Vector Product and an Integrable Dynamical System

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Abstract *We study an autonomous system of first order ordinary differential equations based on the vector product. We show that the system is completely integrable by constructing the first integrals. The connection with Nambu mechanics is established. The extension to higher dimensions is also discussed.*

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Vectors in the vector space $\mathbb{R}^3$ form a simple Lie algebra under the vector product.[1–2] In particular they satisfy $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$ and the Jacobi identity

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{w} \times (\boldsymbol{u} \times \boldsymbol{v}) + \boldsymbol{v} \times (\boldsymbol{w} \times \mathbf{u}) = \mathbf{0}\,.$$

The volume V spanned by the three vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ in $\mathbb{R}^3$ with $\boldsymbol{u}_1 \cdot (\boldsymbol{u}_2 \times \boldsymbol{u}_3) \geq 0$ is given by

$$V = \boldsymbol{u}_1 \cdot (\boldsymbol{u}_2 \times \boldsymbol{u}_3) = \boldsymbol{u}_2 \cdot (\boldsymbol{u}_3 \times \boldsymbol{u}_1) = \boldsymbol{u}_3 \cdot (\boldsymbol{u}_1 \times \boldsymbol{u}_2)\,,$$

where $\cdot$ denotes the scalar product. Such a dreibein appears for electromagnetic fields with the electric field $\boldsymbol{E}$, magnetic induction $\boldsymbol{B}$, and the wave vector $\boldsymbol{k}$ (or Poynting vector $\boldsymbol{S}$).

Let $\boldsymbol{u}_1(t)$, $\boldsymbol{u}_2(t)$, $\boldsymbol{u}_3(t) \in \mathbb{R}^3$. We solve the initial value problem of the nonlinear autonomous system of first order differential equations

$$\frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t} = \boldsymbol{u}_2 \times \boldsymbol{u}_3, \quad \frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = \boldsymbol{u}_3 \times \boldsymbol{u}_1, \quad \frac{\mathrm{d}\boldsymbol{u}_3}{\mathrm{d}t} = \boldsymbol{u}_1 \times \boldsymbol{u}_2, \quad (1)$$

where $\times$ denotes the vector product. Fixed points (time independent solutions) are given by $u_{jk} = c$ for all $j = 1, 2, 3$ and $k = 1, 2, 3$, where $c \in \mathbb{R}$. The divergence of the corresponding vector field of Eq. (1) is 0. This means the Lie derivative of the volume differential form vanishes.

We show that the dynamical system is completely integrable by constructing 8 independent first integrals.

Two first integrals can be found as follows. From scalar multiplications $\boldsymbol{u}_j \cdot \mathrm{d}\boldsymbol{u}_k/\mathrm{d}t$ we obtain

$$\boldsymbol{u}_1 \cdot \frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t} = \boldsymbol{u}_2 \cdot \frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = \boldsymbol{u}_3 \cdot \frac{\mathrm{d}\boldsymbol{u}_3}{\mathrm{d}t} = \boldsymbol{u}_1 \cdot (\boldsymbol{u}_2 \times \boldsymbol{u}_3)\,.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_2 \cdot \boldsymbol{u}_2) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_3 \cdot \boldsymbol{u}_3) = 2V\,.$$

Thus we find the polynomial first integrals

$$I_1(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \boldsymbol{u}_1^2 - \boldsymbol{u}_2^2\,, \quad I_2(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \boldsymbol{u}_1^2 - \boldsymbol{u}_3^2\,.$$

The first integral $\boldsymbol{u}_2^2 - \boldsymbol{u}_3^2$ is dependent on these two first integrals. We define

$$l_1 := \sqrt{\boldsymbol{u}_1 \cdot \boldsymbol{u}_1}\,, \quad l_2 := \sqrt{\boldsymbol{u}_2 \cdot \boldsymbol{u}_2}\,, \quad l_3 := \sqrt{\boldsymbol{u}_3 \cdot \boldsymbol{u}_3}\,.$$

Thus the two first integrals I_1, I_2 provide the constants of motion

$$l_1^2 - l_2^2 = c_1\,, \quad l_1^2 - l_3^2 = c_2\,,$$

where c_1, c_2 are constants. From the scalar multiplications $\boldsymbol{u}_j \cdot \mathrm{d}\boldsymbol{u}_k/\mathrm{d}t$ $(j \neq k)$ we obtain for example for $\boldsymbol{u}_1 \cdot \mathrm{d}\boldsymbol{u}_2/\mathrm{d}t$

$$\boldsymbol{u}_2 \cdot \frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t} + \boldsymbol{u}_1 \cdot \frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = 0\,.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_1 \cdot \boldsymbol{u}_2) = 0\,.$$

Thus we obtain three more polynomial first integrals

$$I_3(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \boldsymbol{u}_1 \cdot \boldsymbol{u}_2\,, \quad I_4(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \boldsymbol{u}_2 \cdot \boldsymbol{u}_3\,,$$
$$I_5(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \boldsymbol{u}_3 \cdot \boldsymbol{u}_1\,.$$

Introducing angles α_{jk} between the vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ provide the constants of motion

$$l_1 l_2 \cos(\alpha_{12}) = c_3\,, \quad l_2 l_3 \cos(\alpha_{23}) = c_4\,,$$
$$l_3 l_1 \cos(\alpha_{31}) = c_5\,.$$

The time evolution of $V(t)$ is given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t} \cdot (\boldsymbol{u}_2 \times \boldsymbol{u}_3) + \frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} \cdot (\boldsymbol{u}_3 \times \boldsymbol{u}_1) + \frac{\mathrm{d}\boldsymbol{u}_3}{\mathrm{d}t} \cdot (\boldsymbol{u}_1 \times \boldsymbol{u}_2)\,.$$

Consequently

$$\frac{\mathrm{d}V}{\mathrm{d}t} = (\boldsymbol{u}_2 \times \boldsymbol{u}_3)^2 + (\boldsymbol{u}_3 \times \boldsymbol{u}_1)^2 + (\boldsymbol{u}_1 \times \boldsymbol{u}_2)^2\,.$$

Next we utilize the identity

$$(\boldsymbol{u}_1 \times \boldsymbol{u}_2)^2 \equiv (\boldsymbol{u}_2 \cdot \boldsymbol{u}_2)(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1) - (\boldsymbol{u}_2 \cdot \boldsymbol{u}_1)(\boldsymbol{u}_1 \cdot \boldsymbol{u}_2)\,.$$

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Introducing angles α_{jk} between the vectors we find

$$\frac{\mathrm{d}V}{\mathrm{d}t} = l_2^2 l_3^2 + l_3^2 l_1^2 + l_1^2 l_2^2 - (c_3^2 + c_4^2 + c_5^2) = F(s_1, s_2, s_3)\,,$$

where $s_1 = l_1^2$, $s_2 = l_2^2$, and $s_3 = l_3^2$. Thus the expression for $\mathrm{d}V/\mathrm{d}t$ does not explicitly contain angles. Thus we can construct an autonomous first order system for s_1, s_2, s_3

$$\frac{\mathrm{d}s_1}{2V} = \frac{\mathrm{d}s_2}{2V} = \frac{\mathrm{d}s_3}{2V} = \frac{\mathrm{d}V}{F(s_1, s_2, s_3)} = \mathrm{d}t\,.$$

Since $s_2 = s_1 - c_1$, $s_3 = s_1 - c_2$ we obtain

$$\begin{aligned} F(s_1, s_2, s_3) &= s_1(s_1 - c_1) + s_1(s_1 - c_2) \\ &\quad + (s_1 - c_1)(s_1 - c_2) - (c_3^2 + c_4^2 + c_5^2)\,, \end{aligned}$$

and therefore

$$\begin{aligned} &(s_1(s_1 - c_1) + s_1(s_1 - c_2) + (s_1 - c_1)(s_1 - c_2) \\ &- (c_3^2 + c_4^2 + c_5^2))\mathrm{d}s_1 = 2V\,\mathrm{d}V\,. \end{aligned}$$

Integration provides the first integral

$$\begin{aligned} I_6(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) &= s_1^3 - (c_1 + c_2)s_1^2 \\ &\quad + (c_1 c_2 - (c_3^2 + c_4^2 + c_5^2))s_1 - V^2\,, \end{aligned}$$

with the constant of motion $I_6 = c_6$, where

$$c_6 = -2c_3c_4c_5 - c_2c_3^2 - c_1c_5^2\,.$$

Thus

$$V = \pm\sqrt{s_1^3 - (c_1 + c_2)s_1^2 + (c_1c_2 - (c_3^2 + c_4^2 + c_5^2))s_1 - c_6}\,.$$

Therefore $s_1(t)$ is inversion of the elliptic integral

$$\frac{\mathrm{d}s_1}{\sqrt{s_1^3 - (c_1 + c_2)s_1^2 + (c_1c_2 - (c_3^2 + c_4^2 + c_5^2))s_1 - c_6}} = \pm 2\mathrm{d}t.$$

We know the lengths of all vectors $l_j(t)$ and angles between them $\alpha_{jk}(t)$.

To find the remaining first integrals we proceed as follows. Consider the matrix M, with columns which are the vectors $\boldsymbol{u}_j(t)$

$$M(t) = (\boldsymbol{u}_1(t)\boldsymbol{u}_2(t)\boldsymbol{u}_3(t))\,.$$

Thus we obtain the symmetric matrix

$$M^{\mathrm{T}}M = \begin{pmatrix} \boldsymbol{u}_1^2 & \boldsymbol{u}_1\cdot\boldsymbol{u}_2 & \boldsymbol{u}_1\cdot\boldsymbol{u}_3 \\ \boldsymbol{u}_2\cdot\boldsymbol{u}_1 & \boldsymbol{u}_2^2 & \boldsymbol{u}_2\cdot\boldsymbol{u}_3 \\ \boldsymbol{u}_3\cdot\boldsymbol{u}_1 & \boldsymbol{u}_3\cdot\boldsymbol{u}_2 & \boldsymbol{u}_3^2 \end{pmatrix},$$

and for MM^{T} the symmetric matrix

$$\begin{pmatrix} u_{11}^2 + u_{21}^2 + u_{31}^2 & u_{11}u_{12} + u_{21}u_{22} + u_{31}u_{32} & u_{11}u_{13} + u_{21}u_{23} + u_{31}u_{33} \\ u_{12}u_{11} + u_{22}u_{21} + u_{32}u_{31} & u_{12}^2 + u_{22}^2 + u_{32}^2 & u_{12}u_{13} + u_{22}u_{23} + u_{32}u_{33} \\ u_{11}u_{13} + u_{21}u_{23} + u_{31}u_{33} & u_{12}u_{13} + u_{22}u_{23} + u_{32}u_{33} & u_{13}^2 + u_{23}^2 + u_{33}^2\,. \end{pmatrix}.$$

We see that $\det(M^{\mathrm{T}}M) = V^2$. Now we have

$$\frac{\mathrm{d}M^{\mathrm{T}}}{\mathrm{d}t}M = VI_3\,, \quad M^{\mathrm{T}}\frac{\mathrm{d}M}{\mathrm{d}t} = VI_3\,,$$

$$\frac{\mathrm{d}M}{\mathrm{d}t}M^{\mathrm{T}} = VI_3\,, \quad M\frac{\mathrm{d}M^{\mathrm{T}}}{\mathrm{d}t} = VI_3\,.$$

Consequently the time derivative of the commutator of M and M^{T} vanishes, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}([M, M^{\mathrm{T}}]) = 0_3\,.$$

Thus the entries of the 3×3 matrix $[M, M^{\mathrm{T}}]$ are first integrals. Since the matrix $MM^{\mathrm{T}} - M^{\mathrm{T}}M$ is symmetric and $\mathrm{tr}(MM^{\mathrm{T}} - M^{\mathrm{T}}M) = 0$ we find 5 more first integrals, namely

$$\begin{aligned} I_6 &= u_{21}^2 + u_{31}^2 - u_{12}^2 - u_{13}^2\,, \\ I_7 &= u_{12}^2 + u_{32}^2 - u_{21}^2 - u_{23}^2\,, \\ I_8 &= (u_{11} - u_{22})(u_{12} - u_{21}) + u_{31}u_{32} - u_{13}u_{23} \\ &= u_{11}u_{12} + u_{22}u_{21} + u_{31}u_{32} - I_3\,, \\ I_9 &= (u_{11} - u_{33})(u_{13} - u_{31}) + u_{21}u_{23} - u_{12}u_{32} \\ &= u_{11}u_{13} + u_{31}u_{33} + u_{21}u_{23} - I_5\,, \\ I_{10} &= (u_{22} - u_{33})(u_{23} - u_{32}) + u_{12}u_{13} - u_{21}u_{31} \\ &= u_{22}u_{23} + u_{33}u_{32} + u_{12}u_{13} - I_4\,. \end{aligned}$$

Two of the first integrals are dependent. Thus we have altogether 8 independent first integrals and the dynamical system (1) is completely integrable.

Since we have 8 first integrals the question arises whether the dynamical system (1) can be reconstructed with Nambu mechanics.[3–5] In Nambu mechanics the equations of motion are given by

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = \frac{\partial(u_j, I_1, \ldots, I_{n-1})}{\partial(u_1, u_2, \ldots, u_n)}\,, \quad j = 1, 2, \ldots, n\,,$$

where $\partial(u_j, I_1, \ldots, I_{n-1})/\partial(u_1, u_2, \ldots, u_n)$ denotes the Jacobian determinant and $I_k : \mathbb{R}^n \to \mathbb{R}$ $(k = 1, \ldots, n-1)$ are $n-1$ smooth functions. Then the I_k $(k = 1, \ldots, n-1)$ are first integrals of the dynamical system. Thus starting from the 8 first integrals given above we find

$$\frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t} = (\boldsymbol{u}_2 \times \boldsymbol{u}_3)f(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)\,,$$

$$\frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = (\boldsymbol{u}_3 \times \boldsymbol{u}_1)f(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)\,,$$

$$\frac{\mathrm{d}\boldsymbol{u}_3}{\mathrm{d}t} = (\boldsymbol{u}_1 \times \boldsymbol{u}_2)f(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)\,,$$

where f is a polynomial in $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$. A computer algebra program for Nambu mechanics was provided by Hardy *et al.*[6]

The dynamical system also passes the Painlevé test.[7] Furthermore system (1) admits a Lax representation. This means it can be written in the matrix form $\mathrm{d}L/\mathrm{d}t = [A, L](t)$. Finally since the system (1) admits first integrals it can be written in skew-gradient form. This can be used to derive a discrete version of (1) which preserves the first integrals.

The vector product is intrinsic to $\mathbb{R}^3$. An extension of this integrable system to higher dimensions is as follows. Using the exterior product $\wedge$ (note that the exterior product is associative) and the f-linear Hodge star operator $*$ with the metric tensor field[2,8–9]

$$g = \mathrm{d}x_1 \otimes \mathrm{d}x_1 + \mathrm{d}x_2 \otimes \mathrm{d}x_2 + \mathrm{d}x_3 \otimes \mathrm{d}x_3 \,,$$

system (1) can be written as

$$\frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t} = *(\boldsymbol{u}_2 \wedge \boldsymbol{u}_3)\,, \quad \frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = *(\boldsymbol{u}_3 \wedge \boldsymbol{u}_1)\,,$$

$$\frac{\mathrm{d}\boldsymbol{u}_3}{\mathrm{d}t} = *(\boldsymbol{u}_1 \wedge \boldsymbol{u}_2)\,.$$

Thus an extension to four dimensions would be

$$\frac{\mathrm{d}\boldsymbol{u}_1}{\mathrm{d}t} = *(\boldsymbol{u}_2 \wedge \boldsymbol{u}_3 \wedge \boldsymbol{u}_4)\,, \quad \frac{\mathrm{d}\boldsymbol{u}_2}{\mathrm{d}t} = *(\boldsymbol{u}_3 \wedge \boldsymbol{u}_4 \wedge \boldsymbol{u}_1)\,,$$

$$\frac{\mathrm{d}\boldsymbol{u}_3}{\mathrm{d}t} = *(\boldsymbol{u}_4 \wedge \boldsymbol{u}_1 \wedge \boldsymbol{u}_2)\,, \quad \frac{\mathrm{d}\boldsymbol{u}_4}{\mathrm{d}t} = *(\boldsymbol{u}_1 \wedge \boldsymbol{u}_2 \wedge \boldsymbol{u}_3)\,,$$

with the metric tensor field

$$g = \mathrm{d}x_1 \otimes \mathrm{d}x_1 + \mathrm{d}x_2 \otimes \mathrm{d}x_2 + \mathrm{d}x_3 \otimes \mathrm{d}x_3 + \mathrm{d}x_4 \otimes \mathrm{d}x_4\,.$$

The extension to higher dimension is now obvious. Other metric tensor fields could also be considered.

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