [DF98, Theorem 2.4] that there exists $P \in A[x, y]$ such that

$$D' = \Delta_P \quad \text{and} \quad \ker D = \ker D' = k[z, P]\,.$$

Obviously, d divides a in $k[z]$. Let $a = da'$. An easy calculation shows that $\mathrm{div}(a')$ is the plinth divisor of $\mathbf{u}'$ and that for all $Q \in A[x, y]$ we have

$$D(Q) = a \quad \text{if and only if} \quad D'(Q) = a'\,. \tag{3}$$

Lemma 5.9. *Let* $A = k[z]$*. If* $Q \in A[x, y]$ *such that* $D(Q) = a$*, then* $E := \Delta_Q$ *is an irreducible locally nilpotent derivation. Moreover,* E *commutes with* D*.*

Proof. Let K be the quotient field of A. The extension of D' to $K[x, y]$ satisfies $D'(Q/a') = 1$. Thus $K[x, y] = K[P, Q]$. If we extend E to a derivation of $K[x, y]$ one easily sees that E is locally nilpotent. Thus E is a locally nilpotent derivation of $A[x, y]$.

By [DF98, Theorem 2.4]) there exists $S \in A[x, y]$ such that $E = h\Delta_S$ and Δ_S is irreducible. Thus $-a' = E(P) = h\Delta_S(P) = -h\Delta_P(S)$. Hence $\Delta_P(S) \in \mathrm{pl}(\Delta_P)$ and thus $\Delta_P(S)$ is a multiple of a'. This implies that $h \in k^*$ and proves that E is irreducible.

If we extend Δ_P and Δ_Q to $K[x, y] = K[P, Q]$, we get $\Delta_P = a'(\partial/\partial Q)$ and $\Delta_Q = -a'(\partial/\partial P)$ where

$$a' = \det \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} \in K\,.$$

Thus E commutes with D'. But since $a = da'$, it follows that $d \in k[z] \subseteq \ker D \cap \ker E$. This implies that E and $D = dD'$ commute. □

Definition 5.10. For any $Q \in k^{[3]}$ with $D(Q) = a$ we call

$$\mathbf{e} := \operatorname{Exp}(E) = \operatorname{Exp}(\Delta_Q)$$

an *admissible complement to* $\mathbf{u}$.

By (3), we get that $\mathbf{e}$ is an admissible complement to $\mathbf{u}$ if and only if $\mathbf{e}$ is an admissible complement to $\mathbf{u}'$. It follows from Lemma 5.9 that $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{e}, \mathbf{u} \rangle} \cdot \mathbf{e}$ is a subgroup of unipotent automorphisms inside $\operatorname{Cent}(\mathbf{u})$ where $\langle \mathbf{e}, \mathbf{u} \rangle$ denotes the subgroup of $\operatorname{Aut}(\mathbb{A}^3)$ that is generated by $\mathbf{e}$ and $\mathbf{u}$.

Remark 5.11. We have $\mathbb{A}^2 \setminus \Gamma = C \times \mathbb{A}^1$ for some curve $C \subseteq \mathbb{A}^1$. The restriction of $\mathbf{u}$ and of $\mathbf{e}$ to the open subset

$$\pi^{-1}(\mathbb{A}^2 \setminus \Gamma) = (C \times \mathbb{A}^1) \times \mathbb{A}^1 = \operatorname{Spec}(k[z]_a[P, Q])$$

are given by $(u, v, w) \mapsto (u, v, w+1)$ and $(u, v, w) \mapsto (u, v+1, w)$ respectively, where (u, v, w) is the coordinate system $(z, -P/a', Q/a)$.

5.4. **The property (Sat).** We introduce in this subsection a property for a subset $S \subseteq \operatorname{Aut}(\mathbb{A}^n)$ and we will show that $\operatorname{Cent}(\mathbf{v})$ satisfies this property for any unipotent automorphism $\mathbf{v} \in \operatorname{Aut}(\mathbb{A}^n)$. This property will then play a key role when we describe the set of unipotent elements inside the centralizer. One can think of this property as a saturation feature on the unipotent elements in S.

Definition 5.12. Let $S \subseteq \operatorname{Aut}(\mathbb{A}^n)$ be a subset. We say that S has the property (Sat) if for all unipotent automorphisms $\mathbf{w} \in \operatorname{Aut}(\mathbb{A}^n)$ and for all $0 \neq f \in \mathcal{O}(\mathbb{A}^n)^{\mathbf{w}}$ we have

$$f \cdot \mathbf{w} \in S \quad \Longrightarrow \quad \mathbf{w} \in S\,. \tag{Sat}$$

Proposition 5.13. *If* $\mathbf{v} \in \operatorname{Aut}(\mathbb{A}^n)$ *is unipotent, then the subgroup* $\operatorname{Cent}(\mathbf{v}) \subseteq \operatorname{Aut}(\mathbb{A}^n)$ *satisfies the property (Sat).*

Proof. Let $\mathbf{v} = \operatorname{Exp}(B)$ and let $\mathbf{w} = \operatorname{Exp}(F)$. Assume that $f \cdot \mathbf{w}$ commutes with $\mathbf{v}$ for some $\mathbf{w}$-invariant $0 \neq f \in \mathcal{O}(\mathbb{A}^n)$. If $\mathbf{v} = \mathbf{id}$ or $\mathbf{w} = \mathbf{id}$, then (Sat) is obviously satisfied. Thus we assume $\mathbf{v} \neq \mathbf{id} \neq \mathbf{w}$. For the Lie-bracket we have

$$0 = [fF, B] = f[F, B] - B(f)F\,. \tag{4}$$

Thus, it is enough to prove that $B(f) = 0$.

First, assume that F is irreducible. By (4), it follows that f divides $B(f)F(g)$ for all $g \in \mathcal{O}(\mathbb{A}^n)$. As F is irreducible, the ideal $(\operatorname{im} F)$ is equal to $\mathcal{O}(\mathbb{A}^n)$. Hence f divides $B(f)$. As B is locally nilpotent, it follows that $B(f) = 0$.

Now, let $F = f'F'$ for some irreducible F'. Thus, $ff'F'$ commutes with B and by the argument above, $B(ff') = 0$. Since $\ker B$ is factorially closed in $\mathcal{O}(\mathbb{A}^n)$, we have $B(f) = 0$. □

5.5. **The subgroup $N \subseteq \operatorname{Cent}(\mathbf{u})$.** Let $\mathbf{u} \in \operatorname{Aut}(\mathbb{A}^3)$ be unipotent. We define in this subsection a subgroup N of $\operatorname{Cent}(\mathbf{u})$ and we gather some facts about this group. In the next subsection, we will prove that N is exactly the set of unipotent automorphisms $\operatorname{Cent}(\mathbf{u})$ if $\mathbf{u}$ is not a modified translation. We treat first the case when $\mathbf{u}$ is irreducible and consider afterwards the general case.

5.5.1. *The case when* $\mathbf{u}$ *is irreducible.*

Definition 5.14. Let $\mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ be unipotent and irreducible. We define the subset $N \subseteq \mathrm{Cent}(\mathbf{u})$ as

$$N = N(\mathbf{u}) := \begin{cases} \mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{e}, \mathbf{u} \rangle} \cdot \mathbf{e} \circ \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} \cdot \mathbf{u} & \text{if the plinth divisor of } \mathbf{u} \text{ is a fence} \\ \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} \cdot \mathbf{u} & \text{otherwise.} \end{cases}$$

where $\mathbf{e}$ is an admissible complement to $\mathbf{u}$ (cf. Subsection 5.3). Moreover, we define

$$M = M(D) := \begin{cases} (\ker E \cap \ker D)E + \ker(D)D & \text{if } \operatorname{rank} D \leq 2, \\ \ker(D)D & \text{otherwise.} \end{cases}$$

where $\mathbf{u} = \mathrm{Exp}(D)$ and $\mathbf{e} = \mathrm{Exp}(E)$ (cf. Lemma 5.9).

Proposition 5.15 (Properties of N)**.** *Let* $\mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ *be unipotent and irreducible. Then:*

i) The set N *consists of unipotent automorphisms and we have* $N = \mathrm{Exp}(M)$*.*

ii) The group $\mathrm{Cent}(\mathbf{u})$ *normalizes* $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} \cdot \mathbf{u}$ *and we have for all* $\mathbf{g} \in \mathrm{Cent}(\mathbf{u})$ *and for all* $f \cdot \mathbf{u} \in \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} \cdot \mathbf{u}$

$$\mathbf{g}^{-1} \circ f \cdot \mathbf{u} \circ \mathbf{g} = \mathbf{g}^*(f) \cdot \mathbf{u}\,.$$

Moreover, N *is a closed normal subgroup of* $\mathrm{Cent}(\mathbf{u})$*.*

iii) Denote by $\mathrm{Iner}(\mathbb{A}^2, \Gamma)_u$ *the set of unipotent automorphisms of* $\mathrm{Iner}(\mathbb{A}^2, \Gamma)$*. Then*

$$1 \longrightarrow \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} \cdot \mathbf{u} \longrightarrow N \xrightarrow{p|_N} \mathrm{Iner}(\mathbb{A}^2, \Gamma)_u \longrightarrow 1\,.$$

is a split short exact sequence of ind-groups. If Γ *is a fence, then* $p|_N$ *induces an isomorphism* $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{e}, \mathbf{u} \rangle} \cdot \mathbf{e} \simeq \mathrm{Iner}(\mathbb{A}^2, \Gamma)_u$ *for any admissible complement* $\mathbf{e}$ *to* $\mathbf{u}$*. Moreover,* N *is independent of the choice of* $\mathbf{e}$*.*

iv) The subgroup $N \subseteq \mathrm{Aut}(\mathbb{A}^3)$ *satisfies the property (Sat).*

Proof. Assume first that Γ is not a fence. Then i) and iv) are clear, ii) follows from Proposition 5.1, iii) follows from Proposition 3.1. Thus we can assume that Γ is a fence.

i) Let $hE + fD \in M$. By induction on $l \geq 1$ one sees that $(hE + fD)^l$ is the sum of terms of the form gE^iD^j where $g \in \ker D$. From this fact, one can deduce that $hE + fD$ is locally nilpotent and hence M consists only of locally nilpotent derivations.

For all $f \in \ker D$ and $h \in \ker D \cap \ker E$ and $q \geq 0$ we have

$$fD\,\mathrm{ad}(hE)^q = (-1)^q h^q E^q(f) D$$

where $A\,\mathrm{ad}(B) = [A, B]$. With the aid of this formula, an application of the Baker-Campbell-Hausdorff formula yields $\mathrm{Exp}(hE) \circ \mathrm{Exp}(fD) = \mathrm{Exp}(L)$ where L is the finite sum

$$\sum_m \sum_{p_i, q_i} \frac{(-1)^{m-1}}{m \sum (p_i + q_i)} \frac{fD\,\mathrm{ad}(fD)^{p_1 - 1}\,\mathrm{ad}(hE)^{q_1} \cdots \mathrm{ad}(fD)^{p_m}\,\mathrm{ad}(hE)^{q_m}}{p_1! q_1! \cdots p_m! q_m!} \in M$$

(see [Jac62, Proposition 1, §5, chp. V]). Hence $N \subseteq \mathrm{Exp}(M)$ which shows in particular, that N consists of unipotent automorphisms. Moreover, $\mathrm{Exp}\, hE$ and $\mathrm{Exp}(fD + hE)$ coincide on $\ker D$. Lemma 5.3 implies $(\mathrm{Exp}\, hE)^{-1} \circ \mathrm{Exp}(fD + hE) = \mathrm{Exp}(f'D)$ for some $f' \in \ker D$ and thus $\mathrm{Exp}(M) \subseteq N$.

ii) The first part is a calculation. One can check that $N = p^{-1}(\mathrm{Iner}(\mathbb{A}^2, \Gamma)_u)$ by using Proposition 5.1. Since $\mathrm{Iner}(\mathbb{A}^2, \Gamma)_u$ is a closed normal subgroup of $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ it follows that N is a closed normal subgroup of $\mathrm{Cent}(\mathbf{u})$.

iii) It is enough to show that the homomorphism $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{e}, \mathbf{u} \rangle} \cdot \mathbf{e} \to \mathrm{Iner}(\mathbb{A}^2, \Gamma)_u$ (induced by p) is an isomorphism of ind-groups. Injectivity follows from the fact that $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{e}, \mathbf{u} \rangle} \cdot \mathbf{e} \cap \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} \cdot \mathbf{u} = \{\mathbf{id}\}$ and surjectivity follows from a straightforward calculation. As $N = p^{-1}(\mathrm{Iner}(\mathbb{A}^2, \Gamma)_u)$ it follows that N is independent of the choice of $\mathbf{e}$.

iv) Let $0 \neq hE + fD \in M$. It is enough to prove that

$$\gcd(h, f) = 1 \quad \Longrightarrow \quad hE + fD \text{ is irreducible} \tag{$\triangle$}$$

where the greatest common divisor is taken in the polynomial ring $\ker D = k[z, P]$. Indeed, let $gB = hE + fD \in M$ for some locally nilpotent derivation $B \neq 0$ and some $0 \neq g \in \ker B$ and let $h = \gcd(h, f)h'$, $f = \gcd(h, f)f'$. Thus B vanishes on $\ker(h'E + f'D)$ and since $h'E + f'D$ is irreducible, there exists $b \in \ker(h'E + f'D)$ such that $B = b(h'E + f'D)$. This implies $gb = \gcd(h, f) \in k[z]$ and therefore $b \in k[z]$. This shows that $B \in M$.

Let us prove ($\triangle$). Since E and D are irreducible (see Lemma 5.9) we can assume that h and f both are non-zero. A calculation shows

$$hE + fD = \Delta_F\,, \quad F = hQ + fP - \int \left(\frac{\partial f}{\partial P} P \right) \mathrm{d}P$$

where the integration is taken inside the polynomial ring $\ker D = k[z, P]$ and Δ_F is taken with respect to $R[x, y]$ where $R = k[z]$. Let $f = \sum_{i=0}^{n} f_i(z) P^i$. Thus we have

$$fP - \int \left(\frac{\partial f}{\partial P} P \right) \mathrm{d}P = \sum_{i=0}^{n} f_i(z) \left(1 - \frac{i}{i+1} \right) P^{i+1}\,.$$

Denote this last polynomial by $G \in k[z, P]$.

Now, assume towards a contradiction that $hE + fD$ is not irreducible. Hence, we have $hE + fD = bB$ for some locally nilpotent derivation B and some non-constant $b \in \ker B$. By plugging in P and Q in $hE + fD = bB$ and using the fact that $\gcd(h, f) = 1$ we see that b divides a (recall that $D(Q) = a$ and $E(P) = -a$). Hence there exists a root z_0 of a such that the induced derivation of $\Delta_F = hE + fD$ on $k[x, y, z]/(z - z_0) \simeq k[x, y]$ vanishes. Thus, there exists a constant $c \in k$ such that

$$h(z_0)Q(x, y, z_0) + \sum_{i=0}^{n} f_i(z_0) \left(1 - \frac{i}{i+1} \right) P^{i+1}(x, y, z_0) = c\,. \tag{$\boxdot$}$$

The polynomial $P(x, y, z_0) \in k[x, y]$ is non-constant, since otherwise $D = \Delta_P$ would have a 2-dimensional fixed point set, contradicting the irreducibility of D (cf. [Dai07, 2.10]). If $h(z_0) = 0$, then we have $f(z_0, P) = 0$ by ($\boxdot$). Hence $\gcd(h, f) \neq 1$, a contradiction. Thus we can assume $h(z_0) \neq 0$. In this case it follows that $Q + h(z_0)^{-1}(G(z, P) - c)$ is divisible by $z - z_0$ inside $k^{[3]}$. Thus,

$$D \left(\frac{Q + h(z_0)^{-1}(G(z, P) - c)}{z - z_0} \right) = \frac{a}{z - z_0}\,.$$

But this contradicts the fact, that a is a generator of $\mathrm{pl}\, D$.

□

5.5.2. *The general case.*

Definition 5.16. Let $\mathbf{id} \neq \mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ be unipotent and let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. We define the group N as

$$N = N(\mathbf{u}) := \begin{cases} N(\mathbf{u}') & \text{if the plinth divisor of } \mathbf{u} \text{ is a fence} \\ \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' & \text{otherwise.} \end{cases}$$

It follows from Subsection 5.1 and Subsection 5.3 that $N(\mathbf{u}) \subseteq \mathrm{Cent}(\mathbf{u})$.

5.6. **The group $\mathrm{Cent}(\mathbf{u})$ as a semi-direct product.** In this subsection, we prove our first main result: There exists an algebraic subgroup $R \subseteq \mathrm{Cent}(\mathbf{u})$ such that $\mathrm{Cent}(\mathbf{u})$ is the semi-direct product of N with R, if $\mathbf{u}$ is not a modified translation. This shows, that the quotient $\mathrm{Cent}(\mathbf{u})/N$ is rather small. If we endow the quotient $\mathrm{Cent}(\mathbf{u})/N$ with the algebraic group structure induced by R, then we will see in the next subsection, that the quotient $\mathrm{Cent}(\mathbf{u})/N$ consists only of semi-simple elements (cf. Corollary 5.20).

Theorem 5.17. *Let $\mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ be unipotent and assume that $\mathbf{u}$ is not a modified translation. Then the subgroup $N \subseteq \mathrm{Cent}(\mathbf{u})$ is closed and normal, and there exists an algebraic subgroup $R \subseteq \mathrm{Cent}(\mathbf{u})$ such that $\mathrm{Cent}(\mathbf{u})$ is the semi-direct product of N and R as an ind-group. Moreover, all elements of $\mathrm{Cent}(\mathbf{u})$ are algebraic.*

First, we prove Theorem 5.17 in a very special case and deduce the general case to it.

Lemma 5.18. *Let $\mathbf{u} = d \cdot \mathbf{u}'$ where $\mathbf{u}' = (x+1, y, z)$ and assume that $d \in \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} = k[y, z]$ is non-constant. Then $N = N(\mathbf{u})$ is a closed normal subgroup of $\mathrm{Cent}(\mathbf{u})$ and there exists a closed subgroup $R \subseteq \mathrm{Cent}(\mathbf{u})$ such that $\mathrm{Cent}(\mathbf{u})$ is the semi-direct product of N and R as an ind-group. Moreover all elements of $\mathrm{Cent}(\mathbf{u})$ are algebraic.*

Proof. By Proposition 5.15, $N = N(\mathbf{u})$ is a closed normal subgroup of $\mathrm{Cent}(\mathbf{u})$.

If Γ is not a fence, then it follows from Proposition 3.1 that $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group. By Proposition 5.5 there exists a closed subgroup of $\mathrm{Cent}(\mathbf{u})$ that is mapped via $p \colon \mathrm{Cent}(\mathbf{u}) \twoheadrightarrow \mathrm{Aut}(\mathbb{A}^2, \Gamma)$ isomorphically onto $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ and $\mathrm{Cent}(\mathbf{u}) \simeq N \rtimes R$.

Now, assume that Γ is a fence. By Lemma 5.6 we can assume after a coordinate change of $\mathbb{A}^2 = \mathbb{A}^3 /\!\!/ \mathbb{G}_a$ that $a \in k[z]$ and $\mathrm{Iner}(\mathbb{A}^2, \Gamma)_u = \{ (y+h, z) \mid h \in k[z],\ a|h \}$. Thus we have a split short exact sequence of ind-groups

$$1 \longrightarrow \mathrm{Iner}(\mathbb{A}^2, \Gamma)_u \hookrightarrow \mathrm{Aut}(\mathbb{A}^2, \Gamma) \overset{q}{\twoheadrightarrow} (k^* \ltimes k[z]/(a)) \rtimes \mathrm{Aut}(\mathbb{A}^1, V(a)) \longrightarrow 1$$

$$(\lambda y + h, \alpha z + \beta) \longmapsto (\lambda, h + (a), \alpha z + \beta)$$

$$(\lambda y + h_0, \alpha z + \beta) \longleftarrow\!\shortmid (\lambda, \sigma, \alpha z + \beta)$$

where $h_0 \in \sigma \subseteq k[z]$ is the representative of σ of minimal degree. Let R be the algebraic group $(k^* \ltimes k[z]/(a)) \rtimes \mathrm{Aut}(V(a), \mathbb{A}^1)$. Since N is generated by $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ and $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{e}, \mathbf{u}' \rangle} \cdot \mathbf{e} \simeq \mathrm{Iner}(\mathbb{A}^2, \Gamma)_u$, we have the desired split short exact sequence of ind-groups

$$1 \longrightarrow N \hookrightarrow \mathrm{Cent}(\mathbf{u}) \overset{q \circ p}{\twoheadrightarrow} R \longrightarrow 1\,.$$

By using Proposition 5.5, Proposition 5.15 ii) and the fact that every element of $\mathrm{Aut}(\mathbb{A}^2, \Gamma)$ is algebraic, one can see, that every element of $\mathrm{Cent}(\mathbf{u})$ is algebraic. □

Proof of Theorem 5.17. Let $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}} = k[\tilde{y}, \tilde{z}]$ and let $\tilde{x} \in k^{[3]}$ such that $D(\tilde{x}) = a$, where a is a generator of the plinth ideal $\operatorname{pl} D$. Of course we can interpret $\tilde{y}$, $\tilde{z}$ as elements of $k^{[3]}$. Let $\tilde{\mathbf{u}} := a \cdot (\tilde{x}+1, \tilde{y}, \tilde{z}) \in \operatorname{Aut}(\mathbb{A}^3)$ where we interpret $a \in k[\tilde{y}, \tilde{z}]$. The morphism $\mathbb{A}^3 \to \mathbb{A}^3$ induced by the inclusion $k[\tilde{x}, \tilde{y}, \tilde{z}] \subseteq k[x,y,z] = \mathcal{O}(\mathbb{A}^3)$ is birational and hence we get an injective homomorphism of groups

$$\eta \colon \operatorname{Cent}(\mathbf{u}) \longrightarrow \operatorname{Cent}(\tilde{\mathbf{u}}) .$$

In fact, η is a homomorphism of ind-groups, due to the following commutative diagram, where $r \colon k[x,y,z] \twoheadrightarrow k[\tilde{x}, \tilde{y}, \tilde{z}]$ is a k-linear retraction

$$\begin{array}{ccccc}
k[x,y,z]^3 & \xrightarrow[\text{morph.}]{\mathbf{g} \mapsto (\mathbf{g}^*(\tilde{x}), \mathbf{g}^*(\tilde{y}), \mathbf{g}^*(\tilde{z}))} & k[x,y,z]^3 & \xrightarrow[\text{lin.}]{r \times r \times r} & k[\tilde{x}, \tilde{y}, \tilde{z}]^3 \\
\uparrow \scriptstyle{\text{loc. closed}} & & & & \scriptstyle{\text{loc. closed}} \uparrow \\
\operatorname{Cent}(\mathbf{u}) & & \xrightarrow{\quad\quad \eta \quad\quad} & & \operatorname{Cent}(\tilde{\mathbf{u}})
\end{array}$$

According to Lemma 5.18, $\operatorname{Cent}(\tilde{\mathbf{u}})$ is the semi-direct product of $N(\tilde{\mathbf{u}})$ with some closed algebraic subgroup $\tilde{R} \subseteq \operatorname{Cent}(\tilde{\mathbf{u}})$. Let $H \subseteq \tilde{R}$ be a closed algebraic subgroup. We claim that $\eta^{-1}(H) \subseteq \operatorname{Cent}(\mathbf{u})$ is an algebraic subgroup. Since $\eta \colon \operatorname{Cent}(\mathbf{u}) \to \operatorname{Cent}(\tilde{\mathbf{u}})$ is a homomorphism of ind-groups, it follows that $\eta^{-1}(H)$ is a closed subgroup. As H is algebraic and thus acts locally finite on $\mathbb{A}^3$, it follows that $\eta^{-1}(H)$ acts also locally finite on $\mathbb{A}^3$ by [KS12, Lemma 3.6]. This implies the claim.

According to the claim all elements of $\operatorname{Cent}(\mathbf{u})$ are algebraic and $R := \eta^{-1}(\tilde{R})$ is algebraic as well. Since η is an injective homomorphism of ind-groups we have the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N(\tilde{\mathbf{u}}) & \hookrightarrow & \operatorname{Cent}(\tilde{\mathbf{u}}) & \twoheadrightarrow & \tilde{R} & \longrightarrow & 1 \\
 & & \scriptstyle{\text{iso. of groups}} \uparrow & & \uparrow \scriptstyle{\eta} & & \uparrow \scriptstyle{\text{cl. embedd.}} & & \\
1 & \longrightarrow & N(\mathbf{u}) & \hookrightarrow & \operatorname{Cent}(\mathbf{u}) & \twoheadrightarrow & R & \longrightarrow & 1
\end{array}$$

As the first column is a split short exact sequence of ind-groups, the second coloumn is also a split short exact sequence of ind-groups. This proves the theorem. ☐

5.7. **The unipotent elements of** $\operatorname{Cent}(\mathbf{u})$**.** The goal of this subsection is to prove our second main result: The unipotent elements of $\operatorname{Cent}(\mathbf{u})$ are exactly N provided $\mathbf{u}$ is not a modified translation. As we know from Proposition 5.15 the set N satisfies the property (Sat). This will be a key ingredient in the proof of the next theorem.

Theorem 5.19. *Let* $\mathbf{u} \in \operatorname{Aut}(\mathbb{A}^3)$ *be unipotent and assume that* $\mathbf{u}$ *is not a modified translation. Then the set of unipotent elements of* $\operatorname{Cent}(\mathbf{u})$ *is equal to* N*.*

Proof. Let $\mathbf{u} = d \cdot \mathbf{u}'$ be a standard decomposition. Let $\mathbf{g} \in \operatorname{Cent}(\mathbf{u})$ be a unipotent automorphism with $\mathbf{g} \neq \mathbf{id}$. If Γ is not a fence, then $\operatorname{Aut}(\mathbb{A}^2, \Gamma)$ contains no unipotent automorphism $\neq \mathbf{id}$ (see Proposition 3.1). By Proposition 5.1 it follows that $\mathbf{g} \in \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' = N$.

Hence we can assume that $\Gamma = \operatorname{div}(a)$ is a fence. Let $z \in \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}}$ be a variable of $k^{[3]}$ such that $a \in k[z]$ (see Lemma 5.6). If $\mathcal{O}(\mathbb{A}^3)^{\mathbf{g}} = \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}}$, then $\mathbf{g}$ is a modification of $\mathbf{u}'$ and therefore $\mathbf{g} \in N$. Now, assume $\mathcal{O}(\mathbb{A}^3)^{\mathbf{g}} \neq \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}}$. Thus $\mathbf{g}$ is not a modification of $\mathbf{u}'$ and hence $\mathbf{id} \neq p(\mathbf{g}) \in \operatorname{Aut}(\mathbb{A}^2, \Gamma)$. Since $p(\mathbf{g})$ is unipotent and $0 \neq a \in k[z]$, it follows that $z \in \mathcal{O}(\mathbb{A}^3)^{\mathbf{g}}$ and thus $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{g}, \mathbf{u} \rangle}$ is an

∞-dimensional k-vector space. By Theorem 5.17 there exists an algebraic subgroup $R \subseteq \mathrm{Cent}(\mathbf{u})$ and a split short exact sequence of ind-groups

$$1 \to N \hookrightarrow \mathrm{Cent}(\mathbf{u}) \overset{\theta}{\twoheadrightarrow} R \to 1\,.$$

If $\mathbf{g} \notin N$, then $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{g}, \mathbf{u} \rangle} \cdot \mathbf{g} \cap N = 1$, since N satisfies the property (Sat). We get an injection $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{g}, \mathbf{u} \rangle} \to \mathrm{Cent}(\mathbf{u}) \twoheadrightarrow R$, $h \mapsto \theta(h \cdot \mathbf{g})$. Choose any filtration by finite dimensional k-subspaces to turn $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{g}, \mathbf{u} \rangle}$ into an ind-group. It follows that $\mathcal{O}(\mathbb{A}^3)^{\langle \mathbf{g}, \mathbf{u} \rangle} \to R$ is an injective homomorphism of ind-groups. But this implies that R has closed algebraic subgroups of arbitrary high dimension, which is absurd. This finishes the proof of the theorem. □

If we endow $\mathrm{Cent}(\mathbf{u})/N$ with the algebraic group structure induced by the semi-direct product decomposition coming from Theorem 5.17, then we get immediately the following corollary from Theorem 5.19.

Corollary 5.20. *If $\mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ is unipotent and $\mathbf{u}$ is not a modified translation, then the algebraic group $\mathrm{Cent}(\mathbf{u})/N$ consists only of semi-simple elements. In particular, the connected component of the neutral element in $\mathrm{Cent}(\mathbf{u})/N$ is a torus.*

5.8. **The subgroup $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \subseteq \mathrm{Cent}(\mathbf{u})$.** The next proposition is an application of Theorem 5.17 and Theorem 5.19.

Proposition 5.21. *Let $\mathbf{u} \in \mathrm{Aut}(\mathbb{A}^3)$ be unipotent and assume that $\mathbf{u}$ is not a modified translation. If $\mathbf{u} = d \cdot \mathbf{u}'$ is a standard decomposition, then the subgroup $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ of $\mathrm{Cent}(\mathbf{u})$ is characteristic.*

Before we prove this proposition, we construct an appropriate coordinate system on the quotient $\mathbb{A}^3 /\!\!/ \mathbb{G}_a = \mathbb{A}^2$ in the case when Γ is a non-empty fence. Let $R \subseteq \mathrm{Cent}(\mathbf{u})$ be a closed algebraic subgroup such that $\mathrm{Cent}(\mathbf{u}) = R \ltimes N$ (see Theorem 5.17). Let $T := R^0$ be the connected component of the neutral element in R. By Corollary 5.20, it is a torus (possibly $\dim T = 0$). The torus T acts faithfully on the quotient $\mathbb{A}^3 /\!\!/ \mathbb{G}_a$, since the kernel of the map $p\colon \mathrm{Cent}(\mathbf{u}) \to \mathrm{Aut}(\mathbb{A}^3 /\!\!/ \mathbb{G}_a, \Gamma)$ consists only of unipotent elements (see Proposition 5.1). Let (z, P) be a coordinate system of $\mathbb{A}^2 = \mathbb{A}^3 /\!\!/ \mathbb{G}_a$ such that $\Gamma \subseteq \mathbb{A}^2$ is given by the standard embedding $F \times \mathbb{A}^1 \subseteq \mathbb{A}^2$ for some 0-dimensional closed subscheme $F \subseteq \mathbb{A}^1$ and $\mathbf{u}' = \mathrm{Exp}(\Delta_P)$ (see Subsection 5.3).

Proposition 5.22. *There exists $q \in k$ and $r \in k[z]$ such that the action of T on $\mathbb{A}^3 /\!\!/ \mathbb{G}_a = \mathbb{A}^2$ is diagonal with respect to the coordinate system $(z+q, P+r)$.*

Proof. As the torus T leaves $\Gamma = V(a) \subseteq \mathbb{A}^2$ invariant and since $a \in k[z]$, it follows that there exists $q \in k$ such that $z + q$ is a semi-invariant for the action of T. By replacing $z+q$ with z we can assume that $\pi\colon \mathbb{A}^2 \to \mathbb{A}^1$, $(z, P) \to z$ is T-equivariant with respect to a suitable T-action on $\mathbb{A}^1$. Due to [KK96, Proposition 1], every lift of a T-action on $\mathbb{A}^1$ to $\mathbb{A}^2$ (with respect to π) is equivalent to a trivial lift (with respect to π). Thus, there exists $r \in k[z]$ such that $P + r$ is a semi-invariant with respect to the action of T. This finishes the proof. □

According to the last proposition we can and will assume that the action of T on the quotient $\mathbb{A}^3 /\!\!/ \mathbb{G}_a = \mathbb{A}^2$ is diagonal with respect to the coordinate system (z, P). Let $A := k[z]$ and let $A \ltimes A[P]$ be the semi-direct product defined by

$$(h, f) \cdot (h', f') := (h + h', f(P + h'a) + f') \tag{5}$$

where $a \in A$ such that $\Gamma = \mathrm{div}(a)$. From Proposition 5.15 it follows that

$$A \ltimes A[P] \xrightarrow{\sim} N(\mathbf{u}) \,, \quad (h, f) \longmapsto h \cdot \mathbf{e} \circ f \cdot \mathbf{u}'$$

is an isomorphism of groups where $\mathbf{e}$ is an admissible complement to $\mathbf{u}$. Under this isomorphism the subgroup $A[P]$ is sent onto $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$.

From the next lemma it follows that $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}'$ is a characteristic subgroup of the connected component of $\mathrm{Cent}(\mathbf{u})$ in the case when Γ is a non-empty fence and $\dim T > 0$.

Lemma 5.23. *Let $A \ltimes A[P]$ be defined as in (5) and let $G := T \ltimes (A \ltimes A[P])$ where T is a torus with $\dim T > 0$. Assume that $A[P] \subseteq G$ is a normal subgroup, that the action of T by conjugation on $A[P]$ induces a faithful diagonal representation on $kz \oplus kP$ and that T acts on $A[P]$ by k-algebra automorphisms. Furthermore we assume that the action of T by conjugation on the quotient $G/A[P]$ is non-trivial and the product in G satisfies*

$$(\lambda, 0, 0) \cdot (1, h, f) = (\lambda, h, f) \,.$$

Then $A[P] = \mathrm{Cent}_G \, G^{(2)}$ where $G^{(2)}$ denotes the second derived group of G.

Proof of Lemma 5.23. As the action by conjuagtion of T on $G/A[P]$ is non-trivial, it follows that the first derived subgroup $G^{(1)}$ is not contained in $A[P]$. As T is abelian it follows that $G^{(1)} \subseteq A \ltimes A[P]$ and as A is abelian we conclude $G^{(2)} \subseteq A[P]$. Thus there exists $(1, h_0, f_0) \in G^{(1)}$ with $h_0 \neq 0$. Denote by $\rho \colon T \hookrightarrow \mathrm{GL}(kz \oplus kP)$ the representation of T on $kz \oplus kP$. By assumption $\rho(\lambda)$ is a diagonal matrix

$$\rho(\lambda) = \begin{pmatrix} \rho_1(\lambda) & 0 \\ 0 & \rho_2(\lambda) \end{pmatrix}$$

and $\ker(\rho_1) \cap \ker(\rho_2) = \{1\}$. As $A[P]$ is abelian, we get $A[P] \subseteq \mathrm{Cent}_G \, G^{(2)}$. Now, we prove $\mathrm{Cent}_G \, G^{(2)} \subseteq A[P]$. We have for all $(1, 0, q) \in G^{(1)}$

$$(1, 0, q - q(P - h_0 a)) = [(1, 0, q), (1, h_0, f_0)] \in G^{(2)}$$

where $[\cdot, \cdot]$ denotes the commutator-bracket. Moreover, $(1, 0, z^i P^j) \in G^{(1)}$ for all $(i, j) \in \mathbb{N}_0^2$ such that $\rho_1^i \rho_2^j$ is not the trivial character, as we have $(1, 0, z^i P^j) = [(\lambda, 0, 0), (1, 0, z^i P^j)]$ for some well chosen $\lambda \in T$. For all $j \geq 0$, the character $\rho_1^i \rho_2^j$ is non-trivial, provided that i is large enough. For all $(1, 0, f) \in A[P]$ we have

$$\begin{aligned} \mathrm{Cent}_G(1, 0, f) &= \{ (\lambda', h', f') \in G \mid (\lambda', h', f') \cdot (1, 0, f) \cdot (\lambda', h', f')^{-1} = (1, 0, f) \} \\ &= \{ (\lambda', h', f') \in G \mid f = \lambda' * f(P - h'a) \} \end{aligned}$$

where $*$ denotes the action by conjugation of T on $A[P]$. Let $(\lambda', h', f') \in \mathrm{Cent}_G \, G^{(2)}$. Since $(\lambda', h', f') \in \mathrm{Cent}_G(1, 0, z^i h_0 a)$ for i sufficiently large, it follows that $\lambda' \in \ker \rho_1$. Moreover, we have $(\lambda', h', f') \in \mathrm{Cent}_G(1, 0, z^i h_0 a(2P - h_0 a))$ for sufficiently large i. Thus, $2\rho_2(\lambda')P - 2h'a = 2P$ and therefore $\lambda' \in \ker \rho_2$, $h' = 0$. Hence we have $(\lambda', h', f') = (1, 0, f') \in A[P]$. This proves $\mathrm{Cent}_G \, G^{(2)} \subseteq A[P]$. ☐

Proof of Proposition 5.21. It follows from Theorem 5.17 that $T \ltimes N \subseteq \mathrm{Cent}(\mathbf{u})$ is a subgroup of finite index. Moreover, $T \ltimes N$ has no proper subgroup of finite index, as this group is generated by groups that have no proper subgroup of finite index. This implies that $T \ltimes N$ is a characteristic subgroup of $\mathrm{Cent}(\mathbf{u})$. Let $G := T \ltimes N$ and let $H := \mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'} \cdot \mathbf{u}' \subseteq N$. We distinguish several cases.

i) Γ is a fence, $\dim T > 0$: From Lemma 5.23 it follows that $H = \mathrm{Cent}_G(G^{(2)})$.

ii) Γ is a fence, $\dim T = 0$: Let $h \cdot \mathbf{e} \circ f \cdot \mathbf{u}' \in \mathrm{Cent}_G(G^{(1)})$. A calculation shows that

$$\mathbf{id} = [h \cdot \mathbf{e} \circ f \cdot \mathbf{u}', [P^2 \cdot \mathbf{u}', \mathbf{e}]] = -2ha^2 \cdot \mathbf{u}' \,.$$

Hence $h = 0$. It follows that $H = \mathrm{Cent}_G(G^{(1)})$.

iii) Γ is not a fence, $\dim T > 0$: There exists a coordinate system (v_1, v_2) of $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'}$ such that the action of T on $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'}$ is diagonal with respect to (v_1, v_2) (see [Kam79]). Let ρ_1 and ρ_2 be the characters of T such that $\mathbf{t}^*(v_i) = \rho_i(\mathbf{t})v_i$ for all $\mathbf{t} \in T$. Let $\mathbf{t} \circ f \cdot \mathbf{u}' \in \mathrm{Cent}_G(G^{(1)})$. A calculation shows that for $i = 1, 2$ we have

$$\mathbf{id} = [\mathbf{t} \circ f \cdot \mathbf{u}', [\mathbf{t}^{-1}, v_i \cdot \mathbf{u}']] = (1 - \rho_i(\mathbf{t}^{-1}))(1 - \rho_i(\mathbf{t}))v_i \cdot \mathbf{u}' \,.$$

Thus $\rho_i(\mathbf{t}) = 1$ for $i = 1, 2$. As the action of T on $\mathcal{O}(\mathbb{A}^3)^{\mathbf{u}'}$ is faithful, it follows that $\mathbf{t} = 1$. It follows that $H = \mathrm{Cent}_G(G^{(1)})$.

iv) Γ is not a fence, $\dim T = 0$: In this case we have $H = G$.

In every case it follows that $H \subseteq G$ is a characteristic subgroup and thus H is a characteristic subgroup of $\mathrm{Cent}(\mathbf{u})$. This finishes the proof. □

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Curriculum Vitae

Immanuel Stampfli, born on July 6th, 1984, in Bern. I am a citizen of Günsberg, Solothurn. My parents are Rudolf and Christiane Stampfli-Rollier.

Education

Aug. 91 - July 97	Primary school in Länggasse
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Aug. 99 - June 03	Maturity (Matura) scientific, Gymnasium Neufeld, Bern
Oct. 03 - Sept. 08	Studies of Mathematics, minor in Computer Science and Physics at the University of Bern
Sept. 07 - Aug. 08	Diploma in Mathematics supervised by Prof. Dr. A. Jeanneret at the University of Bern. Title: *Differenzierbare Strukturen auf* S^7
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