CHARACTERIZATIONS OF SCALARS IN BANACH ALGEBRAS

by

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Introduction

Amongst the most well-behaved and recognizable elements in a Banach algebra are scalars, that is, multiples of the identity. In spectral theory and ordered structures we readily come across such elements. Under the guise of the abstract setting of a Banach algebra, it is often the case that the elements we are dealing with are none-other than scalars. Characterizations of such elements therefore remain of significant importance. In the discussion that follows we give characterizations of the identity and, more generally, scalars.

The first chapter is a recollection of some theory relating to Banach algebras, spectral theory, commutative Banach algebras and C^*-algebras. Although brief, some important results and concepts are discussed here, to facilitate our work in the subsequent chapters.

In the second chapter we introduce some spectral characterizations of scalars in Banach algebras. This chapter follows [7] closely. Amongst the techniques that we employ in this chapter, are those of subharmonic analysis, where the results discussed in [3] were of significant importance. To a large extent, this work is a continuation and generalization of the work done on characterizations of the radical in Banach algebras by R.Brits, [8]. Our main result here, is that if A is a semisimple Banach algebra and $a \in A$ has the property that the number of elements in the spectrum of ax is less than or equal to the number of elements in the spectrum of x for all x in an arbitrary neighbourhood of the identity, then a is a scalar. Furthermore, as a consequence of our main result and others, we obtain new spectral characterizations of commutative Banach algebras. In particular we show that A is commutative if and only if it has the property that the number of elements in the spectrum remains invariant under all permutations of three elements in a neighbourhood of the identity.

In the final chapter we introduce an ordering on the Banach algebra via an algebra cone. In contrast to the subharmonic and spectral tools that were used in the previous chapter, here we rely almost entirely on the structure given by the algebra cone and different forms of norm-boundedness. Here we follow [6] closely. The focus of this chapter are the so-called Gelfand-Hille Theorems, namely conditions under which an element in an ordered Banach algebra with spectrum $\{1\}$ is the identity of the algebra. In particular we show that if an element a and its inverse belong to a closed normal algebra cone, then if a has unit spectrum and is doubly Abel bounded, it is the identity. Furthermore, our main result in this chapter is that if the spectrum of a is $\{1\}$, then a is the identity if and only if some natural power of a is Abel bounded and some natural power of a dominates the identity (relative to a closed proper algebra cone).

Chapter 1

Some reminders about Banach algebras and spectral theory

Before we get started, we give a brief overview of Banach algebras, spectral theory, and related concepts. Included here are some reminders and remarks that will be useful for the ensuing discussion in Chapters 2 and 3. Unfortunately, many beautiful results, not directly related to the work that follows, have been omitted.

1.1 Banach algebras

An *algebra* over a field K is a vector space A over K such that for all $x, y, z \in A$ and $\alpha \in K$:

1. There is a unique product $xy \in A$
2. $(xy)z = x(yz)$ (associative under multiplication)
3. $x(y + z) = xy + xz$ (left distributive over addition)
4. $(x + y)z = xz + yz$ (right distributive over addition)
5. $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

If, in addition, we have that for all $x, y \in A$,

$$xy = yx,$$

then A is said to be *Abelian* or *commutative*. In particular, the set of elements $x \in A$ for which $xz = zx$ for all $z \in A$ is called the *center* of A, denoted by $\mathrm{Z}(A)$ throughout.

If there exists an element $\mathbf{1} \in A$ such that for all $x \in A$

$$\mathbf{1}x = x\mathbf{1} = x$$

then A is said to be an *algebra with identity/unit* and $\mathbf{1}$ is the *identity/unit* of A.

If A is an algebra with identity, then $x \in A$ is said to be *invertible* if there exists an element, written $x^{-1} \in A$ such that

$$xx^{-1} = x^{-1}x = \mathbf{1}.$$

The set of invertible elements of A will henceforth be denoted by A^{-1}.

If the field $K = \mathbb{R}$ or $K = \mathbb{C}$, then A is said to be a *real algebra* or *complex algebra* respectively.

A *subalgebra* B of an algebra A, is a subspace of A that is algebraically closed under the operation of multiplication.

A *normed algebra* is an algebra A equipped with a norm $\|\cdot\|$ such that the norm is submultiplicative i.e. for all $x, y \in A$

$$\|xy\| \leq \|x\| \cdot \|y\|.$$

Moreover, we can assume that $\|\mathbf{1}\| = 1$ since otherwise we can replace the norm $\|\cdot\|$ with an equivalent norm $\|\cdot\|_1$ such that this property holds.

A *Banach algebra* is a complete normed algebra.

If A is a Banach algebra without unit, then A can be transformed into a unital Banach algebra $\tilde{A} = A \times \mathbb{C} = \{(x, \alpha) : x \in A, \alpha \in \mathbb{C}\}$. Addition, scalar multiplication, and multiplication are defined respectively as $(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$; $\lambda(x, \alpha) = (\lambda x, \lambda\alpha)$ and $(x, \alpha) \cdot (y, \beta) = (xy + \beta x + \alpha y, \alpha\beta)$. The norm $\|\cdot\|_{\tilde{A}}$ on $\tilde{A}$ is given by $\|(x, \alpha)\|_{\tilde{A}} = \|x\|_A + |\alpha|$.

Throughout, A will denote a complex unital Banach algebra with unit, $\mathbf{1}$.

1.2 Spectral theory

If A is a Banach algebra, then the *spectrum* of $x \in A$, denoted $\sigma(x)$, is the set

$$\{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin A^{-1}\}.$$

It can be shown that $\sigma(x)$ is compact and nonempty for any x in a complex Banach algebra (see [3, Theorem 3.2.8]).
Furthermore, the mapping $\lambda \mapsto (\lambda\mathbf{1} - x)^{-1}$ is analytic on $\mathbb{C} \setminus \sigma(x)$ and goes to

zero at infinity. $R_\lambda(x) = (\lambda\mathbf{1} - x)^{-1}$ is called the *resolvent* of x.

The following lemma, by N. Jacobson, tells us that $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$ for all $x, y \in A$.

Theorem 1.2.1 (N. Jacobson, [3, Lemma 3.1.2])**.** *Let A be an algebra with $x, y \in A$ and $0 \neq \lambda \in \mathbb{C}$. Then $\lambda\mathbf{1} - xy \in A^{-1}$ if and only if $\lambda\mathbf{1} - yx \in A^{-1}$.*

If A is a Banach algebra, then $x \in A$ is said to be *nilpotent* if $x^n = 0$ for some $n \in \mathbb{N}$.

In a Banach algebra A, of great significance are the elements with trivial spectrum: if $q \in A$ has $\sigma(q) = \{0\}$ then q is said to be *quasinilpotent.* The set of quasinilpotent elements of A will be denoted by $\mathrm{QN}(A)$ throughout. It is well known that in a finite-dimensional Banach algebra all quasinilpotents are nilpotent.

A special subset of the quasinilpotents plays a significant role in spectral theory, since these elements behave in a similar way to 0: $x \in A$ is said to be a *radical element* if $xA \subseteq \mathrm{QN}(A)$.
From Jacobson's Lemma [3, Lemma 3.1.2], mentioned above, it follows that this definition is equivalent to saying that $Ax \subseteq \mathrm{QN}(A)$. The set of radical elements is then referred to as the *radical* of A, denoted by $\mathrm{Rad}(A)$. In particular, it is clear that $\mathrm{Rad}(A) \subseteq \mathrm{QN}(A)$.
In the case where $\mathrm{Rad}(A) = \{0\}$, then A is referred to as *semisimple.*
The majority of the results that follow make use of a semisimple Banach algebra, although if A is not semisimple, then the quotient algebra $A/\mathrm{Rad}(A)$ will suffice as a semisimple replacement for A.

The next theorem is very useful since it gives a series expansion for invertible elements of a specific form; it also tells us that all elements within the unit ball centered at $\mathbf{1}$ are invertible.

Theorem 1.2.2 ([3, Theorem 3.2.1])**.** *Let A be a Banach algebra. If $x \in A$ satisfies $\|x\| < 1$, then $\mathbf{1} - x$ is invertible and*

$$(\mathbf{1} - x)^{-1} = \mathbf{1} + \sum_{j=1}^{\infty} x^j.$$

The *spectral radius*, $\rho(x)$, of an x in A is defined as

$$\rho(x) = \sup_{\lambda \in \sigma(x)} |\lambda|.$$

Remarkably, this algebraic definition of the spectral radius is equivalent to the following, topological definition

$$\rho(x) = \lim_{n \to \infty} \|x^n\|^{1/n}.$$

Moreover, from this definition it is clear that $\rho(x) \leq \|x\|$ for all $x \in A$.

F.F Bonsall and J. Duncan [5, Chapter 1.2, Theorem 9] proved that the above theorem is also true if $\rho(x) < 1$. We will use this result frequently in Chapter 3, so we state it here for reference

Theorem 1.2.3 ([5, Chapter 1.2, Theorem 9]). *Let A be a Banach algebra. If $x \in A$ satisfies $\rho(x) < 1$, then $\mathbf{1} - x$ is invertible and*

$$(\mathbf{1} - x)^{-1} = \mathbf{1} + \sum_{j=1}^{\infty} x^j.$$

In general $\rho(x+y) \neq \rho(x)+\rho(y)$ and $\rho(xy) \neq \rho(x)\rho(y)$ for $x, y \in A$. However, if x and y commute then the spectral radius is subadditive and submultiplicative, that is:

$$\rho(x + y) \leq \rho(x) + \rho(y) \ \text{ and } \ \rho(xy) \leq \rho(x)\rho(y)$$

if $xy = yx$ [3, Corollary 3.2.10].

Let $x \in A$ and suppose that Ω is an open set containing $\sigma(x)$. Furthermore, let Γ be an arbitrary smooth contour included in Ω and surrounding $\sigma(x)$. For functions f, analytic on Ω

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda\mathbf{1} - x)^{-1} d\lambda.$$

Here $f(x)$ is defined via the so-called holomorphic functional calculus, see [3, §III.3].

The following well-known theorem is invaluable in spectral theory.

Theorem 1.2.4 (Spectral Mapping Theorem, [3, Theorem 3.3.3]). *Let A be a Banach algebra and $x \in A$. Then for all functions f analytic on a neighbourhood of $\sigma(x)$,*

$$\sigma(f(x)) = f(\sigma(x)).$$

Finally, we mention some other spectral parameters of an $x \in A$ that will be encountered frequently in the next chapter:

- $\delta(x) = \sup\{|\lambda_1 - \lambda_2| : \lambda_1, \lambda_2 \in \sigma(x)\}$, the *spectral diameter* of x
- $\mathrm{Arg}(x) = \{\mathrm{Arg}(\lambda) : \lambda \in \sigma(x)\}$, the *spectral argument* of x
- $\#\sigma(x)$, the cardinality of the set $\sigma(x)$

1.3 Commutative Banach algebras

Throughout this section we will assume that A is a commutative Banach algebra. A *multiplicative linear functional / character* on A is a nonzero linear functional χ, such that for all $x, y \in A$,

$$\chi(xy) = \chi(x)\chi(y).$$

From the above definition it can be shown (see [3, Chapter IV]) that

- $\chi(\mathbf{1}) = 1$
- $\chi(x) \in \sigma(x)$ for all $x \in A$
- χ is continuous with norm 1.

The collection of all characters on A will be denoted by $\Delta(A)$ or simply by Δ if it is clear what Banach algebra is under consideration.

The following discovery was made by I.M. Gelfand and gives us much to work with when dealing with the spectrum of elements in commutative Banach algebras.

Theorem 1.3.1 (I.M. Gelfand, [3, Theorem 4.1.2])**.** *Let A be a commutative Banach algebra and $x \in A$. Then*

$$\sigma(x) = \{\chi(x) : \chi \in \Delta\}.$$

From the above theorem, it follows that for a commutative Banach algebra A and $x \in A$,

$$\rho(x) = \max\{|\chi(x)| : \chi \in \Delta\}.$$

In general, if B is a subalgebra of a Banach algebra A then $\sigma_A(x) \subseteq \sigma_B(x)$ for all $x \in B$. It is worth mentioning, and we shall use the fact that, if B is a maximal commutative subalgebra then $\sigma_A(x) = \sigma_B(x)$ for all $x \in A$.

1.4 C^*-algebras

If A is an algebra, then an *involution* on A is a map $* : A \to A$ such that for $x, y \in A$ and $\alpha, \beta \in \mathbb{C}$

1. $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$
2. $(x^*)^* = x$
3. $(xy)^* = y^* x^*$

(where $\overline{\alpha}$, $\overline{\beta}$ are the complex conjugates of α and β respectively).
If A is an algebra and $*$ is an involution, then $x \in A$ is said to be

- *normal* if $xx^* = x^*x$
- *self-adjoint* if $x = x^*$
- *unitary* if $xx^* = x^*x = \mathbf{1}$.

For any Banach algebra with involution,

$$\sigma(x^*) = \{\overline{\lambda} : \lambda \in \sigma(x)\}.$$

Hence $\rho(x^*) = \rho(x)$ from the above statement.

A C^*-*algebra* is a Banach algebra A with an involution such that for all $x \in A$

$$\|x^*x\| = \|x\|^2.$$

C^*-algebras naturally induce an ordering $\geq$ given by $x \geq 0$ if $x = x^*$ and $\sigma(x) \subseteq [0, \infty)$. If $x \geq 0$ then it is referred to as *positive*.

In a C^*-algebra A it is always true that

$$xx^* \geq 0$$

for all $x \in A$ [3, Theorem 6.2.11].

If x is self-adjoint in a C^*-algebra, then $\sigma(x) \subseteq \mathbb{R}$.

Chapter 2

Spectral characterizations of scalars in Banach algebras

In many respects the work discussed in this chapter can be seen as a second installment to the work done in [8], where multiplicative spectral characterizations of the Jacobson radical were given.

Throughout this chapter, $B(a, R)$ (and $S(a, R)$) will be used to denote an open ball (and sphere, respectively), centered at a with radius R (for $0 < R \in \mathbb{R}$).

We shall begin this chapter with a brief look at subharmonic functions and some important subharmonic results that will be used in this chapter, all of which can be found in [3].

In the second section we introduce some spectral characterizations using different spectral parameters, including the spectral radius, spectral diameter, the arguments of the spectrum, and the number of elements in the spectrum. In each case the assumption is a form of multiplicative contraction or invariance under these spectral parameters. It is worth noting that for most of these characterizations, the spectral assumption is only required to hold in an arbitrarily small neighbourhood of the identity and not for all elements in the algebra. Our main result is that if A is semisimple and $a \in A$ is such that $\#\sigma(ax) \leq \#\sigma(x)$ for all x in an arbitrary neighbourhood of $\mathbf{1}$ then $a = \alpha\mathbf{1}$ $(\alpha \in \mathbb{C})$.

The third section deals with some interesting examples where we show the necessity of the assumptions in our results of the preceding section. Moreover, we characterize the spectra of elements satisfying forms of multiplicative invariance under the spectral radius, norm and spectral argument.

In our final section, we discuss some characterizations of commutative Banach algebras, which stem from the results in the preceding sections. Specifically, as a consequence of our main result, we have that A is commutative if and only if $\#\sigma(xzy) = \#\sigma(yzx)$ for all x, y, z in a neighbourhood of $\mathbf{1}$.

2.1 A brief word on subharmonic functions

This section contains a very brief look into the beautiful world of subharmonic functions. Many theorems that have been proved using other methods have far more elegant subharmonic proofs. [3] contains some very useful results and applications of subharmonic functions, and all the results mentioned below can be found there.

Firstly, we consider the definition of a subharmonic function

Let D be a domain of $\mathbb{C}$ and ϕ a function, $\phi : D \to \mathbb{R} \cup \{-\infty\}$. ϕ is said to *subharmonic* on D if it is upper semicontinuous on D and satisfies

$$\phi(\lambda_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(\lambda_0 + re^{i\theta}) d\theta$$

for all $\overline{B(\lambda_0, r)} \subseteq D$. This inequality is the so-called *mean inequality.*

Subharmonic functions have some useful properties and are invaluable tools in analysis. [3, Appendix] gives a summary of some of these properties, without proof, including the following

Theorem 2.1.1 ([3, Theorem A.1.2])**.** *If ϕ is a subharmonic function on an open set D, then for every $a \in D$*

$$\phi(a) = \limsup_{\substack{z \to a \\ z \neq a}} \phi(z)$$

Theorem 2.1.2 (Maximum principle for subharmonic functions, [3, Theorem A.1.3])**.** *Let ϕ be a subharmonic function on a domain D of $\mathbb{C}$. If there exists an $a \in D$ such that $\phi(z) \leq \phi(a)$ for all $z \in D$, then $\phi(z) = \phi(a)$ for all $z \in D$.*

Corollary 2.1.3 ([3, Corollary A.1.4])**.** *Let ϕ be subharmonic on a bounded domain D. If there exists an $0 < M \in \mathbb{R}$ such that*

$$\limsup_{\substack{z \to \xi \\ z \in D}} \phi(z) \leq M$$

for every $\xi \in \partial D$, then $\phi(z) < M$ on D or ϕ is constant on D.
For an unbounded domain D, if in addition to the assumptions above, $\limsup_{\substack{z \to \infty \\ z \in D}} \phi(z) \leq M$*, then the same is true.*

Theorem 2.1.4 (Liouville's theorem for subharmonic functions, [3, Theorem A.1.11]). *If ϕ is subharmonic on $\mathbb{C}$ and*

$$\liminf_{r\to\infty} \frac{\max_{0\leq\theta\leq 2\pi} \phi(re^{i\theta})}{\ln r} = 0,$$

then ϕ is constant.

The following result, due to E. Vesentini and given in [3, Theorem 3.4.7], tells us that the spectral radius is subharmonic over analytic functions:

Theorem 2.1.5 (Vesentini, [3, Theorem 3.4.7]). *Let D be a domain of $\mathbb{C}$ and f be an analytic function from D into a Banach algebra A. Then the mapping $\phi : \lambda \mapsto \rho(f(\lambda))$ is subharmonic on D.*

If f is an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A then from subharmonic analysis, several conclusions can be drawn if we know something about the spectrum of $f(\lambda)$ as λ varies in D. For instance, the next result tells us that if the images of D under f have real spectrum, then the spectrum of these images is the same.

Theorem 2.1.6 ([3, Corollary 3.4.12]). *Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. If $\sigma(f(\lambda)) \subseteq \mathbb{R}$ for all $\lambda \in D$ then $\sigma(f(\lambda))$ is constant on D.*

Finally, we mention the so-called Scarcity Theorem, which we shall use extensively in this chapter.
The notion of the *capacity* of a Borel subset of $\mathbb{C}$ is discussed in [3], but is essentially a measure of the size of the set. It is worth mentioning, however, that any non-trivial open ball $B(z_0, r)$ ($z_0 \in \mathbb{C}$, $r > 0$) has nonzero capacity.

Theorem 2.1.7 (Scarcity of elements with finite spectrum, [3, Theorem 3.4.25]). *Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Then either the set of $\lambda \in D$ such that $\sigma(f(\lambda))$ is finite, is a Borel set having zero capacity, or there exists an integer $n \geq 1$ and a closed discrete set $E \subseteq D$ such that $\#\sigma(f(\lambda)) = n$ for $\lambda \in D \setminus E$ and $\#\sigma(f(\lambda)) < n$ for $\lambda \in E$. In the latter case, the n points of $\sigma(f(\lambda))$ are locally holomorphic functions on $D \setminus E$.*

2.2 Some characterizations of scalars

In this section we introduce several characterizations of scalars making use of different spectral parameter assumptions.

According to [3, Theorem 5.3.1] the following statements are equivalent

- $a \in \mathrm{Rad}(A)$,
- There exist $0 < R, C \in \mathbb{R}$ such that $\rho(x) \leq C\|x - a\|$ for all x satisfying $\|x - a\| < R$,

- $\rho(a+q) = 0$ for all $q \in \mathrm{QN}(A)$.

Making use of this result, we have the following two theorems.

Note that the spectrum is said to be *Lipschitzian* at a if there exist $0 < R, C \in \mathbb{R}$ such that $\Delta(\sigma(x), \sigma(a)) \leq C\|x-a\|$ for all x satisfying $\|x-a\| < R$ (where Δ represents the Hausdorff distance).

Theorem 2.2.1. *Let A be a semisimple Banach algebra and $a \in A$. If the spectrum is Lipschitzian at a and $\sigma(a) = \{\alpha\}$, then $a = \alpha\mathbf{1}$.*

Proof. Since the spectrum is Lipschitzian at a it follows that there exists $0 < R, C \in \mathbb{R}$ such that $\Delta(\sigma(x), \sigma(a)) \leq C\|x-a\|$ for all x satisfying $\|x-a\| < R$. Clearly, $\Delta(\sigma(x), \sigma(a)) = \Delta(\sigma(x-\alpha\mathbf{1}), \sigma(a-\alpha\mathbf{1}))$. Since $\sigma(a-\alpha\mathbf{1}) = \{0\}$, taking $x - \alpha\mathbf{1} = y$, from our assumption

$$\rho(y) = \Delta(\sigma(y), \{0\}) = \Delta(\sigma(y), \sigma(a-\alpha\mathbf{1})) \leq C\|x-a\| = C\|y-(a-\alpha\mathbf{1})\|$$

for all y such that $\|y-(a-\alpha\mathbf{1})\| < R$. From [3, Theorem 5.3.1] it follows that $a - \alpha\mathbf{1} \in \mathrm{Rad}(A) = \{0\}$. Thus $a = \alpha\mathbf{1}$. □

If an element in a semisimple Banach algebra has the property that, under multiplication, it leaves all elements in some neighbourhood of the identity spectrally invariant, then clearly that element has analogs with the identity. The following theorem states that only the identity of the algebra has this property. Later, we shall show that the following theorem is a simple corollary of our main result.

Theorem 2.2.2. *Let A be a semisimple Banach algebra and $a \in A$. Then $a = \mathbf{1}$ if and only if $\sigma(ax) = \sigma(x)$ for all x in a neighbourhood of $\mathbf{1}$.*

Proof. Obviously we only need to prove the reverse implication.
Clearly if we take $x = \mathbf{1}$, then $\sigma(a) = \{1\}$ and hence a is invertible. Let $q \in \mathrm{QN}(A)$.
For λ sufficiently small, say $\lambda \in B(0,R)$, then $\lambda a^{-1}q + \mathbf{1}$ will be in the neighbourhood of $\mathbf{1}$ for which the hypothesis holds. Hence, for all such $\lambda \in \mathbb{C}$, from our hypothesis

$$\sigma(\lambda q + a) = \sigma\big(a(\lambda a^{-1}q + \mathbf{1})\big) = \sigma(\lambda a^{-1}q + \mathbf{1}) = \sigma(\lambda a^{-1}q) + 1.$$

Thus $\sigma(\lambda q + a - \mathbf{1}) = \lambda\sigma(a^{-1}q)$ and so

$$\rho\big(q + \frac{1}{\lambda}(a-\mathbf{1})\big) = \rho(a^{-1}q)$$

for all $0 \neq \lambda \in B(0,R)$. Furthermore,

$$\rho\big(q + \frac{1}{\lambda}(a-\mathbf{1})\big) \leq \|q + \frac{1}{\lambda}(a-\mathbf{1})\| \leq \|q\| + \left|\frac{1}{\lambda}\right| \cdot \|a-\mathbf{1}\| \leq \|q\| + \frac{1}{R}\|a-\mathbf{1}\|$$

for all $\lambda \in \mathbb{C} - B(0,R)$. Hence if we combine the above information

$$\rho\big(q + \frac{1}{\lambda}(a-\mathbf{1})\big) \leq M$$

for all $\lambda \in \mathbb{C} - \{0\}$, where $0 < M \in \mathbb{R}$. Furthermore

$$\limsup_{\lambda \to 0} \rho\big(q + \frac{1}{\lambda}(a - \mathbf{1})\big) \leq M.$$

Hence, taking $\mu = \frac{1}{\lambda}$ it follows that the subharmonic function $\phi : \mu \mapsto \rho\big(q + \mu(a - \mathbf{1})\big)$ is such that

$$\phi(\mu) = \rho\big(q + \mu(a - \mathbf{1})\big) \leq M \text{ on } \mathbb{C}$$

and

$$\limsup_{\mu \to \infty} \phi(\mu) \leq M.$$

Hence by [3, Corollary A.1.4] and Liouville's Theorem [3, Theorem A.1.11], ϕ is constant. Taking $\mu = 0$, we have that $\rho(q) = 0$. Hence for $\mu = 1$,

$$\rho\big(q + (a - \mathbf{1})\big) = 0.$$

Since $q \in \mathrm{QN}(A)$ was chosen arbitrarily, it follows from [3, Theorem 5.3.1] that $a - \mathbf{1} \in \mathrm{Rad}(A) = \{0\}$. Thus $a = \mathbf{1}$. □

The proof of the next theorem is fairly obvious, but nonetheless useful in conjunction with the theorems that follow.

Theorem 2.2.3. *Let A be a semisimple Banach algebra and $a \in A$ such that $\sigma(a) = \{\alpha\}$. Then $a = \alpha\mathbf{1}$ if and only if $a \in \mathrm{Z}(A)$.*

Proof. Clearly if $a = \alpha\mathbf{1}$ then $a \in \mathrm{Z}(A)$.
On the other hand, if $a \in \mathrm{Z}(A)$

$$\rho\big((a - \alpha\mathbf{1})x\big) \leq \rho(a - \alpha\mathbf{1})\rho(x) = 0$$

for all $x \in A$ (since $a - \alpha\mathbf{1}$ and x commute). It follows that $a - \alpha\mathbf{1} \in \mathrm{Rad}(A) = \{0\}$. Thus $a = \alpha\mathbf{1}$. □

From the theorem above we see that it is sufficient to show that $a \in A$ having single spectrum belongs to $\mathrm{Z}(A)$ in order to infer that a is a scalar. In [3] some characterizations of the center and scalars of a Banach algebra are given, specifically [3, Theorem 5.2.1, 5.2.2, 5.2.4, 5.3.2]; these theorems, in conjunction with the above theorem, provide useful tools in finding characterizations of scalars. Several of the results that follow are in fact sufficient conditions for an element to belong to the center.

The following theorem follows from, and can be seen as a multiplicative version of, the result in [3, Theorem 5.2.2], namely that if there exists an $M > 0$ such that $\rho(a + x) \leq M(1 + \rho(x))$ for all $x \in A$, then $a \in \mathrm{Z}(A)$.

Theorem 2.2.4. *Let A be a semisimple Banach algebra and $a \in A$ such that $\rho(ax) \leq \rho(x)$ for all $x \in A^{-1}$. Then $a \in \mathrm{Z}(A)$.*

Proof. Let $x \in A$ and $\lambda \in \mathbb{C}$ such that $1 + \rho(x) < |\lambda|$. Then $\text{dist}(\lambda, \sigma(x)) > 1$ and since $\lambda \notin \sigma(x)$ we also have that $\lambda\mathbf{1} - x$ is invertible. Now,

$$\lambda\mathbf{1} - (a + x) = (\lambda\mathbf{1} - x)\big[\mathbf{1} - R_\lambda(x)a\big].$$

But, from the hypothesis

$$\rho(R_\lambda(x)a) \leq \rho(R_\lambda(x)) = \frac{1}{\text{dist}(\lambda, \sigma(x))} < 1$$

(where the last equality follows from [3, Theorem 3.3.5]). Hence we can conclude that $\mathbf{1} - R_\lambda(x)a$ is invertible, and so $\lambda\mathbf{1} - (a+x)$ is invertible i.e. $\lambda \notin \sigma(a+x)$. It follows that if $\mu \in \sigma(a + x)$ then $|\mu| \leq 1 + \rho(x)$. Therefore,

$$\rho(a + x) \leq 1 + \rho(x) \quad \text{for all } \ x \in A.$$

Thus $a \in \mathrm{Z}(A)$ [3, Theorem 5.2.2]. □

In the section hereafter we discuss the spectra of elements satisfying the above theorem, but with the stronger assumption of equality, that is, spectral invariance of the radius.

Notice that if $a \in A$ is such that $\sigma(a) = \{\alpha\}$ and $\rho(ax) \leq \rho(x)$ for all $x \in A^{-1}$ then, taking $x = \mathbf{1}$ we have $\rho(a) \leq 1$. Therefore, as a consequence of the above theorem, together with Theorem 2.2.3 we have the following corollary.

Corollary 2.2.5. *Let A be a semisimple Banach algebra and $a \in A$ such that $\rho(ax) \leq \rho(x)$ for all $x \in A^{-1}$. If $\sigma(a) = \{\alpha\}$ $(\alpha \in \mathbb{C})$ then $a = \alpha\mathbf{1}$, with $|\alpha| \leq 1$.*

If it is known that a is quasinilpotent in a semisimple Banach algebra and $\text{Arg}(ax) \subseteq \text{Arg}(x)$ for all x in a neighbourhood of $\mathbf{1}$, then [8, Theorem 2.8] tells us that $a = 0$. As a generalization of this result we have the following theorem.

Theorem 2.2.6. *Let A be a semisimple Banach algebra and suppose $a \in A$ has $\sigma(a) = \{\alpha\}$. Then $a = \alpha\mathbf{1}$ where $\alpha \in \mathbb{R}^+ \cup \{0\}$ if and only if $\text{Arg}(ax) \subseteq \text{Arg}(x)$ for all x in a neighbourhood of $\mathbf{1}$.*

Proof. Obviously we only need to prove the reverse implication. If $\alpha = 0$ then $a \in \text{QN}(A)$ and the result follows from [8, Theorem 2.8]. Otherwise, from the hypothesis (by taking $x = \mathbf{1}$), we get $\text{Arg}(a) \subseteq \{0\}$ and so $\alpha \in \mathbb{R}^+$. Dividing by α, if necessary, we may assume that $\sigma(a) = \{1\}$ and then prove that $a = \mathbf{1}$.
Let q be an arbitrary quasinilpotent. For $|\lambda|$ sufficiently small, say $|\lambda| < R$, then $\mathbf{1} + \lambda q$ is in a neighbourhood of $\mathbf{1}$ and so from our assumption,

$$\text{Arg}(a(\mathbf{1} + \lambda q)) \subseteq \text{Arg}(\mathbf{1} + \lambda q) = \{0\}.$$

If we consider the analytic function $f : \lambda \mapsto a(\mathbf{1} + \lambda q)$ then the above equation tells us that $\sigma(f(\lambda)) \subseteq \mathbb{R}$ for all $|\lambda| < R$. Hence, by [3, Corollary 3.4.12], it follows that for all $|\lambda| < R$, we have $\sigma\left(a(\mathbf{1} + \lambda q)\right)$ is constant. Taking $\lambda = 0$ it follows that $\sigma(a(\mathbf{1}+\lambda q)) = \sigma(a) = \{1\}$. Therefore $\#\sigma(a(\mathbf{1}+\lambda q)) = 1$ for all $|\lambda| < R$. However, since f is analytic on all of $\mathbb{C}$, and $\{\lambda : |\lambda| < R\}$ is a set having nonzero capacity; the Scarcity Theorem says that $\#\sigma(a(1+\lambda q)) = 1$ for all $\lambda \in \mathbb{C}$, and, moreover there is an entire function g such that $\sigma(a(\mathbf{1}+\lambda q)) = \{g(\lambda)\}$. But g being constant on $B(0,R)$ it must be constant on $\mathbb{C}$, from the Identity Principle. So we obtain for all $\lambda \neq 0$

$$\rho\left(\frac{a}{\lambda} + aq\right) = \frac{1}{|\lambda|}.$$

Since the spectral radius is subharmonic over analytic functions [3, Theorem 3.4.7] we get

$$\rho(aq) = \limsup_{\lambda \to \infty} \rho\left(\frac{a}{\lambda} + aq\right) = 0.$$

Since $q \in \mathrm{QN}(A)$ was chosen arbitrarily, it follows that aq is quasinilpotent for all quasinilpotents q. Let $q \in \mathrm{QN}(A)$ and $\lambda \neq 0$ an arbitrary element of $\mathbb{C}$. Consider

$$\lambda\mathbf{1} - (aq - q) = [\mathbf{1} - aq(\lambda\mathbf{1} + q)^{-1}](\lambda\mathbf{1} + q).$$

If we observe that $q(\lambda\mathbf{1}+q)^{-1}$ is quasinilpotent (from the Spectral Mapping Theorem) then, from the preceding paragraph, it follows that $aq(\lambda\mathbf{1}+q)^{-1} \in \mathrm{QN}(A)$ and so $1 - aq(\lambda\mathbf{1}+q)^{-1} \in A^{-1}$. Consequently $\lambda\mathbf{1} - (a-\mathbf{1})q$ is invertible for all $\lambda \neq 0$, and hence $\sigma((a-\mathbf{1})q) = \{0\}$. Thus $(a-\mathbf{1})q$ is quasinilpotent for all $q \in \mathrm{QN}(A)$ (since $q \in \mathrm{QN}(A)$ was chosen arbitrarily). However, $a - \mathbf{1} \in \mathrm{QN}(A)$, and so from [8, Theorem 2.3] $a = \mathbf{1}$ and the theorem is proved. ☐

S. Grabiner showed that $a \in \mathrm{Z}(A)$ if and only if $\sup_{x\in A} \delta(a - e^x a e^{-x}) < \infty$. A more recent proof of this was given in [3, Theorem 5.2.4], using a subharmonic argument. In the proofs of the theorems to follow, this result provides a valuable tool.

From [18, Theorem 2], we know that if $a \in A$ satisfies $\rho(a(\mathbf{1}+q)) = 0$ for all $q \in QN(A)$ then $a \in \mathrm{Rad}(A)$. Making use of this result, amongst others, we have our main result given by the next theorem. Again, our assumption need only hold for all elements in some arbitrarily small neighbourhood of the identity.

Theorem 2.2.7. *Let A be a semisimple Banach algebra and $a \in A$ such that*

$$\#\sigma(ax) \leq \#\sigma(x) \text{ for all } x \text{ in a neighbourhood of } \mathbf{1}.$$

Then $a \in Z(A)$, a has single spectrum, and so $a = \alpha\mathbf{1}$ $(\alpha \in \mathbb{C})$.

Proof. Fix a $q \in \mathrm{QN}(A)$. Since $\#\sigma(a\mathbf{1}) \leq \#\sigma(\mathbf{1}) = 1$ it follows that $\sigma(a) = \{\alpha\}$ $(\alpha \in \mathbb{C})$.
Furthermore, from our hypothesis,

$$\#\sigma\Big(a(\mathbf{1}+\lambda q)\Big) \leq \#\sigma(\mathbf{1}+\lambda q) = 1$$

for all $\lambda \in \mathbb{C}$ sufficiently small. From the Scarcity Theorem it follows that $\#\sigma\Big(a(\mathbf{1}+\lambda q)\Big) = 1$ for all $\lambda \in \mathbb{C}$. In particular, for $\lambda = 1$, we have $\#\sigma\Big(a(\mathbf{1}+q)\Big) = 1$ and since $q \in \mathrm{QN}(A)$ was chosen arbitrarily, $\#\sigma\Big(a(\mathbf{1}+q)\Big) = 1$ for all $q \in \mathrm{QN}(A)$. Clearly, $a \notin A^{-1}$ or $a \in A^{-1}$, so we consider both cases.
If $0 \in \sigma(a)$ then $0 \in \sigma\Big(a(\mathbf{1}+q)\Big)$ (since $\mathbf{1}+q \in A^{-1}$), and so it follows that $\rho\Big(a(\mathbf{1}+q)\Big) = 0$ for all $q \in \mathrm{QN}(A)$. Hence, from [18, Theorem 2], $a \in \mathrm{Rad}(A) = \{0\}$.
Alternatively, if $0 \notin \sigma(a)$ then $a \in A^{-1}$. Dividing a by α if necessary, we may suppose that $\sigma(a) = \{1\}$. Now, from our assumption, for a fixed $x \in A$ and all $\lambda \in \mathbb{C}$ sufficiently small we have

$$\#\sigma(a - \lambda e^x a e^{-x}) \leq \#\sigma(\mathbf{1} - \lambda a^{-1} e^x a e^{-x}) = \#\sigma(a^{-1} e^x a e^{-x}).$$

Furthermore, from Jacobson's Lemma [3, Lemma 3.1.2], $\sigma(a^{-1}e^x a e^{-x}) = \sigma(ae^{-x}a^{-1}e^x)$. Taking $a^{-1} = \mathbf{1}+[a^{-1}-\mathbf{1}]$ we have $\#\sigma(a^{-1}e^{-x}ae^x) = \#\sigma\left(a(\mathbf{1}+e^{-x}[a^{-1}-\mathbf{1}]e^x\right)$. Noticing that $a^{-1}-\mathbf{1} \in \mathrm{QN}(A)$, it follows that $e^{-x}(a^{-1}-\mathbf{1})e^x \in \mathrm{QN}(A)$ and so combining the information above

$$\#\sigma(a - \lambda e^x a e^{-x}) \leq \#\sigma\Big(a(\mathbf{1}+e^{-x}[a^{-1}-\mathbf{1}]e^x)\Big) = 1$$

(where the last equality follows from $\#\sigma\Big(a(\mathbf{1}+q)\Big) = 1$ for all $q \in \mathrm{QN}(A)$). Hence $\#\sigma(a-\lambda e^x a e^{-x}) = 1$ for all $\lambda \in \mathbb{C}$ sufficiently small. A second application of the Scarcity Theorem yields $\#\sigma(a-\lambda e^x a e^{-x}) = 1$ for all $\lambda \in \mathbb{C}$. In particular for $\lambda = 1$, it follows that $\#\sigma(a - e^x a e^{-x}) = 1$ and hence

$$\delta(a - e^x a e^{-x}) = 0.$$

Since x was chosen arbitrarily, $a \in \mathrm{Z}(A)$ [3, Theorem 5.2.4]. Since $\sigma(a) = \{\alpha\}$ the result follows from Theorem 2.2.3. □

Clearly, if $\sigma(ax) = \sigma(x)$ then $\#\sigma(ax) = \#\sigma(x)$ and so we also have Theorem 2.2.2 as an obvious consequence of our main result (although the proof given earlier did not require the use of the Scarcity Theorem).

The following theorem can be seen as an additive version of our main result, and an alternative version of [3, Theorem 5.3.2], which states that an element a in a semisimple Banach algebra A satisfies $\#\sigma(a+q) = 1$ for all $q \in \mathrm{QN}(A)$ if and only if a is a scalar.

Theorem 2.2.8. *Let A be a semisimple Banach algebra and $a \in A$ such that*

$$\#\sigma(a+x) \leq \#\sigma(x) \text{ for all } x \text{ in a neighbourhood of } \mathbf{1}.$$

Then $a = \alpha\mathbf{1}$ ($\alpha \in \mathbb{C}$).

Proof. Fix a $q \in \mathrm{QN}(A)$. From our assumption,

$$\#\sigma\Big(a + (\mathbf{1} + \lambda q)\Big) \leq \#\sigma(\mathbf{1} + \lambda q) = 1$$

for all $\lambda \in \mathbb{C}$ sufficiently small. From the Scarcity Theorem, it follows that $\#\sigma(a + \mathbf{1} + \lambda q) = 1$ for all $\lambda \in \mathbb{C}$. Hence (for $\lambda = 1$)

$$\#\sigma(a + q) = \#\sigma(a + \mathbf{1} + q) = 1.$$

Since $q \in \mathrm{QN}(A)$ was chosen arbitrarily, from [3, Theorem 5.3.2], the result follows. □

If we consider the number of elements in the spectrum of an element $x \in A$ and the spectral diameter of x - in general little, if anything, can be said about the one parameter given information about the other. However, clearly, $\delta(x) = 0$ if and only if $\#\sigma(x) = 1$. Hence these two spectral parameters are relatable in this special case. From this observation, we can obtain a corresponding result to Theorem 2.2.7 in terms of the spectral diameter. The proof is very similar to that of Theorem 2.2.7, but is included here for completeness.

Theorem 2.2.9. *Let A be a semisimple Banach algebra and $a \in A$ such that*

$$\delta(ax) \leq \delta(x) \text{ for all } x \text{ in a neighbourhood of } \mathbf{1}.$$

Then $a \in Z(A)$, a has single spectrum, and so $a = \alpha\mathbf{1}$ ($\alpha \in \mathbb{C}$). Furthermore $|\alpha| \leq 1$ (unless A is one-dimensional).

Proof. Since $\delta(a\mathbf{1}) \leq \delta(\mathbf{1}) = 0$ it follows that $\sigma(a) = \{\alpha\}$ ($\alpha \in \mathbb{C}$). Let $q \in \mathrm{QN}(A)$.
From our hypothesis,

$$\delta\Big(a(\mathbf{1} + \lambda q)\Big) \leq \delta(\mathbf{1} + \lambda q) = \delta(\lambda q) = |\lambda|\delta(q) = 0$$

for all $\lambda \in \mathbb{C}$ sufficiently small. From the Scarcity Theorem [3, Theorem 3.4.25] it follows that $\#\sigma\Big(a(\mathbf{1} + \lambda q)\Big) = 1$ for all $\lambda \in \mathbb{C}$, in particular for $\lambda = 1$. Hence, since $q \in \mathrm{QN}(A)$ was chosen arbitrarily

$$\#\sigma\Big(a(\mathbf{1} + q)\Big) = 1 \text{ for all } q \in \mathrm{QN}(A).$$

If $0 \in \sigma(a)$ then $0 \in \sigma\Big(a(\mathbf{1} + q)\Big)$ for all $q \in \mathrm{QN}(A)$, and so it follows that $\rho\Big(a(\mathbf{1} + q)\Big) = 0$ for all $q \in \mathrm{QN}(A)$. Hence, from [18, Theorem 2], $a = 0$.
Alternatively, if $0 \notin \sigma(a)$ then $a \in A^{-1}$. Dividing a by α if necessary, we may suppose that $\sigma(a) = \{1\}$. Let x be an arbitrary member of A. Now, from our assumption, for $\lambda \in \mathbb{C}$ sufficiently small we have

$$\delta(a - \lambda e^x a e^{-x}) \leq \delta(\mathbf{1} - \lambda a^{-1} e^x a e^{-x}) = |\lambda|\delta(a^{-1} e^x a e^{-x}).$$

Similarly to the proof of Theorem 2.2.7, if we take $a^{-1} = \mathbf{1} + [a^{-1} - \mathbf{1}]$ and notice that $a^{-1} - \mathbf{1} \in \mathrm{QN}(A)$, we have

$$\delta(a - \lambda e^x a e^{-x}) \leq |\lambda| \delta(a e^{-x} a^{-1} e^x) = |\lambda| \delta\Big(a(\mathbf{1} + e^{-x}[a^{-1} - \mathbf{1}]e^x)\Big) = 0$$

(where the last equality follows from $\#\sigma\Big(a(\mathbf{1}+q)\Big) = 1$ for all $q \in \mathrm{QN}(A)$). Therefore, since x was chosen arbitrarily, $\#\sigma(a - \lambda e^x a e^{-x}) = 1$ for all $x \in A$ and the result of the proof is identical to that of Theorem 2.2.7.
Notice that if there exists an x_0 in a neighbourhood of $\mathbf{1}$ such that $\#\sigma(x_0) > 1$ then $\delta(x_0) \neq 0$ and so $|\alpha|\delta(x_0) = \delta(\alpha\mathbf{1} \cdot x_0) \leq \delta(x_0)$. Hence $|\alpha| \leq 1$. Alternatively, if A is one-dimensional then $\delta(x) = 0$ for all $x \in A$ and so there is no restriction on $\alpha \in \mathbb{C}$. □

The following theorem is an additive version of Theorem 2.2.9, the proof of which is similar to that of Theorem 2.2.8.

Theorem 2.2.10. *Let A be a semisimple Banach algebra and $a \in A$ such that*

$$\delta(a + x) \leq \delta(x) \text{ for all } x \text{ in a neighbourhood of } \mathbf{1}.$$

Then $a = \alpha\mathbf{1}$ $(\alpha \in \mathbb{C})$.

Proof. The proof is the same as for Theorem 2.2.8 if we notice that for a $q \in \mathrm{QN}(A)$, from our assumption,

$$\delta\Big(a + (\mathbf{1} + \lambda q)\Big) \leq \delta(\mathbf{1} + \lambda q) = 0$$

for all $\lambda \in \mathbb{C}$ sufficiently small. Hence $\#\sigma\Big(a + (\mathbf{1} + \lambda q)\Big) = 1$ for all $\lambda \in \mathbb{C}$ sufficiently small. □

Clearly the above results are not only sufficient for an $a \in A$ to be a scalar, but also necessary for $a = \alpha\mathbf{1}$ (with the given restriction on α).

We now explore the situation where $\sigma(a) = \{1\}$ and we have invariance with respect to the norm.
In the theorem that follows $L_a : A \to A$ refers to the operator defined by $L_a : x \mapsto ax$, that is, left multiplication by a. Furthermore, we say that an $a \in A$ is doubly power bounded if there exists a $D > 0$ such that $\|a^{\pm n}\| < D$ for all $n \in \mathbb{N}$. A more detailed discussion of different forms of norm-boundedness can be found in the next chapter.

Theorem 2.2.11. *Let A be a Banach algebra and $a \in A$ such that $\sigma(a) = \{1\}$. Then $a = \mathbf{1}$ if and only if $L_a : A \to A$ is an isometry.*

Proof. If $a = \mathbf{1}$ then clearly $ax = x$ for all $x \in A$, from which the result follows. Conversely, since $\sigma(a) = \{1\}$, a^{-1} exists. Furthermore, since $\|ax\| = \|x\|$ for all $x \in A$ it follows that in particular for the choices $x = \mathbf{1}$ and $x = a^{-1}$, $\|a\| = 1 = \|a^{-1}\|$. Hence

$$\|a^{\pm n}\| \leq \|a^{\pm 1}\|^n = 1$$

for all $n \in \mathbb{N}$. Therefore a is doubly power bounded and since $\sigma(a) = \{1\}$, $a = \mathbf{1}$ [10]. □

2.3 Examples and remarks

In this section we discuss some examples to illustrate our results and assumptions. Furthermore we describe the nature of elements a, and their spectra, that satisfy forms of spectral invariance under different spectral parameters. In particular we consider elements a that satisfy $\rho(ax) = \rho(x)$, or $\|ax\| = \|x\|$, or $\text{Arg}(ax) = \text{Arg}(x)$ for all $x \in A$. Firstly, however, we make the following observations.
Let $\sigma'(x)$ denote the nonzero spectrum of x, that is, $\sigma'(x) = \sigma(x) - \{0\}$.

Theorem 2.3.1. *If A is a Banach algebra and $a \in A$ satisfies $\sigma'(ax) = \sigma'(x)$ for an $x \in A$, then $\sigma(ax) = \sigma(x)$.*

Proof. Clearly if a is invertible, then ax is invertible if and only if x is invertible, that is, $0 \in \sigma(ax)$ if and only if $0 \in \sigma(x)$ for all $x \in A$. Hence the result will follow if we can show that a is invertible. Since $\sigma'(a\mathbf{1}) = \sigma'(\mathbf{1}) = \{1\}$, it follows that $\sigma(a) \subseteq \{0, 1\}$. If $\sigma(a) = \{0, 1\}$ then for $0 \neq \alpha \in \mathbb{C}$,

$$\{0, 1 + \alpha\} = \sigma\big(a(\alpha a + \mathbf{1})\big) \text{ and } \sigma(\alpha a + \mathbf{1}) = \{1, 1 + \alpha\}.$$

Hence $\sigma'\big(a(\alpha a + \mathbf{1})\big) \neq \sigma'(\alpha a + \mathbf{1})$ which contradicts our assumption. Thus $\sigma(a) = \{1\}$, a is invertible and the result follows. □

Making use of the above remarks, we have the following corollary to Theorem 2.2.2.

Corollary 2.3.2. *Let A be a semisimple Banach algebra and $a \in A$. Then $a = \mathbf{1}$ if and only if $\sigma'(ax) = \sigma'(x)$ for all x in a neighbourhood of $\mathbf{1}$.*

Recall from Theorem 2.2.2 that if A is a semisimple Banach algebra and $a \in A$ satisfies $\sigma(ax) = \sigma(x)$ for all x in a neighbourhood of $\mathbf{1}$ then $a = \mathbf{1}$. Clearly the question arises as to whether or not we may consider an arbitrary neighbourhood (not necessarily of $\mathbf{1}$), with the same conclusion. The following example shows the contrary:

Let A be the semisimple Banach algebra of all functions continuous on $\overline{B(0,1)}$, analytic on $B(0,1)$ and defined at the point $2 \in \mathbb{C}$, with multiplication defined pointwise. Let

$$a(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \overline{B(0,1)} \\ 0 & \text{if } \lambda = 2 \end{cases}$$

$$f(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in \overline{B(0,1)} \\ 0 & \text{if } \lambda = 2 \end{cases}$$

We want to show that $\sigma(ag) = \sigma(g)$ for all g in a neighbourhood $B(f, \frac{1}{3})$ of f. Let $g \in B(f, \frac{1}{3})$. Clearly $(ag)(\lambda) = 1 \cdot g(\lambda) = g(\lambda)$ for all $\lambda \in \overline{B(0,1)}$. Hence we must show that $g(2) \in g(\overline{B(0,1)})$ and $0 \in \sigma(g)$ (since $0 \in \sigma(ag)$), from which it will follow that $\sigma(ag) = \sigma(g)$.
Since $g \in B(f, \frac{1}{3})$,

$$|g(\lambda) - f(\lambda)| < \frac{1}{3} \quad \text{for all } \ \lambda \in \overline{B(0,1)} \cup \{2\}.$$

Suppose that $g(2) = c$. In particular then, $|g(2) - f(2)| = |c| < \frac{1}{3}$. Assume to the contrary that $c \notin g(\overline{B(0,1)})$. Then $g(\lambda) - c \neq 0$ for all $\lambda \in \overline{B(0,1)}$. However, $g - c$ is analytic on the interior $B(0,1)$ and so it must assume a minimum on the boundary i.e. on $S(0,1)$ by the Maximum-Modulus Principle.
Note that

$$|g(\lambda) - c - f(\lambda)| \leq |g(\lambda) - f(\lambda)| + |c| < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

for all $\lambda \in \overline{B(0,1)}$. In particular then, the minimum of $g - c$ must be within $\frac{2}{3}$ of $S(0,1)$ (since the minimum occurs on the boundary) i.e.

$$\frac{1}{3} < |g(\lambda_0) - c|$$

where $\lambda_0 \in \overline{B(0,1)}$ is the point at which $g - c$ attains its minimum. Hence

$$\frac{1}{3} < |g(\lambda) - c|$$

for all $\lambda \in \overline{B(0,1)}$. Since $|c| < \frac{1}{3}$ we know that $c \in \overline{B(0,1)}$, and so for $\lambda = c$

$$\frac{1}{3} < |g(c) - c| = |g(c) - f(c)|.$$

This, however, contradicts $g \in B(f, \frac{1}{3})$. Thus $g(2) \in g(\overline{B(0,1)})$. Note that $0 \in \sigma(ag)$ since $a(2) = 0$. Applying a similar argument as the one given above, we may also conclude that $0 \in \sigma(g)$:
Suppose that $0 \notin g(\overline{B(0,1)})$, then $g(\lambda) \neq 0$ for all $\lambda \in \overline{B(0,1)}$. Again, since g is analytic on the interior $B(0,1)$ it assumes it's minimum on the boundary. Since $g \in B(f, \frac{1}{3})$, $|g(\lambda) - f(\lambda)| < \frac{1}{3}$ and so the minimum of g must be within $\frac{1}{3}$ of $S(0,1)$. In particular, $\frac{2}{3} < |g(0)| = |g(0) - f(0)|$. This, however, contradicts $g \in B(f, \frac{1}{3})$ and so $0 \in \sigma(g)$.
To summarize then, $\sigma(ag) = \sigma(g)$ for all g in a neighbourhood of $f \neq \mathbf{1}$, but clearly $a \neq \mathbf{1}$.

If we turn our attention to the spectral radius, then the following remarks can be made.

It is not hard to see that if we replace the condition in Corollary 2.2.5 with the stronger assumption that $\rho(ax) = \rho(x)$ for all $x \in A$, without any restrictions on the spectrum of a, it is not enough for a to be a scalar:
Let S_σ denote a compact subset of the complex unit circle. Consider $A = C(S_\sigma)$, the set of all complex continuous functions on S_σ with the supremum norm. Now let $a \in A$ be defined by $a : \lambda \mapsto \lambda$. Clearly then $\sigma(a) = S_\sigma$. Let $x \in A$. It follows that

$$\begin{aligned}\rho(ax) &= \max_{\lambda \in S_\sigma} |(ax)(\lambda)| \\ &= \max_{\lambda \in S_\sigma} |a(\lambda)| \cdot |x(\lambda)| \\ &= \max_{\lambda \in S_\sigma} |x(\lambda)| \quad (\text{since } |a(\lambda)| = |\lambda| = 1 \text{ for all } \lambda \in S_\sigma) \\ &= \rho(x)\end{aligned}$$

In this case $\sigma(a)$ is any compact subset of the unit circle, $\rho(ax) = \rho(x)$ for all $x \in A$ and A is semisimple, but $a \neq \alpha\mathbf{1}$ for any $\alpha \in \mathbb{C}$.
Now consider the disk algebra B (i.e. the Banach algebra of all functions analytic on the open unit ball and continuous on the boundary). With the same choice of function for a as above, clearly $\sigma(a) = \overline{B(0,1)}$. Let $x \in A$. Then since ax is analytic on $B(0,1)$ it assumes its maximum value on the boundary (by the Maximum-Modulus Principle) and so

$$\begin{aligned}\rho(ax) &= \max_{\lambda \in \overline{B(0,1)}} |(ax)(\lambda)| \\ &= \max_{|\lambda|=1} |(ax)(\lambda)| \\ &= \max_{|\lambda|=1} |a(\lambda)| \cdot |x(\lambda)| \\ &= \max_{|\lambda|=1} |x(\lambda)| \quad (\text{since } |a(\lambda)| = |\lambda| = 1) \\ &= \rho(x)\end{aligned}$$

Here $\sigma(a) = \overline{B(0,1)}$, $\rho(ax) = \rho(x)$ for all $x \in B$ and B is semisimple, but again $a \neq \alpha\mathbf{1}$ for any $\alpha \in \mathbb{C}$.

From the above observations, even with the stronger assumption that $\rho(ax) = \rho(x)$ for all $x \in A$, we cannot conclude in general that a is a scalar, but we can say something about the spectrum of a in this situation. In the first example, the spectrum of the element was a subset of the unit circle, and in the second the spectrum of the element was the closed unit ball. The following theorem shows that this is the general situation.
Firstly, note that if a is invertible and $\rho(ax) = \rho(x)$ for all $x \in A$ then it is easy to show that $\sigma(a) \subseteq S(0,1)$. In general we have the following

Theorem 2.3.3. *Let A be a Banach algebra and $a \in A$ such that $\rho(ax) = \rho(x)$ for all $x \in A$. Then a belongs to the center modulo the radical and*

1. $\sigma(a) \subseteq S(0,1)$ *or*

2. $\sigma(a) = \overline{B(0,1)}$

Proof. Clearly a belongs to the center modulo the radical from Theorem 2.2.4. Since $\rho(a\mathbf{1}) = \rho(\mathbf{1}) = 1$ it follows that $\sigma(a) \subseteq \overline{B(0,1)}$. We want to show that $\partial\sigma(a) \cap B(0,1) = \emptyset$ (where $\partial\sigma(a)$ denotes the boundary of the spectrum of a). Suppose to the contrary that $\lambda_0 \in \partial\sigma(a) \cap B(0,1)$. Then every neighbourhood of λ_0 contains points in and outside of $\sigma(a)$. Hence there exists a $\lambda' \in B(0,1)$ such that $\lambda' \notin \sigma(a)$ and $\operatorname{dist}(\lambda', S(0,1)) > |\lambda' - \lambda_0|$ (i.e. λ' is closer to λ_0 than it is to the set $S(0,1)$).
Now let B be a maximal commutative subalgebra containing a. Then $\sigma_A(x) = \sigma_B(x)$ for all $x \in B$. Moreover $\sigma(x) = \{\chi(x) : \chi \in \Delta\}$ for all $x \in B$ (where Δ represents the collection of all characters, as discussed in 1.3).
From the assumption, $\rho\big(a(\lambda'\mathbf{1} - a)^{-1}\big) = \rho\big((\lambda'\mathbf{1} - a)^{-1}\big)$ i.e.

$$\max_{\chi \in \Delta} \left| \frac{\chi(a)}{\lambda' - \chi(a)} \right| = \max_{\chi \in \Delta} \left| \frac{1}{\lambda' - \chi(a)} \right|.$$

Since $\rho(a) = 1$ it follows that $|\chi(a)| \leq 1$ for all $\chi \in \Delta$. Now let $\chi_0 \in \Delta$ such that

$$\left| \frac{\chi_0(a)}{\lambda' - \chi_0(a)} \right| = \max_{\chi \in \Delta} \left| \frac{\chi(a)}{\lambda' - \chi(a)} \right|.$$

If $|\chi_0(a)| = 1$ then

$$\frac{1}{|\lambda' - \chi_0(a)|} = \max_{\chi \in \Delta} \left| \frac{\chi(a)}{\lambda' - \chi(a)} \right|$$

which implies that λ' is closest to an element of the spectrum lying on the unit circle. This, however, contradicts $\operatorname{dist}(\lambda', S(0,1)) > |\lambda' - \lambda_0|$.
Hence $|\chi_0(a)| < 1$. However, then

$$\max_{\chi \in \Delta} \left| \frac{\chi(a)}{\lambda' - \chi(a)} \right| < \max_{\chi \in \Delta} \left| \frac{1}{\lambda' - \chi(a)} \right|$$

which contradicts our assumption. □

On the other hand, the next example illustrates that the situation is better for $B(X)$ (the Banach algebra of all bounded linear operators on a Banach space X). Notice that we have weakened our spectral radius assumption to inequality, but nonetheless we obtain a stronger conclusion about the nature of such elements in this special algebra.

Example 2.3.4. *Let $A = B(X)$. Then $\rho(TS) \leq \rho(S)$ for all $S \in A$ if and only if $T = \alpha I$ where $\alpha \in \mathbb{C}$ and $|\alpha| \leq 1$.*

Note that $B(X)$ is semisimple. If $\rho(TS) \leq \rho(S)$ for all $S \in A$, then from Theorem 2.2.4, $T \in \mathrm{Z}(A)$.
Now, from Schur's Theorem [3, Theorem 4.2.2], the center of A is isomorphic to $\mathbb{C}$. Hence it follows that $T = \alpha I$ for some $\alpha \in \mathbb{C}$. Since $\rho(TS) \leq \rho(S)$ for all

$S \in A$, taking $S = I$ yields $\rho(T) \leq 1$. Clearly then $|\alpha| \leq 1$.
The converse is obvious.

We require the Banach algebra in the above theorem to have an identity since, for example, the theorem is not true in radical algebras (algebras in which every element belongs to the radical).

Based on Theorem 2.3.3, we know the nature of the spectra of elements a satisfying $\rho(ax) = \rho(x)$ for all $x \in A$. It follows from the proof of Example 2.3.4 that if in the Banach algebra $B(X)$, $\rho(TS) = \rho(S)$ for all $S \in B(X)$, then $T = \alpha I$ (where $|\alpha| = 1$) and so T is invertible. Therefore in $B(X)$ only invertible elements can satisfy such a spectral requirement - the same is true for C^*-algebras as the next example illustrates

Example 2.3.5. *If A is a C^*-algebra then $\rho(ax) = \rho(x)$ for all $x \in A$ implies that $a \in \mathrm{Z}(A)$ and $\sigma(a) \subseteq S(0,1)$.*

Since $\rho(ax) = \rho(x)$ for all $x \in A$, from Theorem 2.3.3, $a \in \mathrm{Z}(A)$ and $\sigma(a) \subseteq S(0,1)$ or $\sigma(a) = \overline{B(0,1)}$. Assume to the contrary that $\sigma(a) = \overline{B(0,1)}$. Then $0 \in \sigma(a)$.
Now, from the hypothesis of the theorem and properties of the involution

$$\rho(aa^*x) = \rho(a^*x) = \rho((a^*x)^*) = \rho(x^*a) = \rho(x^*) = \rho(x)$$

for all $x \in A$. Thus $\rho(aa^*x) = \rho(x)$ for all $x \in A$ and so again from Theorem 2.3.3, applied to the element aa^*, we can conclude that $\sigma(aa^*) \subseteq S(0,1)$ or $\sigma(aa^*) = \overline{B(0,1)}$.
Observe that $aa^* \notin A^{-1}$:
Since $0 \in \sigma(a)$, from properties of involutions, $0 \in \sigma(a^*)$. If $aa^* \in A^{-1}$ then there exists a $b \in A^{-1}$ such that $(aa^*)b = \mathbf{1} = b(aa^*)$; but since $a \in \mathrm{Z}(A)$, it follows that $a^*(ba) = \mathbf{1} = (ba)a^*$, which contradicts $0 \in \sigma(a^*)$.
Therefore $0 \in \sigma(aa^*)$ and so $\sigma(aa^*) = \overline{B(0,1)}$. However, since $aa^* \geq 0$ (properties of C^*-algebras), we can conclude that $\sigma(aa^*) \subseteq [0, \infty)$. Clearly we have reached a contradiction and so $\sigma(a) \subseteq S(0,1)$.

Since aa^* is self-adjoint it follows that $\sigma(aa^*) \subseteq \mathbb{R}$ (properties of C^*-algebras). From the above proof we have $\sigma(aa^*) \subseteq S(0,1)$, but since $aa^* \geq 0$ we know that $-1 \notin \sigma(aa^*)$, hence $\sigma(aa^*) = \{1\}$. From our observations in the above proof and Corollary 2.2.5 $aa^* = \mathbf{1}$. Therefore a is in fact unitary in the above example.

The question arises as to whether or not the above result is true in the more general setting of a semisimple Banach algebra with involution. However, consider the disk algebra A (which is semisimple), with involution defined by $f^*(z) = \overline{f(\overline{z})}$ for $f \in A$, and $z \in \overline{B(0,1)}$. If we take $f \in A$ defined by $f(z) = z$, then f satisfies the spectral radius hypothesis (as we showed in the examples at the beginning of this section), but has spectrum $\sigma(f) = \overline{B(0,1)}$.

As the next proposition illustrates, we also know that divisors of zero cannot satisfy this form of invariance.

Proposition 2.3.6. *If A is a semisimple Banach algebra and $a \in A$ is such that $\rho(ax) = \rho(x)$ for all $x \in A$, then a is not a divisor of zero.*

Proof. Clearly $a \neq 0$. Let $b \in A$ such that $ab = 0$. From the hypothesis of the theorem $\rho(bx) = \rho(abx) = \rho(0) = 0$ for all $x \in A$. Clearly then, $b \in \mathrm{Rad}(A) = \{0\}$. Thus $b = 0$ and the result follows. $\square$

Our attention now turns to elements a that satisfy $\|ax\| = \|x\|$ for all $x \in A$. Again, much can be said about the spectrum of such elements, as the following corollary to Theorem 2.3.3 shows.

Corollary 2.3.7. *Let A be a Banach algebra and $a \in A$ such that $\|ax\| = \|x\|$ for all $x \in A$. Then*

1. $\sigma(a) \subseteq S(0,1)$ *or*
2. $\sigma(a) = \overline{B(0,1)}$

Proof. Taking $x = \mathbf{1}$ yields $\|a\| = 1$ and hence $\rho(a) \leq 1$ from which it follows that $\sigma(a) \subseteq \overline{B(0,1)}$.
Now let B be a maximal commutative subalgebra of A, containing a. Then for all $x \in B$

$$\begin{aligned} \rho(ax) &= \lim_{n\to\infty} \|(ax)^n\|^{1/n} = \lim_{n\to\infty} \|a^n x^n\|^{1/n} \text{ (since } B \text{ is commutative)} \\ &= \lim_{n\to\infty} \|x^n\|^{1/n} \text{ (by assumption)} \\ &= \rho(x) \end{aligned}$$

Thus we have $\rho(ax) = \rho(x)$ for all $x \in B$, and so an identical argument to the one in Theorem 2.3.3 yields the same result. $\square$

One can also prove the above corollary in the following way:
If $\|ax\| = \|x\|$ for all $x \in A$, then L_a is an isometry on A. However, it is well known that $\sigma(a) = \sigma(L_a)$ is either a subset of $S(0,1)$ or it is $\overline{B(0,1)}$ [4, Proposition 1.15 and the discussion preceding it].

Consider $A = M_n(\mathbb{C})$. Then for any non-trivial permutation matrix a we have $\|ax\| = \|x\|$ for all $x \in A$, but $ay \neq ya$ for some $y \in A$. For example, take

$$a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

Then

$$ax = \begin{bmatrix} x_3 & x_4 \\ x_1 & x_2 \end{bmatrix}$$

It follows that $\|x\| = \sup\{|x_1| + |x_2|, |x_3| + |x_4|\} = \|ax\|$. However,

$$ay = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = ya.$$

Hence, in general the assumption $\|ax\| = \|x\|$ for all $x \in A$, is not sufficient for a to belong to the center modulo the radical. However, the following theorem shows that invariance with respect to the spectral arguments is sufficient. It is worth noting from the proof of the theorem that follows, that only invertible elements can satisfy this form of invariance.

Theorem 2.3.8. *Let A be a Banach algebra and $a \in A$ such that*

$$\operatorname{Arg}(ax) = \operatorname{Arg}(x)$$

for all x in some neighbourhood of $\mathbf{1}$. Then a is in the center modulo the radical.

Proof. Taking $x = \mathbf{1}$, we have $\operatorname{Arg}(a) = \operatorname{Arg}(\mathbf{1}) = \{0\}$, and so we can conclude that $\sigma(a) \subseteq \mathbb{R}^+ \cup \{0\}$.
Suppose that $0 \in \sigma(a)$, then $1 \in \sigma(ia + \mathbf{1})$ and $\sigma(ia + \mathbf{1}) - \{1\} \subseteq \mathbb{C} - \mathbb{R}$. Hence $0 \in \operatorname{Arg}(ia + \mathbf{1})$, however $0 \notin \operatorname{Arg}\big(a(ia + \mathbf{1})\big)$, and so we have a contradiction. Therefore $a \in A^{-1}$.
Let $x \in A$. Note that $\operatorname{Arg}(\frac{a}{\alpha}x) = \operatorname{Arg}(x)$ for all x in a neighbourhood of $\mathbf{1}$ and $0 < \alpha \in \mathbb{R}$. Now, by employing the Spectral Mapping Theorem and Jacobson's Lemma [3, Lemma 3.1.2]

$$\sigma(\mathbf{1} - \alpha e^{\lambda x} a^{-1} e^{-\lambda x}) = 1 - \alpha\sigma(e^{\lambda x} a^{-1} e^{-\lambda x}) = 1 - \alpha\sigma(a^{-1}) \subseteq \mathbb{R}$$

(since $\sigma(a) \subseteq \mathbb{R}$). Let $0 < R \in \mathbb{R}$ be arbitrary and $|\lambda| < R$. From our assumption, for all α sufficiently small

$$\begin{aligned}
\sigma(\mathbf{1} - \alpha e^{\lambda x} a^{-1} e^{-\lambda x}) \subseteq \mathbb{R} &\Rightarrow \sigma(\frac{a}{\alpha} - a e^{\lambda x} a^{-1} e^{-\lambda x}) \subseteq \mathbb{R} \\
&\Rightarrow \sigma(\mathbf{1} + \frac{a}{\alpha} - a e^{\lambda x} a^{-1} e^{-\lambda x}) \subseteq \mathbb{R} \\
&\Rightarrow \sigma(\mathbf{1} + \alpha a^{-1} - \alpha e^{\lambda x} a^{-1} e^{-\lambda x}) \subseteq \mathbb{R} \\
&\Rightarrow \sigma(a^{-1} - e^{\lambda x} a^{-1} e^{-\lambda x}) \subseteq \mathbb{R}
\end{aligned}$$

Hence if we consider the analytic function $f : \lambda \mapsto a^{-1} - e^{\lambda x} a^{-1} e^{-\lambda x}$ then $\sigma(f(\lambda)) \subseteq \mathbb{R}$ for all $|\lambda| < R$. Thus, $\sigma(f(\lambda))$ is constant for $|\lambda| < R$ [3, Corollary 3.4.12]. For $\lambda = 0$ we have $a^{-1} - e^{\lambda x} a^{-1} e^{-\lambda x} = 0$, and so $\sigma(a^{-1} - e^{\lambda x} a^{-1} e^{-\lambda x}) = \{0\}$ for $|\lambda| < R$. By the Scarcity Theorem, it follows that $\#\sigma(a^{-1} - e^{\lambda x} a^{-1} e^{-\lambda x}) = 1$ for all $\lambda \in \mathbb{C}$. Since $x \in A$ was chosen arbitrarily, $\#\sigma(a^{-1} - e^{x} a^{-1} e^{-x}) = 1$ for all $x \in A$ and so a is in the center modulo the radical [3, Theorem 5.2.4]. ☐

Notice that even with the stronger assumption of equality in the above theorem (as opposed to Theorem 2.2.6), we may not conclude that a is a scalar unless

we know that a has single spectrum. This idea is illustrated by the following example:
In a similar manner to the example at the beginning of this section, we let A be the semisimple Banach algebra of all functions continuous on $\overline{B(0,1)}$, analytic on $B(0,1)$ and defined at the point $2 \in \mathbb{C}$, with multiplication defined pointwise. Furthermore, we take

$$a(\lambda) = \begin{cases} 2 & \text{if } \lambda \in \overline{B(0,1)} \\ 3 & \text{if } \lambda = 2 \end{cases}$$

Since the range of a is the set $\{2,3\} \subseteq \mathbb{R}$ it follows that $\text{Arg}(ax) = \text{Arg}(x)$ for all $x \in A$ (since multiplication of a complex number by a positive real doesn't alter its principal argument). Clearly, though, a is not a scalar.

2.4 Characterizations of commutative Banach algebras

Some characterizations of commutative Banach algebras in terms of the spectral radius are discussed in [3, Corollary 5.2.3]. Here we introduce some completely different spectral characterizations using the number of elements in the spectrum and the spectral diameter.
As an interesting consequence of Theorem 2.2.7, we obtain the following characterizations of commutative Banach algebras. Jacobson's Lemma [3, Lemma 3.1.2] states that $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$ for all $x, y \in A$. Clearly then if $z \in \text{Z}(A)$, $\sigma(xzy) \cup \{0\} = \sigma(yzx) \cup \{0\}$ for all $x, y \in A$. Interestingly we can show that this is a defining property for commutative algebras, that is, if $\sigma(xzy) \cup \{0\} = \sigma(yzx) \cup \{0\}$ for all $x, y, z \in A$ then A is commutative. Remarkably, if only the number of elements in the spectrum are invariant under this permutation, and only for elements in an arbitrarily small neighbourhood of the identity, then A is commutative. The following result is thus a Jacobson type theorem.

Theorem 2.4.1. *Let A be a semisimple Banach algebra such that*

$$\#\sigma(xzy) = \#\sigma(yzx) \text{ for all } x, y, z \text{ in a neighbourhood of } \mathbf{1}.$$

Then A is commutative.

Proof. Fix $x, y \in A$. Furthermore, fix $\beta \in \mathbb{C}$ sufficiently small so that, from our hypothesis

$$\#\sigma(e^{\lambda x}e^{\beta y}e^{-\lambda x}e^{-\beta y}z) = \#\sigma(e^{-\beta y}ze^{\beta y}e^{-\lambda x}e^{\lambda x}) = \#\sigma(e^{-\beta y}ze^{\beta y}) = \#\sigma(z)$$

for all z in a neighbourhood of $\mathbf{1}$ and $\lambda \in \mathbb{C}$ sufficiently small. Note that, taking $z = \mathbf{1}$ we have $\#\sigma(e^{\lambda x}e^{\beta y}e^{-\lambda x}e^{-\beta y}) = 1$. From the above equation and Theorem 2.2.7, it follows that for λ sufficiently small, $e^{\lambda x}e^{\beta y}e^{-\lambda x}e^{-\beta y} = \alpha_\lambda \mathbf{1}$ $(\alpha_\lambda \in \mathbb{C})$. Thus

$$e^{\lambda x}e^{\beta y} = \alpha_\lambda e^{\beta y}e^{\lambda x}.$$

From Jacobson's Lemma [3, Lemma 3.1.2], $\sigma(e^{\lambda x}e^{\beta y}) = \sigma(e^{\beta y}e^{\lambda x})$ and so clearly $\alpha_\lambda = 1$. It follows that for all λ sufficiently small $e^{\lambda x} - e^{\beta y}e^{\lambda x}e^{-\beta y} = 0$ and so $\delta\left(e^{\lambda x} - e^{\beta y}e^{\lambda x}e^{-\beta y}\right) = 0$. Applying the Scarcity Theorem we have, $\delta\left(e^{\lambda x} - e^{\beta y}e^{\lambda x}e^{-\beta y}\right) = 0$ for all $\lambda \in \mathbb{C}$, in particular for $\lambda = 1$. Hence $\delta\left(e^{x} - e^{\beta y}e^{x}e^{-\beta y}\right) = 0$ for all β sufficiently small (since our fixed β was chosen arbitrarily). A second application of the Scarcity Theorem (to the variable β) tells us that this equation is true for all $\beta \in \mathbb{C}$. Taking $\beta = 1$ we have the following

$$\delta\left(e^{x} - e^{y}e^{x}e^{-y}\right) = 0.$$

Since $y \in A$ was chosen arbitrarily, we have $e^x \in \mathrm{Z}(A)$ by [3, Theorem 5.2.4]. Furthermore, since x was also chosen arbitrarily, $e^x \in \mathrm{Z}(A)$ for all $x \in A$. Therefore every exponential element belongs to $\mathrm{Z}(A)$, but since every element of a Banach algebra can be written as the sum of two exponentials, the result follows. □

In the first section we made the observation that the spectral diameter and number of elements in the spectrum of an element can be related to each other when there is only a single element in the spectrum of that element. Hence the following theorem is an analog of Theorem 2.4.1 in terms of the spectral diameter and can be proved in a similar fashion. Furthermore, in the same way that we made use of Theorem 2.2.7 for the proof of Theorem 2.4.1, Theorem 2.2.9 is most useful in the proof that follows.

Theorem 2.4.2. *Let A be a semisimple Banach algebra such that*

$$\delta(xzy) = \delta(yzx) \text{ for all } x, y, z \text{ in a neighbourhood of } \mathbf{1}.$$

Then A is commutative.

Proof. Let x, y be fixed elements in A. Furthermore, fix $\beta \in \mathbb{C}$ sufficiently small so that, from our hypothesis

$$\delta(e^{\lambda x}e^{\beta y}e^{-\lambda x}e^{-\beta y}z) = \delta(e^{-\beta y}ze^{\beta y}e^{-\lambda x}e^{\lambda x}) = \delta(e^{-\beta y}ze^{\beta y}) = \delta(z)$$

for all z in a neighbourhood of $\mathbf{1}$ and $\lambda \in \mathbb{C}$ sufficiently small. Note that, taking $z = \mathbf{1}$ we have $\delta(e^{\lambda x}e^{\beta y}e^{-\lambda x}e^{-\beta y}) = 0$. From the above equation and Theorem 2.2.9, it follows that for λ sufficiently small, $e^{\lambda x}e^{\beta y}e^{-\lambda x}e^{-\beta y} = \alpha_\lambda \mathbf{1}$ ($\alpha_\lambda \in \mathbb{C}$). Thus

$$e^{\lambda x}e^{\beta y} = \alpha_\lambda e^{\beta y}e^{\lambda x}.$$

The remainder of the proof follows analogously to that of Theorem 2.4.1. □

Chapter 3

Gelfand-Hille type theorems in ordered Banach algebras

In this chapter, we give further structure to our Banach algebra by introducing ordering. The results that follow are quite different from those of the previous chapter, since we do not make use of spectral parameters here, but rather boundedness constraints. Furthermore, our focus here is more towards characterizations of the identity itself. Again, unless otherwise stated, A will denote a complex, unital Banach algebra.

The presence of an ordering within a Banach algebra allows us to weaken the sufficient conditions for an $a \in A$ with unit spectrum to be the identity. The ordering that we introduce is via an algebra cone. For basic properties of ordered Banach algebras see [17], [16] and [12].

A subset C of A is called an *algebra cone* if C satisfies the following

1. $C + C \subseteq C$
2. $\lambda C \subseteq C \quad$ (for $0 \leq \lambda \in \mathbb{R}$)
3. $C \cdot C \subseteq C$
4. $\mathbf{1} \in C$

Any cone C of A induces an ordering $\leq$ on A in the following way:

$$a \leq b \quad \text{if and only if} \quad b - a \in C.$$

Then $x \in C$ is referred to as *positive* and $C = \{x \in A : x \geq 0\}$.

An algebra cone C is said to be

- *proper* if $C \bigcap (-C) = \{0\}$.
- *normal* if there exists $0 < \alpha \in \mathbb{R}$ such that

$$0 \leq x \leq y \quad \Rightarrow \quad \|x\| \leq \alpha \|y\|.$$

- *closed* if it is closed with respect to the norm of A.
- *inverse closed* if for all $x \in A^{-1}$

$$x \in C \quad \Rightarrow \quad x^{-1} \in C.$$

Note that if $x \in C \bigcap (-C)$, then it follows that $-x \in C$ i.e. $x \leq 0$. Hence if C is normal, with normality constant $0 < \alpha \in \mathbb{R}$, then $0 \leq \|x\| \leq \alpha\|0\| = 0$ and so $x = 0$. Therefore every normal cone is proper.
Clearly, normal algebra cones behave very well, since their algebraic structure coincides well with the geometric structure given by the norm. Later we shall show how, together with other constraints, the assumption of a normal cone can be reduced to the weaker property of a proper cone, with the same conclusion.

We shall use (A, C) to denote an ordered Banach algebra with algebra A and algebra cone C.

In the first section, we introduce some boundedness definitions that will aid us in the work to follow; and we discuss some of the work that has been done in this area.

The second section deals with the notion of Abel boundedness. Here, we investigate the relationship between the Abel boundedness of a and natural powers of a amongst others. These results will be useful for the section hereafter.

In the third section, we give a condition on the cone of an ordered Banach algebra under which the notions of Abel boundedness and Cesàro boundedness are equivalent; and we discuss our main result, namely that if C is a closed proper algebra cone contained in A and $a \in A$ has $\sigma(a) = \{1\}$, then $a = \mathbf{1}$ if and only if a^L is Abel bounded and $a^N \geq \mathbf{1}$ for some $L, N \in \mathbb{N}$.

Obviously, if $a - \mathbf{1}$ is nilpotent of order 1 then a is the identity. As a generalization of our work in this chapter, in the fourth section we discuss some results relating to the nilpotency of $a - \mathbf{1}$.

Finally, in the last section, we consider the role played by inverse closed algebra cones in our consideration. Using this slightly different structure, we obtain a sufficient condition for an $a \in A$ to be the identity.

3.1 The role played by boundedness

We mention different forms of norm-bounded constraints on an element $a \in A$ in this section. These different forms of boundedness are very useful constraints for us to infer that an $a \in A$ with unit spectrum is the identity, as we shall see in the sections hereafter.

Quite a bit of groundwork was laid in this area by Grobler and Huijsmans [11]. We summarize some of these boundedness conditions briefly below.
An $a \in A$ is said to be

- *power bounded* if there exists a $D > 0$ such that

$$\|a^n\| \leq D \text{ for all } n \in \mathbb{N}.$$

- *Cesàro bounded* if there exists a $D > 0$ such that

$$\|M_n(a)\| \leq D \text{ for all } n \in \mathbb{N},$$

  where

$$M_n(a) = \frac{\mathbf{1} + a + \dots a^n}{n+1}$$

  is called the n'th *Cesàro mean* of a.

- *Abel bounded* if there exists a $D > 0$ such that

$$\|(1-\theta)\sum_{k=0}^{\infty} \theta^k a^k\| \leq D \text{ for all } \theta \in (0,1).$$

- *uniformly Abel bounded* if there exists a $D > 0$ such that

$$\|(1-\theta)\sum_{k=0}^{n} \theta^k a^k\| \leq D \text{ for all } \theta \in (0,1), n \in \mathbb{N}.$$

The notions of Abel and uniformly Abel bounded can be generalized to *(N)-Abel bounded* and *(N)-uniformly Abel bounded* given by

$$\|(1-\theta)^N \sum_{k=0}^{\infty} \theta^k a^k\| \leq D \text{ for a } D > 0 \text{ and for all } \theta \in (0,1),$$

and

$$\|(1-\theta)^N \sum_{k=0}^{n} \theta^k a^k\| \leq D \text{ for a } D > 0 \text{ and for all } \theta \in (0,1), n \in \mathbb{N}$$

respectively.

If a is invertible, and both a and a^{-1} have one of the above forms of boundedness, then a is referred to as *doubly* bounded of that form; for instance doubly power bounded means that there exists a $D > 0$ such that

$$\|a^{\pm n}\| \leq D \text{ for all } n \in \mathbb{N}.$$

If $a \in A$ is Abel bounded, then clearly $\sum_{k=0}^{\infty} \theta^k a^k$ must converge (for all $\theta \in (0,1)$), so θa must be power bounded i.e. $\rho(\theta a) \leq 1$. Hence $\rho(a) \leq 1$.

For $a \in \mathbb{C}^n$, we use the norm

$$\|a\| = \|(a_{ij})\| = \sup_{i=1,\ldots,n} (|a_{i1}| + \ldots + |a_{in}|).$$

The following simple example illustrates that the notions of power bounded and Cesàro bounded are not the same:
Let

$$T = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}.$$

Note that

$$T^n = \begin{pmatrix} i^n & ni^{n-1} \\ 0 & i^n \end{pmatrix}.$$

Hence $\|T^n\| = 1 + n$ and so T is not power bounded. However,

$$M_n(T) = \frac{1}{n+1} \begin{pmatrix} \sum_{k=0}^n i^k & \sum_{k=0}^n ki^{k-1} \\ 0 & \sum_{k=0}^n i^k \end{pmatrix}.$$

and so $\|M_n(T)\| = \frac{1}{n+1}\left[|\sum_{k=0}^n i^k| + |\sum_{k=0}^n ki^{k-1}|\right]$.
Now, $|\sum_{k=0}^n i^k| \leq \sum_{k=0}^n 1 = n+1$.
Also, $\sum_{k=0}^n kx^{k-1} = \frac{d}{dx}\left(\sum_{k=0}^n x^k\right) = \frac{d}{dx}\left(\frac{1-x^{n+1}}{1-x}\right) = \frac{-(n+1)x^n(1-x)+(1-x^{n+1})}{(1-x)^2}$.
Taking $x = i$, and noting that $(1-i)^2 = -2i$,

$$|\sum_{k=0}^n ki^{k-1}| \leq \frac{(n+1)|i|^n(1+|i|) + (1+|i|^{n+1})}{|-2i|} = \frac{2(n+1)+2}{2} = n+2.$$

It follows then that $\|M_n(T)\| \leq \frac{1}{n+1}[(n+1)+(n+2)] \leq 4$.

Furthermore, as shown in [11, Example 8],

$$T = \begin{pmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{pmatrix}.$$

is Abel bounded but not uniformly Abel bounded.

Most of the following illustrated hierarchy was given by Grobler and Huijsmans [11], which shows the relationship between these different forms of boundedness

Remark 3.1.1.

$$\begin{array}{ccccc}
\textit{Power bounded} & & & & \\
\Downarrow & & & & \\
\textit{Cesàro bounded} & \Leftrightarrow & \textit{Uniformly Abel bounded} & \Rightarrow & \textit{Abel bounded} \\
 & & \Downarrow & & \Downarrow \\
 & & \textit{(N)-Uniformly Abel bounded} & \Rightarrow & \textit{(N)-Abel bounded}
\end{array}$$

It was shown in [11, Theorem 2] that in a Banach algebra a Cesàro bounded element is uniformly Abel bounded. Drissi and Zemánek [9, remarks preceding (10)] raised the question whether the converse is true? That question was answered by Montes-Rodríguez, Sánchez-Álvarez and Zemánek [15, Theorem 3.1] who recently showed that the notions of Cesàro bounded and uniformly Abel bounded are the same. This discovery is significant because it narrows the gap between power boundedness and Abel boundedness: clearly the concept of Cesàro boundedness has a similar form to that of power boundedness, whereas the notion of uniformly Abel boundedness has obvious analogs to that of Abel boundedness.

One can expand the above diagram as follows: Note that it is easy to see that if $a \in A$, where A is a Banach algebra, is Cesàro bounded then $\|M_n(a)\| = o(n^N)$ as $n \to \infty$ (where $N \in \mathbb{N}$). The proof of the next observation is not difficult and will be omitted.

Proposition 3.1.2. *Suppose A is a Banach algebra and $a \in A$ is such that $\|M_n(a)\| = o(n^N)$ as $n \to \infty$. Then a is (N)-uniformly Abel bounded.*

Since Cesàro boundedness is the same as uniformly Abel boundedness, the implication in the above proposition can be reversed.

As the next theorem shows us, in a finite-dimensional Banach algebra, if an element has unit spectrum and is power bounded, then it is the identity. The proof is fairly easy, but employs a useful technique similar to that used in [11].

Theorem 3.1.3. *Let A be a finite-dimensional Banach algebra and $a \in A$. If $\sigma(a) = \{1\}$ and a is power bounded, then $a = \mathbf{1}$.*

Proof. Note that since $\sigma(a) = \{1\}$, $\sigma(a-\mathbf{1}) = \{0\}$. Hence $a-\mathbf{1}$ is quasinilpotent, and therefore nilpotent (since A is finite-dimensional). Thus $(a-\mathbf{1})^N = 0$ for an $N \in \mathbb{N}$. Hence for all $n \geq N$,
$a^n = \big((a-\mathbf{1})+\mathbf{1}\big)^n = \sum_{k=0}^{n} \binom{n}{k}(a-\mathbf{1})^k = \sum_{k=0}^{N-1} \binom{n}{k}(a-\mathbf{1})^k$. Since a is power bounded, for some $D > 0$ and all $n \in \mathbb{N}$ we have $\|a^n\| \leq D$. Hence for all $n \in \mathbb{N}$

$$\left\| \sum_{k=0}^{N-1} \binom{n}{k}(a-\mathbf{1})^k \right\| \leq D \Rightarrow \left\| \sum_{k=0}^{N-2} \binom{n}{k}(a-\mathbf{1})^k + \binom{n}{N-1}(a-\mathbf{1})^{N-1} \right\| \leq D$$

Dividing both sides by $\binom{n}{N-1}$ gives

$$\left\| \sum_{k=0}^{N-2} \frac{(N-1)!}{(n-k)(n-(k+1))\ldots(n-(N-2))k!}(a-\mathbf{1})^k + (a-\mathbf{1})^{N-1} \right\| \leq \frac{D}{\binom{n}{N-1}}$$

(since $k \leq N-2 < N-1$). Considering the limit as n tends to infinity gives $0 \leq \|(a-\mathbf{1})^{N-1}\| \leq 0$, and so $(a-\mathbf{1})^{N-1} = 0$. It follows from induction that $(a-\mathbf{1}) = 0$, from which the result follows. $\square$

More generally, Gelfand [10] showed that if $\sigma(a) = \{1\}$ and a is doubly power bounded then $a = \mathbf{1}$. Hille [13] later elaborated on this result for an a that is doubly power bounded of some order. This pioneering work of Gelfand and Hille is the reason that theorems of this form are often referred to in the literature as Gelfand-Hille type theorems. Allan and Ransford [2, Theorem 1.1] subsequently proved the same result of Gelfand more elegantly, using holomorphic functional calculus.
Mbekhta and Zemánek [14, Theorem 2] showed that if $\sigma(a) = \{1\}$ and a is doubly Cesàro bounded then $a = \mathbf{1}$. In light of our remarks in the previous paragraph, this result also tells us that if $\sigma(a) = \{1\}$ and a is doubly uniformly Abel bounded, then $a = \mathbf{1}$. This result was proved directly in [11, Theorem 7]. Furthermore, Grobler and Huijsmans [11, Theorem 11], showed that a is the identity if $\sigma(a) = \{1\}$ and a is Abel bounded and doubly (N)-uniformly Abel bounded (where $N \in \mathbb{N}$).

Note that the requirement that an element with unit spectrum be doubly (N)-uniformly Abel bounded $(N > 1)$ is insufficient for that element to be the identity, as the following example shows:
Let

$$T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$T^k = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } T^{-k} = \begin{pmatrix} 1 & 0 & -k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\theta \in (0,1)$.

$$\begin{aligned} \|(1-\theta)^2 \sum_{k=0}^{n} \theta^k T^k\| &= (1-\theta)^2 \left[\left| \sum_{k=0}^{n} \theta^k \right| + \theta \left| \sum_{k=0}^{n} k\theta^{k-1} \right| \right] \\ &= (1-\theta)^2 \left[\left| \frac{1-\theta^{n+1}}{1-\theta} \right| + \theta \left| \frac{1-\theta^n}{(1-\theta)^2} - \frac{n\theta^n}{(1-\theta)} \right| \right] \\ &\leq (1-\theta)(1-\theta^{n+1}) + \theta(1-\theta^n) + (1-\theta)n\theta^{n+1} \end{aligned}$$

Noting that $(1-\theta)n\theta^{n+1} \leq (1-\theta)(1+\theta+\ldots+\theta^{n+1}) \leq (1-\theta)(1-\theta)^{-1} = 1$ (since $\theta \in (0,1)$) it follows that $\|(1-\theta)^2 \sum_{k=0}^{n} \theta^k T^k\| \leq 1+1+1 = 3$. Similarly T^{-1} is (2)-uniformly Abel bounded. Hence $\sigma(T) = \{1\}$ and T is doubly (2)-uniformly Abel bounded, but is clearly not the identity. Furthermore, if we consider the cone consisting of 3×3 real matrices with nonnegative entries, notice that T is positive. Hence this example also implies that doubly (N)-uniform Abel boundedness $(N > 1)$ of a cone element with unit spectrum is insufficient for that element to be the identity.

3.2 Abel bounded elements

If we consider the complex series $\sum_{k=0}^{\infty} a_k$, then the Abel sum of the series is defined to be

$$\lim_{\theta \to 1^-} \sum_{k=0}^{\infty} a_k \theta^k$$

if the series converges for all $\theta \in (0,1)$ and the limit exists. Abel's Theorem states that if the series $\sum_{k=0}^{\infty} a_k$ is convergent then the Abel sum exists. However, the converse is not necessarily true. In a similar way, when working in a Banach algebra we have the Abel boundedness condition, namely that an element $a \in A$ is said to be Abel bounded if there exists a $D > 0$ such that for all $\theta \in (0,1)$, $\|(1-\theta)\sum_{k=0}^{\infty} \theta^k a^k\| \leq D$.

Since, in this chapter, we are considering characterizations of the identity, it follows that we are dealing with elements $a \in A$ such that $\sigma(a) = \{1\}$. If $\sigma(a) = \{1\}$ then $a = \mathbf{1} - q$ for some quasinilpotent q. Naturally the question arises as to the nature of a quasinilpotent q such that $\mathbf{1} - q$ is Abel bounded. The theorem that follows shows us that q cannot be nilpotent.

Theorem 3.2.1. *If $q \neq 0$ is nilpotent then neither $\mathbf{1} - q$ nor $(\mathbf{1} - q)^{-1}$ is Abel bounded.*

Proof. Let $\theta \in (0,1)$. Suppose that q is nilpotent of order N, that is $q^N = 0$. Note that from the Spectral Mapping Theorem, $\sigma(q) = \{0\}$ and $\rho(\theta(\mathbf{1} - q)) < 1$. Hence from Theorem 1.2.3, and the nilpotency of q

$$\begin{aligned}(1-\theta)\sum_{k=0}^{\infty} \theta^k (\mathbf{1}-q)^k &= (1-\theta)\big[\mathbf{1} - \theta(\mathbf{1}-q)\big]^{-1} = (1-\theta)\big[(1-\theta)\mathbf{1} + \theta q\big]^{-1} \\ &= \left[\mathbf{1} + \frac{\theta q}{1-\theta}\right]^{-1} = \sum_{k=0}^{\infty}\left[\frac{-\theta q}{1-\theta}\right]^k = \sum_{k=0}^{N-1}\left[\frac{-\theta q}{1-\theta}\right]^k \\ &= \mathbf{1} + \frac{1}{(1-\theta)^{N-1}} \sum_{k=1}^{N-1} (-\theta q)^k (1-\theta)^{N-1-k}\end{aligned}$$

Since the term in q^{N-1} dominates the rest of the sum, there is no danger of cancelation, and so as $\theta \to 1^-$ we get

$$\left\| \mathbf{1} + \lim_{\theta \to 1^-} \frac{(-q)^{N-1}}{(1-\theta)^{N-1}} \right\| = \infty.$$

Hence $\mathbf{1} - q$ is not Abel bounded.
Furthermore,

$$(\mathbf{1} - q)^{-1} = \sum_{j=0}^{N-1} q^j = \mathbf{1} + \sum_{j=1}^{N-1} q^j,$$

and so if we take $q' = -\sum_{j=1}^{N-1} q^j$ we have $(\mathbf{1}-q)^{-1} = \mathbf{1}-q'$. Clearly $\sigma(q') = \{0\}$ (by the Spectral Mapping Theorem) and ${q'}^N = 0$. Therefore we can apply exactly the same argument as the one above to $\mathbf{1} - q'$ with the same result. Thus, $(\mathbf{1} - q)^{-1}$ is also not Abel bounded. □

From the proof of the above theorem if q is nilpotent of order N then

$$(1-\theta)^N \sum_{k=0}^{\infty} \theta^k (\mathbf{1} - q)^k = (1-\theta)^{N-1}\mathbf{1} + \sum_{k=1}^{N-1} (-\theta q)^k (1-\theta)^{N-1-k}.$$

Hence $\|(1-\theta)^N \sum_{k=0}^{\infty} \theta^k (\mathbf{1} - q)^k\| \leq D$ for some $D > 0$ and for all $\theta \in (0, 1)$. Therefore if q is nilpotent of order N then $\mathbf{1} - q$ is doubly (N)-Abel bounded. Furthermore, it follows that the strongest case of Theorem 3.2.1 occurs when q is nilpotent of order 2, in which case $\mathbf{1} - q$ is doubly (2)-Abel bounded.

Something can be said regarding scalar multiples of an Abel bounded element as the next theorem, and the corollary thereof, illustrates.

Theorem 3.2.2. *Let A be a Banach algebra and $a \in A$ such that a is uniformly Abel bounded. Then αa is uniformly Abel bounded for all $\alpha \in [0, 1]$.*

Proof. Obviously the case, $\alpha = 0$ is trivial. Hence, fix $\alpha \in (0, 1]$. Since a is uniformly Abel bounded there exists a $D > 0$ such that $\|(1-\theta)\sum_{k=0}^{n} \theta^k a^k\| \leq D$ for all $\theta \in (0, 1)$, $n \in \mathbb{N}$. In particular then

$$\|(1 - \theta\alpha) \sum_{k=0}^{n} (\theta\alpha)^k a^k\| \leq D$$

for all $\theta \in (0, 1)$, $n \in \mathbb{N}$ (since $0 < \theta\alpha < 1$ for all $\theta \in (0, 1), \alpha \in (0, 1]$). Hence $\frac{(1-\theta\alpha)}{(1-\theta)} \|(1-\theta)\sum_{k=0}^{n} \theta^k (\alpha a)^k\| \leq D$ and so

$$\|(1-\theta) \sum_{k=0}^{n} \theta^k (\alpha a)^k\| \leq \frac{D(1-\theta)}{(1-\theta\alpha)}$$

for all $\theta \in (0, 1)$, $n \in \mathbb{N}$. Since $0 < \alpha \leq 1$, then $1 - \theta \leq 1 - \theta\alpha$ and so

$$\|(1-\theta) \sum_{k=0}^{n} \theta^k (\alpha a)^k\| \leq \frac{D(1-\theta)}{(1-\theta\alpha)} \leq D$$

for all $\theta \in (0, 1)$, $n \in \mathbb{N}$. It follows then that αa is uniformly Abel bounded for all such α. □

If we take the limit as n goes to infinity in the above proof then we have the following obvious corollary to Theorem 3.2.2.

Corollary 3.2.3. *Let A be a Banach algebra and $a \in A$ such that a is Abel bounded. Then αa is Abel bounded for all $\alpha \in [0,1]$.*

We now turn to some useful results that will lead us to our main result in the next section. In particular we show that if $\sigma(a) = \{1\}$ then a is Abel bounded if and only if a^N is Abel bounded for all $N \in \mathbb{N}$.
The next theorem tells us that regardless of any spectral requirements, if some natural power of a is Abel bounded, then a itself is Abel bounded.

Theorem 3.2.4. *Let A be a Banach algebra and $a \in A$ such that a^N is Abel bounded for some $N \in \mathbb{N}$. Then a is Abel bounded.*

Proof. Since a^N is Abel bounded, it follows that there exists a $D > 0$ such that

$$\|(1-\theta)\sum_{k=0}^{\infty}\theta^k a^{Nk}\| \leq D \text{ for all } \theta \in (0,1).$$

Equivalently,

$$\|(1-\theta^N)\sum_{k=0}^{\infty}\theta^{Nk} a^{Nk}\| \leq D \text{ for all } \theta^N \in (0,1)$$

since $0 < \theta < 1 \Rightarrow 0 < \theta^N < 1$. Now,

$$\begin{aligned}(1-\theta^N)\sum_{k=0}^{\infty}\theta^k a^k &= (1-\theta^N)\left[\sum_{k=0}^{\infty}(\theta a)^{Nk} + \ldots + \sum_{k=0}^{\infty}(\theta a)^{Nk+(N-1)}\right] \\ &= \left[\mathbf{1} + \theta a + \ldots + (\theta a)^{N-1}\right](1-\theta^N)\sum_{k=0}^{\infty}(\theta a)^{Nk}.\end{aligned}$$

For some $K > 0$ and for all $\theta \in (0,1)$,

$$\|\mathbf{1} + \theta a + \ldots + (\theta a)^{N-1}\| \leq \sum_{i=0}^{N-1}(\theta\|a\|)^i \leq \sum_{i=0}^{N-1}\|a\|^i \leq K;$$

and since $1 - \theta \leq 1 - \theta^N$

$$\|(1-\theta)\sum_{k=0}^{\infty}\theta^k a^k\| \leq \|(1-\theta^N)\sum_{k=0}^{\infty}\theta^k a^k\| \leq K \cdot \|(1-\theta^N)\sum_{k=0}^{\infty}\theta^{Nk} a^{Nk}\| \leq K \cdot D.$$

Hence a is Abel bounded. □

If we make use of an argument similar to the one above, we have the following analogous result for uniformly Abel bounded elements.

Corollary 3.2.5. *Let A be a Banach algebra and $a \in A$ such that a^N is uniformly Abel bounded for some $N \in \mathbb{N}$. Then a is uniformly Abel bounded.*

Proof. Fix an $n \in \mathbb{N}$. From the Division Algorithm, there exists $q, r \in \mathbb{N}$ such that $n = Nq + r$ where $0 \leq r < N$. Hence

$$(1-\theta^N)\sum_{k=0}^{n}\theta^k a^k = (1-\theta^N)\Big[\sum_{k=0}^{q}(\theta a)^{Nk} + \sum_{k=0}^{q}(\theta a)^{Nk+1} + \ldots + \sum_{k=0}^{q}(\theta a)^{Nk+r}$$

$$+\sum_{k=0}^{q-1}(\theta a)^{Nk+r+1} + \sum_{k=0}^{q-1}(\theta a)^{Nk+r+2} + \ldots + \sum_{k=0}^{q-1}(\theta a)^{Nk+(N-1)}\Big].$$

Thus,

$$(1-\theta^N)\sum_{k=0}^{n}\theta^k a^k = (1-\theta^N)\Bigg(\Big[\mathbf{1} + \theta a + \ldots + (\theta a)^r\Big]\sum_{k=0}^{q}(\theta a)^{Nk}$$

$$+\Big[(\theta a)^{r+1} + (\theta a)^{r+2} + \ldots + (\theta a)^{N-1}\Big]\sum_{k=0}^{q-1}(\theta a)^{Nk}\Bigg).$$

The remainder of the proof follows similarly to that of Theorem 3.2.4. □

On the other hand, if we know that a is Abel bounded and its spectrum lies on the nonnegative real axis, then every natural power of a is Abel bounded:

Theorem 3.2.6. *If $a \in A$ is Abel bounded and $\sigma(a) \subseteq [0, \infty)$, then a^N is Abel bounded for all $N \in \mathbb{N}$.*

Proof. Since a is Abel bounded $\rho(a) \leq 1$ (see the remark following the definition of Abel boundedness), and hence $\rho(\theta a) < 1$ for all $\theta \in (0, 1)$. Therefore, from Theorem 1.2.3

$$(1-\theta^N)\sum_{k=0}^{\infty}(\theta a)^{Nk} = (1-\theta^N)\big(\mathbf{1} - (\theta a)^N\big)^{-1}.$$

Since

$$\mathbf{1} - (\theta a)^N = (\mathbf{1} - \theta a)\big[\mathbf{1} + \theta a + \ldots + (\theta a)^{N-1}\big],$$

it is clear that

$$\big(\mathbf{1} - (\theta a)^N\big)^{-1} = (\mathbf{1} - \theta a)^{-1}\big[\mathbf{1} + \theta a + \ldots + (\theta a)^{N-1}\big]^{-1}$$

and so, putting the above information together

$$(1-\theta^N)\sum_{k=0}^{\infty}(\theta a)^{Nk} = (1-\theta^N)(\mathbf{1} - \theta a)^{-1}\big[\mathbf{1} + \theta a + \ldots + (\theta a)^{N-1}\big]^{-1}.$$

Since $\sigma(a) \subseteq [0, \infty)$, the map $f : \theta \mapsto \big[\mathbf{1} + \theta a + \ldots + (\theta a)^{N-1}\big]^{-1}$ is continuous from $[0, 1]$ into A, and hence $f((0, 1))$ is a bounded subset of A. Hence there exists an $M > 0$ such that $\left\| \left[\mathbf{1} + \theta a + \ldots + (\theta a)^{N-1}\right]^{-1} \right\| \leq M$ for all $\theta \in [0,1]$. Furthermore,

$$(1-\theta^N) = (1-\theta)\big(\mathbf{1} + \theta + \ldots + \theta^{N-1}\big) \leq N(1-\theta), \text{ for all } \theta \in (0,1).$$

This implies that

$$\left\| (1-\theta^N)\sum_{k=0}^{\infty}(\theta a)^{Nk} \right\| \leq N \cdot M(1-\theta)\|(1-\theta a)^{-1}\|$$

for all $\theta \in (0,1)$ and the result follows. □

Note that, in contrast to Theorem 3.2.4, Theorem 3.2.6 has a spectral requirement: Theorem 3.2.4 says that we can move "backwards" to conclude that a is Abel bounded, given that some power of a is Abel bounded; whereas Theorem 3.2.6 says that we can only move "forwards" to conclude that powers of a are Abel bounded if a is Abel bounded and the spectrum of a lies on the nonnegative real axis. As the following example illustrates, we cannot do away with the spectral requirement:
Let

$$T = \begin{pmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{pmatrix}.$$

Then T is Abel bounded, as mentioned previously, but

$$T^4 = \begin{pmatrix} 1 & -4i & -6 \\ 0 & 1 & -4i \\ 0 & 0 & 1 \end{pmatrix}$$

is not by Theorem 3.2.1 since $T^4 = I - q_0$ where

$$q_0 = \begin{pmatrix} 0 & 4i & 6 \\ 0 & 0 & 4i \\ 0 & 0 & 0 \end{pmatrix}$$

is nilpotent. In this case, $\sigma(T) = \{i\} \nsubseteq [0,\infty)$.

3.3 Gelfand-Hille type theorems

Here we discuss our main result, but first we mention another Gelfand-Hille type theorem. We end this section with an example involving the Volterra operator - an operator that has played a significant role as a source of counterexamples in this area of research.

If an $a \in A$ is Cesàro bounded (or uniformly Abel bounded), then it is Abel bounded as illustrated by (3.1.1). Abel bounded elements, though, are not necessarily Cesàro bounded. However, using an argument similar to the one given by Grobler and Huijsmans [11, Theorem 3] for Banach lattices we have the following result for closed normal algebra cones.

Theorem 3.3.1. *Let (A, C) be an ordered Banach algebra, with C a closed normal algebra cone. If $a \in C$ and a is Abel bounded, then a is Cesàro bounded.*

Proof. Assume that C has normality constant α. Since $a \in C$ and C is closed,

$$0 \leq (1-\theta)\sum_{k=0}^{n}\theta^k a^k \leq (1-\theta)\sum_{k=0}^{\infty}\theta^k a^k \text{ for all } \theta \in (0,1), n \in \mathbb{N}.$$

Moreover, since $\theta \in (0,1)$

$$0 \leq (1-\theta)\theta^n\sum_{k=0}^{n} a^k \leq (1-\theta)\sum_{k=0}^{n}\theta^k a^k \text{ for all } \theta \in (0,1), n \in \mathbb{N}.$$

Hence

$$0 \leq (1-\theta)\theta^n\sum_{k=0}^{n} a^k \leq (1-\theta)\sum_{k=0}^{n}\theta^k a^k \leq (1-\theta)\sum_{k=0}^{\infty}\theta^k a^k$$

for all $\theta \in (0,1)$, $n \in \mathbb{N}$. Thus, from the normality of C

$$\|(1-\theta)\theta^n\sum_{k=0}^{n} a^k\| \leq \alpha\|(1-\theta)\sum_{k=0}^{\infty}\theta^k a^k\|$$

for all $\theta \in (0,1)$, $n \in \mathbb{N}$.
Since a is Abel bounded there exists a $D > 0$ such that

$$\|(1-\theta)\theta^n\sum_{k=0}^{n} a^k\| \leq \alpha D \text{ for all } \theta \in (0,1), n \in \mathbb{N}.$$

For a fixed $n \in \mathbb{N}$ we take $\theta = \frac{n}{n+1}$ to obtain

$$\left\|\frac{1}{n+1}\Big(\frac{n}{n+1}\Big)^n\sum_{k=0}^{n} a^k\right\| \leq \alpha D.$$

Thus

$$\left\|\frac{1}{n+1}\sum_{k=0}^{n} a^k\right\| \leq \alpha D\Big(1-\frac{1}{n+1}\Big)^{-n} = \alpha D\frac{n}{n+1}\Big(1-\frac{1}{n+1}\Big)^{-(n+1)}.$$

Therefore

$$\|M_n(a)\| \leq \alpha D\Big(1-\frac{1}{n+1}\Big)^{-(n+1)}.$$

Since $a_n = \Big(1-\frac{1}{n+1}\Big)^{-(n+1)} \to e$ as $n \to \infty$, the sequence (a_n) is bounded by say $0 < M \in \mathbb{R}$. Thus $\|M_n(a)\| \leq \alpha DM$ for all $n \in \mathbb{N}$. □

Returning to the boundedness hierarchy (3.1.1), we notice that Theorem 3.3.1 allows us to reverse the horizontal implications when considering a closed, normal algebra cone C. If we recall the result due to Mbekhta and Zemánek [14, Theorem 2] - that every doubly Cesàro bounded element with single spectrum is the identity - we have the following corollary to Theorem 3.3.1.

Corollary 3.3.2. *Let (A, C) be an ordered Banach algebra, with C a closed normal algebra cone. If $\sigma(a) = \{1\}$, $a, a^{-1} \in C$ and a is doubly Abel bounded, then $a = \mathbf{1}$.*

We can now proceed with our main result, which allows us to weaken the assumption of normality of the cone in Corollary 3.3.2 to a proper cone.

Theorem 3.3.3. *Let (A, C) be an ordered Banach algebra, with closed proper algebra cone C. Let $a \in A$ such that $\sigma(a) = \{1\}$. Then $a = \mathbf{1}$ if and only if*

1. *a^L is Abel bounded and*
2. *$a^N \geq \mathbf{1}$*

for some $L, N \in \mathbb{N}$.

Proof. The forward implication is obvious.
For the reverse implication, assume that a^L is Abel bounded and $a^N \geq 1$. Since a^L is Abel bounded, a is Abel bounded (from Theorem 3.2.4). Furthermore, since $\sigma(a) = \{1\}$ from Theorem 3.2.6 it follows that a^N is Abel bounded. Thus

$$\|(1-\theta)(\mathbf{1} - \theta a^N)^{-1}\| = \|(1-\theta)\sum_{k=0}^{\infty} \theta^k (a^N)^k\| \leq D$$

for some $D > 0$ and for all $\theta \in (0, 1)$. Manipulation gives

$$\left\| \left(\frac{1}{\theta} - 1\right) \left[\left(\frac{1}{\theta} - 1\right)\mathbf{1} - (a^N - \mathbf{1})\right]^{-1} \right\| \leq D$$

Since $\theta \in (0, 1)$, taking $\lambda = \frac{1}{\theta} - 1$ we see that $\lambda > 0$. Let $y = a^N - \mathbf{1}$. Hence

$$\|\lambda(\lambda\mathbf{1} - y)^{-1}\| \leq D$$

for all $\lambda > 0$.
Hence $\lambda^2(\lambda\mathbf{1} - y)^{-1} \to 0$ as $\lambda \to 0^+$. Since $\sigma(y) = \{0\}$ (from the Spectral Mapping Theorem), it follows that $\rho\left(\frac{y}{\lambda}\right) = 0 < 1$ for all $\lambda > 0$. Hence, replacing $(\lambda\mathbf{1} - y)^{-1}$ with its Laurent expansion (by Theorem 1.2.3) results in

$$\lambda^2 \left(\frac{1}{\lambda}\right) \sum_{k=0}^{\infty} \left(\frac{y}{\lambda}\right)^k = \lambda \sum_{k=0}^{\infty} \left(\frac{y}{\lambda}\right)^k = \lambda\mathbf{1} + y + \sum_{k=2}^{\infty} \frac{y^k}{\lambda^{k-1}} \to 0 \text{ as } \lambda \to 0^+.$$

Then it follows that

$$\lambda\mathbf{1} + \sum_{k=2}^{\infty} \frac{y^k}{\lambda^{k-1}} \to -y \text{ as } \lambda \to 0^+.$$

Note that since $a^N \geq \mathbf{1}$, $y \in C$. Hence $\lambda\mathbf{1} + \sum_{k=2}^{\infty} \frac{y^k}{\lambda^{k-1}} \in C$ for all $\lambda > 0$ (C is closed). Then $\lambda\mathbf{1} + \sum_{k=2}^{\infty} \frac{y^k}{\lambda^{k-1}}$ must converge to an element in C as $\lambda \to 0^+$ (again, since C is closed). Therefore $-y \in C$. Hence $y \in C \bigcap (-C) = \{0\}$ (since C is proper). Thus $y = 0$ and so $a^N = \mathbf{1}$. Now $0 = \mathbf{1} - a^N = (\mathbf{1} - a)(\mathbf{1} + a + \ldots + a^{N-1})$ and since $\sigma(a) = \{1\}$ it follows that $\mathbf{1} + a + \ldots + a^{N-1}$ is invertible. Thus $a = \mathbf{1}$. □

If we consider elements $a \in A$ with $\sigma(a) \subseteq [0, \infty)$ such that a^L is Abel bounded and $a^N \geq \mathbf{1}$ for some $L, N \in \mathbb{N}$, the following observations can be made:

- Firstly note that since a is Abel bounded, as mentioned previously, $\rho(a) \leq 1$.
- Secondly, for $\rho(a) \leq 1$, that is $\sigma(a) \subseteq [0, 1]$:
  Using the expansion $\lambda^2(\lambda\mathbf{1} - y)^{-1} = \lambda\mathbf{1} + y + \frac{y^2}{\lambda+2}\sum_{k=0}^{\infty}\left(\frac{y+2}{\lambda+2}\right)^k$ in the proof of Theorem 3.3.3, we again obtain $y = 0$. Hence $a^N = \mathbf{1}$. Now, following a similar argument to the one given in the proof above, since $\sigma(a) \subseteq [0, 1]$ it follows that $a = \mathbf{1}$.

Therefore, in light of the above remarks, Theorem 3.3.3 is the strongest result for elements of this form.

Making use of Theorem 3.3.3, we give the following example of an element that is not Abel bounded, but whose inverse is Abel bounded.
Let (A, C) be an ordered Banach algebra with C a closed proper algebra cone. If $0 \neq q \in C$ and q is quasinilpotent, then since $\mathbf{1} + q \geq \mathbf{1}$, the element $\mathbf{1} + q$ cannot be Abel bounded; otherwise by Theorem 3.3.3 $\mathbf{1} + q = \mathbf{1}$. Thus if q is any positive quasinilpotent, then $\mathbf{1} + q$ is not Abel bounded. In particular, for $1 \leq p \leq \infty$ let $V : L^p[0, 1] \to L^p[0, 1]$ be the Volterra operator, defined by

$$(Vf)(x) = \int_0^x f(t)dt, \text{ for } f \in L^p[0, 1].$$

Then V is a positive operator on $L^p[0, 1]$, with respect to a normal algebra cone. Therefore it follows that $I + V$ is not Abel bounded.
Furthermore, we can show that $(I + V)^{-1}$ is Abel bounded if and only if $I - V$ is Abel bounded:
T.V. Pedersen proved that $I - V = M^{-1}(I + V)^{-1}M$, where $(Mf)(x) = e^{-x}f(x)$ [1, p. 15].
Assume that $I - V$ is Abel bounded. Then there exists a $D > 0$ such that

$$\left\|(1 - \theta)\sum_{k=0}^{\infty}\theta^k\left[M^{-1}(I + V)^{-1}M\right]^k\right\| = \|(1 - \theta)\sum_{k=0}^{\infty}\theta^k(I - V)^k\| \leq D$$

for all $\theta \in (0, 1)$. Observing that $\left[M^{-1}(I + V)^{-1}M\right]^k = M^{-1}(I + V)^{-k}M$, we have

$$\left\|M^{-1}\Big[(1 - \theta)\sum_{k=0}^{\infty}\theta^k(I + V)^{-k}\Big]M\right\| \leq D \text{ for all } \theta \in (0, 1),$$

but since

$$\|(1 - \theta)\sum_{k=0}^{\infty}\theta^k(I + V)^{-k}\| \leq \|M\| \cdot \left\|M^{-1}\Big[(1 - \theta)\sum_{k=0}^{\infty}\theta^k(I + V)^{-k}\Big]M\right\| \cdot \|M^{-1}\|,$$

$(I+V)^{-1}$ is Abel bounded.
For the converse, assume that $(1+V)^{-1}$ is Abel bounded. Then there exists a $D>0$ such that $\|(1-\theta)\sum_{k=0}^{\infty}\theta^k(I+V)^{-k}\| \leq D$ for all $\theta \in (0,1)$. Directly from the fact that the norm is submultiplicative, and the remarks above

$$\begin{aligned}
\left\|(1-\theta)\sum_{k=0}^{\infty}\theta^k\left[1-V\right]^k\right\| &= \left\|(1-\theta)\sum_{k=0}^{\infty}\theta^k\left[M^{-1}(I+V)^{-1}M\right]^k\right\| \\
&= \left\|(1-\theta)\sum_{k=0}^{\infty}\theta^k M^{-1}\left[(I+V)^{-1}\right]^k M\right\| \\
&\leq \|M^{-1}\|\cdot\left\|(1-\theta)\sum_{k=0}^{\infty}\theta^k\left[(I+V)^{-1}\right]^k\right\|\cdot\|M\| \\
&\leq \|M^{-1}\|\cdot D\cdot\|M\|.
\end{aligned}$$

Thus $I-V$ is Abel bounded.
Montes-Rodríguez, Sánchez-Álvarez and Zemánek [15, Theorem 3.3] showed that $I-V$ is Abel bounded, using the resolvent. Hence, from the above argument, $(I+V)^{-1}$ is Abel bounded. However, as stated previously, $I+V$ is not Abel bounded. Thus $I+V$ is not doubly Abel bounded. For related results, see [15].

3.4 Nilpotency

If $a \in A$ has $\sigma(a)=\{1\}$, then as we discussed previously, $a-\mathbf{1}$ is quasinilpotent. In this section we are going to provide conditions under which $a-\mathbf{1}$ is nilpotent. Drissi and Zemánek [9] also provided conditions under which $a-\mathbf{1}$ is nilpotent, in particular we mention one of these conditions towards the end of this section.

Our first result can be seen as a generalization of the main result of this chapter. The technique of the proof is similar to that of Theorem 3.3.3.

Theorem 3.4.1. *Let (A,C) be an ordered Banach algebra, with closed proper algebra cone C. Let $a \in A$ such that $\sigma(a)=\{1\}$. If $a \geq \mathbf{1}$ and if a is (N)-Abel bounded then $(a-\mathbf{1})^N=0$.*

Proof. Assume that a is (N)-Abel bounded and $a \geq 1$. Using the fact that a is (N)-Abel bounded, as well as Theorem 1.2.3

$$\begin{aligned}
\left\|\theta^N\Big(\frac{1}{\theta}-1\Big)^N\frac{1}{\theta}\Big[\Big(\frac{1}{\theta}-1\Big)\mathbf{1}-\big(a-\mathbf{1}\big)\Big]^{-1}\right\| &= \|(1-\theta)^N(\mathbf{1}-\theta a)^{-1}\| \\
&= \|(1-\theta)^N\sum_{k=0}^{\infty}\theta^k a^k\| \leq D
\end{aligned}$$

for some $D > 0$ and for all $\theta \in (0,1)$. Take $\lambda = \frac{1}{\theta} - 1$ and $y = a - \mathbf{1}$. Then

$$\left\| \left(\frac{1}{\lambda+1}\right)^{N-1} \lambda^N [\lambda \mathbf{1} - y]^{-1} \right\| \leq D.$$

Using the identity $\lambda(\lambda\mathbf{1} - y)^{-1} = \mathbf{1} + y(\lambda\mathbf{1} - y)^{-1}$, then for some $D' > 0$

$$\left\| \left(\frac{1}{\lambda+1}\right)^{N-1} \lambda^{N-1} y(\lambda\mathbf{1} - y)^{-1} \right\| \leq D'$$

for all $\lambda > 0$. Hence

$$\lambda \Big[\left(\frac{1}{\lambda+1}\right)^{N-1} \lambda^{N-1} y(\lambda\mathbf{1} - y)^{-1} \Big] \to 0 \text{ as } \lambda \to 0^+.$$

Replacing $(\lambda\mathbf{1} - y)^{-1}$ with its Laurent expansion and simplifying yields

$$\sum_{k=N}^{\infty} \frac{y^k}{\lambda^{k-N+1}} \to -y^N \text{ as } \lambda \to 0^+.$$

Since $a \geq 1$, $y^N \in C$. Hence $\sum_{k=N}^{\infty} \frac{y^k}{\lambda^{k-N+1}} \in C$ for all $\lambda > 0$. This series must converge to an element in C as $\lambda \to 0^+$. Therefore $-y^N \in C$. It follows that $y^N \in C \bigcap (-C) = \{0\}$. Thus $(a - \mathbf{1})^N = 0$. □

If in Theorem 3.3.1 we relax the condition of a being Abel bounded to a being (N)-Abel bounded, we can prove the following:

Theorem 3.4.2. *Let (A, C) be an ordered Banach algebra with C normal and closed. If $a \in C$ is (N)-Abel bounded, then $\|M_n(a)\| = o(n^N)$ as $n \to \infty$.*

Proof. Let α denote the normality constant. Since C is a closed algebra cone and $a \in C$,

$$(1-\theta)^N \sum_{k=0}^{\infty} \theta^k a^k \geq (1-\theta)^N \sum_{k=0}^{n} \theta^k a^k \geq (1-\theta)^N \theta^n \sum_{k=0}^{n} a^k = (1-\theta)^N \theta^n (n+1) M_n(a)$$

for all $\theta \in (0,1)$, $n \in \mathbb{N}$. From the normality of C and since a is (N)-Abel bounded it follows that

$$(1-\theta)^N \theta^n (n+1) \|M_n(a)\| \leq D$$

for some $D > 0$ and for $\theta \in (0,1)$, $n \in \mathbb{N}$. Now, for a fixed n let $\theta = \frac{n}{n+1}$. Then

$$\begin{aligned} \|M_n(a)\| &\leq \frac{D}{n+1}(n+1)^N (1 + \frac{1}{n})^n \\ &= D(n+1)^{N-1}(1 + \frac{1}{n})^n \end{aligned}$$

Now, since $(1 + \frac{1}{n})^n \to e$ as $n \to \infty$, it follows that $\|M_n(a)\| = o(n^N)$ as $n \to \infty$. □

Drissi and Zemánek [9, Theorem 2] showed that if $a \in A$ with $\sigma(a) = \{1\}$ is such that $\|M_n(a)\| = o(n^p)$ as $n \to \infty$ and $\|M_n(a^{-1})\| = o(n^q)$ as $n \to \infty$ for some $p, q \in \mathbb{N}$ then $(a - \mathbf{1})^s = 0$ where $s = \min(p, q)$.
As an immediate consequence of this result and Theorem 3.4.2 we have the following corollary.

Corollary 3.4.3. *Let (A, C) be an ordered Banach algebra with C normal and closed. Let $a \in A$ have $\sigma(a) = \{1\}$. If $a, a^{-1} \in C$ and if a is doubly (N)-Abel bounded then $(a - \mathbf{1})^N = 0$.*

3.5 Inverse closed algebra cones

In this section we are going to investigate what can be said about an element with unit spectrum that belongs to an inverse closed algebra cone. Recall that an algebra cone is said to be inverse closed if whenever $a \in C$ and a is invertible, then $a^{-1} \in C$.

Theorem 3.5.1. *Let (A, C) be an ordered Banach algebra with a closed, proper and inverse closed algebra cone C; and $a \in A$ such that $\sigma(a) = \{1\}$. If $a^N \in C$ for some $N \in \mathbb{N}$ then $a = \mathbf{1}$.*

Proof. Since $\sigma(a) = \{1\}$, from Theorem 1.2.3, for $|\lambda| > 1$

$$(\lambda \mathbf{1} - a^N)^{-1} = \sum_{k=0}^{\infty} \frac{a^{Nk}}{\lambda^{k+1}}.$$

If $\lambda > 1$, since $a^N \geq 0$, it follows that $(\lambda \mathbf{1} - a^N)^{-1} \in C$ (since C is closed). C is also inverse closed, and so we have $\lambda \mathbf{1} - a^N \in C$ for all $\lambda > 1$. If $\lambda \to 1^+$, again since C is closed it follows that

$$\mathbf{1} - a^N \in C.$$

Now, since $\sigma(a) = \{1\}$, it follows that a is invertible. Since $a^N \in C$ and C is inverse closed, $a^{-N} \in C$. From a similar argument applied to a^{-N} we can conclude that $\mathbf{1} - a^{-N} \in C$. Since C is algebraically closed under multiplication,

$$a^N - \mathbf{1} = a^N(\mathbf{1} - a^{-N}) \in C.$$

Thus $\mathbf{1} - a^N \in C \bigcap (-C)$. Hence, since C is proper we must have that $a^N = \mathbf{1}$.
Finally,

$$a^N - \mathbf{1} = (a - \mathbf{1})(a^{N-1} + \ldots + \mathbf{1}) = 0,$$

but since $\sigma(a) = \{1\}$ the Spectral Mapping Theorem implies that $a^{N-1} + \ldots + \mathbf{1}$ is invertible. Therefore $a = \mathbf{1}$. □

It follows from the proof of Theorem 3.5.1 that we can prove the following weaker result. Specifically, it tells us that elements of a closed, inverse closed algebra cone, are dominated by their spectral radii.

Theorem 3.5.2. *Let* (A, C) *be an ordered Banach algebra with* C *closed and inverse closed. If* $a \in C$ *then* $0 \leq a \leq \rho(a)\mathbf{1}$*.*

Proof. Note that for all $|\lambda| > \rho(a)$, from Theorem 1.2.3 we have

$$(\lambda\mathbf{1} - a)^{-1} = \sum_{k=0}^{\infty} \frac{a^k}{\lambda^{k+1}}.$$

Since $a \in C$ and C is closed it follows that for all $\lambda > \rho(a)$,

$$\sum_{k=0}^{\infty} \frac{a^k}{\lambda^{k+1}} \in C.$$

Hence $(\lambda\mathbf{1} - a)^{-1} \in C$ for all $\lambda > \rho(a)$. Since C is inverse closed, it follows that $\lambda\mathbf{1} - a \in C$ for all $\lambda > \rho(a)$. Now if $\lambda \to \rho(a)^+$, since C is closed, $\rho(a)\mathbf{1} - a \in C$. Hence $0 \leq a \leq \rho(a)\mathbf{1}$. □

Corollary 3.5.3. *In any ordered Banach algebra* (A, C) *with* C *a closed, proper and inverse closed algebra cone* $\mathrm{QN}(A) \bigcap C = \{0\}$.

Proof. Clearly, $0 \in \mathrm{QN}(A) \bigcap C$.
Let $a \in \mathrm{QN}(A) \bigcap C$. Since $a \in \mathrm{QN}(A)$ it follows that $\rho(a) = 0$. Hence directly from the preceding theorem we have $0 \leq a \leq 0 \cdot \mathbf{1} = 0$. Hence $a \leq 0$ and $0 \leq a$ and since C is proper we have $a = 0$. □

One of the simplest examples of a closed, proper and inverse closed algebra cone is obtained if we consider the cone of all $n \times n$ diagonal matrices with nonnegative real entries. Clearly, the spectrum of an element belonging to the cone is the set of points on the diagonal. Thus, a quasinilpotent element of the cone has zeros on the main diagonal, as is therefore the zero matrix.

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