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Quadratic Forms and Linear Algebraic Groups

Organised by
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Abstract. Topics discussed at the workshop Quadratic forms and linear algebraic groups included besides the algebraic theory of quadratic and Hermitian forms and their Witt groups several aspects of the theory of linear algebraic groups and homogeneous varieties, as well as some arithmetic aspects pertaining to the theory of quadratic forms over function fields over local fields.

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Introduction by the Organisers

The workshop was organized by Detlev Hoffmann (Nottingham), Alexander Merkurjev (Los Angeles), and Jean-Pierre Tignol (Louvain-la-Neuve), and was attended by 50 participants. Funding from the Leibniz Association within the grant "Oberwolfach Leibniz Graduate Students" (OWLG) provided support toward the participation of five young researchers. Additionally, the "US Junior Oberwolfach Fellows" program of the US National Science Foundation funded travel expenses for two post docs from the USA.

The workshop followed a long tradition of Oberwolfach meetings on algebraic theory of quadratic forms and related structures. In the last 15 years the algebraic theory of quadratic forms was greatly influenced by methods of algebraic geometry and linear algebraic groups.

The schedule of the meeting comprised 21 lectures of 45 minutes each. Highlights of the conference include recent progress of the application of the patching principle, progress on the Grothendieck-Serre conjecture, and the solution of the problem on excellent connections in the motive of a quadratic form.

In his lecture, J-P. Serre remarked that he had given his first talk in Oberwolfach in 1949.

Workshop: Quadratic Forms and Linear Algebraic Groups

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Abstracts

A Local-Global Principle for Algebraic Group Actions and Applications to Quadratic Forms

Julia Hartmann
(joint work with David Harbater, Daniel Krashen)

The u-invariant of a field is the maximal dimension of anisotropic quadratic forms over that field. For a long time, it was an open problem to determine the u-invariant of p-adic function fields. In 2007, Parimala and Suresh were able to prove that the u-invariant of nondyadic p-adic function fields has the expected value 8 (see [5]). This note explains a local global principle for actions of rational linear algebraic groups which yields a new and different proof of their u-invariant result. It should be noted that the case of $p = 2$ has been solved very recently by D. Leep (see [4]).

1. Patching

The origin of our local global principle lies in factorization results occuring naturally in the context of patching. The basic idea of this technique is very simple: To give an algebraic structure over a field, it is sometimes sufficient to give it over a collection of overfields, in a compatible way. This is in effect a descent technique; the name stems from the fact that the fields are of geometric origin, mimicking complex patching or glueing techniques but replacing the metric topology by an adic topology. Versions of this method have been in use for quite a while, mostly in inverse Galois theory. More recently, these methods were modified to work over fields rather than rings [2], which also led to completely new areas of application.

Given a quadruple $F \leq F_1, F_2 \leq F_0$ of fields, there is a base change functor

$$\Theta : \mathrm{Vect}(F) \to \mathrm{Vect}(F_1) \times_{\mathrm{Vect}(F_0)} \mathrm{Vect}(F_2)$$

from the category $\mathrm{Vect}(F)$ of finite dimensional F-vector spaces to the 2-fibre product of the corresponding categories over F_1, F_2 and F_0, respectively. An object in the 2-fibre product is a triple (V_1, V_2, ϕ), where V_i is a vector space over F_i and $\phi : V_1 \otimes_{F_1} F_0 \to V_2 \otimes_{F_2} F_0$ is an isomorphism. Field patching works in situations when Θ is an equivalence of categories; such quadruples of fields where F is the function field of a curve over a complete discretely valued field are given in [2].

We give a simple example of the fields occuring in [2]. Let k be a field, let $X = \mathbb{P}^1_k$, and let $F = k((t))(x)$ be the function field of $X \times_k k[[t]]$. Let F_1 be the fraction field of $k[x][[t]]$ (geometrically, this corresponds to the open subset $X \setminus \{\infty\}$ of X), let F_2 be the fraction field of $k[[x^{-1}, t]]$ (corresponding to the point ∞ of X), and let F_0 equal $k(x)((t))$.

The relation between patching and factorization is explained by the following easy

Proposition 1. *Let $F_1, F_2 \leq F_0$ be fields, and let $F = F_1 \cap F_2$. The base change functor Θ is an equivalence of categories if and only if for all $n \in \mathbb{N}$ and for all $A \in \mathrm{GL}_n(F_0)$, there is a factorization $A = A_1 A_2$ where $A_i \in \mathrm{GL}_n(F_i)$.*

It is not too difficult to see that the fields given in the example above satisfy these conditions.

More generally, if $\mathcal{X}$ is a smooth projective curve over a complete discrete valuation ring with uniformizer t, and P is a closed point of the closed fibre X of $\mathcal{X}$ with complement $U := X \setminus \{P\}$, one defines $\hat{R}_U$ to be the t-adic completion of the ring of functions on $\mathcal{X}$ which are regular on U, one lets $\hat{R}_P$ denote the completion of the local ring of $\mathcal{X}$ at P and $\hat{R}_\wp$ the completion of the localization of $\hat{R}_P$ at $t\hat{R}_P$. These rings have fraction fields $F_U, F_P \leq F_\wp$. Then

Theorem 2 ([2], Theorem 5.9)**.**
(1) *The intersection $F_U \cap F_P$ is the function field F of $\mathcal{X}$.*
(2) *The corresponding base change functor is an equivalence of categories.*

In particular, for such fields one has the above matrix factorization in GL_n. It is a natural question to ask whether the factorization carries over to other linear algebraic groups.

2. Factorization and Local Global Principles for Rational Linear Algebraic Groups

The results of this section can be found in or easily deduced from [3]. We say that a linear algebraic group defined over a field F is *rational* if every component is rational as an F-variety. (Note that in particular, the group of components of a rational group is a constant finite group.)

For such groups, one has

Theorem 3. *Let $F, F_U, F_P, F_\wp$ be as in the previous section. Let G be a rational linear algebraic group defined over F. Then every $A \in G(F_\wp)$ is of the form $A = A_U A_P$ for $A_U \in G(F_U), A_P \in G(F_P)$.*

As an immediate consequence, we obtain a local global principle for the existence of rational points.

Corollary 4. *Let G/F be a rational linear algebraic group, and let H/F be a G-variety such that $G(E)$ acts transitively on $H(E)$ for every extension field $E \geq F$. Then $H(F) \neq \varnothing$ if and only if $H(F_U) \neq \varnothing \neq H(F_P)$.*

We apply this to the projective variety H defined by a quadratic form q with $G = \mathrm{O}(q)$. The rationality of G was already known to Cayley, and G acts transitively on H by the Witt extension theorem.

Corollary 5. *Let q be a regular quadratic form over F (assume* $\mathrm{char}(F) \neq 2$*). Then q is isotropic over F if and only if it is isotropic over F_U and F_P.*

Inductively, this yields

Corollary 6. *The map on Witt groups $W(F) \to W(F_U) \times W(F_P)$ is injective.*

In order to recover the theorem of Parimala and Suresh, these results need to be extended to the case of nonsmooth curves. Here, a collection of overfields is obtained by considering fields corresponding to preimages of $\mathbb{P}^1 \setminus \{\infty\}$ and ∞ under a finite morphism from that curve to $\mathbb{P}^1$. One then obtains analogues of Theorem 3 and Corollary 4 for *connected* rational linear algebraic groups, and consequently an analogue of Corollary 5 for regular quadratic forms of dimension at least 3 (using the connected component $\mathrm{SO}(q)$ in place of $\mathrm{O}(q)$). The latter was shown to imply a local gobal principle in terms of completions with respect to discrete valuations by Colliot-Thélène, Parimala, and Suresh in [1].

From either of the local global principles one can recover the u-invariant result by using residue forms. In fact, one obtains a stronger result that allows for iteration, e.g., to prove that $u(\mathbb{Q}_p((t))(x)) = 16$ $(p \neq 2)$.

Scheiderer [6] considers the dimension of *torsion* quadratic forms and obtains analogous results on the behaviour of the u-invariant e.g. of real Laurent series fields.

We remark that the same local global principle for actions of rational groups also yields results on the so-called *period-index problem*, i.e., the relationship of period and index in the Brauer group (see [3], Section 5 for details).

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Principe local global pour les espaces homogènes sur les corps de fonctions de courbes p-adiques, II (Lokal-global Prinzip für homogene Räume über Funktionenkörpern von p-adischen Kurven, II)

Jean-Louis Colliot-Thélène

J'ai exposé une partie de l'article [1], écrit en collaboration avec R. Parimala et V. Suresh.

Soit A un anneau de valuation discrète de rang 1, complet, de corps des fractions K et de corps résiduel k. Soit F un corps de fonctions d'une variable sur K. A toute valuation discrète v de rang 1 sur F, non nécessairement triviale sur K, associons le complété F_v de F. Soit Y une F-variété qui est un espace homogène d'un F-groupe linéaire connexe G.

Question (ouverte) : si la F-variété Y a des points dans tous les F_v, a -t-elle un point dans F ?

On montre qu'il en est ainsi dans les deux cas suivants :

(1) La variété Y est une quadrique lisse de dimension au moins 1, et la caractéristique de k n'est pas 2.

(2) Le F-groupe G est extension de A à F d'un groupe réductif (connexe) sur A, la F-variété sous-jacente à $G \times_A F$ est F-rationnelle, et Y est un espace principal homogène de G.

De récents théorèmes de recollement de Harbater, Hartmann et Krashen ([2, 3]) jouent un rôle fondamental dans les démonstrations. L'hypothèse de F-rationalité de la variété sous-jacente au groupe connexe G_F est essentielle dans leurs démonstrations, mais on ne sait pas si elle est nécessaire pour leurs énoncés.

Voici un corollaire de l'énoncé (2) :

(3) Le groupe de Brauer de F s'injecte dans le produit des groupes de Brauer des complétés F_v.

Supposons maintenant que le corps k est fini, c'est-à-dire que le corps K est un corps p-adique. Dans ce cas, l'énoncé (3) est proche de théorèmes de Lichtenbaum (et Tate) et de Grothendieck. L'énoncé (1) quant à lui a alors pour conséquence le résultat suivant.

(3) Pour K p-adique avec $p \neq 2$, toute forme quadratique sur F en au moins 9 variables a un zéro.

On reconnaît là un théorème de Parimala et Suresh [4], dont une démonstration radicalement nouvelle est donnée par Harbater, Hartmann et Krashen dans [3]. Ces derniers obtiennent d'ailleurs des généralisations de l'énoncé (3) pour d'autres corps résiduels k que les corps finis.

[Depuis février, un grand progrès sur le théorème (3) a été accompli, par des méthodes radicalement différentes : élimination de la restriction $p \neq 2$, validité d'un énoncé analogue sur les corps de fonctions de d variables (Leep, conséquence d'un récent théorème de Heath-Brown). Ceci fut l'objet de l'exposé de D. Leep à cette rencontre.]

J'avais déjà fait un rapport sur l'article [1] lors de la session "Arithmetik der Körper" à Oberwolfach en février 2009. J'avais alors donné les grandes lignes de la démonstration du résultat (1).

J'ai cette fois-ci donné les grandes lignes de la démonstration du théorème (2).

J'ai aussi, sans démonstration, discuté la nécessité des hypothèses dans le théorème (1). Plus précisément, soient K un corps p-adique et F un corps de fonctions d'une variable sur K. Soit Ω l'ensemble des valuations discrètes de rang 1 sur F, et soit $\Omega_{F/K} \subset \Omega$ l'ensemble des valuations triviales sur K.

Une forme quadratique de rang 2 qui est isotrope sur chaque F_v pour $v \in \Omega$ n'est pas nécessairement F-isotrope (exemple classique, retrouvé lors de la session "Arithmetik der Körper" en février).

Pour $n = 3, 4$, toute forme quadratique en n variables sur F qui est isotrope sur chaque F_v pour $v \in \Omega_{F/K}$ est isotrope sur F (conséquence du théorème de Lichtenbaum).

Pour $n = 5, 6, 7, 8$, une forme quadratique en n variables sur F qui est isotrope sur chaque F_v pour $v \in \Omega_{F/K}$ n'est pas nécessairement F-isotrope. Le cas $n = 5$ est le fruit d'une discussion entre les auteurs lors de la présente session.

A la fin de mon exposé, j'ai brièvement mentionné les liens du théorème (2) avec la question de savoir si pour K p-adique et F comme ci-dessus, l'invariant de Rost

$$H^1(F, G) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

a un noyau trivial. Le corps F est de dimension cohomologique 3. Après mon exposé, A. S. Merkur'ev a montré qu'on ne saurait espérer un tel énoncé pour un corps F de dimension cohomologique 3 arbitraire, sans hypothèse supplémentaire sur le groupe G.

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The u-invariant of p-adic function fields

David Leep

The u-invariant of a field K, $u(K)$, is the maximal dimension of an anisotropic quadratic form defined over K. If the maximum does not exist, then we set $u(K) = \infty$. This definition of the u-invariant of a field is the classical u-invariant, and is useful only for nonreal fields (fields that have no orderings).

Let L be a finitely generated function field of transcendence degree m over a field K. Very little is known about $u(L)$ in terms of $u(K)$.

If K is a C_i-field (in the sense of Tsen and Lang), then $u(K) \leq 2^i$. The Tsen-Lang theory implies that $u(L) \leq 2^{i+m}$. If K has the additional property that $u(E) = u(K) = 2^i$ for all finite algebraic extensions E/K, then it can be shown that $u(L) = 2^{i+m}$.

If K is a hereditarily quadratically closed field, then similar results are also known for $u(L)$. See [EW] for more details. Other than these examples, there are hardly any other known results for this situation.

One particularly interesting case is when K is a p-adic field. (That is, K is a finite algebraic extension of the field $\mathbf{Q}_p$ of p-adic numbers.) In this case, K has some properties that are similar to the C_2-field $\mathbf{F}_p((t))$, but it is known that K is not a C_2-field. (See [G-2].) A p-adic field K has the property that $u(E) = u(K) = 4$ for all finite algebraic extensions E/K. Thus one could conjecture that $u(L) = 2^{m+2}$, as would be the case if K were a C_2-field. (It is not hard to show using arguments from valuation theory that $u(L) \geq 2^{m+2}$.) For a long time there had been a question whether $u(L)$ is finite, even when $m = 1$. The first finiteness results came in the late 1990's. If $m = 1$ and $p \neq 2$, finiteness results for $u(L)$ were established by Merkurjev ($u(L) \leq 26$); Hoffmann and van Geel ($u(L) \leq 22$); and Parimala and Suresh ($u(L) \leq 10$). See [HvG] and [PS-1]. Recently, Parimala and Suresh showed in [PS-2] that $u(L) = 8$. An additional proof of this fact has been given even more recently in [HHK].

Using a recent result of Heath-Brown ([H]), we can now show that $u(L) = 2^{m+2}$ for all $m \geq 0$ and for all primes p, including $p = 2$. In fact, we can compute the u-invariant for a wider class of fields, as shown below.

A field K satisfies property $\mathcal{A}_i$ if every system of r quadratic forms defined over K in n variables, $n > 2^i r$, has a nontrivial common zero in an extension field of odd degree over K.

We will write $K \in \mathcal{A}_i$ to denote that K satisfies property $\mathcal{A}_i$. If K is a C_i-field, then $K \in \mathcal{A}_i$. In this case, the odd degree extension of K can always be chosen as the field K.

Proposition 1. *If $K \in \mathcal{A}_i$, then $u(K) \leq 2^i$.*

Theorem 2. *Let K be a field and let L be a finite algebraic extension of K. If $K \in \mathcal{A}_i$, then $K(x), K((x)) \in \mathcal{A}_{i+1}$ and $L \in \mathcal{A}_i$.*

Theorem 3 (Heath-Brown)**.** *Let K be a p-adic field with residue field F. Let $S = \{Q_1, \ldots, Q_r\}$ be a system of r quadratic forms defined over K in n variables. If $n > 4r$ and $|F| \geq (2r)^r$, then S is isotropic over K.*

Corollary 4. *If K is a p-adic field, then $K \in \mathcal{A}_2$.*

Let L be a field obtained by a finite sequence of the following types of extensions. Let K be an arbitrary field and let

$$K = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_t = L,$$

where for $0 \leq i \leq t-1$ we have either $K_{i+1} \cong K_i(x)$, or $K_{i+1} \cong K((x))$, or K_{i+1} is a simple algebraic extension of K_i (meaning that $K_{i+1} = K_i(\theta)$ where $\theta \in K_{i+1}$ is algebraic over K_i).

Let m denote the number of non-algebraic extensions in the sequence above. Thus $0 \leq m \leq t$. In this situation, we will say that L is a basic field over K of type m.

Lemma 5. *Let K be a field and suppose that $u(E) = u(K)$ for all finite algebraic extensions E/K. Let L be a basic field over K of type m. Then $u(L) \geq 2^m u(K)$.*

Corollary 6. *Let L be a basic field over K of type m and suppose that $K \in \mathcal{A}_i$.*

(1) *Then $L \in \mathcal{A}_{i+m}$. In particular, $u(L) \leq 2^{i+m}$.*
(2) *If $u(E) = u(K) = 2^i$ for all finite algebraic extensions E/K, then $u(L) = 2^{i+m}$.*

Theorem 7. *Let L be a finitely generated function field of transcendence degree m over a p-adic field K. Then $u(L) = 2^{m+2}$.*

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Un théorème de décomposabilité pour les algèbres de degré 8 à involution orthogonale

Anne Quéguiner-Mathieu

(joint work with Alexandre Masquelein, Jean-Pierre Tignol)

Dans cet exposé, basé sur les articles [8] et [9], nous établissons un théorème de décomposabilité pour les algèbres de degré 8 et d'indice au plus 4, munies d'une involution de type orthogonal. Nous présentons ensuite deux applications de ce théorème, l'une en théorie des formes quadratiques, et l'autre concernant l'invariant d'Arason pour les involutions orthogonales.

On travaille sur un corps F de caractéristique différente de 2.

1. Décomposabilité

Soit A une F-algèbre centrale simple de degré 8, munie d'une involution σ de type orthogonal. On sait [6, (42.11)] que l'algèbre à involution (A,σ) est totalement décomposable, c'est-à-dire isomorphe à un produit de trois algèbres de quaternions à involution

$$(A,\sigma) = \otimes_{i=1}^{3}(Q_i,\sigma_i),$$

si et seulement si son discriminant est trivial et son algèbre de Clifford a une composante déployée. De plus, il découle des travaux de Becher [2] que si A est déployée (respectivement semblable à une algèbre de quaternions), alors elle admet une décomposition comme ci-dessus dans laquelle toutes les algèbres de quaternions (respectivement toutes sauf une) sont déployées. En indice 4 en revanche, Sivatski [10, Prop. 5] a montré qu'il n'est pas toujours possible de trouver une décomposition dans laquelle une des algèbres de quaternions est déployée, le produit des deux autres étant alors à division. On peut cependant séparer les parties déployées et à division de l'algèbre, tout en respectant l'involution, à condition de ne pas se limiter aux décompositions en produit de quaternions. C'est l'objet du théorème qui suit:

Théorème 1. *Soit (A,σ) une F-algèbre centrale simple de degré 8 et d'indice au plus 4, munie d'une involution de type orthogonal. On suppose que (A,σ) est totalement décomposable. L'algèbre à involution (A,σ) admet une décomposition de la forme*

$$(A,\sigma) = (D,\theta)\otimes \mathrm{Ad}_{\langle\langle\lambda\rangle\rangle},$$

où (D,θ) est une algèbre à division de degré 4 munie d'une involution orthogonale, et $\mathrm{Ad}_{\langle\langle\lambda\rangle\rangle}$ désigne l'algèbre déployée $M_2(F)$ munie de l'involution adjointe à la forme quadratique $\langle 1,-\lambda\rangle$.

Remarque 2. Notons que (D,θ) est généralement indécomposable. Soit L/F l'extension quadratique étale donnée par le discriminant de θ. Il découle de [6, (15.7)] que $(D,\theta) = N_{L/F}(Q,\gamma)$ pour une certaine algèbre de quaternions Q sur L munie de son involution canonique, où $N_{L/F}$ désigne la corestriction. De plus, un calcul des invariants de (A,σ) permet de montrer que si elle est totalement décomposable, alors il existe $\mu\in L^\times$ tel que $\lambda = N_{L/F}(\mu)$. Ainsi, la décomposition de (A,σ) peut s'écrire:

$$(A,\sigma) = N_{L/F}(Q,\gamma)\otimes \mathrm{Ad}_{\langle\langle N_{L/F}(\mu)\rangle\rangle}.$$

2. Formes quadratiques de dimension 8 dans I^2F

La classification des formes quadratiques dont la classe de Witt appartient à une grande puissance de l'idéal fondamental IF de l'anneau de Witt est une question classique en théorie des formes quadratiques. Ainsi, outre le théorème d'Arason-Pfister qui décrit les formes de dimension au plus 2^n dont la classe de Witt appartient à I^nF, on trouve dans la littérature de nombreux résultats proposant, pour de petites valeurs de n, une description explicite des formes quadratiques de petite dimension dans I^nF.

On s'intéresse ici au résultat suivant dû à Izhboldin et Karpenko:

Théorème 3 ([5, Thm. 16.10]). *Soit φ une forme quadratique de dimension 8 sur F. Les propositions suivantes sont équivalentes:*

(i) Le discriminant de φ est trivial et son algèbre de Clifford est d'indice au plus 4.

(ii) Il existe une extension quadratique étale L/F et une forme ψ sur L, semblable à une 2-forme de Pfister, telles que $\varphi = \mathrm{tr}_\star(\psi)$, où $\mathrm{tr}_\star$ désigne le transfert associé à la trace de l'extension quadratique L/F (voir [7, VII.1.2] *pour une définition).*

Dans [8, §4], nous proposons une nouvelle démonstration de ce résultat, basée sur le théorème 1. Contrairement à la démonstration d'origine, nous n'utilisons pas la description donnée par Rost des formes quadratiques de dimension 14 dont la classe de Witt est dans I^3F [5, Rmk 16.11.2].

3. Invariant d'Arason pour les involutions orthogonales en degré 8

Soit (A,σ) une algèbre de degré 8 à involution orthogonale. On suppose que (A,σ) est totalement décomposable, c'est-à-dire par [6, (42.11)] que son discriminant est trivial et que son algèbre de Clifford a une composante déployée. Si A est déployée, ceci équivaut à dire que σ est l'adjointe d'une forme quadratique de dimension 8 qui est semblable à une 3-forme de Pfister; l'invariant d'Arason de cette forme est un invariant cohomologique de degré 3 de (A,σ).

On souhaiterait étendre fonctoriellement la définition de cet invariant au cas non déployé. Si l'algèbre A est à division, c'est impossible par [1, Thm 3.9]. Plus précisément, si F_A est un corps de déploiement générique de A, l'invariant $e^3(\sigma_{F_A}) \in H^3(F_A,\mu_2)$ est bien défini puisque A_{F_A} est déployée. Mais on peut montrer qu'il n'est jamais dans l'image du morphisme induit par l'extension des scalaires d'un quotient de $H^3(F,M)$ vers $H^3(F_A,\mu_2)$, et ce quel que soit le module de torsion M [11, §3.5].

Dans le cas où l'algèbre A n'est pas à division, en revanche, on peut définir un invariant e^3 de la manière suivante:

Théorème 4. *Soit (A,σ) une algèbre de degré 8 et d'indice au plus 4 à involution orthogonale. On suppose que (A,σ) est totalement décomposable, et on considère une décomposition*

$$(A,\sigma) = N_{L/F}(Q,\gamma) \otimes \mathrm{Ad}_{\langle\langle N_{L/F}(\mu)\rangle\rangle},$$

donnée par le Thm 1 (cf. rem 2). Alors

$$e^3(A,\sigma) = N_{L/F}((\mu)\cdot[Q]) \in H^3(F)/(F^\times \cdot [A]),$$

est un invariant de (A,σ).

En particulier, $e^3(A,\sigma)$ ne dépend pas de la décomposition choisie, qui pourtant n'est pas unique. Notons que cet invariant coïncide avec l'invariant e^3_{hyp} défini par Garibaldi à l'aide de l'invariant de Rost (cf. [3, (3.4)]). En effet, tous deux sont des descentes de l'invariant d'Arason des formes quadratiques. On en déduit que l'invariant de Garibaldi est d'ordre 2 et non pas 4 dans la situation étudiée ici, ce qui découle également de [4, Thm 15.4].

Dans [9], nous étudions les propriétés de cet invariant. En particulier, $e^3(A,\sigma)$ est nul si et seulement si (A,σ) est hyperbolique. En revanche, ce n'est pas un invariant classifiant. Enfin, il découle de Sivatski [10, Prop 2] que $e^3(A,\sigma)$ est représenté par un symbole si et seulement si (A,σ) admet une décomposition comme dans le Thm 1 avec (D,θ) décomposable.

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The Procesi-Schacher conjecture and Hilbert's 17th problem for algebras with involution

Thomas Unger
(joint work with Igor Klep)

Hilbert's 17th problem ("Is every nonnegative real polynomial a sum of squares of rational functions?") was solved in the affirmative by Artin in 1927. Starting with Helton's seminal paper [1], in which he proved that every positive semidefinite real or complex noncommutative polynomial is a sum of hermitian squares of *polynomials*, variants of Hilbert's 17th problem in a *noncommutative* setting have become a topic of current interest with wide-ranging applications. These results are often functional analytic in flavour.

A general framework to deal with this type of problem in an algebraic setting is the theory of central simple algebras with an involution. Let F be a formally real field with space of orderings X_F. Procesi and Schacher [3] consider central simple algebras A, equipped with an involution σ, which is *positive* with respect to an ordering $P \in X_F$ in the sense that its involution trace form $T_\sigma(x) := \mathrm{Trd}(\sigma(x)x)$, $\forall x \in A$ is positive semidefinite with respect to P. An ordering $P \in X_F$ for which σ is positive is called a *σ-ordering.* Let

$$X_F^\sigma := \{P \in X_F \mid P \text{ is a } \sigma\text{-ordering}\}.$$

An element $a \in \mathrm{Sym}(A,\sigma)$ is *σ-positive for $P \in X_F^\sigma$* if the scaled involution trace form $\mathrm{Trd}(\sigma(x)ax)$ is positive semidefinite with respect to $P \in X_F^\sigma$ and *totally σ-positive* if it is σ-positive for all $P \in X_F^\sigma$. It should be noted that Weil made a very detailed study of positive involutions [4]. For $x \in A$, $\sigma(x)x$ is called a *hermitian square.*

Procesi and Schacher prove the following result, which can be considered as a noncommutative analogue of Artin's solution to Hilbert's 17th problem:

Theorem 1. [3, Theorem 5.4] *Let A be a central simple F-algebra with involution σ. Let $T_\sigma \simeq \langle \alpha_1, \dots, \alpha_m \rangle$ (with $\alpha_1, \dots, \alpha_m \in F$) and let $a \in \mathrm{Sym}(A,\sigma)$. The following statements are equivalent:*

(i) *a is totally σ-positive;*

(ii) *there exist $x_{i,\varepsilon} \in A$ with*

$$a = \sum_{\varepsilon \in \{0,1\}^m} \alpha^\varepsilon \sum_i \sigma(x_{i,\varepsilon})x_{i,\varepsilon}.$$

(Here α^ε denotes $\alpha_1^{\varepsilon_1} \cdots \alpha_m^{\varepsilon_m}$.)

In the case $\deg A = 2$ (i.e., quaternion algebras with arbitrary involution), Procesi and Schacher show that the weights α_j are superfluous. They conjecture that this is also the case for $\deg A > 2$:

Conjecture. [3, p. 404] *In a central simple algebra A with involution σ, every totally σ-positive element is a sum of hermitian squares.*

We show that the Procesi-Schacher conjecture is false in general. An elementary counterexample can already be obtained in degree 3:

Theorem 2. [2, Theorem 3.2] *Let F_0 be a formally real field and let $F = F_0(X,Y)$. Let $A = M_3(F)$ and $\sigma = \mathrm{ad}_q$, where $q = \langle X, Y, XY \rangle$. The ($\sigma$-symmetric) element XY is totally σ-positive, but is not a sum of hermitian squares in (A,σ).*

For split central simple algebras equipped with the transpose (orthogonal) or conjugate transpose (unitary) involution the conjecture is true, which has been known since the 1970s. For symplectic involutions, we show that the conjecture is true for split, but false for non-split central simple algebras. The conjecture is also false for non-split central simple algebras with unitary involution.

For more details we refer to [2], where we also apply the results of Procesi and Schacher to study non-dimensionfree positivity of noncommutative polynomials. Our *Positivstellensatz* [2, Theorem 5.4] roughly says that a noncommutative polynomial all of whose evaluations in $n \times n$ matrices (for *fixed* n) are positive semidefinite, is a sum of hermitian squares with denominators and weights.

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On Grothendieck–Serre's conjecture concerning principal G-bundles over reductive group schemes, II

Ivan Panin

A well-known conjecture due to J.-P. Serre and A. Grothendieck [Se, Remarque, p.31], [Gr1, Remarque 3, p.26-27], and [Gr2, Remarque 1.11.a] asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme G over R the map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion of R into K, has trivial kernel.

Let R be a semi-local ring and G be a reductive group scheme over R. Recall an "isotropy condition" for G.

(**I**) *Each R-simple component of the derived group G_{der} of the group G is isotropic over R.*

The hypothesis (**I**) means more precisely the following : consider the derived group $G_{der} = [G, G]$ of the group G and the simply-connected cover G^{sc}_{der} of G_{red}; that group G^{sc}_{der} is a product over R (in a unique way) of R-indecomposable groups, which are required to be isotropic.

Theorem 1. *Let R be regular semi-local domain containing an infinite perfect field or the semi-local ring of finitely many points on a smooth irreducible k-variety X over an infinite field k. Let K be the fraction field of R. Let G be a reductive R-group scheme satisfying the condition* (**I**). *Then the map*

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion R into K, has trivial kernel.

In other words, under the above assumptions on R and G each principal G-bundle P over R which has a K-rational point is itself trivial.

• Clearly, this Theorem extends as the geometric case of the main result of J.-L. Colliot-Thélène and J.-J. Sansuc [C-T/S], so main results of I.Panin, A.Stavrova and N.Vavilov [PSV, Thm.1.1, Thm. 1.2]. However our proof of Theorem 1 **is based heavily on those results and on two purity results** proven in the present talk (see [Pa2, Thm.1.0.3, Thm.12.0.34]).

• The case of arbitrary reductive group scheme over a discrete valuation ring is completely solved by Y.Nisnevich in [Ni].

• The case when G is an arbitrary tori over a regular local ring is done by J.-L. Colliot-Thélène and J.-J. Sansuc in [C-T/S].

• For simple group schemes of **classical** series this result follows from more general results established by the first author, A. Suslin, M. Ojanguren and K. Zainoulline [PS], [OP1], [Z], [OPZ], [Pa1]. In fact, unlike our Theorem 1, **no isotropy hypotheses** was imposed there. However our result **is new, say for simple adjoint groups of exceptional type** (and for many others).

• The case of **arbitrary** simple adjoint group schemes of type E_6 and E_7 is done by the first author, V.Petrov and A.Stavrova in [PPS].

• There exists a folklore result, concerning type G_2. It gives affirmative answer in this case, also independent of isotropy hypotheses, see the paper by V. Chernousov and the first author [ChP].

• The case when the group scheme G comes from the ground field k is completely solved by J.-L. Colliot-Thélène, M. Ojanguren, M. S. Raghunatan and O. Gabber in [C-T/O], when k is perfect, in [R1]; O. Gabber announced a proof for a general ground field k.

For the outline of the case of simply-connected R-group scheme look at [PSV, Sect.2].

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Grothendieck–Serre conjecture for adjoint groups of types E_6 and E_7

Anastasia Stavrova

(joint work with Ivan Panin, Viktor Petrov)

Assume that R is a semi-local regular ring containing an infinite perfect field, or that R is a semi-local ring of several points on a smooth scheme over an infinite field. Let K be the field of fractions of R. Let H be a strongly inner adjoint simple algebraic group of type E_6 or E_7 over R. We prove that the kernel of the map

$$\mathrm{H}^1_{\acute{e}t}(R,H) \to \mathrm{H}^1_{\acute{e}t}(K,H)$$

induced by the inclusion of R into K is trivial. This continues the recent series of papers [PaSV], [Pa] on the Grothendieck–Serre conjecture [Gr, Rem. 1.11].

Recall that a semi-simple algebraic group scheme G over a semi-local ring R is called *strongly inner*, if it is given by a cocycle coming from $\mathrm{H}^1_{\acute{e}t}(R,G^{sc})$, where G^{sc} is the corresponding simply connected group scheme. This is also equivalent to the triviality of all Tits algebras of G (cf. [PS]).

Theorem 1. *Let R be a semi-local domain. Assume moreover that R is regular and contains a infinite perfect field k, or that R is a semi-local ring of several points on a k-smooth scheme over an infinite field k. Let K be the field of fractions of R. Let H be an adjoint strongly inner simple group scheme of type E_6 or E_7 over R. Then the map*

$$\mathrm{H}^1_{\acute{e}t}(R,H) \to \mathrm{H}^1_{\acute{e}t}(K,H)$$

induced by the inclusion of R into K has trivial kernel.

The proof of this Theorem is based, firstly, on the main result of [Pa] which implies that, under the same assumptions on R, the Grothendieck–Serre conjecture holds for any isotropic simple algebraic group scheme G over R. This topic was addressed by Ivan Panin in his talk.

The second ingredient of the proof is the following lemma, which says, roughly speaking, that a (possibly anisotropic) strongly inner simple group scheme of type E_6 or E_7 can be embedded into a larger isotropic simple group scheme. This lemma follows directly from [PS, Th. 2].

Note that by the *type* of a parabolic subgroup of a semi-simple group scheme over a semi-local domain we mean the set of vertices of its Dynkin diagram that constitutes the Dynkin diagram of a Levi subgroup of this parabolic. Our numbering of Dynkin diagrams follows [B].

Lemma. *Let R be a semi-local domain, and let H be a strongly inner adjoint simple group scheme of type E_6 (respectively, E_7) over R. There exists an inner* adjoint *simple group scheme G of type E_7 (respectively, E_8) over R, together with a maximal parabolic subgroup P of type $\{1,2,3,4,5,6\}$ (respectively, $\{1,2,3,4,5,6,7\}$), such that H is isomorphic to the quotient $L/\mathrm{Cent}(L)$ for a Levi subgroup L of P.*

The speaker is much indebted to Jean-Pierre Serre who has pointed out a hole in the proof of the Theorem presented in her talk (now fixed).

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Algebraic structures arising from isotropic groups

Viktor Petrov

(joint work with Anastasia Stavrova)

It is well-known that in many cases reductive groups can be presented as automorphism groups of certain algebraic structures like Azumaya algebras with involutions, Jordan algebras, Cayley algebras or Brown algebras. Such descriptions may be helpful in studying homogeneous varieties or orbits of a group in a representation. We provide a unified framework for these descriptions. Our approach also sheds light on the structure of isotropic groups, their constructions like appearing in the Freudenthal-Tits magic square and so on.

Let's start with an isotropic semisimple adjoint group G over a commutative ring R. Let P be a proper parabolic subgroup of G, L be a Levi subgroup of P; denote by S the maximal split torus in $Cent(L)$. The action of S on the Lie algebra $Lie(G)$ determines a decomposition

$$Lie(G) = Lie(L) \oplus \bigoplus_{A \in \Phi_P} V_A,$$

where V_A are some projective modules parametrized by a finite set Φ_P of characters of S (called the *relative roots*). Assume for simplicity that the type of P appears in the list of Tits indices (actually a weaker purely combinatorial assumption suffices, see [PSt2]); this guarantees that Φ_P is indeed a root system (possibly non-reduced, that is of type BC_r; in this case we assume additionally that $2 \in R^\times$).

$Lie(G)$ naturally lives inside the Hopf algebra $Dist(G)$ of distributions on G (also known as the *hyperalgebra* of G). One can show that over each $v \in V_A$ there exists a unique sequence of *divided powers* $v^{(0)}$, $v^{(1)} = v$, $v^{(2)}$, $v^{(3)}$, ... This means that

$$\Delta(v^{(k)}) = \sum_{i=0}^{k} v^{(i)} \otimes v^{(k-i)}$$

for each k.

Now for each A, $B \in \Phi_P$ and k such that $kA + B \in \Phi_P$ we can define the operation $ad^{(k)}\colon V_A \times V_B \to V_{A+kB}$ by the formula

$$ad_v^{(k)}(u) = \sum_{i=0}^{k} v^{(i)} u(-v)^{(k-i)},$$

which is of degree k in v and linear in u. In the case $\Phi_P = BC_1$ we also need to consider trilinear operations $V_A \times V_{-A} \times V_A \to V_A$ given by the double commutator. We call the system $(V_A)_{A\in\Phi_P}$ equipped with these operation the *Chevalley structure* of G with respect to P. The case $\Phi_P = A_1$ leads to the definition of *Jordan pairs* and was considered by Loos [L] and Faulkner [F]. In the case $\Phi_P \neq BC_r$ the system $\mathcal{V} = (V_A)_{A\in\Phi_P}$ but with operations $ad^{(1)}$ only appeared in Zelmanov's work [Z] under the name of a *Jordan system*.

The Chevalley structures can be characterized axiomatically, and G can be completely recovered from its Chevalley structure. Namely, L is isomorphic to the connected component of $Aut(\mathcal{V})$, and there are relations involving the elements of the unipotent radicals of P and the opposite parabolic subgroup P^-, similar to the Chevalley commutator formula but expressed in terms of $ad^{(k)}$ (cf. [PSt]; finding these relations was our original motivation). The group scheme G can be obtained then via an appropriate generalization of Kostant's construction of Chevalley groups [Ko].

Further, any simple group H of type distinct from E_8, F_4 and G_2 with trivial Tits algebras is isogeneous to some Levi subgroup L [PSt2], so we obtain descriptions of reductive groups as automorphism groups. Simple groups of types F_4, G_2, E_6 and the classical groups appear as subgroups in $Aut(\mathcal{V})$ stabilizing certain elements in V_A.

It's interesting to compare Chevalley structures arising from different parabolic subgroups. If $Q \leq P$ is another parabolic subgroup contained in P with a Levi subgroup $M \leq L$, and $(W_B)_{B\in\Phi_Q}$ is the corresponding structure, then each V_A decomposes as a direct sum $\bigoplus_{B\in\pi^{-1}(A)} W_B$, where $\pi\colon \Phi_Q \to \Phi_P$ is the natural projection map, and the operations in $(V_A)_{A\in\Phi_P}$ can be recovered from the operations in $(W_B)_{B\in\Phi_Q}$. This decomposition can be thought of as a generalized

Pierce decomposition. Many constructions of algebraic structures can be interpreted this way. Say, Freudenthal's construction of a cubic Jordan algebra from an octonion algebra corresponds to the projection $F_4 \to G_2$, the construction of a Freudenthal triple system from a cubic Jordan algebra corresponds to the projection $G_2 \to BC_1$, and so on.

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Gersten-Witt complexes and Steenrod squares mod 2

Baptiste Calmès
(joint work with Jean Fasel)

All schemes are over a fixed base field k and the letter W means Witt group. Let l be a line bundle on a smooth scheme X. There is a Gersten-Witt complex

$$\cdots \to \bigoplus_{x\in X^{(i)}} W\big(k(x)\big) \to \bigoplus_{x\in X^{(i+1)}} W\big(k(x)\big) \to \cdots$$

where the differentials depend on l (for a precise definition, see [1]). It is interesting because it is a row of the first page of a spectral sequence converging to the Witt groups $W^i(X, l)$. However, in general, its cohomology can be very hard to compute. The purpose of this talk is to explain how to compute the cohomology of the Gersten-Witt complex for a smooth scheme X with a regular cellular filtration, first in terms of cellular cohomology with coefficients in the Witt group of the field k and then in terms of a *Steenrod complex* defined using the Steenrod square operation on Chow groups of X modulo 2. As an example, we show how to compute the cohomology of this Steenrod complex in combinatorial terms for projective homogeneous varieties.

First of all, a *regular cellular filtration* on X is a sequence of embedded closed subsets such that the open complements of such a subset in the next one is an affine space, called an open cell, and the closure of such an open cell in X is a union of (smaller) open cells. This is a common condition in topology and, for example, it is satisfied for the cellular filtration given by Bruhat decomposition on a split projective homogeneous variety. For such a regular cellular filtration, we define a cellular Witt complex

$$\cdots \to \bigoplus_{\mathrm{codim} U = d} W(k) \to \bigoplus_{\mathrm{codim} U = d+1} W(k) \to \cdots$$

in an analogous way to the cellular complex in topology, except that the differentials depend on the line bundle l (the U's run over the open cells).

Theorem 1. *For any smooth scheme X with a line bundle l and a regular cellular filtration, there is a canoncial quasi-isomorphism from the cellular Witt complex to the Gersten-Witt complex. In particular, they have the same cohomology. If moreover the closed cells are normal, this cohomology is a free finitely generated $W(k)$-module.*

Let us now explain the Steenrod complex. We consider $Ch^*(X)$, the Chow groups of X modulo 2. The Steenrod square operation $\mathrm{Sq} : Ch^i(X) \to Ch^{i+1}(X)$ satisfies $\mathrm{Sq} \circ \mathrm{Sq} = 0$ and therefore defines a differential on $Ch^*(X)$. We can twist it by the class of a line bundle l in $Ch^1(X)$ and define $\mathrm{Sq}_l : x \mapsto \mathrm{Sq}(x) + x.l$. This still defines a differential. Thus, for any line bundle l, we are given a Steenrod complex

$$\cdots \overset{\mathrm{Sq}_l}{\to} Ch^i(X) \overset{\mathrm{Sq}_l}{\to} Ch^{i+1}(X) \overset{\mathrm{Sq}_l}{\to} \cdots$$

The cohomology of this complex is an $\mathbb{F}_2$-vector space ($Ch^*(X)$ is a graded module over the Chow ring modulo 2 of k, which is $\mathbb{F}_2$). Now let k be an algebraically closed field. Then $W(k) = \mathbb{F}_2$ as a ring.

Theorem 2. *Assume X is a smooth scheme with a regular cellular filtration, over an algebraically closed field k. Sending the generator of the copy of $W(k) = \mathbb{F}_2$ corresponding to the open cell U (in the cellular Witt complex) to the class of the closure of U in $Ch^*(X)$ defines a isomorphism of complexes. In particular the Steenrod complex and the cellular complex have the same cohomology.*

Combining the two theorems for smooth schemes with a regular cellular filtration with normal closed cells, we get a description of the cohomology of the Gersten-Witt complex: it is free over $W(k)$, and the rank can be detected by going to the algebraic closure and computing the cohomology of the Steenrod complex.

Let us now see what this gives for a projective homogeneous variety G/P where G is a split semi-simple simply connected linear algebraic group and P is a parabolic subgroup. As mentionned above, G/P has a regular cellular filtration and the closed cells are known to be normal. Everything can be computed using the usual combinatorics of root systems, which mainly involve the Weyl group of the root system of G (with respect to a given maximal torus) and the Bruhat order on it (with respect to the choice of a Borel subgroup containing the torus).

The Chow groups of G/P have a $\mathbb{Z}$-basis in terms of Schubert varieties, which are exactly the closures of the open cells. They are indexed by a subset of elements of the Weyl group W (minimal elements in certain cosets depending on P). The differential Sq_l can then be described by the following formula:

$$\mathrm{Sq}_l(x^w) = \sum_{\alpha \in R^+(w)} (\langle \rho_l, \alpha^\vee \rangle + 1) x^{ws_\alpha}.$$

Let us explain the notation in this formula. First of all, x^w is the Schubert variety corresponding to w in the Weyl group, in such a way that the length of w is the codimension of the Schubert variety. The reflection s_α is the one associated to the positive root α. Then, $R^+(w)$ is a subset of the positive roots, namely the α's such that $w > ws_\alpha$ for the Bruhat order and the length of ws_α is exactly one more than the length of w. The vector ρ_l lives in the vector space of the root system and is a sum of some fundamental weights (the ones not in l when one identifies $Ch^1(X)$ of G/P with the set of subsets of the set of fundamental weights not in P). The bilinear form $\langle -, \alpha^\vee \rangle$ comes with the root system. Our proof of this formula is not completely obvious and requires the use of desingularizations of the Schubert varieties.

Anyway, the punch-line is that this formula is completely combinatorial, and that for any given G/P, it is explicit enough to describe the Steenrod complex and find the ranks of its cohomology groups in each degree. To give a very simple example, if G is SL_3, which is of type A_2, and P is the Borel, there are four possible line bundles l modulo 2. For the trivial line bundle, the cohomology of the Steenrod complex (and thus of the Gersten-Witt complex) has rank 1 in degrees 0 and 3 and is zero elsewhere. For all other line bundles, the cohomology is 0.

In fact, we hope to prove that for a reasonable class of cellular complexes, the Gersten-Witt spectral sequence degenerates in page 2 and therefore that the cohomology of the Gersten-Witt complex exactly computes the Witt groups. We expect that it is the case for split projective homogeneous varieties. We are able to prove it in an important number of cases, but not in general, yet. In the example of SL_3/B mentioned above, the Witt groups are indeed zero for all non-trivial line bundles since the page 2 of the spectral sequence is already 0.

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Gerbes over henselian discrete valuation rings

Patrick Brosnan

(joint work with Zinovy Reichstein, Angelo Vistoli)

The talk concerns a result about gerbes used in the theory of essential dimension. I begin by reminding the reader what essential dimension is following [1].

Suppose $\mathcal{X}$ is is a stack over a field k. Suppose L and M are field extension of k and a is an object in $\mathcal{X}(L)$. We say that a is *defined over* M if there is an embedding $\sigma : M \to L$ and an object $b \in \mathcal{X}(M)$ such that a is isomorphic to to the base change of b to L via σ. The essential dimension $\operatorname{ed} a$ of a is the infimum of the transcendence degrees of M over k such that a is defined over M. The essential dimension $\operatorname{ed}\mathcal{X}$ (or sometimes $\operatorname{ed}\mathcal{X}/k$) of $\mathcal{X}$ over k is the supremum taken over all $a \in \mathcal{X}(L)$ for all fields L over k of $\operatorname{ed} a$.

For example, if $\mathcal{X} = BG$ for G an algebraic group the $\operatorname{ed}\mathcal{X}$ is the essential dimension of the group G as studied by Buhler and Reichstein [3, 4].

In [2], Reichstein, Vistoli and I computed the essential dimension of the moduli stacks $\mathcal{M}_{g,n}$ of genus g curves with n marked points. The main theorem that allowed us to do this was the following result, which we have called the *genericity theorem*.

Theorem 1. *Let $\mathcal{X}$ be a smooth, connected, Deligne-Mumford stack with finite inertia. Let X be the course moduli space (whose existence is a result of Keel-Mori) with function field $K(X)$. Let $\mathcal{G} = \mathcal{X} \times_X \operatorname{Spec} K(X)$ denote the generic gerbe of $\mathcal{X}$. Then*

$$\operatorname{ed}\mathcal{X}/k = \operatorname{ed}\mathcal{G}/K(X) + \dim X.$$

It turns out that we can do slightly more when $\mathcal{X}$ is itself an étale gerbe over X. In fact, if we prove the theorem in a slightly stronger form for $\mathcal{X}$ a gerbe over a discrete valuation ring, then we can use it together with some essentially geometric reasoning and a devissage to obtain the genericity theorem in general.

Before stating our main result for gerbes let me recall that a Deligne-Mumford stack $\mathcal{X}$ is called *tame* if, for each geometric point $\xi : \operatorname{Spec}\Omega \to \mathcal{X}$, the characteristic of Ω does not divide the order of the inertia of ξ. For example, the stack $B\mathbf{Z}/p$ is tame over $\operatorname{Spec}\mathbf{Q}$, but not over $\operatorname{Spec}\mathbf{Z}_p$.

Theorem 2. *Let $\mathcal{X}$ be a tame étale gerbe over a discrete valuation ring R. Let s denote the closed point of* $\operatorname{Spec} R$ *and η the generic point. Then*

$$\operatorname{ed}\mathcal{X}_s/k(s) \le \operatorname{ed}\mathcal{X}_\eta/k(\eta).$$

Furthermore, if R is an equicharacteristic complete discrete valuation ring, the we have equality, i.e., $\operatorname{ed}\mathcal{X}_s/k(s) = \operatorname{ed}\mathcal{X}_\eta/k(\eta)$.

To prove Theorem 2, the first thing to notice is that R can be replaced by a complete discrete valuation ring for the entire theorem. This is possible because completing R does not change the residue field and extends the fraction field, and essential dimension does not increase under base change. So from now on the main techniques of the proof will work for gerbes over henselian discrete valuation rings.

Once this reduction is made, one way to prove Theorem 2 is to directly relate the sections of the gerbe over $k(\eta)$ with those over $k(s)$. Now if R is henselian, we have an isomorphism $\mathcal{X}(R) \to \mathcal{X}(k(s))$. Coupled with the restriction functor $\mathcal{X}(R) \to \mathcal{X}(k(\eta))$ this gives rise to a functor $r : \mathcal{X}(k(s)) \to \mathcal{X}(k(\eta))$. In fact, this functor can always be split. That is there is a functor $\sigma : \mathcal{X}(k(\eta)) \to \mathcal{X}(k(s))$ such that $r \circ \sigma$ is isomorphic to the identity. In essence, this splitting σ comes from the fact that the inertia exact sequence

$$1 \to I \to \mathrm{Gal}(k(\eta)) \to \mathrm{Gal}(k(s) \to 1$$

can always be split. Now being able to split the functor $r : \mathcal{X}(k(s)) \to \mathcal{X}(k(\eta))$ is not enough to prove Theorem 2. What is needed is essentially needed splitting for every unramified field extension of R which commute with each other. In general it seems that this cannot be achieved. In fact, it seems that the inertia exact sequences themselves cannot be split in a coherent manner. However, if we consider only tame stacks then we only need to consider the tame inertia I_t. That is we, in that case, the choice of roots of a uniformizing parameter splits the tame inertia exact sequences in a coherent way. We obtain the following result which has Theorem 2 as a fairly direct consequence.

Lemma 3. *Let $h : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ be an unramified extension of henselian traits and let $\mathcal{X}$ be a tame étale gerbe over A. Write $j_B : \{\eta_B\} \to B$ (resp. $j_A : \{\eta_A\} \to A$) for the inclusion of the generic points. Then it is possible to find functors $\sigma_B : \mathcal{X}(k(\eta_B)) \to \mathcal{X}(B)$ and $\sigma_A : \mathcal{X}(k(\eta_A)) \to \mathcal{X}(A)$ such that the diagram*

$$\begin{array}{ccccc} \mathcal{X}(B) & \xrightarrow{r} & \mathcal{X}(k(\eta_B)) & \xrightarrow{\sigma_B} & \mathcal{X}(B) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{X}(A) & \xrightarrow{r} & \mathcal{X}(k(\eta_A)) & \xrightarrow{\sigma_A} & \mathcal{X}(A) \end{array}$$

commutes (up to natural isomorphism) and the horizontal compositions are isomorphic to the identity. (Here the upward arrows are the obvious base change functors.)

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Motivic invariants of algebraic groups

Nikita Semenov

One of the main problems in the theory of linear algebraic groups over an arbitrary field is their classification. For this purpose one constructs invariants and tries first to classify them.

The known invariants can be subdivided into the following types:

(1) combinatorial invariants (Dynkin diagram, Tits diagram, ...);
(2) cohomological invariants (discriminant of quadratic forms, Tits algebras, Rost invariant, ...);
(3) motivic invariants.

Let k be a field and G a simple linear algebraic group over k. Consider the variety X of Borel subgroups of G. This is a smooth projective variety defined over k. The following holds:

Theorem 1 (Petrov, Semenov, Zainoulline)**.** *Let p be a prime number. Consider the Chow-motive $\mathcal{M}(X)$ of X with $\mathbb{Z}/p$-coefficients.*

There exists a unique indecomposable direct summand R of $\mathcal{M}(X)$ such that its Poincaré polynomial equals:

$$P(R,t)=\prod_{i=1}^{r}\frac{t^{d_i p^{j_i}}-1}{t^{d_i}-1}\in\mathbb{Z}[t]$$

for some integers r, d_i, j_i.

The assignment $G\mapsto R$ is an example of a motivic invariant.

Next one can show the following:

Theorem 2. *Let Y be a smooth projective irreducible variety of dimension $n>0$ over a field k with $\operatorname{char} k=0$. Assume that Y has no zero-cycles of odd degree and let M be a direct summand of the Chow-motive of Y with $\mathbb{Z}/2$-coefficients such that $M\otimes\mathcal{M}(Y)\simeq\mathcal{M}(Y)\oplus\mathcal{M}(Y)\{n\}$.*

Then $n=2^{s-1}-1$ for some s and there exists a functorial invariant $u\in H^s(k,\mathbb{Z}/2)$ such that for any field extension K/k we have: $u_K=0$ iff Y_K has a zero-cycle of odd degree.

Assume now that $p=2$, $j_l=1$ for some l and all other $j_i=0$. This happens, e.g., for the anisotropic (compact) group of type E_8 over $\mathbb{R}$. Then the motive R of the 1st theorem satisfies $R\otimes\mathcal{M}(X)\simeq\mathcal{M}(X)\oplus\mathcal{M}(X)\{d_l\}$.

Combining the above results we obtain a positive solution of a problem posed by J.-P. Serre in 1999:

Theorem 3. *Let G be a group of type E_8 defined over $\mathbb{Q}$ such that $G_{\mathbb{R}}$ is compact. Then for any field extension K/k we have: G_K splits iff $(-1)^5\in H^5(K,\mathbb{Z}/2)$ is zero.*

Notice that the implication $\Leftarrow$ was shown by M. Rost in 1999.

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Incompressibility of generalized Severi-Brauer varieties

Nikita Karpenko

This talk exposes results of [6].

A smooth complete irreducible variety X over a field F is said to be *incompressible*, if any rational map $X \dashrightarrow X$ is dominant.

An important (at least by the amount of available applications, see, e.g., [8], [4], or [1]) example of an incompressible variety is as follows. Let p be a positive prime integer. Let D be a central division F-algebra of degree a power of p, say, p^n. For any integer i, the *generalized Severi-Brauer variety* $X(i; D)$ is the F-variety of right ideals in D of reduced dimension i. The variety $X(1; D)$ (the usual Severi-Brauer variety of D) is incompressible.

In fact, the variety $X = X(1; D)$ has a stronger property – *p-incompressibility*. The original proof, given in [7] and [8], makes use of Quillen's computation of K-theory of Severi-Brauer varieties. A recent proof, given in [10], makes use of Steenrod operations on the modulo p Chow groups (it better explains the reason of the p-incompressibility but works only over fields of characteristic $\neq p$). We give a third, particularly simple proof.

One of the main results of the present talk is the incompressibility theorem which affirms that the variety $X(p^m; D)$ is p-incompressible also for $m = 1, \dots, n-1$ (in the case of $p = 2$ and $m = n-1$ this was shown earlier by Bryant Mathews [9] using a different method). The remaining results of this talk are either obtained on the way to the main result or are quite immediate consequences of it.

We start in the general context of an arbitrary semisimple affine algebraic group G of inner type over a field F. Let T be the set of the conjugacy classes of the maximal parabolic subgroups in $\bar{G} = G_{\bar{F}}$ (T can be identified with the set of vertices of the Dynkin diagram of G). To each $\tau \subset T$, a projective G-homogeneous F-variety X_τ is associated in a standard way. We define an indecomposable motive M_τ in the category of Chow motives with coefficients in $\mathbb{F}_p$, as the summand in the complete motivic decomposition of X_τ such that the 0-codimensional Chow group of M_τ is non-zero. We show that the motive of any finite direct product of projective G-homogeneous F-varieties decomposes into a sum of shifts of the motives M_τ (with various τ). Therefore we solve the inner case of the following problem: Let $\mathfrak{X}_G$ be the class of all finite direct products of projective homogeneous varieties under an action of a semisimple affine algebraic group G. According to [3], the motive (still with $\mathbb{F}_p$ coefficients, p a fixed prime) of any variety in $\mathfrak{X}_G$ decomposes and in a unique way in a finite direct sum of indecomposable motives. The problem is to describe the indecomposable summands which appear this way.

Our method can also be applied to the groups of outer type. Since we are mainly interested in the inner type A and for the sake of simplicity, we do not work with groups of outer type in this paper.

For $G = \operatorname{Aut} D$, the above result translates as follows. To each integer $m = 0, 1, \ldots, n$, an indecomposable motive $M_{m,D}$ in the category of Chow motives with coefficients in $\mathbb{F}_p$ is associated. This is the summand in the complete motivic decomposition of the variety $X(p^m; D)$ such that the 0-codimensional Chow group of $M_{m,D}$ is non-zero. The motive of any variety in $\mathfrak{X}_D := \mathfrak{X}_G$ decomposes into a sum of shifts of the motives $M_{m,D}$ (with various m).

With this in hand, we prove two structure results concerning the motives $M_{m,D}$. We show that the d-dimensional Chow group of $M_{m,D}$, where $d = \dim X(p^m; D)$, is also non-zero. This result is equivalent to the p-incompressibility of the variety $X(p^m; D)$, so that we get the incompressibility theorem at this point. The second structure result on the motive $M_{m,D}$ is a computation of the p-adic valuation of its rank. In fact, we can not separate the proofs of these two structure results. We prove them simultaneously by induction on $\deg D$ (and using the decomposition theorem).

An immediate consequence of the incompressibility theorem is as follows. We recall the notion of *canonical dimension at* p (or *canonical* p*-dimension*) $\operatorname{cdim}_p(X)$ of a smooth complete irreducible algebraic variety X. This is a certain non-negative integer satisfying $\operatorname{cdim}_p X \leq \dim X$; moreover, $\operatorname{cdim}_p X = \dim X$ if and only if X is p-incompressible. In particular, by our main result, $\operatorname{cdim}_p X(p^m; D) = \dim X(p^m; D) = p^m(p^n - p^m)$. The canonical dimension at p of any variety in $\mathfrak{X}_A := \mathfrak{X}_{\operatorname{Aut} A}$, where A is an arbitrary central simple F-algebra, can be easily computed in terms of $\operatorname{cdim}_p X(p^m; D)$, where D is the p-primary part of a division algebra Brauer-equivalent to A.

In spite of a big number of obtained results, one may say that (the motivic part of) this paper raises more questions than it answers. Indeed, although we show that the motives of the varieties in $\mathfrak{X}_G$ decompose into sums of shifts of M_τ (and find a restriction on τ in terms of a given variety), we do not precisely determine this decomposition (even for G simple of inner type A): we neither know how many copies of M_τ (for a given τ and a given variety) do really appear in the decomposition, nor do we determine the shifting numbers. Moreover, the understanding of the structure of the motives M_τ themselves, which we provide for G simple of inner type A (the case of our main interest here), is not satisfactory. It may happen that $M_{m,D}$ is always the whole motive of the variety $X(p^m, D)$ (that is, the motive of this variety probably is indecomposable): we do not possess a single counter-example. In fact, the variety $X(p^m, D)$ *is* indecomposable for certain values of p, n, and m. Two cases are known for a long time: $m = 0$ (the Severi-Brauer case) and $m = 1$ with $p = 2 = n$ (reducing the exponent of D to 2, we come to the case of an Albert quadric here). We get a generalization of the Albert case. The other values of p, n, m should be studied in this regard.

But the qualitative analysis is done (for instance, the properties of $M_{m,D}$, we establish, show that this motive behaves essentially like the whole motive of the variety $X(p^m, D)$ even if it is "smaller"). And the proofs are not complicated. Combinatorics or complicated formulas do not show up at all, in particular, because we (can) neglect the shifting numbers of motivic summands in most places. The results we are getting this way are less precise but, as we believe, they contain the essential piece of information. They can be (and are) applied (in [5]) to prove the hyperbolicity conjecture on orthogonal involutions.

We conclude the introduction by some remarks on the motivic category we are using. First of all, the category of Chow motives with coefficients in $\mathbb{F}_p$ (or, slightly more general, with coefficients in a finite connected commutative ring Λ), in which we are working in this paper, can be replaced by a simpler category. This simpler category is constructed in exactly the same way as the category of Chow motives with the only difference that one kills the elements of Chow groups which vanish over some extension of the base field. Working with this simpler category, we do not need the nilpotence tricks (the nilpotence theorem and its standard consequences) anymore. This simplification of the motivic category does not harm to any external application of our motivic results. So, this is more a question of taste than a question of necessity that we stay with the usual Chow motives.

On the other hand, somebody may think that our category of usual Chow motives is not honest or usual enough because these are Chow motives with coefficients in $\mathbb{F}_p$ and not in $\mathbb{Z}$. Well, there are at least three arguments here. First, decompositions into sums of indecomposables are not unique for coefficients in $\mathbb{Z}$, even in the case of projective homogenous varieties of inner type A (see [3, Example 32] or [2, Corollary 2.7]). Therefore the question of describing the indecomposables does not seem so reasonable for the integral motives. Second, any decomposition with coefficients in $\mathbb{F}_p$ lifts (and in a unique way) to the coefficients $\mathbb{Z}/p^n\mathbb{Z}$ for any $n \geq 2$, [11, Corollary 2.7]. Moreover, it also lifts to $\mathbb{Z}$ (non-uniquely this time) in the case of varieties in $\mathfrak{X}_G$, where G is a semisimple affine algebraic group of inner type for which p is the unique torsion prime, [11, Theorem 2.16]. And third, may be the most important argument is that the results on motives with coefficients in $\mathbb{F}_p$ are sufficient for the applications.

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Excellent connections in the motives of quadrics

Alexander Vishik

Let Q be smooth projective quadric of dimension n over the field k of characteristic not 2, and $M(Q)$ be its *motive* in the category $Chow(k)$ of Chow motives over k (see [8], or Chapter XII of [1]). Over the algebraic closure $\overline{k}$, our quadric becomes completely split, and so, cellular. This implies that $M(Q|_{\overline{k}})$ becomes isomorphic to a direct sum of Tate motives:

$$M(Q|_{\overline{k}}) \cong \oplus_{\lambda\in\Lambda(Q)}\mathbb{Z}(\lambda)[2\lambda],$$

where $\Lambda(Q) = \Lambda(n)$ is $\{i|\, 0 \leq i \leq [n/2]\} \coprod \{n-i|\, 0 \leq i \leq [n/2]\}$. But over the ground field k our motive could be much less decomposable. The *Motivic Decomposition Type* invariant $MDT(Q)$ measures what kind of decomposition we have in $M(Q)$. Any direct summand N of $M(Q)$ also splits over $\overline{k}$, and $N|_{\overline{k}} \cong \sum_{\lambda\in\Lambda(N)}\mathbb{Z}(\lambda)[2\lambda]$, where $\Lambda(N) \subset \Lambda(Q)$ (see [8] for details). We say that $\lambda, \mu \in \Lambda(Q)$ are *connected*, if for any direct summand N of $M(Q)$, either both λ and μ are in $\Lambda(N)$, or both are out. This is an equivalence relation, and it splits $\Lambda(Q) = \Lambda(n)$ into disjoint union of *connected components*. This decomposition is called the *Motivic Decomposition Type*. It interacts in a nontrivial way with the *Splitting pattern*, and using this interaction one proves many results about both invariants. The (absolute) *Splitting pattern* $\mathfrak{j}(q)$ of the form q is defined as an increasing sequence $\{\mathfrak{j}_0, \mathfrak{j}_1, \ldots, \mathfrak{j}_h\}$ of all possible Witt indices of $q|_E$ over all possible field extensions E/k. Then, the (relative) *Splitting pattern* $\mathfrak{i}(q)$ is defined as $\{\mathfrak{i}_0, \ldots, \mathfrak{i}_h\} := \{\mathfrak{j}_0, \mathfrak{j}_1 - \mathfrak{j}_0, \mathfrak{j}_2 - \mathfrak{j}_1, \ldots, \mathfrak{j}_h - \mathfrak{j}_{h-1}\}$.

Let us denote the elements $\{\lambda|\, 0 \leq \lambda \leq [n/2]\}$ of $\Lambda(n)$ as λ_{lo}, and the elements $\{n-\lambda|\, 0 \leq \lambda \leq [n/2]\}$ as λ^{up}. The principal result relating the splitting pattern and the motivic decomposition type claims that all elements of $\Lambda(Q)$ come in pairs whose structure depends on the splitting pattern.

Proposition 1. ([8, Proposition 4.10], cf. [1, Theorem 73.26]) *Let λ and μ be such that $\mathfrak{j}_{r-1} \leq \lambda, \mu < \mathfrak{j}_r$, where $1 \leq r \leq h$, and $\lambda + \mu = \mathfrak{j}_{r-1} + \mathfrak{j}_r - 1$. Then λ_{lo} is connected to μ^{up}.*

Consequently, any direct summand in the motive of anisotropic quadric consists of even number of Tate-motives when restricted to $\overline{k}$, in particular, of at least two Tate-motives. If it consists of just two Tate-motives we will call it *binary*. It can happen that $M(Q)$ splits into *binary motives*. As was proven by M.Rost and D.Hoffmann ([7, Proposition 4] and [2]), this is the case for *excellent* quadrics, and, hypothetically, it should be the only such case. The *excellent* quadratic forms introduced by M.Knebusch ([5]) are sort of substitutes for the Pfister form in dimensions which are not powers of two. Namely, if you want to construct such a form of dimension, say, m, you need first to present m in the form $2^{r_1} - 2^{r_2} + \ldots + (-1)^{s-1}2^{r_s}$, where $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geq 1$ (it is easy to see that such presentation is unique), and then choose pure symbols $\alpha_i \in \mathrm{K}^M_{r_i}(k)/2$ such that $\alpha_1 \vdots \alpha_2 \vdots \ldots \vdots \alpha_s$. Then the respective excellent form is an m-dimensional form $(\langle\langle\alpha_1\rangle\rangle - \langle\langle\alpha_2\rangle\rangle + \ldots + (-1)^{s-1}\langle\langle\alpha_s\rangle\rangle)_{an}$. In particular, if $m = 2^r$ one gets an r-fold Pfister form. It follows from the mentioned result that the only connections in the motives of excellent quadrics are binary ones coming from Proposition 1. At the same time, the experimental data suggested that in the motive of anisotropic quadric Q of dimension n we should have not only connections coming from the splitting pattern $\mathfrak{i}(Q)$ of Q but also ones coming from the excellent splitting pattern:

Conjecture 2. ([8, Conjecture 4.22]) *Let Q and P be anisotropic quadrics of dimension n with P-excellent. Then we can identify $\Lambda(Q) = \Lambda(n) = \Lambda(P)$, and for $\lambda, \mu \in \Lambda(n)$,*

$$\lambda, \mu \text{ connected in } M(P) \quad \Rightarrow \quad \lambda, \mu \text{ connected in } M(Q).$$

Partial case of this Conjecture, where λ and μ belong to the *outer excellent shell* (that is, $\lambda, \mu < \mathfrak{j}_1(P)$), was proven earlier and presented by the author at the conference in Eilat, Feb. 2004. The proof uses *Symmetric operations*, and the Grassmannian $G(1, Q)$ of projective lines on Q, and is a minor modification of the proof of [9, Theorem 4.4] (assuming $char(k) = 0$). Another proof using Steenrod operations and $Q^{\times 2}$ appears in [1, Corollary 80.13] (here $char(k) \neq 2$).

Now we can prove the whole conjecture for all field of characteristic different from 2.

Theorem 3. *Conjecture 2 is true.*

This Theorem shows that the connections in the motive of an excellent quadric are *minimal* among anisotropic quadrics of a given dimension. Moreover, for a given anisotropic quadric Q we get not just one set of such connections, but $h(Q)$ sets, where $h(Q)$ is a *height* of Q, since we can apply the Theorem not just to q but to $(q|_{k_i})_{an}$ for all fields $k_i, 0 \leq i < h$ from the *generic splitting tower of Knebusch* (see [4]). And the more splitting pattern of Q differs from the excellent splitting pattern, the more nontrivial conditions we get, and the more indecomposable $M(Q)$ will be.

As an application of this philosophy, we get the result bounding from below the rank of indecomposable direct summand in the motive of a quadric in terms of its dimension. For the direct summand N of $M(Q)$ let us denote as $rank(k)$ the cardinality of $\Lambda(N)$ (that is, the number of Tate-motives in $N|_{\overline{k}}$), as $a(N)$ and

$b(N)$ the minimal and maximal element in $\Lambda(N)$, respectively, and as $\dim(N)$ the difference $b(N) - a(N)$.

Theorem 4. *Let N be indecomposable direct summand in the motive of anisotropic quadric with* $\dim(N) + 1 = 2^{r_1} - 2^{r_2} + \ldots + (-1)^{s-1}2^{r_s}$*, where $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geq 1$. Then:*

(1) *$rank(N) \geq 2s$;*
(2) *For $1 \leq k \leq s$, let $d_k = \sum_{i=1}^{k-1}(-1)^{i-1}2^{r_i-1} + \varepsilon(k) \cdot \sum_{j=k}^{s}(-1)^{j-1}2^{r_j}$, where $\varepsilon(k) = 1$, if k is even, and $\varepsilon(k) = 0$, if k is odd. Then*

$$(a(N) + d_k)_{lo} \in \Lambda(N), \qquad \textit{and} \qquad (n - b(N) + d_k)^{up} \in \Lambda(N).$$

This is a generalisation of the binary motive Theorem ([6, Theorem 6.1], see also other proofs in [9, Theorem 4.4] and [1, Corollary 80.11]) which claims that the dimension of a binary direct summand in the motive of a quadric is equal to $2^r - 1$, for some r, and which has many applications in the quadratic form theory. Moreover, we describe which particular Tate-motives must be present in $N|_{\overline{k}}$ depending on the dimension on N. An immediate corollary of this is another proof of the Theorem of Karpenko (formerly known as the Conjecture of Hoffmann) describing possible values of the first higher Witt index of q in terms of $\dim(q)$.

Theorem 5. (N.Karpenko, [3]) *Let q be anisotropic quadratic form of dimension m. Then $(\mathfrak{i}_1(q) - 1)$ is a remainder modulo 2^r of $(m-1)$, for some $r < \log_2(n-1)$.*

But aside from the value of the first Witt index the Theorem 4 gives many other relations on higher Witt indices.

Another application is the characterisation of even-dimensional indecomposable direct summands in the motives of quadrics.

Theorem 6. *Let N be indecomposable direct summand in the motive of anisotropic quadric. Then the following conditions are equivalent:*

(1) *$\dim(N)$ is even;*
(2) *there exist i such that $(\mathbb{Z}(i)[2i] \oplus \mathbb{Z}(i)[2i])$ is a direct summand of $N|_{\overline{k}}$.*

The latter result, in particular, implies that in the motives of even-dimensional quadrics all the Tate-motives living in the shells with higher Witt indices 1 are connected among themselves.

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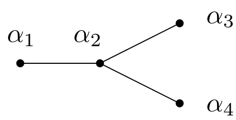
Kneser-Tits problem for trialitarian groups and bounded generation by restricted elements

Philippe Gille, Uzi Vishne

1. Introduction

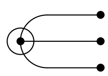
Let F be a field. Let F_s/F be a separable closure of F and denote by Γ_F the Galois group of F_s/F. We consider a semisimple group G/F of absolute type D_4 [9], whose root system can be depicted as



The automorphism group of this Dynkin diagram is S_3, hence G defines a class in $\mathrm{H}^1(F, S_3) = \mathrm{Hom}_{ct}(\Gamma_F, S_3)/S_3$, namely an isomorphism class of cubic étale algebras [4, §18]. If this cubic étale algebra, say K/F, is a field, we say that G is trialitarian. The following result answers the Kneser-Tits problem for those groups.

Theorem 1.1. [3, §6.1] *Let G/F be a semisimple simply connected trialitarian group. If G is isotropic, then the (abstract) group $G(F)$ is simple.*

Since $Z(G) = \ker(R_{K/F}(\mu_2) \to \mu_2)$, note that $Z(G)(F) = 1$. If G is quasi-split (for example in the case of finite fields), this is a special case of Chevalley's theorem [1]. By Tits tables for indices, the only other case to consider is that with Tits index

 .

In the number field case, this has been proven by G. Prasad and M.S. Raghunathan [6]. Our goal is to explain how this result follows from a general statement and how it applies together with Prasad's approach to a nice understanding of generators for the rational points of the anisotropic kernel of G.

2. Invariance under transcendental extensions

Assume for convenience that F is infinite. Let G/F be a semisimple connected group which is absolutely almost simple and isotropic. We denote by $G^+(F)$ the (normal) subgroup of $G(F)$ which is generated by the $R_u(P)(F)$ for P running over the F–parabolic subgroups of G. Tits showed that any proper normal subgroup of $G^+(F)$ is central [8] [10]. So for proving that $G(F)/Z(G)(F)$ is simple, the plan is to show the triviality of the Whitehead group

$$W(F,G) = G(F)/G^+(F).$$

This is the Kneser-Tits problem. Note that by Platonov's work, $W(F,G)$ can be non-trivial, e.g. for special linear groups of central simple algebras [5].

Theorem 2.1. [3, §5.3] *The map $W(F,G) \to W(F(t),G)$ is an isomorphism.*

Corollary 2.2. *If G/F is a F-rational variety, then $W(F,G) = 1$.*

Let us sketch the proof of the Corollary. The idea is to consider the generic element $\xi \in G(F(G))$. Since $F(G)$ is purely transcendental over F, it follows that $\xi \in G(F) \cdot G^+(F(G))$. Since $G^+(F)$ is Zariski dense in G, we can see by specialization that $\xi \in G^+(F(G))$. Therefore there exists an dense open subset U of G such that $U(F) \subset G^+(F)$. But $U(F) \cdot U(F) = G(F)$, thus $W(F,G) = 1$.

Assume now that G/F is trialitarian. Since Chernousov and Platonov have shown that such a group is an F-rational variety [2, §8], we thus conclude that $W(F,G) = 1$.

3. Bounded generation by restricted elements

We assume that $\mathrm{char}(F) \neq 2$ and for convenience that F is perfect and infinite. In [6], Prasad gives an explicit description of $W(F,G)$ in terms of the the Tits algebra of G, which is the Allen algebra $M_2(D)$ for D a quaternion division algebra over K satisfying $\mathrm{cor}_{K/F}[D] = 0 \in \mathrm{Br}(F)$, where K is a cubic étale extension of F. We have

$$W(F,G) = U/\langle R\rangle,$$

where U is the group of elements of the quaternion algebra D/K whose reduced norm is in $F^\times$, and R is the set of elements $x \neq 0$ for which both the reduced norm and the reduced trace are in F. Combined with Theorem 1.1, we get the

Corollary 3.1. *$\langle R\rangle = U$.*

This leaves open the question of bounding the number of generators from R required to express every element of U.

One may consider the same question when K is a cubic étale extension which is not a field, namely, $K = F \times L$ for L a quadratic field extension of F, or $K = F \times F \times F$, and D is an Azumaya algebra over K. In the former case, $D = D_1 \times D_2$ where D_1 is a quaternion algebra over F and D_2 a quaternion algebra over L, with $\mathrm{cor}_{L/F} D_2 \sim D_1$. In the latter, $D = D_1 \times D_2 \times D_3$, where D_i

($i = 1, 2, 3$) are quaternion algebras over F, and $D_1 \otimes_F D_2 \otimes_F D_3 \sim F$. The sets V and R can be defined in the same manner as above.

This is not an artificial generalization: extending scalars from F to $\tilde{F} = K$, the algebra becomes $\tilde{D} = D \otimes_F K$ which is an Azumaya algebra over $\tilde{K} = K \otimes_F K$, and $\tilde{K}$ is a cubic étale extension of $\tilde{F}$, which is not a field.

Theorem 3.2 ([7, §2]). *When K is not a field, every element of U is a product of at most 3 elements of R.*

On the other hand, by means of generic counterexamples, one can show that 3 is the best possible:

Proposition 3.3 ([7, Cor. 4.0.4]). *Let $F = \mathbb{Q}(\eta, \lambda)$, $K = F \times F \times F$, and $D = (\alpha, \eta + 1)_F \times (\alpha, \lambda)_F \times (\alpha, (\eta + 1)\lambda)_F$, where $\alpha = \eta^2 - 4$. Let x_i, y_i ($i = 1, 2, 3$) be standard generators for the i'th component.*

Then the element $v = ((\eta + x_1)(\eta + 2 + 2y_1), \eta(1 + x_2), 2\eta) \in D_1 \times D_2 \times D_3$ is in V, but not in $R \cdot R$. In particular $V \not\subseteq R \cdot R$.

Another explicit counterexample [7, Cor. 4.0.4] shows that $V \not\subset R \cdot R$ when $K = F \times L$. By means of extending scalars [7, §5], it also follows that $V \not\subset R \cdot R$ when K is a field.

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Killing Forms of Lie Algebras

Audrey Malagon

The computation of Killing forms of Lie algebras has been a subject of interest for many years. Over an algebraically closed field or the real numbers of course this is well known from the work of Killing and Cartan. Over arbitrary fields we have some results due to N. Jacobson [5] and J-P. Serre [4]. The computation of Killing forms is not only useful in the classification of Lie algebras, which plays a role in other open conjectures such as Serre's conjecture II, but is also related to other useful invariants. Here we are primarily concerned with computing the Killing form of a Lie algebra of exceptional type over a field of characteristic zero using information contained in its Tits index [2]. We present a method that allows direct computation of the Killing form of a simple isotropic Lie algebra based on the Killing form of a subalgebra containing its anisotropic kernel. This approach allows for streamlined formulas for many Lie algebras of types E_6 and E_7 and yields a unified formula for all Lie algebras of inner type E_6, including the anisotropic ones.

Recall a Lie algebra is a vector space with an additional multiplication called the bracket, which satisfies certain identities. Every Lie algebra acts on itself via the adjoint representation, defined by this bracket operation, i.e. $ad(x)(y) = [xy]$. Roots of a Lie algebra are defined via this adjoint representation [3] and the Tits index encodes both root system data and information about the anisotropic kernel of a Lie algebra [2]. Using properties of root systems and irreducible representations of Lie algebras, we have the following formula for the Killing form of an isotropic Lie algebra in terms of the Killing form of a regular subalgebra.

Theorem 1. *Let L be a simple, isotropic Lie algebra of dimension n defined over a field of characteristic zero with simple roots Δ (all of the same length) and Cartan subalgebra H. Let A be a regular subalgebra of dimension n' with simple roots Δ' containing the anisotropic kernel of L. If $A = \oplus_{i=1}^{n} A_i$ with each A_i simple, then the Killing form κ on L is given by*

$$\kappa = \frac{m(L)}{m(A_1)}\kappa_1 \perp ... \perp \frac{m(L)}{m(A_n)}\kappa_n \perp \kappa|_{Z_H(A)} \perp \frac{n - n' - |\Delta \setminus \Delta'|}{2}\mathcal{H}$$

where m is the Coxeter number of the algebra and κ_i is the Killing form of A_i.

The results extend easily to Lie algebras whose roots have different lengths using a multiple that involves the dual Coxeter number rather than the Coxeter number of the algebra. The Killing form is computed on $Z_H(A)$ using a grading of the Lie algebra and results of [1] to translate the question into one of dimensions of irreducible representations. Using this theorem, we also have the following formula for the Killing form of all inner type Lie algebras of type E_6, including the anisotropic ones, over any field of characteristic not 2 or 3. Here we refer to a modified Rost invariant $r(L)$ described in [6] and we use e_3 to denote the well known Arason invariant.

Theorem 2. *The Killing form of a Lie Algebra of type* $^1E_6$ *is*

$$\kappa = \langle -1 \rangle 4q_0 \perp \langle 2, 6 \rangle \perp 24\mathcal{H}$$

where $1 \perp q_0$ *is a 3-fold Pfister form and* $e_3(1 \perp q_0)$ *is* $r(L)$.

The table below gives the explicit computations of Lie algebras of inner type E_6 based on their Tits index. We also have similar results for certain Lie algebras of type E_7 and outer type E_6.

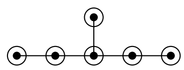
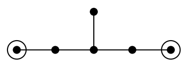
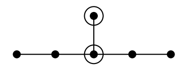
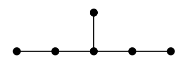
Tits Index	Killing form
	$\langle 1,1,1,1,2,6 \rangle \perp 36\mathcal{H}$
	$\langle -1 \rangle 4q_0 \perp \langle 2, 6 \rangle \perp 24\mathcal{H}$
	$\langle 1,1,1,1,2,6 \rangle \perp 36\mathcal{H}$
	$\langle -1 \rangle 4q_0 \perp \langle 2, 6 \rangle \perp 24\mathcal{H}$

TABLE 1. Results for $^1E_6$

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There is no "Theory of Everything" inside $\mathbf{E_8}$

Skip Garibaldi
(joint work with Jacques Distler)

Recently, the preprint [3] by Garrett Lisi has generated a lot of popular interest, because it boldly claims to be a sketch of a "Theory of Everything". We explain some reasons why an entire class of such models—which include the model in [3]—cannot work.

To build a Theory of Everything along the lines of [3], one fixes a real algebraic group E and considers subgroups $\mathrm{SL}(2,\mathbb{C}) = \mathrm{Spin}(3,1)$ and G of E, where

(T1) G is connected, reductive, compact, and centralizes $\mathrm{SL}(2,\mathbb{C})$.

In order to interpret this setup in terms of physics, one identifies elements of a basis (e.g., a Chevalley basis) of $\mathrm{Lie}(\mathrm{E}) \otimes \mathbb{C}$ with particles as in [3]. The subgroup $\mathrm{SL}(2,\mathbb{C})$ is the local Lorentz group. One views G as the gauge group of the theory; a physicist might want to take G to be the gauge group of the Standard Model, which is isogenous to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

Complexifying $\mathrm{SL}(2,\mathbb{C})$ gives $\mathrm{SL}_{2,\mathbb{C}} \times \mathrm{SL}_{2,\mathbb{C}}$. We decompose $\mathrm{Lie}(\mathrm{E}) \otimes \mathbb{C}$ as a representation of $\mathrm{SL}_{2,\mathbb{C}} \times \mathrm{SL}_{2,\mathbb{C}} \times (G \times \mathbb{C})$ as

$$(*) \qquad \mathrm{Lie}(\mathrm{E}) \otimes \mathbb{C} = \bigoplus_{m,n \geq 1} m \otimes n \otimes V_{m,n}$$

where m and n denote the irreducible representations of $\mathrm{SL}_{2,\mathbb{C}}$ with that dimension and $V_{m,n}$ is a representation of $G \times \mathbb{C}$. The Spin-Statistics Theorem [4] says that the pieces of the direct sum $(*)$ with $m+n$ odd correspond to fermions and those with $m+n$ even correspond to bosons. The Standard Model is a chiral gauge theory, which in terms of mathematics says:

(T2) $V_{1,2}$ is a complex representation of G,

that is, there does *not* exist a semilinear map $J\colon V_{1,2} \to V_{1,2}$ so that $J^4 = \mathrm{Id}_{V_{1,2}}$.

Additionally, a serious result from physics (see sections 13.1, 25.4 of [5]) says that a unitary interacting theory is incompatible with massless particles in higher representations, so we also demand:

(T3) $V_{1,4} = V_{4,1} = 0$ and $V_{m,n} = 0$ for $m + n \geq 6$.

The representations $V_{2,3}$ and $V_{3,2}$ can only be nonzero in the presence of local supersymmetry (in which case basis vectors in these representation correspond to gravitinos). This possibility is not considered in [3], but we allow it here because the extra generality does not cost much. We prove in [1]:

Theorem. *If* E *is a real form of* E_8 *or the (transfer of the) complex form of* E_8*, then there are no subgroups* $\mathrm{SL}(2,\mathbb{C})$ *and* G *of* E *that satisfy (T1), (T2), and (T3).*

Sketch of proof. We sketch only the simpler case where E is a real form of E_8. Hypothesis (T3) gives constraints on the two copies of $\mathrm{SL}_{2,\mathbb{C}}$ in $\mathrm{E} \times \mathbb{C}$ coming from the $\mathrm{SL}(2,\mathbb{C})$ subgroup, in terms of the Dynkin index from [2]; specifically each copy has index 1 or 2. Complex conjugation interchanges the two factors, so both copies have the same index.

In case both have index 1, the centralizer of $\mathrm{SL}(2,\mathbb{C})$ in E has identity component C isomorphic to $\mathrm{Spin}(11,1)$, $\mathrm{Spin}(9,3)$, or $\mathrm{Spin}(7,5)$. In case both have index 2, C is isomorphic to the transfer of $\mathrm{Sp}_{4,\mathbb{C}}$. In each case, one checks that the restriction of $V_{1,2}$ to the maximal compact subgroup $G_{\max}$ of C is not complex, i.e., there is a J that is invariant under $G_{\max}$. No matter what G one picks that satisfies (T1), it is contained in $G_{\max}$, so (T2) fails. □

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On symmetric bilinear forms of height 2 in characteristic 2

Ahmed Laghribi

Our aim is to give some results on the standard splitting of bilinear forms in characteristic 2 [7].

Let F be a field of characteristic 2. All bilinear forms are supposed to be regular, symmetric and of finite dimension. For a bilinear form B, let $\dim B$ (*resp.* B_{an}) denote its dimension (*resp.* its anisotropic part). For $m \geq 1$ an integer, we denote by $BP_m(F)$ (*resp.* $GBP_m(F)$) the set of bilinear forms isometric (*resp.* similar) to m-bilinear Pfister forms. The 0-bilinear Pfister form is the form $\langle 1\rangle$, where for any $a_1, \cdots, a_n \in F^* := F - \{0\}$, we denote by $\langle a_1, \cdots, a_n\rangle$ the bilinear form given by the polynomial $\sum_{i=1}^n a_i x_i y_i$.

To any bilinear form B with underlying F-vector space V, we associate the quadratic form $\widetilde{B}$ defined on V by: $\widetilde{B}(v) = B(v,v)$ for $v \in V$. This quadratic form is uniquely determined by B. The function field of B, denoted by $F(B)$, is by definition the function field of the affine quadric given by $\widetilde{B}$. The standard splitting tower of B is the sequence $(F_i, B_i)_{i\geq 0}$ given as follows:

$$\begin{cases} F_0 = F \quad \text{and} \quad B_0 = B_{\mathrm{an}} \\ \text{For } n \geq 1: \quad F_n = F_{n-1}(B_{n-1}) \quad \text{and} \quad B_n = ((B_{n-1})_{F_n})_{\mathrm{an}}. \end{cases}$$

The *height* of B, denoted by $\mathrm{h}(B)$, is the smallest integer h such that $\dim B_h \leq 1$. The classification of bilinear forms of height 1 is as follows:

Theorem 1. *([6, Th. 4.1]) An anisotropic bilinear form B is of height 1 if and only if there exists a bilinear Pfister form $\pi = \langle 1\rangle \perp \pi'$ such that B is similar to π or π' according as $\dim B$ is even or odd.*

If B is a bilinear form such that $\dim B_{\text{an}} \geq 2$, then $\text{h}(B) \geq 1$. By Theorem 1 we attach to B a numerical invariant, called the *degree* of B, as follows: If $(F_i, B_i)_{i\geq 0}$ is the standard splitting tower of B and $h = \text{h}(B)$, then the form B_{h-1} is of height 1, and thus it corresponds to a bilinear Pfister form π over F_{h-1} in the sense of Theorem 1. The form π is unique, we call it the *leading form* of B. If $\dim B$ is even, then the degree of B, denoted by $\deg(B)$, is the integer d such that $\dim \pi = 2^d$. Otherwise, we put $\deg(B) = 0$. The form B is called *good* if the form π is definable over F, which means that $\pi \simeq \rho_{F_{h-1}}$ for a suitable bilinear form ρ over F ($\simeq$ denotes the isometry of bilinear forms). In this case, we know that ρ is isometric to a unique bilinear Pfister form over F [6, Prop. 5.3]. The following question arises:

Question 1. *Let $h > 0$ and $d \geq 0$ be two integers. Which are the anisotropic bilinear forms over F of height h and degree d?*

Theorem 1 answers this question for $h = 1$ and arbitrary d. Concerning the height 2, we classified in [6, Prop. 5.9, Th. 5.10] good bilinear forms of height 2 and degree d, except in the following case:

$$(\star) \qquad d > 0 \quad \text{and} \quad \dim B = 2^n \ \text{ with } \ n > d + 1.$$

Recently, in collaboration with Rehmann [8, Th. 1.1, 1.2], we gave an answer to $(\star)$ for $d = 1$ or 2, and we classified nongood bilinear forms of height 2 and degree 2. The main result in [7] is the following theorem which gives a complete answer to $(\star)$:

Theorem 2. *Let B be an anisotropic bilinear form of degree $d > 0$ such that $\dim B = 2^n$ with $n > d + 1$. Then, the following statements are equivalent:*
(1) B is good of height 2 and leading form $\tau_{F(B)}$ with $\tau \in BP_d(F)$.
(2) $B \simeq x\theta \otimes (\lambda' \perp \langle y \rangle)$ for suitable $x, y \in F^*$, $\theta \in BP_{d-1}(F)$ and $\lambda = \langle 1 \rangle \perp \lambda' \in BP_{n-d+1}(F)$ such that $\widetilde{B}$ is similar to $\widetilde{\theta \otimes \lambda}$, and $\tau \simeq \langle 1, y \rangle \otimes \theta$.

There is no analogue of this theorem in characteristic not 2, since a result of Hoffmann [3] asserts that if an anisotropic quadratic form φ over a field of characteristic not 2 is good of height 2 and $\dim \varphi$ is a power of 2, then $\dim \varphi = 2^{d+1}$, where d is the degree of φ. Our proof of Theorem 2 is based on a descent argument which uses important computations on Witt kernels of function field extensions, due to me and Aravire-Baeza [2, Cor. 3.3], [5, Th. 1.2], [6, Prop. 4.13]. The divisibility of B by a form in $BP_{d-1}(F)$, given in statement (2) of Theorem 2, is a consequence of the following proposition:

Proposition 1. *Let $C \in BP_m(F)$, $D \in BP_n(F)$ be anisotropic, and $x, y \in F^*$. If the form $xC \perp yD$ is isotropic, then the Witt index of $xC \perp yD$ is equal to 2^k for some $k \geq 0$, and there exist $\theta \in BP_k(F)$, $C_1 \in BP_{m-k}(F)$ and $D_1 \in BP_{n-k}(F)$ such that $C \simeq \theta \otimes C_1$ and $D \simeq \theta \otimes D_1$.*

This proposition extends to bilinear forms in characteristic 2 the classical result of Elman and Lam about the linkage of quadratic Pfister forms in characteristic not 2. We proved it using a recent work of Arason and Baeza concerning the "chain p-equivalence" relation in the case of bilinear forms in characteristic 2 [1, Appendix A]. We also recovered the following linkage result due to Aravire and Baeza:

Proposition 2. *([2, Cor. 2.3]) Let $L = F^2(x_1, \cdots, x_n)$ be such that $[L : F^2] = 2^n$. Let $\pi_1,\ \pi_2 \in BP_n(L)$. Then, there exists a form $\pi_3 \in BP_n(L)$ such that $\pi_1 \perp \pi_2 \equiv \pi_3 \pmod{(IF)^{n+1}}$, where IF is the ideal of even dimensional bilinear forms of the Witt ring $W(F)$ of F.*

Another question studied in [7] concerns the behaviour of good bilinear forms of height 2 over the function fields of their leading forms. An important ingredient that we used is the notion of the norm degree introduced in [4, Section 8]. Recall that for a bilinear form B such that the quadratic form $\widetilde{B}$ represents a nonzero scalar, the *norm degree* of $\widetilde{B}$, denoted by $\operatorname{ndeg}_F(\widetilde{B})$, is the degree of the field $F^2(\alpha\beta \mid \alpha, \beta \in D_F(\widetilde{B}))$ over F^2, where $D_F(\widetilde{B})$ denotes the set of nonzero scalars represented by $\widetilde{B}$.

(1) The case of good bilinear forms of height 2 and degree > 0:

Proposition 3. *Let B be an anisotropic bilinear form which is good of height 2 and degree $d > 0$. Let $\tau \in BP_d(F)$ be such that $\tau_{F(B)}$ is the leading form of B. Then, we have the following statements:*
(1) $B_{F(\tau)}$ is isotropic if and only if $B_{F(\tau)}$ is metabolic.
(2) The form $B_{F(\tau)}$ is isotropic in the following cases: ($\dim B$ is not a power of 2) or ($\dim B = 2^n$ with $n > d+1$) or ($\dim B = 2^{d+1}$ and $\operatorname{ndeg}_F(\widetilde{B}) = 2^{d+1}$).
(3) If $\dim B = 2^{d+1}$ and $\operatorname{ndeg}_F(\widetilde{B}) > 2^{d+1}$, then $B_{F(\tau)} \in GBP_{d+1}(F(\tau))$ is anisotropic, and $2\dim B \leq \operatorname{ndeg}_F(\widetilde{B})$.

The converse of statement (3) of this proposition is given as follows:

Proposition 4. *Let B and C be anisotropic bilinear forms such that $\dim B = 2^{d+1}$ ($d \geq 1$), $\dim C = 2^d$ and $B_{F(C)} \in GBP_{d+1}(F(C))$ is anisotropic. Suppose that $2\dim B \leq \operatorname{ndeg}_F(\widetilde{B})$ or, more weakly, B is not similar to a bilinear Pfister form. Then, B is good of height 2 and degree d, and $\widetilde{\tau}$ is similar to $\widetilde{C}$, where $\tau \in BP_d(F)$ and $\tau_{F(B)}$ is the leading form of B. In this case, $\operatorname{ndeg}_F(\widetilde{B}) > 2^{d+1}$.*

Using Theorem 2 and Proposition 4, we proved the following descent result:

Corollary 1. *Let B and C be anisotropic bilinear forms such that $\dim B = 2^{d+1}$ and $\dim C = 2^d$ ($d \geq 1$). If $B_{F(C)} \in BP_{d+1}(F(C))$ is anisotropic, then there exists $D \in BP_{d+1}(F)$ such that $B_{F(C)} \simeq D_{F(C)}$.*

(2) The case of good bilinear forms of height 2 and degree 0: This case is reduced to Proposition 3 as follows:

Proposition 5. *Let B be an anisotropic bilinear form which is good of height 2 and degree 0. Let $\tau \in BP_d(F)$ be such that $\tau_{F(B)}$ is the leading form of B. Then, $\dim B \geq 2^d + 1$, and we have:*
(1) If $\dim B = 2^d + 1$, then $B_{F(\tau)}$ is isotropic.

(2) If $\dim B > 2^d + 1$*, then* $C := (B \perp \langle \det B \rangle)_{\mathrm{an}}$ *is good of height* 2 *and leading form* $\tau_{F(C)}$*, and* $B_{F(\tau)}$ *is isotropic if and only if* $C_{F(\tau)}$ *is isotropic.*

We also obtained a complete classification of good quadratic forms, singular or not, of height 2 in characteristic 2.

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Using quadratic places for specializing forms

Manfred Knebusch

Let $\lambda : K \to L \cup \{\infty\}$ be a place between fields of any characteristic, $\mathcal{O} := \mathcal{O}_\lambda$ its valuation domain, and $k := \mathcal{O}/\mathfrak{m}$ its residue class field. The place λ induces a field embedding $\bar{\lambda} : k \hookrightarrow L$.

Given a quadratic space $E = (E, q)$ over K we want to "specialize" E to a quadratic space $\lambda_*(E)$ over L. This is possible in a reasonable way if E contains a free $\mathcal{O}$-module M of finite rank such that $E = KM$ and $q(M) \subset \mathcal{O}$, such that the induced quadratic space $(M/\mathfrak{m}M, \bar{q})$ over k has an anisotropic quasilinear part $QL(M/\mathfrak{m}M)$.

We then put

$$\lambda_*(E) := M/\mathfrak{m}M \otimes_{k,\bar{\lambda}} L,$$

and we say that E has *fair reduction (=FR) under* λ. If in this case moreover, the quasilinear part $QL(\lambda_*(E))$ is still anisotropic, then the Witt-index and the anisotropic part of $\lambda_*(E)$ are determined by the generic splitting tower $(K_r \mid 0 \leq r \leq h)$ of E in precisely the same way as known in the easier special case when $char(L) \neq 2$, cf. [1].

A *quadratic place* $\Lambda : K \to L \cup \{\infty\}$ is a triple $\Lambda = (\lambda, H, \chi)$ with $\lambda : K \to L \cup \{\infty\}$ an ordinary place, H a subgroup of $Q(K) = K^*/K^{*2}$ containing all classes ϵK^{*2} with $\epsilon \in \mathcal{O}^*$, and $\chi : H \to Q(L)$ a group homomorphism with $\chi(\epsilon K^{*2}) = \lambda(\epsilon)L^{*2}$ for every $\epsilon \in \mathcal{O}^*$.

In the talk a notion of fair reduction of a quadratic space E under Λ is developed, which generalises the fair reduction for ordinary places. If $\Lambda_*(E)$ has anisotropic quasilinear part, then again the Witt-index and anisotropic quasilinear part of $\Lambda_*(E)$ can be determined by the generic splitting tower $(K_r \mid 0 \leq r \leq h)$ of E via a maximal "extension" $M : K_m \to L \cup \{\infty\}$ of the quadratic place Λ. To prove this, the notion of fair reduction has to be generalised to a notion of "stably conservative reduction", which is complicated; but miraculously then the generic splitting works as well as for ordinary places. The point is that, expanding an ordinary place $\lambda : K \to L \cup \{\infty\}$ suitably to a quadratic place $\Lambda : K \to L \cup \{\infty\}$, a form φ which has bad reduction under λ may become a form with FR under Λ. Then the splitting pattern of $\Lambda_*(\varphi)$ is a coarsening of the splitting pattern of φ.

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Projective homogeneous varieties birational to quadrics

Mark L. MacDonald

Recently Totaro has solved the birational classification problem for a large class of quadrics [To08]. In particular, let ϕ be an r-Pfister form over a field k of characteristic not 2, and $b = \langle b_1, \cdots b_n\rangle$ be a non-degenerate quadratic form with $n \geq 2$.

Proposition 1. *The birational class of the quadric defined by*

$$q = \phi \otimes \langle b_1, \cdots, b_{n-1}\rangle \perp \langle b_n\rangle$$

only depends on the isometry classes of ϕ and $\phi \otimes b$, and not on the choice of diagonalization of b.

The Sarkisov program [Co94] predicts that any birational map between quadrics (in fact between any two Mori fibre spaces) factors as a chain of composites of "elementary links". In [Mac09] the author explicitly factors many of Totaro's birational maps into chains of elementary links, and also proves the following theorem.

Theorem 2. *For $r = 0, 1, 2$ and $n \geq 3$, or $r = 3$ and $n = 3$, for each of the birational equivalences from Prop. 1, there is a birational map which factors into two elementary links, each of which is the blow up of a reduced subscheme followed by a blow down. Furthermore, if $r \neq 1$ or ϕ is not hyperbolic, then the intermediate Mori fibre space of this factorization will be the projective homogeneous variety $X(J)$ of traceless rank one elements in a Jordan algebra J.*

This birational map from a quadric to $X(J)$ is the codimension 1 restriction of a birational map between projective space and the projective variety V_J of rank one elements of J, first written down by Jacobson [Ja85, 4.26].

3 **Motivic decompositions.** Let G be a semisimple linear algebraic group of inner type, and X a projective homogeneous G-variety such that G splits over the function field of X, which is to say, X is generically split (see [PSZ08, 3.6] for a convenient table). Then [PSZ08] gives a direct sum decomposition of the Chow motive $\mathcal{M}(X;\mathbb{Z}/p\mathbb{Z})$ of X. They show that it is the direct sum of some Tate twists of a single indecomposable motive $\mathcal{R}_p(G)$, which generalizes the Rost motive. This work unified much of what was previously known about motivic decompositions of anisotropic projective homogeneous varieties.

In the non-generically split cases less is known. Quadrics are in general not generically split, but much is known by the work of Vishik and others, especially in low dimensions [Vi04].

Theorem 4. *The motive of the projective quadric defined by the quadratic forms in Prop. 1 may be decomposed into the sum, up to Tate twists, of Rost motives and higher forms of Rost motives.*

In [Mac09], the author uses this knowledge of motives of quadrics to produce motivic decompositions for the non-generically split projective homogeneous G-varieties $X(J)$ which appear in Thm. 2. The algebraic groups G are of Lie type ${}^2A_{n-1}$, C_n and F_4, and are automorphism groups of simple reduced Jordan algebras of degree ≥ 3. These varieties $X(J)$ come in four different types which we label $r = 0, 1, 2$ or 3, corresponding to the 2^r dimensional composition algebra of the simple Jordan algebra J; we will describe the type of $X(J)$ as G/P for a parabolic subgroup P.

As an application of Thm. 2, together with our knowledge of the motives of the base loci, we can deduce the following theorem.

Theorem 5. *The motive of $X(J)$ is the direct sum of a higher form of a Rost motive, F_n^r, together with several Tate twisted copies of the Rost motive R^r.*

The $r = 1$ case of this theorem provides an alternate proof of Krashen's motivic equivalence [Kr07, Thm. 3.3]. On the other hand, the $r = 1$ case of this theorem is shown in [SZ08, Thm. (C)] by using Krashen's result.

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Invariants, torsion indices and oriented cohomology of complete flags

Kirill Zainoulline
(joint work with Baptiste Calmès, Victor Petrov)

Let H^* be an algebraic cohomology theory endowed with characteristic classes c_i such that for any line bundles L_1 and L_2 over a variety X we have

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2). \tag{1}$$

The basic example of such a theory is the Chow group $CH^*(X)$ of algebraic cycles modulo rational equivalence relation. Using the language of formal group laws one says that H^* corresponds to the additive formal group law.

Let G be a split semi-simple simply-connected linear algebraic group over a field and let T be a split maximal torus inside G contained in a Borel subgroup B. Consider the variety G/B of Borel subgroups of G with respect to T. In two classical papers [2] and [3] Demazure studied the cohomology ring $H^*(G/B)$ and provided an algorithm to compute $H^*(G/B)$ in terms of generators and relations.

The main object of his considerations was the so called characteristic map

$$c\colon S^*(M) \to H^*(G/B), \tag{2}$$

where $S^*(M)$ is the symmetric algebra of the characters group M of T. In [2] Demazure interpreted this map from the point of view of invariant theory of the Weyl group W of G by identifying its kernel with the ideal generated by non-constant invariants $S^*(M)^W$. The cohomology ring $H^*(G/B)$ was then replaced by a certain algebra constructed in terms of operators and defined in purely combinatorial terms. Observe that the characteristic map c is not surjective in general. The size of its cokernel is measured by the torsion index of G. The latter is defined to be the index of the image of c in $H^{\dim G/B}(G/B) = \mathbb{Z}$.

In the present paper, we generalize most of the results of papers [2] and [3] to the case of an arbitrary algebraic oriented cohomology theory h, i.e. where the right hand side of (1) is replaced by an arbitrary formal group law F

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

Such theories were extensively studied by Levine-Morel [4], Panin-Smirnov [6], Merkurjev [5] and others. Apart from the Chow ring, classical examples are algebraic K-theory and algebraic cobordism Ω.

To generalize the characteristic map (2), we first introduce a substitute for the symmetric algebra $S^*_R(M)$ over the coefficient ring R of h. This algebraic object called a *formal group ring* and denoted by $R[[M]]_F$ plays the central role in our paper. The Weyl group W acts naturally on $R[[M]]_F$ and we have a characteristic map

$$c_{G/B}\colon R[[M]]_F \to h(G/B).$$

As in [2] we introduce a subalgebra $D(M)_F$ of the R-linear endomorphisms of $R[[M]]_F$ generated by certain Δ-operators and taking its dual we obtain the respective combinatorial substitute $\mathcal{H}(M)_F$ for the cohomology ring $h(G/B)$. The characteristic map (2) then turns into the map

$$c\colon R[[M]]_F \to \mathcal{H}(M)_F,$$

and the main result of our paper says that

Theorem. *If the torsion index is invertible in R and R has no 2-torsion, then the characteristic map is surjective and its kernel is the ideal of the formal group ring $R[[M]]_F$ generated by W-invariant elements of the augmentation ideal.*

As an application of the developed techniques we provide an efficient algorithm for computing the cohomology ring $h(G/B;\mathbb{Z})$. To do this we generalize the Bott-Samelson approach introduced in [3].

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Le groupe quaquaversal, vu comme groupe S-arithmétique

Jean-Pierre Serre

1. Les groupes $G(m,n)$

Les pavages non périodiques de $\mathbf{R}^3$ décrits dans [1] et [2] font intervenir certains groupes de rotations, notés $G(m,n)$. Rappelons comment sont définis ces groupes. On se donne deux entiers $m,n \geqslant 3$ et l'on note x_m une rotation de $\mathbf{R}^3$ d'angle $2\pi/m$ autour d'un axe D_x; on note de même y_n une rotation de $2\pi/n$ autour d'un axe D_y perpendiculaire à D_x. Le groupe $G(m,n)$ est défini comme le sous-groupe $\langle x_m, y_n\rangle$ de $\mathbf{SO}_3(\mathbf{R})$ engendré par x_m et y_n. A part le cas $m = n = 4$ où $G(m,n)$ est isomorphe au groupe symétrique S_4, ces groupes sont infinis, et l'on peut les décrire comme des amalgames de deux groupes finis (cf. [3] , [4]); ils possèdent des sous-groupes libres d'indice fini. Ainsi, par exemple, le groupe $G(6,4)$, qui correspond au pavage "quaquaversal" de Conway-Radin [1], est isomorphe à l'amalgame $D_6 *_{D_2} D_4$, où D_i désigne un groupe diédral d'ordre $2i$.

2. S-arithméticité des groupes $G(m,n)$

On peut se demander si certains groupes $G(m,n)$ sont des sous-groupes arithmétiques (ou plutôt S-arithmétiques) du groupe algébrique $\mathbf{SO}_3$. Cette question a été posée par G.R.Robinson [5] à propos des groupes $G(4,2^n), n > 2$. On va voir que l'on peut y répondre, à la fois pour les $G(4,2^n)$, et pour le groupe quaquaversal $G(6,4)$.

Ce dernier cas est celui qui s'énonce le plus simplement. Si l'on choisit bien les coordonnées dans $\mathbf{R}^3$, on constate que $G(6,4)$ est contenu dans le groupe $\mathbf{SO}_{3,q}(\mathbf{Z}[\frac{1}{2}])$, où $\mathbf{SO}_{3,q}$ désigne le groupe spécial orthogonal de la forme quadratique $q(x,y,z) = x^2 + 3y^2 + 3z^2$.

Théorème 1. *L'inclusion $G(6,4) \to \mathbf{SO}_{3,q}(\mathbf{Z}[\frac{1}{2}])$ est une égalité.*

En particulier, $G(6,4)$ est un sous-groupe S-arithmétique de $\mathbf{SO}_{3,q}$, où $S = \{2\}$.

Le cas de $G(4,2^n), n \geqslant 3$, est différent. Tout d'abord, on doit remplacer $\mathbf{Q}$ par le corps $K_n = \mathbf{Q}(w_{2^n} + w_{2^n}^{-1})$, où w_{2^n} désigne une racine primitive 2^n-ième de l'unité (c'est le plus grand sous-corps réel du corps cyclotomique $\mathbf{Q}(w_{2^n})$). On a par exemple $K_3 = \mathbf{Q}(\sqrt{2}), K_4 = \mathbf{Q}(\sqrt{2+\sqrt{2}})$, etc. Soit $A_n = \mathbf{Z}[w_{2^n} + w_{2^n}^{-1}]$ l'anneau des entiers de K_n. Le groupe $G(4,2^n)$ se plonge de façon naturelle dans le groupe $\mathbf{SO}_3(A_n[\frac{1}{2}])$ relatif à la forme quadratique standard $x^2 + y^2 + z^2$.

Théorème 2. a) *Si $n = 3$ ou 4, l'inclusion $G(4,2^n) \to \mathbf{SO}_3(A_n[\frac{1}{2}])$ est une égalité.*

b) *Si $n \geqslant 5$, $G(4,2^n)$ est un sous-groupe d'indice infini de $\mathbf{SO}_3(A_n[\frac{1}{2}])$.*

3. Indications sur les démonstrations

Pour le th.1, on utilise l'action du groupe $\Gamma = \mathbf{SO}_{3,q}(\mathbf{Z}[\frac{1}{2}])$ sur l'arbre de Bruhat-Tits X du groupe $\mathbf{SO}_{3,q}$ relativement au corps local $\mathbf{Q}_2$, cf. [7], chap.II, §1 (noter que $\mathbf{SO}_{3,q}$ est isomorphe à $\mathbf{PGL}_2$ sur $\mathbf{Q}_2$). L'action de Γ sur X a pour domaine fondamental une demi-arête $P - P'$, le fixateur de P (resp. P', resp. $P - P'$) étant D_6 (resp. D_4, resp. D_2); cela montre (cf. [7], chap.I, §4.1) que $\Gamma = D_6 *_{D_2} D_4$; comme D_6 et D_4 sont contenus dans $G(6,4)$, on en déduit que $\Gamma = G(6,4)$.

La démonstration du th.2.a) est analogue à celle du th.1.

Pour le th.2.b), on utilise les caractéristiques d'Euler-Poincaré ([6],§1) des deux groupes $G(4,2^n)$ et $\Gamma_n = \mathbf{SO}_3(A_n[\frac{1}{2}])$. On a:

$\chi(G(4,2^n)) = -1/12 + 1/2^{n+1}$,

et

$\chi(\Gamma_n) = -2^{-2^{n-2}}\zeta_{K_n}(-1) = -\operatorname{disc}(K_n)^{3/2}(2\pi)^{-2^{n-1}}\zeta_{K_n}(2)$,

où ζ_{K_n} désigne la fonction zêta du corps K_n, et $\operatorname{disc}(K_n)$ est le discriminant de K_n.

La première égalité se déduit de l'isomorphisme $G(4,2^n) = S_4 *_{D_4} D_{2^n}$, démontré dans [4], th.1, et la seconde résulte d'un calcul de volume basé sur la valeur du nombre de Tamagawa de $\mathbf{SO}_3$, cf. [8] et [7], chap.II, §1.5.

[Exemples : pour $n = 3$ (resp. 4) les deux caractéristiques d'Euler-Poincaré sont égales à $-1/48$ (resp. à $-5/96$); pour $n = 5$, celle de $G(4,2^n)$ est égale à $-13/192 = -0,067...$ et celle de Γ_n à $-2^{-6}.3.5.97 = -22,734...$]

Pour $n \geqslant 5$, on constate que $|\chi(\Gamma_n)| > |\chi(G(4,2^n))|$, ce qui ne serait pas possible si l'indice de $G(4,2^n)$ dans Γ_n était fini.

Remarque. En fait, $|\chi(\Gamma_n)|$ tend vers l'infini avec n; on en déduit que le nombre minimum de générateurs de Γ_n a la même propriété.

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