DOMINATION RESULTS:
VERTEX PARTITIONS AND EDGE WEIGHT FUNCTIONS

by

Justin Gilfillan Southey

Submitted in fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics
in the
Faculty of Science
at the
University of Johannesburg

Supervisor: Prof. Michael A. Henning

January 2012

To the many who have seen beauty in mathematics;

and the few who have seen both in me.

Like Poetry, Mathematics is Beautiful

Timidly I ask
each one I meet if they
find mathematics beautiful
or useful, and each one dares to say,
"Useful, of course. I use it every day."
And if I seem to want a proof,
they all go on to tell
that daily they subtract and add
to keep a checkbook; sometimes also
they multiply to find how many squares
they need to tile the kitchen floor.

Mathematics is not only plus
and minus, not just counting one,
two, three. There are rules to bend
defiantly, so parallels
will meet before infinity. Look
at the magic of unending terms
that converge to a finite sum:
start with one-half plus half of one-half
plus half of the last again and again.
Though we go on forever, we never
pass one. Do you find me difficult? Oh, dear!

Suppose, instead, I ask
if poetry is beautiful
or useful. Will each person say,
"Useful, of course. I use it every day."
And if I seem to want a proof,
will they go on to say that they
use rhymes to call to mind the days
of a month - like "Thirty hath
September" - and to remember
how to spell words with 'i' and 'e'.

I have a faint, enduring hope
that someday folks will see
mathematics to be
as lovely
as poetry.

JoAnne Growney (Silver Spring, MD)
American Mathematical Monthly **101** (1994) 484

Preface

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [45, 46]. In this thesis, we continue the study of domination, by adding to the theory; improving a number of known bounds and solving two previously published conjectures.

With the exception of the introduction, each chapter in this thesis corresponds to a single paper already published or submitted as a journal article. Despite the seeming disparity in the content of some of these articles, there are two overarching goals achieved in this thesis. The first is an attempt to partition the vertex set of a graph into two sets, each holding a specific domination-type property. The second is simply to improve known bounds for various domination parameters. In particular, an edge weighting function is presented which has been useful in providing some of these bounds.

Although the research began as two separate areas of focus, there has been a fair degree of overlap and a number of the results contained in this thesis bridge the gap quite pleasingly. Specifically, Chapter 11 uses the edge weighting function to prove a bound on one of the sets in our most fundamental partitions, while the improvement on a known bound presented in Chapter 7 was inspired by considering the possible existence of another partition. This latter proof relies implicitly on the 'almost' existence of such a partition.

In Chapter 1, we outline the results of the thesis and introduce some basic notation. We prove an existence result for "dominating, total dominating, partitionable" graphs in Chapter 2, characterize all such graphs in Chapter 3, and then examine the case when such a partition is exhaustive in Chapter 4. We prove a similar existence result for "dominating paired-dominating partitionable" graphs in Chapter 5 and again characterize all such graphs in Chapter 6. In Chapter 7 we improve on a published upper bound on the total restrained domination number in cubic graphs and in Chapter 8 we investigate the ratio of the independent domination number to the domination number in cubic graphs. We then introduce an edge weighting function on dominating sets in Chapter 9 and apply it to provide bounds on the upper domination number and the upper total domination number in regular graphs. In Chapter 10, we solve a conjectured bound on the total domination number in claw-free cubic graphs using a modified edge weighting function. Finally, in Chapter 11 we use this weighting argument to provide a bound on one of the sets in the partition presented in Chapter 2.

Chapters 2, 3, 4, 5, 6, 7, 8, 10, and 11 have been published or accepted for publication in [65], [66], [61], [87], [88], [89], [90], [91] and [92], respectively, while Chapter 9 has been submitted for journal consideration; see [93]. In addition, though not directly linked to the topics presented in this thesis, the author has been involved in four further journal articles accepted or submitted for publication; see [37], [55], [56], and [57].

Acknowledgement

I wish to thank the South African National Research Foundation for their financial support of my studies. I offer special thanks to my supervisor, Professor Michael Henning, for his constant patience and guidance and for proposing that first and gripping problem, just on the edge of my untried ability. I am grateful to everyone who has encouraged and enabled my journey; but most particularly my parents, whose unfailing belief in me continues to dare me to dream.

Contents

Chapter 1

Introduction and Overview

In this chapter, we provide an overview of the thesis and then introduce some standard definitions and notation. Specific notation required only sporadically or in one chapter will be introduced as required. Similarly, non-standard terminology used in proofs or to simplify reading will be presented at a convenient proximity to its usage.

In Chapter 2, we show that the vertex set of every graph with minimum degree at least two and no 5-cycle component can be partitioned into a dominating set and a total dominating set. Exceedingly simple to state, this almost surprising existence sowed the seed for many of the ideas presented in this thesis. In Chapter 3 we go on to provide a constructive characterization of first the trees, and then the graphs, whose vertex set can be partitioned into a dominating set and a total dominating set. We then examine, in Chapter 4, the question of when such a partition necessarily contains the entire vertex set. We answer the question for all graphs with minimum degree at least two and that have no induced five cycle.

The situation is not quite as straightforward when attempting to partition the vertices of a graph into a dominating set and a paired-dominating set. In fact, we demonstrate that no minimum degree is sufficient to guarantee the existence of such a partition in

Chapter 5. However, we prove that the vertex set of every cubic graph can be thus partitioned. In Chapter 6, we provide a constructive characterization of first the trees, and then the graphs, whose vertex set can be partitioned into a dominating set and a paired-dominating set.

Here the thesis diverges temporarily to look at upper bounds on various domination type parameters in various classes of graphs, most frequently cubic. The first of these, however, is implicitly linked to the idea that the vertices in a cubic graph can be partitioned into a total dominating set and an 'almost' total dominating set. Jiang, Kang and Shan [2] showed that the minimum cardinality of a total restrained dominating set of a connected cubic graph of order n is at most $13n/19$. In Chapter 7, we improve this upper bound to $(n+4)/2$ and demonstrate that our new improved bound is essentially best possible. Staying with connected cubic graphs we show, in Chapter 8, that the ratio of the independent domination number to the domination number is at most $4/3$, except in the case of $K(3,3)$. Furthermore, we characterize the graphs achieving this bound.

We introduce the useful edge weighting function on dominating sets in Chapter 9 and show that if we impose a regularity condition on a graph, then upper bounds on both the upper domination number and the upper total domination number can be greatly improved. We show that these bounds are sharp and characterize the infinite families of graphs that achieve equality in both cases. In Chapter 10, we use the same edge weighting function, with additional weight discharging rules, to solve the conjecture posed in [30] that for connected claw-free cubic graphs of order $n \geq 10$, the total domination number is at most $4n/9$.

Chapter 11 brings the thesis full circle, and uses a weighting argument to provide a bound for cubic graphs on one of the sets in the partition presented in Chapter 2. In particular we show that every connected cubic graph on n vertices has a total dominating set whose complement contains a dominating set such that the cardinality of the total dominating set is at most $(n+2)/2$, and this bound is essentially best possible.

Although each chapter covers the content of a single journal article, the thesis has been assembled in such a way that it can be read from cover to cover with a through-running theme. Alternatively, each chapter may be read individually, with all necessary notation and specific terminology required for the presented results included in the relevant chapter. To avoid the construction of artificially unique and cumbersome labels, some function or family names have been recycled in later chapters. The meanings, however, should be clear in the context of the chapter, and hopefully make for simpler reading.

1.1 General Notation

For notation and graph theory terminology we in general follow [45]. Specifically, let $G = (V, E)$ be a simple undirected graph with vertex set $V(G)$ of order $n(G) = |V(G)|$ and edge set $E(G)$ of size $m(G) = |E(G)|$. If the graph G is clear from context, we abbreviate $V(G)$ to V, $E(G)$ to E, $n(G)$ to n and $m(G)$ to m. Let $S \subseteq V$ be a subset of vertices in G and let u and v be vertices in V.

We denote the degree of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$). We call a vertex of degree k a *degree-k vertex*. A graph is k-regular if every vertex in the graph has degree k. A 3-regular graph is also called a *cubic graph*. We denote the number of vertices of S adjacent to v in G by $d_S(v)$. In particular, $d_V(v) = d_G(v)$.

If G is a connected graph, then the *distance* $d_G(u, v)$ between u and v is the length of a shortest u–v path in G. The *eccentricity* $e(v)$ of the vertex v is the distance between v and a vertex farthest from v in G. The maximum eccentricity among the vertices of G is its *diameter*, which is denoted by $\text{diam}(G)$. If $e(v) = \text{diam}(G)$, then v is called a *diametrical vertex*. A *u–v walk* is an alternating sequence of vertices and edges, starting with u and ending with v, and with each edge being incident to the vertices immediately preceding and succeeding it in the sequence.

By a proper subgraph of a graph G we mean a subgraph of G that is different from G. The subgraph induced by S is denoted by $G[S]$, or simply by G_S, while the graph $G - S$ is the graph obtained from G by deleting the vertices in S and all edges incident with S. For a set $M \subseteq E$, the graph $G - M$ is the graph obtained from G by deleting all the edges in M. If X and Y are two subsets of V, we denote the set of all edges of G that join a vertex of X and a vertex of Y by $[X, Y]$.

The *open neighborhood* of v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. For the set S, its *open neighborhood* is the set $N_G(S) = \cup_{v \in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from context, we simply write $N(v)$, $N[v]$, $N(S)$, and $N[S]$.

For the following definitions let v be a vertex in S. The *S-private neighborhood* of v is defined by $\text{pn}[v, S] = \{w \in V \mid N_G[w] \cap S = \{v\}\}$, while its *open S-private neighborhood* is defined by $\text{pn}(v, S) = \{w \in V \mid N_G(w) \cap S = \{v\}\}$. We remark that the sets $\text{pn}[v, S] \setminus S$ and $\text{pn}(v, S) \setminus S$ are equivalent and define the *S-external private neighborhood* of v to be this set, abbreviated $\text{epn}[v, S]$ or $\text{epn}(v, S)$. The *S-internal private neighborhood* of v is defined by $\text{ipn}[v, S] = \text{pn}[v, S] \cap S$ and its *open S-internal private neighborhood* is defined by $\text{ipn}(v, S) = \text{pn}(v, S) \cap S$. We define an *S-external private neighbor* of v to be a vertex in $\text{epn}(v, S)$ and an *S-internal private neighbor* of v to be a vertex in $\text{ipn}(v, S)$. We remark that either v is isolated in $G[S]$, in which case $\text{ipn}[v, S] = \{v\}$, or v has at least one neighbor in S, in which case $\text{ipn}[v, S] = \emptyset$. Thus, $\text{ipn}[v, S] \in \{\emptyset, \{v\}\}$.

A *matching* in a graph G is a set of independent edges in G. If M is a matching in G, an *M-matched vertex* is a vertex incident with an edge in M while an *M-unmatched vertex* is a vertex not incident with an edge in M. An *M-alternating path* of G is a path whose edges are alternately in M and not in M. A *perfect matching* M in G is a matching in G such that every vertex of G is incident to an edge of M.

Let X and Y be two subsets of V. The set X *dominates* Y in G if $Y \subseteq N[X]$, while X *totally dominates* Y in G if $Y \subseteq N(X)$. In particular, if X dominates V, then X is called a *dominating set* of G, abbreviated DS. If X totally dominates V, then X is called a *total dominating set* of G, abbreviated TDS. Hence, S is a DS of G if $N[S] = V$, while S is a TDS of G if $N(S) = V$. If S totally dominates V and $G[S]$ contains a perfect matching M (not necessarily induced), then S is called a *paired-dominating set* of G, abbreviated PDS. Two vertices joined by an edge of M are said to be *paired* and are also called *partners* in S. The set S is a a *total restrained dominating set*, abbreviated TRDS, of G if S is a TDS and, in addition, every vertex of $V \setminus S$ is adjacent to a vertex in $V \setminus S$. An *independent dominating set* of G, abbreviated ID-set, is a set that is both dominating and independent in G.

The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a DS of G. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS of G. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS of G. The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of an ID-set of G. A DS of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-set, a TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set, a TRDS of G of cardinality $\gamma_{tr}(G)$ is called a $\gamma_{tr}(G)$-set, and an ID-set of G of cardinality $i(G)$ is called an $i(G)$-set. A DS (resp., TDS, TRDS, PDS, ID-set) S is said to be minimal if, for all vertices $v \in S$, we have that $S \setminus \{v\}$ is not a DS (resp., TDS, TRDS, PDS, ID-set).

The *upper domination number*, $\Gamma(G)$, of a graph G is the maximum cardinality of a minimal DS in G and we call a minimal DS of cardinality $\Gamma(G)$ a $\Gamma(G)$-set. Similarly, the *upper total domination number*, $\Gamma_t(G)$, of a graph G is the maximum cardinality of a minimal TDS in G and we call a minimal TDS of cardinality $\Gamma_t(G)$ a $\Gamma_t(G)$-set.

A *rooted tree* distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique r–v path, while a *child* of v is any other neighbor of v. We let $C(v)$ denote the set of children of v. A *descendant* of v is a vertex

u such that the unique r–u path contains v. Thus, every child of v is a descendant of v. A vertex of degree one is called a *leaf* and its neighbor is called a *support* vertex. A *strong support vertex* is adjacent to at least two leaves.

A *path* on n vertices is denoted by P_n and a *cycle* on n vertices by C_n. By a P_n-component (resp., C_n-component) of a graph we mean a component of the graph isomorphic to a path (resp., cycle) on n vertices. We say that a graph is F*-free* if it does not contain F as an induced subgraph. In particular, if $F = C_5$, then we say that the graph is C_5*-free*. Further, if $F = K_{1,3}$, then we say that the graph is *claw-free*.

Chapter 2

The Existence of DTDP Graphs

A simple yet fundamental observation in domination theory made by Ore [80] is that every graph of minimum degree at least one contains two disjoint dominating sets. Thus, the vertex set of every graph without isolated vertices can be partitioned into two dominating sets. In contrast to that, Zelinka [99, 100] showed that no minimum degree is sufficient to guarantee the existence of three disjoint dominating sets or of two disjoint total dominating sets. Clearly, if the domatic number [100] of a graph G is at least $2k$, then, by definition, G contains $2k$ disjoint dominating sets and hence also k disjoint total dominating sets. Therefore, the results of Calkin et al. [7] and Feige et al. [32] imply that a sufficiently large minimum degree and small maximum degree together imply the existence of arbitrarily many disjoint (total) dominating sets.

To see that no minimum degree is sufficient to guarantee the existence of two total dominating sets, consider the bipartite graph G_n^k formed by taking as one partite set a set A of n elements, and as the other partite set all the k-element subsets of A, and joining each element of A to those subsets it is a member of. Then G_n^k has minimum degree k. As observed in [99], if $n \geq 2k-1$ then in any 2-coloring of A at least k vertices must receive the same color, and these k are the neighborhood of some vertex.

In contrast, results of Calkin and Dankelmann [7] and Feige et al. [32] show that if the maximum degree is not too large relative to the minimum degree, then sufficiently large minimum degree does suffice.

Heggernes and Telle [51] showed that the decision problem to decide if there is a partition of $V(G)$ into two total dominating sets is NP-complete, even for bipartite graphs. Broere et al. [6] considered the question of how many edges must be added to G to ensure a partition of V into two total dominating sets in the resulting graph. They denote this minimum number by $td(G)$. It is clear that $td(G)$ can only exist for graphs with at least four vertices. In particular, it was shown that if T is a tree with ℓ leaves, then $\ell/2 \leq td(T) \leq \ell/2+1$. Dorfling et al. [24] showed that given a graph of order $n \geq 4$ with minimum degree at least 2, one can add at most $(n-2\sqrt{n}\,)/4+O(\log n)$ edges such that the resulting graph has two disjoint total dominating sets, and this bound is best possible.

In this chapter we give an exchange argument for a result which is somehow located between Ore's positive and Zelinka's negative observations. More specifically, we consider the question of whether the vertex set of every graph with minimum degree at least two can be partitioned into a dominating set and a total dominating set. In future chapters, we shall call such a graph a DTDP-graph (standing for "dominating, total dominating, partitionable graph").

2.1 DTDP Existence Result

Clearly the vertex set of a 5-cycle C_5 cannot be partitioned into a dominating set and a total dominating set. We show that this is the only exception. Before presenting the result we introduce the following notation for this chapter. For $S \subseteq V$ and $v \in S$, we say that v is an S-*bad* vertex if $N[v] \subseteq S$. Further, we say that a vertex $u \in S$ is an S-*weak* vertex if u has degree 1 in $G[S]$ and its neighbor in S is an S-bad vertex. We now prove:

Theorem 2.1 *If $G = (V, E)$ is a graph with $\delta(G) \geq 2$ that contains no C_5-component, then V can be partitioned into a dominating set and a total dominating set.*

Proof. Among all total dominating sets of G, let S be chosen so that

(1) the number of S-bad vertices is minimized, and

(2) subject to (1), the number of S-weak vertices is minimized.

Assume that there is at least one S-bad vertex. Let v be such a vertex. If v has no S-weak neighbor, then $S' = S \setminus \{v\}$ is a total dominating set of G with fewer S'-bad vertices than S-bad vertices, contradicting our choice of S. Hence we may assume that every S-bad vertex has at least one S-weak neighbor.

Let w be an S-weak vertex. Since $\delta(G) \geq 2$, w is adjacent to at least one vertex in $V \setminus S$. If $\text{epn}(w, S) = \emptyset$, then $S' = S \setminus \{w\}$ is a total dominating set of G with fewer S'-bad vertices than S-bad vertices, contradicting our choice of S. Hence, $|\text{epn}(w, S)| \geq 1$. For each S-weak vertex w, let $w' \in \text{epn}(w, S)$. Since $\delta(G) \geq 2$, w' is adjacent to at least one vertex in $V \setminus S$ and $N[w'] \setminus \{w\} \subseteq V \setminus S$.

We show next that every S-weak vertex has degree 2 in G. As defined earlier, let w be an S-weak vertex and suppose that $\deg w \geq 3$. Then, $S' = S \cup \{w'\}$ is a total dominating set of G that satisfies condition (1), but with fewer S'-weak vertices than S-weak vertices, contradicting our choice of S. Hence, every S-weak vertex has degree 2.

As defined earlier, let v be an S-bad vertex. Then, v has at least one S-weak neighbor. For $k \geq 1$, let $W = \{w_1, \ldots, w_k\}$ be the set of all S-weak neighbors of v. Then, $N(w_i) = \{v, w'_i\}$ for $i = 1, \ldots, k$. Let $W' = \{w'_1, \ldots, w'_k\}$.

If every vertex in W' is adjacent to a vertex in $V \setminus (S \cup W')$, then $S' = (S \cup W') \setminus \{v\}$ is a total dominating set of G with fewer S'-bad vertices than S-bad vertices, contradicting our choice of S. Hence, renaming vertices if necessary, we may assume that $N[w'_1] \subseteq W' \cup \{w_1\}$ and that $w'_1 w'_2$ is an edge of G.

If $\deg v \geq 3$, then $S' = (S \cup \{w_1', w_2'\}) \setminus \{w_1, w_2\}$ is a total dominating set of G with fewer S'-bad vertices than S-bad vertices, contradicting our choice of S. Hence each of v, w_1, w_1' and w_2 has degree 2 in G and $C\colon v, w_1, w_1', w_2', w_2, v$ is an induced 5-cycle in G.

Since G contains no C_5-component, the vertex w_2' is adjacent to some vertex not in the 5-cycle C. But then $S' = (S \cup \{w_1', w_2'\}) \setminus \{v, w_1\}$ is a total dominating set of G with fewer S'-bad vertices than S-bad vertices, contradicting our choice of S. We deduce, therefore, that the total dominating set S contains no S-bad vertices. Hence, $V \setminus S$ is a dominating set of G, and we are done. $\square$

We close the chapter with the remark that the minimum degree condition of Theorem 2.1 cannot be relaxed to $\delta(G) \geq 1$. Some examples are given at the beginning of the next chapter.

Chapter 3

Characterizing DTDP Graphs

In Chapter 2, we showed that every graph with minimum degree at least two that contains no C_5-component is a DTDP-graph. (Recall that DTDP-graph stands for "dominating, total dominating, partitionable graph".)

Not every graph with minimum degree one is a DTDP-graph. The simplest such counterexample is a star $K_{1,n}$. The graph obtained from the corona $\text{cor}(H)$ of an arbitrary graph H (denoted $H \circ K_1$ in [45] and defined to be the graph obtained from H by adding a pendant edge to each vertex of H) by subdividing at least one of the added pendant edges is another example of a graph that is not a DTDP-graph and whose diameter can be made arbitrarily large (by choosing H to have large diameter).

3.1 Graph Labelings

Our aim in this chapter is to provide a constructive characterization of DTDP-graphs. The key to our constructive characterization is to find a labeling of the vertices that indicates the role each vertex plays in the sets associated with both parameters. This idea of labeling the vertices is introduced in [25], where trees with equal domination and independent domination numbers as well as trees with equal domination and total domination numbers are characterized.

We define a *labeling* of a graph G as a partition $S = (S_A, S_B)$ of $V(G)$. The *label* or *status* of a vertex v, denoted $\text{sta}(v)$, is the letter $x \in \{A, B\}$ such that $v \in S_x$. Our aim is to describe a procedure to build DTDP-graphs in terms of labelings. By a *labeled-P_4*, we shall mean a P_4 with the two central vertices labeled A and the two leaves labeled B.

3.1.1 The Graph Family $\mathcal{T}$

Let $\mathcal{T}$ be the minimum family of labeled trees that: (i) contains a labeled-P_4; and (ii) is closed under the four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ listed below, which extend a labeled tree T by attaching a tree to the vertex $v \in V(T)$.

- **Operation $\mathcal{O}_1$.** Assume $\text{sta}(v) = A$. Add a vertex u_1 and the edge vu_1. Let $\text{sta}(u_1) = B$.

- **Operation $\mathcal{O}_2$.** Assume $\text{sta}(v) = A$. Add a path u_1u_2 and the edge vu_1. Let $\text{sta}(u_1) = A$ and $\text{sta}(u_2) = B$.

- **Operation $\mathcal{O}_3$.** Assume $\text{sta}(v) = B$. Add a path $u_1u_2u_3$ and the edge vu_1. Let $\text{sta}(u_1) = \text{sta}(u_2) = A$ and $\text{sta}(u_3) = B$.

- **Operation $\mathcal{O}_4$.** Assume $\text{sta}(v) = B$. Add a path $u_1u_2u_3u_4$ and the edge vu_1. Let $\text{sta}(u_1) = \text{sta}(u_4) = B$ and $\text{sta}(u_2) = \text{sta}(u_3) = A$.

These four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ are illustrated in Figure 3.1.

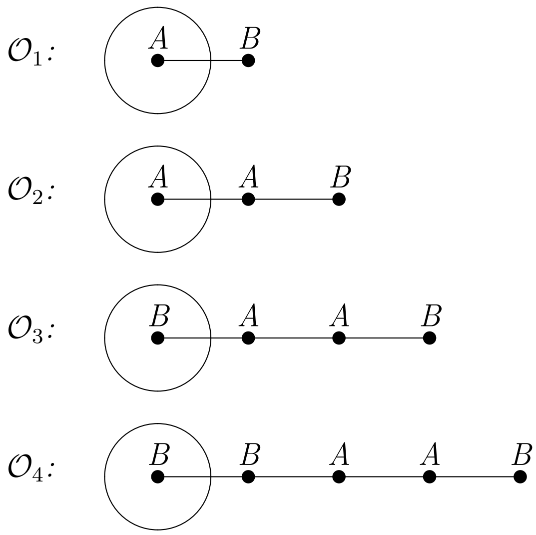


Figure 3.1: The four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$.

3.1.2 The Graph Family $\mathcal{G}$

Let $\mathcal{O}_5$, $\mathcal{O}_6$, and $\mathcal{O}_7$ be the three operations listed below, which extend a labeled graph G as follows:

- **Operation $\mathcal{O}_5$.** Let u and v be two nonadjacent vertices in G. Add the edge uv.
- **Operation $\mathcal{O}_6$.** Let $v \in V(G)$ and assume $\text{sta}(v) = B$. Add a path u_1u_2 and the edges vu_1 and vu_2. Let $\text{sta}(u_1) = \text{sta}(u_2) = A$.
- **Operation $\mathcal{O}_7$.** Let u and v be distinct vertices of G. Assume $\text{sta}(u) = \text{sta}(v) = B$. Add a path u_1u_2 and the edges uu_1 and vu_2. Let $\text{sta}(u_1) = \text{sta}(u_2) = A$.

These three operations are illustrated in Figure 3.2.

Let $\mathcal{G}$ be the minimum family of labeled graphs that: (i) contains a labeled-P_4; and (ii) is closed under the seven operations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_7$ described earlier.

By construction, the family $\mathcal{T}$ is a subfamily of the family $\mathcal{G}$. We shall need the following observation which follows from the way in which the family $\mathcal{G}$ is constructed.

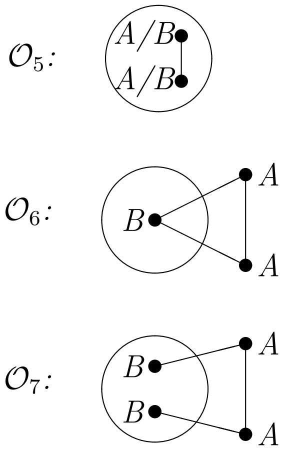


Figure 3.2: The three operations $\mathcal{O}_5$, $\mathcal{O}_6$ and $\mathcal{O}_7$.

Observation 3.1 *Let $(G, S) \in \mathcal{G}$ for some labeling $S = (S_A, S_B)$. Then the following properties hold:*

(a) *Every vertex of status A is adjacent to a vertex of status A and to a vertex of status B.*

(b) *Every vertex of status B is adjacent to a vertex of status A.*

(c) *S_A is a TDS of G, while S_B is a DS of G.*

(d) *If $(G, S) \in \mathcal{T}$, then every leaf of G has status B and every support vertex has status A.*

3.2 DTDP Characterization Results

In this chapter, we have two immediate aims. Our first aim is to determine which trees are DTDP-trees. For this purpose, we establish the following constructive characterization of DTDP-trees that uses labelings, a proof of which is presented in Section 3.2.1.

Theorem 3.2 *The DTDP-trees are precisely those trees T such that $(T, S) \in \mathcal{T}$ for some labeling S.*

Our second aim is to determine which connected graphs with minimum degree one are DTDP-graphs. We remark that if a connected graph has a spanning DTDP-tree, then it is a DTDP-graph. However, a connected DTDP-graph does not necessarily have a spanning DTDP-tree. For example, let G_k be obtained from the disjoint union of $k \geq 1$ copies of K_3 by adding a path P_3 and joining a leaf of the added path to one vertex from each copy of K_3. The graph G_3 is illustrated in Figure 3.3. Then, G_k is a DTDP-graph but G_k does not have a spanning DTDP-tree, a proof of which can be found in Section 3.2.3. We remark that we could have replaced some or all of the copies of K_3 in G_k with copies of C_6 or C_9.

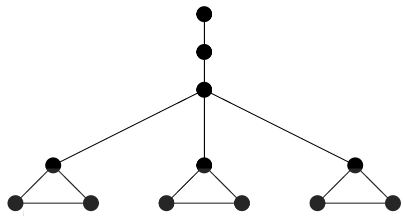


Figure 3.3: The graph G_3.

Every DTDP-graph has order at least 3. Trivially, the only DTDP-graph of order 3 is the complete graph K_3. Our main result is the following constructive characterization of DTDP-graphs of order at least 4 that uses labelings, a proof of which is presented in Section 3.2.2.

Theorem 3.3 *The connected DTDP-graphs of order at least* 4 *are precisely those graphs* G *such that* $(G, S) \in \mathcal{G}$ *for some labeling* S.

3.2.1 Proof of Theorem 3.2

Since every TDS in a tree contains all the support vertices, we have the following observation.

Observation 3.4 *Let T be a rooted DTDP-tree and let $D = (D_1, D_2)$ be a partition of $V(T)$ into a TDS D_1 and a DS D_2. Then the following properties hold:*

(a) *Every leaf belongs to D_2 while every support vertex belongs to D_1.*

(b) *If every child of a vertex is a leaf, then its parent belongs to D_1.*

Recall the statement of Theorem 3.2.

Theorem 3.2. *The DTDP-trees are precisely those trees T such that $(T, S) \in \mathcal{T}$ for some labeling S.*

Proof. Suppose first that T is a tree and $(T, S) \in \mathcal{T}$ for some labeling S. By Observation 3.1(c), (S_A, S_B) is a partition of $V(T)$ into a TDS S_A and a DS S_B, and so T is a DTDP-tree. This establishes the sufficiency.

To prove the necessity, we proceed by induction on the order n of a DTDP-tree T. Since every star $K_{1,n-1}$ is not a DTDP-tree, we have that $n \geq 4$ and $\text{diam}(T) \geq 3$. If $n = 4$, then $T = P_4$ and $(T, S) \in \mathcal{T}$, where S is the labeling of a labeled-P_4. This establishes the base case. For the inductive hypothesis, let $n \geq 5$ and assume that for every DTDP-tree T' of order less than n there exists a labeling S' such that $(T', S') \in \mathcal{T}$.

Let T be a DTDP-tree of order n. Let $D = (D_1, D_2)$ be a partition of $V(T)$ into a TDS D_1 and a DS D_2. We now root the tree T at a diametrical vertex r. Necessarily, r is a leaf. Let u be a vertex at maximum distance from r. Necessarily, u is a leaf. Let v be the parent of u, let w be the parent of v, and let x be the parent of w (possibly, $x = r$). Since u is at maximum distance from the root r, every child of v is a leaf. Then, by Observation 3.4, we observe that $C(v) \subset D_2$ and $\{v, w\} \subseteq D_1$. In particular, $u \in D_2$.

Suppose that T has a strong support vertex z. Let z_1 and z_2 be two leaf-neighbors of z in T. By Observation 3.4, we observe that $\{z_1, z_2\} \subseteq D_2$ and $z \in D_1$. Let $T' = T - z_1$. Then, $(D_1, D_2 \setminus \{z_1\})$ is a partition of $V(T')$ into a TDS D_1 and a DS $D_2 \setminus \{z_1\}$. Hence, T' is a DTDP-tree. Applying the inductive hypothesis to T', there exists a labeling

$S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. By Observation 3.1(d), $z \in S'_A$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_1$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A, S'_B \cup \{z_1\})$. Hence, if T has a strong support vertex, then $(T, S) \in \mathcal{T}$ for some labeling S, as desired. Hence we may assume that T has no strong support vertex. In particular, $d(v) = 2$.

Suppose $d(w) \geq 3$. Let $v' \in C(w) \setminus \{v\}$. Suppose $d(v') \geq 2$. By our choice of the vertex u, every child of v' is a leaf. Since T has no strong support vertex, $d(v') = 2$. Let u' be the child of v'. Then, u' is a leaf. By Observation 3.4, $\{u, u'\} \subseteq D_2$ and $\{v, v', w\} \subseteq D_1$. Let $T' = T - \{u', v'\}$. Then, $(D_1 \setminus \{v'\}, D_2 \setminus \{u'\})$ is a partition of $V(T')$ into a TDS $D_1 \setminus \{v'\}$ and a DS $D_2 \setminus \{u'\}$. Hence, T' is a DTDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. By Observation 3.1, $\{v, w\} \subseteq S'_A$ and $u \in S'_B$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_2$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v'\}, S'_B \cup \{u'\})$. Hence, if $d(v') \geq 2$, then $(T, S) \in \mathcal{T}$ for some labeling S, as desired. Therefore we may assume that every child of w, different from v, is a leaf. Thus since T has no strong support vertex, $d(w) = 3$ and $C(w) = \{v, v'\}$, where v' is a leaf. By Observation 3.4, $\{u, v'\} \subseteq D_2$ and $\{v, w\} \subseteq D_1$.

Suppose $x \in D_1$. Let $T' = T - \{u, v\}$. Then, $(D_1 \setminus \{v\}, D_2 \setminus \{u\})$ is a partition of $V(T')$ into a TDS $D_1 \setminus \{v\}$ and a DS $D_2 \setminus \{u\}$. Hence, T' is a DTDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. By Observation 3.1, $v' \in S'_B$ and $w \in S'_A$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_2$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v\}, S'_B \cup \{u\})$. Hence, if $x \in D_1$, then $(T, S) \in \mathcal{T}$ for some labeling S, as desired. Thus we may assume that $x \in D_2$.

We now let $T' = T - v'$. Then, $(D_1, D_2 \setminus \{v'\})$ is a partition of $V(T')$ into a TDS D_1 and a DS $D_2 \setminus \{v'\}$. Hence, T' is a DTDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. By Observation 3.1,

$\{v, w\} \subseteq S'_A$ and $u \in S'_B$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_1$ to T'. Hence, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A, S'_B \cup \{v'\})$. We have therefore shown that if $d(w) \geq 3$, then $(T, S) \in \mathcal{T}$ for some labeling S, as desired. Thus we may assume that $d(w) = 2$. Since $n \geq 5$, the vertex x is not the root r of the rooted tree T. Let y be the parent of x.

By Observation 3.4, $u \in D_2$ and $\{v, w\} \subseteq D_1$. Since $D = (D_1, D_2)$ is a partition of $V(T)$ into a TDS D_1 and a DS D_2, we must have that $x \in D_2$. Hence, by Observation 3.4, the vertex x is not a support vertex. In particular, no child of x is a leaf.

Suppose $d(x) \geq 3$. Let $w' \in C(x) \setminus \{w\}$. Since no child of x is a leaf, $d(w') \geq 2$. By our choice of the vertex u, the vertex w' is either a support vertex or is the parent of a support vertex. Suppose w' is not the parent of a support vertex. Then, since T has no strong support vertex, $d(w') = 2$ and the child v' of w' is a leaf. However by Observation 3.4, this would imply that $v' \in D_2$ and $\{w', x\} \in D_1$, contradicting the fact that $x \in D_2$. Hence, w' must be the parent of a support vertex v'. Let u' be a child of v'. An identical argument as shown with the vertex w, shows that we may assume $d(w') = d(v') = 2$. Hence by Observation 3.4, $u' \in D_2$ and $\{v', w'\} \subseteq D_1$. Thus, $x \in D_2$ is adjacent to a vertex of D_1 different from w. We now consider the tree $T' = T - \{u, v, w\}$. Then, $(D_1 \setminus \{v, w\}, D_2 \setminus \{u\})$ is a partition of $V(T')$ into a TDS $D_1 \setminus \{v, w\}$ and a DS $D_2 \setminus \{u\}$. Hence, T' is a DTDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. By Observation 3.1, $\{u', x\} \subseteq S'_B$ and $\{v', w'\} \subseteq S'_A$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_3$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v, w\}, S'_B \cup \{u\})$. Hence, if $d(x) \geq 3$, then $(T, S) \in \mathcal{T}$ for some labeling S, as desired. Therefore we may assume that $d(x) = 2$. As observed earlier, $\{u, x\} \subseteq D_2$ and $\{v, w\} \subseteq D_1$.

Suppose $y \in D_1$. We now consider the tree $T' = T - \{u, v, w\}$. Then, $(D_1 \setminus \{v, w\}, D_2 \setminus \{u\})$ is a partition of $V(T')$ into a TDS $D_1 \setminus \{v, w\}$ and a DS $D_2 \setminus \{u\}$. Hence, T' is a DTDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$

such that $(T', S') \in \mathcal{T}$. By Observation 3.1, the leaf $x \in S'_B$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_3$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v, w\}, S'_B \cup \{u\})$. Hence, if $y \in D_1$, then $(T, S) \in \mathcal{T}$ for some labeling S, as desired. Therefore we may assume that $y \in D_2$.

We now consider the tree $T' = T - \{u, v, w, x\}$. Then, $(D_1 \setminus \{v, w\}, D_2 \setminus \{u, x\})$ is a partition of $V(T')$ into a TDS $D_1 \setminus \{v, w\}$ and a DS $D_2 \setminus \{u, x\}$. Hence, T' is a DTDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. If $y \in S'_B$, then we can restore the tree T by applying Operation $\mathcal{O}_4$ to T'. If $y \in S'_A$, then we can restore the tree T by first applying Operation $\mathcal{O}_1$ to T' and then Operation $\mathcal{O}_3$ to the resulting tree. In both cases, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v, w\}, S'_B \cup \{u, x\})$. Thus, $(T, S) \in \mathcal{T}$ for some labeling S, as desired. This completes the necessity, and the proof of Theorem 3.2 is complete. □

3.2.2 Proof of Theorem 3.3

Recall the statement of Theorem 3.3.

Theorem 3.3. *The connected DTDP-graphs of order at least 4 are precisely those graphs G such that $(G, S) \in \mathcal{G}$ for some labeling S.*

Proof. Suppose first that G is a connected graph and $(G, S) \in \mathcal{G}$ for some labeling S. By Observation 3.1(c), (S_A, S_B) is a partition of $V(G)$ into a TDS S_A and a DS S_B, and so G is a connected DTDP-graph. This establishes the sufficiency.

To prove the necessity we proceed by induction on the order $n \geq 4$ of a connected DTDP-graph G. Since every star $K_{1,n-1}$ is not a DTDP-graph and since $n \geq 4$, we have that $\text{diam}(G) \geq 3$, and so G contains P_4 as a subgraph. If $n = 4$, then let $G' = P_4$ be a subgraph of G (possibly, $G' = G$) obtained from G by removing zero, one, two or three edges. Then, $(G', S) \in \mathcal{G}$, where S is the labeling of a labeled-P_4 and we can restore the graph G from G' by repeated applications (including the possibility of none) of Operation $\mathcal{O}_5$. Thus, $(G,S) \in \mathcal{G}$. This establishes the base case. For the inductive hypothesis, let $n \geq 5$ and assume that for every DTDP-graph G' of order less than n there exists a labeling S' such that $(G',S') \in \mathcal{G}$.

Let G be a connected DTDP-graph of order n. Among all partitions $D = (D_1, D_2)$ of $V(G)$ into a TDS D_1 and DS D_2 of G and among all spanning connected subgraphs H of G such that $D = (D_1, D_2)$ is a partition of $V(H)$ into a TDS D_1 and DS D_2 of H (possibly, $H = G$), let the partition $D = (D_1, D_2)$ and the graph H be chosen so that

(1) $|D_1|$ is a minimum.

(2) Subject to (1), $|E(H)|$ is minimized.

If there are two adjacent vertices u and v in H that both belong to the DS D_2, then the edge uv could have been removed from H, contradicting the minimality of H. Hence the set D_2 is an independent set in H.

If H is a tree, then by Theorem 3.2, there exists a labeling $S = (S_A, S_B)$ such that $(H,S) \in \mathcal{T} \subset \mathcal{G}$. Thus, we can restore the graph G from H by repeated applications (including the possibility of none) of Operation $\mathcal{O}_5$. Hence, $(G,S) \in \mathcal{G}$. We may therefore assume that H is not a tree, for otherwise the desired result follows.

Since H is not a tree, H must contain a cycle. Let $C\colon v_1v_2v_3\ldots v_kv_1$, $k \geq 3$, be a shortest cycle in H (of length k). We proceed further with the following three claims.

Claim 1 *The cycle C has the following properties:*

(a) $V(C) \cap D_2 \neq \emptyset$.

(b) *Every vertex of C in D_1 is adjacent in H to some other vertex of C in D_1.*

(c) *No three consecutive vertices on C are all in D_1.*

(d) $k \equiv 0 \pmod 3$, *and we may assume that $v_i \in D_2$ for $i \equiv 1 \pmod 3$ and $v_i \in D_1$ for $i \equiv 0, 2 \pmod 3$.*

Proof. (a) If $V(C) \subseteq D_1$, then for any edge $e \in E(C)$, the edge e could be removed from H; that is, $H - e$ is a connected graph, D_1 is a TDS of $H - e$, and D_2 is a DS of $H - e$. This contradicts the minimality of H.

(b) Assume that there is a vertex v of C in D_1 with both its neighbors on C in D_2. For notational convenience, we may assume that $v = v_2$. Thus, $v_1 \in D_2$, $v_2 \in D_1$ and $v_3 \in D_2$. Since D_2 is an independent set in H, we have that $k \geq 4$ and that $v_4 \in D_1$. But then the edge v_2v_3 could be removed from H, contradicting the minimality of H.

(c) Assume that there are three consecutive vertices on C in D_1. For notational convenience, we may assume that $\{v_1, v_2, v_3\} \subseteq D_1$. By (a), $k \geq 4$. If $v_4 \in D_1$, then the edge v_2v_3 could be removed from H, contradicting the choice of H. Hence $v_4 \in D_2$. Since D_2 is an independent set in H, we have that either $k = 4$ or $k \geq 5$ and $v_5 \in D_1$. Suppose $d_H(v_3) \geq 3$. Then v_3 has a neighbor u in $V(H) \setminus \{v_2, v_4\}$. If $u \in D_1$, the edge v_2v_3 could be removed from H, while if $u \in D_2$, the edge v_3v_4 could be removed from H. In both cases we contradict the choice of H. Hence, $d_H(v_3) = 2$. But then $(D_1 \setminus \{v_3\}, D_2 \cup \{v_3\})$ is a partition of $V(H)$ (and hence $V(G)$) into a TDS $D_1 \setminus \{v_3\}$ and DS $D_2 \cup \{v_3\}$, contradicting Condition (1) of the choice of our partition D.

(d) By (a), at least one vertex of C belongs to D_2. For notational convenience, we may assume that $v_1 \in D_2$. Since D_2 is an independent set in H, $v_2 \in D_1$. By (b), $v_3 \in D_1$. If $k = 3$, then the desired result follows. Hence we may assume that $k \geq 4$. By (c), $v_4 \in D_2$. Since D_2 is an independent set in H, $k \geq 5$ and $v_5 \in D_1$. By (b), $k \geq 6$ and $v_6 \in D_1$. If $k = 6$, then the desired result follows. Hence we may assume that $k \geq 7$. Continuing in this way, we have that $k \equiv 0 \pmod 3$ and that $v_i \in D_2$ for $i \equiv 1 \pmod 3$ and $v_i \in D_1$ for $i \equiv 0, 2 \pmod 3$. □

Claim 2 *If $k = 3$, then $(G, S) \in \mathcal{G}$ for some labeling S.*

Proof. Suppose $k = 3$. By Claim 1(d), $v_1 \in D_2$ and $\{v_2, v_3\} \subseteq D_1$. Suppose $d_H(v_2) \geq 3$

and $d_H(v_3) \geq 3$. Then, v_2 has a neighbor u_2 in $V(H) \setminus \{v_1, v_3\}$ and v_3 has a neighbor u_3 in $V(H) \setminus \{v_1, v_2\}$ (possibly $u_2 = u_3$). If $u_2 \in D_2$ we could have removed the edge v_1v_2, contradicting the choice of H. Hence, $u_2 \in D_1$. Similarly, $u_3 \in D_1$. But then we could have removed the edge v_2v_3, contradicting the choice of H. Hence at least one of v_2 and v_3 has degree 2 in H. Without loss of generality, we may assume that $d_H(v_3) = 2$. Suppose $d_H(v_2) \geq 3$. Then, v_2 has a neighbor u_2 in $V(H) \setminus \{v_1, v_3\}$ and, as before, $u_2 \in D_1$. But then $(D_1 \setminus \{v_3\}, D_2 \cup \{v_3\})$ is a partition of $V(H)$ (and hence $V(G)$) into a TDS $D_1 \setminus \{v_3\}$ and DS $D_2 \cup \{v_3\}$, contradicting Condition (1) of the choice of our partition D. Hence $d_H(v_2) = d_H(v_3) = 2$.

Since $n \geq 4$ and H is connected, $d_H(v_1) \geq 3$. If $N_H(v_1) \setminus \{v_2, v_3\} \subset D_2$, let $D'_1 = (D_1 \setminus \{v_2\}) \cup \{v_1\}$ and let $D'_2 = (D_2 \setminus \{v_1\}) \cup \{v_2\}$. Then, $D' = (D'_1, D'_2)$ is a partition of $V(G)$ into a TDS D'_1 and DS D'_2 of G. Further, let $H' = H - v_1v_2$. Then, H' is a spanning connected subgraph of G such that $D' = (D'_1, D'_2)$ is a partition of $V(H')$ into a TDS D'_1 and DS D'_2 of H'. However since $|D'| = |D|$ and $|E(H')| < |E(H)|$, this contradicts our choice of the partition $D = (D_1, D_2)$ and the graph H. Hence at least one vertex in $N_H(v_1) \setminus \{v_2, v_3\}$ belongs to the set D_1.

Let $G' = H - \{v_2, v_3\}$. Then, $(D_1 \setminus \{v_2, v_3\}, D_2\})$ is a partition of $V(G')$ into a TDS $D_1 \setminus \{v_2, v_3\}$ and DS D_2. Hence, G' is a DTDP-graph. Applying the inductive hypothesis to G', there exists a labeling $S' = (S'_A, S'_B)$ such that $(G', S') \in \mathcal{G}$. If $v_1 \in S'_A$, we can restore the graph H from G' by first applying Operation $\mathcal{O}_2$ and then Operation $\mathcal{O}_5$. We can then restore the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A \cup \{v_2\}, S'_B \cup \{v_3\})$. If $v_1 \in S'_B$, we can restore the graph H from G' by applying Operation $\mathcal{O}_6$. We can then restore the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A \cup \{v_2, v_3\}, S'_B)$. $\square$

Claim 3 *If $k > 3$, then $(G, S) \in \mathcal{G}$ for some labeling S.*

Proof. Suppose $k > 3$. By Claim 1(d), $k \equiv 0 \pmod 3$, and $v_i \in D_2$ for $i \equiv 1 \pmod 3$ and $v_i \in D_1$ for $i \equiv 0, 2 \pmod 3$. An identical argument used in the proof of Claim 2, shows that $d_H(v_2) = d_H(v_3) = 2$. Let $G' = H - \{v_2, v_3\}$. Then, $(D_1 \setminus \{v_2, v_3\}, D_2\})$ is a partition of $V(G')$ into a TDS $D_1 \setminus \{v_2, v_3\}$ and DS D_2. Hence, G' is a DTDP-graph. Applying the inductive hypothesis to G', there exists a labeling $S' = (S'_A, S'_B)$ such that $(G', S') \in \mathcal{G}$.

If $v_1 \in S'_A$, we can restore the graph H from G' by first applying Operation $\mathcal{O}_2$ and then Operation $\mathcal{O}_5$. We can then restore the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A \cup \{v_2\}, S'_B \cup \{v_3\})$. Similarly, if $v_4 \in S'_A$, then $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A \cup \{v_3\}, S'_B \cup \{v_2\})$. Hence we may assume that $\{v_1, v_4\} \subseteq S'_B$. In this case, we can restore the graph H from G' by applying Operation $\mathcal{O}_7$. We can then restore the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A \cup \{v_2, v_3\}, S'_B)$. □

We now return to the proof of Theorem 3.3. By Claim 2 and Claim 3, $(G, S) \in \mathcal{G}$ for some labeling S, as desired. This completes the necessity and also the proof of Theorem 3.3. □

3.2.3 Proof that G_k contains no spanning DTDP-tree

Proof. Recall that G_k is the graph obtained from the disjoint union of $k \geq 1$ copies of K_3 by adding a path P_3 and joining a leaf of the path to one vertex from each copy of K_3. For $i = 1, \ldots, k$, let $u_i v_i w_i u_i$ be the k original copies of K_3 and let uvw be the added path P_3, where u is joined to u_i for every i. The graph G_3 is illustrated in Figure 3.4.

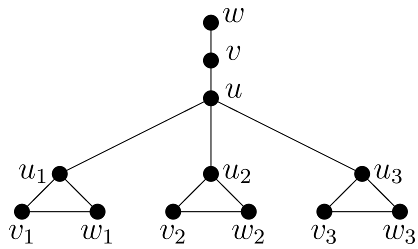


Figure 3.4: The graph G_3.

Then, G_k is a DTDP-graph. We show that G_k does not have a spanning DTDP-tree. Assume, to the contrary, that G_k has a spanning tree T_k. Let $D = (D_1, D_2)$ be a partition of $V(T_k)$ into a TDS D_1 and a DS D_2. Then, uvw is a path in T_k where w is a leaf and $d(v) = 2$. Thus, $w \in D_2$ while $\{u, v\} \in D_1$. If exactly one of v_i and w_i is a leaf in T_k, say w_i, then $uu_iv_iw_i$ is a path in T_k where $d(u_i) = d(v_i) = 2$. Thus, $w_i \in D_2$, $v_i \in D_1$, $u_i \in D_1$, and $u \in D_2$, a contradiction. Hence both v_i and w_i are leaves in T_k with u_i as their common neighbor. Thus, $\{v_i, w_i\} \subset D_2$ while $u_i \in D_1$. But then $N[u] = D_1$, and so no vertex in D_1 is adjacent with u, a contradiction. Hence, G_k has no spanning DTDP-tree. □

Chapter 4

Exhaustive DTDP Graphs

In Chapter 2, we showed that if G is a graph of minimum degree at least 2 with no C_5-component, then $V(G)$ can be partitioned into a dominating set D and a total dominating set T (see Theorem 2.1). A characterization of all graphs with disjoint dominating and total dominating sets was given in Chapter 3.

Recently, several authors have studied the cardinalities of pairs of disjoint dominating sets in graphs (see, for example, [20, 35, 50, 58, 75, 77]). The context of this research motivates the question for which graphs Theorem 2.1 is best-possible in the sense that the union $D \cup T$ of the two sets necessarily contains all vertices of the graph G. The following recent result in [60] gives a partial answer to this question.

Theorem 4.1 ([60]) *If G is a graph of minimum degree at least 3 with at least one component different from the Petersen graph, then G contains a dominating set D and a total dominating set T which are disjoint and satisfy $|D| + |T| < |V(G)|$.*

A *DT-pair* of a graph G is a pair (D, T) of disjoint sets of vertices of G such that D is a dominating set and T is a total dominating set of G. A DT-pair (D, T) in G is *exhaustive* if $|D| + |T| = |V(G)|$. Thus a DT-pair (D, T) in G is non-exhaustive if

$|D| + |T| < |V(G)|$. Note that Theorem 2.1 implies that every graph with minimum degree at least 2 and with no C_5-component, has an exhaustive DT-pair. Using the notation of Hedetniemi et al. [50], for a graph G we define $\gamma\gamma_t(G)$ as follows:

$$\gamma\gamma_t(G) = \min\{|D| + |T| \colon (D,T) \text{ is DT-pair of } G\}.$$

We call a DT-pair (D,T) whose union $D \cup T$ has cardinality $\gamma\gamma_t(G)$ a $\gamma\gamma_t(G)$-pair. By Theorem 2.1, $\gamma\gamma_t(G)$ exists for every graph G with minimum degree at least 2 and with no C_5-component. Hence we have the following immediate consequence of Theorem 2.1.

Theorem 4.2 *If G is a graph with minimum degree at least 2 and with no C_5-component, then $\gamma\gamma_t(G) \le |V(G)|$.*

In this chapter, we characterize the graphs that achieve equality in the upper bound in Theorem 4.2 and that have no induced C_5 subgraph.

Recall that a graph is *F-free* if it does not contain F as an induced subgraph. In particular, if $F = C_5$, then we say that the graph is *C_5-free*. The graph obtained from a complete graph K_n, where $n \ge 4$, by subdividing every edge once, is denoted by K_n^*. We note that $|V(K_n^*)| = |V(K_n)| + |E(K_n)| = n + \binom{n}{2}$. We now define the families $\mathcal{C}$ and $\mathcal{K}^*$ of particular cycles and subdivided complete graphs as follows:

$$\begin{aligned} \mathcal{C} &= \{C_n : n \ge 3 \text{ and } n \ne 5\} \\ \mathcal{K}^* &= \{K_n^* : n \ge 4\}. \end{aligned}$$

We define a vertex as *small* if it has degree 2, and *large* if it has degree greater than 2. For a graph G, we let $\mathcal{L}(G)$ and $\mathcal{S}(G)$ denote the set of all large and small vertices of G, respectively. For notational convenience, we simply write $\mathcal{L}$ and $\mathcal{S}$ when G is clear from the context.

4.1 The Problematic 5-Cycle

In this chapter we study graphs that achieve equality in the upper bound in Theorem 4.2. If we restrict our attention to graphs with minimum degree at least 3, then a characterization of graphs is given by Theorem 4.1 which shows the only component is the Petersen graph.

However the situation becomes much more complicated when we relax the degree condition from minimum degree at least 3 to minimum degree at least 2. In this case a characterization seems difficult to obtain since there are several families each containing infinitely many graphs that achieve equality in Theorem 4.2. For example, consider the following four families of connected graphs different from the 5-cycle with minimum degree at least 2 that satisfy the property that every DT-pair is exhaustive.

- **The Family $\mathcal{D}$:** For $k \geq 2$, let $\mathcal{D}_k$ be the connected graph that can be constructed from k disjoint 5-cycles by identifying a set of k vertices, one from each cycle, into one vertex. Let $\mathcal{D} = \{\mathcal{D}_k : k \geq 2\}$. The family $\mathcal{D}$ is depicted in Figure 4.1(a). We remark that a graph in the family $\mathcal{D}$ is called a *daisy* in the literature.

- **The Family $\mathcal{D}_b$:** For $k \geq 0$, we define $D_b(k)$ to be the connected graph obtained from two disjoint 5-cycles by joining a vertex from one of the cycles to a vertex in the other and subdividing the resulting edge k times. Let $\mathcal{D}_b = \{\mathcal{D}_b(k) : k \geq 0\}$. The family $\mathcal{D}_b$ is depicted in Figure 4.1(b). We remark that a graph in the family $\mathcal{D}_b$ is called a *dumb-bell* in the literature.

- **The Family $\mathcal{D}_1$:** For $k \geq 1$, let $\mathcal{D}_1(k)$ be the connected graph that can be constructed from k disjoint 5-cycles and a dumb-bell $D_b(3)$, defined above, by identifying a set of $k+1$ vertices, one from each cycle and the central vertex of the dumb-bell, into one vertex. Let $\mathcal{D}_1 = \{\mathcal{D}_1(k) : k \geq 1\}$. The family $\mathcal{D}_1$ is depicted in Figure 4.1(c).

- **The Family $\mathcal{D}_2$:** For $k \geq 1$ and $\ell \geq 1$, let $\mathcal{D}_2(k,\ell)$ be the connected graph that can be constructed from $k+\ell$ disjoint 5-cycles by identifying a set of k vertices, one from each of k cycles, into one vertex u and identifying a set of ℓ vertices, one from each of the remaining ℓ cycles, into one vertex v and then adding a path of length 2 joining u and v. Let $\mathcal{D}_2 = \{\mathcal{D}_2(k) : k \geq 1 \text{ and } \ell \geq 1\}$. The family $\mathcal{D}_2$ is depicted in Figure 4.1(d).

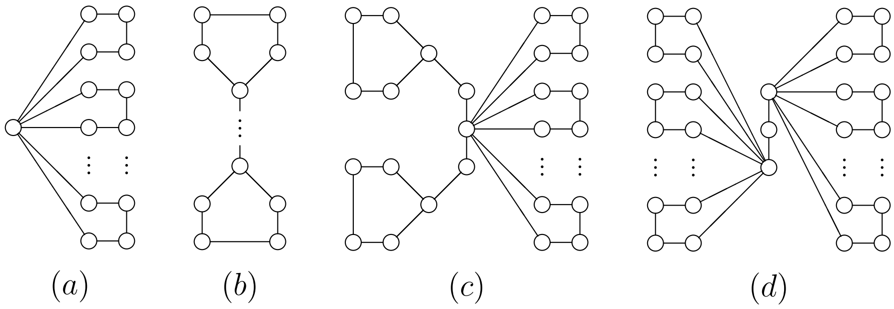


Figure 4.1: Graphs containing no non-exhaustive DT-pairs.

It is a routine exercise to check that if $G \in \mathcal{D} \cup \mathcal{D}_b \cup \mathcal{D}_1 \cup \mathcal{D}_2$, then $\gamma\gamma_t(G) = |V(G)|$. We note, however, that such a graph G contains an induced 5-cycle. Several other graphs G that contain induced 5-cycles and satisfy $\gamma\gamma_t(G) = |V(G)|$ can readily be constructed. These families demonstrate that a characterization of general graphs that achieve equality in Theorem 4.2 seems difficult to obtain. We therefore restrict our attention to graphs with no induced 5-cycle.

4.2 Exhaustive DTDP Result

Our aim in this chapter is to characterize the C_5-free graphs which achieve equality in Theorem 4.2. We shall prove:

Theorem 4.3 *Let G be a connected C_5-free graph with $\delta(G) \geq 2$. Then, $\gamma\gamma_t(G) = |V(G)|$ if and only if $G \in \mathcal{C} \cup \mathcal{K}^*$.*

We will refer to a graph G as an *n-minimal graph* if G has order n and G is edge-minimal with respect to satisfying the following three conditions: (i) $\delta(G) \geq 2$, (ii) G is connected, (iii) $\gamma\gamma_t(G) = n$. We shall need the following key lemma which shows that removing edges can never lead to a violation of condition (iii) above.

Lemma 4.4 *Let G be a connected C_5-free graph of order n with $\delta(G) \geq 2$ and $\gamma\gamma_t(G) = n$. If G is not n-minimal, then G contains an n-minimal spanning subgraph with no induced 5-cycle.*

The following result characterizes n-minimal C_5-free graphs and is useful in the proof of our main result.

Theorem 4.5 *Let G be a C_5-free graph of order n. Then, G is n-minimal if and only if $G \in \mathcal{C} \cup \mathcal{K}^*$.*

We note that every graph $G \in \mathcal{D} \cup \mathcal{D}_b \cup \mathcal{D}_1 \cup \mathcal{D}_2$ is an n-minimal graph but, as remarked earlier, such graphs are not C_5-free. We shall proceed as follows. We first prove a number of useful preliminary results in Section 4.2.1. We then prove Lemma 4.4 in Section 4.2.2 and Theorem 4.5 in Section 4.2.3, before presenting a proof of our main result, namely Theorem 4.3, in Section 4.2.4.

4.2.1 Preliminary Results

In this section, we present several preliminary results that will prove to be useful. We begin with a proof of our key lemma.

Lemma 4.6 *If $G = C_n$, where $n \neq 5$, and (D, T) is a DT-pair in G, then $|D| + |T| = n$.*

Proof. Let G be the cycle $v_1v_2 \dots v_n$, where $n \neq 5$. By Theorem 2.1, G has a DT-pair. For the sake of contradiction, suppose that G has a DT-pair (D, T) such that $|D| + |T| < n$. Renaming vertices, if necessary, we may assume that $v_2 \notin D \cup T$. Then, $|D \cap \{v_1, v_3\}| \geq 1$ and $|T \cap \{v_1, v_3\}| \geq 1$. We may assume that $v_1 \in D$ and $v_3 \in T$. If $n = 3$, then v_3 is not totally dominated by T, a contradiction. Hence, $n \geq 4$. If $v_4 \notin D$, then v_3 is not dominated by D, a contradiction. Hence, $v_4 \in D$. But then v_3 is not totally dominated by T, a contradiction. □

Lemma 4.7 *If $G \in \mathcal{K}^*$ and (D, T) is a DT-pair in G, then $|D| + |T| = |V(G)|$.*

Proof. Let $G \in \mathcal{K}^*$. Then, G may be obtained from the complete graph K_ℓ, for some $\ell \geq 4$, by subdividing every edge exactly once. By Theorem 2.1, there exists a DT-pair (D, T) in G. If there are two vertices in $\mathcal{L}$ that do not belong to T, then the vertex in $\mathcal{S}$ with these two vertices as its neighbors is not totally dominated by T, a contradiction. Hence, T contains all vertices in $\mathcal{L}$, except possibly one. If $\mathcal{L} \subseteq T$, then since every degree-2 vertex is dominated by D, we have that $\mathcal{S} \subseteq D$. But then no vertex in $\mathcal{L}$ is totally dominated by T, a contradiction. Hence, exactly one vertex, v say, in $\mathcal{L}$ is not in T. Since every vertex in $\mathcal{S} \setminus N(v)$ has both its neighbors in T, and since $\mathcal{S} \setminus N(v)$ is dominated by D, we have that $\mathcal{S} \setminus N(v) \subseteq D$. Furthermore, in order for T to totally dominate $\mathcal{L} \setminus \{v\}$ we have that $N(v) \subset T$. But then $v \in D$ in order for the set D to dominate $N(v)$. Thus, $D = (\mathcal{S} \setminus N(v)) \cup \{v\}$ and $T = (\mathcal{L} \setminus \{v\}) \cup N(v)$, and so $|D| + |T| = |\mathcal{S}| + |\mathcal{L}| = |V(G)|$, as desired. □

The following observation follows from the proofs of Lemmas 4.6 and 4.7.

Observation 4.8 *Let $G \in \mathcal{C} \cup \mathcal{K}^*$ and let $v \in V(G)$. Then, G has the following properties.*

(a) *There exist DT-pairs (D_1, T_1) and (D_2, T_2) with $v \in D_1$ and with $v \in T_2$.*

(b) *If $G \in \mathcal{C}$ and $uv \in E(G)$, there exist DT-pairs (D_1, T_1) and (D_2, T_2) with $\{u, v\} \subseteq T_1$ and with $u \in D_2$ and $v \in T_2$.*

(c) *If $G \in \mathcal{K}^*$ and $v \in \mathcal{L}$, then there exists a DT-pair (D, T) with $v \in D$ and $N(v) \subset T$. Furthermore, every vertex in $\mathcal{L} \setminus \{v\}$ belongs to T and has exactly one neighbor in T with the remaining neighbors all in D.*

Lemma 4.9 *Let $G = (V, E)$ be a cycle C_n, where $n \neq 5$, and let $v \in V$. Then there exists a pair (D, T) of disjoint sets of vertices in G such that $|D| + |T| < n$, $v \in T$, and either*

(i) *D dominates V and T totally dominates $V \setminus \{v\}$, or*

(ii) *D dominates $V \setminus \{v\}$ and T totally dominates V.*

Proof. Let G be the cycle $v_1 v_2 \ldots v_n v_1$, where $n \neq 5$ and where $v = v_1$. If $n = 3$, let $D = \{v_2\}$ and $T = \{v_1\}$, while if $n = 4$, let $D = \{v_3\}$ and $T = \{v_1, v_2\}$. If $n \geq 6$ and $n \equiv 0 \pmod 3$, let $v_i \in D$ if $i \equiv 0 \pmod 3$ and let $v_i \in T$ if $i \equiv 1, 2 \pmod 3$ and $i \neq 2$. If $n \geq 6$ and $n \equiv 1 \pmod 3$, let $v_i \in D$ if $i \equiv 0 \pmod 3$ and let $v_i \in T$ if $i \equiv 1, 2 \pmod 3$ and $i \notin \{2, n\}$, and let $v_n \in D$. If $n \geq 6$ and $n \equiv 2 \pmod 3$, let $v_i \in D$ if $i \equiv 0 \pmod 3$ and let $v_i \in T$ if $i \equiv 1, 2 \pmod 3$ and $i \notin \{2, n-1\}$, and let $v_{n-1} \in D$. In all cases, the pair (D, T) satisfies the requirements of the lemma. □

Lemma 4.10 *Let $F \neq C_5$ be a connected graph with $\delta(F) \geq 2$ and let G be obtained from F by subdividing an edge of F three times. If $\gamma\gamma_t(G) = |V(G)|$, then $\gamma\gamma_t(F) = |V(F)|$.*

Proof. We use a proof by contrapositive. Suppose that $\gamma\gamma_t(F) < |V(F)|$. We show that $\gamma\gamma_t(G) < |V(G)|$. Let (D_F, T_F) be a $\gamma\gamma_t(F)$-pair in F. Then, $|D_F| + |T_F| = \gamma\gamma_t(F) < |V(F)|$. Let $e = uv$ be the edge of F that is subdivided three times to produce the path $uv_1v_2v_3v$ in G. Note that u and v are not adjacent in G.

Suppose that $T_F \cap \{u, v\} \neq \emptyset$. Renaming vertices, if necessary, we may assume that $u \in T_F$. If $v \in T_F$, let $D = D_F \cup \{v_2\}$ and let $T = T_F \cup \{v_1, v_3\}$. If $v \in D_F$, let $D = D_F \cup \{v_1\}$

and let $T = T_F \cup \{v_2, v_3\}$. If $v \notin D_F \cup T_F$, let $D = D_F \cup \{v_2\}$ and let $T = T_F \cup \{v, v_3\}$. Then, (D, T) is a DT-pair in G with $|D| + |T| = |D_F| + |T_F| + 3 < |V(F)| + 3 = |V(G)|$. Hence, $\gamma\gamma_t(G) < |V(G)|$, as desired. Thus we may assume that $T_F \cap \{u, v\} = \emptyset$.

Suppose that $D_F \cap \{u, v\} \neq \emptyset$. Renaming vertices, if necessary, we may assume that $u \in D_F$. In this case, let $D = D_F \cup \{v_3\}$ and let $T = T_F \cup \{v_1, v_2\}$, and once again (D, T) is a DT-pair in G with $|D| + |T| < |V(G)|$. Thus we may assume that $D_F \cap \{u, v\} = \emptyset$. Now, $|D_F| + |T_F| \leq |V(F)| - 2$. We note that each of u and v is adjacent to a vertex in D_F and to a vertex in T_F. We now let $D = D_F \cup \{v, v_1\}$ and let $T = T_F \cup \{v_2, v_3\}$. Then, (D, T) is a DT-pair in G with $|D| + |T| = |D_F| + |T_F| + 4 \leq |V(F)| + 2 < |V(G)|$. Hence, $\gamma\gamma_t(G) < |V(G)|$. □

We remark that the converse of Lemma 4.10 is not necessarily true.

Lemma 4.11 *Let G be the graph obtained from $k \geq 2$ disjoint cycles $F_1, F_2, \ldots, F_k$ of lengths $n_1, n_2, \ldots, n_k$, respectively, by identifying a set of k vertices, one from each cycle, into one vertex called v. If $n_i \neq 5$ for $i = 1, 2, \ldots, k$, then G has a non-exhaustive DT-pair.*

Proof. Let G be the graph defined in the statement of the lemma. For $i \in \{1, 2, \ldots, k\}$, let v_i be the vertex of F_i that was identified into the vertex v. Let (D_1, T_1) be a pair of disjoint sets of vertices in F_1 that satisfies the requirements of Lemma 4.9 for the graph F_1 with v_1 the specified vertex in the cycle. Then, $v_1 \in T_1$, $|D_1| + |T_1| < n_1$, and either (i) D_1 dominates $V(F_1)$ and T_1 totally dominates $V(F_1) \setminus \{v_1\}$ or (ii) D_1 dominates $V(F_1) \setminus \{v_1\}$ and T_1 totally dominates $V(F_1)$. For each $i \in \{2, \ldots, k\}$, $F_i \in \mathcal{C}$ and hence, by Observation 4.8(a), there exists a DT-pair (D_i, T_i) in F_i such that $v_i \in T_i$. Let

$$D = \bigcup_{i=1}^{k} D_i \quad \text{and} \quad T = \left(\bigcup_{i=1}^{k} (T_i \setminus \{v_i\}) \right) \cup \{v\}.$$

Then, (D, T) is a non-exhaustive DT-pair in G. $\square$

Lemma 4.12 *If $G \neq C_n$ is a C_5-free hamiltonian graph of order n, then $\gamma\gamma_t(G) < n$.*

Proof. Let $G \neq C_n$ be a C_5-free hamiltonian graph of order n and let C be a hamiltonian cycle in G. Thus, every edge in $E(G) \setminus E(C)$ is a chord of C in G. Among all chords of C, let uv be chosen so that $k = d_C(u, v)$ is minimized. Since a chord of C is not an edge of C, we note that $k \geq 2$. Let $P\colon u_0u_1 \dots u_k$ be a shortest u-v path in C, where $u = u_0$ and $v = u_k$, and let C' be the cycle $u_0u_1 \dots u_ku_0$. By our choice of uv, C' is an induced cycle in G. If $k = 4$, then C' is an induced 5-cycle in G, contradicting the fact that G is C_5-free. Hence, $C' \in \mathcal{C}$.

Let $v_0v_1 \dots v_\ell$ be the v-u path in C not containing u_1, where $v = v_0$ and $u = v_\ell$. Thus, C is the cycle $u_0u_1 \dots u_kv_1v_2 \dots v_\ell$ and $n = k + \ell$. Since $k = d_C(u, v)$, we note that $\ell \geq k \geq 2$. We now apply Observation 4.8(b) to the cycle $C' \in \mathcal{C}$ as follows. If $\ell \equiv 0, 1 \pmod 3$, let (D', T') be a DT-pair in C' such that $\{u, v\} = \{u_0, u_k\} \subseteq T'$, while if $\ell \equiv 2 \pmod 3$, let (D', T') be a DT-pair in C' such that $u = u_0 \in D'$ and $v = u_k \in T'$. Let $D'' = \{v_i \mid i \equiv 2 \pmod 3 \text{ and } 1 < i < \ell\}$ and let $T'' = \{v_i \mid i \equiv 0, 1 \pmod 3 \text{ and } 1 < i < \ell\}$. Let $D = D' \cup D''$ and let $T = T' \cup T''$. We note that $v_1 \notin D \cup T$ and that (D, T) is a DT-pair in $C + uv$. Hence, (D, T) is a non-exhaustive DT-pair in $C + uv$ and therefore in G, and so $\gamma\gamma_t(G) < n$. $\square$

Lemma 4.13 *Let G be a connected C_5-free graph of order n. If there exists a spanning proper subgraph F of G such that $F \in \mathcal{K}^*$, then $\gamma\gamma_t(G) < n$.*

Proof. Let G be a connected C_5-free graph of order n and suppose there exists a spanning proper subgraph F of G such that $F \in \mathcal{K}^*$. Among all edges in $E(G) \setminus E(F)$, let the edge uv be chosen so that $d_F(u) + d_F(v)$ is maximized and, subject to that, the number of common neighbors of u and v in F is maximized. Let $F' = F + uv$.

By definition of the family $\mathcal{K}^*$, we note that $\mathcal{L}(F) \geq 4$. Suppose $\{u, v\} \subset \mathcal{L}(F)$. Let $w \in \mathcal{L}(F) \setminus \{u, v\}$. Let u' be the common neighbor of u and w in F, and let v' be the common neighbor of v and w in F. By Observation 4.8(c), there exists a DT-pair (D, T) in F such that $w \in D$, $\{u', v'\} \subset N(w) \subset T$ and $\{u, v\} \subset T$. Now $(D, T \setminus \{u'\})$ is a non-exhaustive DT-pair in F' and therefore in G, and so $\gamma\gamma_t(G) < n$. Hence we may assume, without loss of generality, that $d_F(u) = 2$.

Suppose $v \in \mathcal{L}(F)$. Since $uv \notin E(F)$, we note that $v \notin N(u)$. Let $w \in N(u)$. Then, $w \in \mathcal{L}(F)$. Let v' be the common neighbor of v and w. By Observation 4.8(c), there exists a DT-pair (D, T) in F such that $w \in D$, $\{u, v'\} \subset N(w) \subset T$ and $v \in T$. Now $(D, T \setminus \{v'\})$ is a non-exhaustive DT-pair in F' and therefore in G, and so $\gamma\gamma_t(G) < n$. Hence we may assume that $d_F(v) = 2$.

Let $N_F(u) = \{u_1, u_2\}$ and let $N_F(v) = \{v_1, v_2\}$. Then, $\{u_1, u_2\} \subset \mathcal{L}(F)$ and $\{v_1, v_2\} \subset \mathcal{L}(F)$. Suppose that u and v have no common neighbor in F. Then, $\{u_1, u_2\} \cap \{v_1, v_2\} = \emptyset$. Let w be the common neighbor of u_1 and v_1 in F. Then, C': uu_1wv_1vu is a 5-cycle in F' and hence in G. By our choice of the edge uv, the cycle C' is an induced 5-cycle in G, contradicting the fact that G is C_5-free. Hence, u and v have a common neighbor in F and we may assume that $u_1 = v_1$. By Observation 4.8(c), there exists a DT-pair (D, T) in F such that $u_1 \in D$, $\{u, v\} \subset N(u_1) \subset T$ and $\{u_2, v_2\} \subset T$. Furthermore, we note that every neighbor of u_2 in F, different from u, is totally dominated by $T \setminus \{u_2\}$. Thus, $(D, T \setminus \{u_2\})$ is a non-exhaustive DT-pair in F' and therefore in G, and so $\gamma\gamma_t(G) < n$. $\square$

We now combine Lemma 4.12 and Lemma 4.13 into the following result.

Lemma 4.14 *Let G be a connected C_5-free graph of order n. If there exists a spanning proper subgraph F of G such that $F \in \mathcal{C} \cup \mathcal{K}^*$, then $\gamma\gamma_t(G) < n$.*

4.2.2 Proof of Lemma 4.4

Recall the statement of Lemma 4.4.

Lemma 4.4. *Let G be a connected C_5-free graph of order n with $\delta(G) \geq 2$ and $\gamma\gamma_t(G) = n$. If G is not n-minimal, then G contains an n-minimal spanning subgraph with no induced 5-cycle.*

Proof. Let $G = (V, E)$ be the graph defined in the statement of the lemma such that G is not n-minimal. By removing edges from G, we can obtain an n-minimal spanning subgraph of G. From all such subgraphs, choose F so that the number of induced 5-cycles in F is minimized. For the sake of contradiction, suppose that F contains the induced 5-cycle $C\colon v_1v_2v_3v_4v_5v_1$. If $n = 5$, then since G is C_5-free we may assume, relabeling vertices if necessary, that $v_1v_3 \in E$. But then $(\{v_3, v_4\}, \{v_1, v_5\})$ is a non-exhaustive DT-pair in G, a contradiction. Hence, $n \neq 5$ and since F is connected, we may assume $d_F(v_1) \geq 3$. By the minimality of F, $d_F(v_2) = d_F(v_5) = 2$.

For the sake of contradiction, suppose that $d_F(v_3) \geq 3$. Then by the minimality of F, $d_F(v_4) = 2$. If $v_2v_4 \in E$, then the graph obtained from F by adding this edge and removing the edge v_1v_2 is an n-minimal spanning subgraph of G containing fewer induced 5-cycles than F, contradicting the choice of F. Hence, $v_2v_4 \notin E$. Similarly, $v_2v_5 \notin E$. If $v_1v_4 \in E$, then the graph obtained from F by adding this edge and removing the edge v_3v_4 is an n-minimal spanning subgraph of G with fewer induced 5-cycles than F, contradicting the choice of F. Hence, $v_1v_4 \notin E$ and, by a similar argument, $v_3v_5 \notin E$. If $v_1v_3 \in E$, let $F' = F + v_1v_3$. By Theorem 2.1, there exists a DT-pair (D', T') in F'. To totally dominate v_2 we may assume, without loss of generality, that $v_1 \in T'$. If $v_3 \in D'$, then $((D' \setminus \{v_2, v_5\}) \cup \{v_4\}, (T' \setminus \{v_2, v_4\}) \cup \{v_5\})$ is a non-exhaustive DT-pair in F' and hence in G, a contradiction. Hence, $v_3 \in T'$. To dominate v_2, we therefore have that $v_2 \in D'$. But then $((D' \setminus \{v_4\}) \cup \{v_5\}, T' \setminus \{v_4, v_5\})$ is a non-exhaustive DT-pair in F' and hence in G, again a contradiction. Thus, $v_1v_3 \notin E$. Hence, C is an induced 5-cycle in G, contradicting the fact that G is C_5-free. Therefore, $d_F(v_3) = 2$. Similarly, $d_F(v_4) = 2$.

If $v_2v_i \in E$ for some $i \in \{4, 5\}$, then the graph obtained from F by adding this edge and removing the edge v_1v_2 is an n-minimal spanning subgraph of G containing fewer induced 5-cycles than F, contradicting the choice of F. Hence, $v_2v_5 \notin E$ and $v_2v_4 \notin E$. By a similar argument, $v_3v_5 \notin E$. If $v_1v_3 \in E$, let $F' = F + v_1v_3$. By Theorem 2.1, there exists a DT-pair (D', T') in F'. If $v_1 \in T'$, then $((D' \setminus \{v_2, v_5\}) \cup \{v_3, v_4\}, (T' \setminus \{v_2, v_3, v_4\}) \cup \{v_5\})$ is a non-exhaustive DT-pair in F' and hence in G, a contradiction. Hence, $v_1 \in D'$. But then $((D' \setminus \{v_2, v_3, v_4\}) \cup \{v_5\}, (T' \setminus \{v_2, v_5\}) \cup \{v_3, v_4\})$ is a non-exhaustive DT-pair in F' and hence in G, again a contradiction. Hence, $v_1v_3 \notin E$. Similarly, $v_1v_4 \notin E$. Thus, C is an induced 5-cycle in G, contradicting the fact that G is C_5-free. □

4.2.3 Proof of Theorem 4.5

We are now in a position to prove our key preliminary result, namely Theorem 4.5. Recall that a graph G is an n-minimal graph if G has order n and G is edge-minimal with respect to satisfying the following three conditions: (i) $\delta(G) \geq 2$, (ii) G is connected, (iii) $\gamma\gamma_t(G) = n$. Recall the statement of Theorem 4.5.

Theorem 4.5. *Let G be a C_5-free graph of order n. Then, G is n-minimal if and only if $G \in \mathcal{C} \cup \mathcal{K}^*$.*

Proof. If $G \in \mathcal{C} \cup \mathcal{K}^*$, then, by definition of the families $\mathcal{C}$ and $\mathcal{K}^*$, $\delta(G) \geq 2$ and G is connected. By Lemmas 4.6 and 4.7, $\gamma\gamma_t(G) = n$. Furthermore, $\delta(G - e) = 1$ for any edge e in G, and so G is n-minimal. This establishes the sufficiency.

To prove the necessity, we proceed by induction on the order n of an n-minimal C_5-free graph G. If $n \in \{3, 4\}$, then $G = C_n \in \mathcal{C}$. Suppose $n = 5$. Since $G \neq C_5$, either G contains a C_3, in which case G can be obtained from two disjoint 3-cycles by identifying a vertex from each cycle into one vertex, or G contains a C_4 but no C_3, in which case

$G = K_{2,3}$. In both cases, there exists a non-exhaustive (D, T)-pair in G, contradicting the fact that G is n-minimal. Hence, $n \neq 5$. This establishes the base cases.

Let $n \geq 6$ and assume that the result is true for all n'-minimal C_5-free graphs, where $3 \leq n' < n$. Let $G = (V, E)$ be an n-minimal C_5-free graph. Before proceeding further, we present two observations that will be useful in what follows. If e is an edge of G, then $\gamma\gamma_t(G - e) \geq \gamma\gamma_t(G)$. Hence, by the minimality of G, we have the following observation.

Observation 4.15 *If $e \in E$, then either e is a bridge of G or $\delta(G - e) = 1$.*

Observation 4.16 *If G' is a connected subgraph of G of order $n' < n$ with $\delta(G') \geq 2$, then either $G' \in \mathcal{C} \cup \mathcal{K}^*$ or $\gamma\gamma_t(G') < n'$.*

Proof. Let G' be a connected subgraph of G of order $n' < n$ with $\delta(G') \geq 2$. Suppose $\gamma\gamma_t(G') = n'$. Then, G' contains a spanning subgraph G'' which is n'-minimal. By induction, $G'' \in \mathcal{C} \cup \mathcal{K}^*$. If G'' is a proper subgraph of G', then Lemma 4.4 implies a contradiction. Hence, $G' = G''$, and so $G' \in \mathcal{C} \cup \mathcal{K}^*$. □

In what follows, we simply write $\mathcal{L}$ rather than $\mathcal{L}(G)$ and $\mathcal{S}$ rather than $\mathcal{S}(G)$ when G is clear from the context. If $|\mathcal{L}| = 0$, then $G = C_n$ and, since G is C_5-free, $G \in \mathcal{C}$ and we are done. Hence, we may assume that $|\mathcal{L}| \geq 1$. If $|\mathcal{L}| = 1$, then G satisfies the conditions of Lemma 4.11 and thus has a non-exhaustive DT-pair, contradicting the fact that G is n-minimal. Hence, $|\mathcal{L}| \geq 2$. We prove the following claim about the set $\mathcal{L}$ of large vertices in G.

Claim A *$\mathcal{L}$ is an independent set in G.*

Proof. For the sake of contradiction, suppose that $\{u, v\} \subseteq \mathcal{L}$ with $uv \in E$. Then, by Observation 4.15, uv is a bridge of G. Let G_u and G_v denote the components of

$G - uv$ containing u and v respectively. We note that $\gamma\gamma_t(G) \leq \gamma\gamma_t(G_u) + \gamma\gamma_t(G_v)$. If $\gamma\gamma_t(G_u) < |V(G_u)|$ or $\gamma\gamma_t(G_v) < |V(G_v)|$, then $\gamma\gamma_t(G) < n$, a contradiction. Hence, $\gamma\gamma_t(G_u) = |V(G_u)|$ and $\gamma\gamma_t(G_v) = |V(G_v)|$. Therefore, by Observation 4.16, $\{G_u, G_v\} \subset \mathcal{C} \cup \mathcal{K}^*$. If $G_u \in \mathcal{C}$, then, by Lemma 4.9, there exists a pair (D_1, T_1) of disjoint sets of vertices in G_u such that $u \in T_1$, $|D_1| + |T_1| < |V(G_u)|$, and either (i) D_1 dominates $V(G_u)$ and T_1 totally dominates $V(G_u) \setminus \{u\}$ or (ii) D_1 dominates $V(G_u) \setminus \{u\}$ and T_1 totally dominates $V(G_u)$. Using Observation 4.8(a), let (D_2, T_2) be a DT-pair in G_v with $v \in T_2$ if (i) holds and $v \in D_2$ if (ii) holds. In both cases, $(D_1 \cup D_2, T_1 \cup T_2)$ is a non-exhaustive DT-pair in G, a contradiction. Hence, $G_u \in \mathcal{K}^*$. Similarly, $G_v \in \mathcal{K}^*$.

Let u' be a neighbor of u in G_u. Since uu' is not a bridge in G_u, the edge uu' is not a bridge in G, and so, by Observation 4.15, $\delta(G - uu') = 1$. Since $d_G(u) \geq 3$, we note that $d_{G-uu'}(u) \geq 2$, implying that $d_G(u') = 2$ and so $d_{G_u}(u') = 2$. Let u'' be the neighbor of u' distinct from u. Since every edge in G_u is incident with a vertex of large degree and a vertex of small degree, $d_{G_u}(u) \geq 3$ and $d_{G_u}(u'') \geq 3$. Therefore, by Observation 4.8(c), there exists a DT-pair (D_1, T_1) such that $u'' \in D_1$, $u' \in N(u'') \subset T_1$ and $u \in T_1$. By Observation 4.8(a), there exists a DT-pair (D_2, T_2) in G_v with $v \in T_2$. Thus, $(D_1 \cup D_2, T_1 \cup T_2 \setminus \{u'\})$ is a non-exhaustive DT-pair in G, a contradiction. Hence, we conclude that $\mathcal{L}$ is an independent set in G. $\square$

Let R be any component of $G - \mathcal{L}$ and note that R is a path. If R has only one vertex, or has at least two vertices with the two ends of R adjacent in G to different large vertices, then we say that R is a 2-*path*. Otherwise we say that R is a 2-*handle*.

Claim B *Every 2-path in G contains at most two vertices.*

Proof. Let $P\colon v_1 \ldots v_k$ be a longest 2-path in G and let v_0 and v_{k+1} be the large vertices that are adjacent in G to v_1 and v_k, respectively. By definition of a 2-path, we note that $v_0 \neq v_{k+1}$. For the sake of contradiction, suppose that $k \geq 3$. Let F be the graph obtained from G by deleting the vertices v_1, v_2 and v_3 and adding the edge v_0v_4. Then G can be obtained from F by subdividing an edge of F three times. Since $\mathcal{L}(G) = \mathcal{L}(F)$ and $|\mathcal{L}(G)| \geq 2$, we note that F is not a cycle. In particular, $F \neq C_5$. By construction, F is a connected graph with $\delta(F) \geq 2$. Hence, by Lemma 4.10, $\gamma\gamma_t(F) = |V(F)|$. We proceed further with a subclaim showing that F is C_5-free.

Subclaim B1 *F is C_5-free.*

Proof. Suppose that F contains an induced 5-cycle C. Since G is C_5-free, we note that C contains the edge v_0v_4 and therefore $k \in \{3, 4, 5\}$. Suppose that $k = 3$. Let C be the cycle $v_0w_1w_2w_3v_4v_0$. We note that either $w_1w_2w_3$ is a 2-path in G or $w_2 \in \mathcal{L}$. We now consider the graph $F' = F - v_0v_4$ and note that F' is a connected subgraph of G with $\delta(F') \geq 2$ and $V(F') = V(F)$. Further, $|V(F')| \geq \gamma\gamma_t(F') \geq \gamma\gamma_t(F) = |V(F)|$, and so $\gamma\gamma_t(F') = |V(F')|$. By Observation 4.16, $F' \in \mathcal{C} \cup \mathcal{K}^*$. We note that $v_0w_1w_2w_3v_4$ is a path in F'. If $F' \in \mathcal{C}$, then, by our choice of P we have that $F' \in \{C_6, C_7, C_8\}$. In all three cases, we can find a DT-pair (D', T') in F' such that $\{v_0, v_4\} \subset T'$. If $F' \in \mathcal{K}^*$, then since w_1 and w_3 have degree 2 in both G and F', we note that $\{v_0, v_4, w_2\} \subset \mathcal{L}(F')$ and by Observation 4.8(c), there exists a DT-pair (D', T') in F such that $w_2 \in D'$ and $\{v_0, v_4\} \subset T'$. But then $(D' \cup \{v_2\}, T' \cup \{v_1\})$ is a non-exhaustive DT-pair in G, a contradiction. Hence, $k \in \{4, 5\}$.

Suppose that $k = 4$. Let C be the cycle $v_0w_1w_2v_5v_4v_0$. We note that, since $\mathcal{L}$ is an independent set, w_1w_2 is a 2-path in G. We now consider the graph $F' = F - v_4$ and note that F' is a connected subgraph of G with $\delta(F') \geq 2$. If $\gamma\gamma_t(F') < |V(F')|$, let (D', T') be a $\gamma\gamma_t(F')$-pair. But then $(D' \cup \{v_1, v_4\}, T' \cup \{v_2, v_3\})$ is a non-exhaustive DT-pair in G, a contradiction. Hence, $\gamma\gamma_t(F') = |V(F')|$, and so by Observation 4.16, $F' \in \mathcal{C} \cup \mathcal{K}^*$. Since both ends of the edge $w_1w_2 \in E(F')$ are small vertices in F', we note that $F' \notin \mathcal{K}^*$. Hence, $F' \in \mathcal{C}$. By Observation 4.8(b), there exists a DT-pair (D', T') in F' such that $\{v_0, w_1\} \subseteq T'$. Necessarily, $w_2 \in D'$. If $v_5 \in T'$, let $D = D' \cup \{v_2, v_3\}$ and

$T = (T' \setminus \{w_1\}) \cup \{v_1, v_4\}$. If $v_5 \in D'$, let $D = D' \cup \{v_2\}$ and $T = T' \cup \{v_3, v_4\}$. In both cases, (D, T) is a non-exhaustive DT-pair in G, a contradiction. Hence, $k = 5$.

Let C be the cycle $v_0v_4v_5v_6v'v_0$. We note that, since $\mathcal{L}$ is an independent set, $v' \in \mathcal{S}(G)$ and $N(v') = \{v_0, v_6\}$. We now consider the graph $F' = F - \{v_4, v_5\}$ and note that F' is a connected graph with $\delta(F') \geq 2$. Furthermore, F' is a subgraph of G and hence $F' \neq C_5$. Let (D', T') be a $\gamma\gamma_t(F')$-pair. In order to totally dominate the vertex v' in F', $|\{v_0, v_6\} \cap T'| \geq 0$. We may assume, without loss of generality, that $v_0 \in T'$. But then $(D' \cup \{v_2, v_5\}, T' \cup \{v_3, v_4\})$ is a non-exhaustive DT-pair in G, a contradiction. This completes the proof of Subclaim B1. □

We now return to the proof of Claim B. By Subclaim B1, the graph F is a connected C_5-free graph with $\delta(F) \geq 2$ that satisfies $\gamma\gamma_t(F) = |V(F)|$. Let $n' = n - 3$, and so $|V(F)| = n'$. If F is not n'-minimal, then by Lemma 4.4, F contains an n'-minimal spanning subgraph F' with no induced 5-cycle. But then, by the induction hypothesis, $F' \in \mathcal{C} \cup \mathcal{K}^*$ and therefore, by Lemma 4.14, $\gamma\gamma_t(F) < n' = |V(F)|$, a contradiction. Hence, F is n'-minimal, and by the induction hypothesis, $F \in \mathcal{C} \cup \mathcal{K}^*$. As observed earlier, F is not a cycle, and so $F \in \mathcal{K}^*$. Since $\mathcal{L}(G) = \mathcal{L}(F)$, we note that $v_0 \in \mathcal{L}(F)$ and that $k = 4$. Let w be a large vertex different from v_0 and v_5. Let v_0' be the common neighbor of v_0 and w in F, and let v_5' be the common neighbor of v_5 and w in F. By Observation 4.8(c), there exists a DT-pair (D', T') such that $w \in D'$, $\{v_0', v_5'\} \subset N(w) \subset T'$ and $\{v_0, v_5\} \subset T'$. But now $((D' \setminus \{v_4\}) \cup \{v_2, v_3\}, (T' \setminus \{v_0'\}) \cup \{v_1, v_4\})$ is a non-exhaustive DT-pair in G, a contradiction. □

Claim C *Every 2-path in G contains exactly one vertex.*

Proof. Let $P\colon v_1 \dots v_k$ be a longest 2-path in G and let v_0 and v_{k+1} be the large vertices that are adjacent in G to v_1 and v_k, respectively. We show that $k = 1$. By Claim B, $k \leq 2$. For the sake of contradiction, suppose that $k = 2$. Let $F = G - \{v_1, v_2\}$.

Suppose that F is disconnected. Let F_1 and F_2 denote the components containing v_0 and v_3, respectively. Then, $F = F_1 \cup F_2$. We consider first the case where $\gamma\gamma_t(F_1) < |V(F_1)|$ and $\gamma\gamma_t(F_2) < |V(F_2)|$. Let (D_1, T_1) and (D_2, T_2) be non-exhaustive DT-pairs in F_1 and F_2, respectively. If $v_0 \notin D_1$ then $(D_1 \cup D_2 \cup \{v_2\}, T_1 \cup T_2 \cup \{v_0, v_1\})$ is a non-exhaustive DT-pair in G, a contradiction. Therefore, $v_0 \in D_1$. Similarly, $v_3 \in D_2$. But then $(D_1 \cup D_2, T_1 \cup T_2 \cup \{v_1, v_2\})$ is a non-exhaustive DT-pair in G, again a contradiction. Hence, without loss of generality, we may assume that $\gamma\gamma_t(F_1) = |V(F_1)|$. By Observation 4.16, $F_1 \in \mathcal{C} \cup \mathcal{K}^*$ and therefore, by Observation 4.8(a), there is a DT-pair (D_1, T_1) in F_1 with $v_0 \in T_1$. If $\gamma\gamma_t(F_2) = |V(F_2)|$, then, similarly, $F_2 \in \mathcal{C} \cup \mathcal{K}^*$ and there is a DT-pair (D_2, T_2) in F_2 with $v_3 \in T_2$. But then $(D_1 \cup D_2 \cup \{v_1\}, T_1 \cup T_2)$ is a non-exhaustive DT-pair in G, a contradiction. Thus, $\gamma\gamma_t(F_2) < |V(F_2)|$. As before, let (D_2, T_2) be a non-exhaustive DT-pair in F_2. But then $(D_1 \cup D_2 \cup \{v_2\}, T_1 \cup T_2 \cup \{v_1\})$ is a non-exhaustive DT-pair in G, again a contradiction. Hence, F is connected.

Suppose $\gamma\gamma_t(F) < |V(F)|$. Let (D, T) be a $\gamma\gamma_t(F)$-pair. If $v_0 \in T$, then $(D \cup \{v_2\}, T \cup \{v_1\})$ is a non-exhaustive DT-pair in G, a contradiction. Therefore, $v_0 \notin T$. Similarly, $v_3 \notin T$. In order to totally dominate v_0 in F, there is a vertex $x \in N(v_0) \cap T$. Since $\mathcal{L}$ is an independent set in G, $d_G(x) = d_F(x) = 2$. Let $N(x) = \{v_0, y\}$. In order to totally dominate x, we note that $y \in T$. In order to dominate x, we note that $v_0 \in D$. Similarly, $v_3 \in D$. But then $(D, T \cup \{v_1, v_2\})$ is a non-exhaustive DT-pair in G, a contradiction. Hence, $\gamma\gamma_t(F) = |V(F)|$.

By Observation 4.16, $F \in \mathcal{C} \cup \mathcal{K}^*$. Suppose $F \in \mathcal{K}^*$. Since every neighbor of v_0 is a degree-2 vertex in G and hence in F, we note that $v_0 \in \mathcal{L}(F)$. Similarly, $v_3 \in \mathcal{L}(F)$. We note that v_0v_3 is not an edge of F. Let v' be the common neighbor of v_0 and v_3 in F. But then $v_0v_1v_2v_3v'v_0$ is an induced 5-cycle in G, contradicting the fact that G is C_5-free. Hence, $F \notin \mathcal{K}^*$, and so $F \in \mathcal{C}$. Since G is C_5-free, we note that v_0 and v_3 have no common neighbor in F. Hence, by the choice of P, we note that $F = C_6$ and that $d_F(v_0, v_3) = 3$. Let F be the cycle $w_0w_1 \ldots w_5w_0$ where $w_0 = v_0$ and $w_3 = v_3$. Then,

$(\{w_1, w_4, v_1\}, \{w_0, w_2, w_3, w_5\})$ is a non-exhaustive DT-pair in G, a contradiction. □

Claim D *There is no 2-handle in G.*

Proof. For the sake of contradiction, suppose that there is a 2-handle in G. Among all 2-handles in G, let $P\colon v_1v_2 \dots v_k$ have maximum length. Let v be the common neighbor of v_1 and v_k. We note that $v \in \mathcal{L}$. Further, we note that $k \geq 2$ and since G is C_5-free, $k \neq 4$. Let C be the cycle $vv_1v_2 \dots v_kv$ and let v' be a neighbor of v not on C. Since $\mathcal{L}$ is an independent set in G, $d_G(v') = 2$.

Suppose $d_G(v) \geq 4$. Let $F = G - V(P)$. Then, F is a connected C_5-free graph with $\delta(F) = 2$. If $\gamma\gamma_t(F) < |V(F)|$, let (D_1, T_1) be a $\gamma\gamma_t(F)$-pair. If $\gamma\gamma_t(F) = |V(F)|$, then by Observation 4.16, $F \in \mathcal{C} \cup \mathcal{K}^*$ and let (D_1, T_1) be a DT-pair in F such that v in T_1. We note that such a pair exists by Observation 4.8(a). If $v \in D_1$, let (D_2, T_2) be a DT-pair in C such that $v \in D_2$. Once again, such a pair exists by Observation 4.8(a). If $v \in T_1$, let (D_2, T_2) be a pair of disjoint sets of vertices in C such that $|D_2|+|T_2| < |V(C)|$, $v \in T_2$, and either (i) D_2 dominates $V(C)$ and T_2 totally dominates $V(C) \setminus \{v\}$, or (ii) D_2 dominates $V(C) \setminus \{v\}$ and T_2 totally dominates $V(C)$. In all cases, $(D_1 \cup D_2, T_1 \cup T_2)$ is a non-exhaustive DT-pair in G, a contradiction. Hence, $d_G(v) = 3$, and so $N(v) = \{v_1, v_k, v'\}$.

We note that, since vv' is a bridge in G, the degree-2 vertex v' belongs to a 2-path in G. Let $N(v') = \{v, w\}$. By Claim C, $w \in \mathcal{L}$. Let $F = G - (V(C) \cup \{v'\})$. Then, F is a connected C_5-free graph with $\delta(F) = 2$. Let (D_1, T_1) be a $\gamma\gamma_t(F)$-pair. If $w \in D_1$, let (D_2, T_2) be a DT-pair in C such that $v \in T_2$. If $w \in T_1$, let (D_2, T_2) be a DT-pair in C such that $v \in D_2$. In both cases, we note that such a pair exists by Observation 4.8(a). Furthermore, in both cases, $(D_1 \cup D_2, T_1 \cup T_2)$ is a non-exhaustive DT-pair in G, a contradiction. Hence, $w \notin D_1 \cup T_1$ and (D_1, T_1) is a non-exhaustive DT-pair in F. We now let (D_2, T_2) be a DT-pair in C such that $v \in T_2$. Then, $(D_1 \cup D_2 \cup \{v'\}, T_1 \cup T_2)$ is a non-exhaustive DT-pair in G, a contradiction. □

The following result is an immediate consequence of Claims C and D.

Claim E *The graph G is a bipartite graph with partite sets $\mathcal{L}$ and $\mathcal{S}$.*

We show next that a common neighbor of two large vertices is unique.

Claim F *Every two vertices in $\mathcal{L}$ have at most one common neighbor.*

Proof. For the sake of contradiction, suppose that $\{u, v\} \subseteq \mathcal{L}$ and w and w' are distinct common neighbors of u and v. Let $F = G - w'$. Then, F is a connected C_5-free graph with $\delta(F) = 2$. Suppose $\gamma\gamma_t(F) < |V(F)|$. Let (D, T) be a $\gamma\gamma_t(F)$-pair. Since T totally dominates w, $\{u, v\} \cap T \neq \emptyset$. But then $(D \cup \{w'\}, T)$ is a non-exhaustive DT-pair in G, a contradiction. Hence, $\gamma\gamma_t(F) = |V(F)|$, and so, by Observation 4.16, $F \in \mathcal{C} \cup \mathcal{K}^*$. If $F \in \mathcal{K}^*$ then, since $d_F(w) = 2$, we have that $\{u, v\} \subset \mathcal{L}(F)$. Therefore, by Observation 4.8(c), there exists a DT-pair (D,T) in F such that $u \in D$ and $v \in T$. But then (D, T) is a non-exhaustive DT-pair in G, a contradiction. Hence, $F \notin \mathcal{K}^*$, and so $F \in \mathcal{C}$. But then $F = C_4$, and so $n = 5$, a contradiction. □

Claim G *Every two vertices in $\mathcal{L}$ have exactly one common neighbor.*

Proof. By Claim F, every two vertices in $\mathcal{L}$ have at most one common neighbor. Hence it suffices to show that every two vertices in $\mathcal{L}$ have a common neighbor. For the sake of contradiction, suppose that $\{u, v\} \subseteq \mathcal{L}$ and that u and v have no common neighbor. Let $N(u) = \{u_1, u_2, \ldots, u_r\}$, and so $d_G(u) = r$. By Claim E, we note that $N(u) \subseteq \mathcal{S}$. For $i = 1, 2, \ldots, r$, let $N(u_i) = \{u, v_i\}$. By Claim E, we note that $v_i \in \mathcal{L}$ for each such i. By Claim F, $v_i \neq v_j$ for $1 \le i < j \le r$. Let $F = G - N[u]$. Then, F is a C_5-free graph with $\delta(F) = 2$. We note that F is possibly disconnected.

Suppose $\gamma\gamma_t(F) < |V(F)|$. Let (D, T) be a $\gamma\gamma_t(F)$-pair. For $i = 1, 2, \ldots, r$, let w_i be a neighbor of v_i in T. By Claim E, $w_i \in \mathcal{S}$. Hence, since D dominates and T

totally dominates w_i, we note that $v_i \in D \cup T$. If $v_i \in D$ for some i, $1 \le i \le r$, then $(D \cup (N(u) \setminus \{u_i\}), T \cup \{u, u_i\})$ is a non-exhaustive DT-pair in G, a contradiction. Therefore, $\{v_1, v_2, \ldots, v_r\} \subset T$. But then $(D \cup \{u\}, T \cup \{u_1\})$ is a non-exhaustive DT-pair in G, again a contradiction. Hence, $\gamma\gamma_t(F) = |V(F)|$.

Suppose F is disconnected. Let $F_1, F_2, \ldots, F_t$ be the components in F. By assumption, $t \ge 2$. Since $\gamma\gamma_t(F) = |V(F)|$, we note that $\gamma\gamma_t(F_i) = |V(F_i)|$ for all $i = 1, 2, \ldots, t$. Hence, by Observation 4.16, $F_i \in \mathcal{C} \cup \mathcal{K}^*$. Switching indices if necessary, we may assume that $v_i \in F_i$ for $i = 1, 2, \ldots, t$. For each such i, let (D_i, T_i) be a DT-pair in F_i such that $v_i \in D_i$. We note that such pairs exist by Observation 4.8(a). Let $D = \bigcup_{i=1}^{t} D_i$ and let $T = \bigcup_{i=1}^{t} T_i$. Then, (D, T) is a DT-pair in F and $(D \cup (N(u) \setminus \{u_1, u_2\}), T \cup \{u, u_1\})$ is a non-exhaustive DT-pair in G, a contradiction. Hence, F is connected.

By Observation 4.16, $F \in \mathcal{C} \cup \mathcal{K}^*$. Since $d_F(v) = d_G(v) \ge 3$, F is not a cycle and therefore $F \in \mathcal{K}^*$. By Claim E, the set $\mathcal{L}(G) \setminus \{u\} = \mathcal{L}(F)$. In particular, each vertex $v_i \in \mathcal{L}(F)$ for $i = 1, 2, \ldots, r$. By Observation 4.8(c), there exists a DT-pair (D, T) in F such that $v \in D$ and $\{v_1, v_2, \ldots, v_r\} \subset T$. But then $(D \cup \{u\}, T \cup \{u_1\})$ is a non-exhaustive DT-pair in G, a contradiction. □

We now return to the proof of Theorem 4.5. By Claims E and G, the graph G is a bipartite graph with partite sets $\mathcal{L}$ and $\mathcal{S}$ where every two vertices in $\mathcal{L}$ have exactly one common neighbor. Hence, $G \in \mathcal{K}^*$. This completes the necessity and the proof of Theorem 4.5. □

4.2.4 Proof of Theorem 4.3

We are now in a position to present a proof of our main result, namely Theorem 4.3. Recall the statement of Theorem 4.3.

Theorem 4.3. *Let G be a connected C_5-free graph with $\delta(G) \geq 2$. Then, $\gamma\gamma_t(G) = |V(G)|$ if and only if $G \in \mathcal{C} \cup \mathcal{K}^*$.*

Proof. The sufficiency follows from Lemmas 4.6 and 4.7. To prove the necessity, let G be a connected C_5-free graph of order n with $\delta(G) \geq 2$ such that $\gamma\gamma_t(G) = n$. Suppose that $G \notin \mathcal{C} \cup \mathcal{K}^*$. Then, by Theorem 4.5, G is not an n-minimal graph. Hence, by Lemma 4.4, G contains an n-minimal spanning subgraph F with no induced 5-cycle. By Theorem 4.5, $F \in \mathcal{C} \cup \mathcal{K}^*$. Therefore, by Lemma 4.14, $\gamma\gamma_t(G) < n$, a contradiction. Hence, $G \in \mathcal{C} \cup \mathcal{K}^*$. □

Chapter 5

The Existence of DPDP Graphs

Paired-domination was introduced by Haynes and Slater [48, 49] as a model for assigning backups to guards for security purposes and is studied in [9, 21, 22, 28, 33, 47, 48, 49, 52, 53, 63, 64, 67, 81, 98] inter alia.

We recall the results of Zelinka [99, 100] which showed that no minimum degree is sufficient to guarantee the existence of two disjoint total dominating sets. Since every paired-dominating set is a total dominating set, Zelinka's result is also true for paired-dominating sets. We therefore ask a similar question to that of Chapter 2; that is, which graphs contain disjoint dominating and paired-dominating sets?

Unlike the result of Theorem 2.1 in Chapter 2, where the vertex set of all connected graphs with minimum degree at least 2 can be partitioned into a dominating set and a total dominating set (with the exception of the 5-cycle), the situation now becomes much more complex. Our aim in this chapter is twofold: first to show that no minimum degree is sufficient to guarantee the existence of a partition of the vertex set into a dominating set and a paired-dominating set; secondly, to prove that every cubic graph contains a disjoint dominating set and paired-dominating set.

In Chapter 2, a graph whose vertex set can be partitioned into a DS and a TDS is called a DTDP-*graph* (standing for "dominating, total dominating, partitionable graph"). Hence Theorem 2.1 can be restated as follows.

Theorem 2.1 *Every connected graph with minimum degree at least* 2 *that is different from a* 5*-cycle is a DTDP-graph.*

Following this notation, we call a graph whose vertex set can be partitioned into a DS and a PDS a DPDP-*graph* (standing for "dominating, paired-dominating, partitionable graph"). A TD-*pair* of a graph G is a pair (T, D) of disjoint sets of vertices of G such that T is a TDS and D is a DS of G, while a PD-*pair* is a pair (P, D) of disjoint sets such that P is a PDS and D is a DS of G. Every PD-pair in a graph is also a TD-pair in the graph, and so every DPDP-graph is a DTDP-graph. The converse, however, is not true in general. The simplest such counterexample is obtained from a star $K_{1,n}$ by subdividing at least two of the edges.

5.1 DPDP Existence Results

As remarked earlier, unlike the result of Theorem 2.1, it is not enough to forbid the 5-cycle and guarantee the existence of the desired partition. We shall prove the following two results, proofs of which can be found in Section 5.2.

Theorem 5.1 *No minimum degree is sufficient to guarantee the existence of a disjoint dominating set and paired-dominating set.*

Theorem 5.2 *There exist infinite families of connected graphs with minimum degree two and maximum degree three that are not DPDP-graphs.*

Although for every positive integer $\delta \geq 1$ there are infinite families of graphs with minimum degree δ whose vertex set cannot be partitioned into a DS and a PDS, our main result shows that the vertex set of every cubic graph can be partitioned into a DS and PDS. We shall prove the following result, a proof of which can be found in Section 5.3.

Theorem 5.3 *Every cubic graph is a DPDP-graph.*

5.2 Non-Existence Proofs

Recall the statement of Theorem 5.1.

Theorem 5.1. *No minimum degree is sufficient to guarantee the existence of a disjoint dominating set and paired-dominating set.*

Proof. Let $k \geq 2$ be an arbitrary fixed integer. We shall show that there exists a graph G_k with minimum degree k that is not a DPDP-graph. Let G_k be the graph on $(k^k + k - 1)(k + 1)$ vertices constructed as follows. Let F be the graph of $(k - 1)$ disjoint copies of $K_{1,k}$, and so $F = (k - 1)K_{1,k}$. Label the $k - 1$ degree-k vertices in F by $u_1, u_2, \ldots, u_{k-1}$ and for $i = 1, 2, \ldots, k - 1$, let $N(u_i) = \{v_1^i, v_2^i, \ldots, v_k^i\}$. We construct the index set $I = \{(i_1, i_2, \ldots, i_{k-1}) : 1 \leq i_1, i_2, \ldots, i_{k-1} \leq k\}$ and, for each $\xi \in I$, we let F_ξ be the graph comprising k disjoint copies of $K_{1,k}$, and so $F_\xi = kK_{1,k}$. Let X_ξ be the set of k vertices in F_ξ with degree k. Now we let G_k be the graph obtained from the disjoint union $(\bigcup_{\xi \in I} F_\xi) \cup F$ as follows: For every $\xi = (i_1, i_2, \ldots, i_{k-1}) \in I$ and for every $j = 1, 2, \ldots, k-1$, join $v_{i_j}^j$ to each vertex with degree 1 in F_ξ. Note that $\delta(G) = k$. When $k = 3$, the graph G_k is sketched in Figure 5.1.

For the sake of contradiction suppose that G_k is a DPDP-graph. Let (P, D) be a PD-pair in G. Thus, (P, D) is a pair of disjoint sets such that P is a PDS and D is a DS of G. Since the set P totally dominates $\{u_1, u_2, \ldots, u_{k-1}\}$, we may assume, reassigning

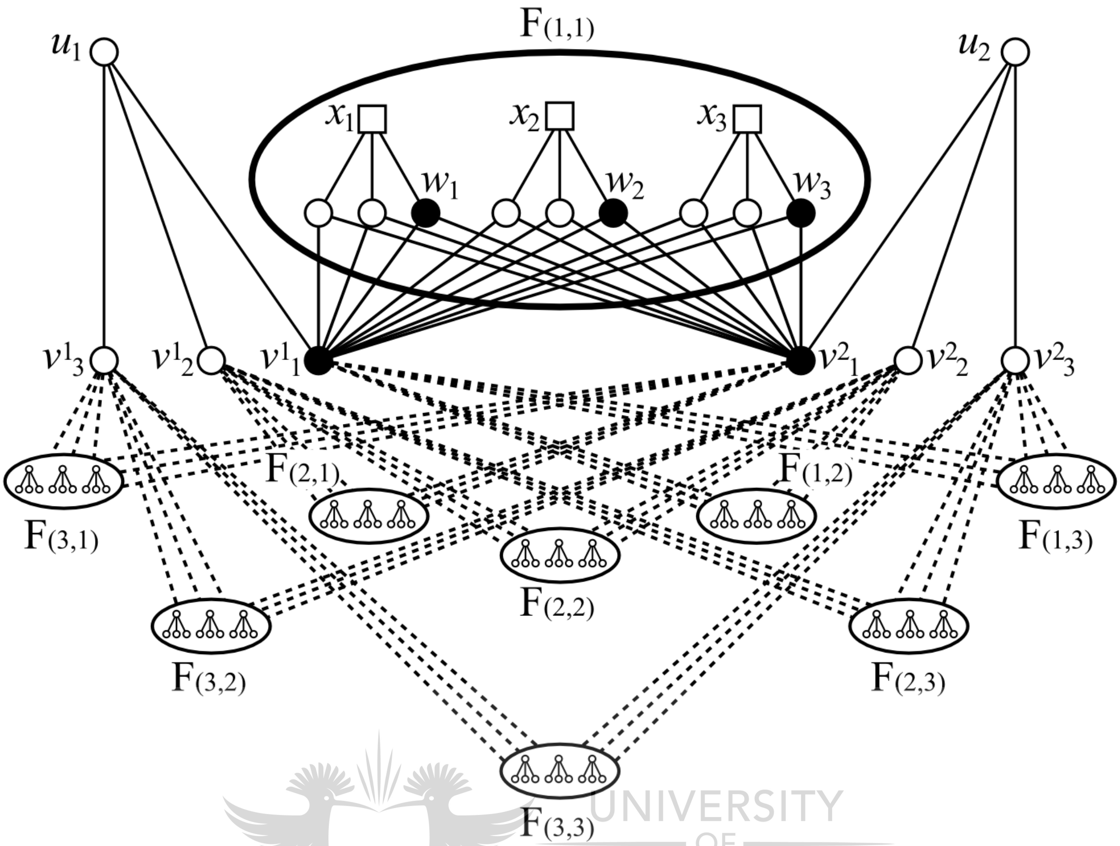


Figure 5.1: A sketch of G_3.

indices if necessary, that $\{v_1^1, v_1^2, \ldots, v_1^{k-1}\} \subset P$. Let $\varphi = (1, 1, \ldots, 1) \in I$ and let $X_\varphi = \{x_1, x_2, \ldots, x_k\}$. Since P totally dominates X_φ, for each $i \in \{1, 2, \ldots, k\}$ there is a vertex $w_i \in N(x_i)$ that belongs to the set P. By construction, we note that for each such $i \in \{1, 2, \ldots, k\}$, $N(w_i) = \{v_1^1, v_1^2, \ldots, v_1^{k-1}, x_i\}$, implying that $x_i \in D$. Further, w_i is paired with v_1^j for some $j \in \{1, 2, \ldots, k-1\}$. But then by the Pigeonhole Principle, there is an $\ell \in \{1, 2, \ldots, k-1\}$ such that v_1^ℓ is paired with two or more vertices from the set $\{w_1, w_2, \ldots, w_k\}$, a contradiction. Hence, G_k is not a DPDP-graph.

The x_i and w_i labels are included in Figure 5.1 for the case when $k = 3$. Vertices in P and D are represented by shaded circles and hollow squares, respectively. □

Recall the statement of Theorem 5.2.

Theorem 5.2. *There exist infinite families of connected graphs with minimum degree two and maximum degree three that are not DPDP-graphs.*

Proof. For $k \geq 1$ an integer, let G_k be the graph obtained from a path P on $2k+1$ vertices as follows: For each vertex z of the path P, add a 5-cycle and join z to one vertex of this cycle. The graph G_2 is illustrated in Figure 5.2. We note that if $uvwxyu$ is a 5-cycle in G_k such that $d(u) = d(v) = d(w) = d(x) = 2$ and $d(y) = 3$ with $N(y) = \{u, x, z\}$, then for any TD-pair (T, D) in G_k where T is a TDS and D is a DS of G_k, we have either:

(i) $\{u, x, y\} \subset T$ *and* $\{v, w, z\} \subset D$, *or*
(ii) $\{v, w, z\} \subset T$ *and* $\{u, x\} \subset D$.

If (i) holds for some such 5-cycle in G_k, then the subgraph of G_k induced by the TDS T contains the path uyx as a component and hence has no perfect matching. In this case, the (T, D)-pair would not be a (P, D)-pair. We may therefore assume that for every such 5-cycle in G_k, (ii) holds and so $V(P) \subset T$. In order to totally dominate the set $V(P)$, the set of $2k+1$ degree-3 vertices in G_k not on the path P all belong to D. In the graph G_2, illustrated in Figure 5.2, this partition is represented with the vertices in T depicted by shaded circles and the vertices in D by hollow circles. However since P is a path on an odd number of vertices, we note that the subgraph of G_k induced by the TDS T has no perfect matching. Hence the (T, D)-pair is not a (P, D)-pair. Since every (P, D)-pair is a (T, D)-pair, the graph G_k is not a DPDP-graph. □

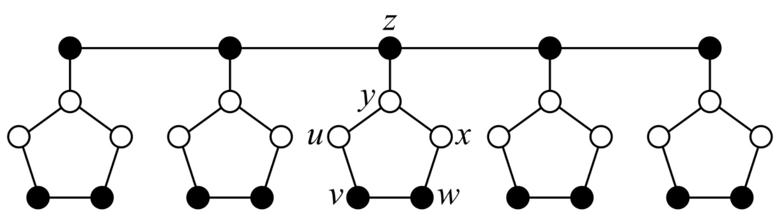


Figure 5.2: The graph G_2.

5.3 Existence Proof

Before proceeding to the proof of Theorem 5.3, we introduce the following additional notation and definitions. Throughout this section, we restrict our attention to cubic graphs unless otherwise stated. By Theorem 2.1, every cubic graph has a TD-pair. For a given TD-pair $\mathcal{D} = (T, D)$ in a (cubic) graph G, we let $\varphi(\mathcal{D})$ be the number of M-unmatched vertices in a maximum matching M of the subgraph $G[T]$ induced by T. We note that $\mathcal{D}$ is a PD-pair if and only if $\varphi(\mathcal{D}) = 0$. Furthermore, we let $\xi(\mathcal{D})$ be the number of edges in $G[T]$. We say that the TD-pair $\mathcal{D} = (T, D)$ is an *optimal TD-pair* in G if among all TD-pairs in G the following two conditions hold:

(1) $\varphi(\mathcal{D})$ *is minimized.*

(2) *Subject to* (1)*,* $\xi(\mathcal{D})$ *is minimized.*

Let $\mathcal{D} = (T, D)$ be an *optimal TD-pair* in G, and let M be an arbitrary maximum matching in $G[T]$. We say that an M-unmatched vertex w' in T is $\mathcal{D}_M$*-desirable* if there exists a subset $\{u, v, w, x\} \subset V(G)$ such that $\{u, v\} \subseteq D$, $\{w', w, x\} \subseteq T$, $u \in \text{epn}(w', T)$, $v \in \text{ipn}(u, D)$, $N(v) = \{u, w, x\}$, and the component of $G[T]$ containing w is an M-alternating w-x path (possibly, of length 1) that starts and ends with edges of M and every vertex in this component has a T-epn in G. A graphical sketch of a $\mathcal{D}_M$-desirable vertex w' is given in Figure 5.3. Vertices in T and D are represented by shaded and hollow circles, respectively. We proceed further by proving the following two lemmas.

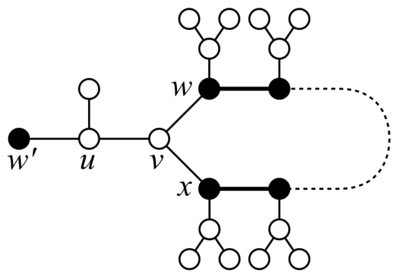


Figure 5.3: A $\mathcal{D}_M$-desirable vertex w'.

Lemma 5.4 *Let $\mathcal{D} = (T, D)$ be an optimal TD-pair in a cubic graph G and let M be a maximum matching in $G[T]$. If w is an M-unmatched vertex in T, then the component of $G[T]$ containing w is an odd cycle and every vertex in this component has a T-*epn *in G.*

Proof. Let G, $\mathcal{D}$ and M be defined as in the statement of the lemma and suppose w is an M-unmatched vertex in T. Let U be the set of all M-unmatched vertices in T and let $S = T \setminus U$. We note that U is an independent set and that M is a perfect matching in $G[S]$. Since T is a TDS in G, the vertex w has a neighbor in T. Since U is an independent set, such a neighbor of w belongs to S. Let $P\colon w_0v_1w_1 \dots v_kw_k$ be a longest M-alternating path in $G[T]$ that starts at $w = w_0$. We note that $v_iw_i \in M$ for $i = 1, 2, \dots, k$. Further, by the maximality of M, we note that $N(w_i) \cap T \subseteq S \cup \{w_0\}$. In particular, $\text{ipn}(w_i, T) = \emptyset$.

If $|\text{epn}(w_i, T)| = 0$ for some $i \in \{0, 1, \dots, k\}$, then $\mathcal{D}' = (D \cup \{w_i\}, T \setminus \{w_i\})$ is a TD-pair with $\varphi(\mathcal{D}') < \varphi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence for all $i = 0, 1, \dots, k$, $|\text{epn}(w_i, T)| > 0$ and we let $w_i' \in \text{epn}(w_i, T)$. If $|\text{epn}(v_i, T)| = 0$ for some $i \in \{1, 2, \dots, k\}$ then $\mathcal{D}' = ((D \setminus \{w_{i-1}', w_i'\}) \cup \{v_i\}, (T \setminus \{v_i\}) \cup \{w_{i-1}', w_i'\})$ is a TD-pair with $\varphi(\mathcal{D}') < \varphi(\mathcal{D})$, again contradicting our choice of $\mathcal{D}$. Hence $|\text{epn}(v_i, T)| > 0$ for all $i = 0, 1, \dots, k$. We note, therefore, that since G is a cubic graph, each internal vertex on the path P has degree 2 in $G[T]$ and is adjacent in $G[T]$ only to the vertices immediately preceding it and succeeding it on P.

Let $N(w_k) = \{v_k, w_k', x\}$. If $x \in D$, then $\mathcal{D}' = (D \setminus \{w_k'\}, T \cup \{w_k'\})$ is a TD-pair with $\varphi(\mathcal{D}') < \varphi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence, $x \in T$. As observed earlier, $x \in S \cup \{w_0\}$. If $x \in S$, then $xx' \in M$ for some $x' \notin V(P)$. But then $w_0v_1w_1 \dots v_kw_kxx'$ is an M-alternating path in $G[T]$ that starts at w_0 and has length exceeding that of P, contradicting our choice of P. Hence, $x = w_0$ and the desired result follows. $\square$

Lemma 5.5 *If $\mathcal{D} = (T, D)$ is an optimal TD-pair in a cubic graph G and M is a maximum matching in $G[T]$, then every M-unmatched vertex in T is $\mathcal{D}_M$-desirable.*

Proof. Let G, $\mathcal{D}$ and M be defined as in the statement of the lemma. Let U be the set of all M-unmatched vertices in T and let $S = T \setminus U$. We note that U is an independent set and that M is a perfect matching in $G[S]$. By Lemma 5.4, every vertex in U has two neighbors in S and one neighbor in D. Hence we have the following claim.

Claim 1 *If $\{a, b\} \subseteq T$ and $a \in \mathrm{ipn}(b, T)$, then $ab \in M$. In particular, if $b \in U$, then $\mathrm{ipn}(b, T) = \emptyset$.*

Let $w_0 \in U$. We show that w_0 is a $\mathcal{D}_M$-desirable vertex. By Lemma 5.4, the component of $G[T]$ containing w is an odd cycle and every vertex in this component has a T-epn in G. Let $u \in \mathrm{epn}(w_0, T)$. Let $N(u) = \{v, v', w_0\}$ and note that $\{v, v'\} \subset D$. Let $N(v) = \{u, w, x\}$. Since T totally dominates v, we may assume $w \in T$. Let $N(w) = \{v, w_1, w_2\}$. Since T totally dominates w, we may assume $w_1 \in T$.

If $w \in U$, then by Lemma 5.4, $\{w_1, w_2\} \subset T$ and $v \in \mathrm{epn}(w, T)$. Furthermore, by Claim 1, $\mathrm{ipn}(w, T) = \emptyset$. But then $\mathcal{D}' = ((D \setminus \{u, v\}) \cup \{w, w_0\}, (T \setminus \{w, w_0\}) \cup \{u, v\})$ is a TD-pair in G with $\varphi(\mathcal{D}') < \varphi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence, $w \in S$ and we may assume that $ww_1 \in M$. We show next that $x \in T$.

Claim 2 *$x \in T$.*

Proof. For the sake of contradiction suppose that $x \in D$. If $w_2 \in D$, then $\mathcal{D}' = ((D \setminus \{v\}) \cup \{w_0\}, (T \setminus \{w_0\}) \cup \{v\})$ is a TD-pair in G with $\varphi(\mathcal{D}') = \varphi(\mathcal{D})$ but $\xi(\mathcal{D}') < \xi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence, $w_2 \in T$. By Claim 1, $w_2 \notin \mathrm{ipn}(w, T)$, and so $|N(w_2) \cap T| = 2$. Also by Claim 1, $\mathrm{ipn}(w_1, T) = \emptyset$. If $\mathrm{epn}(w_1, T) = \emptyset$, then $\mathcal{D}' = ((D \setminus \{v\}) \cup \{w_0, w_1\}, (T \setminus \{w_0, w_1\}) \cup \{v\})$ is a TD-pair in G with $\varphi(\mathcal{D}') < \varphi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence, $|\mathrm{epn}(w_1, T)| \geq 1$. Let $w' \in \mathrm{epn}(w_1, T)$.

If $w' = x$, then $\mathcal{D}' = ((D \setminus \{x\}) \cup \{w\}, (T \setminus \{w\}) \cup \{x\})$ is a TD-pair in G with $\varphi(\mathcal{D}') = \varphi(\mathcal{D})$ but with $\xi(\mathcal{D}') < \xi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence, $w' \neq x$. But now $\mathcal{D}' = ((D \setminus \{u, v, w'\}) \cup \{w, w_0\}, (T \setminus \{w, w_0\}) \cup \{u, v, w'\})$ is a TD-pair in G with $\varphi(\mathcal{D}') < \varphi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. We conclude that $x \in T$. □

By Claim 2, $x \in T$. Let $N(x) = \{v, x_1, x_2\}$. Since T totally dominates x, we may assume $x_1 \in T$. If $x \in U$, then by Lemma 5.4, $\{x_1, x_2\} \subset T$ and $v \in \text{epn}(x, T)$, a contradiction since v is also adjacent to the vertex $w \in T$. Hence, $x \in S$ and we may assume that $xx_1 \in M$. We note that possibly $x = w_1$.

Claim 3 *$w_2 \in D$ or $x_2 \in D$.*

Proof. For the sake of contradiction, suppose that $w_2 \in T$ and $x_2 \in T$. By Claim 1, $w_2 \notin \text{ipn}(w, T)$ and thus $|N(w_2) \cap T| = 2$. Similarly, $x_2 \notin \text{ipn}(x, T)$ and thus $|N(x_2) \cap T| = 2$. If $w_1 = x$, then $x_1 = w$ and $\mathcal{D}' = ((D \setminus \{v\}) \cup \{w_0, x\}, (T \setminus \{w_0, x\}) \cup \{v\})$ is a TD-pair in G with $\varphi(\mathcal{D}') < \varphi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence, $w_1 \neq x$.

If w_1 has a T-epn, w' say, then $\mathcal{D}' = ((D \setminus \{w'\}) \cup \{w\}, (T \setminus \{w\}) \cup \{w'\})$ is a TD-pair in G with $\varphi(\mathcal{D}') = \varphi(\mathcal{D})$ but with $\xi(\mathcal{D}') < \xi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence, $\text{epn}(w_1, T) = \emptyset$. Similarly, $\text{epn}(x_1, T) = \emptyset$. Furthermore, by Claim 1, $\text{ipn}(w_1, T) = \text{ipn}(x_1, T) = \emptyset$.

Suppose there exists a vertex $y \in D$ such that $N(y) \cap T = \{w_1, x_1\}$. Then, $\mathcal{D}' = ((D \setminus \{v, y\}) \cup \{w_0, w_1, x\}, (T \setminus \{w_0, w_1, x\}) \cup \{v, y\})$ is a TD-pair in G with $\varphi(\mathcal{D}') < \varphi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence for every vertex $y \in D$, we have that $N(y) \cap T \not\subseteq \{w_1, x_1\}$. But then $\mathcal{D}' = ((D \setminus \{v\}) \cup \{w_0, w_1, x_1\}, (T \setminus \{w_0, w_1, x_1\}) \cup \{v\})$ is a TD-pair in G such that $\varphi(\mathcal{D}') = \varphi(\mathcal{D})$ and $\xi(\mathcal{D}') < \xi(\mathcal{D})$, again contradicting our choice of $\mathcal{D}$. □

Claim 4 *$w_2 \in D$ and $x_2 \in D$.*

Proof. By Claim 3, $w_2 \in D$ or $x_2 \in D$. Renaming vertices if necessary, we may assume that $x_2 \in D$. For the sake of contradiction, suppose that $w_2 \in T$. By Claim 1, $w_2 \notin \text{ipn}(w, T)$ and thus $|N(w_2) \cap T| = 2$. If w_1 has a T-epn, w' say, then $\mathcal{D}' = ((D \setminus \{w'\}) \cup \{w\}, (T \setminus \{w\}) \cup \{w'\})$ is a TD-pair in G with $\varphi(\mathcal{D}') = \varphi(\mathcal{D})$ but with $\xi(\mathcal{D}') < \xi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence, $\text{epn}(w_1, T) = \emptyset$. Furthermore, by Claim 1, $\text{ipn}(w_1, T) = \emptyset$. But then $\mathcal{D}' = ((D \setminus \{v\}) \cup \{w_0, w_1\}, (T \setminus \{w_0, w_1\}) \cup \{v\})$ is a TD-pair in G with $\varphi(\mathcal{D}') < \varphi(\mathcal{D})$, contradicting our choice of $\mathcal{D}$. Hence, $w_2 \in D$. □

Claim 5 *The component of $G[T]$ containing w is an M-alternating w-x path that starts and ends with edges of M. Moreover, every vertex in this component has a T-*epn *in G.*

Proof. Let $D' = (D \setminus \{v\}) \cup \{w_0\}$ and let $T' = (T \setminus \{w_0\}) \cup \{v\}$. We note that if $z \in T \cap T'$, then $\text{epn}(z, T) = \text{epn}(z, T')$. Furthermore, $\mathcal{D}' = (D', T')$ is a TD-pair in G such that $\varphi(\mathcal{D}') = \varphi(\mathcal{D})$ and $\xi(\mathcal{D}') = \xi(\mathcal{D})$. Since M is a maximum matching in $G[T']$ and v is an M-unmatched vertex in T', the component of $G[T']$ containing v is an odd cycle and every vertex in this component has a T'-epn in G by Lemma 5.4. The desired result follows. □

By Claim 5, w_0 is a $\mathcal{D}_M$-desirable vertex. This completes the proof of Lemma 5.5. □

We are now in a position to present a proof of our main result. Recall the statement of Theorem 5.3.

Theorem 5.3. *Every cubic graph is a DPDP-graph.*

Proof. Let G be a cubic graph and suppose, for the sake of contradiction, that G is not a DPDP-graph. By Theorem 2.1, G is a DTDP-graph. Let $\mathcal{D} = (T, D)$ be an optimal

TD-pair in G and let M be a maximum matching in $G[T]$. Since $\mathcal{D}$ is not a PD-pair, $\varphi(\mathcal{D}) > 0$. Let w_0 be an M-unmatched vertex in T.

We now choose k to be the largest integer such that $w_0u_1v_1w_1u_2v_2w_2 \dots u_kv_kw_k$ is a path in G satisfying the following properties: For each $i \in \{1, 2, \dots, k\}$, $\{u_i, v_i\} \subset D$, $w_i \in T$, $u_i \in \text{epn}(w_{i-1}, T)$, $v_i \in \text{ipn}(u_i, D)$, $N(v_i) = \{u_i, w_i, x_i\}$ and the component of $G[T]$ containing w_i is an M-alternating w_i-x_i path, P_i say, that starts and ends with edges of M and every vertex in this component has a T-epn in G. By Lemma 5.5, $k \geq 1$. Let

$$D' = \left(D \setminus (\bigcup_{i=1}^{k}\{v_i\})\right) \cup \left(\bigcup_{i=0}^{k-1}\{w_i\}\right) \quad \text{and} \quad T' = \left(T \setminus (\bigcup_{i=0}^{k-1}\{w_i\})\right) \cup \left(\bigcup_{i=1}^{k}\{v_i\}\right).$$

We note that if $z \in T \cap T'$, then $\text{epn}(z, T) = \text{epn}(z, T')$. For $i = 1, 2, \dots, k$, let $M_i = E(P_i) \cap M$ and let $M'_i = (E(P_i) \setminus M) \cup \{v_ix_i\}$. We now consider the matching M' in $G[T']$ defined by

$$M' = \left(M \setminus (\bigcup_{i=1}^{k} M_i)\right) \cup \left(\bigcup_{i=1}^{k} M'_i\right).$$

We note that $|M| = |M'|$ and that $\mathcal{D}' = (D', T')$ is a TD-pair in G. Furthermore, since $|T| = |T'|$ and $|M| = |M'|$, we have that $\varphi(\mathcal{D}') = \varphi(\mathcal{D})$. Additionally, $\xi(\mathcal{D}') = \xi(\mathcal{D})$. Thus by the choice of $\mathcal{D}$, $\mathcal{D}'$ is an optimal TD-pair in G and M' is a maximum matching in $G[T']$. Since w_k is an M'-unmatched vertex in T', w_k is a $\mathcal{D}'_{M'}$-desirable vertex by Lemma 5.4. Hence there exist vertices $\{u_{k+1}, v_{k+1}, w_{k+1}, x_{k+1}\} \subset V(G)$ such that $\{u_{k+1}, v_{k+1}\} \subset D'$, $\{w_{k+1}, x_{k+1}\} \subset T'$, $u_{k+1} \in \text{epn}(w_k, T')$, $v_{k+1} \in \text{epn}(u_{k+1}, D')$, $N(v_{k+1}) = \{u_{k+1}, w_{k+1}, x_{k+1}\}$ and the component of $G[T']$ containing w_{k+1} is an M'-alternating w_{k+1}-x_{k+1} path that starts and ends with edges of M' and every vertex in this component has a T'-epn in G.

But now, by the construction of $\mathcal{D}'$ and M', $w_0u_1v_1w_1 \dots u_kv_kw_ku_{k+1}v_{k+1}w_{k+1}$ is a path in G satisfying the following properties: For each $i \in \{1, 2, \dots, k+1\}$, $\{u_i, v_i\} \subset D$, $w_i \in T$, $u_i \in \text{epn}(w_{i-1}, T)$, $v_i \in \text{ipn}(u_i, T)$, $N(v_i) = \{u_i, w_i, x_i\}$ and the component of

$G[T]$ containing w_i is an M-alternating w_i-x_i path that starts and ends with edges of M and every vertex in this component has a T-epn in G. This, however, contradicts our choice of k. We deduce, therefore that the graph G is a DPDP-graph. □

Chapter 6

Characterizing DPDP Graphs

A characterization of graphs whose vertex set can be partitioned into a dominating set and a total dominating set is given in Chapter 3. The context of this research motivates the question of which graphs have disjoint dominating and paired-dominating sets. In the previous chapter we showed that DPDP-graphs are more difficult to pin down than DTDP-graphs when the minimum degree is at least 2. Our aim in this chapter is to provide a constructive characterization of all graphs whose vertex set can be partitioned into a dominating set and a paired-dominating set.

Recall that a graph whose vertex set can be partitioned into a dominating set and a total dominating set is called a DTDP-graph and a graph whose vertex set can be partitioned into a dominating set and a paired-dominating set a DPDP-graph. A TD-pair of a graph G is a pair (T, D) of disjoint sets of vertices of G such that T is a total dominating set and D is a dominating set of G, while a PD-pair is a pair (P, D) of disjoint sets such that P is a paired-dominating set and D is a dominating set of G.

As noted in the previous chapter, every PD-pair in a graph is also a TD-pair in the graph, and so every DPDP-graph is a DTDP-graph. The converse, however, is not true in general, with the simplest counterexample obtained from a star $K_{1,n}$ by subdividing at least two of the edges. More generally, let G be the graph obtained from an arbitrary graph H by attaching two pendant edges to each vertex of H and then, for each vertex in H, subdividing exactly one of the added pendant edges. The graph obtained from G by attaching an additional pendant edge to any of the vertices from the original graph H and subdividing this edge is a DTDP-graph, but not a DPDP-graph, whose diameter can be made arbitrarily large (by choosing H to have large diameter).

Moreover, unlike the result of Theorem 2.1, which proves that all connected graphs with minimum degree at least 2 (except the 5-cycle) are DTDP-graphs, the situation becomes more complex for DPDP-graphs. Indeed there are infinite families of connected graphs of minimum degree at least 2 that are not DPDP-graphs. The simplest such family consists of graphs $D_k(5)$ that can be constructed from $k \geq 2$ disjoint 5-cycles by identifying a set of k vertices, one from each cycle, into one new vertex v.

Observation 6.1 *For $k \geq 2$, the graph $D_k(5)$ is not a DPDP-graph.*

Proof. For the sake of contradiction, suppose that $G = D_k(5)$ is a DPDP-graph for some $k \geq 2$. Let (P, D) be a PD-pair in G. We note that P is also a total dominating set in G. If $v \in D$, then in order to totally dominate each neighbor of v, every vertex at distance 2 from v belongs to P. In order to dominate these vertices at distance 2 from v, every neighbor of v therefore belongs to D. But then v is not totally dominated by P, a contradiction. Hence, $v \in P$. In order to totally dominate v, let u be a neighbor of v in P. Let $uvwxyu$ be the 5-cycle containing u. To dominate u, we must have that $y \in D$. To totally dominate x, we therefore have that $w \in P$. Since the subgraph induced by P contains a perfect matching, we have that $x \in P$. But then w is not dominated by D, a contradiction. Hence, G contains no PD-pair; that is, G is not a DPDP-graph. $\square$

We also remark that there exist graphs with minimum degree at least 2 and arbitrarily large diameter that are not DPDP-graphs.

6.1 Graph Labelings

Our aim in this chapter is to provide a constructive characterization of DPDP-graphs. As in Chapter 3, where we characterize DTDP-graphs, the key to our constructive characterization is to find a labeling of the vertices that indicates the role each vertex plays in the sets associated with both parameters. We define a *labeling* of a graph G as a partition $S = (S_A, S_B)$ of $V(G)$. The *label* or *status* of a vertex v, denoted $\mathrm{sta}(v)$, is the letter $x \in \{A, B\}$ such that $v \in S_x$. Our aim is to describe a procedure to build DPDP-graphs in terms of labelings. By a *labeled-*P_4, we shall mean a P_4 with the two central vertices labeled A and the two leaves labeled B.

6.1.1 The Graph Family $\mathcal{T}$

Let $\mathcal{T}$ be the minimum family of labeled trees that: (i) contains a labeled-P_4; and (ii) is closed under the four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ listed below, which extend a labeled tree T by attaching a tree to the vertex $v \in V(T)$.

- **Operation $\mathcal{O}_1$.** Let v be a vertex with $\mathrm{sta}(v) = A$. Add a vertex u_1 and the edge vu_1. Let $\mathrm{sta}(u_1) = B$.
- **Operation $\mathcal{O}_2$.** Let v be a vertex with $\mathrm{sta}(v) = A$. Add a path $u_1u_2u_3u_4$ and the edge vu_2. Let $\mathrm{sta}(u_1) = \mathrm{sta}(u_4) = B$ and $\mathrm{sta}(u_2) = \mathrm{sta}(u_3) = A$.
- **Operation $\mathcal{O}_3$.** Let v be a vertex with $\mathrm{sta}(v) = B$. Add a path $u_1u_2u_3$ and the edge vu_1. Let $\mathrm{sta}(u_1) = \mathrm{sta}(u_2) = A$ and $\mathrm{sta}(u_3) = B$.
- **Operation $\mathcal{O}_4$.** Let v be a vertex with $\mathrm{sta}(v) = B$. Add a path $u_1u_2u_3u_4$ and the edge vu_1. Let $\mathrm{sta}(u_1) = \mathrm{sta}(u_4) = B$ and $\mathrm{sta}(u_2) = \mathrm{sta}(u_3) = A$.

These four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ are illustrated in Figure 6.1.

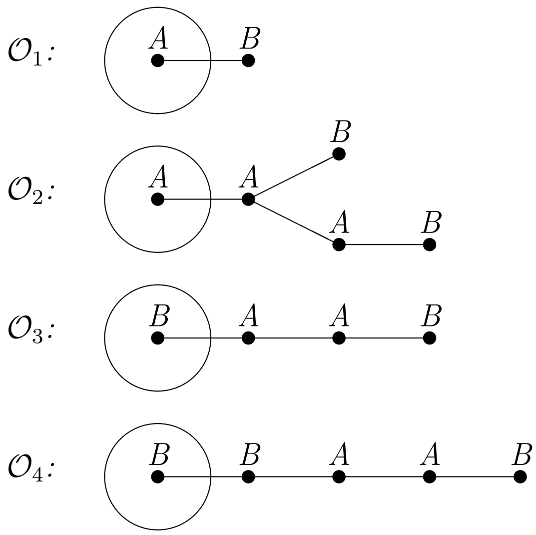


Figure 6.1: The four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$.

6.1.2 The Graph Family $\mathcal{G}$

Let $\mathcal{O}_5$, $\mathcal{O}_6$, $\mathcal{O}_7$ and $\mathcal{O}_8$ be the four operations listed below, which extend a labeled graph G as follows:

- **Operation $\mathcal{O}_5$.** Let u and v be two nonadjacent vertices in G. Add the edge uv.
- **Operation $\mathcal{O}_6$.** Let v be a vertex with $\text{sta}(v) = B$. Add a path u_1u_2 and the edges vu_1 and vu_2. Let $\text{sta}(u_1) = \text{sta}(u_2) = A$.
- **Operation $\mathcal{O}_7$.** Let u and v be distinct vertices of G with $\text{sta}(u) = \text{sta}(v) = B$. Add a path u_1u_2 and the edges uu_1 and vu_2. Let $\text{sta}(u_1) = \text{sta}(u_2) = A$.
- **Operation $\mathcal{O}_8$.** Let v be a vertex with $\text{sta}(v) = A$. Add a cycle $u_1u_2u_3u_1$ and the edge vu_1. Let $\text{sta}(u_1) = \text{sta}(u_2) = A$ and $\text{sta}(u_3) = B$.

These four operations are illustrated in Figure 6.2.

Let $\mathcal{G}$ be the minimum family of labeled graphs that: (i) contains a labeled-P_4; and (ii) is closed under the eight operations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_8$ described earlier. By construction,

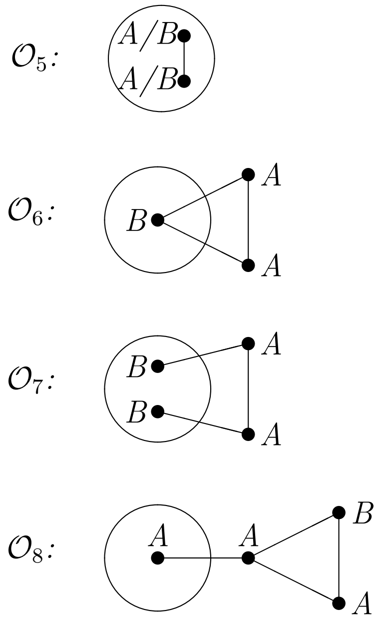


Figure 6.2: The four operations $\mathcal{O}_5$, $\mathcal{O}_6$, $\mathcal{O}_7$ and $\mathcal{O}_8$.

the family $\mathcal{T}$ is a subfamily of the family $\mathcal{G}$.

We shall need the following observation which follows from the way in which the family $\mathcal{G}$ is constructed.

Observation 6.2 *Let* $(G, S) \in \mathcal{G}$ *for some labeling* $S = (S_A, S_B)$*. Then the following properties hold:*

(a) *Every vertex of status* A *is adjacent to a vertex of status* A *and to a vertex of status* B*;*

(b) *Every vertex of status* B *is adjacent to a vertex of status* A*;*

(c) *Since each operation adds exactly zero or two adjacent vertices of status* A*, the subgraph induced by* S_A *contains a perfect matching comprising exactly those edges incident with both status* A *vertices added at each operation (with the exception of* $\mathcal{O}_1$ *and* $\mathcal{O}_5$*) as well as the edge incident with both status* A *vertices of the labeled-*P_4*. Hence* S_A *is a PDS of* G*, while* S_B *is a DS of* G*.*

(d) *If* $v \in V(G)$ *and* $d(v) = 1$*, then* v *has status* B *and the neighbor of* v *has status* A*.*

6.2 DPDP Characterization Results

In this chapter, we have two immediate aims. Our first aim is to determine which trees are DPDP-trees. For this purpose, we establish the following constructive characterization of DPDP-trees that uses labelings, a proof of which is presented in Section 6.2.1.

Theorem 6.3 *The DPDP-trees are precisely those trees* T *such that* $(T, S) \in \mathcal{T}$ *for some labeling* S*.*

Our second aim is to determine which connected graphs with minimum degree one are DPDP-graphs. We remark that if a connected graph has a spanning DPDP-tree, then it is a DPDP-graph. However, a connected DPDP-graph does not necessarily have a spanning DPDP-tree. For example, let G_k be obtained from the disjoint union of $k \geq 1$ copies of K_3 by adding a path P_3 and joining a leaf of the added path to one vertex from each copy of K_3. The graph G_3 is illustrated in Figure 3.3 in Chapter 3. Then, G_k is a DPDP-graph but G_k does not have a spanning DPDP-tree. We remark that we could have replaced some or all of the copies of K_3 in G_k with copies of C_6 or C_9.

Every DPDP-graph has order at least 3. Trivially, the only DPDP-graph of order 3 is the complete graph K_3. Our main result is the following constructive characterization of DPDP-graphs of order at least 4 that uses labelings, a proof of which is presented in Section 6.2.2.

Theorem 6.4 *The connected DPDP-graphs of order at least* 4 *are precisely those graphs* G *such that* $(G, S) \in \mathcal{G}$ *for some labeling* S*.*

6.2.1 Proof of Theorem 6.3

Recall that a PD-pair in a graph G is a pair (P, D) of disjoint sets such that P is a PDS and D is a DS of G. Since every PDS in a tree contains all the support vertices, we have the following observation.

Observation 6.5 *Let T be a rooted DPDP-tree and let (D_1, D_2) be a PD-pair in T. Then the following properties hold:*

(a) *Every leaf belongs to D_2 while every support vertex belongs to D_1.*

(b) *If every child of a vertex is a leaf, then its parent belongs to D_1.*

Recall the statement of Theorem 6.3.

Theorem 6.3. *The DPDP-trees are precisely those trees T such that $(T, S) \in \mathcal{T}$ for some labeling S.*

Proof. Suppose first that T is a tree and $(T, S) \in \mathcal{T}$ for some labeling S. By Observation 6.2(c), (S_A, S_B) is a PD-pair in T, and so T is a DPDP-tree. This establishes the sufficiency.

To prove the necessity, we proceed by induction on the order n of a DPDP-tree T. Since every star $K_{1,n-1}$ is not a DPDP-tree, we have that $n \geq 4$ and $\text{diam}(T) \geq 3$. If $n = 4$, then $T = P_4$ and $(T, S) \in \mathcal{T}$, where S is the labeling of a labeled-P_4. This establishes the base case. For the inductive hypothesis, let $n \geq 5$ and assume that for every DPDP-tree T' of order less than n there exists a labeling S' such that $(T', S') \in \mathcal{T}$.

Let T be a DPDP-tree of order n. Let $D = (D_1, D_2)$ be a a PD-pair in T. Let $T_1 = T[D_1]$ be the subgraph of T induced by D_1 and let M be a perfect matching in T_1. We now root the tree T at a diametrical vertex r. Necessarily, r is a leaf. Let u be a vertex at maximum distance from r. Necessarily, u is a leaf. Let v be the parent of u, let w be the parent of v, and let x be the parent of w (possibly, $x = r$). Since u is at maximum distance from the root r, every child of v is a leaf. By Observation 6.5, we observe that $C(v) \subset D_2$ and $\{v, w\} \subseteq D_1$. In particular, $u \in D_2$. Furthermore, $d_{T_1}(v) = 1$ and hence $vw \in M$; that is, v and w are paired in D_1. We proceed further with a series of claims that we may assume the tree T satisfies.

Claim A *T has no strong support vertex.*

Proof. Suppose that T has a strong support vertex z. Let z_1 and z_2 be two leaf-neighbors of z in T. By Observation 6.5, we note that $\{z_1, z_2\} \subseteq D_2$ and $z \in D_1$. Let $T' = T - z_1$. Then, $(D_1, D_2 \setminus \{z_1\})$ is a PD-pair in T', and so T' is a DPDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. By Observation 6.2(d), $z \in S'_A$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_1$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A, S'_B \cup \{z_1\})$. Hence, if T has a strong support vertex, then $(T, S) \in \mathcal{T}$ for some labeling S, as desired. □

By Claim A, we note that $d_T(v) = 2$.

Claim B $d_T(w) = 2$.

Proof. Suppose $d_T(w) \geq 3$. Let $v' \in C(w) \setminus \{v\}$. Suppose $d_T(v') \geq 2$. By our choice of the vertex u, every child of v' is a leaf. Since T has no strong support vertex, $d_T(v') = 2$. Let u' be the child of v'. Then, u' is a leaf. By Observation 6.5, $u' \in D_2$ and $v' \in D_1$. Thus, v' and w are paired in D_1, contradicting the fact that v and w are paired in D_1. Hence every child of w, different from v, is a leaf. Thus since T has no strong support vertex, $d_T(w) = 3$ and $C(w) = \{v, v'\}$, where v' is a leaf. Thus by Observation 6.5, $\{u, v'\} \subseteq D_2$ and $\{v, w\} \subseteq D_1$, with v and w paired in D_1.

Suppose $x \in D_1$. Since v and w are paired in D_1, the partner of x in D_1 is different from w. We also note that since $\{x, w\} \subseteq D_1$, x is adjacent to a vertex of D_2 different from w. Let $T' = T - \{u, v, v', w\}$. Then, $(D_1 \setminus \{v, w\}, D_2 \setminus \{u, v'\})$ is a PD-pair in T', and so T' is a DPDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. If $x \in S'_A$, then we can restore the tree T by applying Operation $\mathcal{O}_2$ to T'. If $x \in S'_B$, then we can restore the tree T by first applying Operation $\mathcal{O}_3$ to T' and then Operation $\mathcal{O}_1$ to the resulting tree. In both cases, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v, w\}, S'_B \cup \{u, v'\})$. Hence, if $x \in D_1$, then $(T, S) \in \mathcal{T}$ for some labeling S, as desired. Thus we may assume that $x \in D_2$.

We now let $T' = T - v'$. Then, $(D_1, D_2 \setminus \{v'\})$ is a PD-pair in T', and so T' is a DPDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. By Observation 6.2, $\{v, w\} \subseteq S'_A$ and $u \in S'_B$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_1$ to T'. Hence, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A, S'_B \cup \{v'\})$. □

By Claim B, we have that $d_T(w) = 2$. Since $n \geq 5$, the vertex x is not the root r of the rooted tree T. Let y be the parent of x. As remarked earlier, $u \in D_2$ and $\{v, w\} \subseteq D_1$ with v and w paired in D_1. In order to dominate w, we have that $x \in D_2$.

Claim C $d_T(x) = 2$.

Proof. Suppose $d_T(x) \geq 3$. Let $w' \in C(x) \setminus \{w\}$. By Observation 6.5, the vertex x is not a support vertex. Thus, no child of x is a leaf. In particular, $d_T(w') \geq 2$. By our choice of the vertex u, every descendant of w' is a leaf or is at distance 2 from w'. Suppose every child of w' is a leaf. Then, since T has no strong support vertex, $d_T(w') = 2$. Let v' denote the child of w', and so v' is a leaf. By Observation 6.5, $v' \in D_2$ and $\{w', x\} \subseteq D_1$, contradicting the fact that $x \in D_2$. Hence, w' has a descendant u' at distance 2 from w'. As shown in Claim B, we may assume that $d_T(w') = 2$. By Observation 6.5, $u' \in D_2$ and $\{v', w'\} \subseteq D_1$ with v' and w' paired in D_1.

We now consider the tree $T' = T - \{u, v, w\}$. Then, $(D_1 \setminus \{v, w\}, D_2 \setminus \{u\})$ is a PD-pair in T', and so T' is a DPDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. By Observation 6.2, $\{u', x\} \subseteq S'_B$ and $\{v', w'\} \subseteq S'_A$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_3$ to T'. Hence, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v, w\}, S'_B \cup \{u\})$. □

By Claim C, we have that $d_T(x) = 2$.

Claim D $y \in D_2$.

Proof. Suppose $y \in D_1$. We now consider the tree $T' = T - \{u, v, w\}$. Then, $(D_1 \setminus \{v, w\}, D_2 \setminus \{u\})$ is a PD-pair in T', and so T' is a DPDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. By Observation 6.2, the leaf $x \in S'_B$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_3$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v, w\}, S'_B \cup \{u\})$. □

We now return to the proof of Theorem 6.3. By Claim D, we have that $y \in D_2$. We now consider the tree $T' = T - \{u, v, w, x\}$. Then, $(D_1 \setminus \{v, w\}, D_2 \setminus \{u, x\})$ is a PD-pair in T', and so T' is a DPDP-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. If $y \in S'_B$, then we can restore the tree T by applying Operation $\mathcal{O}_4$ to T'. If $y \in S'_A$, then we can restore the tree T by first applying Operation $\mathcal{O}_1$ to T' and then Operation $\mathcal{O}_3$ to the resulting tree. In both cases, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v, w\}, S'_B \cup \{u, x\})$. Thus, $(T, S) \in \mathcal{T}$ for some labeling S, as desired. This completes the necessity, and the proof of Theorem 6.3. □

6.2.2 Proof of Theorem 6.4

Recall the statement of Theorem 6.4.

Theorem 6.4. *The connected DPDP-graphs of order at least* 4 *are precisely those graphs* G *such that* $(G,S) \in \mathcal{G}$ *for some labeling* S.

Proof. Suppose first that G is a connected graph and $(G,S) \in \mathcal{G}$ for some labeling S. By Observation 6.2(c), (S_A, S_B) is a PD-pair in G, and so G is a connected DPDP-graph. This establishes the sufficiency.

To prove the necessity we proceed by induction on the order $n \geq 4$ of a connected DPDP-graph G. If $n = 4$, then since no star is a DPDP-graph, the graph G contains P_4 as a subgraph. Let $G' = P_4$ be a subgraph of G (possibly, $G' = G$) obtained from G by removing zero, one, two or three edges. Then, $(G', S) \in \mathcal{G}$, where S is the labeling of a labeled-P_4 and we can restore the graph G from G' by repeated applications (including the possibility of none) of Operation $\mathcal{O}_5$. Thus, $(G,S) \in \mathcal{G}$. This establishes the base case. For the inductive hypothesis, let $n \geq 5$ and assume that for every DPDP-graph G' of order less than n there exists a labeling S' such that $(G', S') \in \mathcal{G}$.

Let G be a connected DPDP-graph of order n. Among all PD-pairs $\mathcal{D} = (D_1, D_2)$ in G and among all spanning connected subgraphs H of G such that (D_1, D_2) is a PD-pair in H (possibly, $H = G$), let the partition (D_1, D_2) and the graph H be chosen so that

(1) $|D_1|$ is minimized.
(2) Subject to (1), $|E(H)|$ is minimized.
(3) Subject to (2), $\sum_{v \in D_1} d_H(v)$ is minimized.

Let M be a perfect matching in $G[D_1]$ that is used to determine the pairing of vertices in the PDS D_1.

Claim E *If* H *has a strong support vertex, then* $(G,S) \in \mathcal{G}$ *for some labeling* S.

Proof. Suppose H has a strong support vertex v. Let v_1 and v_2 be two leaf-neighbors of

v in H. By Observation 6.5, we note that $\{v_1, v_2\} \subseteq D_2$ and $v \in D_1$. Let $H' = H - v_1$. Then, $(D_1, D_2 \setminus \{v_1\})$ is a PD-pair in H', and so H' is a DPDP-graph. Applying the inductive hypothesis to H', there exists a labeling $S' = (S'_A, S'_B)$ such that $(H', S') \in \mathcal{G}$. By Observation 6.2(d), $v \in S'_A$. Thus, we can restore the graph H by applying Operation $\mathcal{O}_1$ to H'. We can then restore G from H by repeated applications of Operation $\mathcal{O}_5$. Therefore, $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A, S'_B \cup \{v_1\})$. □

Hence, by Claim E, we may assume that H has no strong support vertex. We proceed further with the following useful lemma, called the Cycle Lemma, that we may assume the graph H satisfies.

Cycle Lemma *For $k \geq 3$, if $C\colon v_1v_2v_3 \ldots v_kv_{k+1} = v_1$ is a cycle in H, then the following properties hold:*

(a) *No two adjacent vertices on C both belong to D_2.*

(b) $V(C) \cap D_2 \neq \emptyset$.

(c) *Every vertex of C in D_1 is adjacent in H to some other vertex of C in D_1.*

(d) *No three consecutive vertices on C are all in D_1.*

(e) $k \equiv 0 \pmod 3$, *and* $v_i \in D_2$ *for* $i \equiv 1 \pmod 3$ *and* $v_i \in D_1$ *for* $i \equiv 0, 2 \pmod 3$.

(f) $v_iv_{i+1} \in M$ *for* $i \equiv 2 \pmod 3$.

(g) *The cycle C is chordless.*

(h) *Every vertex in D_1 on the cycle C is adjacent in H to exactly one vertex in D_2.*

(i) $d_H(v_i) = 2$ *or* $d_H(v_{i+1}) = 2$ *for* $i \equiv 2 \pmod 3$.

(j) *If $v_i \in D_1$ and $d_H(v_i) \geq 3$, then every edge incident with v_i not on the cycle C is a bridge of H and does not belong to M.*

Proof. (a) For the sake of contradiction, suppose there are two adjacent vertices u and v in C that both belong to the DS D_2. But then the graph $H' = H - uv$ is a spanning connected subgraph of G and (D_1, D_2) in a PD-pair in H', contradicting the minimality condition (2) of H. (In what follows, we will simply say that the edge uv could be removed from H, contradicting the minimality of H.)

(b) For the sake of contradiction, suppose $V(C) \subseteq D_1$. Let $e \in E(C) \setminus M$. But then the edge e could be removed from H, contradicting the minimality of H.

(c) For the sake of contradiction, suppose that there is a vertex v of C in D_1 with both its neighbors on C in D_2. For notational convenience, we may assume that $v = v_2$. Thus, $\{v_1, v_3\} \subseteq D_2$ and $v_2 \in D_1$. By part (a), we have that $k \geq 4$ and that $v_4 \in D_1$. But then the edge v_2v_3 could be removed from H, contradicting the minimality of H.

(d) For the sake of contradiction, suppose that there are three consecutive vertices on C in D_1. For notational convenience, we may assume that $\{v_1, v_2, v_3\} \subseteq D_1$. If $v_1v_2 \notin M$, then the edge v_1v_2 could be removed from H, contradicting the minimality of H. Hence, $v_1v_2 \in M$. But then $v_2v_3 \notin M$, and so the edge v_2v_3 could be removed from H, contradicting the minimality of H.

(e) By (b), at least one vertex of C belongs to D_2. For notational convenience, we may assume that $v_1 \in D_2$. By (a), $v_2 \in D_1$. By (c), $v_3 \in D_1$. If $k = 3$, then the desired result follows. Hence we may assume that $k \geq 4$. By (d), $v_4 \in D_2$. By (a), $k \geq 5$ and $v_5 \in D_1$. By (c), $k \geq 6$ and $v_6 \in D_1$. If $k = 6$, then the desired result follows. Hence we may assume that $k \geq 7$. Continuing in this way, we have that $k \equiv 0 \pmod 3$ and that $v_i \in D_2$ for $i \equiv 1 \pmod 3$ and $v_i \in D_1$ for $i \equiv 0, 2 \pmod 3$.

(f) By (e), $\{v_i, v_{i+1}\} \subseteq D_1$ for $i \equiv 2 \pmod 3$. Further, if the edge $v_iv_{i+1} \notin M$, then it could be removed from H, contradicting the minimality of H. Thus any edge on C incident with two vertices from D_1 is in M.

(g) If there is a chord in the cycle C (that does not join two consecutive vertices on C), then it could be removed from H, contradicting the minimality of H.

(h) For the sake of contradiction, suppose $v_i \in D_1$ is adjacent to two or more vertices in D_2. By (e), $i \equiv 0, 2 \pmod 3$. If $i \equiv 2 \pmod 3$, then the edge $v_{i-1}v_i$ could be removed from H. If $i \equiv 0 \pmod 3$, then the edge v_iv_{i+1} could be removed from H. Both cases contradict the minimality of H.

(i) Let $i \equiv 2 \pmod 3$. By part (g), the cycle C is an induced cycle in H, and so $d_H(v_i) \geq d_C(v_i) = 2$ and $d_H(v_{i+1}) \geq d_C(v_{i+1}) = 2$. For the sake of contradiction, suppose that $d_H(v_i) \geq 3$ and $d_H(v_{i+1}) \geq 3$. Let w and x be neighbors of v_i and v_{i+1}, respectively, not on C. Possibly, $w = x$. By part (h), $w \in D_1$ and $x \in D_1$. By part (f), v_i and v_{i+1} are paired in D_1. Let w' and x' be the partners of w and x, respectively, in D_1. Then, $w' \notin \{v_i, v_{i+1}\}$ and $x' \notin \{v_i, v_{i+1}\}$. If $k \geq 6$, then $(D_1 \setminus \{v_i, v_{i+1}\}, D_2 \cup \{v_i, v_{i+1}\})$ is a PD-pair of H, and hence of G, contradicting condition (1) of the choice of our partition $\mathcal{D}$. Hence, $k = 3$ and $i = 2$.

If v_1 is adjacent to a vertex in $D_1 \setminus \{v_2, v_3\}$, then $(D_1 \setminus \{v_2, v_3\}, D_2 \cup \{v_2, v_3\})$ is a PD-pair of H, contradicting condition (1) of the choice of our partition $\mathcal{D}$. Hence, $N(v_1) \setminus \{v_2, v_3\} \subseteq D_2$. Thus if $d_H(v_1) \geq 3$, then $((D_1 \setminus \{v_2\}) \cup \{v_1\}, (D_2 \setminus \{v_1\}) \cup \{v_2\})$ is a PD-pair in $H - v_1v_2$, contradicting the minimality of H. Therefore, $d_H(v_1) = 2$. But then $((D_1 \setminus \{v_2\}) \cup \{v_1\}, (D_2 \setminus \{v_1\}) \cup \{v_2\})$ is a PD-pair of H that satisfies conditions (1) and (2) but contradicts condition (3) of the choice of our partition $\mathcal{D}$. Hence, $d_H(v_i) = 2$ or $d_H(v_{i+1}) = 2$, as desired.

(j) Suppose $v_i \in D_1$ and $d_H(v_i) \geq 3$. By part (e), $i \equiv 0, 2 \pmod 3$. By part (g), the cycle C is an induced cycle in H. Let w be a neighbor of v_i that is not on the cycle C. By part (h), $w \in D_1$. By part (f), $v_iw \notin M$. Hence if v_iw is a cycle edge, it could be removed from H, contradicting the minimality of H. Therefore, v_iw is a bridge of H. □

We now introduce some additional notation. For any graph F, if $e = ab$ is a bridge in F, we let $F_a^{(e)}$ and $F_b^{(e)}$ denote the components of $F - e$ that contain a and b, respectively. If the edge e is clear from context, we simply denote $F_a^{(e)}$ by F_a and $F_b^{(e)}$ by F_b. We call a bridge of a graph with at least one of its ends contained in a cycle a *cycle-bridge*. If, in addition, the removal of the cycle-bridge produces a graph containing a P_3-component, then we call the bridge a P_3*-cycle-bridge*. For any graph F, let $\xi(F)$ denote the number of cycle-bridges in F that are not P_3-cycle-bridges. Further if $F' = F$ or if F' is a component of $F - f$, where f is a cycle-bridge that is not a P_3-cycle-bridge in F, we call F' a ξ*-subgraph* of F.

Among all ξ-subgraphs of H, let H' be chosen so that

(i) $\xi(H')$ is minimized.

(ii) Subject to (i), $|V(H')|$ is minimized.

We note that if $\xi(H) = 0$ then $H' = H$. If $H' \neq H$, let $e = ab$ be the cycle-bridge in H such that $H' = H_a^{(e)}$. We note further that any cycle-bridge in H' is also a cycle-bridge in H. The following claim proves some desirable properties about the ξ-subgraph H'.

Claim F $\xi(H') = 0$ *and* $|V(H')| \geq 3$.

Proof. If $\xi(H) = 0$ then $H' = H$ and both results follow readily. Thus we may assume $\xi(H) \geq 1$ and $H' \neq H$. Hence, $e = ab$ is the cycle-bridge in H such that $H' = H_a^{(e)}$. For the sake of contradiction, suppose $\xi(H') \geq 1$. Then, H' contains a cycle-bridge $f = cd$ that is not a P_3-cycle-bridge. We may assume, renaming the vertices c and d if necessary, that a and c are in different components of $H' - f$. But now $H_c^{(f)}$ is a ξ-subgraph of H with $\xi(H_c^{(f)}) < \xi(H')$, contradicting our choice of H'. Hence, $\xi(H') = 0$.

Suppose $|V(H')| = 1$. Then, a is the only vertex in H' and hence $d_H(a) = 1$. Therefore, $a \in D_2$ and $b \in D_1$. But, since e is a cycle-bridge, the vertex b lies on some cycle in H and so, by part (h) of the Cycle Lemma, $a \in D_1$, a contradiction. Hence, $|V(H')| \geq 2$. Suppose $|V(H')| = 2$. Then, $d_{H'}(a) = 1$. Let a' be the neighbor of a in H' and note that $d_H(a') = 1$ while $d_H(a) = 2$. Necessarily, $a' \in D_2$ and $\{a, b\} \subseteq D_1$ with a and b paired in D_1 (and so, $ab \in M$). But, since e is a cycle-bridge, the vertex b lies on some cycle in H and so, by part (j) of the Cycle Lemma, $ab \notin M$, a contradiction. Hence, $|V(H')| \geq 3$. □

Tree Lemma *If H' is a tree, then $(G, S) \in \mathcal{G}$ for some labeling S.*

Proof. Suppose H' is a tree. If $\xi(H) = 0$ then $H' = H$ and H is a tree. Then by Theorem 6.3, there exists a labeling $S = (S_A, S_B)$ such that $(H, S) \in \mathcal{T} \subset \mathcal{G}$. Thus, we can restore the graph G from H by repeated applications (including the possibility of none) of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, as desired. Thus we may assume $\xi(H) \geq 1$ and $H' \neq H$. Hence, $e = ab$ is the cycle-bridge in H such that $H' = H_a$.

By Claim F, $|V(H')| \geq 3$. Since $H_a = H' \neq P_3$, we have that $|V(H_a)| \geq 4$. Since b lies on a cycle, $|V(H_b)| \geq 3$. Suppose that $|V(H_b)| = 3$. Then, H_b is a 3-cycle and thus H_b is a ξ-subgraph of H with $\xi(H_b) = 0 \leq \xi(H')$ and with $|V(H_b)| < |V(H')|$, contradicting our choice of H'. Hence, $|V(H_b)| \geq 4$.

We now root the tree H_a at the vertex a and let u be a vertex at maximum distance from a. Necessarily, u is a leaf. If $d_{H_a}(a, u) = 1$, then H_a is a star with at least three leaves, contradicting the fact that H has no strong support vertices. Hence, $d_{H_a}(a, u) \geq 2$. Let v be the parent of u in H_a.

Suppose $d_{H_a}(a, u) = 2$. Then, a is the parent of v in H_a. Since H has no strong support vertex, u is the only child of v, and so $N_H(v) = \{a, u\}$. Hence, $\{a, v\} \subseteq D_1$ with $av \in M$. This implies that every child of a besides v is a leaf. Since $|V(H_a)| \geq 4$, $d_{H_a}(a) \geq 2$. If $d_{H_a}(a) > 2$, then a is a strong support vertex, a contradiction. Hence, $d_{H_a}(a) = 2$, and so $|V(H_a)| = 4$. Let a' be the child of a in H_a distinct from v. Thus, H_a is the path $a'avu$, and $\{a', u\} \subseteq D_2$. Since b lies on a cycle, by the Cycle Lemma, at least one neighbor of b on the cycle is in D_1 and so $(D_1 \cap V(H_b), D_2 \cap V(H_b))$ is a PD-pair in H_b, and so H_b is a DPDP-graph. Applying the inductive hypothesis to H_b, there exists a labeling $S' = (S'_A, S'_B)$ such that $(H_b, S') \in \mathcal{G}$. If $b \in S'_A$, we can restore the graph H from H_b by applying Operation $\mathcal{O}_2$. If $b \in S'_B$, we can restore the graph H from G' by first applying Operation $\mathcal{O}_3$ and then applying Operation $\mathcal{O}_1$. We can then restore the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling

$S = (S'_A \cup \{a, v\}, S'_B \cup \{a', u\})$. Thus we may assume that $d_{H_a}(a, u) \geq 3$.

Let w be the parent of v in the rooted tree H_a and let x be the parent of w in H_a (possibly, $x = a$). Proceeding now exactly as in the proof of Theorem 6.3, we have that $(H, S) \in \mathcal{G}$ for some labeling S. We can then restore G from H by repeated applications of Operation $\mathcal{O}_5$ and therefore $(G, S) \in \mathcal{G}$. □

By the Tree Lemma, we may assume that H' is not a tree.

Small Order Lemma *If $|V(H')| = 3$, then $(G, S) \in \mathcal{G}$ for some labeling S.*

Proof. Suppose $|V(H')| = 3$. Then $H' \neq H$ and hence $e = ab$ is the cycle-bridge in H such that $H' = H_a$. Then, since $H' \neq P_3$ we must have that $H' = C_3$. Let H' be given by the cycle aa_1a_2a. We note that each of a, a_1 and a_2 has degree 2 in H'. If $a \in D_1$, then by the Cycle Lemma, we may assume that $a_1 \in D_1$ and $a_2 \in D_2$. Furthermore, $aa_1 \in M$ and $b \in D_1$. But then $((D_1 \setminus \{a\}) \cup \{a_2\}, (D_2 \setminus \{a_2\}) \cup \{a\})$ is a PD-pair in H that satisfies conditions (1) and (2) but contradicts condition (3) of the choice of our partition $\mathcal{D}$. Hence, $a \in D_2$.

Suppose $b \in D_2$. Then, $((D_1 \setminus \{a_1\}) \cup \{a\}, (D_2 \setminus \{a\}) \cup \{a_1\})$ is a PD-pair in H and hence G. Furthermore, it is a PD-pair in $H - aa_1$ which is a spanning connected subgraph of G, contradicting condition (2) of our choice of H. Hence, $b \in D_1$.

Let b' be the partner of b in D_1, and let c be a neighbor of b' in D_2. Note that, since ab is a bridge, $\{b', c\} \subset V(H_b)$. Let $G' = H - \{a_1, a_2\}$ and note that $|V(G')| \geq |\{a, b, b', c\}| = 4$. Then, $(D_1 \setminus \{a_1, a_2\}, D_2\})$ is a PD-pair in G', and so G' is a DPDP-graph. Applying the inductive hypothesis to G', there exists a labeling $S' = (S'_A, S'_B)$ such that $(G', S') \in \mathcal{G}$. Since a is a leaf in G', Observation 6.2 implies that $a \in S'_B$. Hence we can restore the graph H from G' by applying Operation $\mathcal{O}_6$. We can then restore the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling

$(S'_A \cup \{a_1, a_2\}, S'_B)$. □

By the Small Order Lemma, we may assume that $|V(H')| \geq 4$. We are now able to prove the following desirable properties about the ξ-subgraph H'.

Claim G *The ξ-subgraph H' has the following properties.*

(a) *Every P_3-cycle-bridge in H' is a P_3-cycle-bridge in H.*

(b) *Every cycle-bridge f in H' belongs to $[D_1, D_2]$ with the end of f that lies on a cycle in D_2.*

(c) *At least one vertex of H' belongs to D_2.*

(d) *$D_2 \cap V(H')$ is an independent set in H'.*

(e) *If $x \in D_1 \cap V(H')$, then $d_{H'}(x) = 2$ (with one neighbor of x in D_1 and the other in D_2).*

Proof. (a) If it exists, let $f = cd$ be a P_3-cycle-bridge in H' where d lies on a cycle. Then, $H'^{(f)}_c$ is a P_3-component in $H' - f$. For the sake of contradiction, suppose f is a not a P_3-cycle-bridge in H. Then $H^{(f)}_c \neq H'^{(f)}_c$ and, thus, $H' \neq H$. Recall that $e = ab$ is the cycle-bridge in H such that $H' = H^{(e)}_a$. Necessarily, $a \in V(H'^{(f)}_c)$. But then $H^{(f)}_d$ is a ξ-subgraph of H with $\xi(H^{(f)}_d) = \xi(H')$ but $|V(H^{(f)}_d)| < |V(H')|$, contradicting our choice of H'. This establishes part (a).

(b) If it exists, let $f = cd$ be a cycle-bridge in H' where d lies on a cycle. Since $\xi(H') = 0$, f is a P_3-cycle-bridge in H' and thus, by part (a), in H. Therefore, $H^{(f)}_c$ is a P_3-component of $H - f$ and since H contains no strong support vertex, the vertex c is a leaf in this P_3-component. Let $H^{(f)}_c$ be given by the path cc_1c_2. We note that $d_H(c_2) = 1$ and $d_H(c) = d_H(c_1) = 2$. Hence, $\{c_2, d\} \subseteq D_2$ and $\{c, c_1\} \subseteq D_1$. In particular, $f \in [D_1, D_2]$ and $d \in D_2$. This establishes part (b).

(c) If $|D_2 \cap V(H')| = 0$, then $H' \neq H$ and every vertex in H' is adjacent to some vertex in $H - V(H')$, contradicting the fact $e = ab$ is a cycle-bridge in H such that $H' = H_a^{(e)}$. Hence, $|D_2 \cap V(H')| \geq 1$.

(d) For the sake of contradiction, suppose that ww' is an edge of H', where $\{w, w'\} \subseteq D_2$. By the Cycle Lemma, ww' is a bridge and therefore, by part (b), neither w nor w' lies on a cycle in H'. Among all vertices lying on some cycle in H', choose u so that the distance $d_{H'}(u, w)$ is minimum. Let v be the vertex adjacent to u on the unique shortest u–w path (possibly, $v \in \{w, w'\}$). By the choice of u, we have that uv is a bridge. We note that uv is a cycle-bridge. Since $\xi(H') = 0$, uv is a P_3-cycle-bridge in H' and thus, by part (a), in H. Therefore, $H_v^{(uv)}$ is a P_3-component of $H - uv$ and since H contains no strong support vertex, v is a leaf in this P_3-component. We note that $\{w, w'\} \subset V(H_v^{(uv)})$. Let $H_v^{(uv)}$ be given by the path vv_1v_2. We note that $d_H(v_2) = 1$ and $d_H(v) = d_H(v_1) = 2$. Hence, $v_2 \in D_2$ and $\{v, v_1\} \subseteq D_1$. Thus, $H_v^{(uv)}$ has exactly one vertex in D_2, contradicting the fact that $\{w, w'\} \subseteq D_2 \cap V(H_v^{(uv)})$.

(e) For the sake of contradiction, suppose that $x \in D_1 \cap V(H')$ and $d_{H'}(x) \geq 3$. Suppose that C is a cycle in H' containing x. By the Cycle Lemma, one neighbor of x on C is paired with x in D_1 and the other neighbor of x on C belongs to D_2. Since $d_{H'}(x) \geq 3$, there is a cycle-bridge incident with x. By part (b), the vertex x, which lies on a cycle, belongs to D_2, a contradiction. Hence, every edge incident with x in H' is a bridge in H'. Let x' be the partner of x in D_1, and let y be a neighbor of x in D_2. Let z be a neighbor of x distinct from x' and y. Among all vertices that belong to a cycle in H', choose u' so that the distance $d_{H'}(u', x)$ is minimum. Let v' be the vertex adjacent to u' on the unique shortest u'–x path. By the choice of u', we note that $u'v'$ is a cycle-bridge. Since $\xi(H') = 0$, $u'v'$ is a P_3-cycle-bridge in H' and thus, by part (a), in H. Therefore, $H_{v'}^{(u'v')}$ is a P_3-component of $H - u'v'$. Since H contains no strong support vertices, v' is a leaf in this P_3-component, and so $d_{H'}(v') \leq d_H(v') = 2$. In particular, we note that $v' \neq x$, and so $\{x, x', y, z\} \subseteq V(H_{v'}^{(u'v')})$. Thus the component $H_{v'}^{(u'v')}$ contains at least four vertices, a contradiction. □

We now proceed by labeling some (or possibly all) of the vertices in H' as follows. If $H = H'$, then select the vertex a to be any vertex in D_2. If $H' \neq H$, then $e = ab$ is the cycle-bridge in H such that $H' = H_a$. Let $k = |D_2 \cap V(H')|$ and label the vertices in $D_2 \cap V(H')$ by $w_1, w_2, \ldots, w_k$ so that $d_{H'}(a, w_i) \leq d_{H'}(a, w_j)$ for $1 \leq i < j \leq k$ (possibly, $a = w_1$).

If $k \geq 2$ then for each $i \in \{2, 3, \ldots, k\}$, let v_i be the vertex preceding w_i on a shortest a–w_i path in H' and let u_i be the vertex preceding v_i on the same a–w_i path. We note that since $D_2 \cap V(H')$ is an independent set in H', we must have that $v_i \in D_1$. By Claim G(e), each vertex in $D_1 \cap V(H')$ has degree 2 in H' and has one neighbor in D_1. Hence, $v_i \neq v_j$ for $2 \leq i < j \leq k$. Further for each $i \in \{2, 3, \ldots, k\}$, $u_i \in D_1$ and u_i has exactly one other neighbor in H' besides v_i, necessarily w_j for some $j < i$. If $a \in D_1$ and a lies on a cycle of length 3, then assign it the label u_1 and assign its neighbor in D_1 that belongs to this cycle the label v_1. We note, by our choice of labels, no vertex in $V(H')$ is assigned more than one label from the set of labels $\bigcup_{1 \leq i \leq k}\{u_i, v_i, w_i\}$. We note that either $a = w_1$ or $a = u_1$ or $a = u_2$.

We call a vertex in H' that is not assigned a label from $\bigcup_{1 \leq i \leq k}\{u_i, v_i, w_i\}$ an unlabeled vertex, and we let U be the set of unlabeled vertices in H' (possibly $|U| = 0$). We note that by Claim G(e), every vertex in U belongs to D_1 and is adjacent (in H') to exactly one other unlabeled vertex from D_1 and exactly one labeled vertex from D_2. Let H_k be the graph $H - U$. If $k \geq 2$ then for $i = 1, 2, \ldots, k - 1$, let H_i be the graph $H_{i+1} - \{u_{i+1}, v_{i+1}, w_{i+1}\}$.

Claim H *If $H = H'$, then $(G, S) \in \mathcal{G}$ for some labeling S.*

Proof. Suppose $H = H'$. Then, $a \in D_2$ and $a = w_1$. If $k = 1$, then $D_1 = V(H) \setminus \{a\}$. In this case, since $n \geq 5$, we note by Claim G(e) that H can be constructed from $t \geq 2$ disjoint 3-cycles by identifying a set of t vertices, one from each cycle, into one vertex called a. Thus, $H - a = tK_2$ with the vertices in each copy of K_2 partners in D_1. Let

$axya$ be a 3-cycle containing a. Then, $(\{a, x\}, D_1 \setminus \{b\})$ is a PD-pair of H, and hence of G, contradicting condition (1) of the choice of our partition $\mathcal{D}$. Hence, $k \geq 2$. Let $G' = H_2$ and note that $G' = P_4$. Let $S' = (S'_A, S'_B)$, where $S'_A = \{u_2, v_2\}$ and $S'_B = \{w_1, w_2\}$. Then, $(G', S') \in \mathcal{G}$. If $k \geq 3$, then for each $i = 3, \ldots, k$, we can restore the graph H_i from H_{i-1} by applying Operation $\mathcal{O}_3$ and noting that $\text{sta}(u_i) = \text{sta}(v_i) = A$ and $\text{sta}(w_i) = B$. We can then restore the graph H from H_k by repeated applications of Operations $\mathcal{O}_6$ and $\mathcal{O}_7$ and the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A \cup U \cup (\bigcup_{3 \leq i \leq k} \{u_i, v_i\}), S'_B \cup (\bigcup_{3 \leq i \leq k} \{w_i\}))$. $\square$

By Claim H, we may assume that $H \neq H'$ and hence $e = ab$ is the cycle-bridge in H such that $H' = H_a$. We now present two final claims that consider the cases when $a \in D_2$ and $a \in D_1$, respectively.

Claim I *If $a \in D_2$ then $(G, S) \in \mathcal{G}$ for some labeling S.*

Proof. Suppose that $a \in D_2$. Then, $a = w_1$. If $b \in D_1$, let b' be the partner of b in D_1, and let c be a neighbor of b' in D_2. Note that, since ab is a bridge, $\{b', c\} \subset V(H_b)$. Let $G' = H_1$ and note that $d_{G'}(a) = 1$. Furthermore, since $\{a, b, b', c\} \subseteq V(G')$, $|V(G')| \geq 4$. Then, $(D_1 \cap V(H_b), (D_2 \cap V(H_b)) \cup \{a\})$ is a PD-pair of G', and so G' is a DPDP-graph. We note that $|V(G')| < |V(H)|$ and thus, applying the inductive hypothesis to G', there exists a labeling $S' = (S'_A, S'_B)$ such that $(G', S') \in \mathcal{G}$. By Observation 6.2(d), $a \in S'_B$. For each $i = 2, 3, \ldots, k$, we can restore the graph H_i from H_{i-1} by applying Operation $\mathcal{O}_3$ and noting that $\text{sta}(u_i) = \text{sta}(v_i) = A$ and $\text{sta}(w_i) = B$. We can then restore the graph H from H_k by repeated applications of Operations $\mathcal{O}_6$ and $\mathcal{O}_7$ and the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A \cup U \cup (\bigcup_{2 \leq i \leq k} \{u_i, v_i\}), S'_B \cup (\bigcup_{2 \leq i \leq k} \{w_i\}))$. Hence we may assume that $b \in D_2$, for otherwise we have the desired result.

Let b_1 be a neighbor of b in D_1, and let b_2 be the partner of b_1 in D_1. Let b_3 be a neighbor of b_2 in D_2 (possibly, $b = b_3$). Note that, since ab is a bridge, $\{b_1, b_2, b_3\} \subseteq V(H_b)$. Suppose $|V(H_b)| = 3$. Then, $b = b_3$ and H_b is the cycle bb_1b_2b. We note that in this case, $d_H(b_1) = d_H(b_2) = 2$. But then, $((D_1 \setminus \{b_1\}) \cup \{b\}, (D_2 \setminus \{b\}) \cup \{b_1\})$ is a PD-pair in H and hence G. Furthermore, it is a PD-pair in $H - bb_1$ which is a spanning connected subgraph of G, contradicting condition (2) of our choice of H. Hence, we may assume that $|V(H_b)| \geq 4$.

Suppose $k = 1$. Then, $D_1 = V(H') \setminus \{a\}$. Then, since $|V(H')| \geq 4$, we note by Claim G(e) that H' can be constructed from $t \geq 2$ disjoint 3-cycles by identifying a set of t vertices, one from each cycle, into one vertex called a. Let $axya$ be a 3-cycle containing a. Then, $((D_1 \setminus V(H')) \cup \{a, x\}, (D_2 \cup V(H')) \setminus \{a, x\})$ is a PD-pair of H, and hence of G, contradicting condition (1) of the choice of our partition $\mathcal{D}$. Hence, $k \geq 2$.

Now, $(D_1 \cap V(H_b), D_2 \cap V(H_b))$ is a PD-pair in H_b, and so H_b is a DPDP-graph. Applying the inductive hypothesis to H_b, there exists a labeling $S' = (S'_A, S'_B)$ such that $(H_b, S') \in \mathcal{G}$. If $b \in S'_B$, we can restore the graph H_2 from H_b by applying Operation $\mathcal{O}_4$. If $b \in S'_A$, we can restore the graph H_2 from H_b by first applying Operation $\mathcal{O}_1$ and then applying Operation $\mathcal{O}_3$. In both cases, $\text{sta}(u_2) = \text{sta}(v_2) = A$ and $\text{sta}(w_1) = \text{sta}(w_2) = B$. If $k > 2$, then for each $i = 3, \dots, k$, we can restore the graph H_i from H_{i-1} by applying Operation $\mathcal{O}_3$ and noting that $\text{sta}(u_i) = \text{sta}(v_i) = A$ and $\text{sta}(w_i) = B$. We can then restore the graph H from H_k by repeated applications of Operations $\mathcal{O}_6$ and $\mathcal{O}_7$ and finally restore the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A \cup U \cup (\bigcup_{2 \leq i \leq k}\{u_i, v_i\}), S'_B \cup (\bigcup_{1 \leq i \leq k}\{w_i\}))$. This completes the proof of Claim I. □

Claim J *If $a \in D_1$, then $(G, S) \in \mathcal{G}$ for some labeling S.*

Proof. Suppose that $a \in D_1$. If $a = u_2$, then $k \geq 2$ and we note then that the component of $H_2 - ab$ containing a consists of the path $w_1u_2v_2w_2$, and so both w_1 and w_2 have degree 1 in H_2. Since H' is not a tree and since, by Claim G(e), $d_{H'}(u_2) = d_{H'}(v_2) = 2$, we must have $d_{H'}(w_1) > 1$ or $d_{H'}(w_2) > 1$. By the Cycle Lemma, $w_1w_2 \notin E(H')$ and hence $|V(H')| > 4$. Consequently, $|V(H_2)| < |V(H)|$. Now, $((D_1 \cap V(H_b)) \cup \{u_2, v_2\}, (D_2 \cap V(H_b)) \cup \{w_1, w_2\})$ is a PD-pair in H_2, and so H_2 is a DPDP-graph. Applying the inductive hypothesis to H_2, there exists a labeling $S' = (S'_A, S'_B)$ such that $(H_2, S') \in \mathcal{G}$. By Observation 6.2, $\{w_1, w_2\} \subseteq S'_B$ and $\{u_2, v_2\} \subseteq S'_A$. If $k > 2$, then for each $i = 3, \ldots, k$, we can restore the graph H_i from H_{i-1} by applying Operation $\mathcal{O}_3$ and noting that $\text{sta}(u_i) = \text{sta}(v_i) = A$ and $\text{sta}(w_i) = B$. We can then restore the graph H from H_k by repeated applications of Operations $\mathcal{O}_6$ and $\mathcal{O}_7$ and finally restore the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A \cup U \cup (\bigcup_{3 \le i \le k}\{u_i, v_i\}), S'_B \cup (\bigcup_{3 \le i \le k}\{w_i\}))$. Hence, we may assume that $a = u_1$.

Since $a = u_1$, we have that $w_1v_1u_1w_1$ is a cycle in H' containing the vertex a. By Claim G(e), $d_{H'}(a) = 2$ and $d_{H'}(v_1) = 2$. Thus, a and v_1 are partners in D_1. Further, $d_H(a) = 3$ and $N_H(a) = \{b, v_1, w_1\}$. By part (h) of the Cycle Lemma, $b \in D_1$. Let b_1 be the partner of b in D_1. Let b_2 be a neighbor of b in D_2 and let b_3 be a neighbor of b_1 in D_2.

Suppose $b_2 = b_3$. By condition (2) of the choice of our partition $\mathcal{D}$, we note that (D_1, D_2) is not a PD-pair in $H - b_1b_2$, and so b_2 is the only vertex in D_2 adjacent to b_1. But then $(D_1 \setminus \{b_1, v_1\}, D_2 \cup \{b_1, v_1\})$ is a PD-pair in H, contradicting condition (1) of the choice of our partition $\mathcal{D}$. Hence, $b_2 \neq b_3$. Furthermore, since ab is a bridge, $\{b, b_1, b_2, b_3\} \subseteq V(H_b)$, and so $|V(H_b)| \ge 4$.

Now, $(D_1 \cap V(H_b), D_2 \cap V(H_b))$ is a PD-pair in H_b, and so H_b is a DPDP-graph. Applying the inductive hypothesis to H_b, there exists a labeling $S' = (S'_A, S'_B)$ such that $(H_b, S') \in \mathcal{G}$. If $b \in S'_B$, we can restore the graph H_1 from H_b by first applying Operation $\mathcal{O}_3$ and then applying Operation $\mathcal{O}_5$. If $b \in S'_A$, we can restore the graph H_1 from H_b by applying Operation $\mathcal{O}_8$. In both cases, $\text{sta}(u_1) = \text{sta}(v_1) = A$ and $\text{sta}(w_1) = B$. If $k \ge 2$, then for each $i = 2, \ldots, k$, we can restore the graph H_i from H_{i-1} by applying Operation $\mathcal{O}_3$ and noting that $\text{sta}(u_i) = \text{sta}(v_i) = A$ and $\text{sta}(w_i) = B$. We can then restore the graph H from H_k by repeated applications of Operations $\mathcal{O}_6$ and $\mathcal{O}_7$ and finally restore the graph G from H by repeated applications of Operation $\mathcal{O}_5$. Hence, $(G, S) \in \mathcal{G}$, where S is the labeling $(S'_A \cup U \cup (\bigcup_{1\leq i\leq k}\{u_i, v_i\}), S'_B \cup (\bigcup_{1\leq i\leq k}\{w_i\}))$. This completes the proof of Claim J. □

We have thus demonstrated that $(G, S) \in \mathcal{G}$ for some labeling S. This completes the necessity, and the proof of Theorem 6.4. □

Chapter 7

Total Restrained Domination

In this chapter, we continue the study of total restrained domination in graphs, a concept introduced by Telle and Proskurowksi [95] as a vertex partitioning problem. Recent papers on total restrained domination in graphs can be found in [8, 40, 41, 62, 72, 79, 82, 83, 101]. We improve on a previously published bound in the case of cubic graphs.

Partitioning the vertices of a graph into sets holding various domination properties can quickly provide simple bounds on the corresponding domination parameters. As an example, the now familiar observation made by Ore [80] that every graph of minimum degree at least one contains two disjoint dominating sets yields an upper bound of half the order on the domination number. We observe that a similar bound would hold for the total domination number if it were not for Zelinka's observation regarding the less frequent existence of a partition of the vertices into two total dominating sets. In fact, if such a partition always existed, the bound would also hold for the total restrained domination number, since both sets would be not only total dominating sets, but also total restrained dominating sets.

Even in the restricted case of cubic graphs, such a partition is not guaranteed. However, in the case when no such partition exists it is, loosely put, 'a very near miss'. It is this 'almost' partition that lies at the heart of the results in this chapter.

Before examining the total restrained domination number, we note that an upper bound on the total domination number of a cubic graph follows directly from a more general result due to several authors, including Archdeacon et al. [2], Chvátal and McDiarmid [15], Thomassé and Yeo [96], and Tuza [97], that every graph with minimum degree at least three has total domination number at most one-half its order.

Theorem 7.1 ([2, 15, 96, 97]) *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq n/2$.*

7.1 Improving a Published Bound

Using intricate and clever counting arguments, Jiang, Kang and Shan [72] established the following upper bound on the total restrained domination number of a cubic graph.

Theorem 7.2 ([72]) *If G is a connected cubic graph of order n, then $\gamma_{\rm tr}(G) \leq 13n/19$.*

Our aim is to improve the upper bound given in Theorem 7.2. We shall prove:

Theorem 7.3 *If G is a connected cubic graph of order n, then $\gamma_{\rm tr}(G) \leq (n+4)/2$.*

We show that our new improved bound is essentially best possible by providing two infinite families of connected cubic graphs G of order n with $\gamma_{\rm tr}(G) = n/2$.

7.1.1 Preliminary Results

As a special case of König's [73] result that every regular bipartite graph has a perfect matching, we have the following result.

Observation 7.4 ([73]) *Every cubic bipartite graph contains a perfect matching.*

Lemma 7.5 *If $G = (V, E)$ is a connected non-bipartite graph and $\{u, v\} \subset V$, then there exists a u-v walk in G of even length.*

Proof. Let $G = (V, E)$ be a connected non-bipartite graph and let $\{u, v\} \subset V$. Let C be an odd cycle in G and let $w \in V(C)$. Let P_u be a shortest u-w path and P_v a shortest v-w path in G. Let W_1 be the u-v walk which traverses the path P_u from u to w and then the path P_v from w to v. Let W_2 be the u-v walk which traverses the path P_u from u to w, then the cycle C, and finally the path P_v from w to v. We note that W_1 is of even length if, and only if, W_2 is of odd length. In either case, the desired result follows. □

Lemma 7.6 *Let $G = (V, E)$ be a cubic graph of order n and $v \in V$. If there exists a TRDS $S \subset V$ such that $V \setminus S$ dominates $V \setminus \{v\}$, then $\gamma_{\mathrm{tr}}(G) \leq (n + 2)/2$.*

Proof. Let $G = (V, E)$ be a cubic graph of order n with $v \in V$ and suppose that there exists a TRDS $S \subset V$ such that $V \setminus S$ dominates $V \setminus \{v\}$. We may assume that $|S| > (n + 2)/2$ for otherwise, the desired result follows. Hence, $|V \setminus S| < (n - 2)/2$. If $V \setminus S$ dominates V then $V \setminus S$ is a TRDS and the desired result follows. We may therefore assume that $N(v) \subseteq S$. Since S is a TRDS, the subgraph $G[V \setminus S]$ contains no isolated vertices, and so we must have that $v \in S$. Let $u \in N(v)$ and let $N(u) = \{v, w, x\}$. Since $V \setminus S$ dominates u, we may assume that $w \in V \setminus S$. If $x \in V \setminus S$, then $S' = (V \setminus S) \cup \{u\}$ is a TRDS with $|S'| < n/2$ and the desired result follows. Hence, we may assume that $x \in S$. If $d_{G[S]}(x) > 1$ then, again, $S' = (V \setminus S) \cup \{u\}$ is a total restrained dominating set with $|S'| < n/2$ and the desired result follows. Hence we may assume that $d_{G[S]}(x) = 1$. But now $S'' = (V \setminus S) \cup \{u, x\}$ is a TRDS with $|S''| < (n + 2)/2$, as desired. □

The following lemma shows the existence of a useful partition of one of the partite sets in a cubic bipartite graph.

Lemma 7.7 *Let G be a connected cubic bipartite graph of order n with partite sets X and Y. For any specified vertex $y \in Y$ there exists a partition of X into X_1 and X_2 such that X_1 dominates Y and X_2 dominates $Y \setminus \{y\}$.*

Proof. Let $G = (V, E)$ be a connected cubic bipartite graph of order n with partite sets X and Y. Let y be an arbitrary vertex in Y. By Observation 7.4, G contains a perfect matching M. Let $k = n/2$ and note that $|M| = |X| = |Y| = k$. Let $H = G - M$. Since every vertex in G is incident with exactly one edge in M, we have that H is a 2-regular bipartite graph. Hence, every component in H is a cycle of even length. Let c be the number of components in H (possibly $c = 1$). Let x_0 be the vertex that is M-matched to y and let H_0 be the graph consisting of only the vertex x_0 and zero edges. Let $S_0 = S'_0 = \emptyset$ and let $i = 1$. We perform the following iterative construction while $i \leq c$.

Let y_0^i be the vertex M-matched to x_{i-1}. We note that $y_0^i \notin V(H_{i-1})$. Let C^i : $y_0^i x_1^i y_1^i x_2^i y_2^i \ldots x_{k_i}^i y_{k_i}^i = y_0^i$ be the cycle component of H containing y_0^i where $k_i = |V(C^i)|/2$. If $x_{i-1} \notin S_{i-1}$ then let $S_i = S_{i-1} \cup \{x_j^i \mid j \equiv 1 \pmod 2\}$ and let $S'_i = S'_{i-1} \cup \{x_j^i \mid j \equiv 0 \pmod 2\}$. Otherwise if $x_{i-1} \in S_{i-1}$, let $S_i = S_{i-1} \cup \{x_j^i \mid j \equiv 0 \pmod 2\}$ and let $S'_i = S'_{i-1} \cup \{x_j^i \mid j \equiv 1 \pmod 2\}$. Let $H_i = G[\bigcup_{j=1}^{i} V(C^i)]$ and note that $S_i \cup S'_i = V(H_i) \cap X$, $S_i \cap S'_i = \emptyset$, S_i dominates $V(H_i) \cap Y$, and S'_i dominates $(V(H_i) \cap Y) \setminus \{y_0^1\}$. We note further, that for all $v \in V(H_i)$ we must have $2 \leq d_{H_i}(v) \leq 3$. If for all $v \in V(H_i)$ we have that $d_{H_i}(v) = 3$, then since G is connected, $H_i = G$ and $i = c$. In this case, our iterative construction is complete. Hence, we may assume that $i < c$ and that there exists a vertex $x_i \in V(H_i)$ such that $d_{H_i}(x_i) = 2$. Additionally, since H_i is a bipartite graph with partite sets of equal size, we may choose such an x_i to be from X. Necessarily, $x_i \in S_i \cup S'_i$ and the vertex in Y that is M-matched to x_i is not in $V(H_i)$ and we repeat the iterative step after incrementing i by 1.

By construction, $H_c = G$. Furthermore, $S_c \cup S'_c = X$, $S_c \cap S'_c = \emptyset$, S_c dominates Y, and S'_c dominates $Y \setminus \{y_0^1\}$. But $y = y_0^1$ and so, letting $X_1 = S_c$ and $X_2 = S'_c$, the desired result follows. □

Our final preliminary result uses Lemma 7.6 and Lemma 7.7 to establish a bound on the total restrained domination number in the case of connected cubic bipartite graphs.

Lemma 7.8 *If G is a connected cubic bipartite graph of order n, then $\gamma_{\rm tr}(G) \le (n+2)/2$.*

Proof. Let G be a cubic bipartite graph of order n with partite sets X and Y. We note that $|X| = |Y| = n/2$. Let $x \in X$ and $y \in Y \cap N(x)$. By Lemma 7.7, there exists a partition of X into X_1 and X_2 such that X_1 dominates Y and X_2 dominates $Y \setminus \{y\}$. Similarly, there exists a partition of Y into Y_1 and Y_2 such that Y_1 dominates X and Y_2 dominates $X \setminus \{x\}$.

If X_2 dominates Y and Y_2 dominates X, then $X_1 \cup Y_1$ and $X_2 \cup Y_2$ are disjoint TDSs in G and hence also, TRDSs in G. Letting S be the smaller of $X_1 \cup Y_1$ and $X_2 \cup Y_2$ (or the former, in the case of equality), we have that $|S| \le n/2$ and the desired result follows. Hence we may assume, without loss of generality, that X_2 does not dominate Y. We note that $x \in N(y) \subseteq X_1$.

If Y_2 dominates X, then, switching the labels Y_1 and Y_2 if necessary, we may assume that $y \in Y_1$. We now let $S = X_1 \cup Y_1$ and note that S is a TRDS in G and $V \setminus S = X_2 \cup Y_2$ dominates $V \setminus \{y\}$. Then, by Lemma 7.6, $\gamma_{\rm tr}(G) \le (n+2)/2$, as desired. Hence we may assume that Y_2 does not dominate X. We note that $y \in N(x) \subseteq Y_1$.

If $|X_1 \cup Y_2| \le |X_2 \cup Y_1|$, we let $S = X_1 \cup Y_2 \cup \{y\}$. Conversely, if $|X_2 \cup Y_1| < |X_1 \cup Y_2|$, we let $S = X_2 \cup Y_1 \cup \{x\}$. In both cases, we note that S is a TRDS and $|S| \le (n+2)/2$, as desired. $\Box$

7.1.2 Proof of Theorem 7.3

We are now in a position to prove our main result, namely Theorem 7.3. Recall the statement of the Theorem 7.3.

Theorem 7.3. *If G is a connected cubic graph of order n, then $\gamma_{\rm tr}(G) \le (n+4)/2$.*

Proof. Let $G = (V, E)$ be a connected cubic graph of order n. If G is a bipartite graph, then the result follows from Lemma 7.8. Thus we may assume that G is not a bipartite graph. Let $V = \{v_1, v_2, \ldots, v_n\}$. Let $G' = (V', E')$ be the bipartite graph obtained from G as follows. Let G' have partite sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ and let $E' = \{x_i y_j \mid v_i v_j \in E\}$. We note that G' is a cubic bipartite graph on $2n$ vertices and that $x_i y_i \notin E'$ for all $i \in \{1, 2, \ldots, n\}$.

We show first that G' is connected. It suffices to show that for all $1 \le i < j \le n$, there exists an x_i-x_j walk in G'. Since G is non-bipartite, by Lemma 7.5, there exists a v_i-v_j walk of even length in G. Let $v_i = v_{\ell_0} v_{\ell_1} v_{\ell_2} \ldots v_{\ell_{2k}} = v_j$ be such a v_i-v_j walk. But now $x_i = x_{\ell_0} y_{\ell_1} x_{\ell_2} y_{\ell_3} \ldots x_{\ell_{2k}} = x_j$ is an x_i-x_j walk in G'. Hence, G' is connected. Therefore, by Lemma 7.7, there exists a partition of X into X_1 and X_2 such that X_1 dominates Y and X_2 dominates $Y \setminus \{y_1\}$.

We consider the set $S = \{v_i \in V \mid x_i \in X_1\}$ and show that S is a TDS in G. Let v_j be an arbitrary vertex in V. Since X_1 dominates Y, there exists a vertex $x_i \in X_1$ such that $i \neq j$ and $x_i y_j \in E'$. By our construction of G', $v_i v_j \in E$. By definition of the set S, $v_i \in S$. Hence, every vertex in V is adjacent to some vertex in S, and so S is a TDS in G as claimed. If X_2 dominates Y, then by a similar argument, $V \setminus S$ is a TDS in G. But then each of S and $V \setminus S$ is a TRDS, and so $\gamma_{\rm tr}(G) \le \min(|S|, |V \setminus S|) \le n/2$, and we are done. We may therefore assume that X_2 does not dominate Y, and so $N(y_1) \subseteq X_1$. Hence, by our construction of G' and definition of S, $N(v_1) \subseteq S$.

We show next that every vertex in V is adjacent to a vertex in $V \setminus S$, with the exception of v_1. Let v_j be an arbitrary vertex in V such that $j \neq 1$. Since X_2 dominates $Y \setminus \{y_1\}$, there exists a vertex $x_i \in X_2$ such that $i \neq j$ and $x_i y_j \in E'$. By our construction of G', $v_i v_j \in E$. By definition of the set S, we have that $v_i \in V \setminus S$, and so v_j is adjacent to a vertex in $V \setminus S$, as desired. Consequently, $G[V \setminus S]$ contains no isolated vertices, except possibly v_1. We note, therefore, that $S \cup \{v_1\}$ is a TRDS in G (possibly, $v_1 \in S$). If $|S| \leq n/2$ then $\gamma_{\rm tr}(G) \leq |S| + 1 \leq (n+2)/2$ and the desired result follows. We may therefore assume that $|S| > n/2$ or, equivalently, $|V \setminus S| < n/2$. But since G is cubic, n is even and hence $|V \setminus S| \leq (n-2)/2$.

Let $u \in N_G(v_1)$ and let $N_G(u) = \{v_1, w_1, w_2\}$. Note that $u \in S$. If $v_1 \in S$, then $G[V \setminus S]$ contains no isolated vertices, and so S is a TRDS in G such that $V \setminus S$ dominates $V \setminus \{v_1\}$. The desired result now follows from Lemma 7.6. Hence, we may assume that $v_1 \in V \setminus S$.

Since S totally dominates V, we may assume that $w_1 \in S$ in order to totally dominate u. Suppose that $w_2 \in V \setminus S$. If $d_{G[S]}(w_1) > 1$, then $(V \setminus S) \cup \{u\}$ is a TRDS and $\gamma_{\rm tr}(G) \leq |V \setminus S| + 1 \leq n/2$, as desired. We may therefore assume $d_{G[S]}(w_1) = 1$. But now $(V \setminus S) \cup \{u, w_1\}$ is a TRDS and $\gamma_{\rm tr}(G) \leq |V \setminus S| + 2 \leq (n+2)/2$, as desired. Hence, if $w_2 \in V \setminus S$, the desired result follows and so we may assume that $w_2 \in S$.

If $d_{G[S]}(w_1) > 1$ and $d_{G[S]}(w_2) > 1$, then $(V \setminus S) \cup \{u\}$ is a TRDS and the desired result follows. We may therefore assume, without loss of generality, that $d_{G[S]}(w_1) = 1$. If $d_{G[S]}(w_2) > 1$ then $(V \setminus S) \cup \{u, w_1\}$ is a TRDS and, again, the desired result follows. Hence, we may assume that $d_{G[S]}(w_2) = 1$. But now $(V \setminus S) \cup \{u, w_1, w_2\}$ is a TRDS and so $\gamma_{\rm tr}(G) \leq |V \setminus S| + 3 \leq (n+4)/2$. This concludes the proof of Theorem 7.3. □

7.1.3 Examples Showing the Tightness of our Result

Let G be a connected cubic graph of order n. In this chapter, we improved the upper bound on $\gamma_{\rm tr}(G)$ established by Jiang, Kang and Shan [72] in Theorem 7.2 from $13n/19$ to $(n+4)/2$. We will now show that our result is essentially best possible.

The generalized Petersen graph G_{16} of order $n = 16$ shown in Figure 7.1 achieves equality in Theorem 7.1.

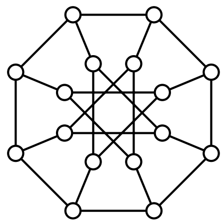


Figure 7.1: The generalized Petersen graph G_{16} of order 16.

Two infinite families $\mathcal{G}$ and $\mathcal{H}$ of connected cubic graphs (described below) with total domination number one-half their orders are constructed in [31]. For $k \geq 2$ consider two copies of the path P_{2k} with respective vertex sequences $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ and $c_1, d_1, c_2, d_2, \ldots, c_k, d_k$. For each $i \in \{1, 2, \ldots, k\}$, join a_i to d_i and b_i to c_i. To complete the construction of graphs in $\mathcal{G}$ ($\mathcal{H}$, respectively), join a_1 to c_1 and b_k to d_k (a_1 to b_k and c_1 to d_k, respectively). Two graphs G and H in the families $\mathcal{G}$ and $\mathcal{H}$ are illustrated in Figure 7.2.

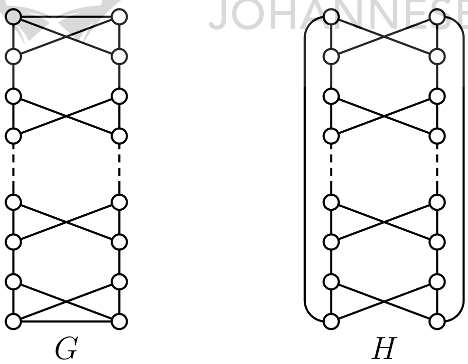


Figure 7.2: Cubic graphs $G \in \mathcal{G}$ and $H \in \mathcal{H}$ of order n with $\gamma_t(G) = \gamma_t(H) = n/2$.

We remark that in [69] it is shown that there are no other extremal connected graphs achieving the bound in Theorem 7.1; that is, if G is a connected graph of order n with $\delta(G) \geq 3$ and $\gamma_t(G) = n/2$, then $G \in \mathcal{G} \cup \mathcal{H}$ or $G = G_{16}$.

If $G \in \mathcal{G} \cup \mathcal{H}$ has order $n = 4k$, then using the notation described earlier to construct the families $\mathcal{G}$ and $\mathcal{H}$, we note that the set $S = \{a_1, a_2, \ldots, a_k\} \cup \{b_1, b_2, \ldots, b_k\}$ is a

TRDS in G, and so $\gamma_{\rm tr}(G) \le n/2$. Further if $G = G_{16}$, then G has order $n = 16$ and the vertices on the outer 8-cycle of G_{16} as drawn in Figure 7.1 form a TRDS of G, and so $\gamma_{\rm tr}(G) \le n/2$. Hence if $G \in \mathcal{G} \cup \mathcal{H} \cup \{G_{16}\}$ has order n, then $\gamma_{\rm tr}(G) \le n/2$. As remarked earlier, if $G \in \mathcal{G} \cup \mathcal{H} \cup \{G_{16}\}$ has order n, then $\gamma_t(G) = n/2$. Since every TRDS is a TDS, we note that $\gamma_t(G) \le \gamma_{\rm tr}(G)$ for every graph G. Consequently, we have the following observation.

Observation 7.9 *If $G \in \mathcal{G} \cup \mathcal{H}$ or G is the generalized Petersen graph G_{16} shown in Figure 7.1 and G has order n, then $\gamma_{\rm tr}(G) = n/2$.*

Chapter 8

Independent Domination

Recall that an independent dominating set in a graph is a set that is both dominating and independent. Equivalently, an independent dominating set is a maximal independent set. The theory of independent domination was formalized by Berge [4] and Ore [80] in 1962. The independent domination number was introduced by Cockayne and Hedetniemi in [18]. Independent dominating sets are now extensively studied in the literature; see, for example, [1, 44, 85, 94] and the two books by Haynes, Hedetniemi, and Slater [45, 46]. Independent dominating sets in regular graphs, and in cubic graphs in particular, are also well studied; see for example [42, 43, 74, 78] and elsewhere. In this chapter, we consider the ratio of the independent domination number to the domination number in a cubic graph.

In 1991, Barefoot, Harary, and Jones [3] gave a class of 2-connected cubic graphs for which the difference between i and γ is unbounded and conjectured that for any 3-connected cubic graph the difference is at most 1. Their conjecture was disproved in multiple papers, including [19, 74, 102, 103], who showed collectively that there are cubic graphs that are 3-connected with γ and i arbitrarily far apart. We consider the ratio i/γ in a connected cubic graph.

The question of best possible bounds for cubic graphs remains unresolved. Lam, Shiu, and Sun [76] gave a proof of the following result.

Theorem 8.1 ([76]) *For a connected cubic graph G on n vertices, $i(G) \leq 2n/5$ except for $K(3,3)$.*

We note that equality in Theorem 8.1 holds for the prism $C_5 \times K_2$ but it is not known if this is the only cubic graph achieving this bound. In [37], the authors provide a simple counting argument to show that the ratio of the independence number and the domination number in a cubic graph cannot be too large as is evident from the following result.

Theorem 8.2 ([37]) *If G is a connected cubic graph, then $i(G)/\gamma(G) \leq 3/2$, with equality if and only if $G = K(3,3)$.*

The following open question is posed in [37].

Question 1 ([37]) *If $G \neq K(3,3)$ is a connected cubic graph, then is it true that $i(G)/\gamma(G) \leq 4/3$?*

8.1 Ratio Result

Our aim in this chapter is to improve the bound given in Theorem 8.2 by answering Question 1 in the affirmative and, in addition, to characterize the graphs achieving this improved bound of 4/3. In particular, we shall prove the following result, a proof of which can be found in Section 8.3.

Theorem 8.3 *If $G \neq K(3,3)$ is a connected cubic graph, then $i(G)/\gamma(G) \leq 4/3$, with equality if and only if $G = C_5 \times K_2$.*

8.2 Useful Notation and Preliminary Results

For the remainder of the chapter we assume that G is a connected cubic graph of order n and $G \neq K(3,3)$. We introduce some useful notation and preliminary results.

If D is a dominating set in G such that $\Delta(G_D) \leq 1$, we call D a *near independent dominating set*, abbreviated *NID-set*. We remark that if D is a NID-set, then every component in G_D is isomorphic to either K_1 or K_2 and that every ID-set is a NID-set.

Lemma 8.4 *Suppose that D is a NID-set in G and let k denote the number of components in G_D that are isomorphic to K_2. Then, $i(G) \leq |D| + k$.*

Proof. We proceed by induction on k. If $k = 0$, then D is an ID-set and the result is immediate. This establishes the base case. Suppose $k \geq 1$ and let $v \in D$ such that $d_D(v) = 1$. Let $u \in N_G(v) \setminus D$. We note that $|\text{epn}(v,D)| \leq 2$. If $|\text{epn}(v,D)| = 2$ and $G[\text{epn}(v,D)] = K_2$, then let $D' = (D \setminus \{v\}) \cup \{u\}$. Otherwise, let $D' = (D \setminus \{v\}) \cup \text{epn}(v,D)$. In both cases, $|D'| \leq |D| + 1$. Furthermore, D' is a NID-set in G and there are precisely $k-1$ components in $G_{D'}$ that are isomorphic to K_2. Hence, by the inductive hypothesis, $i(G) \leq |D'| + (k-1) \leq |D| + k$. □

Lemma 8.5 *Suppose that D is a NID-set in G and let k denote the number of components in G_D that are isomorphic to K_2. If there exists a vertex $v \in D$, such that $d_D(v) = 1$ and $|\text{epn}(v,D)| \leq 1$, then $i(G) \leq |D| + k - 1$.*

Proof. Let $D' = (D \setminus \{v\}) \cup \text{epn}(v,D)$. Since $|\text{epn}(v,D)| \leq 1$, we have $|D'| \leq |D|$. Furthermore, D' is a NID-set in G and there are precisely $k-1$ components in $G_{D'}$ that are isomorphic to K_2. Hence, by Lemma 8.4, $i(G) \leq |D'| + (k-1) \leq |D| + k - 1$. □

If D is a $\gamma(G)$-set and for every $\gamma(G)$-set D' we have that $m(G_D) \leq m(G_{D'})$, then we say D is an *edge minimal* $\gamma(G)$*-set*. Thus an edge minimal $\gamma(G)$-set is a $\gamma(G)$-set that induces a subgraph of minimum size. The following lemma will prove useful.

Lemma 8.6 *Let D be an edge minimal $\gamma(G)$-set with $v \in D$. If v is not an isolated vertex in G_D, then $d_D(v) = 1$ and $|\text{epn}(v, D)| = 2$.*

Proof. Suppose that $N(v) \cap D \neq \emptyset$. Let $N(v) = \{v_1, v_2, v_3\}$ where that $v_3 \in D$. If $\text{epn}(v, D) = \emptyset$, then $D \setminus \{v\}$ is a dominating set, contradicting the fact that D is a $\gamma(G)$-set. Hence, $|\text{epn}(v, D)| \geq 1$. Switching labels for v_1 and v_2, if necessary, we can assume that $v_1 \in \text{epn}(v, D)$. If $v_2 \notin \text{epn}(v, D)$, then $D' = (D \setminus \{v\}) \cup \{v_1\}$ is a $\gamma(G)$-set with $m(G_{D'}) < m(G_D)$, contradicting our choice of the set D. Hence, $v_2 \in \text{epn}(v, D)$. □

Lemma 8.6 motivates the following definitions. If D is an edge minimal $\gamma(G)$-set, we define $D_1 = \{v \in D \mid d_D(v) = 0\}$ and $D_2 = \{v \in D \mid d_D(v) = 1\}$. We note that $D_1 \cap D_2 = \emptyset$ and by Lemma 8.6 we have $D_1 \cup D_2 = D$. Furthermore, every vertex in D_2 has precisely one neighbor in D_2 and two D-external private neighbors. We define $k_1 = |D_1|$ and $k_2 = |D_2|/2$ and note that $\gamma(G) = |D| = |D_1| + |D_2| = k_1 + 2k_2$. Our next lemma further clarifies the structure in $N[D_2]$.

Lemma 8.7 *Suppose that $D_2 \neq \emptyset$ for some edge minimal $\gamma(G)$-set D. Let $uv \in E(G_{D_2})$, $N(u) = \{v, u', u''\}$, $N(v) = \{u, v', v''\}$, $V' = \{u', u'', v', v''\}$ and $E' = E(G_{V'})$. Relabeling vertices if necessary, we may assume that precisely one of the following three properties holds:*

(i) $E' = \emptyset$.

(ii) $E' = \{u'v'\}$.

(iii) $E' = \{u'v', u'v''\}$.

Proof. By Lemma 8.6, $\text{epn}(u, D) = \{u', u''\}$ and $\text{epn}(v, D) = \{v', v''\}$. If $u'u'' \in E'$, then $D' = (D \setminus \{u\}) \cup \{u'\}$ is a $\gamma(G)$-set with $m(G_{D'}) < m(G_D)$, contradicting the fact that D is an edge minimal $\gamma(G)$-set. Thus, $u'u'' \notin E'$ and analogously, $v'v'' \notin E'$. Hence, $|E'| \leq 4$. If $E' = \emptyset$, then property (i) holds and we are done. Relabeling vertices, if necessary, we may therefore assume that $u'v' \in E'$. If $|E'| = 1$, then property (ii) holds and we are done. Thus we may assume that $|E'| \geq 2$. If $|E'| = 4$, then $G = K(3,3)$, a contradiction. Thus, $|E'| \leq 3$ and we can assume, relabeling vertices if necessary, that $u''v' \notin E'$. If $u''v'' \in E'$, then $D'' = (D \setminus \{u, v\}) \cup \{u'', v'\}$ is a $\gamma(G)$-set with $m(G_{D''}) < m(G_D)$, contradicting our choice of D. Hence, $u''v'' \notin E'$, implying that $E' = \{u'v', u'v''\}$ and property (iii) holds. □

Motivated by Lemma 8.7, we provide some final definitions and labels for vertices in an edge minimal $\gamma(G)$-set, D. If $k_2 \neq 0$, let $E(G_D) = \{u_1v_1, u_2v_2, \ldots, u_{k_2}v_{k_2}\}$ and note that $D_2 = \{u_1, v_1, u_2, v_2, \ldots, u_{k_2}, v_{k_2}\}$. For $i \in \{1, \ldots, k_2\}$, let $N(u_i) = \{v_i, u_i', u_i''\}$, let $N(v) = \{u_i, v_i', v_i''\}$, let $V_i' = \{u_i', u_i'', v_i', v_i''\}$, and let $E_i' = E(G_{V_i'})$. Relabeling vertices if necessary, we may assume by Lemma 8.7 that $E_i' \in \{\emptyset, \{u_i'v_i'\}, \{u_i'v_i', u_i'v_i''\}\}$ for each $i \in \{1, \ldots, k_2\}$. For each such i, let $V_i = N[\{u_i, v_i\}]$, and let $G_i = G_{V_i}$. We call G_i a *unit* of G. More specifically, if $E_i' = \emptyset$ we call G_i a 0-*unit*, if $E_i' = \{u_i'v_i'\}$ we call G_i a 1-*unit*, and if $E_i' = \{u_i'v_i', u_i'v_i''\}$ we call G_i a 2-*unit*. For $j \in \{0, 1, 2\}$, let ℓ_j be the number of j-units in G. For each $i \in \{1, \ldots, k_2\}$ define u_i^* as follows. If G_i is a 0-unit , let $u_i^* = u_i$; otherwise, let $u_i^* = u_i''$. Let

$$A = \bigcup_{i=1}^{k_2} \{u_i^*, v_i', v_i''\}.$$

Note that $|A| = 3k_2$. If $k_1 > 0$, let $D_1 = \{w_1, w_2, \ldots, w_{k_1}\}$ and for $i \in \{1, \ldots, k_1\}$, let $B_i = N(w_i) = \{w_i^1, w_i^2, w_i^3\}$. Let $B = N(D_1)$. For $i \in \{1, \ldots, k_1\}$, let $E_i \subseteq [A, B]$ such that $e \in E_i$ if and only if e is incident with a vertex in B_i. Further let $A_i \subseteq A$ such that $a \in A_i$ if and only if a is incident with an edge in E_i. We note that $|A_i| \leq |E_i| \leq 6$.

We define $\xi(D)$ to be the number of edges in $G[N(D_2) \setminus D_2]$. If D is an edge minimal $\gamma(G)$-set and $\xi(D) \leq \xi(D')$ for every edge minimal $\gamma(G)$-set D', then we say that D is a *desirable* $\gamma(G)$*-set*.

8.3 Proof of Ratio Result

We are now in a position to prove our main result, namely Theorem 8.3. Throughout the proof of Theorem 8.3 we use the notation and vertex labeling introduced in Section 8.2. Recall the statement of Theorem 8.3.

Theorem 8.3 *If $G \neq K(3,3)$ is a connected cubic graph, then $i(G)/\gamma(G) \leq 4/3$, with equality if and only if $G = C_5 \times K_2$.*

Proof. Let $G \neq K(3,3)$ be a connected cubic graph and let $G = (V, E)$. Let D be a desirable $\gamma(G)$-set, and recall that by definition, the set D is also an edge minimal $\gamma(G)$-set. We proceed with the following three claims.

Claim A *If $k_1 < k_2$, then $i(G)/\gamma(G) < 4/3$.*

Proof. By Lemma 8.6, every vertex in D_2 has precisely one neighbor in D_2 and two D-external private neighbors. Hence, we have $|N[D_2]| = 3|D_2| = 6k_2$. Furthermore, every vertex not in $N[D_2]$ is necessarily dominated by D_1 and therefore $V \setminus N[D_2] \subseteq N[D_1]$. Thus we have the following inequality chain,

$$n - 6k_2 = |V \setminus N[D_2]| \leq |N[D_1]| \leq 4|D_1| = 4k_1.$$

Equivalently, $n \leq 4k_1 + 6k_2$. Therefore, since $k_1 < k_2$ we have $n < 10k_2$. Further, $n - 6k_2 \leq 4|D_1|$ and so $|D_1| \geq (n - 6k_2)/4$. Hence, $\gamma(G) = |D| = |D_1| + |D_2| \geq (n - 6k_2)/4 + 2k_2 = (n + 2k_2)/4$. Since G is a cubic graph and $G \neq K(3,3)$ we have, by

Theorem 8.1, that $i(G) \leq 2n/5$. Therefore,

$$\begin{aligned}
\frac{i(G)}{\gamma(G)} &\leq \frac{2n/5}{(n+2k_2)/4} \\
&= \frac{8n}{5n+10k_2} \\
&< \frac{8n}{5n+n} \qquad \text{(since } n < 10k_2\text{)} \\
&= 4/3,
\end{aligned}$$

as desired. □

Claim B *If $k_2 \leq k_1$, then $i(G)/\gamma(G) \leq 4/3$.*

Proof. Suppose $k_2 \leq k_1$. Since D is a $\gamma(G)$-set, we have that $\gamma(G) = |D| = |D_1| + |D_2| = k_1 + 2k_2$. Furthermore, D is a NID-set in G and k_2 is the number of components in G_D that are isomorphic to K_2. If $k_2 = 0$, then D is an ID-set, and so $i(G) \leq |D| = \gamma(G)$. Consequently, $i(G) = \gamma(G)$, or, equivalently, $i(G)/\gamma(G) = 1$. Thus we may assume that $k_2 > 0$. By Lemma 8.4 we have that $i(G) \leq |D| + k_2 = k_1 + 3k_2$. Therefore,

$$\begin{aligned}
\frac{i(G)}{\gamma(G)} &\leq \frac{k_1+3k_2}{k_1+2k_2} \\
&= 1 + \frac{k_2}{k_1+2k_2} \\
&\leq 1 + \frac{k_2}{3k_2} \qquad \text{(since } k_2 \leq k_1\text{)} \\
&= 4/3,
\end{aligned}$$

as desired. □

The following two properties follow immediately by replacing the relevant inequality signs with strict inequality signs in the proof of Claim B.

Claim C *The following two properties hold.*

(i) *If* $i(G) < k_1 + 3k_2$*, then* $i(G)/\gamma(G) < 4/3$.

(ii) *If* $k_2 < k_1$*, then* $i(G)/\gamma(G) < 4/3$.

From Claims A and B we get $i(G)/\gamma(G) \leq 4/3$. This proves the desired bound and we turn our attention to proving the second part of Theorem 8.3, namely the characterization of graphs achieving this bound. We suppose now that $i(G)/\gamma(G) = 4/3$ and show that $G = C_5 \times K_2$. By Claim C, we have $i(G) = k_1 + 3k_2$ and $k_1 = k_2$. To simplify notation in the remainder of the proof, we let $k = k_1 = k_2$ and so we have $i(G) = 4k$ and $\gamma(G) = |D| = 3k$. Additionally, let $I = \{1, \ldots, k\}$. We proceed with a series of claims, culminating in the desired result. Recall that a *packing* in G is a set of vertices that are pairwise at distance at least 3 apart in G.

Claim D *The set* D_1 *is a packing in* G.

Proof. Suppose, for the sake of contradiction, that D_1 is not a packing in G. Thus since D_1 is an independent set, there are two vertices x and y in D_1 that have a common neighbor. For each $v \in V$ we have $|N[v]| = 4$ and so $|N[D_1]| \leq 4|D_1| = 4k$. But since $N(x) \cap N(y) \neq \emptyset$, we have $|N[D_1]| < 4k$. As in the proof of Claim A, we have $|N[D_2]| = 6k$ and $V \setminus N[D_2] \subseteq N[D_1]$. We now get the following inequality chain,

$$n - 6k = |V \setminus N[D_2]| \leq |N[D_1]| < 4k.$$

Hence, $n < 10k$ or, equivalently, $k > n/10$. Therefore, $\gamma(G) = 3k > 3n/10$. By Theorem 8.1 we have $i(G) \leq 2n/5$, and so $i(G)/\gamma(G) < (2n/5)/(3n/10) = 4/3$, a contradiction. □

Claim D shows that every vertex in D_1 has three D-external private neighbors. Combining this with Lemma 8.6, we have that every vertex not in D is a D-external private neighbor for some vertex in D. Equivalently, $G[V \setminus D]$ is a 2-regular graph and is therefore a disjoint union of cycles.

Claim E *No edge of G is incident with vertices from two distinct units.*

Proof. Suppose, to the contrary, that for some $\{i, j\} \subseteq I$ we have $x \in \{u_i, v_i\}$, $y \in \{u_j, v_j\}$, $x' \in \text{epn}(x, D)$ and $y' \in \text{epn}(y, D)$ such that $x'y' \in E$. Let $D' = (D \setminus \{x\}) \cup \text{epn}(x, D)$. Note that $|D'| = |D| + 1$ and D' is a NID-set with $k-1$ copies of K_2 in $G_{D'}$. Furthermore, since $x' \in D'$ we have $y' \notin \text{epn}(y, D')$ and so $|\text{epn}(y, D')| \leq 1$. But now D' is a NID-set with $k-1$ copies of K_2 in $G_{D'}$ and y is a vertex in D' such the $d_{D'}(y) = 1$ and $|\text{epn}(y, D')| \leq 1$. Hence, by Lemma 8.5 we have $i(G) \leq |D'|+k-2 = |D|+k-1 = 4k-1$. But this contradicts the fact that $i(G) = 4k$. □

Claim F *The following hold.*

(a) *$A \cup D_1$ is an $i(G)$-set.*

(b) *$|[A, B]| = 4\ell_0 + 5\ell_1 + 4\ell_2$.*

Proof. By Claim E we have that if $\{i, j\} \subseteq I$, then $[V_i, V_j] = \emptyset$. We therefore observe that $A \cup D_1$ is an ID-set in G of cardinality $4k$. This establishes Part (a). Further each 1-unit contributes exactly five edges to $[A, B]$, whilst every other unit contributes exactly four edges to $[A, B]$. Hence, $|[A, B]| = 4\ell_0 + 5\ell_1 + 4\ell_2$. □

Claim G *For $i \in I$, the set $N(B_i)$ contains vertices from at most two units in G.*

Proof. For the sake of contradiction, suppose that for some $i \in I$ we have $\{x'_1, x'_2, x'_3\} \subseteq N(B_i)$ such that x'_1, x'_2 and x'_3 each lie in a different unit of G. For $\ell \in \{1, 2, 3\}$, let x_ℓ be the unique neighbor of x'_ℓ such that $x_\ell \in D_2$ and $x'_\ell \in \text{epn}(x_\ell, D)$. Necessarily, x_1, x_2 and x_3 each lie in a different unit of G. Recall that $|\text{epn}(x_\ell, D)| = 2$ for each $\ell \in \{1, 2, 3\}$.

Suppose that B_i is an independent set. Let $D' = (D \setminus \{w_i\}) \cup B_i$. Now, $|D'| = |D| + 2$. Furthermore, $x'_\ell \notin \text{epn}(x_\ell, D')$ for $\ell \in \{1,2,3\}$ and so $|\text{epn}(x_\ell, D')| \leq 1$. For $\ell \in \{1,2,3\}$, let $X_\ell = \text{epn}(x_\ell, D')$. Let $D'' = (D' \setminus \{x_1, x_2, x_3\}) \cup X_1 \cup X_2 \cup X_3$. Note that $|D''| \leq |D'|$ and D'' is a NID-set with $k-3$ copies of K_2 in $G_{D''}$. By Lemma 8.4, $i(G) \leq |D''| + (k-3) \leq |D'| + (k-3) = (|D|+2) + (k-3) = 4k-1$, contradicting the fact that $i(G) = 4k$. Thus, B_i is not an independent set.

We may assume, relabeling vertices if necessary, that $w_i^2 w_i^3 \in E(G)$. Since $\{x'_1, x'_2, x'_3\} \subseteq N(B_i)$ and G is cubic, there are no further edges in G_{B_i}. Since w_i^3 is adjacent to at most one vertex in $\{x'_1, x'_2, x'_3\}$, we may further assume, relabeling vertices if necessary, that neither x'_1 nor x'_2 is adjacent to w_i^3. Let $D^* = (D \setminus \{w_i\}) \cup \{w_i^1, w_i^2\}$. Now, $|D^*| = |D| + 1$. Furthermore, $x_\ell \notin \text{epn}(x_\ell, D^*)$ for $\ell \in \{1,2\}$ and so $|\text{epn}(x_\ell, D^*)| \leq 1$. For $\ell \in \{1,2\}$, let $X_\ell^* = \text{epn}(x_\ell, D^*)$. Let $D^{**} = (D^* \setminus \{x_1, x_2\}) \cup X_1^* \cup X_2^*$. Note that $|D^{**}| \leq |D^*|$ and D^{**} is a NID-set with $k-2$ copies of K_2 in $G_{D^{**}}$. By Lemma 8.4, $i(G) \leq |D^{**}| + (k-2) \leq |D^*| + (k-2) = (|D|+1) + (k-2) = 4k-1$, which is a contradiction and the desired result follows. □

Our next claim provides some additional structure in the graph G whenever any $N(B_i)$ contains vertices from two distinct units in G.

Claim H *Let $i \in I$ and suppose $N(B_i) \cap V_{j_1} \neq \emptyset$ and $N(B_i) \cap V_{j_2} \neq \emptyset$ for some $\{j_1, j_2\} \subseteq I$. Then, $|N(x) \cap N(B_i)| \leq 1$ for every $x \in \{u_{j_1}, u_{j_2}, v_{j_1}, v_{j_2}\}$.*

Proof. It suffices to show that for $x \in \{u_{j_1}, v_{j_1}, u_{j_2}, v_{j_2}\}$, we have $\text{epn}(x, D) \not\subseteq N(B_i)$. Suppose, to the contrary, that $\text{epn}(x, D) \subseteq N(B_i)$. Switching j_1 and j_2 if necessary, we may assume $x \in \{u_{j_1}, v_{j_1}\}$. Let $y' \in N(B_i) \cap V_{j_2}$ and let y be the unique vertex in D_2 such that $y' \in \text{epn}(y, D)$. Note that y' is adjacent to at least one vertex in B_i. Let $D' = (D \setminus \{w_i\}) \cup B_i$. Now, $|D'| = |D| + 2$. Furthermore, $\text{epn}(x, D') = \emptyset$ and $y' \notin \text{epn}(y, D')$. Hence, $|\text{epn}(y, D')| \leq 1$. Let $D'' = (D' \setminus \{x, y\}) \cup \text{epn}(y, D')$ and note that $|D''| \leq |D'| - 1$.

If B_i is an independent set, then D'' is a NID-set with $k-2$ copies of K_2 in $G_{D''}$ and so by Lemma 8.4 we have $i(G) \le |D''| + (k-2) \le (|D'|-1) + (k-2) = |D| + k - 1 = 4k-1$, a contradiction. Thus, B_i is not an independent set.

We may assume, relabeling vertices if necessary, that $w_i^2 w_i^3 \in E(G)$. Since $|\text{epn}(x, D)| = 2$ and $\text{epn}(x, D) \cup \{y'\} \subseteq N(B_i)$, there are no further edges in G_{B_i}. Note that w_i^2 and w_i^3 belong to the same K_2-component of $G_{D''}$. Further, $w_i \notin \text{epn}(w_i^3, D'')$ and so $|\text{epn}(w_i^3, D'')| \le 1$. But now D'' is a NID-set with $k-1$ copies of K_2 in $G_{D''}$ and w_i^3 is a vertex in D'' such that $d_{D''}(w_i^3) = 1$ and $|\text{epn}(w_i^3, D'')| \le 1$. Hence, by Lemma 8.5, we have $i(G) \le |D''| + k - 2 \le (|D'|-1) + (k-2) = |D| + k - 1 = 4k-1$, a contradiction. Therefore, $\text{epn}(x, D) \not\subseteq N(B_i)$ and the desired result follows. □

Claim I *Suppose that G_i is a 2-unit for some $i \in I$ and u_i'' has two neighbors in B_j for some $j \in I$. If w is the vertex in B_i not adjacent to u_i'', then w has no neighbors in $N(D_2)$.*

Proof. We may assume, relabeling vertices if necessary, that G_1 is a 2-unit and $\{u_1''w_1^1, u_1''w_1^2\} \subseteq E$. If $\{v_1'w_1^3, v_1''w_1^3\} \subseteq E$, then $D^* = (D \setminus \{v_1, w_1\}) \cup \{u_1'', w_1^3\}$ is a $\gamma(G)$-set with $m(G_{D^*}) = m(G_D)$ but with $\xi(D^*) < \xi(D)$, contradicting our choice of D. Thus, we may assume, switching labels for v_1' and v_1'' if necessary, that $v_1''w_1^3 \notin E$. If $v_1'w_1^3 \in E$, then $D' = (D \setminus \{u_1, v_1, w_1\}) \cup \{u_1'', v_1'', w_1^3\}$ is a $\gamma(G)$-set with $m(G_{D'}) < m(G_D)$, contradicting the fact that D is an edge minimal $\gamma(G)$-set. Hence, $v_1'w_1^3 \notin E$. Suppose, for the sake of contradiction, that w_1^3 has a neighbor in $N(D_2)$, say y'. Necessarily, y' is in a unit different to G_1. Let y be the unique vertex in D_2 such that $y' \in \text{epn}(y, D)$. Let $D'' = (D \setminus \{u_1, w_1\}) \cup \{u_1', u_1'', w_1^3\}$. Now, $|D''| = |D| + 1$. Furthermore, $y' \notin \text{epn}(y, D'')$, and so $|\text{epn}(y, D'')| \le 1$. But now D'' is a NID-set with $k-1$ copies of K_2 in $G_{D''}$ and y is a vertex in D'' such the $d_{D''}(y) = 1$ and $|\text{epn}(y, D'')| \le 1$. Hence, by Lemma 8.5, we have $i(G) \le |D''| + k - 2 = (|D|+1) + k - 2 = 4k-1$, a contradiction. Hence, w_1^3 has no neighbor in $N(D_2)$. □

Recall that for $i \in \{1, \ldots, k\}$, E_i is the set of edges in $[A, B]$ that are incident with a vertex in B_i. Further, A_i is the set of vertices in A that are incident with an edge in E_i. As observer earlier, $|A_i| \le |E_i| \le 6$.

Claim J *For each $i \in I$, $|E_i| \le 5$.*

Proof. Suppose, to the contrary, that $|E_i| = 6$ for some $i \in I$. Since the vertices in each unit in G are incident with at most five edges from $[A, B]$, we have that $A_i \subseteq N(B_i)$ contains vertices from at least two units in G. Hence, by Claim G, the set A_i contains vertices from exactly two units. Relabeling vertices, if necessary, we may assume that these units are G_1 and G_2 and that at least three edges in E_i are incident with vertices in G_1. We remark that $A_i \subseteq \{u_1^*, v_1', v_1'', u_2^*, v_2', v_2''\}$.

Suppose v_1'' is incident with two edges in E_i. Necessarily, G_1 is a 0-unit or a 1-unit and we may assume, relabeling the vertices of B_i if necessary, that $\{v_1''w_i^1, v_1''w_i^2\} \subseteq E_i$. Consider the two neighbors of w_i^3 different from w_i. By Claim H, neither of them is v_1'. Furthermore, at least one of them, x' say, is different from u_1''. Hence, x' is in G_2. Let x be the unique vertex in D_2 such that $x' \in \mathrm{epn}(x, D)$. Let $D' = (D \setminus \{v_1, w_i\}) \cup \{v_1', v_1'', w_i^3\}$. Then, $|D'| = |D| + 1$. Furthermore, $x' \notin \mathrm{epn}(x, D')$ and so $|\mathrm{epn}(x, D')| \le 1$. But now D' is a NID-set with $k - 1$ copies of K_2 in $G_{D'}$ and x is a vertex in D' such that $d_{D'}(x) = 1$ and $|\mathrm{epn}(x, D')| \le 1$. Hence, by Lemma 8.5, we have that $i(G) \le |D'| + k - 2 = (|D| + 1) + k - 2 = 4k - 1$, a contradiction. Hence, v_1'' is incident with at most one edge from E_i.

If G_1 is a 0-unit, then v_1'' is incident with exactly one edge from E_i and v_1' is incident with exactly two edges from E_i, contradicting Claim H. Hence, G_1 is a 1-unit or a 2-unit. Therefore, $u_1'v_1' \in E$, and so v_1' is incident with at most one edge from E_i. By Claim H, at most one of v_1' and v_1'' is incident with an edge in E_i, implying that at most one edge from E_i is incident with a vertex in $\{v_1', v_1''\}$. However by our choice of G_1, there are at least three edges in E_i incident with vertices in G_1. Hence, u_1'' is incident with two edges from E_i and exactly one of v_1' or v_1'' is incident with one edge from E_i. We may assume, relabeling the vertices of B_i if necessary, that $\{u_1''w_i^1, u_1''w_i^2\} \subseteq E_i$. Consider again the two neighbors of w_i^3 different from w_i. At least one of them, y' say, is not in $\{v_1', v_1''\}$. Since $u_1' \notin A$, we note that $u_1' \notin N(B_i)$. Hence, y' is in G_2. Let y be the unique vertex in D_2 such that $y' \in \text{epn}(y, D)$. Let $D'' = (D \setminus \{u_1, w_i\}) \cup \{u_1', u_1'', w_i^3\}$. Now, $|D''| = |D| + 1$. Furthermore, $y' \notin \text{epn}(y, D'')$ and so $|\text{epn}(y, D'')| \leq 1$. But now D'' is a NID-set with $k-1$ copies of K_2 in $G_{D''}$ and y is a vertex in D'' such the $d_{D''}(y) = 1$ and $|\text{epn}(y, D'')| \leq 1$. Hence, by Lemma 8.5, we have that $i(G) \leq |D''| + k - 2 = (|D| + 1) + k - 2 = 4k - 1$, a contradiction. We conclude that $|E_i| \leq 5$. □

Claim K *If $|E_i| \leq 4$ for each $i \in I$, then the following hold.*

(a) *No unit in G is a 1-unit.*

(b) *$|E_i| = 4$ for each $i \in I$.*

(c) *If G_i is a 0-unit and $[\{u_i', u_i''\}, B_j] \neq \emptyset$ for some $i, j \in I$, then $|[\{v_i', v_i''\}, B_j]| \geq 1$.*

(d) *If G_i is a 0-unit and $[\{u_i', u_i''\}, B_j] \neq \emptyset$ for some $i, j \in I$, then $|[\{v_i', v_i''\}, B_j]| \geq 2$.*

(e) *If G_i is a 0-unit for some $i \in I$, then $A_i \not\subseteq V_j$ for any $j \in I$.*

(f) *If G_i is a 0-unit for some $i \in I$, then $|\{u_i', u_i''\} \cap N(B_j)| \leq 1$ for all $j \in I$.*

(g) *If G_i is a 0-unit for some $i \in I$, then $(N(u_i') \cap B) \not\subseteq B_j$ for any $j \in I$.*

(h) *If G_i is a 0-unit for some $i \in I$, then $(N(u_i'') \cap B) \not\subseteq B_j$ for any $j \in I$.*

(i) *No unit in G is a 0-unit.*

(j) *For each $i \in I$, one of the vertices in B_i is not incident with any edge in E_i.*

Proof. (a) Since $|E_i| \leq 4$ for each $i \in I$, we have $\sum_{i=1}^{k} |E_i| \leq 4k$, with equality if and only if $|E_i| = 4$ for each $i \in I$. Recall that $|[A, B]| = 4\ell_0 + 5\ell_1 + 4\ell_2 = 4k + \ell_1$. Hence,

$$4k + \ell_1 = |[A, B]| = \sum_{i=1}^{k} |E_i| \leq 4k,$$

and so $\ell_1 = 0$ and $|E_i| = 4$ for $i \in I$. Since $\ell_1 = 0$, every unit in G is a 0-unit or a 2-unit.

(b) The result follows from the proof of Part (a).

(c) By Part (a), every unit in G is a 0-unit or a 2-unit. For the sake of contradiction, we may assume, relabeling vertices if necessary, that G_1 is a 0-unit, $u_1'w_1^1 \in E$, and $[\{v_1', v_1''\}, B_1] = \emptyset$. Thus, no edge in E_1 is incident with either v_1' or v_1''. Further, since G_1 is a 0-unit, no edge in E_1 is incident with a vertex in V_1. We may assume, relabeling vertices if necessary, that one of the edges in E_1 is incident with a vertex in V_2. Thus, $N(B_1)$ contains vertices from both V_1 and V_2. Hence by Claim G, the set $N(B_1)$ contains vertices from only V_1 and V_2. But then each of the four edges in E_1 is incident with a vertex in V_2. But now, whether G_2 is a 0-unit or a 2-unit in G, we have $\{v_2', v_2''\} \subseteq N(B_1)$, contradicting Claim H and the desired result follows.

(d) By Part (a), every unit in G is a 0-unit or a 2-unit. For the sake of contradiction, we may assume, relabeling vertices if necessary, that G_1 is a 0-unit, $[\{u_1', u_1''\}, B_1] \neq \emptyset$, and that three of the four edges in E_1 are not incident with either v_1' or v_1''. By Part (c), at least one of the four edges in E_1, e_1 say, is incident with either v_1' or v_1''. By Claim G we may assume, relabeling vertices if necessary, that each of the edges in E_1 different from e_1 is incident with a vertex in V_2. By Claim H, $\{v_2', v_2''\} \not\subseteq N(B_1)$. Necessarily then, G_2 is a 2-unit with two edges from $E_1 \setminus \{e_1\}$ incident with u_2'' and the third incident with either v_2' or v_2''. We may assume, relabeling vertices if necessary, that u_2'' is adjacent to both w_1^1 and w_1^2. By Claim I, the vertex w_1^3 has no neighbors in $N(D_2)$. Thus, $|[B_1, N(D_2)]| \leq 4$. But, $|[B_1, N(D_2)]| \geq |E_1| + |[\{u_1', u_1''\}, B_1]| \geq 5$, a contradiction.

(e) For the sake of contradiction, we may assume, relabeling vertices if necessary, that G_1 is a 0-unit and $A_1 \subseteq V_1$. But then $E_1 = [\{v_1', v_1''\}, B]$ and so $|[\{v_1', v_1''\}, B_1]| = |E_1| = 4$. Since $|[\{u_1', u_1'', v_1', v_1''\}, B_1]| \leq 6$, we have that either u_1' or u_1'' has a neighbor in $B \setminus B_1$. We may assume (relabeling vertices, if necessary) that w_2^1 is such a neighbor. But now no edge in E_2 is incident with either v_1' or v_1''. We may assume, relabeling vertices if necessary, that one of the edges in E_2 is incident with a vertex in V_2. Therefore, by Claim G, each of the four edges in E_2 is incident with a vertex in V_2. But now, whether

G_2 is a 0-unit or a 2-unit in G, we have $\{v_2', v_2''\} \subseteq N(B_2)$, contradicting Claim H.

(f) For the sake of contradiction, we may assume, relabeling vertices if necessary, that G_1 is a 0-unit and that both u_1' and u_1'' have a neighbor in B_1. Thus, $\{u_1', u_1''\} \subseteq N(B_1)$ and so by Claim H, we have $N(B_1) \cap V_i = \emptyset$ for each $i \in I \setminus \{1\}$. Therefore, each of the four edges in E_1 is incident with either v_1' or v_1'', and so $A_1 = \{v_1', v_1''\} \subseteq V_1$. But this contradicts Part (e).

(g) For the sake of contradiction, we may assume, relabeling vertices if necessary, that G_1 is a 0-unit and $N(u_1') = \{u_1, w_1^1, w_1^2\}$. Since $|E_1| = 4$, both the neighbors of w_1^3 different from w_1 are in A. We note, therefore, that $u_1''w_1^3 \notin E$. If $\{v_1', v_1''\} \subseteq N(B_1)$, then by Claim H we have $N(B_1) \cap V_i = \emptyset$ for each $i \in I \setminus \{1\}$. But then $A_1 = \{v_1', v_1''\} \subseteq V_1$, contradicting Part (e), and so w_1^3 has a neighbor, x' say, in a different unit to G_1. Let x be the unique vertex in D_2 such that $x' \in \text{epn}(x, D)$. Let $D' = (D \setminus \{u_1, w_1\}) \cup \{u_1', u_1'', w_1^3\}$. Now, $|D'| = |D| + 1$. Furthermore, $x' \notin \text{epn}(x, D')$ and so $|\text{epn}(x, D')| \leq 1$. But now D' is a NID-set with $k-1$ copies of K_2 in $G_{D'}$ and x is a vertex in D' such that $d_{D'}(x) = 1$ and $|\text{epn}(x, D')| \leq 1$. Hence, by Lemma 8.5 we have that $i(G) \leq |D'| + k - 2 = (|D| + 1) + k - 2 = 4k - 1$, a contradiction.

(h) By symmetry of u_1' and u_1'', the proof is analogous to Part (g).

(i) For the sake of contradiction, we may assume, relabeling vertices if necessary, that G_1 is a 0-unit. By Parts (f)-(h), no two of the four neighbors of u_1' and u_1'' in B have w_i as a common neighbor for any $i \in I$. We may assume, relabeling vertices if necessary, that $\{u_1'w_1^1, u_1'w_2^1, u_1''w_3^1, u_1''w_4^1\} \subseteq E$. By Part (d), $|[\{v_1', v_1''\}, B_i]| \geq 2$ for each $i \in \{1, \ldots, 4\}$. But now we have

$$4 = |[\{v_1', v_1''\}, B]| \geq \left| \left[\{v_1', v_1''\}, \bigcup_{i=1}^{4} B_i \right] \right| = \sum_{i=1}^{4} |[\{v_1', v_1''\}, B_i]| \geq 8,$$

a contradiction.

(j) By Parts (a) and (i), every unit in G is a 2-unit. Suppose, for the sake of contradiction, relabeling vertices if necessary, that $E_1 = \{w_1^1x_1, w_1^1x_2, w_1^2x_3, w_1^3x_4\}$. By Claim F, $A \cup D_1$ is an $i(G)$-set. If $x_3 \in \{v_i', v_i''\}$ for some $i \in I$, then $((A \cup D_1) \setminus \{x_3, w_1\}) \cup \{w_1^2\}$ is an ID-set of size $4k-1$, a contradiction. Therefore, $x_3 \notin \{v_i', v_i''\}$ for any $i \in I$ and similarly, $x_4 \notin \{v_i', v_i''\}$ for any $i \in I$. If $x_3 = x_4 = u_i''$ for some $i \in I$, then by Claim I, w_1^1 has no neighbors in $N(D_2)$, a contradiction. Hence, we may assume, relabeling vertices if necessary, that $x_3 = u_1''$ and $x_4 = u_2''$. By Claim G, the neighbors of w_1^1 different from w_1 are both in $V_1 \cup V_2$. By Claim I, w_1^1 is not adjacent to either u_1'' or u_2''. By Claim H, $\{v_1', v_1''\} \not\subseteq N(w_1^1)$ and $\{v_2', v_2''\} \not\subseteq N(w_1^1)$. Hence, we may assume (relabeling vertices, if necessary) that w_1^1 is adjacent to v_1' and v_2'. Let $D' = (D \setminus \{u_1, v_1, u_2, v_2, w_1\}) \cup \{u_1'', v_1'', u_2'', v_2'', w_1^1\}$. Now $|D'| = |D|$ and D' is a $\gamma(G)$-set with $m(G_{D'}) < m(G_D)$, contradicting the fact that D is an edge minimal $\gamma(G)$-set. □

Claim L *For some $i \in I$ we have $|E_i| = 5$.*

Proof. By Claim J we have $|E_i| \leq 5$ for each $i \in I$. Suppose then, for the sake of contradiction, that $|E_i| \leq 4$ for each $i \in I$. By Claim K(a) and Claim K(i), every unit in G is a 2-unit. By Claim K(b), $|E_i| = 4$ for all $i \in I$. Let $B' = \{w_1^3, \ldots, w_k^3\}$ and let $B'' = B \setminus B'$. By Claim K(j), we may assume, relabeling vertices if necessary, that for every $i \in I$, the vertex w_i^3 has no neighbors in A and hence no neighbors in $N(D_2)$. Consequently, $G[B']$ is a 2-regular graph. Let W be an ID-set in $G[B]$. Since any cycle requires at most half its vertices to independently dominate it, we have $|W| \leq k/2$. But now $D' = W \cup \{u_1, \ldots, u_k\} \cup B''$ is an ID-set with $|D'| \leq k/2 + 3k < 4k$, a contradiction. Therefore, $|E_i| = 5$ for some $i \in I$. □

By Claim L, $|E_i| = 5$ for some $i \in I$. Renaming vertices if necessary, we may assume that $|E_1| = 5$. We provide one final claim before completing our characterization of the graph G.

Claim M *$A_1 \subseteq V_j$ for some $j \in I$.*

Proof. Suppose, for the sake of contradiction, that $A_1 \cap V_{j_1} \neq \emptyset$ and $A_1 \cap V_{j_2} \neq \emptyset$ for some $\{j_1, j_2\} \subseteq I$. Relabeling vertices, if necessary, we may assume that $j_1 = 1$ and $j_2 = 2$. Let $A^* = A \cap (V_1 \cup V_2)$. Thus, $A^* = \{u_1^*, v_1', v_1'', u_2^*, v_2', v_2''\}$. By Claim G, we have that $A_1 \subseteq V_1 \cup V_2$ and so $A_1 \subseteq A^*$. We may assume, relabeling the vertices of B_1 if necessary, that both w_1^1 and w_1^2 are incident with two edges in E_1 and w_1^3 is incident with one edge in E_1. Let v be the unique vertex in A^* adjacent to w_1^3.

Suppose $v \in \{v_1', v_1'', v_2', v_2''\}$. Renaming vertices if necessary, we may assume that $v \in \{v_1', v_1''\}$. On the one hand suppose that $v = v_1'$ and G_1 is a 1-unit. Then both neighbors of w_1^1 and w_1^2 in A^* differ from v. In this case, let $D' = (D \setminus \{v_1, u_2, v_2, w_1\}) \cup \{v_1'', u_2^*, v_2', v_2'', w_1^3\}$. Note that in this case, both w_1^1 and w_1^2 are dominated by D'. On the other hand, suppose that $v = v_1'$ and G_1 is not a 1-unit or $v = v_2''$. Then both w_1^1 and w_1^2 have at least one neighbor in A^* different from v. In this case, let $D' = (D \setminus \{u_1, v_1, u_2, v_2, w_1\}) \cup (A^* \setminus \{v\}) \cup \{w_1^3\}$. Note that in the second case the vertex u_1' is dominated by u_1^*, v_1' or by v_1'' from the set D'. In both cases, $|D'| = |D| + 1$ and D' is a NID-set with $k - 2$ copies of K_2 in $G_{D'}$. By Lemma 8.4, $i(G) \leq |D'| + (k - 2) = (|D| + 1) + (k - 2) = 4k - 1$, a contradiction. Hence, $v \in \{u_1^*, u_2^*\}$. Renaming vertices if necessary, we may assume that $v = u_1^*$. This implies that G_1 is a 1-unit or a 2-unit and $v = u_1''$. Thus, $A^* = \{u_1'', v_1', v_1'', u_2^*, v_2', v_2''\}$.

Suppose $|N(u_1'') \cap B_1| \geq 2$. We may assume, switching the labels of w_1^1 and w_1^2 if necessary, that $w_1^2 \in N(u_1'') \cap B_1$. We now consider the two neighbors of w_1^1 in A^*. By Claim H, at most one of them is in $\{v_1', v_1''\}$. Hence, w_1^1 must have a neighbor in $\{u_2^*, v_2', v_2''\} \subseteq V_2$. Let y' be this neighbor and let y be the unique vertex in D_2 such that $y' \in \text{epn}(y, D)$. Let $D'' = (D \setminus \{u_1, w_1\}) \cup \{u_1', u_1'', w_1^1\}$. Now, $|D''| = |D| + 1$. Furthermore, $y' \notin \text{epn}(y, D'')$, and so $|\text{epn}(y, D'')| \leq 1$. But now D'' is a NID-set with $k - 1$ copies of K_2 in $G_{D''}$ and y is a vertex in D'' such the $d_{D''}(y) = 1$ and $|\text{epn}(y, D'')| \leq 1$. Hence, by Lemma 8.5 we have $i(G) \leq |D''| + k - 2 = (|D| + 1) + k - 2 = 4k - 1$, a contradiction.

Therefore, $N(u_1'') \cap B_1 = \{w_1^3\}$.

Again, we consider the two neighbors of w_1^1 in $A^* = \{u_1'', v_1', v_1'', u_2^*, v_2', v_2''\}$. Since w_1^3 is the only neighbor of u_1'' in B_1, neither neighbor of w_1^1 in A^* is u_1'' and by Claim H, at most one of them is in $\{v_1', v_1''\}$. Hence, w_1^1 must have a neighbor in $\{u_2^*, v_2', v_2''\}$. Similarly, w_1^2 must also have a neighbor in $\{u_2^*, v_2', v_2''\}$. We note that $w_1^3 u_1' \notin E$ by Claim H. Let $D^* = (D \setminus \{u_1, u_2, v_2, w_1\}) \cup \{u_1', u_2^*, v_2', v_2'', w_1^3\}$. Now, $|D^*| = |D| + 1$ and D^* is a NID-set with $k-2$ copies of K_2 in G_{D^*}. By Lemma 8.4, $i(G) \leq |D^*| + (k-2) = (|D|+1) + (k-2) = 4k - 1$, a contradiction. The desired result follows. □

We now return to the proof of Theorem 8.3 one last time. By our earlier assumption, $|E_1| = 5$. By Claim M, we have $A_1 \subseteq V_j$ for some $j \in I$. We may assume, renaming vertices if necessary, that $A_1 \subseteq V_1$. Since the vertices of G_1 are incident with all five edges in $E_1 \subseteq [A, B]$, we have that G_1 is a 1-unit. We may assume, relabeling the vertices of B_1 if necessary, that $\{v_1'' w_1^2, v_1'' w_1^3\} \subseteq E_1$. If $v_1' w_1^1 \in E_1$, then $D' = (D \setminus \{v_1, w_1\}) \cup \{v_1'', w_1^1\}$ is a $\gamma(G)$-set with $m(G_{D'}) < m(G_D)$, contradicting the fact that D is an edge minimal $\gamma(G)$-set. Hence, $v_1' w_1^1 \notin E_1$ and we may assume, switching the labels of w_1^2 and w_1^3 if necessary, that $v_1' w_1^3 \in E_1$. But now, since u_1'' is incident with two edges from E_1, we must have $\{u_1'' w_1^1, u_1'' w_1^2\} \subseteq E_1$. If $u_1' w_1^1 \notin E$ then $D'' = (D \setminus \{u_1, v_1, w_1\}) \cup \{u_1', v_1'', w_1^1\}$ is a $\gamma(G)$-set with $m(G_{D''}) < m(G_D)$, contradicting the fact that D is an edge minimal $\gamma(G)$-set. Hence $u_1' w_1^1 \in E$ and, since G is a connected cubic graph, $G = C_5 \times K_2$. □

8.4 A Further Conjecture

As a consequence of our main result, namely Theorem 8.3, we have that if G is a connected cubic graph of order $n \geq 12$, then $i(G)/\gamma(G) < 4/3$. We close the chapter with the following conjecture.

Conjecture 8.8 *If G is a connected cubic graph of sufficiently large order, then $i(G)/\gamma(G) \le 6/5$.*

We remark that if Conjecture 8.8 is true, then the result is best possible. For this purpose, we shall need the following two infinite families $\mathcal{G}_{\rm cubic}$ and $\mathcal{H}_{\rm cubic}$ of connected cubic graphs constructed in [37] as follows.

For $k \ge 1$, define graph G_k as described below. Consider two copies of the path P_{4k} with respective vertex sequences $a_1b_1c_1d_1 \dots a_kb_kc_kd_k$ and $w_1x_1y_1z_1 \dots w_kx_ky_kz_k$. For each $1 \le i \le k$, join a_i to w_i, b_i to x_i, c_i to z_i, and d_i to y_i. To complete G_k join a_1 to d_k and w_1 to z_k. Let $\mathcal{G}_{\rm cubic} = \{ G_k : k \ge 1 \}$.

For $k \ge 1$, define H_k as follows. Consider a copy of the cycle C_{3k} with vertex sequence $a_1b_1c_1 \dots a_kb_kc_ka_1$. For each $1 \le i \le k$, add the vertices $\{w_i, x_i, y_i, z_i^1, z_i^2\}$, and join a_i to w_i, b_i to x_i, and c_i to y_i. To complete the construction of H_k, for each $1 \le i \le k$ and $j \in \{1,2\}$, join z_i^j to each of the vertices w_i, x_i, and y_i. Let $\mathcal{H}_{\rm cubic} = \{ H_k : k \ge 1 \}$.

Graphs in the families $\mathcal{G}_{\rm cubic}$ and $\mathcal{H}_{\rm cubic}$ are illustrated in Figure 8.1.

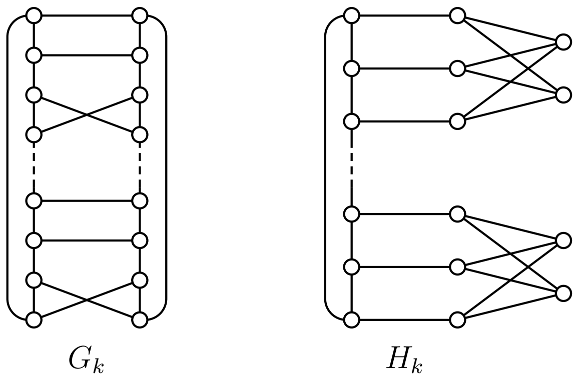


Figure 8.1: Graphs $G_k \in \mathcal{G}_{\rm cubic}$ and $H_k \in \mathcal{H}_{\rm cubic}$.

It is shown in [37] that if $G \in \mathcal{G}_{\rm cubic} \cup \mathcal{H}_{\rm cubic}$ has order n, then $\gamma(G) = \lceil 5n/16 \rceil$ and $i(G) = 3n/8$, implying that $i(G)/\gamma(G) \le 6/5$. In particular, if $n \equiv 0 \pmod{16}$, then $i(G)/\gamma(G) = 6/5$. Hence we have the following result.

Corollary 8.9 ([37]) *There exist connected cubic graphs G of arbitrarily large order satisfying* $i(G)/\gamma(G) = 6/5$.

Hence if Conjecture 8.8 is true, then by Corollary 8.9 the result is best possible.

Chapter 9

Edge Weighting Functions

In this chapter we introduce an edge weighting function which has been useful in achieving bounds on domination parameters similar to the previous two chapters. In order to demonstrate this technique we present the bounds achieved on the upper domination number and the upper total domination number in regular graphs. We also characterize graphs achieving these bounds. In the following two chapters we use a similar modified weighting function to prove additional bounds on different domination parameters in cubic graphs.

Recall that the upper domination number, $\Gamma(G)$, of a graph G is the maximum cardinality of a minimal dominating set in G and that the upper total domination number, $\Gamma_t(G)$, of a graph G is the maximum cardinality of a minimal total dominating set in G. We observe that if G has at least one vertex that is not isolated and if v is such a vertex of G, then $V(G) \setminus \{v\}$ is a dominating set in G, implying that the set $V(G)$ is not a minimal dominating set. Hence we have the following observation.

Observation 9.1 *If G is a graph of order n with maximum degree greater than 0, then $\Gamma(G) \leq n - 1$.*

That the trivial upper bound in Observation 9.1 is sharp, may be seen by taking G to be a star $K_{1,n-1}$ where $n \geq 3$. The set of $n-1$ leaves in the star form a minimal dominating set in G, and so $\Gamma(G) \geq n-1$. Hence by Observation 9.1, $\Gamma(G) = n-1$.

A similar observation may be made for the upper total domination number in a graph with maximum degree greater than 1. If G is such a graph, then G necessarily contains a vertex, v say, such that no neighbor of v has degree 1. Further, $V(G) \setminus \{v\}$ is a total dominating set in G, implying that the set $V(G)$ is not a minimal total dominating set. Hence we have the following observation.

Observation 9.2 *If G is a graph of order n with maximum degree greater than 1, then $\Gamma_t(G) \leq n-1$.*

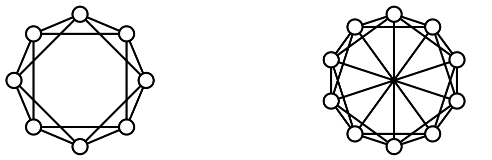
Again, the trivial upper bound in Observation 9.2 is sharp, as may be seen by taking G to be the graph obtained by subdividing every edge in the star $K_{1,(n-1)/2}$ exactly once, where n is odd and $n \geq 5$. The set of $n-1$ leaves and support vertices form a minimal total dominating set in G, and so $\Gamma_t(G) \geq n-1$. Hence by Observation 9.2, $\Gamma_t(G) = n-1$.

Our aim in this chapter is to show that if we impose a regularity condition on the graph, then using edge weighting functions on dominating sets these bounds can be greatly improved. We establish sharp upper bounds on both the upper domination number and the upper total domination number of a graph, and we characterize the extremal graphs that achieve equality in these bounds.

9.1 The Families $\mathcal{B}$, $\mathcal{F}$ and $\mathcal{G}$

A circulant graph $C_n\langle L\rangle$ with a given list $L \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ is a graph on n vertices in which the ith vertex is adjacent to the $(i+j)$th and $(i-j)$th vertices for each j in the list L and where addition is taken modulo n. More precisely, if $L =$

$\{j_1, j_2, \ldots, j_r\} \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, then the circulant graph $C_n\langle L\rangle$ is the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $\{v_i v_{i+j (\bmod n)} \mid i \in \{0, 1, \ldots, n-1\}$ and $j \in \{j_1, j_2, \ldots, j_r\}\}$. For $k \geq 4$ even and $\ell \geq k$ with ℓ even, let $L_{k,\ell} = \{1, 2, \ldots, k/2 - 1, \ell/2\}$, while for $k \geq 3$ odd and $\ell \geq k$, let $L_{k,\ell} = \{1, 2, \ldots, (k-1)/2\}$. In both cases, the circulant graph $C_\ell\langle L_{k,\ell}\rangle$ is a $(k-1)$-regular graph on ℓ vertices. For example, the circulant graph $C_8\langle L_{5,8}\rangle = C_8\langle 1, 2\rangle$ shown in Figure 9.1(a) is a 4-regular graph on 8 vertices, while the circulant graph $C_{10}\langle L_{6,10}\rangle = C_{10}\langle 1, 2, 5\rangle$ shown in Figure 9.1(b) is a 5-regular graph on 10 vertices.



(a) $C_8\langle 1, 2\rangle$ (b) $C_{10}\langle 1, 2, 5\rangle$

Figure 9.1: Graphs in the Family $\mathcal{F}$.

We remark that given any positive integer ℓ_1 the empty graph on ℓ_1 vertices is 0-regular, and given any positive even integer ℓ_2 the graph comprising $\ell_2/2$ copies of K_2 is 1-regular. In view of this remark, and the circulant graphs $C_\ell\langle L_{k,\ell}\rangle$ constructed above, we observe that for every two positive integers k and ℓ where $\ell \geq k$ and where ℓ is even whenever k is even, there always exist $(k-1)$-regular graphs on ℓ vertices. Conversely, since every graph has an even number of vertices of odd degree, every $(k-1)$-regular graph on ℓ vertices satisfies $\ell \geq k$ with ℓ even whenever k is even. Hence we have the following result.

Observation 9.3 *Let k and ℓ be two positive integers. Then there exists a $(k-1)$-regular graph on ℓ vertices if and only if $\ell \geq k$ and where ℓ is even whenever k is even.*

The Family $\mathcal{B}$. Let $\mathcal{B}$ be the family of connected bipartite regular graphs.

The Family $\mathcal{F}$. Let $\mathcal{F}$ be the family of connected regular graphs constructed as follows. Let $k \geq 1$ and $\ell \geq k$ be arbitrary fixed integers, provided that ℓ is even whenever k is even. By Observation 9.3, for every such pair of integers k and ℓ there exist $(k-1)$-regular graphs on ℓ vertices. Let F_1 and F_2 be disjoint $(k-1)$-regular graphs (not necessarily connected) on ℓ vertices with $V(F_1) = \{u_1, u_2, \ldots, u_\ell\}$ and $V(F_2) = \{v_1, v_2, \ldots, v_\ell\}$. Let F be the graph obtained from the disjoint union $F_1 \cup F_2$ by joining u_i to v_i for each $i \in \{1, 2, \ldots, \ell\}$. Let $\mathcal{F}$ be the family of all graphs thus constructed which are, in addition, connected.

A $(\mathbf{k}, \mathbf{s}, \mathbf{t})$-triple. We define a (k, s, t)-*triple* as three non-negative integers k, s and t satisfying the following four conditions.

- $2s + t \geq k \geq 1$.
- $2(s + t) = \ell k$ for some positive integer ℓ.
- If $k = 1$, then $t = 0$.
- If $t > 0$, then $t \geq k$ where t is even whenever k is even.

A $(\mathbf{k}, \mathbf{s}, \mathbf{t})$-graph. Given a (k, s, t)-triple, if $k = 1$ we define a (k, s, t)-*graph* to be the empty graph on $2s$ vertices; otherwise $k > 1$ and we define a (k, s, t)-*graph* to be any bipartite graph, G, with partite sets $X = X_1 \cup X_2$ and Y such that $|X| = 2s + t$, $|X_1| = 2s$, $|X_2| = t$, $|Y| = 2s + t - \ell$, and for all $x_1 \in X_1$, $x_2 \in X_2$ and $y \in Y$ we have $d_G(x_1) = k - 1$, $d_G(x_2) = k - 2$, and $d_G(y) = k$. We remark that (k, s, t)-graphs exist for every (k, s, t)-triple. As an example, consider the following construction of a (k, s, t)-graph from an empty graph with vertex set $X \cup Y$, where $X = \{x_1, \ldots, x_{2s+t}\}$, and $Y = \{y_1, \ldots, y_{2s+t-\ell}\}$. For $i = 1, 2, \ldots, 2s + t - \ell$, let

$$N_i = \bigcup_{j=1}^{k} \{x_{(i-1)k+j}\}$$

where addition is taken modulo $2s+t$, and join the vertex y_i to each vertex in the set N_i. Thus the edges between X and Y are distributed equitably among the vertices in X. By construction, the resulting graph is bipartite and each vertex in $X_1 = \{x_1, \ldots, x_{2s}\}$ has degree $k-1$, each vertex in $X_2 = \{x_{2s+1}, \ldots, x_{2s+t}\}$ has degree $k-2$, and each vertex in $Y = \{y_1, \ldots, y_{2s+t-\ell}\}$ has degree k. The $(3,2,4)$-graph so constructed from the $(3,2,4)$-triple is given in Figure 9.2(a). In Figure 9.2(b) we give a non-isomorphic $(3,2,4)$-graph. We remark that, in general, there are many non-isomorphic (k,s,t)-graphs associated with a given (k,s,t)-triple. However every (k,s,t)-graph has $4s+2t-2(s+t)/k$ vertices.

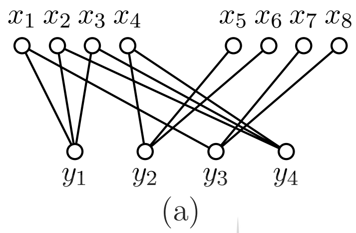


Figure 9.2: Non-isomorphic $(3,2,4)$-graphs.

The Family $\mathcal{G}$. Let $\mathcal{G}$ be the family of regular graphs (not necessarily connected) constructed as follows. Let G_1 be a (k,s,t)-graph. Let $U = \{u_1, v_1, u_2, v_2, \ldots, u_s, v_s\}$ be the set of $2s$ vertices in G_1 of degree $k-1$, let $W = \{w_1, w_2, \ldots, w_t\}$ be the set of t vertices in G_1 of degree $k-2$, and let $Z = \{z_1, z_2, \ldots, z_{2s+t-\ell}\}$ be the set of $2s+t-\ell$ vertices in G_1 of degree k. We remark that if $s=0$ then $U=\emptyset$, and if $t=0$ then $W=\emptyset$. If $s=0$, let $E_U=\emptyset$, while if $s \geq 1$, let $E_U = \{u_1v_1, \ldots, u_sv_s\}$. If $t=0$, let G be the k-regular graph obtained from G_1 by adding the edges from the set E_U. For $t \geq 1$, let G_2 and G_3 be disjoint $(k-1)$-regular graphs (not necessarily connected) of order t with $X = V(G_2) = \{x_1, x_2, \ldots, x_t\}$ and $Y = V(G_3) = \{y_1, y_2, \ldots, y_t\}$, and let $E_W = \bigcup_{i=1}^{t}\{x_iw_i, w_iy_i\}$. If $t \geq 1$, let G be the k-regular graph obtained from the disjoint union $G_1 \cup G_2 \cup G_3$ by adding the edges from the set $E_U \cup E_W$. Let $\mathcal{G}$ be the family of all graphs G thus constructed.

9.2 Main Results

Our aim is to use edge weighting functions on dominating sets to show that if we impose a regularity condition on the graph, then bounds on both its upper domination and upper total domination numbers can be greatly improved. We shall prove the following two results, proofs of which are provided in Sections 9.4.2 and 9.4.3, respectively.

Theorem 9.4 *For every regular graph G of order n with no isolates, $\Gamma(G) \le n/2$, with equality if and only if every component of G belongs to the family $\mathcal{B} \cup \mathcal{F}$.*

Theorem 9.5 *For every k-regular graph G of order n with no isolates, $\Gamma_t(G) \le n/(2 - \frac{1}{k})$, with equality if and only if $G \in \mathcal{G}$.*

9.3 Preliminary Observations

Let G be a graph, let D be a minimal dominating set in G with $u \in D$, and let T be a minimal total dominating set in G with $v \in T$. If $\text{ipn}[u, D] = \text{epn}[u, D] = \emptyset$, then $D \setminus \{u\}$ is a dominating set in G, contradicting the minimality of D. Similarly, if $\text{ipn}(v, T) = \text{epn}(v, T) = \emptyset$, then $T \setminus \{v\}$ is a total dominating set in G, contradicting the minimality of T. We therefore have the following two useful observations.

Observation 9.6 *If D is a minimal dominating set of a graph, then for every $u \in D$ we have* $\text{ipn}[u, D] \ne \emptyset$ *or* $\text{epn}[u, D] \ne \emptyset$.

Observation 9.7 *If T is a minimal total dominating set of a graph, then for every $v \in T$ we have* $\text{ipn}(v, T) \ne \emptyset$ *or* $\text{epn}(v, T) \ne \emptyset$.

We observe next that each graph in the family $\mathcal{B} \cup \mathcal{F}$ has upper domination at least one-half its order. To see this, consider first a graph $G \in \mathcal{B}$. Then, G is a connected bipartite regular graph. Let G have partite sets B_1 and B_2. We remark that $|B_1| = |B_2|$ and that B_1 is a dominating set in G. Furthermore, since $B_1 \setminus \{v\}$ does not dominate v for any $v \in B_1$, we have that B_1 is a minimal dominating set in G. Therefore, $\Gamma(G) \geq |B_1| = |V(G)|/2$. Consider next a graph $F \in \mathcal{F}$ that is constructed as described in Section 9.1. We remark that for such a graph F, the set $V(F_1)$ is a dominating set. Furthermore, since $V(F_1)\setminus\{u_i\}$ does not dominate v_i for any $i \in \{1, 2, \ldots, \ell\}$, we have that $V(F_1)$ is a minimal dominating set. Therefore, $\Gamma(F) \geq |V(F)|/2$. Hence we have the following observation.

Observation 9.8 *If $G \in \mathcal{B} \cup \mathcal{F}$ has order n, then $\Gamma(G) \geq n/2$.*

We now consider a graph $G \in \mathcal{G}$ that is constructed as described in Section 9.1. We remark that for such a graph G, the set $T = U \cup W \cup X$ is a total dominating set. Now, if $s > 0$ then $T \setminus \{u_i\}$ does not totally dominate v_i and $T \setminus \{v_i\}$ does not totally dominate u_i for any $i \in \{1, \ldots, s\}$. Further, if $t > 0$ then $T \setminus \{x_j\}$ does not totally dominate w_j and $T \setminus \{w_j\}$ does not totally dominate y_j for any $j \in \{1, \ldots, t\}$. Hence T is a minimal total dominating set. We remark further that $|T| = |U| + |W| + |X| = 2s + 2t$ and, since $|V(G)| = |V(G_1)| + |V(G_2)| + |V(G_3)|$, we have $|V(G)| = 4s + 2t - 2(s+t)/k + 2t = (2s + 2t)(2 - \frac{1}{k})$. We conclude that $\Gamma_t(G) \geq |T| = |V(G)|/(2 - \frac{1}{k})$. Hence we have the following observation.

Observation 9.9 *If $G \in \mathcal{G}$ is a k-regular graph of order n, then $\Gamma_t(G) \geq n/(2 - \frac{1}{k})$.*

9.4 Proof of Main Results

In order to prove our main results, we first define an edge weight function on a dominating set in a graph.

9.4.1 An Edge Weight Function on a Dominating Set

For disjoint subsets X and Y of vertices in a graph $G = (V, E)$, we denote the set of edges between X and Y by $[X, Y]$. Let S be a dominating set of G. The *edge weight function* of $S \subseteq V$ is defined to be the function $\psi_S : E \to [0, 1]$ that assigns to each edge in $G[S]$ and each edge in $G[V \setminus S]$ a weight of 0 and that assigns to each edge in $[S, V \setminus S]$ a weight in $(0, 1]$ in such a way that for each vertex $v \in V \setminus S$, the weight 1 is shared among the edges joining v to S. Thus if e is an edge joining $v \in V \setminus S$ to S, then $\psi(e) = 1/d_S(v)$, where we recall that $d_S(v)$ denotes the number of vertices in S adjacent to v. Thus since S is a dominating set in G, for each $v \in V \setminus S$, the sum of the weights of the edges incident with v is 1. Further, since each edge in $[S, V \setminus S]$ is incident with exactly one vertex in $V \setminus S$ we have that

$$\sum_{e \in [S, V \setminus S]} \psi_S(e) = \sum_{v \in V \setminus S} \left(\sum_{e \in [\{v\}, S]} \psi_S(e) \right) = \sum_{v \in V \setminus S} 1 = |V \setminus S| = n - |S|. \tag{9.1}$$

Next we define the *vertex weight function* of S, denoted ϕ_S, that assigns to each vertex $v \in S$ the sum of the weights of the edges incident with v. Since every edge in $G[S]$ has weight 0, we have that

$$\phi_S(v) = \sum_{e \in [\{v\}, V \setminus \{v\}]} \psi_S(e) = \sum_{e \in [\{v\}, V \setminus S]} \psi_S(e).$$

Since each edge in $[S, V \setminus S]$ is incident with exactly one vertex in S we have the following equation.

$$\sum_{e \in [S, V \setminus S]} \psi_S(e) = \sum_{v \in S} \left(\sum_{e \in [\{v\}, V \setminus S]} \psi_S(e) \right) = \sum_{v \in S} \phi_S(v). \tag{9.2}$$

Finally, we define the *vertex weight sum* of S, denoted $\xi(S)$, to be the sum over all vertices in S of the weights assigned by ϕ_S; that is,

$$\xi(S) = \sum_{v \in S} \phi_S(v).$$

Hence, from Equations (9.1) and (9.2), the following equation holds for every dominating set S in the graph G.

$$\xi(S) = n - |S|. \tag{9.3}$$

We note that a related concept of vertex weights was used by Slater [86] in his introductory paper on single-fault-tolerant locating-dominating sets in infinite grids. In his paper, however, a total weight of n is distributed amongst the vertices in a given dominating set S rather than a total weight of $n - |S|$.

9.4.2 Proof of Theorem 9.4

We are now in a position to prove our upper domination result, namely Theorem 9.4. Recall the statement of Theorem 9.4.

Theorem 9.4 *For every regular graph G of order n with no isolates, $\Gamma(G) \le n/2$, with equality if and only if every component of G belongs to the family $\mathcal{B} \cup \mathcal{F}$.*

Proof. Let G be a k-regular graph on n vertices where $k \ge 1$ and let D be a $\Gamma(G)$-set. We use the edge weight function ψ_D and vertex weight function ϕ_D to count the number of vertices in D relative to n. Recall that if e is an edge joining $v \in V \setminus D$ to D, then $\psi_D(e) = 1/d_D(v)$. Thus, $\frac{1}{k} \le \psi_D(e) \le 1$ for every edge $e \in [D, V \setminus D]$.

We show that $\phi_D(v) \ge 1$ for each $v \in D$. Let A be the set of isolated vertices in $G[D]$ and let $B = D \setminus A$. Each vertex $a \in A$ is joined by k edges to $V \setminus D$. Thus since $\psi_D(e) \ge \frac{1}{k}$ for every edge joining D to $V \setminus D$, we have $\phi_D(a) \ge k(\frac{1}{k}) = 1$ for each $a \in A$. For every vertex $b \in B$, we have that b is not an isolated vertex in $G[D]$, and so $\text{ipn}[b, D] \neq \{b\}$. Therefore, since $\text{ipn}[v, D] \in \{\emptyset, \{v\}\}$ for every $v \in D$, we must have $\text{ipn}[b, D] = \emptyset$. Hence, by Observation 9.6, we have $\text{epn}[b, D] \neq \emptyset$. But every edge that joins b to a vertex in $\text{epn}[b, D]$ is assigned weight 1 under the function ψ_D, and so we have $\phi_D(b) \geq 1$ for each $b \in B$. Thus, $\phi_D(v) \geq 1$ for each $v \in D$, and so $\xi(D) \geq |D|$. Recall that $n - |D| = \xi(D)$, by Equation (9.3), and so $n - |D| \geq |D|$. Thus, $\Gamma(G) = |D| \leq n/2$. This establishes the desired upper bound.

Next we characterize the regular graphs with no isolated vertex and with upper domination number exactly one-half their order. If such a graph is disconnected, then each of its components is a regular graph with no isolated vertex and with upper domination number exactly one-half its order. Therefore without loss of generality, we restrict our attention to connected regular graphs.

If $G \in \mathcal{B} \cup \mathcal{F}$ has order n, then by Observation 9.8, $\Gamma(G) \geq n/2$. As shown earlier, every regular graph with no isolated vertex has upper domination number at most one-half its order. In particular, $\Gamma(G) \leq n/2$. Consequently, $\Gamma(G) = n/2$.

Conversely, suppose that $G = (V, E)$ is a connected k-regular graph on n vertices where $k \geq 1$ satisfying $\Gamma(G) = n/2$. We show that $G \in \mathcal{B} \cup \mathcal{F}$. If $k = 1$, then $G = K_2 \in \mathcal{B}$. Hence we may assume that $k \geq 2$. Let D be a $\Gamma(G)$-set and let $\overline{D} = V \setminus D$. We again use the edge weight function ψ_D and vertex weight function ϕ_D to count the number of vertices in D relative to n. As shown in our earlier proof which establishes the upper bound of $n/2$, we have $\phi_D(v) \geq 1$ for each $v \in D$, and so $\xi(D) \geq |D|$. If $\phi_D(v) > 1$ for some vertex $v \in D$, then $\xi(D) > |D|$, and so, by Equation (9.3), we have $n - |D| = \xi(D) > |D|$. But then $\Gamma(G) = |D| < n/2$, a contradiction. Hence, $\phi_D(v) = 1$ for every vertex $v \in D$.

Let $D_1 \subseteq D$ such that if $v \in D_1$, then v is isolated in $G[D]$ and every vertex in $N_G(v)$ is isolated in $G[\overline{D}]$. Let $D_2 \subseteq D$ such that every vertex in D_2 has precisely one neighbor in $\overline{D}$ and this neighbor is a D-external private neighbor. Since $k \geq 2$, we note that

$D_1 \cap D_2 = \emptyset$. We proceed further with the following three claims.

Claim A *Every isolated vertex in $G[D]$ belongs to the set D_1, while every non-isolated vertex in $G[D]$ belongs to the set D_2.*

Proof. Let $v \in D$ and let $N_G(v) = \{v_1, v_2, \ldots, v_k\}$. On the one hand, suppose that v is isolated in $G[D]$. Then, $N_G(v) = \{v_1, v_2, \ldots, v_k\} \subseteq \overline{D}$. Then since $\psi_D(e) \geq \frac{1}{k}$ for every edge $e \in [D, \overline{D}]$, we have that

$$1 = \phi_D(v) = \sum_{i=1}^{k} \psi_D(vv_i) \geq \sum_{i=1}^{k} (\frac{1}{k}) = 1.$$

Hence we must have equality throughout the above inequality chain, implying that $\psi_D(vv_i) = \frac{1}{k}$ for all $i \in \{1, 2, \ldots, k\}$. But then for all $i \in \{1, 2, \ldots, k\}$, we have $|N_G(v_i) \cap D| = k$, and so v_i is isolated in $G[\overline{D}]$. Therefore, $v \in D_1$.

Suppose, on the other hand, that v is not isolated in $G[D]$. Then, $\text{ipn}[v, D] \neq \{v\}$ and consequently, $\text{ipn}[v, D] = \emptyset$. Hence, by Observation 9.6, we have that $\text{epn}[v, D] \neq \emptyset$. Renaming vertices if necessary, we may assume $v_1 \in \text{epn}[v, D]$, and so $\psi_D(vv_1) = 1$. Thus since $\psi_D(e) \geq \frac{1}{k}$ for every edge $e \in [D, \overline{D}]$, we have that

$$1 = \phi_D(v) = \sum_{i=1}^{k} \psi_D(vv_i) \geq 1 + \sum_{i=2}^{k} \psi_D(vv_i) \geq 1.$$

Hence we must have equality throughout the above inequality chain, implying that $\psi_D(vv_i) = 0$ for all $i \in \{2, \ldots, k\}$. This in turn implies that $v_i \in D$ for all $i \in \{2, \ldots, k\}$ and therefore that $v \in D_2$. $\square$

Claim B $D = D_1 \cup D_2$.

Proof. By Claim A, we have that every vertex $v \in D$ belongs to either the set D_1 or the set D_2, implying that $D \subseteq D_1 \cup D_2$. By definition of the sets D_1 and D_2, we have that $D_1 \cup D_2 \subseteq D$. Consequently, $D = D_1 \cup D_2$. □

Claim C *Either $D = D_1$ or $D = D_2$.*

Proof. By Claim B, we have that $D = D_1 \cup D_2$. As observed earlier, $D_1 \cap D_2 = \emptyset$. Suppose, for the sake of contradiction, that $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. Let $w \in D_1$ and $x \in D_2$ and consider a shortest w-x path in G, say $y_1y_2 \ldots y_\ell$ where $w = y_1$ and $x = y_\ell$. Let i be the largest index in $\{1, 2, \ldots, \ell - 1\}$ such that $y_i \in D_1$ (possibly, $i = 1$). By the definition of the set D_1, every neighbor of y_i is an isolated vertex in $G[\overline{D}]$. In particular, y_{i+1} is an isolated vertex in $G[\overline{D}]$, and so $y_{i+2} \in D$. If $y_{i+2} \in D_2$, then by the definition of the set D_2, the vertex y_{i+2} has precisely one neighbor in $\overline{D}$ and this neighbor is a D-external private neighbor. However since y_{i+1} is a neighbor of y_{i+2} in $\overline{D}$, this unique neighbor of y_{i+2} in $\overline{D}$ must be y_{i+1}. However, y_{i+1} is a common neighbor of at least two vertices in D, namely y_i and y_{i+2}, and therefore is not a D-external private neighbor, a contradiction. Hence, $y_{i+2} \in D_1$. But then this contradicts our choice of i. Therefore, either $D = D_1$ or $D = D_2$. □

We now return to the proof of Theorem 9.4. By Claim C, either $D = D_1$ or $D = D_2$. If $D = D_1$, then by the definition of D_1, every vertex in D is isolated in $G[D]$ and every vertex in $\overline{D}$ is isolated in $G[\overline{D}]$. Therefore, G is a regular bipartite graph with partite sets D and $\overline{D}$, and so $G \in \mathcal{B}$ as desired. Hence we may assume that $D = D_2$. Let $D = \{z_1, z_2, \ldots, z_{n/2}\}$. For every $i \in \{1, 2, \ldots, n/2\}$ we have, by the definition of D_2, that z_i has exactly one neighbor in $\overline{D}$ and this neighbor is a D-external private neighbor. For each z_i, let z_i' be this unique D-external private neighbor. We note that $\overline{D} = \{z_1', z_2', \ldots, z_{n/2}'\}$. Furthermore, $G[D]$ and $G[\overline{D}]$ are disjoint $(k-1)$-regular graphs and G is the graph obtained from the disjoint union $G[D] \cup G[\overline{D}]$ by joining z_i to z_i' for each $i \in \{1, 2, \ldots, n/2\}$. But this is precisely the definition of a graph in the family $\mathcal{F}$. We conclude that $G \in \mathcal{F}$, as desired. □

9.4.3 Proof of Theorem 9.5

We are now in a position to prove our upper domination result, namely Theorem 9.5. Recall the statement of Theorem 9.5.

Theorem 9.5 *For every k-regular graph G of order n with no isolates, $\Gamma_t(G) \le n/(2-\frac{1}{k})$, with equality if and only if $G \in \mathcal{G}$.*

Proof. Let G be a k-regular graph on n vertices where $k \ge 1$ and let T be a $\Gamma_t(G)$-set. We use the edge weight function ψ_T and vertex weight function ϕ_T to count the number of vertices in T relative to n. Recall that if $e \in [T, V \setminus T]$, then $\psi_T(e) = 1/d_T(v)$, and so $\frac{1}{k} \le \psi_T(e) \le 1$.

We show that, on average, $\phi_T(v) \ge 1 - \frac{1}{k}$ for each vertex $v \in T$. Let $A = \{v \in V \mid \text{ipn}(v,T) \ne \emptyset\}$ and let $B = T \setminus A$. By Observation 9.7, $\text{epn}(v,T) \ne \emptyset$ or $\text{ipn}(v,T) \ne \emptyset$ for each vertex $v \in T$, and so for each $v \in B$, we have $\text{epn}(v,T) \ne \emptyset$. For $X \in \{A, B\}$, let $X_1 = \{v \in X \mid v \in \text{ipn}(u,T) \text{ for some } u \in T\}$ and let $X_2 = X \setminus X_1$. We remark that A_1, A_2, B_1 and B_2 are pairwise disjoint and that $T = A_1 \cup A_2 \cup B_1 \cup B_2$. We consider the weight assigned by the function ϕ_T to vertices from each of these sets in turn.

If $v \in A_1$, then $v \in \text{ipn}(u,T)$ for some $u \in T$. Hence, v has exactly $k-1$ neighbors in $V \setminus T$ and we have $\phi_T(v) \ge (k-1)(\frac{1}{k}) = 1 - \frac{1}{k}$. If $v \in A_2$, then possibly v has no neighbors in $V \setminus T$ and we have $\phi_T(v) \ge 0$. If $v \in B_1$, then $v \in \text{ipn}(u,T)$ for some $u \in T$ and hence v has exactly $k-1$ neighbors in $V \setminus T$. Furthermore, $\text{epn}(v,T) \ne \emptyset$. Therefore under the function ψ_T, at least one edge joining v to $V \setminus T$ is assigned weight 1 and each of the remaining $k-2$ edges joining v to $V \setminus T$ is assigned weight at least $\frac{1}{k}$. Thus, $\phi_T(v) \ge 1 + (k-2)(\frac{1}{k}) = 2(1 - \frac{1}{k})$. Finally if $v \in B_2$, then $\text{epn}(v,T) \ne \emptyset$ and so at least one edge incident with v has weight 1. Thus, $\phi_T(v) \ge 1 > 1 - \frac{1}{k}$. Summing the weights over all vertices in T we therefore obtain the following inequality.

$$\xi(T) = \sum_{v \in T} \phi_T(v) \geq \left(1 - \frac{1}{k}\right) (|A_1| + 2|B_1| + |B_2|). \tag{9.4}$$

We now show that $|B_1| \geq |A_2|$. Let $t = |A_2|$. If $t = 0$, the result is immediate. Hence we may assume that $t \geq 1$. Let $A_2 = \{a_1, \ldots, a_t\}$. For $i \in \{1, \ldots, t\}$ we remark that $\text{ipn}(a_i, T) \neq \emptyset$ and we let $b_i \in \text{ipn}(a_i, T)$. Since a_i is the unique neighbor of b_i in T, we have $b_i \neq b_j$ for $i \neq j$. For any $i \in \{1, \ldots, t\}$, if $b_i \in A$, then $\text{ipn}(b_i, T) \neq \emptyset$, and so necessarily $a_i \in \text{ipn}(b_i, T)$, contradicting the fact that $a_i \in A_2$. Hence, $b_i \in B$ for all $i \in \{1, \ldots, t\}$ and since $b_i \in \text{ipn}(a_i, T)$ we have that $b_i \in B_1$. But now $\{b_1, \ldots, b_t\} \subseteq B_1$, and so $|B_1| \geq t = |A_2|$, as desired. Therefore, $|A_1| + 2|B_1| + |B_2| \geq |A_1| + |A_2| + |B_1| + |B_2| = |T|$ and so, from Inequality (9.4), we get

$$\xi(T) \geq \left(1 - \frac{1}{k}\right) |T|. \tag{9.5}$$

By Equation (9.3), $n - |T| = \xi(T)$, and so by Inequality (9.5) we have $n - |T| \geq (1 - \frac{1}{k})|T|$. Thus, $\Gamma_t(G) = |T| \leq n/(2 - \frac{1}{k})$. This establishes the desired upper bound.

Next we characterize the k-regular graphs with no isolated vertex and with upper total domination number exactly $1/(2 - \frac{1}{k})$ times their order. If $G \in \mathcal{G}$ is a k-regular graph of order n, then by Observation 9.9, $\Gamma_t(G) \geq n/(2 - \frac{1}{k})$. As shown earlier, every k-regular graph with no isolates has upper total domination number at most $1/(2 - \frac{1}{k})$ times its order. In particular, $\Gamma_t(G) \leq n/(2 - \frac{1}{k})$. Consequently, $\Gamma_t(G) = n/(2 - \frac{1}{k})$.

Conversely, suppose that $G = (V, E)$ is a k-regular graph of order n with no isolates such that $\Gamma_t(G) = n/(2 - \frac{1}{k})$. We show that $G \in \mathcal{G}$. If $k = 1$, then G comprises $n/2$ copies of K_2. Furthermore, the empty graph with vertex set $\{u_1, v_1, u_2, v_2, \ldots, u_{n/2}, v_{n/2}\}$ is a (k, s, t)-graph with $k = 1$, $s = n/2$ and $t = 0$, and G can be obtained from this graph by adding the edges $\{u_1v_1, u_2v_2, \ldots, u_sv_s\}$. Thus, $G \in \mathcal{G}$. We may therefore assume that

$k \geq 2$.

Let T be a $\Gamma_t(G)$-set and let $\overline{T} = V \setminus T$. We again use the edge weight function ψ_T and vertex weight function ϕ_T defined earlier to count the number of vertices in T relative to n. By Inequality (9.5), we have $\xi(T) \geq (1 - \frac{1}{k})|T|$. If $\xi(T) > (1 - \frac{1}{k})|T|$, then by Equation (9.3), we have $n - |T| > (1 - \frac{1}{k})|T|$. But then $\Gamma_t(G) = |T| < n/(2 - \frac{1}{k})$, a contradiction. Hence we must have equality in Inequality (9.5); that is,

$$\xi(T) = (1 - \frac{1}{k})|T|. \tag{9.6}$$

Let A_1, A_2, B_1 and B_2 be defined as before. Recall that if $v \in A_1$, then $\phi_T(v) \geq 1 - \frac{1}{k}$, if $v \in A_2$, then $\phi_T(v) \geq 0$, if $v \in B_1$, then $\phi_T(v) \geq 2(1 - \frac{1}{k})$, and if $v \in B_2$, then $\phi_T(v) \geq 1 = (1 - \frac{1}{k}) + \frac{1}{k}$. Furthermore, $T = A_1 \cup A_2 \cup B_1 \cup B_2$ and $|B_1| \geq |A_2|$. Using these facts, and summing the weights over all vertices in T, we have

$$\begin{aligned} \xi(T) &\geq \left(1 - \tfrac{1}{k}\right)(|A_1| + 2|B_1| + |B_2|) + \tfrac{1}{k}|B_2| \\ &\geq \left(1 - \tfrac{1}{k}\right)(|A_1| + |A_2| + |B_1| + |B_2|) + \tfrac{1}{k}|B_2| \\ &\geq \left(1 - \tfrac{1}{k}\right)|T| + \tfrac{1}{k}|B_2|. \end{aligned}$$

Thus if $|B_2| \neq 0$, then $\xi(T) > (1 - \frac{1}{k})|T|$, contradicting Equation (9.6). Hence, $B_2 = \emptyset$ and $T = A_1 \cup A_2 \cup B_1$. Now, if $|B_1| > |A_2|$ and we again sum the weights over all vertices in T, we have

$$\begin{aligned} \xi(T) &\geq (1 - \tfrac{1}{k})(|A_1| + 2|B_1|) \\ &> (1 - \tfrac{1}{k})(|A_1| + |A_2| + |B_1|) \\ &= (1 - \tfrac{1}{k})|T|, \end{aligned}$$

which, again, contradicts Equation (9.6). Hence, $|A_2| = |B_1|$.

We now define the function $\phi' : T \to \{1 - \frac{1}{k}, 0, 2(1 - \frac{1}{k})\}$ so that $\phi' : A_1 \to \{1 - \frac{1}{k}\}$,

$\phi' : A_2 \to \{0\}$, and $\phi' : B_1 \to \{2(1-\frac{1}{k})\}$. We remark that for all $v \in T$ we have $\phi_T(v) \geq \phi'(v)$, and so

$$\begin{aligned}
\xi(T) &= \sum_{v \in T} \phi_T(v) \\
&\geq \sum_{v \in T} \phi'(v) \\
&= (1-\tfrac{1}{k})|A_1| + 0|A_2| + 2(1-\tfrac{1}{k})|B_1| \\
&= (1-\tfrac{1}{k})(|A_1| + 2|B_1|) \\
&= (1-\tfrac{1}{k})(|A_1| + |A_2| + |B_1|) \\
&= (1-\tfrac{1}{k})|T|.
\end{aligned}$$

But by Equation (9.6), $\xi(T) = (1-\frac{1}{k})|T|$. Hence we must have equality throughout the above inequality chain. Thus,

$$\sum_{v \in T} \phi_T(v) = \sum_{v \in T} \phi'(v),$$

and therefore, since $\phi_T(v) \geq \phi'(v)$ for every $v \in T$, we must have $\phi_T(v) = \phi'(v)$ for every $v \in T$. Thus, under the function ϕ_T, every vertex in A_1 is assigned a weight of exactly $1-\frac{1}{k}$, every vertex in A_2 is assigned a weight of zero, and every vertex in B_2 is assigned a weight of exactly $2(1-\frac{1}{k})$.

Let $Y \subseteq \overline{T}$ such that $y \in Y$ if and only if $y \in \text{epn}(v,T)$ for some $v \in T$. Let Z be the set of all vertices in $\overline{T}$ which are isolated in $G[\overline{T}]$. We now examine various properties of vertices in the sets A_1, A_2, B_1, Y and Z, respectively, in the following series of claims. We remark that some of these sets are possibly empty.

Claim I *Every vertex in A_1 has exactly one neighbor in A_1 and exactly $k-1$ neighbors in Z.*

Proof. If $A_1 = \emptyset$, then the result is vacuously true and so we may assume $A_1 \neq \emptyset$. Let $u \in A_1$. By the definition of A_1, we have that $u \in \mathrm{ipn}(v, T)$ for some $v \in T$ and that $\mathrm{ipn}(u, T) \neq \emptyset$. Now, since v is the only neighbor of u in T and since $\mathrm{ipn}(u, T) \subseteq T$, we have $v \in \mathrm{ipn}(u, T)$, and so by definition $v \in A_1$. Furthermore, since $v \in \mathrm{ipn}(u, T)$ and $u \in \mathrm{ipn}(v, T)$, we have $N_G(\{u, v\}) \setminus \{u, v\} \subseteq \overline{T}$. We remark therefore that every component of $G[A_1]$ is isomorphic to K_2.

Let $N_G(u) = \{v, z_1, z_2, \ldots, z_{k-1}\}$ and note then that $\{z_1, z_2, \ldots, z_{k-1}\} \subseteq \overline{T}$. Let e be the edge joining u to v and for each $i \in \{1, \ldots, k-1\}$, let e_i be the edge joining u to z_i. Recall that $\psi_T(e) = 0$ and $\psi_T(e_i) \geq \frac{1}{k}$. Hence,

$$\phi_T(u) = \psi_T(e) + \sum_{i=1}^{k-1} \psi_T(e_i) \geq (k-1)\frac{1}{k} = 1 - \frac{1}{k}.$$

But $\phi_T(u) = 1 - \frac{1}{k}$, and we so we must have equality throughout the above inequality chain. Hence,

$$\sum_{i=1}^{k-1} \psi_T(e_i) = (k-1)\frac{1}{k},$$

and therefore, since $\psi_T(e_i) \geq \frac{1}{k}$ for each $i \in \{1, \ldots, k-1\}$, we deduce that $\psi_T(e_i) = \frac{1}{k}$. By the definition of the function ψ_T, this implies that each z_i has exactly k neighbors in T. Therefore since $d_G(z_i) = k$, we have $N_G(z_i) \subseteq T$ for each $i \in \{1, \ldots, k-1\}$. Since u was chosen to be an arbitrary vertex in A_1, the desired result follows. $\Box$

Motivated by Claim I we let $s = |A_1|/2$ and, if $s > 0$, we let $A_1 = \{u_1, v_1, u_2, v_2, \ldots, u_s, v_s\}$, where u_i is joined to v_i in $G[A_1]$ for each $i \in \{1, \ldots, s\}$. Let $t = |A_2|$ and, if $t > 0$, let $A_2 = \{x_1, x_2, \ldots, x_t\}$. If $t > 0$ then for each $i \in \{1, \ldots, t\}$ we note that $\mathrm{ipn}(x_i, T) \neq \emptyset$ and we let $w_i \in \mathrm{ipn}(x_i, T)$. By definition of the sets A_1 and B_1, we have $w_i \in A_1 \cup B_1$. If $w_i \in A_1$, then by Claim I we must have $x_i \in A_1$, a contradiction. Hence, $w_i \in B_1$ for all $i \in \{1, \ldots, t\}$. Furthermore, since x_i is the unique neighbor of w_i in T, we have

$w_i \neq w_j$ for $i \neq j$. Thus, $\{w_1, w_2, \ldots, w_t\} \subseteq B_1$ and since $|B_1| = |A_2| = t$ we deduce that $B_1 = \{w_1, w_2, \ldots, w_t\}$.

Claim II *If $t > 0$, then $N_G(A_2) \subseteq T$ and $G[A_2]$ is a $(k-1)$-regular graph.*

Proof. Assume $t > 0$ and let $i \in \{1, \ldots, t\}$. Recall that $x_i \in A_2$ and $w_i \in \text{ipn}(x_i, T)$. Since every vertex in A_2 is assigned a weight of zero under the function ϕ_T, we have that $N_G(x_i) \subseteq T$. Let x_i' be a neighbor of x_i distinct from w_i. If $x_i' \in A_1$, then by Claim I we must have $x_i \in A_1$, a contradiction. If $x_i' \in B_1$, then $x_i' = w_j$ for some $j \in \{1, \ldots, t\} \setminus \{i\}$. But $w_j \in \text{ipn}(x_j, T)$, implying that $x_i = x_j$, contradicting the fact that $i \neq j$. Hence, $x_i' \in A_2$. This is true for every vertex $x_i \in A_2$ and for each of the $k-1$ neighbors of x_i different from w_i. Therefore the graph $G[A_2]$ is a $(k-1)$-regular graph. □

Claim III *Every vertex in B_1 has exactly one neighbor in A_2, exactly one neighbor in Y and exactly $k-2$ neighbors in Z.*

Proof. If $t = 0$, then the result is vacuously true and so we may assume $t > 0$. Let $i \in \{1, \ldots, t\}$ and consider the vertex $w_i \in B_1$. Recall that $w_i \in \text{ipn}(x_i, T)$, where $x_i \in A_2$. Further since $w_i \in B_1$, we have $\text{ipn}(w_i, T) = \emptyset$, and so $\text{epn}(w_i, T) \neq \emptyset$. Let $y \in \text{epn}(w_i, T)$ and note that $y \in Y$. Let $N_G(w_i) = \{x_i, y, z_1, z_2, \ldots, z_{k-2}\}$ and note that since $w_i \in \text{ipn}(x_i, T)$ we have $\{y, z_1, \ldots, z_{k-2}\} \subseteq \overline{T}$. Let e' be the edge joining w_i to x_i, let e be the edge joining w_i to y and, for each $j \in \{1, \ldots, k-2\}$, let e_j be the edge joining w_i to z_j. Recall that $\psi_T(e') = 0$, $\psi_T(e) = 1$ and $\psi_T(e_j) \geq \frac{1}{k}$. Hence,

$$\phi_T(w_i) = \psi_T(e') + \psi_T(e) + \sum_{j=1}^{k-2} \psi_T(e_j) \geq 1 + (k-2)\frac{1}{k} = 2(1 - \frac{1}{k}).$$

But $\phi_T(w_i) = 2(1 - \frac{1}{k})$, and so we must have equality throughout the above inequality chain. Hence,

$$\sum_{j=1}^{k-2} \psi_T(e_j) = (k-2)\frac{1}{k},$$

and therefore, since $\psi_T(e_j) \geq \frac{1}{k}$ for each $j \in \{1,\ldots,k-2\}$, we deduce that $\psi_T(e_j) = \frac{1}{k}$. By the definition of the function ψ_T, we have that each z_j has exactly k neighbors in T. Therefore since $d_G(z_j) = k$, we have $N_G(z_j) \subseteq T$ for each $j \in \{1,\ldots,k-2\}$, and so $z_j \in Z$. Since w_i was chosen to be an arbitrary vertex in B_1, the desired result follows. □

Motivated by Claim III, for $t > 0$ and for each $i \in \{1,\ldots,t\}$, we let y_i be the unique T-external private neighbor of w_i, and so $\text{epn}(w_i, T) = \{y_i\}$. Since each w_i is the unique neighbor of y_i in T, we have $y_i \neq y_j$ for $i \neq j$. We note that $\{y_1,\ldots,y_t\} \subseteq Y$.

Claim IV *If $t > 0$, then $Y = \{y_1,\ldots,y_t\}$ and $G[Y]$ is a $(k-1)$-regular graph.*

Proof. Let $y \in Y$. Then, $y \in \text{epn}(v,T)$ for some $v \in T$. By Claim I, $N_G(A_1) \subseteq A_1 \cup Z$, and so $v \notin A_1$. By Claim II, $N_G(A_2) \subseteq T$, and so $v \notin A_2$. Hence, $v \in B_1$, and so, by Claim III, we have that $\text{epn}(v,T) = \{y\}$, and so $v = w_i$ and $y = y_i$ for some $i \in \{1,\ldots,t\}$. Therefore, $Y = \{y_1,\ldots,y_t\}$, as desired.

Let $i \in \{1,\ldots,t\}$ and consider the vertex $y_i \in Y$. Let y_i' be a neighbor of y_i distinct from w_i. Then, $y_i' \in \overline{T}$. Since y_iy_i' is an edge in $G[\overline{T}]$, the vertex y_i' is not isolated in $G[\overline{T}]$, and so $y_i' \notin Z$. Since T is a total dominating set in G, there is a vertex $v \in T$ that is adjacent to y_i'. By Claim I, $v \notin A_1$ and by Claim II, $v \notin A_2$. Hence, $v \in B_1$. Since $y_i' \notin Z$, we have that $y_i' \in Y$ and that $\text{epn}(v,T) = \{y_i'\}$ by Claim III. Thus, $v = w_j$ and $y_i' = y_j$ for some $j \in \{1,\ldots,t\} \setminus \{i\}$. In particular, $y_i' \in Y$. This is true for every vertex $y_i \in Y$ and for each of the $k-1$ neighbors of y_i different from w_i. Therefore the graph $G[Y]$ is a $(k-1)$-regular graph. □

Claim V $\overline{T} = Y \cup Z$.

Proof. By construction, $Y \cup Z \subseteq \overline{T}$. It remains to show that $\overline{T} \subseteq Y \cup Z$. Let $u \in \overline{T}$. Since T is a total dominating set in G, there is a vertex $v \in T$ that is adjacent to u. By

Claim II, $N_G(A_2) \subseteq T$, and so $v \notin A_2$. If $v \in A_1$, then by Claim I, $v \in Z$. If $v \in B_1$, then by Claim III, $v \in Z$ or $v \in Y$. In both cases, $v \in Z \cup Y$, and the desired result follows. □

Let G_1 be the graph constructed from $G[A_1 \cup B_1 \cup Z]$ by removing the edges in $G[A_1]$; that is, by removing the edges $u_i v_i$ for all $i = \{1, \dots, s\}$.

Claim VI *G_1 is a (k, s, t)-graph.*

Proof. Recall that by our earlier assumption, $k \geq 2$. By definition of the set Z, we have that $N_G(Z) \subseteq T$ and by Claim II, we have that $N_G(A_2) \subseteq T$. Consequently, $N_G(Z) \subseteq A_1 \cup B_1$. In particular, for any $z \in Z$ we have $N_G(z) \subseteq A_1 \cup B_1$, and so $k = |N_G(z)| \leq |A_1 \cup B_1| = |A_1| + |B_1| = 2s + t$. Thus the condition $2s + t \geq k \geq 1$ is satisfied. Counting the edges joining Z to $A_1 \cup B_1$ in two ways, we get $k|Z| = |[A_1 \cup B_1, Z]| = (k-1)|A_1| + (k-2)|B_1| = (k-1)(2s) + (k-2)t$, or, equivalently, $|Z| = (2s+t) - 2(s+t)/k$. Since $|Z|$ is an integer, we have that $2(s+t) = \ell k$ for some positive integer ℓ (and so, $|Z| = 2s + t - \ell$). If $t > 0$, then by Claim II we have $G[A_2]$ is a $(k-1)$-regular graph, and so, by Observation 9.3, we have that $t \geq k$ where t is even whenever k is even. Hence the integers k, s and t form a (k, s, t)-triple.

By Claim I and Claim III and the fact that G_1 does not contain any edge in the set $\{u_1 v_1, \dots, u_s v_s\}$, we have that G_1 is a bipartite graph with partite sets $U = A_1 \cup B_1$ and Z such that $|U| = 2s + t$, $|A_1| = 2s$, $|B_1| = t$, $|Z| = 2s + t - \ell$, and for all $a \in A_1$, $b \in B_1$ and $z \in Z$ we have $d_{G_1}(a) = k - 1$, $d_{G_1}(b) = k - 2$, and $d_{G_1}(z) = k$. But this is precisely the definition of a (k, s, t)-graph and the desired result follows. □

By Claim VI, G_1 is a (k, s, t)-graph. Furthermore, $A_1 = \{u_1, v_1, u_2, v_2, \dots, u_s, v_s\}$ is the set of $2s$ vertices in G_1 of degree $k-1$, $B_1 = \{w_1, w_2, \dots, w_t\}$ is the set of t vertices in G_1 of degree $k-2$, and Z is the set of $s + t - \ell$ vertices in G_1 of degree k. Let $G_2 = G[A_2]$ and let $G_3 = G[Y]$. By Claims II and IV, both G_2 and G_3 are $(k-1)$-regular graphs.

Recall that $A_2 = V(G_2) = \{x_1, x_2, \ldots, x_t\}$ and $Y = V(G_3) = \{y_1, y_2, \ldots, y_t\}$. But now G can be reconstructed from the disjoint union $G_1 \cup G_2 \cup G_3$ by joining u_i to v_i for each $i \in \{1, \ldots, s\}$ and by joining x_j to w_j and w_j to y_j for each $j \in \{1, \ldots, t\}$. But this is precisely the definition of a graph in the family $\mathcal{G}$. Hence, $G \in \mathcal{G}$ as desired. this completes the proof of Theorem 9.5. $\square$

Chapter 10

Proof of a Conjecture

In the previous chapter we presented an edge weighting function and put it to use to prove upper bounds on the upper domination number and upper total domination number in regular graphs. In this chapter we use a modified version to answer a published conjecture on the total domination number in claw-free cubic graphs. Our proof assigns weights to the edges and uses discharging rules to determine the average sum of the edge weights incident to each vertex, and then uses counting arguments to establish the desired upper bound.

Recall that a graph is F-free if it does not contain F as an induced subgraph. In particular, if $F = K_{1,3}$, then we say that the graph is claw-free. An excellent survey of claw-free graphs has been written by Flandrin, Faudree, and Ryjáček [34]. Chudnovsky and Seymour have recently attracted considerable interest in claw-free graphs due to their excellent series of papers on this topic (see, [10, 11, 12, 13, 14]).

10.1 Known Results

A TDS S of a graph G is minimal if no proper subset of S is a TDS of G. The following property of minimal TDSs is established by Cockayne, Dawes, and Hedetniemi [16].

Proposition 10.1 ([16]) *If S is a minimal TDS of a connected graph G, then for each vertex $v \in S$, we have that $|\text{epn}(v, S)| \geq 1$ or $|\text{ipn}(v, S)| \geq 1$.*

The authors in [29] established the following upper bound on the total domination number of a connected claw-free graph with minimum degree at least two.

Theorem 10.2 ([29]) *If G is a connected claw-free graph of order n with $\delta(G) \geq 2$, then $\gamma_t(G) \leq (n + 2)/2$ with equality if and only if G is a cycle of length congruent to 2 modulo 4.*

Cockayne, Favaron, and Mynhardt [17] showed that every claw-free cubic graph has total domination number at most one-half its order.

Theorem 10.3 ([17]) *If G is a claw-free cubic graph of order n, then $\gamma_t(G) \leq n/2$.*

The result of Theorem 10.3 also follows from a more general result due to several authors, including Archdeacon et al. [2], Chvátal and McDiarmid [15], Thomassé and Yeo [96], and Tuza [97], that every graph with minimum degree at least three has total domination number one-half its order. The connected claw-free cubic graphs that achieve equality in the bound of Theorem 10.3 are characterized in [28]. This characterization also follows from a more general result in [69] in which connected graphs with minimum degree at least three and total domination number exactly one-half their order are characterized.

G_1 :

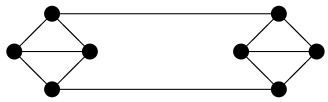


Figure 10.1: A claw-free cubic graph G_1 with $\gamma_t(G_1) = n/2$.

Theorem 10.4 ([28, 69]) *If G is a connected claw-free cubic graph of order n, then $\gamma_t(G) \leq n/2$ with equality if and only if $G = K_4$ or $G = G_1$ where G_1 is the graph shown in Figure 10.1.*

Favaron and Henning [30] showed that the upper bound on the total domination number of the graph G in Theorem 10.4 decreases from one-half its order to five-elevenths its order if the order is at least ten.

Theorem 10.5 ([30]) *If G is a connected claw-free cubic graph of order $n \geq 10$, then $\gamma_t(G) \leq 5n/11$.*

In [30], the authors believed that the bound of five-elevenths the order is not sharp and give the following conjecture.

Conjecture 10.6 ([30]) *Every connected claw-free cubic graph of order at least ten has total domination number at most four-ninths its order.*

10.2 Conjecture Proof

Our aim in this chapter is to prove Conjecture 10.6. The proof methods used in [30] to prove Theorem 10.5 do not suffice to prove Conjecture 10.6. Hence a proof of Conjecture 10.6, if true, requires completely different methods from those used to prove the result of Theorem 10.5. We prove the conjecture by assigning weights to edges and using discharging rules to determine the average sum of the edge weights incident to each vertex. Using counting, we then establish the desired upper bound. We shall prove:

Theorem 10.7 *If G is a connected claw-free cubic graph of order $n \geq 10$, then $\gamma_t(G) \leq 4n/9$.*

The bound of Theorem 10.7 is tight as may be seen by considering the connected claw-free cubic graphs F and H shown in Figure 10.2 with total domination number four-ninths their orders. In each case, an example of a minimum total dominating set is indicated by darkened vertices. We note that by a computer search these are the only examples on 18 vertices. Furthermore, although this bound is tight, higher order graphs achieving the bound are elusive, suggestion perhaps a slightly smaller bound that is asymptotically approached as the order increases. The aim of this chapter, however is to prove the conjecture published in [30].

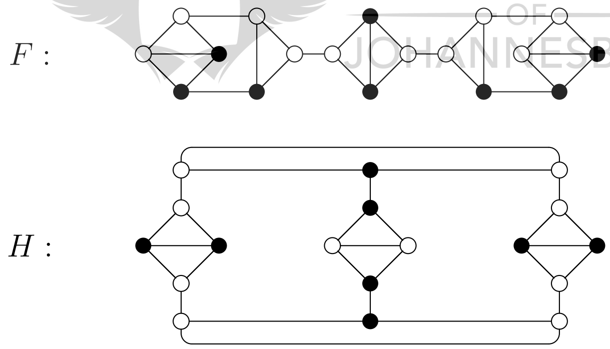


Figure 10.2: Claw-free cubic graphs with total domination numbers four-ninths their orders.

We shall proceed as follows. In Section 10.2.1, we carefully choose a minimum TDS S that, amongst other conditions, induces a subgraph with the minimum number of edges and, subject to this condition, minimizes the number of vertices not in S having all three neighbors in S. Basic properties of the TDS S are then discussed. In Section 10.2.2, we assign weights on all the edges that join S to $V \setminus S$ and weights to the components in $G[S]$. In Section 10.2.3, we show that the average weight of every vertex in S is at least $5/4$. From this we deduce that S contains at most four-ninths the vertices.

10.2.1 The Total Dominating Set S

Let $G = (V, E)$ be a connected claw-free cubic graph of order $n \geq 10$. Let $\overline{G} = (V, \overline{E})$ be the complement of G. For a subset $S \subseteq V$, let $\lambda(S)$ be the number of edges in $G[S]$ and let $\iota(S)$ be the number of isolated vertices in $G[V \setminus S]$. Let $\mathcal{P}$ be the set of P_2-components in $G[S]$ in which neither vertex has an S-external private neighbor in G. Let $\mathcal{P}_1 \subseteq \mathcal{P}$ be the subset of $\mathcal{P}$ consisting of those P_2-components whose vertices have exactly one common neighbor in G. Let $\mathcal{P}_2 \subseteq \mathcal{P}$ be the subset of $\mathcal{P}$ consisting of those P_2-components whose vertices have two common neighbors in G. Let $\mathcal{T}$ be the set of P_3-components in $G[S]$ such that no vertex in the component has two S-external private neighbors in G. Further, let $\beta(S) = |\mathcal{P}|$, $\xi(S) = |\mathcal{P}_2|$, $\varphi(S) = |\mathcal{P}_1|$ and $\alpha(S) = |\mathcal{T}|$.

Among all minimum TDS of G, let S be chosen so that

(1) $\lambda(S)$ *is minimized.*

(2) *Subject to* (1), $\iota(S)$ *is minimized.*

(3) *Subject to* (2), $\beta(S)$ *is minimized.*

(4) *Subject to* (3), $\xi(S)$ *is minimized.*

(5) *Subject to* (4), $\varphi(S)$ *is minimized.*

(6) *Subject to* (5), $\alpha(S)$ *is minimized.*

Necessarily, S is a minimal TDS of G. We define a weak partition (A, B) of the set S (where by *weak partition* we mean that some of the subsets may be empty) as follows. Let A consist of all vertices of S that have an S-external private neighbor. Let B consist of all vertices of S that have an S-internal private neighbor but no S-external private neighbor; that is,

$$A = \{ v \in S : |\mathrm{epn}(v,S)| \geq 1 \}$$
$$B = \{ v \in S : |\mathrm{epn}(v,S)| = 0 \text{ and } |\mathrm{ipn}(v,S)| \geq 1 \}.$$

By Proposition 10.1, every vertex in S belongs to A or B. For $X \in \{A, B\}$, we define an X-neighbor of a vertex $v \in V$ to be a neighbor of v that belongs to the set X. Further, we define a vertex to be an X-vertex if it belongs to X. Since G is a cubic graph, each vertex of A has either one or two S-external private neighbors. For $i = 1, 2$, let $A_i = \{v \in S : |\mathrm{epn}(v,S)| = i \}$. Thus, (A_1, A_2) is a weak partition of the set A.

We shall prove three key properties of the set S. We begin with the following property, a proof of which can be found in Subsection 10.2.1.

Property 1 *Every component in $G[S]$ is either a P_2-component or a P_3-component. Further, every P_3-component consists of a B-vertex with two A-neighbors.*

We call two vertices that induce a P_2-component of $G[S]$ a *pair* in S, while three vertices that induce a P_3-component of $G[S]$ we call a *triple* in S. Motivated by Property 1, we define a triple in S to be an *ABA-triple*. Further, we define a pair in S to be:

an *A-pair* if both vertices belong to A;

an *AB-pair* if one belongs to A and the other to B; and

a *B-pair* if they both belong to B.

If an A-pair is joined in G to an isolated vertex in $G[V \setminus S]$, then we call it a *weak A-pair*; otherwise, we call it a *strong A-pair*. If the A-vertex in an AB-pair belongs to A_2, then we call the AB-pair a *strong AB-pair*; otherwise, we call it a *weak AB-pair*. If at least one of the vertices in a B-pair is adjacent in G to an isolated vertex in $G[V \setminus S]$, then we call the B-pair a *weak B-pair*; otherwise, we call it a *strong B-pair.* If one of the A-vertices in an ABA-triple belongs to A_2, then we call the ABA-triple a *strong ABA-triple*; otherwise, we call it a *weak ABA-triple*.

Note that condition (3) of our choice of S minimizes the number of B-pairs in S. Furthermore, condition (4) minimizes the number of B-pairs in which the vertices have two common neighbors, condition (5) minimizes the number of B-pairs in which the vertices have exactly one common neighbor, and condition (6) minimizes the weak ABA-triples.

Our second key property of the set S is that two distinct B-pairs are at distance at least 3 apart. A proof of Property 2 can be found in Subsection 10.2.1.

Property 2 *Every two distinct B-pairs are at distance at least 3 apart.*

Our third key property of the set S is the following structural result about a subgraph of G that contains a vertex in $V \setminus S$ with all three neighbors in S. A proof of Property 3 can be found in Subsection 10.2.1. Throughout this chapter, whenever we give a diagram of a subgraph of G we indicate vertices of S by darkened vertices and vertices of $V \setminus S$ by circled vertices.

Property 3 *If $G[V \setminus S]$ contains an isolated vertex u, then two neighbors of u belong to an A-pair, while the third neighbor belongs to a B-pair. Furthermore, the vertex u belongs to the subgraph shown in Figure 10.3(a) or Figure 10.3(b), where the darkened vertices are labeled A or B depending on whether they belong to the set A or B, respectively.*

Proof of Property 1

Property 1 is an immediate consequence of Claims 1 and 2 presented in this section.

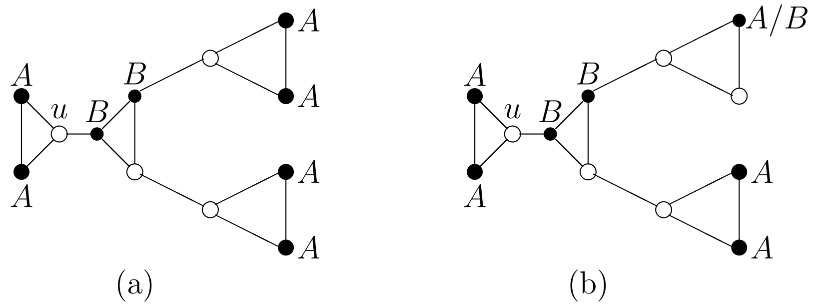


Figure 10.3: The two possible subgraphs of G containing u.

Claim 1 *Every B-vertex with at least two neighbors inside S has exactly two neighbors inside S, both of which are A-vertices with exactly one neighbor inside S.*

Proof. Let $v \in B$ have degree at least 2 in $G[S]$. Let $u \in \text{ipn}(v, S)$. Since v is the only vertex in S adjacent to u, the vertex u has two neighbors outside S. If u is a B-vertex, then $v \in \text{ipn}(u, S)$, contradicting the fact that v has at least two neighbors in S. Hence, by Proposition 10.1, u is an A-vertex. Let $\overline{u} \in \text{epn}(u, S)$ and let $T = (S \setminus \{v\}) \cup \{\overline{u}\}$. If T is a TDS of G, then T is a minimum TDS of G with $\lambda(T) < \lambda(S)$, contradicting our choice of S. Hence, T is not a TDS of G. Let w be a vertex not totally dominated by T. Since every vertex in $V \setminus S$ is totally dominated by $S \setminus \{v\} \subset T$, we have that $w \in S$ and $w \in \text{ipn}(v, S)$. Since u is totally dominated by $\overline{u} \in T$, the vertices u and w are distinct. Hence, $\{u, w\} \subseteq \text{ipn}(v, S)$. Thus both u and w are A-neighbors of v that are adjacent to no vertex of S other than to v. That is, both u and w have degree 1 in $G[S]$ and are adjacent only to v in S. By the claw-freeness of G, the third neighbor of v that is different from u and w, must lie outside S and be adjacent to at least one of u and w. $\square$

Claim 2 *Every A-vertex has exactly one neighbor inside S.*

Proof. Let $v \in A$ and suppose that vertex v has two neighbors, u and w, inside S. Let $\overline{v}$ be the neighbor of v outside S. Then, $\overline{v} \in \text{epn}(v, S)$ and, by the claw-freeness of G, $uw \in E$. Let x and y be the two neighbors of $\overline{v}$ different from v. Then, $\{x, y\} \subset V \setminus S$.

By the claw-freeness of G, the vertices x and y are adjacent. Thus each of x and y is adjacent to exactly one vertex of S. Let $T = (S \setminus \{v\}) \cup \{x\}$. Then, T is a minimum TDS of G with $\lambda(T) < \lambda(S)$, contradicting our choice of S. Hence, v has degree 1 in $G[S]$. □

Proof of Property 2

Suppose that two distinct B-pairs, $\{a, b\}$ and $\{c, d\}$, are at distance 2 apart. Renaming vertices, if necessary, we may assume that b and c have a common neighbor e (necessarily, $e \in V \setminus S$). Let $T = (S \setminus \{a, d\}) \cup \{e\}$. Since $|T| < |S|$, the set T is not a TDS of G. Thus there exists a vertex v totally dominated by S but not by T. Since T totally dominates the set $S \cup \{e\}$, we have that $v \in V \setminus S$ and $N(v) \cap S = \{a, d\}$. Let $N(v) = \{a, d, u\}$. Then, $u \in V \setminus S$. By the claw-freeness of G, we may assume that $au \in E$.

Suppose $bu \in E$. If $de \in E$, then $(S \setminus \{b, c\}) \cup \{v\}$ is a TDS of G, contradicting the minimality of S. Hence, $de \notin E$. By the claw-freeness of G, the vertices c, d and e have a common neighbor and $G = G_1$, where G_1 is the graph shown in Figure 10.1. But then $n = 8$, contradicting our assumption that $n \geq 10$. Hence, $bu \notin E$.

Let $N(b) = \{a, e, f\}$. By the claw-freeness of G, $ef \in E$. If $cf \in E$, then $du \in E$ and, again, $G = G_1$, a contradiction. Hence, $cf \notin E$. Let $N(c) = \{d, e, g\}$. By the claw-freeness of G, $dg \in E$. Since $\{a, b\}$ is a B-pair, there are vertices $f' \in S$ and $u' \in S$ such that $\{ff', uu'\} \subset E$. But then $(S \setminus \{a, b, d\}) \cup \{e, u\}$ is a TDS of G, contradicting the minimality of S. □

Proof of Property 3

Suppose that $G[V \setminus S]$ contains an isolated vertex u. Thus all three neighbors of u are in S. Let $N(u) = \{v, w, x\}$. By the claw-freeness of G, we may assume that $vw \in E$. We proceed with a series of claims that culminate in a contradiction.

Claim 3 *The vertex u does not belong to a $K_4 - e$.*

Proof. Suppose that u belongs to a subgraph G_u of G, where $G_u = K_4 - e$. Suppose u has degree 2 in G_u. Then, v and w are the two neighbors of u in G_u. Let y be the remaining vertex of G_u. By Property 1, $\{v, w\}$ is a B-pair and $y \notin S$. Let $N(y) = \{v, w, z\}$. If $z \in S$, then $(S \setminus \{v, w\}) \cup \{u\}$ is a TDS of G, contradicting the minimality of S. Hence, $z \notin S$.

Suppose that $xz \in E$. Let a be the common neighbor of x and z, and let b be the remaining neighbor of a. Let $N(b) = \{a, c, d\}$. By the claw-freeness of G, $cd \in E$. To totally dominate x, we have that $a \in S$. Thus, $x \in B$. If $a \in B$, then $(S \setminus \{a, v\}) \cup \{u\}$ is a TDS of G, contradicting the minimality of S. Hence, $a \in A$, and so $\{b, c, d\} \subset V \setminus S$. But then $T = (S \setminus \{x\}) \cup \{b\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. Hence, $xz \notin E$.

Let $N(x) = \{a, b, u\}$. By the claw-freeness of G, $ab \in E$. To totally dominate x, we may assume that $a \in S$. By Property 1, $b \notin S$. If $a \in B$, then $(S \setminus \{a, v\}) \cup \{u\}$ is a TDS of G, contradicting the minimality of S. Hence, $a \in A$. Let $\text{epn}(a, S) = \{a'\}$. If $a'b \in E$, then let $T = (S \setminus \{x\}) \cup \{b\}$, while if $a'b \notin E$, then let $T = (S \setminus \{x\}) \cup \{a'\}$. In both cases, T is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. Hence, u must be a degree-3 vertex in G_u.

We may assume that $wx \in E$. By Property 1, $\{v, w, x\}$ is an ABA-triple in S where $w \in B$. Let $\text{epn}(v, S) = \{v'\}$ and let $\text{epn}(x, S) = \{x'\}$. Suppose $v'x' \in E$. Let y be the common neighbor of v' and x' and let z denote the remaining neighbor of y. Let $N(z) = \{a, b, y\}$. Then, $ab \in E$ and $\{v', x', y\} \subset V \setminus S$. However, $T = (S \setminus \{x\}) \cup \{v'\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. Hence, $v'x' \notin E$.

Let $N(v') = \{a, b, v\}$. Since $\text{epn}(v, S) = \{v'\}$, we have that $\{a, b, v'\} \subset V \setminus S$ and by the claw-freeness of G, $ab \in E$. Let $N(a) = \{b, c, v'\}$. To totally dominate a, we have that $c \in S$, and so $c \neq x'$. But then $T = (S \setminus \{v\}) \cup \{a\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. □

By Claim 3, we may assume that $G[\{v, w, x\}] = K_2 \cup K_1$.

Claim 4 *The vertex u does not belong to a 4-cycle.*

Proof. Suppose that u belongs to a 4-cycle $uxywu$. By Property 1, $y \notin S$. Let $N(v) = \{u, w, v'\}$. Let z be the common neighbor of x and y. To totally dominate x, we have that $z \in S$. We note that $\{w, x\} \subseteq B$. Property 1 implies that $\{v, w\}$ is an AB-pair or a B-pair. If $v \in B$, then $(S \setminus \{v, x\}) \cup \{y\}$ is a TDS of G, contradicting the minimality of S. Hence, $v \in A$, and so $\mathrm{epn}(v, S) = \{v'\}$. But then $T = (S \setminus \{w\}) \cup \{v'\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. □

We now begin the final steps for the proof of Property 3. Let $N(x) = \{u, y, z\}$. Since G is claw-free, $yz \in E$. To totally dominate x, we may assume $y \in S$. Since $G[S]$ is K_3-free, $z \notin S$. Suppose $y \in A$. Then, $\mathrm{epn}(y, S) = \{y'\}$. If $y'z \in E$, let $T = (S \setminus \{x\}) \cup \{z\}$. If $y'z \notin E$, let $T = (S \setminus \{x\}) \cup \{y'\}$. In both cases, T is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. Hence, $y \in B$, and so $\{x, y\}$ is a B-pair in S. By Property 1, y is adjacent to neither v nor w, while by Claim 4, z is adjacent to neither v nor w.

If v and y have a common neighbor, then $v \in B$. Therefore, by Property 1 and Property 2, $w \in A$. Let $\mathrm{epn}(w, S) = \{w'\}$. Then, $T = (S \setminus \{v\}) \cup \{w'\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. Hence, v and y have no common neighbor.

Now, if $v \in B$, then $(S \setminus \{v, y\}) \cup \{u\}$ is a TDS of G, contradicting the minimality of S. Hence, $v \in A$. Similarly, $w \in A$. Hence, by Property 1, $\{v, w\}$ is an A-pair. Since this

A-pair is joined to the isolated vertex u in $G[V \setminus S]$, it is a weak A-pair. Thus we have shown that if $G[V \setminus S]$ contains an isolated vertex u, then two neighbors of u belong to a weak A-pair, while the third neighbor belongs to a B-pair.

Let $N(y) = \{x, y', z\}$ and let $N(z) = \{x, y, z'\}$ (possibly, $y' = z'$). If $z' \in S$, then $(S \setminus \{x, y\}) \cup \{z\}$ is a TDS of G, contradicting the minimality of S. Hence, $z' \notin S$. Since $\{x, y\}$ is a B-pair, $y' \notin S$. Suppose $y' = z'$. Let $N(z') = \{a, y, z\}$ and let $N(a) = \{b, c, z'\}$. By the claw-freeness of G, $bc \in E$. If $a \notin S$, then $T = (S \setminus \{x\}) \cup \{z\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. Hence, $a \in S$. To totally dominate a, we may assume $b \in S$. Since $G[S]$ is K_3-free, $c \notin S$. If $b \in B$, then $(S \setminus \{b, x\}) \cup \{z'\}$ is a TDS of G, contradicting the minimality of S. Hence, $b \in A$. Let $\text{epn}(b, S) = \{b'\}$. If $b'c \in E$, let $T = (S \setminus \{a, x\}) \cup \{c, z\}$. If $b'c \notin E$, let $T = (S \setminus \{a, x\}) \cup \{b', z\}$. In both cases, T is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. Hence, $y' \neq z'$.

If $y'z' \in E$, then $T = (S \setminus \{x\}) \cup \{z\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. Hence, $y'z' \notin E$. Let $N(y') = \{a, b, y\}$ and let $N(z') = \{c, d, z\}$. By the claw-freeness of G, $\{ab, cd\} \subset E$. Since $\{x, y\}$ is a B-pair, we may assume that $a \in S$.

If $\{c, d\} \not\subset S$, then $T = (S \setminus \{x\}) \cup \{z\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ but with $\iota(T) < \iota(S)$, contradicting our choice of S. Hence, $\{c, d\} \subset S$. If $c \in B$, then $(S \setminus \{c, x, y\}) \cup \{z, z'\}$ is a TDS of G, contradicting the minimality of S. Hence, $c \in A$. Similarly, $d \in A$. Hence, by Property 1, $\{c, d\}$ is an A-pair, and furthermore, a strong A-pair. If $b \notin S$, then u belongs to the subgraph shown in Figure 10.3(b). If $b \in S$, then y' is an isolated vertex in $G[V \setminus S]$. Thus, as established earlier, $\{a, b\}$ is a weak A-pair, and so u belongs to the subgraph shown in Figure 10.3(a). $\square$

10.2.2 Defining the Weights and Discharging Rules

The general strategy is to define a weight on all the edges that join S to $V \setminus S$. This weight is defined so that for each vertex in $V \setminus S$, the total weight of the edges incident with it sums to 1. Thus the total weight is exactly $|V \setminus S|$. At the same time, we sum the weights of the edges incident with each pair and each triple in S, and after a suitable redistribution using discharging rules, we show that each pair has associated with it a weight of at least 5/2 while each triple has associated with it a weight of at least 15/4. Thus on average each vertex in S, after the redistribution of weights, has a weight of at least 5/4. It follows that the total weight is at least $5|S|/4$. Thus, $n - |S| = |V \setminus S| \geq 5|S|/4$, whence $\gamma_t(G) \leq |S| \leq 4n/9$.

We define a function $\omega : [S, V \setminus S] \to [0, 1]$ that assigns to each edge in $[S, V \setminus S]$ a weight. The simplest idea for such a function is that, for each vertex x in $V \setminus S$, weight 1 is shared among the one, two or three edges joining x to S. Thus for each vertex $x \in V \setminus S$, the function ω assigns the weight $1/d$ to each edge from x to S where d is the number of edges from x to S. Hence if e is an edge joining $x \in V \setminus S$ to S, then $\omega(e) \in \{\frac{1}{3}, \frac{1}{2}, 1\}$ and the sum of the weights assigned to the edges joining x to S is 1. We now define a function ψ that assigns to each subset $S' \subseteq S$ the sum of the weights of the edges from S' to $V \setminus S$; that is,

$$\psi(S') = \sum_{e \in [S', V \setminus S]} \omega(e).$$

If $S' = S$, then $\psi(S)$ is the sum of the weights of all edges in $[S, V \setminus S]$ (namely, $|V \setminus S|$). Using Property 1 and Property 3, the following observation follows readily.

Observation 10.8 *Let $S' \subseteq S$. Then the following properties hold.*

(a) *If S' is a weak A-pair, then $\psi(S') = \frac{8}{3} = 2\left(\frac{5}{4}\right) + \frac{1}{6}$.*

(b) *If S' is a strong A-pair, then $\psi(S') \geq 3 = 2\left(\frac{5}{4}\right) + \frac{1}{2}$.*

(c) *If S' is a weak AB-pair, then $\psi(S') = \frac{5}{2} = 2\left(\frac{5}{4}\right)$.*

(d) *If S' is a strong AB-pair, then* $\psi(S') = 3 = 2\left(\frac{5}{4}\right) + \frac{1}{2}$.

(e) *If S' is a weak B-pair, then* $\psi(S') = \frac{5}{3} = 2\left(\frac{5}{4}\right) - \frac{5}{6}$ *or* $\psi(S') = \frac{11}{6} = 2\left(\frac{5}{4}\right) - \frac{2}{3}$.

(f) *If S' is a strong B-pair, then* $\psi(S') = 2 = 2\left(\frac{5}{4}\right) - \frac{1}{2}$.

(g) *If S' is a weak ABA-triple, then* $\psi(S') = \frac{7}{2} = 3\left(\frac{5}{4}\right) - \frac{1}{4}$.

(h) *If S' is a strong ABA-triple, then* $\psi(S') = 4 = 3\left(\frac{5}{4}\right) + \frac{1}{4}$.

Our aim is for every pair S' in S to have weight $\psi(S') \geq 5/2$ and for every triple S' in S to have weight $\psi(S') \geq 15/4$. So the next step is to redistribute the excess from A-pairs, strong AB-pairs and strong ABA-triples to boost the weight of B-pairs and weak ABA-triples. This redistribution is done by a set of **discharging rules**. These eleven discharging rules are illustrated in Figure 10.4.

Rule 1. If there is a weak A-pair with a common neighbor that is adjacent to a vertex in a B-pair, then discharge a weight of $\frac{1}{6}$ from the weak A-pair to the B-pair.

Rule 2. If there is a strong A-pair with a common neighbor that is adjacent to a common neighbor of a B-pair or if each vertex in a strong A-pair has a common neighbor with one of the vertices of a single B-pair, then discharge a weight of $\frac{1}{2}$ from the strong A-pair to the B-pair.

Rule 3. If there is a strong AB-pair that has two common neighbors with a B-pair, then discharge a weight of $\frac{1}{2}$ from the strong AB-pair to the B-pair.

Rule 4. If there is a strong A-pair with a common neighbor that is at distance 2 from an AB-pair that belongs to a $K_4 - e$ and this AB-pair is itself at distance 2 from a B-pair, then discharge a weight of $\frac{1}{2}$ from the strong A-pair to the B-pair.

Rule 5. If the common neighbor of one of the A-vertices and the B-vertex in a strong ABA-triple is at distance 2 from an AB-pair that belongs to a $K_4 - e$ and this AB-pair is itself at distance 2 from a B-pair, then discharge a weight of $\frac{1}{4}$ from the strong ABA-triple to the B-pair.

Rule 6. If one of the vertices in an A-pair is in A_2 and the other has a common neighbor with a B-pair, then discharge a weight of $\frac{1}{2}$ from the A-pair to the B-pair. We note that the A-pair is necessarily a strong A-pair.

Rule 7. If there is a strong A-pair with a common neighbor that is at distance 2 from the A-vertex of a strong AB-pair and this AB-pair is itself at distance 2 from a B-pair which has exactly one common neighbor with the AB-pair, then discharge a weight of $\frac{1}{2}$ from the strong A-pair to the AB-pair. Discharge an additional weight of $\frac{1}{2}$ from the strong AB-pair to the B-pair and a final weight of $\frac{1}{2}$ from the strong AB-pair to the other pair or triple at distance 2 from the AB-pair.

Rule 8. If the common neighbor of one of the A-vertices and the B-vertex in a strong ABA-triple and the common neighbor of one of the A-vertices and the B-vertex in another strong ABA-triple are both at distance 2 from the A-vertex of a strong AB-pair and this AB-pair is itself at distance 2 from a B-pair which has exactly one common neighbor with the AB-pair, then discharge a weight of $\frac{1}{4}$ from each of the strong ABA-triples to the AB-pair. Discharge an additional weight of $\frac{1}{2}$ from the strong AB-pair to the B-pair and a final weight of $\frac{1}{2}$ from the strong AB-pair to the other pair or triple at distance 2 from the AB-pair.

Rule 9. If there is a strong A-pair with a common neighbor that is at distance 2 from an AB-pair that belongs to a $K_4 - e$ and this AB-pair is itself at distance 2 from a weak ABA-triple whose S-external private neighborhood set contains two adjacent vertices, then discharge a weight of $\frac{1}{4}$ from the strong A-pair to the weak ABA-triple.

Rule 10. If the common neighbor of one of the A-vertices and the B-vertex in a strong ABA-triple is at distance 2 from an AB-pair that belongs to a $K_4 - e$ and this AB-pair is itself at distance 2 from a weak ABA-triple whose S-external private neighborhood set contains two adjacent vertices, then discharge a weight of $\frac{1}{4}$ from the strong ABA-triple to the weak ABA-triple.

Rule 11. If there is a strong A-pair with a common neighbor that is at distance 1 from a $K_4 - e$ that contains the B-vertex of a weak ABA-triple, then discharge a weight of $\frac{1}{4}$ from the strong A-pair to the weak ABA-triple.

Let ζ be the resulting function obtained from ψ by discharging the weights according to the discharging rules defined above. We remark that the only possible pairs or triples from which weights are discharged are A-pairs, strong AB-pairs, or strong ABA-triples and that there is at most one discharge from each such pair or triple. The latter remark is made apparent by the fact that each discharge moves in a unique direction; that is, away from any external private neighbors of a pair or triple. We note further that the purpose of each discharge is to bring the weight of a deficient pair or triple up to the desired threshold and in certain cases some additional excess weight remains with its original pair or triple. In fact, any graph not achieving the bound in Theorem 10.7 will yield pairs or triples which retain some or all of their initial excess weight.

10.2.3 The Weight of each Pair and Triple

We consider the three different types of pairs, namely an A-pair, an AB-pair, and a B-pair as well as the ABA-triple. We show that each pair has weight of at least 5/2 under ζ and each triple has a weight of at least 15/4 under ζ.

Claim 5 *Suppose that there is an isolated vertex u in $G[V \setminus S]$. Let S' be the A-pair that has u as a common neighbor and let S'' be the B-pair that contains a vertex adjacent to u. Then, $\zeta(S') = \zeta(S'') = 5/2$.*

Proof. By Property 3, the vertex u belongs to the subgraph shown in Figure 10.3(a) or Figure 10.3(b). In both cases, $\psi(S') = 8/3$ and by Discharging Rule 1, we have that $\zeta(S') = \psi(S') - 1/6 = 5/2$. If u belongs to the subgraph shown in Figure 10.3(a), then $\psi(S'') = 5/3$ and by Discharging Rule 1 and Rule 2, we have that $\zeta(S'') = \psi(S'') + 1/6 +$

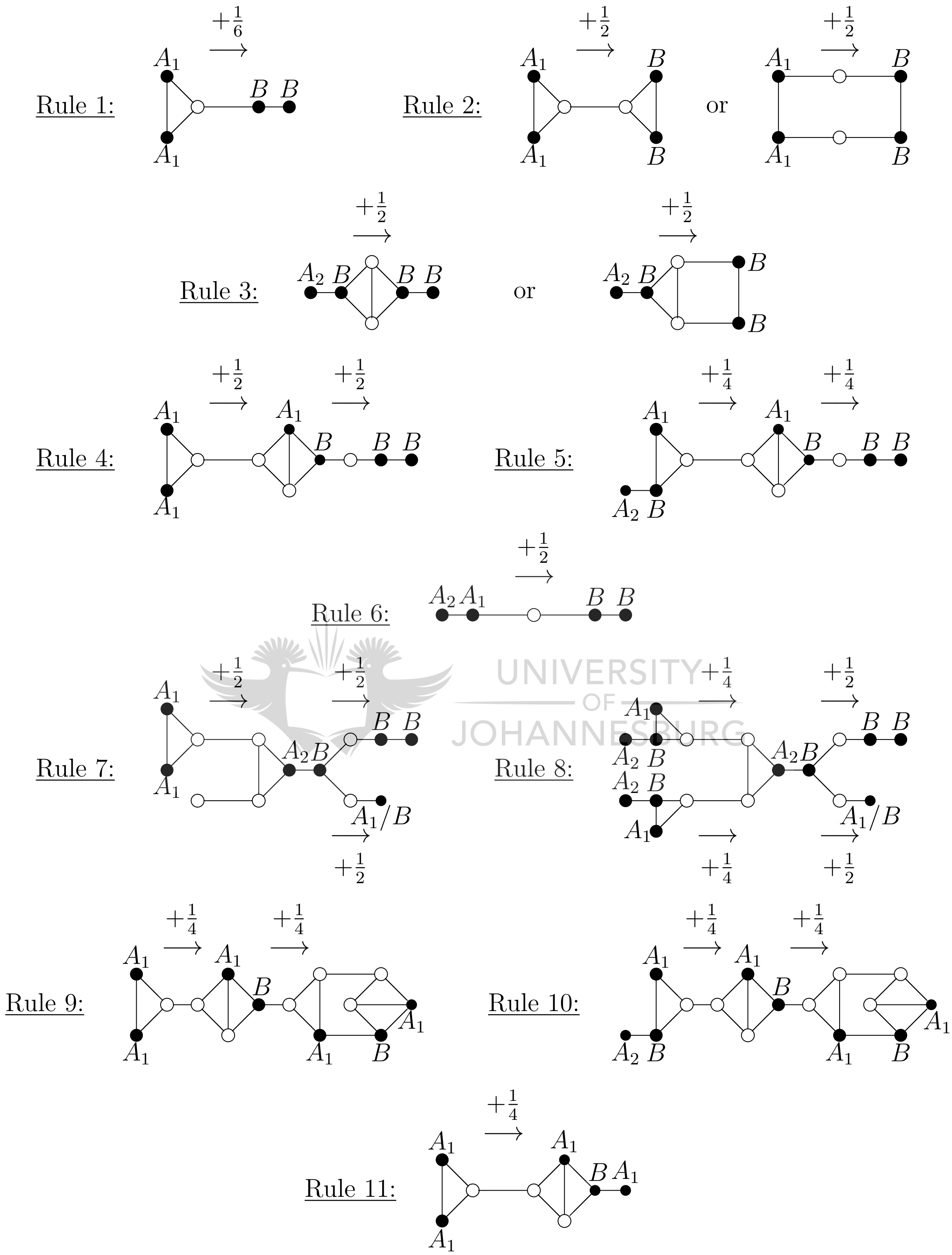


Figure 10.4: The eleven discharging rules.

$1/6+1/2 = 5/2$. If u belongs to the subgraph shown in Figure 10.3(b), then $\psi(S'') = 11/6$ and by Discharging Rule 1 and Rule 2, we have that $\zeta(S'') = \psi(S'') + 1/6 + 1/2 = 5/2$. In both cases, $\zeta(S'') = 5/2$. □

By Claim 5, we may assume that if S' is a pair in S, then no vertex of S' is adjacent with an isolated vertex in $G[V \setminus S]$ (for otherwise, $\zeta(S') = 5/2$, as desired).

For the proof of each of the following four claims, we provide an accompanying reference diagram. Each figure depicts the subgraphs necessarily resulting from the given constraints and each subgraph is drawn, and labeled, to correspond with a point in the proof at which a discharging rule is referenced. It is our intention that the reader wishing to skip the in-depth case analysis may gain an overview of each of the four proofs by examining the figures, while the more particular reader may wish to refer to the figures whilst examining the details. In either case, we note that the figures are not a substitute for the rigorous detail presented in each proof.

Claim 6 *If the two vertices in a B-pair S' have no common neighbor, then $\zeta(S') \geq 5/2$.*

Proof. Suppose that $S' = \{u, v\}$ is a B-pair in S but u and v have no common neighbor. Let $N(u) = \{a, b, v\}$ and let $N(v) = \{c, d, u\}$. By the claw-freeness of G, $\{ab, cd\} \subset E$. Note that $\{a, b, c, d\} \subset V \setminus S$ and that there is no edge between $\{a, b\}$ and $\{c, d\}$. Further, $\psi(S') = 2$. Let $N(a) = \{a', b, u\}$ and $N(b) = \{a, b', u\}$ (possibly, $a' = b'$). Since $u \in B$, we have that $a' \in S$ and $b' \in S$. Let $N(c) = \{c', d, v\}$ and $N(d) = \{c, d', v\}$ (possibly, $c' = d'$). Since $v \in B$, we have that $c' \in S$ and $d' \in S$.

Suppose that $a' = b'$. Let $N(a') = \{a, a_1, b\}$. To totally dominate a', we have that $a_1 \in S$. Hence, $a' \in B$. By Property 2, $a_1 \in A$ and so $\{a', a_1\}$ is an AB-pair. Let $N(a_1) = \{a', a_2, a_3\}$ where $a_2 \in \text{epn}(a_1, S)$, and so $a_2 \notin \{c, d\}$. By the claw-freeness of G, $a_2a_3 \in E$, and so $a_3 \notin \{c, d\}$. If $a_3 \notin \text{epn}(a_1, S)$, then $(S \setminus \{a', a_1, v\}) \cup \{a, a_3\}$ is a TDS

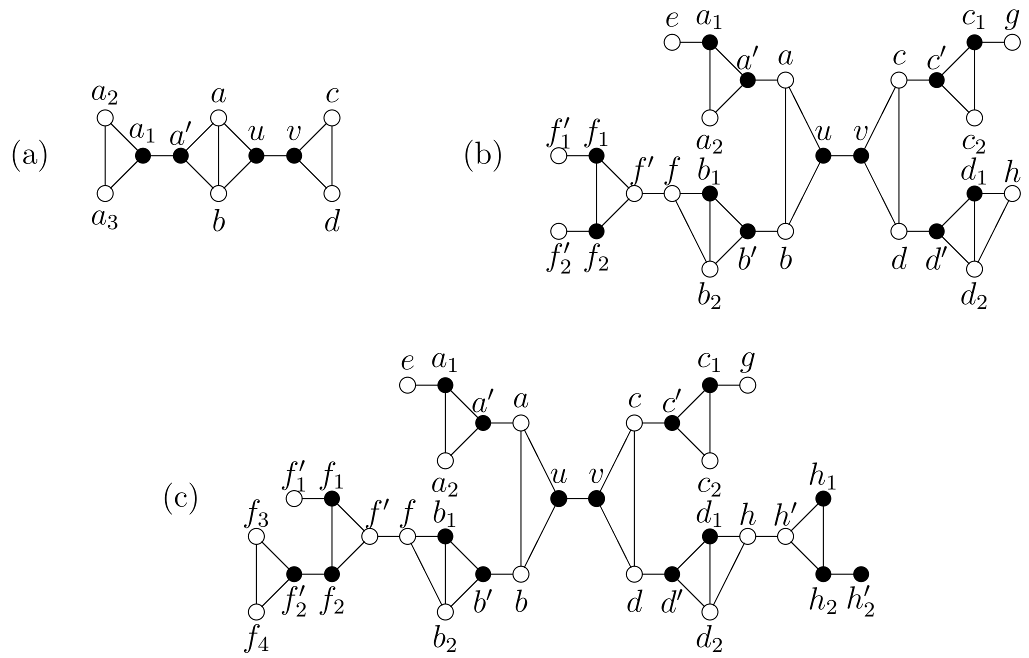


Figure 10.5: The three possible subgraphs containing a B-pair with no common neighbors.

of G, contradicting the minimality of S. Hence, $\text{epn}(a_1, S) = \{a_2, a_3\}$. Thus, $\{a', a_1\}$ is a strong AB-pair with $a' \in B$ and $a_1 \in A_2$. By Rule 3, we discharge a weight of $\frac{1}{2}$ from the strong AB-pair to the B-pair. Hence, $\zeta(S') \geq \psi(S') + \frac{1}{2} = \frac{5}{2}$, as desired. (See Figure 10.5(a).) Hence we may assume that $a' \neq b'$. Similarly, we may assume $c' \neq d'$. Since G is claw-free, the sets $\{a', b'\}$ and $\{c', d'\}$ are disjoint.

If $a'b' \in E$, then let e be the common neighbor of a' and b'. By Property 1, $e \notin S$. But then $\{a', b'\}$ is a B-pair at distance 2 from $\{u, v\}$, contradicting Property 2. Hence, $a'b' \notin E$. Similarly, $c'd' \notin E$. If $a'c' \in E$, then let e be the common neighbor of a' and c'. Again, by Property 1, $e \notin S$ and $\{a', c'\}$ is a B-pair at distance 2 from $\{u, v\}$, contradicting Property 2. Hence, $a'c' \notin E$ and, similarly, $\{a'd', b'c', b'd'\} \subset \overline{E}$.

Let $N(a') = \{a, a_1, a_2\}$, $N(b') = \{b, b_1, b_2\}$, $N(c') = \{c, c_1, c_2\}$ and $N(d') = \{d, d_1, d_2\}$. By the claw-freeness of G, $\{a_1a_2, b_1b_2, c_1c_2, d_1d_2\} \subset E$. To totally dominate a', b', c' and d', we may assume that $\{a_1, b_1, c_1, d_1\} \subset S$. Hence, $\{a', b', c', d'\} \subset B$. By Property 1 and Property 2, $\{a_1, b_1, c_1, d_1\} \subset A$ and so $\{a', a_1\}$, $\{b', b_1\}$, $\{c', c_1\}$ and $\{d', d_1\}$ are AB-pairs.

Let $N(a_1) = \{a', a_2, e\}$ and note that $\mathrm{epn}(a_1, S) = \{e\}$. Similarly, let $N(b_1) = \{b', b_2, f\}$, $N(c_1) = \{c', c_2, g\}$ and $N(d_1) = \{d', d_2, h\}$. Then, $\mathrm{epn}(b_1, S) = \{f\}$, $\mathrm{epn}(c_1, S) = \{g\}$ and $\mathrm{epn}(d_1, S) = \{h\}$.

If $ef \in E$, then $(S \setminus \{a', b_1, v\}) \cup \{b, e\}$ is a TDS of G, contradicting the minimality of S. Hence, $ef \notin E$. Similarly, $gh \notin E$. If $b_2e \in E$, then $(S \setminus \{a_1, b', v\}) \cup \{a, b_2\}$ is a TDS of G, contradicting the minimality of S. Hence, $b_2e \notin E$. By the same argument, $\{a_2f, c_2h, d_2g\} \subset \overline{E}$.

We now proceed with four sub-claims regarding the edges in G.

Claim 6.1 *The following properties hold in G:*

(a) *If $N(e) \cap \{c_2, d_2, g, h\} = \emptyset$, then $a_2e \in E$.*
(b) *If $N(f) \cap \{c_2, d_2, g, h\} = \emptyset$, then $b_2f \in E$.*
(c) *If $N(g) \cap \{a_2, b_2, e, f\} = \emptyset$, then $c_2g \in E$.*
(d) *If $N(h) \cap \{a_2, b_2, e, f\} = \emptyset$, then $d_2h \in E$.*

Proof. Suppose $N(e) \cap \{c_2, d_2, g, h\} = \emptyset$ and assume that $a_2e \notin E$. Let $N(e) = \{a_1, e_1, e_2\}$. By the claw-freeness of G, $e_1e_2 \in E$ and, since $e \in \mathrm{epn}(a_1, S)$, we have that $\{e_1, e_2\} \subset V \setminus S$. Let $N(e_1) = \{e, e_1', e_2\}$ and $N(e_2) = \{e, e_2', e_2\}$ (possibly, $e_1' = e_2'$). In order to totally dominate e_1 and e_2, $\{e_1', e_2'\} \subset S$ and furthermore, $\{e_1', e_2'\} \subset A$. We show first that $e_1' \neq e_2'$ and then that $e_1'e_2' \notin E$.

Suppose $e_1' = e_2'$ and let $N(e_1') = \{e_1, e_2, e_3\}$. To totally dominate e_1', we have that $e_3 \in S$. Since $\{e, e_1, e_2\} \subset V \setminus S$, we note that $\mathrm{epn}(e_1', S) = \{e_1, e_2\}$. If $e_3 \in A$, then by Property 1, $\{e_1', e_3\}$ is an A-pair. But then $T = (S \setminus \{a'\}) \cup \{e\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, $e_3 \in B$. If $\{e_1', e_3\}$ is an AB-pair, then $(S \setminus \{a_1, e_3, v\}) \cup \{a, e_1\}$ is a TDS of G, contradicting the minimality of S. Hence, e_1' and e_3 are part of an ABA-triple. But then $(S \setminus \{a', e_1'\}) \cup \{e\}$ is a TDS of G, contradicting the minimality of S. Hence, $e_1' \neq e_2'$.

Suppose $e_1'e_2' \in E$. Let e_3 be the common neighbor of e_1' and e_2'. Then by Property 1, $e_3 \notin S$ and $(S \setminus \{a_1, e_2', v\}) \cup \{a, e_1\}$ is a TDS of G, contradicting the minimality of S. Hence, $e_1'e_2' \notin E$.

Let $N(e_1') = \{e_1, e_3, e_4\}$ and $N(e_2') = \{e_2, e_5, e_6\}$. By the claw-freeness of G, $\{e_3e_4, e_5e_6\} \subset E$. To totally dominate e_1' and e_2', we may assume $\{e_3, e_5\} \subset S$. If $\{e_3, e_5\} \subset A$, then by Property 1, $\{e_1', e_3\}$ and $\{e_2', e_5\}$ are both A-pairs and $T = (S \setminus \{a'\}) \cup \{e\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, we may assume $e_3 \in B$. If $\{e_1', e_3\}$ is an AB-pair, then $(S \setminus \{a_1, e_3, v\}) \cup \{a, e_1\}$ is a TDS of G, contradicting the minimality of S. Hence, e_1' and e_3 are part of an ABA-triple. But then $(S \setminus \{a', e_1'\}) \cup \{e\}$ is a TDS of G, contradicting the minimality of S. We deduce, therefore, that $a_2e \in E$, as required. This establishes part (a). The proofs of (b), (c) and (d) are analogous. □

Claim 6.2 *The following properties hold in G:*

(a) *If $eg \in E$, then $N(f) \cap \{c_2, d_2, g, h\} = N(h) \cap \{a_2, b_2, e, f\} = \emptyset$.*

(b) *If $eh \in E$, then $N(f) \cap \{c_2, d_2, g, h\} = N(g) \cap \{a_2, b_2, e, f\} = \emptyset$.*

(c) *If $fg \in E$, then $N(e) \cap \{c_2, d_2, g, h\} = N(h) \cap \{a_2, b_2, e, f\} = \emptyset$.*

(d) *If $fh \in E$, then $N(e) \cap \{c_2, d_2, g, h\} = N(g) \cap \{a_2, b_2, e, f\} = \emptyset$.*

Proof. Suppose $eg \in E$. Recall that $\{ef, gh\} \subset \overline{E}$. By the claw-freeness of G, $fg \notin E$. If $fh \in E$, then $(S \setminus \{a_1, b', c', d_1, u, v\}) \cup \{a, d, f, g\}$ is a TDS of G, contradicting the minimality of S. Hence, $fh \notin E$. If $d_2f \in E$, then $(S \setminus \{a', b_1, c_1, d', u, v\}) \cup \{b, c, d_2, e\}$ is a TDS of G, contradicting the minimality of S. Hence, $d_2f \notin E$. If $c_2f \in E$, then $c_2g \notin E$, and so, by the claw-freeness of G, e and g have a common neighbor, g_1 say, with $g_1 \notin S$. Let $N(g_1) = \{e, g, g_2\}$. In order to totally dominate g_1, we have that $g_2 \in S$. But then $(S \setminus \{a_1, c_1, u, v\}) \cup \{a, c, g_1\}$ is a TDS of G, contradicting the minimality of S. Hence, $c_2f \notin E$ and thus $N(f) \cap \{c_2, d_2, g, h\} = \emptyset$. By the same reasoning, $N(h) \cap \{a_2, b_2, e, f\} = \emptyset$. This establishes part (a). The proofs of (b), (c) and (d) are analogous. □

Claim 6.3 *The following properties hold in G:*

(a) *If $c_2e \in E$ or $a_2g \in E$, then $N(f) \cap \{c_2, d_2, g, h\} = N(h) \cap \{a_2, b_2, e, f\} = \emptyset$.*

(b) *If $d_2e \in E$ or $a_2h \in E$, then $N(f) \cap \{c_2, d_2, g, h\} = N(g) \cap \{a_2, b_2, e, f\} = \emptyset$.*

(c) *If $c_2f \in E$ or $b_2g \in E$, then $N(e) \cap \{c_2, d_2, g, h\} = N(h) \cap \{a_2, b_2, e, f\} = \emptyset$.*

(d) *If $d_2f \in E$ or $b_2h \in E$, then $N(e) \cap \{c_2, d_2, g, h\} = N(g) \cap \{a_2, b_2, e, f\} = \emptyset$.*

Proof. (a) Suppose $c_2e \in E$. By the claw-freeness of G, $a_2e \in E$, and so $N(e) = \{a_1, a_2, c_2\}$. Clearly, $a_2 \notin N(h)$ and $c_2 \notin N(f)$. Thus since $c_2e \in E$, we have by Claim 6.2(c) that $fg \notin E$ and by Claim 6.2(d) that $fh \notin E$. If $d_2f \in E$, then by the claw-freeness of G, $b_2f \in E$. But then $N(g) \cap \{a_2, b_2, e, f\} = \emptyset$ and therefore by Claim 6.1(c), $c_2g \in E$. But $N(c_2) = \{c', c_1, e\}$, a contradiction. Hence, $d_2f \notin E$. If $b_2h \in E$, then by the claw-freeness of G, $d_2h \in E$ and $(S \setminus \{a_1, b', c', d_1, u, v\}) \cup \{a, b_2, c_2, d\}$ is a TDS of G, contradicting the minimality of S. Hence, $b_2h \notin E$. Thus, $N(f) \cap \{c_2, d_2, g, h\} = N(h) \cap \{a_2, b_2, e, f\} = \emptyset$. A similar argument follows if $a_2g \in E$. This establishes part (a). The proofs of (b), (c) and (d) are analogous. □

Claim 6.4 *We have that $N(e) \cap \{c_2, d_2, g, h\} = \emptyset$ or $N(f) \cap \{c_2, d_2, g, h\} = \emptyset$, and that $N(g) \cap \{a_2, b_2, e, f\} = \emptyset$ or $N(h) \cap \{a_2, b_2, e, f\} = \emptyset$.*

Proof. If $\{eg, eh, fg, fh\} \not\subset \overline{E}$, then the result follows from Claim 6.2. If $\{a_2g, a_2h, b_2g, b_2h, c_2e, c_2f, d_2e, d_2f\} \not\subset \overline{E}$, then the result follows from Claim 6.3. We may therefore assume that $\{a_2g, a_2h, b_2g, b_2h, c_2e, c_2f, d_2e, d_2f, eg, eh, fg, fh\} \subset \overline{E}$. But then $N(e) \cap \{c_2, d_2, g, h\} = N(f) \cap \{c_2, d_2, g, h\} = N(g) \cap \{a_2, b_2, e, f\} = N(h) \cap \{a_2, b_2, e, f\} = \emptyset$. □

We now return to the proof of Claim 6. By Claim 6.4, we may assume, without loss of generality, that $N(f) \cap \{c_2, d_2, g, h\} = N(h) \cap \{a_2, b_2, e, f\} = \emptyset$. Then, by Claim 6.1,

$\{b_2f, d_2h\} \subset E$. Let $N(f) = \{b_1, b_2, f'\}$ and $N(f') = \{f, f_1, f_2\}$. Since $f \in \text{epn}(b_1, S)$, $f' \notin S$ and since G is claw-free, $f_1f_2 \in E$. To totally dominate f', $f_1 \in S$. Let $N(f_1) = \{f', f_1', f_2\}$ and $N(f_2) = \{f', f_1, f_2'\}$ (possibly $f_1' = f_2'$).

Suppose $f_2 \notin S$. In order to totally dominate f_1, we have that $f_1' \in S$. If $f_1' \in A$, then by Property 1, $\{f_1, f_1'\}$ is an A-pair and $T = (S \setminus \{b'\}) \cup \{f\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Therefore, $f_1' \in B$. If $\{f_1, f_1'\}$ is an AB-pair, then $(S \setminus \{b_1, f_1', v\}) \cup \{b, f'\}$ is a TDS of G, contradicting the minimality of S. Hence, f_1 and f_1' are part of an ABA-triple. If $f_1 \in A_2$, then $T = (S \setminus \{b'\}) \cup \{f\}$ is a minimum TDS of G, with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, $f_1 \in A_1$. Since $f' \in \text{epn}(f_1, S)$, $f_2' \in S$. But then $(S \setminus \{b', f_1\}) \cup \{f\}$ is a TDS of G, contradicting the minimality of S. Hence, $f_2 \in S$.

If $\{f_1, f_2\} \subset A$, then, by Property 1, $\{f_1, f_2\}$ is an A-pair. Using Rule 4, we discharge a weight of $\frac{1}{2}$ from the A-pair $\{f_1, f_2\}$ to the AB-pair $\{b', b_1\}$ and then a weight of $\frac{1}{2}$ from this AB-pair to the B-pair $\{u, v\}$ so that $\zeta(S') \geq \psi(S') + \frac{1}{2} = \frac{5}{2}$, as desired. (See Figure 10.5(b).) Therefore we may assume that $f_2 \in B$. If $\{f_1, f_2\}$ is either an AB-pair or a B-pair, then $(S \setminus \{b_1, f_2, v\}) \cup \{b, f'\}$ is a TDS of G, contradicting the minimality of S. Hence, by Property 1, we may assume that f_1 and f_2 are part of an ABA-triple and $f_2' \in A$. Let $N(f_2') = \{f_2, f_3, f_4\}$. By the claw-freeness of G, we have that $f_3f_4 \in E$. If $f_2' \in A_1$, then we can assume that $f_4 \notin \text{epn}(f_2', S)$. But then $(S \setminus \{b_1, f_2, f_2', v\}) \cup \{b, f', f_4\}$ is a TDS of G, contradicting the minimality of S. Hence, $f_2' \in A_2$ and $\{f_1, f_2, f_2'\}$ is a strong ABA-triple. Using Rule 5, we discharge a weight of $\frac{1}{4}$ from the strong ABA-triple $\{f_1, f_2, f_2'\}$ to the AB-pair $\{b', b_1\}$ and then a weight of $\frac{1}{4}$ from this AB-pair to the B-pair $\{u, v\}$.

By an identical argument to the one above, we can assume that $N(h) = \{d_1, d_2, h'\}$, $N(h') = \{h, h_1, h_2\}$, $N(h_2) = \{h', h_1, h_2'\}$, $h' \notin S$ and $\{h_1, h_2, h_2'\}$ is a strong ABA-triple with $h_1 \in A_1$ and $h_2' \in A_2$. Again, by Rule 5, we discharge an additional weight of $\frac{1}{4}$

from the strong ABA-triple $\{h_1, h_2, h_2'\}$ to the AB-pair $\{d', d_1\}$ and then a weight of $\frac{1}{4}$ from this AB-pair to the B-pair $\{u, v\}$. Hence, $\zeta(S') \geq \psi(S') + \frac{1}{4} + \frac{1}{4} = \frac{5}{2}$, as desired. (See Figure 10.5(c).) This completes the proof of Claim 6. □

Claim 7 *If the two vertices in a B-pair S' have two common neighbors, then $\zeta(S') \geq \frac{5}{2}$.*

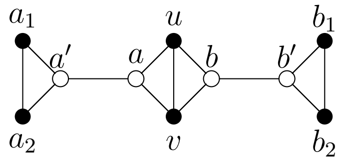


Figure 10.6: The only possible subgraph containing a B-pair with two common neighbors.

Proof. Suppose that $S' = \{u, v\}$ is a B-pair in S and that u and v have two common neighbors, a and b, say. Let $N(b) = \{b', u, v\}$ and $N(a) = \{a', u, v\}$. By Property 1 and Property 3, $\{a, a', b, b'\} \subseteq V \setminus S$. Note that $\psi(S') = 2$. Let $N(b') = \{b, b_1, b_2\}$ (possibly, $a' \in \{b_1, b_2\}$). By the claw-freeness of G, $b_1b_2 \in E$. To totally dominate b', we may assume that $b_1 \in S$.

Suppose that $b_2 \notin S$. Let $N(b_1) = \{b', b_2, c\}$ (possibly, $b_2c \in E$). To totally dominate b_1, $c \in S$. If $c \in A$, then by Property 1, $\{b_1, c\}$ is an A-pair and $(S \setminus \{v\}) \cup \{b\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, $c \in B$. If $\{b_1, c\}$ is an AB-pair, then $(S \setminus \{v\}) \cup \{b\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$ and $\beta(T) = \beta(S)$ but with $\xi(T) < \xi(S)$, contradicting the choice of S. Hence, b_1 and c are part of an ABA-triple. If $b_1 \in A_1$, then $(S \setminus \{b_1, v\}) \cup \{b\}$ is a TDS of G, contradicting the minimality of S. Therefore, $b_1 \in A_2$. But then $T = (S \setminus \{v\}) \cup \{b\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, $b_2 \in S$.

Since $a' \notin S$, we have that $a' \notin \{b_1, b_2\}$. Let $N(a') = \{a_1, a_2\}$. By the claw-freeness of G, $a_1a_2 \in E$. If $\{a_1, a_2\} = \{b_1, b_2\}$, then $n = 8$, a contradiction. Hence, $\{a_1, a_2\} \neq \{b_1, b_2\}$. To dominate a', we may assume that $a_1 \in S$. Suppose that $a_2 \notin S$. If $a_1b_1 \in E$, then $(S \setminus \{a_1, b_1, u, v\}) \cup \{a, a', b'\}$ is a TDS of G, contradicting the minimality of S. Hence, $a_1b_1 \notin E$ and similarly, $a_1b_2 \notin E$. Let $N(a_1) = \{a', a_2, d\}$. To totally dominate a_1, we have that $d \in S$. We now use an identical argument as in the previous paragraph to show that $\{a_1, d\}$ is not an A-pair, an AB-pair or part of an ABA-triple. Hence, $a_2 \in S$.

If both $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are B-pairs, then $(S \setminus \{a_1, b_1, u, v\}) \cup \{a, a', b'\}$ is a TDS of G, contradicting the minimality of S (note that if a_1 and b_1 have a common neighbor x, then by the claw-freeness of G, such a neighbor is adjacent to a_2 or b_2). Hence we may assume that $b_1 \in A$. We proceed further with the following sub-claim.

Claim 7.1 *$b_2 \in A$.*

Proof. Assume, to the contrary, that $b_2 \in B$. Let $N(b_1) = \{b', b_2, c\}$ and $N(c) = \{b_1, c_1, c_2\}$. Then, $c \in \text{epn}(b_1, S)$, $\{c_1, c_2\} \subset V \setminus S$ and, by the claw-freeness of G, $c_1c_2 \in E$. Let $N(c_1) = \{c, c_2, e_1\}$ and note that $e_1 \in A$ with $c_1 \in \text{epn}(e_1, S)$ (possibly, $e_1 \in \{a_1, a_2\}$). Let $N(c_2) = \{c, c_1, e_2\}$ and note that $e_2 \in A$ with $c_2 \in \text{epn}(e_2, S)$ (possibly, $e_2 \in \{a_1, a_2, e_1\}$). If $b_2e_1 \in E$, then by claw-freeness of G, $e_1 = e_2$ and $T = (S \setminus \{b_1, v\}) \cup \{b, c_1\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Therefore, $b_2e_1 \notin E$. Similarly, $b_2e_2 \notin E$.

Suppose that $e_1 = e_2$. Let $N(e_1) = \{c_1, c_2, e_3\}$. To totally dominate e_1, we have $e_3 \in S$. Suppose that $\{b_1, b_2\}$ is an AB-pair. If $e_3 \in A$, then by Property 1, $\{e_1, e_3\}$ is an A-pair. But then $T = (S \setminus \{b_2, v\}) \cup \{b, c\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, $e_3 \in B$. We remark that by the claw-freeness of G, b_2 and e_3 have no common neighbor. If $\{e_1, e_3\}$ is an AB-pair, then $(S \setminus \{b_1, b_2, e_3, v\}) \cup \{b, b', c_1\}$ is a TDS of G, contradicting the minimality of S. Hence, e_1 and e_3 are part of an ABA-triple. Note that $e_1 \in A_2$ since $\text{epn}(e_1, S) = \{c_1, c_2\}$. Now the set $(S \setminus \{b_2, e_1\}) \cup \{c\}$ is a TDS of G, contradicting the minimality of S. Hence, $\{b_1, b_2\}$ is not an AB-pair and thus b_1 and b_2 are part of an

ABA-triple.

Let $N(b_2) = \{b', b_1, f\}$. Then, $f \in A$. Let $N(f) = \{b_2, f_1, f_2\}$ and note that $\{f_1, f_2\} \subset V \setminus S$. By the claw-freeness of G, $f_1f_2 \in E$. If $\{e_1, e_3\}$ is an AB-pair, then $(S \setminus \{b_1, e_3\}) \cup \{c_1\}$ is a TDS of G, contradicting the minimality of S. If e_1 and e_3 are part of an ABA-triple, then $T = (S \setminus \{b_1, e_1\}) \cup \{c, c_1\}$ is a minimum TDS of G with $\lambda(T) < \lambda(S)$, contradicting the choice of S. Hence, $\{e_1, e_3\}$ is an A-pair. But then $T = (S \setminus \{b_1, v\}) \cup \{b, c_1\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, $e_1 \neq e_2$.

Let $N(e_1) = \{c_1, e_3, e_4\}$ and $N(e_2) = \{c_2, e_5, e_6\}$. To totally dominate e_1 and e_2, we may assume that $e_3 \in S$ and $e_5 \in S$. We remark that if $e_1e_2 \in E$, then $e_2 = e_3$, $e_1 = e_5$ and $e_4 = e_6$.

Suppose that $\{b_1, b_2\}$ is an AB-pair. If $e_1e_2 \in E$, then by Property 1, $\{e_1, e_2\}$ is an A-pair and $(S \setminus \{b_1, b_2, e_2, v\}) \cup \{b, b', c_1\}$ is a TDS of G, contradicting the minimality of S. Hence, $e_1e_2 \notin E$. If $e_3 \in A$ and $e_5 \in A$, then by Property 1, $\{e_1, e_3\}$ and $\{e_2, e_5\}$ are both A-pairs. But then $T = (S \setminus \{b_2, v\}) \cup \{b, c\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence we may assume that $e_3 \in B$. If b_2 and e_3 have a common neighbor, then by the claw-freeness of G, such a common neighbor must be the vertex e_4. But then $N(e_4) = \{b_2, e_1, e_3\}$, contradicting Property 3. Hence, b_2 and e_3 have no common neighbor. If $\{e_1, e_3\}$ is an AB-pair, then $(S \setminus \{b_1, b_2, e_3, v\}) \cup \{b, b', c_1\}$ is a TDS of G, contradicting the minimality of S. Hence, e_1 and e_3 are part of an ABA-triple. Since $e_3e_4 \in E$ (by the claw-freeness of G), we note that $e_1 \in A_1$. Thus, $(S \setminus \{b_2, e_1\}) \cup \{c\}$ is a TDS of G, contradicting the minimality of S. Hence, $\{b_1, b_2\}$ is not an AB-pair and thus b_1 and b_2 are part of an ABA-triple.

Let $N(b_2) = \{b', b_1, f\}$. Then, $f \in A$. Let $N(f) = \{b_2, f_1, f_2\}$ and note that $\{f_1, f_2\} \subset V \setminus S$. By the claw-freeness of G, $f_1f_2 \in E$. Proceeding as in the third paragraph of the proof of this subclaim, we have that $\{e_1, e_3\}$ is an A-pair. Similarly, $\{e_2, e_5\}$ is an

A-pair (possibly the same pair). If $e_1e_2 \in E$, then $(S \setminus \{b_1, e_2\}) \cup \{c_1\}$ is a TDS of G, contradicting the minimality of S. Hence, $e_1e_2 \notin E$. But then $T = (S \setminus \{b_1, v\}) \cup \{b, c_1\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. This completes the proof of Subclaim 7.1. □

We now return to the proof of Claim 7. By Subclaim 7.1, $b_2 \in A$ and $\{b_1, b_2\}$ is an A-pair. Using Rule 2, we discharge a weight of $\frac{1}{2}$ from it to the B-pair $\{u, v\}$, whence $\zeta(S') \geq \psi(S') + \frac{1}{2} = \frac{5}{2}$, as desired. (See Figure 10.6.) □

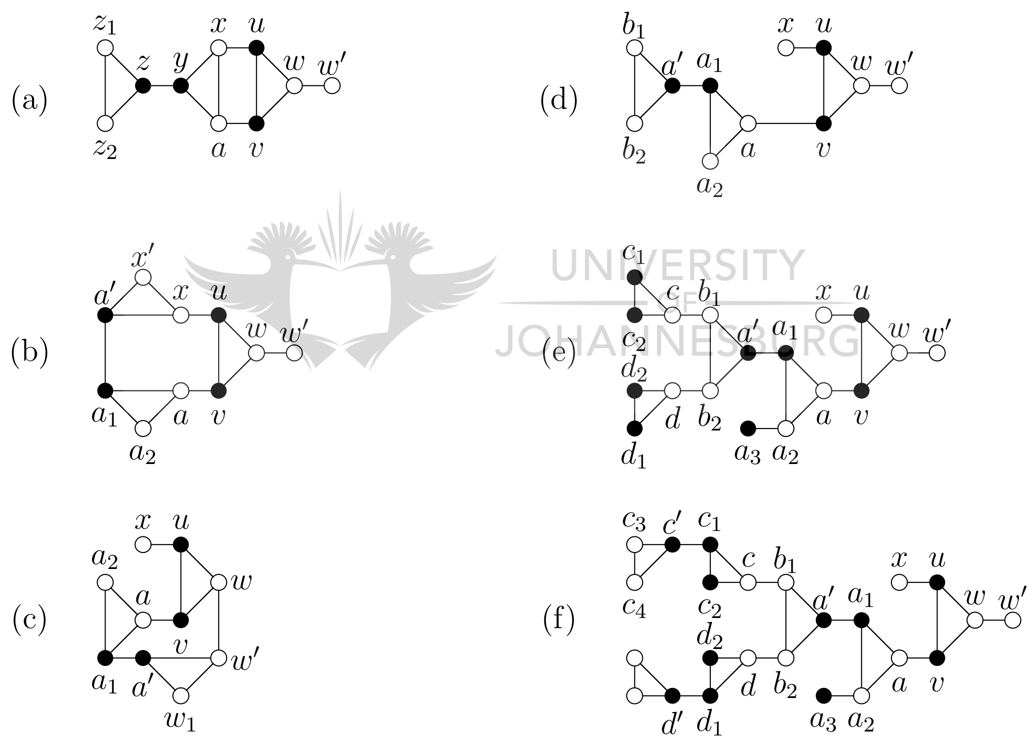


Figure 10.7: The six subgraphs containing a B-pair with exactly one common neighbor.

Claim 8 *If the two vertices in a B-pair S' have only one common neighbor, then $\zeta(S') \geq \frac{5}{2}$.*

Proof. Suppose that $S' = \{u, v\}$ is a B-pair in S and that u and v have exactly one common neighbor, w say. Let $N(w) = \{w', u, v\}$. By Property 1 and Property 3,

$\{w, w'\} \subset V \setminus S$. Let $N(v) = \{a, u, w\}$ and $N(u) = \{v, w, x\}$ (possibly, $w' \in \{a, x\}$). Since $a \neq x$, we may assume that $a \neq w'$. By Property 1, $\{a, x\} \subset V \setminus S$.

Claim 8.1 *If $ax \in E$, then $\zeta(S') \geq \frac{5}{2}$.*

Proof. Suppose $ax \in E$. Then S' is necessarily a strong B-pair. If $w' = x$, then $x \in \text{epn}(u, S)$, contradicting the fact that $u \in B$. Hence, $w' \neq x$. Let y be the common neighbor of a and x. Since $\{u, v\} \subset B$, we have that $y \in S$. Note that $\psi(S') = 2$. Let $N(y) = \{a, x, z\}$. In order to totally dominate y, $z \in S$. By Property 1 and Property 2, $\{y, z\}$ is an AB-pair with $y \in B$. Let $N(z) = \{y, z_1, z_2\}$. By the claw-freeness of G, we have that $z_1z_2 \in E$. If $z \in A_1$, then we may assume that $z_2 \notin \text{epn}(z, S)$. But then $(S \setminus \{u, y, z\}) \cup \{a, z_2\}$ is a TDS of G, contradicting the minimality of S. Hence, $z \in A_2$. We now use Rule 3 to discharge a weight of $\frac{1}{2}$ from the strong AB-pair $\{y, z\}$ to the B-pair $\{u, v\}$ so that $\zeta(S') \geq \psi(S') + \frac{1}{2} = \frac{5}{2}$, as desired. (See Figure 10.7(a).) $\square$

By Claim 8.1, we may assume that $ax \notin E$. Let $N(a) = \{a_1, a_2, v\}$. By the claw-freeness of G, $a_1a_2 \in E$. Since $v \in B$, we may assume that $a_1 \in S$. By Claim 5 we may assume that $a_2 \notin S$. By symmetry, the same arguments apply to the neighbors of x and hence S' is a strong B-pair. Therefore $\psi(S') = 2$. Let $N(a_1) = \{a, a', a_2\}$. To totally dominate a_1, we have $a' \in S$.

Claim 8.2 *If $a'x \in E$, then $\zeta(S') \geq \frac{5}{2}$.*

Proof. Suppose $a'x \in E$. If $w' = x$, then, by the claw-freeness of G, $a'a_2 \in E$ and hence $n = 8$, a contradiction. Hence, $w' \neq x$. Let x' be the common neighbor of a' and x. If $x' \in S$, then $\{a_1, a', x'\}$ is an ABA-triple with $a' \in B$. But then x is an isolated vertex in $G[V \setminus S]$ with two B-neighbors, contradicting Property 3. Hence, $x' \notin S$. If $a_1 \in B$ then $(S \setminus \{a', a_1, v\}) \cup \{a_2, x\}$ is a TDS of G, contradicting the minimality of S. If $a' \in B$, then

$(S \setminus \{a', a_1, u\}) \cup \{a, x'\}$ is a TDS of G, contradicting the minimality of S. Thus, $\{a_1, a'\}$ is an A-pair. We now use Rule 2 to discharge a weight of $\frac{1}{2}$ from this A-pair $\{a_1, a'\}$ to the B-pair $\{u, v\}$ so that $\zeta(S') \geq \psi(S') + \frac{1}{2} = \frac{5}{2}$, as desired. (See Figure 10.7(b).) □

By Claim 8.2, we may assume that $a'x \notin E$.

Claim 8.3 *If $a'w' \in E$, then $\zeta(S') \geq \frac{5}{2}$.*

Proof. Suppose $a'w' \in E$. If $w' = x$, then, $a'a_2 \in E$ and $n = 8$, a contradiction. Hence, $w' \neq x$. Let w_1 be the common neighbor of a' and w'. Suppose $w_1 \in S$. Then by Property 1, $\{a_1, a', w_1\}$ is an ABA-triple with $a' \in B$. But then $(S \setminus \{a_1, u\}) \cup \{a\}$ is a TDS of G, contradicting the minimality of S. Hence, $w_1 \notin S$, and so $w' \in \text{epn}(a', S)$. If $a_1 \in B$ then by Property 1, $\{a_1, a'\}$ is an AB-pair. But then $a_2 \notin \text{epn}(a_1, S)$ and $(S \setminus \{a_1, a', u, v\}) \cup \{a_2, w, w'\}$ is a TDS of G, contradicting the minimality of S. Therefore, by Property 1, $\{a_1, a'\}$ is an A-pair. Thus, $\text{epn}(a_1, S) = \{a_2\}$. If $a' \in A_1$, then $w_1 \notin \text{epn}(a', S)$ and $(S \setminus \{a_1, a', u\}) \cup \{a, w_1\}$ is a TDS of G, contradicting the minimality of S. Hence, $a' \in A_2$. We now use Rule 6 to discharge a weight of $\frac{1}{2}$ from the A-pair $\{a_1, a'\}$ to the B-pair $\{u, v\}$ so that $\zeta(S') \geq \psi(S') + \frac{1}{2} = \frac{5}{2}$, as desired. (See Figure 10.7(c).) □

By Claim 8.3, we may assume that $a'w' \notin E$.

Claim 8.4 *$a'a_2 \notin E$.*

Proof. Suppose $a'a_2 \in E$ and let $N(a') = \{a_1, a_2, b\}$. By Property 1 and Property 2, $\{a', a_1\}$ is an AB-pair. Let $N(b) = \{a', b_1, b_2\}$. By the claw-freeness of G, $b_1b_2 \in E$. Since $b \in \text{epn}(a', S)$, we have that $\{b_1, b_2\} \subset V \setminus S$. If $w' \in \{b_1, b_2\}$, then w' is not dominated by the set S, a contradiction. If $x \in \{b_1, b_2\}$, then $x \in \text{epn}(u, S)$, contradicting the fact that $u \in B$. Hence, the sets $\{w', x\}$ and $\{b_1, b_2\}$ are disjoint. Let $N(b_1) = \{b, b_2, c\}$.

To totally dominate b_1, $c \in S$. Let $N(c) = \{b_1, c_1, c_2\}$. To totally dominate c, we may assume $c_1 \in S$.

Suppose $cb_2 \in E$. Then, $b_2 = c_2$. If $c_1 \in A$, then by Property 1, $\{c, c_1\}$ is an A-pair and $T = (S \setminus \{a_1\}) \cup \{b\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting our choice of S. Hence, $c_1 \in B$. If c and c_1 form part of an ABA-triple, then $(S \setminus \{a_1, c\}) \cup \{b\}$ is a TDS of G, contradicting the minimality of S. Hence, $\{c, c_1\}$ is an AB-pair. But then $T = (S \setminus \{a_1\}) \cup \{b\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$ and $\xi(T) = \xi(S)$ but with $\varphi(T) < \varphi(S)$, contradicting the choice of S. Hence, $b_2c \notin E$ and, by the claw-freeness of G, $c_1c_2 \in E$.

If $b_2c_1 \in E$, then $(S \setminus \{a', c_1, u\}) \cup \{a, b_1\}$ is a TDS of G, contradicting the minimality of S. Hence, $b_2c_1 \notin E$. Let $N(b_2) = \{b, b_1, d\}$. To totally dominate b_2, $d \in S$. Let $N(d) = \{b_2, d_1, d_2\}$. To totally dominate d, we may assume $d_1 \in S$. By the claw-freeness of G, $d_1d_2 \in E$. By Property 3, $\{c_1, c_2\} \neq \{d_1, d_2\}$.

If $\{c_1, d_1\} \subset A$, then, by Property 1, $\{c, c_1\}$ and $\{d, d_1\}$ are both A-pairs and $T = (S \setminus \{a_1\}) \cup \{b\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence we may assume that $c_1 \in B$. If c and c_1 are part of an ABA-triple, then $(S \setminus \{a_1, c\}) \cup \{b\}$ is a TDS of G, contradicting the minimality of S. Hence, $\{c, c_1\}$ is an AB-pair.

If $c_1x \notin E$, then $(S \setminus \{a', c_1, u\}) \cup \{a, b_1\}$ is a TDS of G, contradicting the minimality of S. Hence, $c_1x \in E$. Suppose $d_1 \in A$. Let $T = (S \setminus \{a_1, c, u\}) \cup \{a, b, x\}$. Then, T is a minimum TDS of G. If $w' = x$, then $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$ and $\xi(T) = \xi(S)$ but with $\varphi(T) < \varphi(S)$. If $w' \neq x$, then $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$. Since both cases contradict the choice of S, we deduce that $d_1 \in B$. If d and d_1 are part of an ABA-triple, then $(S \setminus \{a_1, d\}) \cup \{b\}$ is a TDS of G, contradicting the minimality of S. Hence, $\{d, d_1\}$ is an AB-pair. By the claw-freeness of G, $d_1x \notin E$ and therefore $(S \setminus \{a', d_1, u\}) \cup \{a, b_2\}$ is a TDS of G, contradicting the minimality of S.

This completes the proof of Claim 8.4 $\square$

We now return to the proof of Claim 8. Let $N(a') = \{a_1, b_1, b_2\}$ (possibly, $w' \in \{b_1, b_2\}$). By the claw-freeness of G, $b_1b_2 \in E$. If a' and a_1 are part of an ABA-triple, then $(S \setminus \{a_1, u\}) \cup \{a\}$ is a TDS of G, contradicting the minimality of S. Hence, by Property 1 and Property 2, we may assume $\{a', a_1\}$ is either an A-pair or an AB-pair. If $a' \in B$, then $(S \setminus \{a', u\}) \cup \{a\}$ is a TDS of G, contradicting the minimality of S. Hence, $a' \in A$. Suppose $a' \in A_1$. We may assume that $\text{epn}(a', S) = \{b_1\}$. But then $(S \setminus \{a', a_1, u\}) \cup \{a, b_2\}$ is a TDS of G, contradicting the minimality of S. Thus, $a' \in A_2$ and $\text{epn}(a', S) = \{b_1, b_2\}$.

If $a_1 \in A$, then $\{a', a_1\}$ is an A-pair with $a' \in A_2$ and $a_1 \in A_1$. Using Rule 6, we discharge a weight of $\frac{1}{2}$ from the A-pair $\{a', a_1\}$ to the B-pair $\{u, v\}$ so that $\zeta(S') \geq \psi(S') + \frac{1}{2} = \frac{5}{2}$, as desired. (See Figure 10.7(d).) Hence we may assume that $a_1 \in B$. Let $N(a_2) = \{a, a_1, a_3\}$. Then, $a_3 \in S$.

Let $N(b_1) = \{a', b_2, c\}$ and $N(b_2) = \{a', b_1, d\}$ (possibly, $c = d$). We note that $\{c, d\} \subset V \setminus S$. Let $N(c) = \{b_1, c_1, c_2\}$ and $N(d) = \{b_2, d_1, d_2\}$. To totally dominate c and d, we may assume $c_1 \in S$ and $d_1 \in S$ (possibly, $c_1 = d_1$).

Claim 8.5 *The following properties hold in G:*

(a) $c \neq d$ *and* $\{c_1c_2, d_1d_2\} \subset E$.

(b) $cd \notin E$.

(c) $\{c_2, d_2\} \subset S$.

Proof. (a) Suppose $c = d$. Then, $b_1 = d_2$, $b_2 = c_2$ and $c_1 = d_1$. Let $N(c_1) = \{c, c_3, c_4\}$. To totally dominate c_1, we may assume that $c_3 \in S$. By the claw-freeness of G, $c_3c_4 \in E$. If $c_3 \in A$, then by Property 1, $\{c_1, c_3\}$ is an A-pair and $T = (S \setminus \{a_1\}) \cup \{b_1\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, $c_3 \in B$. If c_1 and c_3 are part of an ABA-triple, then $(S \setminus \{a_1, c_1\}) \cup \{b_1\}$

is a TDS of G, contradicting the minimality of S. Hence, $\{c_1, c_3\}$ is an AB-pair. Suppose $c_3x \in E$. If $w' \neq x$, then by the claw-freeness of G, $c_4 = x$. But then since $a_3 \in S$, we have that $(S \setminus \{a_1, a', c_1, c_3, v\}) \cup \{a_2, b_1, c, x\}$ is a TDS of G, contradicting the minimality of S. Hence, $w' = x$. But then $T = (S \setminus \{a_1, c_1, u\}) \cup \{a, b_1, x\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$ and $\xi(T) = \xi(S)$ but with $\varphi(T) < \varphi(S)$, contradicting the choice of S. Hence, $c_3x \notin E$. But then $(S \setminus \{a', c_3, u\}) \cup \{a, c\}$ is a TDS of G, contradicting the minimality of S. We conclude that $c \neq d$. Thus, by the claw-freeness of G, $\{c_1c_2, d_1d_2\} \subset E$. This establishes part (a).

(b) Suppose $cd \in E$. Then, $c = d_2$, $d = c_2$ and $c_1 = d_1$. But then $T = (S \setminus \{a_1\}) \cup \{b_1\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. This establishes part (b).

(c) Suppose $\{c_2, d_2\} \not\subset S$. We may assume that $c_2 \notin S$. Let $N(c_1) = \{c, c_2, c'\}$. To totally dominate c_1, $c' \in S$. If $c' \in A$, then by Property 1, $\{c', c_1\}$ is an A-pair and $T = (S \setminus \{a_1\}) \cup \{b_1\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, $c' \in B$. If c' and c_1 are part of an ABA-triple, then either $c_1 \in A_1$ or $c_1 \in A_2$. If $c_1 \in A_1$ then $\text{epn}(c_1, S) = \{c\}$ and $(S \setminus \{a_1, c_1\}) \cup \{b_1\}$ is a TDS of G, contradicting the minimality of S. If $c_1 \in A_2$ then $\text{epn}(c_1, S) = \{c, c_2\}$ and $T = (S \setminus \{a_1\}) \cup \{b_1\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, $\{c', c_1\}$ is an AB-pair. Suppose $c'x \in E$. If $w' \neq x$, then, by the claw-freeness of G, c' and x have a common neighbor, x' say, and $T = (S \setminus \{a_1\}) \cup \{b_1\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$ and $\xi(T) = \xi(S)$ but with $\varphi(T) < \varphi(S)$, contradicting the choice of S. Hence, $w' = x$. By the claw-freeness of G, $c'c_2 \in E$ and we have that $T = (S \setminus \{a_1, c_1, u\}) \cup \{a, b_1, x\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$ and $\xi(T) = \xi(S)$ but with $\varphi(T) < \varphi(S)$, contradicting the choice of S. Hence, $c'x \notin E$. But then $(S \setminus \{a', a_1, c', u\}) \cup \{a, b_1, c\}$ is a TDS of G, contradicting the minimality of S. Hence, $c_2 \in S$ and, similarly, $d_2 \in S$. This establishes part (c). □

If $\{c_1, c_2\} \subset A$, then by Property 1, $\{c_1, c_2\}$ is an A-pair and using Rule 7, we discharge a weight of $\frac{1}{2}$ to the strong AB-pair $\{a', a_1\}$ and then a weight of $\frac{1}{2}$ to the B-pair $\{u, v\}$. Thus, $\zeta(S') \geq \psi(S') + \frac{1}{2} = \frac{5}{2}$, as desired. (See Figure 10.7(e).) We may therefore assume that $c_1 \in B$.

Suppose $\{c_1, c_2\}$ is an AB-pair or a B-pair. Suppose $c_1x \in E$. If $w' \neq x$, then by the claw-freeness of G we have that $c_2x \in E$, contradicting Property 3. Hence, $w' = x$. But then $T = (S \setminus \{c_1\}) \cup \{c\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$ and $\iota(T) = \iota(S)$ but with $\beta(T) < \beta(S)$, contradicting the choice of S. Hence, $c_1x \notin E$. But then $(S \setminus \{a', a_1, c_1, u\}) \cup \{a, b_1, c\}$ is a TDS of G, contradicting the minimality of S. Hence, c_1 and c_2 are part of an ABA-triple.

Let $N(c_1) = \{c, c', c_2\}$ and $N(c') = \{c_1, c_3, c_4\}$. Note that $c' \in A$ and $\{c_3, c_4\} \subset V \setminus S$. By the claw-freeness of G, $c_3c_4 \in E$. If $c' \in A_1$, we may assume that $c_4 \in \text{epn}(c', S)$. But then $(S \setminus \{a', a_1, c', c_1, u\}) \cup \{a, b_1, c, c_3\}$ is a TDS of G, contradicting the minimality of S. Hence, $c' \in A_2$ and therefore $\{c', c_1, c_2\}$ is a strong ABA-triple. By the same argument, we may assume that $N(d_1) = \{d, d', d_2\}$ and that $\{d', d_1, d_2\}$ is a strong ABA-triple with $d' \in A_2$, $d_1 \in B$ and $d_2 \in A_1$. Using Rule 8, we discharge a weight of $\frac{1}{4}$ from each of these strong ABA-triples to the strong AB-pair $\{a', a_1\}$ and then a weight of $\frac{1}{2}$ from this AB-pair to the B-pair $\{u, v\}$ so that $\zeta(S') \geq \psi(S') + \frac{1}{2} = \frac{5}{2}$, as desired. (See Figure 10.7(f).) □

Claim 9 *If S' is a weak ABA-triple, then $\zeta(S') \geq \frac{15}{4}$.*

Proof. Suppose that $S' = \{u, v, w\}$ is a weak ABA-triple in S with $\{u, w\} \subset A_1$ and $v \in B$. We note that $\psi(S') = \frac{7}{2}$. By the claw-freeness of G, we may assume that u and v have a common neighbor, x say. Let $N(u) = \{a, v, x\}$ and note that $a \in \text{epn}(u, S)$. Let $N(w) = \{b, c, v\}$ with $b \in \text{epn}(w, S)$ and $c \notin \text{epn}(w, S)$. By Property 3, $x \notin N(w)$

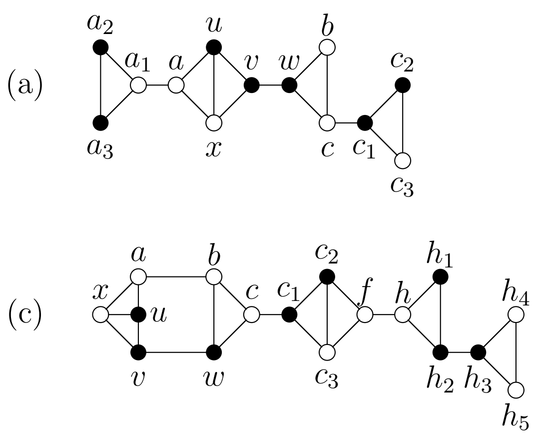


Figure 10.8: The three possible subgraphs containing a weak ABA-triple.

and hence, since G is claw-free, $bc \in E$. Let $N(c) = \{b, c_1, w\}$ and note that $c_1 \in S$. Let $N(c_1) = \{c, c_2, c_3\}$. To totally dominate c_1, we may assume that $c_2 \in S$. By the claw-freeness of G, $c_2c_3 \in E$. Thus, $c_1 \in B$. If $c_2 \in B$, then $\{c_1, c_2\}$ is a B-pair and $(S \setminus \{c_2, w\}) \cup \{c\}$ is a TDS of G, contradicting the minimality of S. Hence, $c_2 \in A$ and therefore, by Property 1, $\{c_1, c_2\}$ is an AB-pair. If $ac_3 \in E$, then by the claw-freeness of G, $N(a) = \{c_3, u, x\}$ and $(S \setminus \{c_1, u\}) \cup \{c_3\}$ is a TDS of G, contradicting the minimality of S. Hence, $ac_3 \notin E$. We proceed further with the following two claims.

Claim 9.1 $ax \in E$.

Proof. For sake of contradiction, suppose that $ax \notin E$. By the claw-freeness of G, $ab \notin E$. Let $N(a) = \{a_1, a_2, u\}$. By the claw-freeness of G, $a_1a_2 \in E$ and since $a \in \text{epn}(u, S)$ we have that $\{a_1, a_2\} \subset V \setminus S$. Let $N(a_1) = \{a, a_2, d\}$ and $N(a_2) = \{a, a_1, e\}$ (possibly, $d = e$). In order to totally dominate a_1 and a_2, we have that $d \in S$ and $e \in S$. Let $N(d) = \{a_1, d_1, d_2\}$ and $N(e) = \{a_2, e_1, e_2\}$. To totally dominate d and e, we may assume that $d_1 \in S$ and $e_2 \in S$ (possibly, $d_1 = e_1$).

Suppose $d = e$. Then $a_1 = e_2$, $a_2 = d_2$ and $d_1 = e_1$. If $d_1 \in A$, then by Property 1, $\{d, d_1\}$ is an A-pair and $T = (S \setminus \{v, w\}) \cup \{a, c\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$, $\xi(T) = \xi(S)$ and $\varphi(T) = \varphi(S)$ but with $\alpha(T) <$

$\alpha(S)$, contradicting the choice of S. Hence, $d_1 \in B$. If $\{d, d_1\}$ is an AB-pair, then $(S \setminus \{d_1, u\}) \cup \{a_1\}$ is a TDS of G, contradicting the minimality of S. Hence, d and d_1 are part of an ABA-triple. But then $(S \setminus \{d, v, w\}) \cup \{a, c\}$ is a TDS of G, contradicting the minimality of S. Hence $d \neq e$.

Suppose $de \in E$. Then, $d = e_1$, $e = d_1$ and $d_2 = e_2$. But then $(S \setminus \{e, u\}) \cup \{a_1\}$ is a TDS of G, contradicting the minimality of S. Hence, $de \notin E$. By the claw-freeness of G, $\{d_1d_2, e_1e_2\} \subset E$.

Suppose $d_1 \neq e_1$. If $\{d_1, e_1\} \subset A$, then by Property 1, $\{d, d_1\}$ and $\{e, e_1\}$ are both A-pairs and $T = (S \setminus \{v, w\}) \cup \{a, c\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$, $\xi(T) = \xi(S)$ and $\varphi(T) = \varphi(S)$ but with $\alpha(T) < \alpha(S)$, contradicting the choice of S. Hence we may assume that $d_1 \in B$. If $\{d, d_1\}$ is an AB-pair, then $(S \setminus \{d_1, u\}) \cup \{a_1\}$ is a TDS of G, contradicting the minimality of S. Hence, d and d_1 are part of and ABA-triple. But then $(S \setminus \{d, v, w\}) \cup \{a, c\}$ is a TDS of G, contradicting the minimality of S. Hence, $d_1 = e_1$. Thus, by the claw-freeness of G, $d_2 = e_2$. But then $(S \setminus \{d, v, w\}) \cup \{a, c\}$ is a TDS of G, contradicting the minimality of S. We deduce, therefore, that $ax \in E$. □

Claim 9.2 *If $ab \notin E$, then $\zeta(S') \geq \frac{15}{4}$.*

Proof. Suppose that $ab \notin E$. Let $N(a) = \{a_1, u, x\}$ and note that since $u \in A$ we have that $a_1 \notin S$. Let $N(a_1) = \{a, a_2, a_3\}$. To totally dominate a_1, we may assume that $a_2 \in S$. By the claw-freeness of G, $a_2a_3 \in E$ (possibly, $\{a_2, a_3\} = \{c_2, c_3\}$).

Suppose $a_3 \notin S$. Let $N(a_2) = \{a_1, a_2, a_4\}$. To totally dominate a_2, we have that $a_4 \in S$. If $a_4 \in A$, then by Property 1 $\{a_2, a_4\}$ is an A-pair and so $\{a_2, a_3\} \neq \{c_2, c_3\}$. But then $T = (S \setminus \{v, w\}) \cup \{a, c\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$, $\xi(T) = \xi(S)$ and $\varphi(T) = \varphi(S)$ but with $\alpha(T) < \alpha(S)$, contradicting the choice of S. Hence, $a_4 \in B$. If $\{a_2, a_4\}$ is an AB-pair, then $(S \setminus \{a_4, u\}) \cup \{a_1\}$ is a TDS of G, contradicting the minimality of S. Hence a_2 and a_4 are part of an ABA-triple. But then $T = (S \setminus \{a_2, u\}) \cup \{a, a_1\}$ is a minimum TDS of G with $\lambda(T) < \lambda(S)$, contradicting the choice of S. Hence, $a_3 \in S$.

Suppose $\{a_2, a_3\} \not\subset A$. We may assume then that $a_3 \in B$. Let $N(a_3) = \{a_1, a_2, a_5\}$. If $\{a_2, a_3\}$ is an AB-pair or a B-pair, then $(S \setminus \{a_3, u\}) \cup \{a_1\}$ is a TDS of G, contradicting the minimality of S. Hence, $\{a_2, a_3, a_5\}$ must be an ABA-triple. Let $a_6 \in \text{epn}(a_5, S)$. Then $T = (S \setminus \{a_3, u\}) \cup \{a_1, a_6\}$ is a minimum TDS of G with $\lambda(T) < \lambda(S)$, contradicting the choice of S. Therefore, $\{a_2, a_3\} \subset A$ and so, by Property 1, $\{a_2, a_3\}$ is an A-pair. Using Rule 11, we discharge a weight of $\frac{1}{4}$ from this A-pair to the weak ABA-triple $\{u, v, w\}$ so that $\zeta(S') \geq \psi(S') + \frac{1}{4} = \frac{15}{4}$, as desired. (See Figure 10.8(a).) $\square$

By Claim 9.2, we may assume that $ab \in E$. Let $N(c_2) = \{c_1, c_3, f\}$. Since $c_2 \in A$, we note that $\text{epn}(c_2, S) = \{f\}$. Let $N(c_3) = \{c_1, c_2, g\}$. By Property 3, we have that $g \notin S$.

Claim 9.3 $f = g$.

Proof. Our proof of Claim 9.3 is a modified argument to the proof of Claim 9.1. For sake of contradiction, suppose that $f \neq g$. If $fg \in E$, let h be the common neighbor of f and g. But then to totally dominate g, we have that $h \in S$, contradicting the fact that $c_2 \in A$. Hence, $fg \notin E$. Let $N(f) = \{c_2, f_1, f_2\}$. By the claw-freeness of G, $f_1 f_2 \in E$ and since $f \in \text{epn}(c_2, S)$ we have that $\{f_1, f_2\} \subset V \setminus S$. Let $N(f_1) = \{f, f_2, d\}$ and $N(f_2) = \{f, f_1, e\}$ (possibly, $d = e$). In order to totally dominate f_1 and f_2, we have that $d \in S$ and $e \in S$. Let $N(d) = \{f_1, d_1, d_2\}$ and $N(e) = \{f_2, e_1, e_2\}$. To totally dominate d and e, we may assume that $d_1 \in S$ and $e_1 \in S$.

Suppose $d = e$. Then, $f_1 = e_2$, $f_2 = d_2$ and $d_1 = e_1$. If $d_1 \in A$, then by Property 1, $\{d, d_1\}$ is an A-pair and $T = (S \setminus \{c_1\}) \cup \{f\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$, $\xi(T) = \xi(S)$ and $\varphi(T) = \varphi(S)$ but with $\alpha(T) < \alpha(S)$, contradicting the choice of S. Hence, $d_1 \in B$. If $\{d, d_1\}$ is an AB-pair, then $(S \setminus \{c_2, d_1, w\}) \cup \{c, f_1\}$ is a TDS of G, contradicting the minimality of S. Hence, d and d_1 are part of an ABA-triple. But then $(S \setminus \{c_1, d\}) \cup \{f\}$ is a TDS of G, contradicting the minimality of S. Hence $d \neq e$.

Suppose $de \in E$. Then, $d = e_1$, $e = d_1$ and $d_2 = e_2$. But then $(S \setminus \{c_2, e, w\}) \cup \{c, f_1\}$ is a TDS of G, contradicting the minimality of S. Hence, $de \notin E$. By the claw-freeness of G, $\{d_1d_2, e_1e_2\} \in E$.

Suppose $d_1 \neq e_1$. If $\{d_1, e_1\} \subset A$, then by Property 1, $\{d, d_1\}$ and $\{e, e_1\}$ are both A-pairs and $T = (S \setminus \{c_1\}) \cup \{f\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$, $\xi(T) = \xi(S)$ and $\varphi(T) = \varphi(S)$ but with $\alpha(T) < \alpha(S)$, contradicting the choice of S. Therefore we may assume that $d_1 \in B$. If $\{d, d_1\}$ is an AB-pair, then $(S \setminus \{c_2, d_1, w\}) \cup \{c, f_1\}$ is a TDS of G, contradicting the minimality of S. Hence, d and d_1 are part of an ABA-triple. But then $(S \setminus \{c_1, d\}) \cup \{f\}$ is a TDS of G, contradicting the minimality of S. Hence $d_1 = e_1$. Thus, by the claw-freeness of G, $d_2 = e_2$. But then $(S \setminus \{d, c_1\}) \cup \{f\}$ is a TDS of G, contradicting the minimality of S. We deduce, therefore, that $f = g$. □

We now return to the proof of Claim 9. By Claim 9.3, $f = g$. Let $N(f) = \{c_2, c_3, h\}$. Since $f \in \text{epn}(c_2, S)$, we have that $h \notin S$. Let $N(h) = \{f, h_1, h_2\}$. By the claw-freeness of G, $h_1h_2 \in E$. To totally dominate h, we may assume that $h_1 \in S$.

Suppose $h_2 \notin S$. Let $N(h_1) = \{h, h_2, h_3\}$. To totally dominate h_1, we have that $h_3 \in S$. If $h_3 \in A$, then by Property 1, $\{h_1, h_3\}$ is an A-pair and $T = (S \setminus \{c_1\}) \cup \{f\}$ is a minimum TDS of G with $\lambda(T) = \lambda(S)$, $\iota(T) = \iota(S)$, $\beta(T) = \beta(S)$, $\xi(T) = \xi(S)$ and $\varphi(T) = \varphi(S)$ but with $\alpha(T) < \alpha(S)$, contradicting the choice of S. Hence, $h_3 \in B$. If $\{h_1, h_3\}$ is an AB-pair, then $(S \setminus \{c_2, h_3, w\}) \cup \{c, h\}$ is a TDS of G, contradicting the minimality of S. Hence, h_1 and h_3 are part of an ABA-triple. But then $T = (S \setminus \{c_2, h_1, w\}) \cup \{c, f, h\}$ is a minimum TDS of G with $\lambda(T) < \lambda(S)$, contradicting the choice of S. Hence, $h_2 \in S$.

If $\{h_1, h_2\} \subset A$, then, by Property 1, $\{h_1, h_2\}$ is an A-pair. Using Rule 9, we discharge a weight of $\frac{1}{4}$ from the A-pair $\{h_1, h_2\}$ to the AB-pair $\{c_1, c_2\}$ and a weight of $\frac{1}{4}$ from this AB-pair to the weak ABA-triple $\{u, v, w\}$ so that $\zeta(S') \geq \psi(S') + \frac{1}{4} = \frac{15}{4}$, as desired. (See Figure 10.8(b).) Hence, $\{h_1, h_2\} \not\subset A$. We may assume that $h_2 \in B$. Let $N(h_2) = \{h, h_1, h_3\}$. If $\{h_1, h_2\}$ is an AB-pair or a B-pair, then $(S \setminus \{c_2, h_2, w\}) \cup \{c, h\}$ is a TDS of G, contradicting the minimality of S. Hence, $\{h_1, h_2, h_3\}$ is an ABA-triple. Let $N(h_3) = \{h_2, h_4, h_5\}$. If $h_3 \in A_1$, then we may assume that $\text{epn}(h_3, S) = \{h_4\}$. But then $T = (S \setminus \{c_2, h_2, h_3, w\}) \cup \{c, h, h_5\}$ is a TDS of G, contradicting the minimality of S. Therefore, $h_3 \in A_2$. Using Rule 10, we discharge a weight of $\frac{1}{4}$ from the strong ABA-triple $\{h_1, h_2, h_3\}$ to the AB-pair $\{c_1, c_2\}$ and a weight of $\frac{1}{4}$ from this AB-pair to the weak ABA-triple $\{u, v, w\}$ so that $\zeta(S') \geq \psi(S') + \frac{1}{4} = \frac{15}{4}$, as desired. (See Figure 10.8(c).) This completes the proof of Claim 9. □

We conclude the section with the following claim.

Claim 10 *The average weight under g of every vertex in S is at least $\frac{5}{4}$.*

Proof. We show that each pair in S has weight of at least 5/2 under g and each triple in S has a weight of at least 15/4 under g. Let $S' \subset S$. If S' is a weak A-pair or a B-pair, then the result follows from Claims 5 to 8. If S' is a weak AB-pair, then no discharging rule alters the weight assigned to the pair, and so $\zeta(S') = \psi(S') = \frac{5}{2}$. If S' is a strong AB-pair, then a maximum weight of $\frac{1}{2}$ is discharged from S' and hence $\zeta(S') \geq \psi(S') - \frac{1}{2} = \frac{5}{2}$. If S' is a strong A-pair, then a maximum weight of $\frac{1}{2}$ is discharged from S' and hence $\zeta(S') \geq \psi(S') - \frac{1}{2} \geq \frac{5}{2}$. If S' is a weak ABA-triple, then the result follows from Claim 9. Finally, if S' is a strong ABA-triple, then a maximum weight of $\frac{1}{4}$ is discharged from S' and hence $\zeta(S') \geq \psi(S') - \frac{1}{4} = \frac{15}{4}$. □

Chapter 11

A Partition and a Bound

In our final chapter, we combine the partition first presented in Chapter 2 with the edge weighting function used in the previous two chapters and present a new bound. More specifically, we show that every connected cubic graph on n vertices has a total dominating set whose complement contains a dominating set such that the cardinality of the total dominating set is at most $(n+2)/2$, and this bound is essentially best possible.

Recently, several authors studied the cardinalities of pairs of disjoint dominating sets in graphs (see, for example, [20, 35, 50, 58, 59, 60, 61, 75, 77]), which serves to motivate this research into the cardinality of a total dominating set whose complement is a dominating set. We restrict our attention to cubic graphs.

11.1 Total Domination in Cubic Graphs

As presented in previous chapters, several authors, including Archdeacon et al. [2], Chvátal and McDiarmid [15], Thomassé and Yeo [96], and Tuza [97], established the following upper bound for the total domination number of a graph with minimum degree at least three.

Theorem 11.1 ([2, 15, 96, 97]) *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq n/2$.*

As an immediate consequence of Theorem 11.1, we have that the total domination number of a cubic graph is at most one-half its order. The generalized Petersen graph G_{16} of order $n = 16$ shown in Figure 11.1 achieves equality in Theorem 11.1.

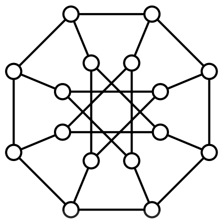


Figure 11.1: The generalized Petersen graph G_{16} of order 16.

Two infinite families $\mathcal{G}$ and $\mathcal{H}$ of connected cubic graphs (described below) with total domination number one-half their orders are constructed in [31]. For $k \geq 2$ consider two copies of the path P_{2k} with respective vertex sequences $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ and $c_1, d_1, c_2, d_2, \ldots, c_k, d_k$. For each $i \in \{1, 2, \ldots, k\}$, join a_i to d_i and b_i to c_i. To complete the construction of graphs in $\mathcal{G}$ ($\mathcal{H}$, respectively), join a_1 to c_1 and b_k to d_k (a_1 to b_k and c_1 to d_k, respectively). Two graphs G and H from the families $\mathcal{G}$ and $\mathcal{H}$ are illustrated in Figure 11.2.

Theorem 11.2 ([69]) *Let G be a connected graph of order n with $\delta(G) \geq 3$. Then, $\gamma_t(G) \leq n/2$, with equality if and only if $G \in \mathcal{G} \cup \mathcal{H}$ or G is the generalized Petersen graph G_{16} shown in Figure 11.1.*

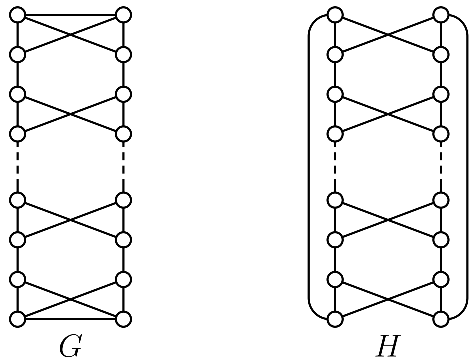


Figure 11.2: Cubic graphs $G \in \mathcal{G}$ and $H \in \mathcal{H}$ of order n with $\gamma_t(G) = \gamma_t(H) = n/2$.

11.2 DT-Pair Total Dominating Sets

Recall that a DT-pair of a graph G, if it exists, is a pair (D, T) of disjoint sets of vertices of G such that D is a DS and T is a TDS of G. We define a *DT-pair total dominating set*, abbreviated DT-pair TDS, to be a total dominating set $T \subseteq V$ such that $V \setminus T$ contains a dominating set. Following the previous notation in the literature, we define the *DT-pair total domination number* of G, denoted by $\gamma\gamma_t^*(G)$, to be the minimum cardinality of a DT-pair TDS of G. A DT-pair TDS of G of cardinality $\gamma\gamma_t^*(G)$ is called a $\gamma\gamma_t^*(G)$-set.

Since every DT-pair TDS of G is a TDS of G, we observe that $\gamma_t(G) \le \gamma\gamma_t^*(G)$. This inequality may be strict. To see that, consider for example the Petersen graph P shown in Figure 11.3. Every $\gamma_t(P)$-set is of the form $N[v]$, where v is an arbitrary vertex in P, but the set $V(P) \setminus N[v]$ is not a DS in P. Thus no $\gamma_t(P)$-set is a DT-pair TDS of P, and so $\gamma\gamma_t^*(P) > \gamma_t(P) = 4$. On the other hand, taking T to be the set of five vertices on the outer cycle of P (as drawn in Figure 11.3), we have a DT-pair TDS of P, and hence $\gamma\gamma_t^*(P) \le |T| = 5$. Consequently, the Petersen graph is a cubic graph of order $n = 10$ with $\gamma_t(P) = 4$ but with $\gamma\gamma_t^*(P) = 5 = n/2$. Consider also the cubic graph P' of order $n = 20$ constructed from two copies of the Petersen graph by removing an edge from each copy and adding the two edges shown in Figure 11.3. Then, $\gamma_t(P') = 8$, but $\gamma\gamma_t^*(P') = 9$.

We remark that if we restrict our attention to connected cubic graphs of girth at least 5, then a computer search produces three graphs G of order $n = 20$ with $\gamma_t(G) < \gamma\gamma_t^*(G)$, while there are 835 such graphs of order $n = 22$ with $\gamma_t(G) < \gamma\gamma_t^*(G)$, and 5890 such graphs of order $n = 24$ with $\gamma_t(G) < \gamma\gamma_t^*(G)$.

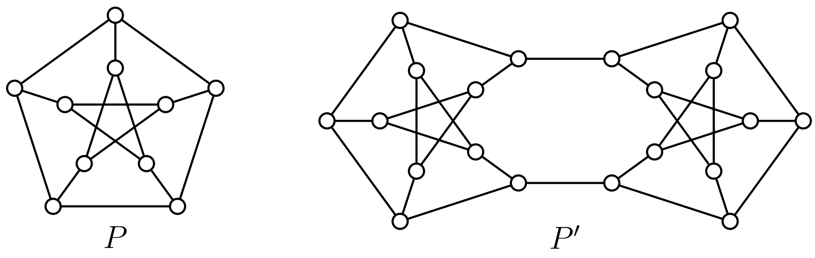


Figure 11.3: The Petersen Graph P and the constructed graph P' of order 20.

Our aim in this chapter is to establish an upper bound on the DT-pair total domination number of a connected cubic graph in terms of its order. We shall prove the following result, a proof of which can be found in Section 11.5.

Theorem 11.3 *If G is a connected cubic graph of order n, then $\gamma\gamma_t^*(G) \le (n+2)/2$.*

The bound of Theorem 11.3 is almost sharp since there exist two infinite families of connected cubic graphs G of order n such that $\gamma\gamma_t^*(G) = n/2$, as may be seen by the following result.

Proposition 11.4 *If $G \in \mathcal{G} \cup \mathcal{H} \cup \{G_{16}\} \cup \{P\}$ has order n, where G_{16} is the generalized Petersen graph shown in Figure 11.1 and P is the Petersen graph shown in Figure 11.3, then $\gamma\gamma_t^*(G) = n/2$.*

Proof. If $G = P$, then G has order $n = 10$ and as observed earlier (see Section 11.2), $\gamma\gamma_t^*(G) = n/2$. Suppose, then, that $G \in \mathcal{G} \cup \mathcal{H} \cup \{G_{16}\}$. If $G = G_{16}$, then G has order $n = 16$ and the vertices on the outer 8-cycle of G as drawn in Figure 11.1 form a DT-pair TDS of G, and hence $\gamma\gamma_t^*(G) \le n/2$. Suppose $G \in \mathcal{G} \cup \mathcal{H}$ has order $n = 4k$. Using the notation described earlier (see Section 11.1) to construct the families $\mathcal{G}$ and

$\mathcal{H}$, the set $S = \{a_1, a_2, \ldots, a_k\} \cup \{b_1, b_2, \ldots, b_k\}$ is a DT-pair TDS in G, and hence $\gamma\gamma_t^*(G) \le 2k = n/2$. Hence if $G \in \mathcal{G} \cup \mathcal{H} \cup \{G_{16}\}$ has order n, then $\gamma\gamma_t^*(G) \le n/2$. By Theorem 11.2, $\gamma_t(G) = n/2$. Consequently, since $\gamma_t(F) \le \gamma\gamma_t^*(F)$ for all graphs F, we have that $\gamma\gamma_t^*(G) = n/2$. □

11.3 Hypergraph Notation and Results

A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called *vertices*, together with a finite multiset $E = E(H)$ of arbitrary subsets of V, called *hyperedges* or simply *edges*. A k-edge in H is an edge of cardinality k in H. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. The *degree* of a vertex v in H, denoted $d_H(v)$ or simply $d(v)$ if H is clear from the context, is the number of edges of H which contain v. The hypergraph H is k-regular if every vertex has degree k in H. For a set F of edges in H, the hypergraph $H - F$ denotes the hypergraph obtained from H by deleting the edges from F. If F consists of a single edge e, we simply write $H - e$ rather than $H - \{e\}$.

Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$. Further, x and y are *connected* if there is a sequence $x = v_0, v_1, v_2 \ldots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1, 2, \ldots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A *(connected) component* of a hypergraph H is a maximal connected subhypergraph of H.

For a graph $G = (V, E)$, we denote by H_G the *open neighborhood hypergraph* of G; that is, H_G is the hypergraph with vertex set $V(H_G) = V$ and with edge multiset $E(H_G) = \{N_G(x) \mid x \in V(G)\}$ consisting of all the open neighborhoods of vertices in G.

A subset T of vertices in a hypergraph H is a *transversal* in H if T has a nonempty intersection with every edge of H. A transversal is also called an *edge cover* or *hitting set* in the literature. Much of the recent interest in total domination in graphs arises from the fact that a total dominating set in a graph G corresponds to a transversal in its open neighborhood hypergraph H_G. This idea of using transversals in hypergraphs to obtain results on total domination in graphs first appeared in a paper by Thomassé and Yeo [96], and subsequently in several other papers, including [68, 69, 70].

A hypergraph H is *bipartite* if its vertex set can be partitioned into two sets such that every hyperedge intersects both partite sets. Equivalently, H is bipartite if it is 2-*colorable*; that is, there is a 2-coloring of the vertices with no monochromatic hyperedge. By definition, every partite set in such a partition in a bipartite hypergraph H is a transversal of H. A hypergraph H is *minimally non-bipartite* if H is not bipartite but every subhypergraph of H, different from H itself, is bipartite. Seymour [84] proved the following property of minimally non-bipartite hypergraphs.

Theorem 11.5 ([84]) *Every minimally non-bipartite hypergraph has at least as many hyperedges as vertices.*

Using Seymour's Theorem 11.5 one can readily prove (or see [71]) the following result.

Corollary 11.6 ([71]) *Every connected* 3*-regular* 3*-uniform hypergraph is either bipartite or becomes bipartite on deleting any hyperedge from it.*

We shall need the following lemma from [69].

Lemma 11.7 ([69]) *If G is a connected bipartite graph, then H_G contains exactly two components* (*which are induced by the two partite sets of G*)*. If G is a connected non-bipartite graph, then H_G contains exactly one component.*

11.4 Preliminary Results

In order to prove our main theorem, namely Theorem 11.3, we first present a number of preliminary results.

Lemma 11.8 *Let G be a connected bipartite cubic graph and let uv be an edge in G. Then there exists a partition (D, T) of the vertices of G such that T totally dominates $V(G)$ and D totally dominates $V(G) \setminus \{u, v\}$.*

Proof. Let $N_G(u) = \{v, v_1, v_2\}$ and let $N_G(v) = \{u, u_1, u_2\}$. Let U and V be the partite sets of G containing u and v, respectively, and note that $\{u_1, u_2\} \subset U$ and $\{v_1, v_2\} \subset V$. We now consider the open neighborhood hypergraph H_G of G. Since G is a cubic graph, H_G is a 3-regular, 3-uniform hypergraph. Further since G is bipartite, by Lemma 11.7, we have that H_G contains two components, one with vertex set U and the other with vertex set V. For $X \in \{U, V\}$, let H_X be the component of H_G with vertex set X. Necessarily, each of H_U and H_V is a 3-regular, 3-uniform hypergraph.

Let $e_u = \{v, v_1, v_2\}$ and $e_v = \{u, u_1, u_2\}$ be the hyperedges of H corresponding to the open neighborhoods of u and v, respectively, in G. Then, $e_u \in E(H_V)$ and $e_v \in E(H_U)$. We now consider the hypergraphs $H'_U = H_U - e_v$ and $H'_V = H_V - e_u$. By Corollary 11.6, both H'_U and H'_V are bipartite. Let U_1 and U_2 be the partite sets in some bipartition of H_U and let V_1 and V_2 be the partite sets in some bipartition of H_V. Renaming sets if necessary, we may assume that $u \in U_1$ and $v \in V_1$.

Let $T = U_1 \cup V_1$ and let $D = U_2 \cup V_2$. Now, since U_1 and U_2 are both transversals in H'_U and since V_1 and V_2 are both transversals in H'_V, we have that T intersects every hyperedge in H_G and D intersects every hyperedge in H_G with the possible exceptions of the hyperedges e_u and e_v. Hence, in the graph G, the set T totally dominates $V(G)$ and the set D totally dominates $V(G) \setminus \{u, v\}$. $\Box$

Lemma 11.9 *Let G be a connected non-bipartite cubic graph and let $v \in V(G)$. Then there exists a partition (D, T) of the vertices of G such that T totally dominates $V(G)$ and D totally dominates $V(G) \setminus \{v\}$.*

Proof. Consider the open neighborhood hypergraph H_G of G. Since G is a cubic graph, H_G is a 3-regular, 3-uniform hypergraph. Further since G is non-bipartite, by Lemma 11.7, we have that H_G is connected. Let $e_v = \{v_1, v_2, v_3\}$ be the hyperedge of H corresponding to the open neighborhood of v in G, and consider the hypergraph $H'_G = H_G - e_v$. By Corollary 11.6, H'_G is bipartite. Let D and T be the partite sets in some bipartition of H_G. Renaming sets if necessary, we may assume that $v_1 \in T$. Now, since D and T are both transversals in H'_G, we have that T intersects every hyperedge in H_G and D intersects every hyperedge in H_G with the possible exception of the hyperedge e_v. Hence in the graph G, the set T totally dominates $V(G)$ and the set D totally dominates $V(G) \setminus \{v\}$. $\square$

We now introduce some additional notation which will be useful in the proofs of the lemmas that follow. For a graph G, let $\iota(G) = \{v \in V(G) \mid d_G(v) = 0\}$; that is, $\iota(G)$ is the set of isolated vertices in G.

Lemma 11.10 *Let $G = (V, E)$ be a connected cubic graph and let $v \in V$. If (D, T) is a partition of V such that T totally dominates V in G, D totally dominates $V \setminus \{v\}$, and $N[v] \subseteq T$, then there exists a DT-pair in G such that the subgraph induced by the dominating set in the DT-pair contains at most seven isolated vertices.*

Proof. Let (D, T) be a partition of V as defined in the statement of the lemma. Then, every vertex in V except for the vertex v has a neighbor in both T and D. In particular, $\iota(G[D]) = \emptyset$. Further, since D does not dominate the vertex v, the set D is not a dominating set of G. Let $N(v) = \{w, x, y\}$, $N(w) = \{v, w_1, w_2\}$, $N(x) = \{v, x_1, x_2\}$ and

$N(y) = \{v, y_1, y_2\}$. We note that the sets $\{w, w_1, w_2\}$, $\{x, x_1, x_2\}$ and $\{y, y_1, y_2\}$ are not necessarily pairwise disjoint.

If $T \setminus \{v\}$ totally dominates V, then $(D \cup \{v\}, T \setminus \{v\})$ is a DT-pair and $\iota(G[D \cup \{v\}]) = \{v\}$. We may therefore assume that $T \setminus \{v\}$ does not totally dominate V, for otherwise the desired result follows. Hence, renaming vertices if necessary, we may assume that $\{w_1, w_2\} \subseteq D$. If $T \setminus \{w\}$ totally dominates V, then $(D \cup \{w\}, T \setminus \{w\})$ is a DT-pair with $\iota(G[D \cup \{w\}]) = \emptyset$, and the desired result follows. We may therefore assume that $T \setminus \{w\}$ does not totally dominate V and, renaming vertices if necessary, that $N(w_1) \cap T = \{w\}$. Let $N(w_1) \cap D = \{w_1', w_2'\}$.

Let $D_1 = D \setminus \{w_1\}$ and let $T_1 = T \cup \{w_1\}$.Then, T_1 totally dominates V and D_1 dominates $V \setminus \{v\}$. Furthermore, $\iota(G[D_1]) \subseteq \{w_1', w_2'\}$. If $T_1 \setminus \{v\}$ totally dominates V, then $(D_1 \cup \{v\}, T_1 \setminus \{v\})$ is a DT-pair, $\iota(G[D_1 \cup \{v\}]) \subseteq \{v, w_1', w_2'\}$, and the desired result follows. We may therefore assume that $T_1 \setminus \{v\}$ does not totally dominate V. The only possible vertices not totally dominated by $T_1 \setminus \{v\}$ are x and y. Renaming vertices if necessary, we may assume that $\{x_1, x_2\} \subseteq D_1$, and so x is not totally dominated by $T_1 \setminus \{v\}$. If $T_1 \setminus \{x\}$ totally dominates V, then $(D_1 \cup \{x\}, T_1 \setminus \{x\})$ is a DT-pair, $\iota(G[D_1 \cup \{x\}]) \subseteq \{w_1', w_2'\}$, and the desired result follows. Hence we may assume that $T_1 \setminus \{x\}$ does not totally dominate V and, renaming vertices if necessary, that $N(x_1) \cap T_1 = \{x\}$. Let $N(x_1) \cap D_1 = \{x_1', x_2'\}$.

Let $D_2 = D_1 \setminus \{x_1\}$ and let $T_2 = T_1 \cup \{x_1\}$. Then, T_2 totally dominates V and D_2 dominates $V \setminus \{v\}$. Furthermore, $\iota(G[D_2]) \subseteq \{w_1', w_2', x_1', x_2'\}$. If $T_2 \setminus \{v\}$ totally dominates V, then $(D_2 \cup \{v\}, T_2 \setminus \{v\})$ is a DT-pair, $\iota(G[D_2 \cup \{v\}]) \subseteq \{v, w_1', w_2', x_1', x_2'\}$, and the desired result follows. We may therefore assume that $T_2 \setminus \{v\}$ does not totally dominate V. The only possible vertex not totally dominated by $T_2 \setminus \{v\}$ is y, and so $\{y_1, y_2\} \subseteq D_2$. If $T_2 \setminus \{y\}$ totally dominates V, then $(D_2 \cup \{y\}, T_2 \setminus \{y\})$ is a DT-pair, $\iota(G[D_2 \cup \{y\}]) \subseteq \{w_1', w_2', x_1', x_2'\}$, and the desired result follows. Hence we may assume that $T_2 \setminus \{y\}$ does not totally dominate V and, renaming vertices if necessary, that $N(y_1) \cap T_2 = \{y\}$. Let $N(y_1) \cap D_2 = \{y_1', y_2'\}$.

Let $D_3 = D_2 \setminus \{y_1\}$ and let $T_3 = T_2 \cup \{y_1\}$. Then, T_3 totally dominates V and D_3 dominates $V \setminus \{v\}$. Furthermore, $\iota(G[D_3]) \subseteq \{w_1', w_2', x_1', x_2', y_1', y_2'\}$. But now $T_3 \setminus \{v\}$ totally dominates V, and so $(D_3 \cup \{v\}, T_3 \setminus \{v\})$ is a DT-pair, $\iota(G[D_3 \cup \{v\}]) \subseteq \{v, w_1', w_2', x_1', x_2', y_1', y_2'\}$, and the desired result follows. □

Lemma 11.11 *If G is a connected non-bipartite cubic graph, then there exists a DT-pair in G such that the subgraph induced by the dominating set in the DT-pair contains at most seven isolated vertices.*

Proof. Let $G = (V, E)$ be a connected non-bipartite cubic graph and let $v \in V$. By Lemma 11.9 there exists a partition (D, T) of the vertices of G such that T totally dominates V and D totally dominates $V \setminus \{v\}$. If D totally dominates V, then D and T are both total dominating sets, and so (D, T) is a DT-pair with $\iota(G[D]) = \emptyset$. Hence we may assume that D does not totally dominate the vertex v. Therefore, $N(v) \subseteq T$. If $v \in D$, then since D totally dominates $V \setminus \{v\}$, we have that D is a dominating set, and so (D, T) is a DT-pair and $\iota(G[D]) = \{v\}$ and the desired result follows. We may therefore assume that $v \in T$. But now we have that (D, T) is a partition of V such that T totally dominates V in G, the set D totally dominates $V \setminus \{v\}$, and $N[v] \subseteq T$. The desired result now follows from Lemma 11.10. □

Lemma 11.12 *If G is a connected cubic graph, then there exists a DT-pair in G such that the subgraph induced by the dominating set in the DT-pair contains at most seven isolated vertices.*

Proof. Let $G = (V, E)$ be a connected cubic graph. If G is non-bipartite, then the result follows from Lemma 11.11. We may therefore assume that G is bipartite. Let $uv \in E$. By Lemma 11.8 there exists a partition (D, T) of the vertices of G such that T

totally dominates V and D totally dominates $V \setminus \{u, v\}$. Let $N(u) = \{v, w, x\}$ and let $N(v) = \{u, y, z\}$.

If D totally dominates V, then D and T are both total dominating sets and the desired result follows since (D, T) is a DT-pair and $\iota(G[D]) = \emptyset$. Hence we may assume that D does not totally dominate $\{u, v\}$. Renaming vertices if necessary, we may assume that $\{u, y, z\} \subseteq T$, and so v is not totally dominated by D. If $v \in D$, then D dominates V, and so (D, T) is a DT-pair with $\iota(G[D]) = \{v\}$, implying the desired result. Hence we may assume that $v \in T$. If $\{w, x\} \cap D \neq \emptyset$, then (D, T) is a partition of V such that T totally dominates V in G, the set D totally dominates $V \setminus \{v\}$, and $N[v] \subseteq T$. The desired result then follows from Lemma 11.10. We may therefore assume that $\{w, x\} \subset T$. Thus, $\{u, v, w, x, y, z\} \subseteq T$.

We note that $\iota(G[D]) = \emptyset$. However, D dominates neither u nor v and is therefore not a dominating set in G. Let $N(w) = \{u, w_1, w_2\}$, $N(x) = \{u, x_1, x_2\}$, $N(y) = \{v, y_1, y_2\}$, and $N(z) = \{v, z_1, z_2\}$. We note that the sets $\{w, w_1, w_2\}$, $\{x, x_1, x_2\}$, $\{y, y_1, y_2\}$ and $\{z, z_1, z_2\}$ are not necessarily pairwise disjoint but that G has no odd cycles since it is bipartite.

If $T \setminus \{u\}$ totally dominates V, then $(D \cup \{u\}, T \setminus \{u\})$ is a DT-pair, $\iota(G[D \cup \{u\}]) = \{u\}$, and the desired result follows. We may therefore assume that $T \setminus \{u\}$ does not totally dominate V. The only possible vertices not totally dominated by $T \setminus \{u\}$ are w and x. Renaming vertices if necessary, we may assume that $\{w_1, w_2\} \subseteq D$, and so w is not totally dominated by $T \setminus \{u\}$. By symmetry, we may assume that $T \setminus \{v\}$ does not totally dominate V and that $\{y_1, y_2\} \subseteq D$. Since G is bipartite, we note that $\{w_1, w_2\} \cap \{y_1, y_2\} = \emptyset$.

If $T \setminus \{w, y\}$ totally dominates V, then $(D \cup \{w, y\}, T \setminus \{w, y\})$ is a DT-pair, $\iota(G[D \cup \{w, y\}]) = \emptyset$, and the desired result follows. We may therefore assume that $T \setminus \{w, y\}$ does not totally dominate V. The only possible vertices not totally dominated by $T \setminus \{w, y\}$

are neighbors of w and y different from u and v. Renaming vertices if necessary, we may assume that $N(w_1) \cap T = \{w\}$, and so w_1 is not totally dominated by $T \setminus \{w, y\}$. Let $N(w_1) \cap D = \{w'_1, w'_2\}$.

Let $D_1 = D \setminus \{w_1\}$ and let $T_1 = T \cup \{w_1\}$. Then, T_1 totally dominates V and D_1 dominates $V \setminus \{u, v\}$. Furthermore, $\iota(G[D_1]) \subseteq \{w'_1, w'_2\}$. If $T_1 \setminus \{u\}$ totally dominates V, then $(D_1 \cup \{u\}, T_1 \setminus \{u\})$ is a DT-pair, $\iota(G[D_1 \cup \{u\}]) \subseteq \{u, w'_1, w'_2\}$, and the desired result follows. We may therefore assume that $T_1 \setminus \{u\}$ does not totally dominate V. The only possible vertex not totally dominated by $T_1 \setminus \{u\}$ is the vertex x, implying that $\{x_1, x_2\} \subset D_1$. Since $D_1 = D \setminus \{w_1\} \subset D$, we have $\{x_1, x_2\} \subset D$.

If $T \setminus \{x\}$ totally dominates V, then $(D \cup \{x\}, T \setminus \{x\})$ is a partition of V such that $T \setminus \{x\}$ totally dominates V in G, the set $D \cup \{x\}$ totally dominates $V \setminus \{v\}$, and $N[v] \subseteq T \setminus \{x\}$. The desired result then follows from Lemma 11.10. We may therefore assume that $T \setminus \{x\}$ does not totally dominate V. The only possible vertices not totally dominated by $T \setminus \{x\}$ are neighbors of x different from u. Renaming vertices if necessary, we may assume that $N(x_1) \cap T = \{x\}$, and so x_1 is not totally dominated by $T \setminus \{x\}$.

Let $N(x_1) \cap D = \{x'_1, x'_2\}$. Let $D_2 = (D \setminus \{w_1, x_1\}) \cup \{u\}$ and let $T_2 = (T \cup \{w_1, x_1\}) \setminus \{u\}$. Then, T_2 totally dominates V and D_2 dominates V. Thus, (D_2, T_2) is a DT-pair. Furthermore, $\iota(G[D_2]) \subseteq \{u, w'_1, w'_2, x'_1, x'_2\}$, and the desired result follows. $\square$

11.5 Proof of Theorem 11.3

We are now ready to prove our main result, namely Theorem 11.3. Let us recall its statement.

Theorem 11.3. *If G is a connected cubic graph of order n, then $\gamma\gamma_t^*(G) \leq (n + 2)/2$.*

Proof. Let $G = (V, E)$ be a connected cubic graph of order n. By Lemma 11.12, there exists a partition (D', T') of V so that D' dominates V, the set T' totally dominates V and $|\iota(G[D'])| \le 7$. Let T be the smallest subset of T' (possibly, $T = T'$) such that T totally dominates V. Let $D = V \setminus T$ and note that $D' \subseteq D$ with equality if and only if $T = T'$. Since D' dominates V, so does D. Thus, (D, T) is a DT-pair in G.

We show that every isolated vertex in $G[D]$ is an isolated vertex in $G[D']$. Let $v \in \iota(G[D])$. Then $N(v) \subseteq T \subseteq T' = V \setminus D'$. Since D' dominates V, we must have that $v \in D'$ and thus, $v \in \iota(G[D'])$. Consequently, $\iota(G[D]) \subseteq \iota(G[D'])$, and so $|\iota(G[D])| \le |\iota(G[D'])| \le 7$. Let $D_1 = \iota(G[D])$ and let $D_2 = D \setminus D_1$. Then $|D_1| \le 7$.

We now use an edge-weighting argument on the edges that join D to T. For this purpose, we define a function ψ_1 that assigns a weight to each vertex $v \in D$ as follows. To each vertex that is isolated in $G[D]$ we assign a weight of 3/2 and to every other vertex in D we assign a weight of 1; that is,

$$\psi_1(v) = \begin{cases} 3/2 & \text{if } v \in D_1 \\ 1 & \text{if } v \in D_2. \end{cases}$$

Then,

$$\sum_{v \in D} \psi_1(v) = 3|D_1|/2 + |D_2| = |D| + |D_1|/2 \le |D| + 7/2. \tag{11.1}$$

We now define a function $\psi_2 : [T, D] \to [0, 1]$ that assigns a weight to each edge in $[T, D]$. For each vertex $v \in D$, the weight $\psi_1(v)$ is equally distributed among the edges joining v to T. Thus if e is an edge joining $v \in D_1$ to T, then $\psi_2(e) = \psi_1(v)/3 = 1/2$ and the sum of the weights assigned to the three edges joining v to T is 3/2. If e is an edge joining $v \in D_2$ to T, then $\psi_2(e) = 1/d_T(v)$, where $d_T(v)$ denotes the number of vertices in T adjacent to v. In this case, $\psi_2(e) \in \{\frac{1}{2}, 1\}$ and the sum of the weights assigned to the edges joining v to T is 1. By our construction,

$$\sum_{e\in[T,D]} \psi_2(e) = \sum_{v\in D} \psi_1(v). \tag{11.2}$$

Finally, we define a function ψ_3 that assigns to each subset $T^* \subseteq T$ the sum of the weights of the edges from T^* to D; that is,

$$\psi_3(T^*) = \sum_{e\in[T^*,D]} \psi_2(e).$$

If $T^* = T$, then $\psi_3(T^*)$ is the sum of the weights of all edges in $[T, D]$. We proceed further with the following claim.

Claim $\psi_3(T) \geq |T|$.

Proof. Let G^* be a component of $G[T]$ and let $T^* = V(G^*)$. It suffices to show that $\psi_3(T^*) \geq |T^*|$. Since (D, T) is a DT-pair in G, every vertex in T has degree 1 or 2 in $G[T]$. Hence G^* is either a cycle, or a path on at least two vertices.

Suppose that G^* is a cycle. Then, $|[T^*, D]| = |T^*|$. Let $e^* = xy \in [T^*, D]$, where $x \in T^*$ and $y \in D$. If $d_T(y) > 1$, then $T \setminus \{x\}$ is a subset of T that totally dominates V, contradicting the minimality of T. Hence, $d_T(y) = 1$ and $N_G(y) \cap T = \{x\}$. Thus, $d_D(y) = 2$, and so $\psi_1(y) = \psi_2(e^*) = 1$. Therefore, $\psi_3(T^*) = \sum_{e\in[T^*,D]} \psi_3(e) = |T^*|$, as desired. We may therefore assume that G^* is a path on at least two vertices.

Let G^* be the path $x_1x_2 \ldots x_k$, where $k = |T^*|$. Let $N_G(x_1) = \{x_2, y_1, y_1'\}$ and let $N_G(x_k) = \{x_{k-1}, y_k, y_k'\}$. Necessarily, $\{y_1, y_1'\} \subseteq D$ and $\{y_k, y_k'\} \subseteq D$. If $k = 2$, then $|[T^*, D]| = 4$ and since $\psi_3(e) \geq 1/2$ for each $e \in [T, D]$, we have that $\psi_3(T^*) \geq 2 = |T^*|$, as desired. We may therefore assume that $k \geq 3$. For $i \in \{2, \ldots, k-1\}$, let $N_G(x_i) = \{x_{i-1}, x_{i+1}, y_i\}$ and note that $y_i \in D$. For $i = 1, 2, \ldots, k$, let $e_i = x_iy_i$. Further, let

$e'_1 = x_1y'_1$ and let $e'_k = x_ky'_k$.

If $d_T(y_1) > 1$ and $d_T(y'_1) > 1$, then $T \setminus \{x_1\}$ is a subset of T that totally dominates V, contradicting the minimality of T. Hence, renaming vertices if necessary, we may assume that $d_T(y_1) = 1$, and so $d_D(y_1) = 2$. Thus, $\psi_1(y_1) = \psi_2(e_1) = 1$. By a similar argument we may assume that $d_T(y_k) = 1$, and so $\psi_1(y_k) = \psi_2(e_k) = 1$. If $k = 3$, then $[T^*, D] = \{e_1, e'_1, e_2, e_3, e'_3\}$. Since $\psi_2(e_1) = \psi_2(e_3) = 1$, while $\psi_3(e) \geq 1/2$ for each $e \in [T, D] \setminus \{e_1, e_3\}$, we have that $\psi_3(T^*) \geq 7/2 > 3 = |T^*|$. If $k = 4$, then $|[T^*, D]| = 6$ and by the same reasoning we have that $\psi_3(T^*) \geq 4 = |T^*|$. Hence we may assume that $k \geq 5$, for otherwise $\psi_3(T^*) \geq |T^*|$, as desired.

For $i \in \{3, \ldots, k-2\}$, if $d_T(y_i) > 1$, then $T \setminus \{x_i\}$ is a subset of T that totally dominates V, contradicting the minimality of T. Hence, $d_T(y_i) = 1$, and so $\psi_1(y_i) = \psi_2(e_i) = 1$ for $i \in \{3, \ldots, k-2\}$. As observed earlier, $\psi_2(e_1) = \psi_2(e_k) = 1$. Moreover, $\psi_3(e) \geq 1/2$ for each $e \in \{e'_1, e_2, e_{k-1}, e'_k\}$. Thus since $[T^*, D] = \{e'_1, e'_k\} \cup \{e_1, e_2, \ldots, e_k\}$, we have that $\psi_3(T^*) \geq k = |T^*|$. This completes the proof of the claim. □

We now return to the proof of Theorem 11.3. By definition of the function $\psi_3(T)$, Inequality (11.1), Equality (11.2) and the above claim, we have that

$$|T| \leq \psi_3(T) = \sum_{e \in [T,D]} \psi_2(e) = \sum_{v \in D} \psi_1(v) \leq |D| + 7/2.$$

Thus, since $|D| = n - |T|$, we get $|T| \leq n/2 + 7/4$. However, every cubic graph has an even number of vertices, and hence $n/2$ is an integer. Thus since $|T|$ is an integer, we have that $|T| \leq n/2 + 1$. This completes the proof of Theorem 11.3. □

Bibliography

[1] S. Ao, E. J. Cockayne, G. MacGillivray, and C. M. Mynhardt, Domination critical graphs with higher independent domination numbers. *J. Graph Theory* **22** (1996), 9-14.

[2] D. Archdeacon, J. Ellis-Monaghan, D. Fischer, D. Froncek, P. C. B. Lam, S. Seager, B. Wei, and R. Yuster, Some remarks on domination. *J. Graph Theory* **46** (2004), 207–210.

[3] C. Barefoot, F. Harary, and K. F. Jones, What is the difference between the domination and independent domination numbers of cubic graph? *Graphs Combin.* **7** (1991), 205-20.

[4] C. Berge. *Theory of Graphs and its Applications.* Methuen, London, 1962.

[5] B. Brešar, Vizing-like conjecture for the upper domination of Cartesian products of graphs – the proof. *Electron. J. Combin.* **12** (2005), Note 12, 6 pp.

[6] I. Broere, M. Dorfling, W. Goddard, J. H. Hattingh, M. A. Henning, and E. Ungerer, Augmenting trees to have two disjoint total dominating sets. *Bull. Institute Combin. Applic.* **42** (2004), 12–18.

[7] N. J. Calkin and P. Dankelmann, The domatic number of regular graphs. *Ars Combin.* **73** (2004), 247-255.

[8] X. G. Chen, W. C. Shiu, and H. Y. Chen, Trees with equal total domination and total restrained domination numbers. *Discuss. Math. Graph Theory* **28** (2008), 59–66.

[9] T. C. E. Cheng, L. Y. Kang, and C. T. Ng, Paired domination on interval and circular-arc graphs. *Discrete Appl. Math.* **155** (2007), 2077–2086.

[10] M. Chudnovsky and P. Seymour, Claw-free graphs. I. Orientable prismatic graphs. *J. Combin. Theory Ser. B* **97** (2007), no. 6, 867–903.

[11] M. Chudnovsky and P. Seymour, Claw-free graphs. II. Non-orientable prismatic graphs. *J. Combin. Theory Ser. B* **98** (2008), no. 2, 249–290.

[12] M. Chudnovsky and P. Seymour, Claw-free graphs. III. Circular interval graphs. *J. Combin. Theory Ser. B* **98** (2008), no. 4, 812–834.

[13] M. Chudnovsky and P. Seymour, Claw-free graphs. IV. Decomposition theorem. *J. Combin. Theory Ser. B* **98** (2008), no. 5, 839–938.

[14] M. Chudnovsky and P. Seymour, Claw-free graphs. V. Global structure. *J. Combin. Theory Ser. B* **98** (2008), no. 6, 1373–1410.

[15] V. Chvátal and C. McDiarmid, Small transversals in hypergraphs. *Combinatorica* **12** (1992), 19–26.

[16] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, Total domination in graphs. *Networks* **10** (1980), 211–219.

[17] E. J. Cockayne, O. Favaron, C. M. Mynhardt, Total domination in claw-free cubic graphs. *J. Combin. Math. Combin. Comput.* **43** (2002), 219–225.

[18] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs. *Networks* **7** (1977), 247–261.

[19] E. J. Cockayne and C. M. Mynhardt, Independence and domination in 3-connected cubic graphs. *J. Combin. Math. Combin. Comput.* **10** (1991), 173-182.

[20] G. S. Domke, J. E. Dunbar, and L. R. Markus, The inverse domination number of a graph. *Ars Combin.* **72** (2004), 149-160.

[21] P. Dorbec and S. Gravier, Paired-domination in P_5-free graphs. *Graphs Combin.* **24** (2008), 303–308.

[22] P. Dorbec, S. Gravier, and M. A. Henning, Paired-domination in generalized claw-free graphs. *J. Combin. Optim.* **14** (2007), 1–7.

[23] P. Dorbec, M. A. Henning, and D. F. Rall, On the upper total domination number of Cartesian products of graphs. *J. Comb. Optim.* **16** (2008), 68-80.

[24] M. Dorfling, W. Goddard, J. H Hattingh, and M. A. Henning, Augmenting a graph of minimum degree 2 to have two disjoint total dominating sets. *Discrete Math.* **300** (2005), 82–90.

[25] M. Dorfling, W. Goddard, M. A. Henning, and C. M. Mynhardt, Construction of trees and graphs with equal domination parameters. *Discrete Math.* **306** (2006), 2647–2654.

[26] Q. Fang, On the computational complexity of upper total domination. *Discrete Appl. Math.* **136** (2004), 13-22.

[27] O. Favaron and M. A. Henning, Upper total domination in claw-free graphs. *J. Graph Theory* **44** (2003), 148–158.

[28] O. Favaron and M. A. Henning, Paired domination in claw-free cubic graphs. *Graphs Combin.* **20** (2004), 447–456.

[29] O. Favaron and M. A. Henning, Total domination in claw-free graphs with minimum degree two. *Discrete Math.* **308** (2008), 3213–3219.

[30] O. Favaron and M. A. Henning, Bounds on total domination in claw-free cubic graphs. *Discrete Math.* **308** (2008), 3491–3507.

[31] O. Favaron, M. A. Henning, C. M. Mynhardt, and J. Puech, Total domination in graphs with minimum degree three. *J. Graph Theory* **34** (2000), 9–19.

[32] U. Feige, M. M. Halldórsson, G. Kortsarz, and A. Srinivasan, Approximating the domatic number. *SIAM J. Comput.* **32** (2002), 172–195.

[33] S. Fitzpatrick and B. Hartnell, Paired-domination. *Discuss. Math. Graph Theory* **18** (1998), 63–72.

[34] E. Flandrin, R. Faudree, and Z. Ryjáček, Claw-free graphs—a survey. *Discrete Math.* **164** (1997), 87–147.

[35] A. Frendrup, M. A. Henning, P. D. Vestergaard, and B. Randerath, On a conjecture about inverse domination in graphs. *Ars Combin.* **97A** (2010), 129–143.

[36] W. Goddard and M. A. Henning, A characterization of cubic graphs with paired-domination number three-fifths their order. *Graphs Combin.* **25** (2009), 675-692.

[37] W. Goddard, M. A. Henning, J. Lyle, and J. Southey, On the independent domination number of regular graphs. To appear in *Annals Combin.*

[38] P. J. P. Grobler and C. M. Mynhardt, Vertex criticality for upper domination and irredundance. *J. Graph Theory* **37** (2001), 205-212.

[39] G. Gutin and V. E. Zverovich, Upper domination and upper irredundance perfect graphs. *Discrete Math.* **190** (1998), 95-105

[40] J. H. Hattingh, E. Jonck, E. J. Joubert, and A. R. Plummer, Total restrained domination in trees. *Discrete Math.* **307** (2007), 1643–1650.

[41] J. H. Hattingh, E. Jonck, E. J. Joubert, and A. R. Plummer, Nordhaus-Gaddum results for restrained domination and total restrained domination in graphs. *Discrete Math.* **308** (2008), 1080–1087.

[42] J. Haviland, Independent domination in regular graphs. *Discrete Math.* **143** (1995), 275–280.

[43] J. Haviland, Upper bounds for independent domination in regular graphs. *Discrete Math.* **307** (2007), 2643–2646.

[44] P. Haxell, B. Seamone, and J. Verstraete, Independent dominating sets and Hamiltonian cycles. *J. Graph Theory* **54** (2007), 233-244.

[45] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[46] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[47] T. W. Haynes and M. A. Henning, Trees with large paired-domination number. *Utilitas Mathematica* **71** (2006), 3–12.

[48] T. W. Haynes and P. J. Slater, Paired-domination and the paired-domatic number. *Congr. Numer.* **109** (1995), 65–72.

[49] T. W. Haynes and P. J. Slater, Paired-domination in graphs. *Networks* **32** (1998), 199–206.

[50] S. M. Hedetniemi, S. T. Hedetniemi, R. C. Laskar, L. Markus, and P. J. Slater, Disjoint dominating sets in graphs. Discrete mathematics, 87–100, *Ramanujan Math. Soc. Lect. Notes Ser.*, 7, Ramanujan Math. Soc., Mysore, 2008.

[51] P. Heggernes and J. A. Telle, Partitioning graphs into generalized dominating sets. *Nordic J. Comput.* **5** (1998), 128–142.

[52] M. A. Henning, Trees with equal total domination and paired-domination numbers. *Utilitas Math.* **69** (2006), 207–218.

[53] M. A. Henning, Graphs with large paired-domination number. *J. Combin. Optim.* **13** (2007), 61–78.

[54] M. A. Henning, Recent results on total domination in graphs: A survey. *Discrete Math.* **309** (2009), 32–63.

[55] M. A. Henning, E. J. Joubert, and J. Southey, Nordhaus-Gaddum bounds for total domination. *Applied Mathematics Letters* **24** (2011), 987–990.

[56] M. A. Henning, E. J. Joubert, and J. Southey, Nordhaus-Gaddum type results for total domination. To appear in *Discrete Mathematics & Theoretical Computer Science.*

[57] M. A. Henning, E. J. Joubert, and J. Southey, Multiple factor Nordhaus-Gaddum type results for domination and total domination. To appear in *Discrete Applied Mathematics.*

[58] M. A. Henning, C. Löwenstein, and D. Rautenbach, Remarks about disjoint dominating sets. *Discrete Math.* **309** (2009), 6451–6458.

[59] M. A. Henning, C. Löwenstein, and D. Rautenbach, An independent dominating set in the complement of a minimum dominating set of a tree. *Applied Mathematics Letters* **23** (2010), 79–81.

[60] M. A. Henning, C. Löwenstein, and D. Rautenbach, Partitioning a graph into a dominating set, a total dominating set, and something else. *Discussiones Mathematicae Graph Theory* **30**(4) (2010), 563–574.

[61] M. A. Henning, C. Löwenstein, D. Rautenbach, and J. Southey, Disjoint dominating and total dominating sets in graphs. *Discrete Applied Math.* **158** (2010), 1615–1623

[62] M. A. Henning and J. E. Maritz, Total restrained domination in graphs with minimum degree two. *Discrete Math.* **308** (2008), 1909–1920.

[63] M. A. Henning and C. M. Mynhardt, The diameter of paired-domination vertex critical graphs. *Czechoslovak Math. J.* **58** (2008), 887–897.

[64] M. A. Henning and M. D. Plummer, Vertices contained in all or in no minimum paired-dominating set of a tree. *J. Combin. Optim.* **10** (2005), 283–294.

[65] M. A. Henning and J. Southey, A note on graphs with disjoint dominating and total dominating sets. *Ars Combin.* **89** (2008), 159–162.

[66] M. A. Henning and J. Southey, A characterization of graphs with disjoint dominating and total dominating sets. *Quaestiones Mathematicae* **32** (2009), 119-129.

[67] M. A. Henning and P. D. Vestergaard, Trees with paired-domination number twice their domination number. *Utilitas Math.* **74** (2007), 187–197.

[68] M. A. Henning and A. Yeo, Total domination in graphs with given girth. *Graphs Combin.* **24** (2008), 333-348.

[69] M. A. Henning and A. Yeo, Hypergraphs with large transversal number and with edge sizes at least three. *J. Graph Theory* **59** (2008), 326–348.

[70] M. A. Henning and A. Yeo, Total domination in 2-connected graphs and in graphs with no induced 6-cycles. *J. Graph Theory* **60** (2009), 55-79.

[71] M. A. Henning and A. Yeo, 2-Colorings in k-regular k-uniform hypergraphs, manuscript.

[72] H. Jiang, L. Kang, and E. Shan, Total restrained domination in cubic graphs. *Graphs Combin.* **25** (2009), 341–350.

[73] König D., Über Graphen und ihre Anwendung auf Determinantheorie und Mengenlehre. *Math. Ann.* **77** (1916), 453–465.

[74] A. V. Kostochka, The independent domination number of a cubic 3-connected graph can be much larger than its domination number. *Graphs Combin.* **9** (1993), 235–237.

[75] V. R. Kulli and S. C. Sigarkanti, Inverse domination in graphs. *Nat. Acad. Sci. Lett.* **14** (1991), 473-475.

[76] P. C. B. Lam, W. C. Shiu, and L. Sun, On independent domination number of regular graphs. *Discrete Math.* **202** (1999), 135–144.

[77] C. Löwenstein and D. Rautenbach, Pairs of disjoint dominating sets and the minimum degree of graphs. *Graphs Combin.* **26** (2010), 407–424.

[78] J. Lyle and W. Goddard, Independent domination in regular graphs. In preparation.

[79] D. X. Ma, X. G. Chen, and L. Sun, On total restrained domination in graphs. *Czechoslovak Math. J.* **55** (2005), 165–173.

[80] O. Ore, *Theory of graphs*, *Amer. Math. Soc. Transl.* **38** (Amer. Math. Soc., Providence, RI, 1962), 206-212.

[81] H. Qiao, L. Kang, M. Cardei, and Ding-Zhu. Du, Paired-domination of trees. *J. Global Optim.* **25** (2003), 43–54.

[82] J. Raczek and J. Cyman, On the total restrained domination number of a graph. *Australas. J. Combin.* **36** (2006), 91–100.

[83] J. Raczek and J. Cyman, Total restrained domination numbers of trees. *Discrete Math.* **308** (2008), 44–50.

[84] P. D. Seymour, On the two coloring of hypergraphs. *Quart. J. Math. Oxford Ser.* **25** (1974), 303–312.

[85] W. C. Shiu, X. Chen, and W. H. Chan, Triangle-free graphs with large independent domination number. *Discrete Optim.* **7** (2010), 86-92.

[86] P. J. Slater, Fault-tolerant locating-dominating sets. *Discrete Math.* **249** (2002), 179–189.

[87] J. Southey and M. A. Henning, Graphs with disjoint dominating and paired-dominating sets. *Central European Journal of Mathematics* **8** (2010), 459–467.

[88] J. Southey and M. A. Henning, A characterization of graphs with disjoint dominating and paired-dominating sets. *J. Combin. Optim.* **22** (2011), 217–234.

[89] J. Southey and M. A. Henning, An improved upper bound on the total restrained domination number in cubic graphs. To appear in *Graphs Combin.* (Online first: 2 June 2011)

[90] J. Southey and M. A. Henning, Domination versus independent domination in cubic graphs. To appear in *Discrete Math.*

[91] J. Southey and M. A. Henning, On a conjecture on total domination in claw-free cubic graphs. *Discrete Math.* **310** (2010), 2984–2999.

[92] J. Southey and M. A. Henning, Dominating and total dominating partitions in cubic graphs. *Central European Journal of Mathematics* **9** (2011), 699–708.

[93] J. Southey and M. A. Henning, Edge weighting functions on dominating set. To appear in *J. Graph Theory.*

[94] L. Sun and J. Wang, An upper bound for the independent domination number. *J. Combin. Theory Ser. B* **76** (1999), 240-246.

[95] J. Telle and A. Proskurowski, Algorithms for vertex partitioning problems on partial k-trees. *SIAM J. Discrete Math.* **10** (1997), 529–550.

[96] S. Thomassé and A. Yeo, Total domination of graphs and small transversals of hypergraphs. *Combinatorica* **27** (2007), 473–487.

[97] Z. Tuza, Covering all cliques of a graph. *Discrete Math.* **86** (1990), 117–126.

[98] Chen Xue-Gang, Sun Liang, and Xing Hua-Ming, Paired-domination numbers of cubic graphs (Chinese). *Acta Math. Sci. Ser. A Chin. Ed.* **27** (2007), 166–170.

[99] B. Zelinka, Total domatic number and degrees of vertices of a graph. *Math. Slovaca* **39** (1989), 7–11.

[100] B. Zelinka, Domatic numbers of graphs and their variants: A survey, in *Domination in Graphs: Advanced Topics*, T.W. Haynes et al. eds, Marcel Dekker, New York, 1998, 351–377.

[101] B. Zelinka, Remarks on restrained domination and total restrained domination in graphs. *Czechoslovak Math. J.* **55** (2005), 393–396.

[102] J. Žerovnik and J. Oplerova, A counterexample to conjecture of Barefoot, Harary, and Jones. *Graphs Combin.* **9** (1993), 205-207.

[103] I. E. Zverovich and V. E. Zverovich. Disproof of a conjecture in the domination theory. *Graphs Combin.* **10** (1994), 389-396.

[104] I. E. Zverovich and V. E. Zverovich, A semi-induced subgraph characterization of upper domination perfect graphs. *J. Graph Theory* **31** (1999), 29-49.