ON THE ROLE OF SUBHARMONIC FUNCTIONS IN THE SPECTRAL THEORY OF GENERAL BANACH ALGEBRAS

by

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Introduction

A Banach space which is equipped with a binary operation of multiplication and satisfying the multiplicative inequality $||xy|| \leq ||x|| \cdot ||y||$ is called a Banach algebra. Numerous Banach spaces in functional analysis are at the same time Banach algebras under a multiplication operation, for example the space $C[a, b]$ of all continuous complex-valued functions on an interval $[a, b]$.
Although examples from analysis were one of the main driving forces for mathematicians to study Banach spaces, the interest in Banach algebras was rather late. One reason was the absence of appropriate algebraic tools, since a great deal of early studies in algebra was based on conditions of finiteness, hence ruling out the greater deal of interesting examples in analysis.

It is due to I. M. Gelfand's pioneering work that the foundations of Banach algebras were laid. His simple proof of the well-known Wiener Theorem (which states that the reciprocal of a non-vanishing absolutely convergent Fourier series is also an absolutely convergent Fourier series) attracted a great deal of attention to Banach algebras. But the first important theorem of the general theory of Banach algebras was the famous Gelfand-Mazur Theorem which states that a normed division algebra (over the complex field) must be isomorphic to the complex field. The latter theorem was the starting point of Gelfand's entire theory of Banach algebras.

Today, Banach algebra theory has a wide range of applications, such as in harmonic analysis, operator theory and function algebras. But Banach algebras are also an important research field on its own. It has a great number of results, mainly through the use of techniques of Banach space and ring theory, as well as holomorphic function theory.

The aim of this dissertation is to give an account of the principal methods and results in spectral theory of general Banach algebras by means of subharmonic functions. We assume that the reader of this dissertation has some knowledge of the fundamentals of complex analysis, as well as functional analysis. The reader should also be familiar with algebraic concepts such as isomorphism and homomorphism.

The main body of this dissertation is divided into five chapters: Chapter 1 is an outline of the general theory of Banach algebras which includes invertibility, spectra of elements, the powerful Holomorphic Functional Calculus, a basic introduction to roots, logarithms and exponents, and lastly representation theory and the radical of an algebra.

Chapter 2 deals with the most important theory and results of subharmonic functions which are fundamental to chapters 3 to 5. Among the results of Chapter 2 which play a crucial role, are the Maximum Principle of Subharmonic Functions, Liouville's Theorem for Subharmonic Functions, the Beckenbach-Saks Theorem and Cartan's Theorem with its relation to capacity and polar sets.
The third chapter, along with the fourth and fifth, are the most important in the dissertation. It contains a great number of applications of the theory of Chapter 2 to spectral theory. Moreover, in Chapter 3, the author discusses in depth the proofs of a number of significant theorems due to B. Aupetit, as given in his book; A Primer on Spectral Theory.
In chapters 4 and 5 we extend the use of subharmonic functions and our newly obtained results from Chapter 3 to the (Jacobson) radical of a Banach algebra, as well as to the advanced field of Jordan-Banach algebras, in particular to the characterization of the radical of these algebras – the McCrimmon radical.

The author has also included an appendix on Borel measures (Appendix A), since these measures are used in the theory of capacity (cf. Chapter 2), although it is assumed that the reader has some knowledge on measure and integration theory. Appendix B is a quick reference to the most important and used concepts in topology, while in Appendix C the proof of Radó's Extension Theorem is discussed, since this theorem plays an important role in Chapter 3. Lastly, Appendix D is a short introduction to the n'th spectral diameter and one of its important properties which is used in the proof of the Scarcity Theorem, Theorem 3.26 (cf. Chapter 3).

Throughout the text $\mathbb{C}$ and $\mathbb{R}$ denote the complex and real field respectively. Scalars are always denoted by the following lowercase Greek letters: $\alpha, \beta, \gamma, \zeta, \eta, \lambda, \upsilon$, ξ. In some cases the italic lowercase letter z will denote a complex variable, but this will be clear from the context. Sets and spaces will be indicated by the capital italic letters X, Y, Z, A, B. For X a Banach space, we denote its topological dual by X'. We denote A a subset of X by $A \subset X$, which is seen as that A is either a proper subset of X or $A = X$. The notation $A \subsetneq X$ is used to indicate that A is a proper subset of X. We also use the following symbols for set theoretic operations with A, B subsets of a set X : $A \cup B$ denotes the union, $A \cap B$ the intersection, and $A \backslash B$ is the set difference of A and B, that is, $A \backslash B = \{x \in X : x \in A \text{ and } x \notin B\}$. The symbols $\bigcup_\alpha A_\alpha$, $\bigcup_{\alpha \,\in\, U} A$, or $\bigcup U = \left(\bigcup_{A \in U} A\right)$ will stand for infinite unions; similarly for intersections.
The complement of $A \subset X$ will be written $X \backslash A$, $(A)^c$, or A^c. We also use $\{x_1, x_2, x_3, \ldots\}$ to denote the set consisting of elements $x_1, x_2, x_3, \ldots$ and $\emptyset$ is the empty set. The notation $n \geq 1$ means $n = 1, 2, 3, \ldots$ where the latter is also indicated by $n = \{1, 2, 3, \ldots\}$.

We also write $\#A$ to denote the number of elements in the set A. In chapters 3 – 5, $A^\wedge$ denotes the polynomial convex hull of A. The latter notation is used to denote the full spectrum of x, given by $\sigma(x)^\wedge$ (cf. Appendix B). Further, we adopt the notation:

$$AB = \{ab : a \in A, b \in B\} \quad \text{and} \quad \alpha A + \beta B = \{\alpha a + \beta b : a \in A, b \in B, \alpha, \beta \in \mathbb{C}\}.$$

Open sets (in topological terms) are denoted by U, V, W, while in general, neighborhoods of an element(s) by Ω and Δ. In Chapter 3, Δ is also used to denote the Hausdorff distance between two sets A and B and is given by $\Delta(A, B)$. No confusion should arise in this case since the Hausdorff distance will always be followed by a bracket containing the sets in question.
For maps and functions between sets and spaces, the notation $\mapsto$ is used. As an example, $x \mapsto f(x)$ means that the element $x \in X$ is mapped onto the element $f(x)$ in the range $f(X)$. Lowercase italic letters f, g, h mostly denote complex-valued functions, unless otherwise stated. The Greek letters ϕ, φ are used to denote subharmonic functions (cf. chapters 2 – 5), while π is reserved for representations (cf. §1.5, Chapter 1). In some cases ϕ is also used to denote a linear functional of X', but this will be clear form the context.
When (x_n) is a sequence in a space X such that x_n converges to an element $x \in X$ we use the notation $x_n \to x$, as $n \to \infty$. We also use the symbols $-\infty$ and $+\infty$ throughout the text. The meaning of both should be clear to the reader. Further, $\mathbb{C}_\infty$ denotes the extended complex field.

Terms, names, and theorems which are defined or stated for the first time are given in **boldface** type. Notes or more information on a specific concept is indicated by superscript $^{(nr.)}$. For example $^{(\mathbf{1})}$ refers to note (**1**) which can be found at the end of the specific chapter. We use =: to mean *equals by definition*. This is used whenever a term is defined by means of a formula. Further, the notation §1.3 indicates the third section of Chapter 1, while Theorem 1.3.5 refers to fifth theorem of section 3 of Chapter 1, and Theorem B.5 refers to the fifth theorem of Appendix B. Note that for chapters 3 to 5 we do not have subsections. So, Theorem 3.2 refers to the second theorem of Chapter 3. Displayed equations/expressions in the proof of a theorem are marked accordingly to the specific theorem, for example (3.14.3) refers to the third equation used in the proof of Theorem 3.14. When working with equations/expressions in the notes at the end of a chapter, the equation/expression will be numbered by a roman numeral, for example (i). In the latter case there will be no ambiguity as to which equation we are referring to. Throughout the text, proofs always end with **Q.E.D.**. The author has also included an index as a reference to the most important terms, definitions, and theorems that appear in the text.

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Chapter 1
Banach Algebras

1.1 Introduction to Banach Algebras

To define a Banach algebra we first consider the definition of an algebra. Note, that all algebras are to be considered over the complex field. The reason for this will later be apparent.

A **complex algebra** A is a vector space over the complex field $\mathbb{C}$, endowed with an algebraic operation, called multiplication. Multiplication is denoted by juxtaposition xy for x, y in A, where xy is called the **product** of x and y. This multiplication operation satisfies the following rules for x, y, and z in A and $\alpha \in \mathbb{C}$:

$$x(yz) = (xy)z,$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \quad \text{and}$$

$$\alpha(xy) = (\alpha x)y = x(\alpha y)$$

([RUD1]; p. 227).

Further, if $B \subset A$ and B is closed under the (same) vector space operations and multiplication of A, then B is called a **subalgebra** of A.

A **normed algebra** A is a normed space which is an algebra, such that for all x, y in A

$$||xy|| \leq ||x|| \cdot ||y||.$$

The latter is called the **multiplicative inequality.**
If a normed algebra A is complete with respect to the norm on A, then it is called a **Banach algebra**[(1)] ([KRE]; p. 395).
The **identity** element (or **unit**) $\mathbf{1}$ is the unique element in A such that for each $x \in A$ we have $x\mathbf{1} = x = \mathbf{1}x$. If A has an identity, then A is said to be **unital**. One can always assume that $||\mathbf{1}|| = 1$, otherwise we can replace $||\cdot||$ by an equivalent norm $|||\cdot|||$ such that $|||\mathbf{1}||| = 1$ ([AUP1]; p. 30).

If A has no identity one can always adjoin one to A. This results in a new Banach algebra $\widetilde{A}$, called the **standard unitization** of A, where $\widetilde{A} = A \times \mathbb{C}$. Further, A is isometrically imbedded into $\widetilde{A}$, via the injective homomorphism $x \mapsto (x, 0)$ from

A into $\widetilde{A}$. Thus, elements in $\widetilde{A}$ have the form (x, α) for $x \in A$ and $\alpha \in \mathbb{C}$, with the algebraic operations on $\widetilde{A}$ defined by:

$$(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta), \quad \beta(x, \alpha) = (\beta x, \beta\alpha), \quad \text{and}$$

$$(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta),$$

for $x, y \in A$ and $\alpha, \beta \in \mathbb{C}$. From this, it is evident that the identity of $\widetilde{A}$ is the element $(0, 1)$. The norm on $\widetilde{A}$ is given by

$$||(x, \alpha)|| = ||x|| + |\alpha|$$

([CON2]; p. 192).
Although the standard unitization is helpful, all Banach algebras will be assumed unital. The ardent reader is referred to [ZEL]; p. 22 to 28, for further reading on the subject of non-unital algebras.

Considering the multiplicative inequality again, it follows that the latter makes multiplication a continuous operation on A. That is, if $x_n \rightarrow x$ and $y_n \rightarrow y$, then $x_n y_n \rightarrow xy$[**(2)**]. In particular, multiplication is **left-continuous** and **right-continuous**, that is:

$$x_n y \rightarrow xy \quad \text{and} \quad xy_n \rightarrow xy.$$

Consequently, we have the following interesting fact:

> *"If A is Banach space and an algebra, with identity $\mathbf{1} \neq 0$, such that multiplication is left- and right-continuous, then we can always find a norm on A, inducing the same topology on A as the given one, such that A is a Banach algebra."*

([AUP1]; p. 66, and [RUD1]; p. 228, 229).

An algebraic property which not all Banach algebras share is commutativity. A Banach algebra is **commutative** if multiplication is commutative, that is, if for all x, y in A it holds true that $xy = yx$. Clearly the identity element commutes with every element of A ([KRE]; p. 394, and [RUD1]; p. 228).
The spaces $\mathbb{R}$ and $\mathbb{C}$ are both examples of commutative Banach algebras. Another example is $C[a, b]$. By definition $C[a, b]$ consists of all continuous complex-valued functions defined on a closed interval $[a, b]$, with product defined as $(xy)(t) = x(t)y(t)$ for $t \in [a, b]$. The identity is the constant function $\mathbf{1}(t) = 1$, and the norm is given by $||x|| = \max_{t \in J} |x(t)|$, where $J = [a, b]$ ([KRE]; p. 396).

An example of a non-commutative Banach algebra is the algebra of all complex $n \times n$ matrices, with $n > 1$ fixed. It is denoted by $M_n(\mathbb{C})$ and an arbitrary element

x by (α_{jk}). The standard (natural) norm used by most authors, and as given by [KRE]; p. 103, is:

$$||x|| = \max_j \sum_{k=1}^{n} |\alpha_{jk}|,$$

with $j = 1, 2, \ldots, n$. The identity is the $n \times n$ identity matrix, with entries 1 on the main diagonal and all other entries 0 ([KRE]; p. 396).

The Banach space $\mathcal{B}(X)$ (or $\mathcal{B}(X, X)$) of all bounded linear operators T on a complex Banach space $X \neq \{0\}$, is a Banach algebra. For $\dim(X) \neq 1$, $\mathcal{B}(X)$ is non-commutative. If $\dim(X) = n < \infty$, then $\mathcal{B}(X)$ is isomorphic to $M_n(\mathbb{C})$. The identity element is the identity operator I on X, and multiplication is defined as the composition of operators. For $x \in X$, the norm on $\mathcal{B}(X)$ will be referred to as the **operator-norm**, and is given by:

$$||T|| = \sup_{||x||=1} ||Tx||$$

([KRE]; p. 92, 396). One can also construct new Banach algebras from old ones ([ZEL]; p. 14). The most well-known, and useful construction in this regard is the notion of a quotient algebra. To discuss this we remind ourselves of an ideal of an algebra.

If A is an algebra, then a **left-ideal** of A is a subalgebra $\mathcal{M}$ of A such that $ax \in \mathcal{M}$ whenever $x \in \mathcal{M}$ and $a \in A$. A **right-ideal** of A is a subalgebra $\mathcal{M}$ of A such that $xa \in \mathcal{M}$ whenever $x \in \mathcal{M}$ and $a \in A$.
If $\mathcal{I}$ is a subalgebra of A, such that $\mathcal{I}$ is both a left- and right-ideal, then $\mathcal{I}$ is called an (**two-sided** or **bilateral**) **ideal**. For $\mathcal{I}$ a closed ideal of a Banach algebra A, it is meant that $\mathcal{I}$ is an ideal of A which is closed with respect to the topology, induced by the norm, on A. Further, if $\mathcal{I}$ is an ideal of an algebra A, then $\mathcal{I}$ is a **proper** ideal of A if $\mathcal{I} \neq A$. $\mathcal{I}$ is said to be **maximal**, if it is a proper ideal of A and $\mathcal{I}$ is not contained in any larger proper ideal of A. That is, if $\mathcal{M}$ is also an ideal of A with $\mathcal{I} \subset \mathcal{M} \subset A$, then the only possibility is that $\mathcal{I} = \mathcal{M}$ or $\mathcal{M} = A$ ([RIC]; p. 41, and [RUD1]; p. 263).

We now consider the **quotient algebra** $A/\mathcal{I}$, with $\mathcal{I}$ a closed ideal of a Banach algebra A. The elements $\widehat{x} \in A/\mathcal{I}$, called **cosets**, are the subsets of A of the form $x + \mathcal{I},\ x \in A$. We define the algebraic operations on $A/\mathcal{I}$ as:

$$\widehat{x} + \widehat{y} = \{x + y + \mathcal{I} : x, y \in A\} = \widehat{x + y},$$

$$\alpha\widehat{x} = \{\alpha x + \mathcal{I} : x \in A, \alpha \in \mathbb{C}\} = \widehat{\alpha x}, \text{ and}$$

$$\widehat{x}\widehat{y} = \{xy + \mathcal{I} : x, y \in A\} = \widehat{xy}.$$

Note that these operations are well-defined in $A/\mathcal{I}$ since $\mathcal{I}$ is an ideal. Further, by the above operations, it follows easily that $A/\mathcal{I}$ is an algebra with additive identity $\mathcal{I} = 0 + \mathcal{I}$, and multiplicative identity $\widehat{\mathbf{1}} = \mathbf{1} + \mathcal{I}$, where $\mathbf{1}$ and 0 are the identity and zero element of A respectively ([GOF]; p. 256, 257).
The norm on $A/\mathcal{I}$ is defined by:

$$|||\widehat{x}||| = \inf\{||x + u|| : u \in \mathcal{I}\},$$

where $x \in A$ is fixed, and $||\cdot||$ is the norm on A ([AUP1]; p. 33). With this norm $A/\mathcal{I}$ is then complete ([GOF]; p. 256, 257).

Proving $|||\cdot|||$ is a norm on $A/\mathcal{I}$, note that $||\widehat{x}|| = 0$ if and only if $\widehat{x} = \mathcal{I}$. Also, if $\widehat{x} \in A/\mathcal{I}$ and $\alpha \in \mathbb{C}$, then, since $\mathcal{I}$ is an ideal,

$$|||\alpha\widehat{x}||| = \inf\{\, ||\alpha x + u|| : u \in \mathcal{I}\,\} = |\alpha| \inf\{\, ||x + u|| : u \in \mathcal{I}\,\} = |\alpha| \cdot |||\widehat{x}|||.$$

For every $\widehat{x}, \widehat{y} \in A/\mathcal{I}$,

$$|||\widehat{x} + \widehat{y}||| = \inf\{\, ||x + y + u + v|| : u, v \in \mathcal{I}\,\} \leq \inf\{\, ||x + u|| + ||y + v|| : u, v \in \mathcal{I}\,\}.$$

Consequently by

$$\inf\{||x + u|| + ||y + v|| : u, v \in \mathcal{I}\} = \inf\{||x + u|| : u \in \mathcal{I}\} + \inf\{||y + v|| : v \in \mathcal{I}\},$$

it follows that

$$|||\widehat{x} + \widehat{y}||| \leq \inf\{\, ||x + u|| : u \in \mathcal{I}\,\} + \inf\{\, ||y + v|| : v \in \mathcal{I}\,\} = |||\widehat{x}||| + |||\widehat{y}|||.$$

To prove the multiplicative inequality of $|||\cdot|||$, consider $\widehat{x}, \widehat{y} \in A/\mathcal{I}$ arbitrary. Then, since $\mathcal{I}$ is a two-sided ideal, we have

$$|||\widehat{x}\widehat{y}||| = \inf\{\, ||xy + u + v|| : u, v \in \mathcal{I}\,\} \leq \inf\{\, ||x + u|| \cdot ||y + v|| : u, v \in \mathcal{I}\,\},$$

and hence

$$|||\widehat{x}\widehat{y}||| \leq \inf\{\, ||x + u|| : u \in \mathcal{I}\,\} \cdot \inf\{\, ||y + v|| : v \in \mathcal{I}\,\} = |||\widehat{x}||| \cdot |||\widehat{y}|||$$

([GOF]; p. 256, 257).

It was mentioned at the beginning of this section that all algebras are to be considered over the complex field. The reason: we shall use a great number of results on holomorphic function theory, which is inefficient in the case of the field of real numbers (cf. §1.4). Also, the topological differences between $\mathbb{C}$ and $\mathbb{R}$ play a role ([RUD1]; p 230).

1.2 Spectra and its Properties

The spectrum is surely one of the most useful and important concepts in the theory of Banach algebras. But when one thinks of the word spectrum the field of physics comes to mind.
All substances (elements or bodies) in nature emits or absorbs radiation. Further, in particular physicists know that each substance has its own unique or characteristic way of emitting or absorbing radiation. Through special methods this emission or absorption can then be made visible to the human eye – what one sees is called the spectrum of the substance.
One can almost say that the spectrum of a substance is like a shadow. This "shadow" helps to identify the substance since, as mentioned previously, each substance has its own unique, characteristic spectrum. Ironically the word spectrum has its origin in the word shadow.

Spectrum, in the mathematical sense, is mainly used to differentiate between the general theory of algebras and that of rings. In the study of Banach algebras it is an aid to the study of invertibility. Moreover, in our work, it will be used in the characterization of elements satisfying a given algebraic property. In particular we will be interested in the set of radical elements. The latter is contained in the set of quasi-nilpotent elements, which is the set of elements in A with spectrum consisting of zero only.

By definition, the spectrum is a set of complex numbers. Although this may seem insignificant, it is actually a benefit to our research, since we can visualize the complex plane. Thus, by means of the unique shadow of an element, the problem of characterizing algebraic properties of elements can be geometrically interpreted.

Invertibility

To understand spectra we first discuss the notion of invertibility.
Let A be an algebra with identity $\mathbf{1}$. Then $x \in A$ is said to be **invertible** if there exists a unique element $x^{-1} \in A$, called the **inverse** of x, such that

$$xx^{-1} = x^{-1}x = \mathbf{1}$$

([RUD1]; p. 231).

We shall denote the set of invertible elements of A by $G(A)$. The capital G is used since $G(A)$ is in fact a multiplicative group containing the identity. It is also an open set in A ([AUP1]; p. 35).

Theorem 1.2.1 Properties of Invertibility
([AUP1]; p. 35, 36, [PAL]; p. 195, and [RUD1]; p. 231)

Let A be a Banach algebra with identity $\mathbf{1}$.

(a) *If $||\mathbf{1} - a|| < 1$, then $a \in G(A)$. Moreover, $a^{-1} = \sum_{j=0}^{\infty} (\mathbf{1} - a)^j$.*
(b) *If $a \in G(A)$ and $x \in A$ satisfies $||x - a|| \leq 1/||a^{-1}||$, then x is invertible.*
(c) *The mapping $x \mapsto x^{-1}$ is a homeomorphism from $G(A)$ onto $G(A)$.*
(d) *If h is a complex homomorphism on A, then $h(\mathbf{1}) = 1$ and $h(x) \neq 0$ for each x invertible.*[(3)]
(e) *If B is a Banach algebra and $h : A \to B$ a homomorphism, then $h(G(A)) \subset G(B)$ and $h(a^{-1}) = (h(a))^{-1}$ for all $a \in G(A)$.*
(f) *If B is a subalgebra of A containing the identity $\mathbf{1}$, then $G(B) \subset G(A)$.*

In §1.1 we defined a proper ideal. We now return to the latter, since ideals are closely linked to invertibility, as shown in the following important property:

Let A be an algebra and $\mathcal{I}$ an arbitrary proper ideal of A. Then

$$x \in G(A) \text{ if and only if } x \notin \mathcal{I}$$

([GOF]; Proposition 4, p. 256).

By Theorem 1.2.1 (a), it follows that every left (respectively right, respectively bilateral) ideal of A is disjoint from the open ball with center $\mathbf{1}$ and radius 1. Moreover, this implies that every maximal left (respectively right, respectively bilateral) ideal of A is closed. Further, for A an algebra with identity $\mathbf{1}$, it follows that every left (respectively right) ideal of A is contained in a maximal left (respectively right) ideal ([AUP1]; p. 33, 35).

There are Banach algebras such that all non-zero elements of the algebra are invertible – for example the complex field. A unital algebra with the property that every non-zero element is invertible is called a **division algebra** ([KRE]; p. 403). This brings us to the following famous theorem. It states that the only complete normed division algebra is the complex field.

Theorem 1.2.2 Gelfand-Mazur Theorem
([AUP1]; p. 39, and [ZEL]; p. 17)

If A is a Banach algebra as well as a division algebra, then A is isometrically isomorphic to the complex field.

Basic Properties of Spectra

Let A be a Banach algebra over the complex field and $x \in A$.
The **spectrum** of x, denoted by $\sigma(x)$, is the set of all complex numbers such that $\lambda\mathbf{1} - x$ is not invertible, that is,

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin G(A)\}.$$

Strictly speaking one should denote $\sigma(x)$ by $\sigma_A(x)$, indicating in which algebra we are working. If there is no confusion the spectrum of x will be denoted by $\sigma(x)$ ([RUD1]; p. 234). As an example consider $T \in \mathcal{B}(X)$ (cf. §1.1). By definition of the spectrum we have that $\lambda \in \sigma(T)$ if and only if at least one of the following conditions hold: (i) the range of $\lambda I - T$ is not all of X, or (ii) $\lambda I - T$ is not injective. If (ii) holds, then λ is called an **eigenvalue** of T ([AUP1]; p. 15). Note that if A is an arbitrary algebra and $x \in A$, such that $0 \in \sigma(x)$, then $x \notin G(A)$.

The **resolvent** of x is the complement of $\sigma(x)$. Thus, by definition of $\sigma(x)$, it is the set of all $\lambda \in \mathbb{C}$ such that $(\lambda\mathbf{1} - x)^{-1}$ exists. Further, the unique number

$$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\},$$

is called the **spectral radius** of x. It is the radius of the smallest, closed circular disk in $\mathbb{C}$, with center 0 and containing $\sigma(x)$ ([RUD1]; p. 234).
The following two theorems summarize the properties of the spectrum and spectral radius.

Theorem 1.2.3 Properties of the Spectrum
([AUP1]; p. 36, [BON]; p. 20, [PAL]; p. 197, and [RUD1]; p. 235)

Let A be Banach algebra.

(a) *If $x, y \in A$ arbitrary, then $\sigma(xy)\backslash\{0\} = \sigma(yx)\backslash\{0\}$, or equivalently $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.*
(b) *If A is unital and h is a homomorphism from A into a Banach algebra B, such that $h(\mathbf{1})$ is the identity of B, then*

$$\sigma_B(h(x)) \subset \sigma_A(x), \text{ for all } x \in A.$$

(c) *If B is a subalgebra of A containing the identity $\mathbf{1}$ of A, then for all $b \in B$*

$$\sigma_A(b) \subset \sigma_B(b).$$

(d) *For $x \in A$, we have that $\sigma(x)$ is a non-empty compact set.*

We can actually say more about property (c) of Theorem 1.2.3; for if B is a closed subalgebra of a Banach algebra A, then for $b \in B$, $\sigma_B(b)$ is the union of $\sigma_A(b)$ and a (possibly empty) collection of bounded components of $\mathbb{C}\backslash\sigma_A(b)$. In particular $\partial\sigma_B(b) \subset \partial\sigma_A(b)$. For $\sigma_A(b) = \sigma_B(b)$ to be true, it is sufficient that $\sigma_A(b)$ fails to separate the complex plane ([AUP1]; p. 41, 42, and [RUD1]; p. 238).
Property (d) is another reason why we consider the scalar field of A to be $\mathbb{C}$ rather than $\mathbb{R}$; since in the case of real Banach algebras it might happen that $\sigma(x) = \emptyset$. As an example consider $M_2(\mathbb{R})$, the 2×2 matrices over $\mathbb{R}$. The matrix

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

has no real eigenvalues, hence its spectrum is empty.

Theorem 1.2.4 Properties of the Spectral Radius
([RIC]; p. 10, and [RUD1]; p. 235)

Let A be a Banach algebra. Then for each $x, y \in A$:

(a) *The spectral radius $\rho(x)$ satisfies the* **spectral radius formula**:

$$\rho(x) = \lim_{n\to\infty} ||x^n||^{1/n} = \inf_{n\geq 1} ||x^n||^{1/n}.$$

(b) $0 \leq \rho(x) \leq ||x||$.
(c) $\rho(xy) = \rho(yx)$.
(d) $\rho(\alpha x) = |\alpha|\rho(x)$ *for all* $\alpha \in \mathbb{C}$, *and* $\rho(x^k) = \rho(x)^k$ *for* $k \in \{1, 2, \ldots\}$.
(e) *If* $xy = yx$, *then* $\rho(xy) \leq \rho(x)\rho(y)$ *and* $\rho(x + y) \leq \rho(x) + \rho(y)$.

A frequently used concept, also for future work, is that of an analytic function f, from the complex plane into a Banach space X. To define such a function let U be an open subset of $\mathbb{C}$ and X a Banach space. Also, consider the derivative of f at a point $z_0 \in \mathbb{C}$, denoted by $f'(z_0)$, that is given by

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

if the limit exists. As a consequence, $f : U \to X$ is then said to be **analytic** on U if for every continuous linear functional ϕ of the dual of X, we have that $z \mapsto (\phi \circ f)(z) = \phi(f(z))$ is holomorphic on U ([BON]; p. 23, 24, and [CON1]; p. 203, 204). Also, if f is analytic, then the derivative of $z \mapsto \phi(f(z))$ at a point z_0 is given by $\phi(f'(z_0))$, which is an easy consequence by the continuity and linearity of ϕ ([CON1]; p. 203, 204). So, the term holomorphic is to be used in the usual sense of Complex analysis, while the term analytic is reserved for a Banach space valued function. A function $f : U \to X$ such that $f(z) \in X$, will be referred to as a **Banach space valued function,** where $z \in U \subset \mathbb{C}$ and X is Banach space. Thus, Banach space valued functions are functions whose ranges belong to a general complex Banach space X. The foregoing terminology obviously applies to Banach algebras as well.

Theorem 1.2.5
([AUP1]; p. 38 and [BON]; p. 24)

Let A be a Banach algebra and $x \in A$. Then the mapping $\lambda \mapsto (\lambda\mathbf{1} - x)^{-1}$ is analytic from $\mathbb{C}\backslash\sigma(x)$ into A.

Again, consider U an open subset of $\mathbb{C}$ and A a Banach algebra. Further let $P(U)$ denote the set of all complex polynomials on U. Then for $p \in P(U)$, where

$$p(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_n z^n \quad (z \in U),$$

the corresponding element $p(x)$ of A is

$$p(x) = \alpha_0\mathbf{1} + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n \quad (x \in A),$$

with $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and the convention $x^0 = \mathbf{1}$. Moreover, the mapping $p \mapsto p(x)$ is a homomorphism from $P(U)$ into A ([BON]; p. 21).
This brings us to the following important theorem, which is actually a precursor to the famous Spectral Mapping Theorem (cf. §1.3, Theorem 1.3.2).

Theorem 1.2.6
([BON]; Proposition 5, p. 21)

Let A be a unital Banach algebra, $x \in A$, and U an open set containing $\sigma(x)$. Then for $p \in P(U)$

$$\sigma(p(x)) = p(\sigma(x)),$$

where $p(\sigma(x)) = \{p(\lambda) : \lambda \in \sigma(x)\}$.

By Theorem 1.2.6, we have for example

$$\sigma(x^2) = \sigma(x)^2 = \{\lambda^2 : \lambda \in \sigma(x)\}.$$

One can actually extend Theorem 1.2.6 by considering a rational function $f = p/q$, with $p, q \in P(U)$. For if q has no zeros in U, then it follows by Theorem 1.2.6 that $0 \notin \sigma(q(x))$. This implies $q(x) \in G(A)$ and we can define $f(x)$ by $f(x) = p(x)q(x)^{-1}$.

1.3 The Holomorphic Functional Calculus

In §1.2, for a Banach algebra A and $f(\lambda) = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_n\lambda^n$ a non-constant complex polynomial, we defined the corresponding Banach algebra element $f(x) = \alpha_0\mathbf{1} + \alpha_1 x + \ldots + \alpha_n x^n$. Actually, we can go even further. For example, considering $f(\lambda) = \sum_{k=0}^{\infty} \alpha_k\lambda^k$ an entire function on $\mathbb{C}$, we then have $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$, with $f(x) \in A$. This series converges in A, since $\sum_{k=0}^{\infty} |\alpha_k| \cdot ||x^k||$ is convergent and A is complete ([KRE]; p. 68). Further, if f is a complex series holomorphic on a disk $|\lambda| < R$, with $R > 0$ constant, and $\rho(x) < R$, then for similar reasons the series $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ exists in A with $f(x) \in A$ ([AUP1]; p. 42). The main question now is: what happens when f is an arbitrary function which is only holomorphic on a neighborhood of $\sigma(x)$? Is it then possible to define $f(x)$?

To define f by means of a series does not hold in general, since it is necessary that $\sigma(x)$ is either an annulus, or at least a disk. For example if $\sigma(x)$ is the union of disjoint sets, then certainly f can not be defined by means of a single series, since how would one "fuse" the different pieces together?

Thus we need to introduce a new tool. To illustrate this idea consider the function $f(\lambda) = (\alpha - \lambda)^{-1}$ on a neighborhood Ω of $\sigma(x)$. By the discussion following Theorem 1.2.6, the natural way of defining $f(x)$ is by $f(x) = (\alpha\mathbf{1} - x)^{-1}$, where $\alpha \notin \Omega$. From this one is led to the conjecture that for f an arbitrary function which is only holomorphic on a neighborhood of $\sigma(x)$, we can define $f(x)$ in A. This is indeed true, and the desired result can be obtained by means of the famous Cauchy formula for contours.

To extend the Cauchy formula to general Banach algebras consider K a compact subset of $\mathbb{C}$ and $f : K \to A$ a continuous function from K into a Banach algebra A. If μ is a complex Borel measure on K, then

$$\int_K f(\lambda)\ d\mu$$

can be defined and exists in A: the element $y = \int_K f(\lambda)\ d\mu$ is the unique element in A such that for all $\phi \in A'$ it holds

$$\phi(y) = \phi\Big(\int_K f(\lambda)\ d\mu\Big) = \int_K \phi(f(\lambda))\ d\mu.$$

For the existence and the uniqueness of y we refer the reader to [RUD1]; p. 74, 75, 240, 241.

Let us start with the building blocks. Suppose A is a Banach algebra, $x \in A$, and Ω is a neighborhood of $\sigma(x)$. Further, let $\alpha \notin \sigma(x)$ and Γ be a smooth contour contained in Ω and surrounding $\sigma(x)$, but not α. Then, for the function $f(\lambda) = (\alpha - \lambda)^n$, $\lambda \in \Omega$, and $n = 0, \pm 1, \pm 2, \dots$ we have

$$(\alpha \mathbf{1} - x)^n = \frac{1}{2\pi i} \int_\Gamma (\alpha - \lambda)^n (\lambda \mathbf{1} - x)^{-1} d\lambda.$$

In particular, for $r(\lambda)$ a rational function on Ω, having no poles surrounded by Γ, it follows that

$$r(x) = \frac{1}{2\pi i} \int_\Gamma r(\lambda)(\lambda \mathbf{1} - x)^{-1} d\lambda.$$

([AUP1]; Lemma 3.3.1, p. 43, and [RUD1]; Lemma 10.24, p. 242).
Consequently by Runge's Theorem ([CON1]; p. 86), for f a holomorphic function on an open set Ω containing $\sigma(x)$, and Γ a smooth contour surrounding $\sigma(x)$ in Ω,

$$f(x) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda \mathbf{1} - x)^{-1} d\lambda$$

is a well-defined element of A ([RUD1]; p. 243).
If we denote by $\mathrm{H}(\Omega)$ the collection of complex functions which are holomorphic on the open set Ω containing $\sigma(x)$, then $\mathrm{H}(\Omega)$ is an algebra and $f \mapsto f(x)$ is a mapping from $\mathrm{H}(\Omega)$ into A. The so-called Holomorphic Functional Calculus is the common name given to the properties of this mapping:

Theorem 1.3.1 Holomorphic Functional Calculus
([AUP1]; Theorem 3.3.3, p. 45, [CON1]; p. 206, and [RUD1]; p. 244)

Let A be a Banach algebra and $x \in A$. Suppose that Ω is an open set containing $\sigma(x)$ and that Γ is an arbitrary smooth contour included in Ω and surrounding $\sigma(x)$. Then the mapping

$$f \mapsto f(x) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda \mathbf{1} - x)^{-1} d\lambda,$$

from $\mathrm{H}(\Omega)$ into A, has the following properties:

(a) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$.
(b) $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) = f_2(x) \cdot f_1(x)$.
(c) *If $f(\lambda) = 1$, then $f(x) = \mathbf{1}$.*
(d) *If $f(\lambda) = \lambda$, then $f(x) = x$.*
(e) *If $f_n \in \mathrm{H}(\Omega)$ $(n = 1, 2, 3, \dots)$ and f_n converges to f uniformly on compact subsets of Ω, then $f(x) = \lim_{n \to \infty} f_n(x)$.*

Note that (a), (b), and (c) express the fact that $f \mapsto f(x)$ is an algebra homomorphism from $\mathrm{H}(\Omega)$ into A, which maps the identity of $\mathrm{H}(\Omega)$ onto the identity of A.

The following important Theorem 1.3.2 is an extension of Theorem 1.2.6.

Theorem 1.3.2 Spectral Mapping Theorem
([CON1]; p. 208)

With the hypothesis of Theorem 1.3.1 *we have*

$$\sigma\big(f(x)\big) = f\big(\sigma(x)\big),$$

where $f\big(\sigma(x)\big) = \{f(\lambda) : \lambda \in \sigma(x)\}$.

It is noteworthy that the back-bone of the proof of Theorem 1.3.2 is the following:

$f(x)$ is invertible if and only if $f(\lambda) \neq 0$ for every $\lambda \in \sigma(x)$

([RUD1]; Theorem 10.28 (a), p. 244).

Many of the results that are valid for complex functions carry over to Banach algebra valued functions. The most striking example is Theorem 1.3.1. Related to this theorem, one may under certain conditions, find a Laurent series representation of a Banach algebra valued function of a complex variable. Two other examples are Liouville's and Morera's Theorem which are proved in the notes of this chapter[**4**]. One theorem's proof that we shall include at this stage of our discussion is that of Theorem 1.3.3. It is one of the most frequently used results in chapters to follow.

Theorem 1.3.3
([AUP1]; Theorem 3.3.5, p. 46)

Let A be Banach algebra. Suppose that $x \in A$ and $\alpha \notin \sigma(x)$. Then we have

$$\operatorname{dist}\big(\alpha, \sigma(x)\big) = \frac{1}{\rho\big((\alpha\mathbf{1} - x)^{-1}\big)}.$$

Proof:

Let Ω be an open set containing $\sigma(x)$, but not α. Clearly the function $f(\lambda) = (\alpha - \lambda)^{-1}$ is holomorphic on Ω. By Theorem 1.3.2 it then follows that

$$\sigma\big((\alpha\mathbf{1} - x)^{-1}\big) = \big\{(\alpha - \lambda)^{-1} : \lambda \in \sigma(x)\big\}.$$

In particular

$$\rho\big((\alpha\mathbf{1} - x)^{-1}\big) = \sup\big\{|\alpha - \lambda|^{-1} : \lambda \in \sigma(x)\big\}.$$

Thus, by the properties of the supremum and infimum

$$\rho\big((\alpha\mathbf{1} - x)^{-1}\big) = \frac{1}{\inf\big\{|\alpha - \lambda| : \lambda \in \sigma(x)\big\}},$$

and consequently

$$\rho\big((\alpha\mathbf{1} - x)^{-1}\big) = \frac{1}{\mathrm{dist}(\alpha, \sigma(x))},$$

from which the desired result follows.
Q.E.D.

1.4 Roots, Exponentials and the Group G(A)

In the prologue of his book, Real and Complex Analysis, Walter Rudin focuses entirely on the theory behind the complex exponential function. He states: "*This is undoubtedly the most important function in mathematics.*" ([RUD2]; p. 1). For this reason we felt that the exponential function, in general Banach algebras, should also be placed in the limelight. The basis of its properties is nothing else than the Holomorphic Functional Calculus. By means of the latter, we can go even further; for one is also richly supplied in the knowledge of logarithms and roots in general Banach algebras. These mathematical concepts will play a fundamental role, not only in the applications of subharmonic functions, but also in the analysis of the radical of Jordan-Banach algebras (cf. Chapter 5, Corollary 5.12).

Before proceeding with our discussion on these elements and functions, we consider some definitions.

Let A be a Banach algebra. To say that $x \in A$ has an **n'th root** in A means that $x = y^n$ $(n \geq 1)$ for some $y \in A$ ([RUD1]; p. 246).
For $y \in A$, define e^y by the series

$$\sum_{n=0}^{\infty} \frac{y^n}{n!}\ ,$$

where $y^0 = \mathbf{1}$. Obviously this series converges absolutely for all $y \in A$. The function $f : A \to A$ given by $f(y) = e^y$ is called the **exponential function** ([ZEL]; p. 29). Note that $f(y) = e^y$, as obtained by the Holomorphic Functional Calculus, coincides with the definition of e^y in terms of the above series representation ([BON]; p. 38).

Further, if $x = e^y$ for some $y \in A$, then y is called a **logarithm** of x ([ZEL]; p. 29). For x to have a logarithm y it is necessary, but not sufficient, that x must be invertible. On the other hand, for $\log x$ to exist, it is sufficient, but not necessary, that $\rho(\mathbf{1} - x) < 1$. Let

$$y = -\sum_{n=1}^{\infty} \frac{1}{n}(\mathbf{1} - x)^n.$$

This series converges absolutely if $\rho(\mathbf{1} - x) < 1$, and it can be shown that $y = \log x$ is a solution of the equation $e^y = x$ ([BON]; Corollary 4, p. 40, [RIC]; p. 14, and [ZEL]; p. 29).

The following theorem, which is a consequence of the Holomorphic Functional Calculus, gives a sufficient condition under which a logarithm and roots of a Banach algebra element exist. Surprisingly, the result relies on a spectral property of the element.

Theorem 1.4.1
([AUP1]; Theorem 3.3.6, p. 46, and [RUD1]; Theorem 10.30, p. 246)

Suppose A is a Banach algebra, $x \in A$, and the spectrum $\sigma(x)$ of x does not separate 0 from ∞. Then

(a) *x has a logarithm in A, that is, by definition there exists $y \in A$ such that $x = e^y$, and*
(b) *x has roots of all orders in A, that is, for every $n \geq 1$ there exists $z \in A$ such that $z^n = x$.*

Returning to the definition of the exponential function, it holds true that if $x, y \in A$ commute, then $e^{x+y} = e^x e^y$ (a complete proof is given in [BON]; p. 39). In particular for $y = -x$, we have $e^x e^{-x} = \mathbf{1}$, and consequently $(e^x)^{-1} = e^{-x}$ ([BON]; p. 39, [RUD1]; p. 257, and [ZEL]; p. 29). Hence $e^x \in G(A)$, and $x \mapsto e^x$ maps A onto a subset of $G(A)$ ([ZEL]; p. 29). This subset is denoted by $\exp(A)$, with

$$\exp(A) = \{e^x : x \in A\}.$$

In particular, $\exp(A)$ contains the open ball $B(\mathbf{1}, 1) = \{x \in A : ||x - \mathbf{1}|| < 1\}$. Furthermore, $a \in \exp(A)$ if and only if a belongs to some connected abelian subgroup of $G(A)$ ([PAL]; Theorem 2.1.12 (d), p. 201). Since $\exp(A)$ is the set of all elements which have logarithms, the previous fact can be equivalently stated: a has a logarithm if and only if it is contained in some connected abelian subgroup of $G(A)$ ([RIC]; Theorem 1.4.12, p. 14).

Denote by G_1 the component of $G(A)$ that contains the identity element $\mathbf{1}$ of A. We shall refer to G_1 as the **principal component** of $G(A)$. For example, an element belongs to G_1 if it has finite order ([RIC]; Lemma 1.4.13, p. 15). Further, it holds true that every element of G_1 has a logarithm (hence $\exp(A) = G_1$) if and only if $\exp(A)$ is a group. In particular, if A is commutative, then $\exp(A) = G_1$ ([RIC]; Corollary 1.4.11, p. 14, and [ZEL]; Corollary 7.4, p. 30). It is interesting to note that if A is a commutative Banach algebra, then $G(A)$ is either connected or has infinitely many components[(5)].

1.5 Representation Theory and the Radical of an Algebra

It is important, in any mathematical theory, to compare general abstract structures with well understood and concrete examples. For rings and algebras one successful way of attempting such a comparison is by representation theory. When trying to set up a representation theory for general algebras it is advantageous to consider a relation between the latter and a class of algebras which is well understood. One such algebra, which has turned out to be particularly suitable for representation theory, is the algebra of linear maps from some linear space into itself (these spaces need not be finite dimensional).
In this section we discuss a specific form of homomorphism from a general Banach algebra into an algebra of linear operators. Such a homomorphism is referred to as a representation of the Banach algebra in question. We also note some results concerning representations; the famous Jacobson's Density Theorem and Sinclair's Theorem. After this discussion we turn our focus to the radical of an algebra.

Let A be Banach algebra and X a vector space of dimension greater or equal to one. A **representation** π of A is a non-trivial homomorphism from A into the algebra of linear operators on X, denoted by $\mathcal{L}(X)$. Thus, $\pi(a)$ is a linear operator in $\mathcal{L}(X)$ for each $a \in A$. If Y is a linear subspace of X such that $\pi(a)Y \subset Y$ for each $a \in A$, then Y is said to be **invariant under** $\pi(a)$. A representation is said to be **irreducible** if the only linear subspaces of X invariant under $\pi(a)$ are $\{0\}$ and X, for each $a \in A$. Further, a representation is bounded if X is Banach space and $\pi(a)$ is a bounded linear operator on X for all $a \in A$. Consequently, π is continuous if it is bounded, and if there exists a constant $C > 0$ such that $||\pi(a)|| \leq C||a||$ for all $a \in A$ ([AUP1]; p. 80). The following theorem states the relation between the spectrum of an element in A and the spectra of the corresponding representations of A.

Theorem 1.5.1
([AUP1]; Theorem 4.2.1 (iii), p. 81)

Let A be Banach algebra. Then for every $a \in A$ the spectrum of a is the union of all the spectra of the $\pi(a)$ in the corresponding algebras $\pi(A)$ for all continuous irreducible representations π.

The Gelfand-Mazur Theorem (Theorem 1.2.2) stated that the only complete normed division algebra is $\mathbb{C}$. Theorem 1.5.2 is a consequence of Theorem 1.2.2 concerning representations.

Theorem 1.5.2 (I. Schur)
([AUP1]; Theorem 4.2.2, p. 82)

Let A be a Banach algebra and π be a continuous irreducible representation of A on a Banach space X. Then

$$\mathcal{F} = \{T : T \in \mathcal{B}(X),\ T\pi(a) = \pi(a)T,\ \textit{for all } a \in A\}$$

is isomorphic to $\mathbb{C}$.

An important result by N. Jacobson is his so-called Density Theorem. It should be noted that this result also holds true for the general case of non-commutative rings ([AUP1]; p. 82).

Theorem 1.5.3 Jacobson Density Theorem
([AUP1]; Theorem 4.2.5, p. 83)

Let A be a Banach algebra and π a continuous irreducible representation of A on a Banach space X. If $\xi_1, \xi_2, \ldots, \xi_n$ are linearly independent in X and if $\eta_1, \eta_2, \ldots, \eta_n$ are in X, then there exists $a \in A$ such that $\pi(a)\xi_i = \eta_i$ for $i = 1, 2, \ldots, n$.

The latter can be extended as seen in the following corollary due to Sinclair.

Corollary 1.5.4 (A. Sinclair)
([AUP1]; Corollary 4.2.6, p. 84)

With the same hypothesis of Theorem 1.5.3, *suppose further that $\eta_1, \eta_2, \ldots, \eta_n$ are linearly independent. Then there exists an invertible $a \in A$ such that $\pi(a)\xi_i = \eta_i$ for $i = 1, 2, \ldots, n$.*

We also include a theorem by Kaplansky that will be used in the spectral characterizations of the radical. Recall that a linear operator T is **algebraic of degree n** if there exists a polynomial p of degree n such that $p(T) = 0$, and $q(T) \neq 0$ for all non-zero polynomials q of degree $\leq n-1$ ([AUP1]; p. 37).

Theorem 1.5.5 (I. Kaplansky)
([AUP1]; Theorem 4.2.7, p. 84)

Let X be complex vector space and T a linear operator from X into X. Suppose there exists an integer $n \geq 1$ such that $\xi, T\xi, T^2\xi, \ldots, T^n\xi$ are linearly dependent for all $\xi \in X$. Then T is algebraic of degree less than or equal to n.

The Radical of an Algebra

The radical of a Banach algebra A, denoted by $\operatorname{Rad} A$, plays a fundamental role in spectral theory and representation theory. As mentioned in §1.2 we shall focus on spectral characterizations of radical elements.
Although there are many characterizations of the radical (which can hence also be seen as definitions), the original definition, due to the algebraist N. Jacobson, is given in terms of continuous irreducible representations ([PAL]; p. 474).

An example of a characterization (which will be taken as a definition) is the following: if $\rho(ax) = 0$ for each $x \in A$, then a is in the radical of A. Intuitively one feels that $a = 0$ which is not necessarily true. This is one example of the strange, almost "bizarre", nature of the radical that can be referred to as a pathology. It is precisely this nature that is analyzed, by application of subharmonic functions, in the forthcoming chapters.

We start with the original definition.

Definition 1.5.6
([AUP1]; Theorem 4.2.1 (ii), p. 81 and [PAL]; Definition 4.3.1, p. 474)

The **radical**, *or* **Jacobson radical**, *of a Banach algebra A is the intersection of the kernels of all continuous irreducible representations of A.*

The following Theorem 1.5.7 is equivalent to Jacobson's definition, and is sometimes given as the definition of the radical. The theorem states that the radical of A is a two-sided ideal of A with equivalent properties (a) – (e). The equivalence of (a), (b), and (e) are due to Einar Hille (1948).

Theorem 1.5.7
([AUP1]; Theorem 3.1.3, p. 34)

Let A be a Banach algebra with identity $\mathbf{1}$. Then the following sets are identical:

(a) *the intersection of all maximal left ideals of A,*
(b) *the intersection of all maximal right ideals of A,*
(c) *the set of x such that $\mathbf{1} - zx$ is invertible in A, for all $z \in A$,*
(d) *the set of x such that $\mathbf{1} - xz$ is invertible in A, for all $z \in A$,*
(e) $\operatorname{Rad} A$.

We shall henceforth, as a direct consequence of Theorem 1.5.7, take the following Corollary 1.5.8 as the definition of the radical.

Corollary 1.5.8

In any Banach algebra A the radical of A is equivalent to the following set:

$$\{x \in A : \rho(xy) = 0 \text{ for all } y \in A\}.$$

Corollary 1.5.8 implies that $\operatorname{Rad} A$ is contained in the set of all **quasi-nilpotent** elements of A, denoted by Q_A. By definition, Q_A is the set of all $a \in A$ such that $\rho(a) = 0$. Note that, in general, the radical is strictly contained in the set of quasi-nilpotent elements, although in the case of a commutative Banach algebras the two sets coincide ([AUP1]; p. 36, 71).

Algebras with the property that $\operatorname{Rad} A = \{0\}$ are referred to as **semi-simple**. For example, if X is a Banach space, then $\mathcal{B}(X)$ is semi-simple (cf. [AUP1]; p. 34). Also, if A is a ring with multiplicative identity $\mathbf{1}$, then $A \backslash \operatorname{Rad} A$ is semi-simple. Further, it holds that $\widehat{x}$ is invertible in $A \backslash \operatorname{Rad} A$ if and only if x is invertible in A. Consequently, $\sigma(\widehat{x}) = \sigma(x)$, for $\widehat{x}$ the coset of x in $A \backslash \operatorname{Rad} A$ ([AUP1]; p. 34, 35).

The following characterization of semi-simple algebras is evident from Corollary 1.5.8:

> *A Banach algebra A is semi-simple if and only if $\sigma(xz) = \{0\}$ for all $z \in A$ implies $x = 0$.*

(Of course we can replace $\sigma(xz)$ by $\sigma(zx)$ by Theorem 1.5.7.)

Given A a semi-simple algebra, and B a subalgebra of A, one should note that it does not necessarily imply that B is also semi-simple. Consider $A = M_2(\mathbb{C})$ which is semi-simple ([AUP1]; p. 34). Define the subalgebra, B of A, as all matrices of the form:

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$$

where α, β, and $\gamma \in \mathbb{C}$.
We prove, using Corollary 1.5.8, that

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is an element of $\operatorname{Rad} B$, implying that $\operatorname{Rad} B \neq \{0\}$, and hence B is not semi-simple.
Choose y arbitrary in B where

$$y = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}.$$

It then follows that

$$xy = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}.$$

Determining the eigenvalues of xy, that is the spectrum of xy, we have

$$\sigma(xy) = \{0\}.$$

Thus $x \in \operatorname{Rad} B$, proving the desired result.

Notes on Chapter 1

(**1**) To quote the authors in [BON]; p. 4:

> *"The name 'Banach algebra' is appropriate only because, as a normed linear space, a Banach algebra is a Banach space. However this name is too firmly established in the literature to be changed at this time. Given a free choice we should like to call complete normed algebras 'Gelfand algebras' in recognition of the distinguished pioneering work of I.M. Gelfand in this field."*

(**2**) The continuity can be easily deduced by the identity:

$$x_n y_n - xy = (x_n - x)y_n + x(y_n - y).$$

(**3**) For Theorem 1.2.1; (d) to (f), it is sufficient that A and B need only be unital algebras, they need not be Banach algebras.

(**4**) **Liouville's Theorem for Banach Algebras:**

If A is a Banach algebra and $f : \mathbb{C} \to A$ is a bounded entire function, then f is constant.

Proof:

Let ϕ be an arbitrary element of the dual of A. Consider the composition $\phi \circ f : \mathbb{C} \to \mathbb{C}$. Since ϕ is linear and continuous we have that $\phi \circ f$ is a bounded and an entire function from $\mathbb{C}$ into $\mathbb{C}$. Hence by Liouville's Theorem for complex functions ϕ is constant. Thus $\frac{d}{d\lambda}\phi(f(\lambda)) = \phi(f'(\lambda)) = 0$ for all $\lambda \in \mathbb{C}$ (cf. §1.2) and $\phi(f(\lambda))$ is constant for all $\phi \in A'$. Suppose that f is not constant. Then there exists λ_1 and λ_2 such that $f(\lambda_1) \neq f(\lambda_2)$. As a consequence of the Hahn-Banach Theorem we can find $\varphi \in A'$ such that $\varphi(f(\lambda_1)) \neq \varphi(f(\lambda_2))$. Clearly this is a contradiction and f is constant.
Q.E.D.

Morera's Theorem for Banach Algebras

Let G be a simply connected domain of $\mathbb{C}$ and A be a Banach algebra. Suppose further that $f : G \to A$ is a continuous function. If

$$\int_\gamma f(\lambda)\, d\lambda = 0$$

for every closed curve γ in G, then f is differentiable on G.

Proof:

From $\int_\gamma f(\lambda)\ d\lambda = 0$ it follows that $F(\lambda) = \int_{\lambda_0}^{\lambda} f(\alpha)\ d\alpha$ is independent of the curve between λ_0 and λ as long as the curve belongs to G.
We show that F is differentiable on G and $F'(\lambda) = f(\lambda)$ for every $\lambda \in G$. Let λ_0 be a fixed point of G and define

$$F(\lambda) = \int_{\lambda_0}^{\lambda} f(\alpha)\ d\alpha.$$

Consider a point $\lambda_1 \in G$. Since G is open, there exists an open ball $B = B(\lambda_1, r)$ properly contained in G. If $\lambda \in B$, then the line segment joining λ_1 and λ also belongs to B and we have

$$F(\lambda) - F(\lambda_1) = \int_{\lambda_0}^{\lambda} f(\alpha)\ d\alpha - \int_{\lambda_0}^{\lambda_1} f(\alpha)\ d\alpha = \int_{\lambda_1}^{\lambda} f(\alpha)\ d\alpha,$$

where, in the last integral, we integrate along the line segment between λ_1 and λ. It then follows that

$$\frac{F(\lambda) - F(\lambda_1)}{\lambda - \lambda_1} - f(\lambda_1) = \frac{1}{\lambda - \lambda_1} \int_{\lambda_1}^{\lambda} (f(\alpha) - f(\lambda_1))\ d\alpha.$$

Let $\varepsilon > 0$ be given. Since f is continuous at λ_1 there exists $\delta > 0$ such that

$$|\lambda - \lambda_1| < \delta\ \Rightarrow \|\ f(\lambda) - f(\lambda_1)\ \| < \varepsilon.$$

So, if $|\lambda - \lambda_1| < \delta$, then for all α on the line segment joining λ_1 and λ, we have $|\alpha - \lambda_1| < \delta$. Consequently

$$\|\ \frac{F(\lambda) - F(\lambda_1)}{\lambda - \lambda_1}\ -\ f(\lambda_1)\ \| = \frac{1}{|\lambda - \lambda_1|}\ \|\ \int_{\lambda_1}^{\lambda} (f(\alpha) - f(\lambda_1))\ d\alpha\ \|\ .$$

Hence, since

$$\frac{1}{|\lambda - \lambda_1|}\ \|\ \int_{\lambda_1}^{\lambda} (f(\alpha) - f(\lambda_1))\ d\alpha\ \| \le \frac{1}{|\lambda - \lambda_1|}\ \cdot\ \varepsilon\ |\lambda - \lambda_1| = \varepsilon,$$

we have

$$\|\ \frac{F(\lambda) - F(\lambda_1)}{\lambda - \lambda_1}\ -\ f(\lambda_1)\ \| \le \varepsilon.$$

Thus, we have shown that

$$\lim_{\lambda \to \lambda_1} \frac{F(\lambda) - F(\lambda_1)}{\lambda - \lambda_1} = f(\lambda_1).$$

Since λ_1 was arbitrary in G it follows that F is differentiable on G and, moreover that for $\lambda \in G$ we have $F'(\lambda) = f(\lambda)$. But if a function is differentiable on a simple connected domain, then it has derivatives of all orders on that domain (this holds true by Cauchy formulas for derivatives). Thus we may conclude that f is analytic on G.
Q.E.D.

(**5**) This result is referred to as Lorch's Theorem. It is not known if the latter holds true in the non-commutative case ([ZEL]; p. 31).

Chapter 2
Subharmonic Function Theory

The following chapter is a basic introduction to the theory of subharmonic functions. Proofs are omitted since the main focus is on the application of this theory to general Banach algebras. Furthermore, only the most important results that will be used in our work are mentioned in this chapter.
The reader is referred to [AUP1] and [RAN] for a more in depth study of the topic.

By a **domain** we will always mean a non-empty, connected, open subset of the complex plane $\mathbb{C}$. For a given subset U of $\mathbb{C}$, $\overline{U}$ will denote its closure relative to $\mathbb{C}_\infty$, and ∂U the topological boundary of U relative to $\mathbb{C}_\infty$ ([CON2]; p. 8, and [RAN]; p. 1).

For $\lambda_0, \lambda \in \mathbb{C}$ and $r > 0$,

$$B(\lambda_0, r) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\}$$

$$\overline{B}(\lambda_0, r) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq r\},$$

denote an open and closed disk, respectively, both with center λ_0.

Further, the letter Δ will be used to denote the Laplacian (unless otherwise indicated). Thus, the Laplacian of $f : \mathbb{C} \to \mathbb{R}$, is by definition $\Delta f = f_{xx} + f_{yy}$ ([RAN]; p. 1).

2.1 Harmonic Functions

Since there is a close link between subharmonic and harmonic functions a short introduction is included on the latter functions. Harmonic functions also have properties which are reminiscent of those of holomorphic functions. It is worth noting the following two results that are used in forthcoming chapters.

(1) A function $f = u + iv$ on a domain D is holomorphic if and only if Re$f = u$ and Im$f = v$ are harmonic functions satisfying the Cauchy-Riemann equations.

(2) A domain D is simply connected if and only if for each harmonic function u on D there is a harmonic function v on D such that $f = u + iv$ is holomorphic on D.
([CON2]; p. 254).

Our definition of a harmonic function, as given in [RAN]; p. 3, is as follows:

Let U be an open subset of $\mathbb{C}$. Then a function $h : U \to \mathbb{R}$ is called **harmonic**, *if it has continuous second partial derivatives on U, and $\Delta h = 0$ on U.*

A function $f : U \to \mathbb{R}$ has the **Mean Value Property** (MVP) if whenever $\overline{B}(\lambda_0, r) \subset U$, then

$$f(\lambda_0) = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda_0 + re^{i\theta})d\theta$$

(CON2; p. 255).

Harmonic functions exhibit the MVP as shown in the following Theorem 2.1.1, which is an easy consequence of Cauchy's Integral Formula.

Theorem 2.1.1 Mean Value Property of Harmonic Functions
([CON2]; Definition 1.5, p. 255, and [RAN]; Theorem 1.1.6, p. 5)

Let U be an open subset of $\mathbb{C}$, $h : U \to \mathbb{R}$ a harmonic function, and $\overline{B}(\lambda_0, r)$ a closed disk contained in U. Then

$$h(\lambda_0) = \frac{1}{2\pi} \int_0^{2\pi} h(\lambda_0 + re^{i\theta})d\theta.$$

An important law that all harmonic functions obey is the so-called Maximum Principle.

Theorem 2.1.2 Maximum Principle of Harmonic Functions
([RAN]; Theorem 1.1.8, p. 6)

Let h be a harmonic function on a domain D of $\mathbb{C}$. If h attains a local maximum, then h is constant.

2.2 Subharmonic Functions

To define the notion of a subharmonic function we first need to discuss the concept of upper semi-continuity. We shall consider the following definition obtained from [RAN]; Definition 2.1.1, p. 25:

Let X be a Banach space. A function $u : X \to [-\infty, \infty)$ is **upper semi-continuous** *if the set $\{x \in X : u(x) < \alpha\}$ is open in X for each $\alpha \in \mathbb{R}$. Also $v : X \to (-\infty, \infty]$ is* **lower semi-continuous** *if $-v$ is upper semi-continuous.*

A useful characterization of upper semi-continuity is:

> *u is upper semi-continuous if and only if* $\limsup_{y \to x} u(y) \leq u(x)$
> *for $x \in X$*

([RAN]; p. 25).

Further, in particular, it holds that a function u is continuous if and only if it is both upper and lower semi-continuous ([RAN]; p. 25).

Definition 2.2.1
([AUP1]; p. 52, and [RAN]; Definition 2.2.1, p. 28, Corollary 2.4.2, p. 35)

Let U be an open subset of the complex plane. A function $\varphi : U \to [-\infty, \infty)$ is called **subharmonic** *if:*

(a) *φ is upper semi-continuous, and*
(b) *φ satisfies the* **Mean Value Inequality** (MVI), *that is, if given $\lambda_0 \in U$ and $\overline{B}(\lambda_0, r)$ is a closed disk contained in U, then*

$$\varphi(\lambda_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(\lambda_0 + re^{i\theta}) \, d\theta.$$

Note that a function h is harmonic if and only if both h and $-h$ are subharmonic.

The following theorem is a summary of the most important properties and characterizations of subharmonic functions.

Theorem 2.2.2
([AUP1]; p. 174, 175, and [RAN]; p. 28, 37, 43, 44, 46)

Let U be an open subset of $\mathbb{C}$.

(a) *If φ_1 and φ_2 are subharmonic on U and $\alpha, \beta \geq 0$, then $\alpha\varphi_1$, $\beta\varphi_2$, and $\varphi_1 + \varphi_2$ are subharmonic on U.*
(b) *If (φ_n) $(n \geq 1)$ is a decreasing sequence of subharmonic functions on U, then $\varphi = \lim_{n\to\infty} \varphi_n$ is subharmonic on U.*
(c) *If φ is subharmonic on U and f is a real, convex*[**(1)**]*, and increasing function on $\mathbb{R}$, then the composition $f \circ \varphi$ is subharmonic on U.*
(d) *If (φ_n) is a sequence of subharmonic functions converging uniformly to φ on each compact subset of U, then the limit φ is subharmonic on U.*
(e) *If φ is subharmonic on U, then so is* $\exp\varphi$, (exp *denotes the exponential function*).
(f) *If φ_1 and φ_2 are positive functions such that $\log\varphi_1$ and $\log\varphi_2$ are subharmonic on U, then $\log(\varphi_1 + \varphi_2)$ is subharmonic on U.*
(g) *Let ϕ be subharmonic on U, $a \in U$, and $r > 0$, such that $\overline{B}(a, r) \subset U$. Further, let*

$$N(a, r, \phi) = \frac{1}{2\pi}\int_0^{2\pi} \phi(a + re^{i\theta})\, d\theta, \text{ and}$$

$$M(a, r, \phi) = \max_{0\leq\theta\leq 2\pi} \phi(a + re^{i\theta}).$$

Then we have that

$$\phi(a) = \limsup_{z\to a} \phi(z) = \lim_{r\to 0} N(a, r, \phi) = \lim_{r\to 0} M(a, r, \phi),$$

for $r > 0$ and $z \neq a$.

Subharmonic functions also share a property which is referred to as non-thinness. The definition is as follows:

Let S be a subset of $\mathbb{C}$ and $\zeta \in \mathbb{C}$. Then S is **non-thin**[**(2)**] *at ζ if $\zeta \in \overline{S\backslash\{\zeta\}}$ and if for every subharmonic function φ defined on a neighborhood of ζ,*

$$\limsup_{z\to\zeta} \varphi(z) = \varphi(\zeta), \quad z \in S\backslash\{\zeta\}.$$

If the latter does not hold true, then S is **thin** at ζ ([RAN]; Definition 3.8.1, p. 79).

Theorem 2.1.2 stated that if a harmonic function attains a local maximum on a domain D, then the function is constant. A similar result holds true for subharmonic functions.

Theorem 2.2.3 Maximum Principle of Subharmonic Functions
([AUP1]; Theorem A.1.3, p. 175, [CON2]; p. 266, and [RAN]; Theorem 2.3.1, p. 29)

Let φ be subharmonic on a domain D in $\mathbb{C}$. Suppose there exists λ_0 in D such that $\varphi(\lambda_0) \geq \varphi(\lambda)$ for all λ in D. Then $\varphi(\lambda_0) = \varphi(\lambda)$ for all λ in D, that is, φ is constant on D.

Amongst the collection of subharmonic functions on $\mathbb{C}$, our next result characterizes those that are constant:

Theorem 2.2.4 Liouville's Theorem for Subharmonic Functions
([AUP1]; Theorem A.1.11, p. 176, and [RAN]; Corollary 2.3.4, p. 31)

Let φ be subharmonic on $\mathbb{C}$. Then the following are equivalent:

(a)

$$\limsup_{z \to \infty} \frac{\varphi(z)}{\log |z|} \leq 0,$$

(b)

$$\liminf_{r \to \infty} \frac{\mathrm{M}(r, \varphi)}{\log r} = 0,$$

where $\mathrm{M}(r, \varphi) = \max_{0 \leq \theta \leq 2\pi} \varphi(re^{i\theta})$, *with* $r > 0$,

(c) *φ is constant on $\mathbb{C}$.*

In particular, every subharmonic function on $\mathbb{C}$ that is bounded above must be constant.

One advantage of working with holomorphic functions is that one can construct subharmonic functions from them. For example, if f is holomorphic on an open subset U of $\mathbb{C}$, then $\log |f|$ is subharmonic on U ([RAN]; Theorem 2.2.2, p. 28). This leads to the following question: *given a function $\phi : U \to [0, \infty)$, when is $\log \phi$ subharmonic on U?* The following theorem by E. F. Beckenbach and S. Saks settles the problem.

Theorem 2.2.5 Beckenbach-Saks Theorem
([AUP1]; Theorem A.1.8, p. 176, and [RAN]; Theorem 2.6.5, p. 44)

Let $\phi : U \to [0, \infty)$ be a function on an open subset U of $\mathbb{C}$. Then $\log \phi$ is subharmonic on U if and only if $z \mapsto |e^{p(z)}|\phi(z)$ is subharmonic on U for each complex polynomial p.

2.3 Potentials, Polar sets and Capacity

Our main focus in this section is the notion of capacity and its relationship with respect to polar sets. Throughout the discussions to follow, we shall only consider finite Borel measures with compact support. The reader is referred to Appendix A for a brief discussion on the latter.

First we consider the concepts of potential and energy.

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. The **potential** *of μ is the function $p_\mu : \mathbb{C} \to [-\infty, \infty)$ defined by*

$$p_\mu(z) = \int \log|z - w| \, d\mu(w), \text{ for } z \in \mathbb{C}$$

([RAN]; Definition 3.1.1, p. 53).

It is interesting to note that potentials are examples of subharmonic functions. In particular, it can be shown that p_μ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C}\backslash(\text{supp } \mu)$, where supp (μ) denotes the support of μ ([RAN]; Theorem 3.1.2, p. 53).

With the above definition we now define the energy of a finite Borel measure on the complex plane.

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. Its **energy**, $I(\mu)$, *is given by*

$$I(\mu) = \int \int \log|z - w| \, d\mu(z) \, d\mu(w) = \int p_\mu(z) \, d\mu(z)$$

([RAN]; Definition 3.2.1, p. 55).

One can almost think of μ as (an electric) charge distribution on $\mathbb{C}$. Thus, $p_\mu(z)$ represents the potential energy at z due to μ, and consequently the total energy due to the charge distribution is $\int p_\mu(z)\ dz$, that is, $I(\mu)$. So, $I(\mu)$ takes into account all the potentials of the different charges.

Let E be a subset of $\mathbb{C}$. E is said to be **polar** if $I(\mu) = -\infty$ for all finite Borel measures $\mu \neq 0$, for which the support of μ is a compact subset of E. Thus sets which only support measures of infinite (negative) energy are referred to as polar sets. An example of polar sets are the singletons.
If $I(\mu) > -\infty$, then $\mu(E) = 0$ for every Borel polar set E. Further, every Borel polar set has Lebesgue measure zero ([RAN]; p. 56).
To discuss the link between polar sets and capacity we introduce yet another concept – equilibrium measures. These are Borel probability measures μ on a compact set K such that the energy $I(\mu)$ is maximized.

Definition 2.3.1
([RAN]; Definition 3.3.1, p. 58)

Let K be a compact subset of $\mathbb{C}$, and denote by $\mathcal{P}(K)$ the collection of all Borel probability measures on K. If there exists $\nu \in \mathcal{P}(K)$ such that

$$I(\nu) = \sup_{\mu \in \mathcal{P}(K)} I(\mu),$$

then ν is called an **equilibrium measure** *for K.*

It can be shown that every compact subset of the complex plane always has an equilibrium measure ([RAN]; Theorem 3.3.2, p. 58). Further, note that the equilibrium measure is unique, provided that K is non-polar. If K is polar, then every $\mu \in \mathcal{P}(K)$ is an equilibrium measure since they all satisfy $I(\mu) = -\infty$ ([RAN]; p. 58).

It is of interest to know how "near" a set is to being polar. The energy $I(\nu)$ of the equilibrium measure is used to determine this desired result. By taking exponentials of the latter energies we are able to define the capacity of a set.

Definition 2.3.2
([RAN]; Definition 5.1.1, p. 127)

The **logarithmic capacity**, *or henceforth just the* **capacity**, *of a subset E of $\mathbb{C}$ is given by*

$$c(E) = \sup_{\mu} e^{I(\mu)},$$

where the supremum is taken over all Borel probability measures μ on $\mathbb{C}$, whose support is a compact subset of E.

In particular, since all compact subsets K of $\mathbb{C}$ have an equilibrium measure ν, it follows that $c(K) = e^{I(\nu)}$ by definition of $I(\nu)$.
We now show that a set is polar if and only if it has capacity zero. In order to do so, define $e^{-\infty} = 0$.
If E is polar, then $c(E) = 0$, since $\sup_\mu e^{I(\mu)} = -\infty$. To prove the converse, assume E is non-polar. That is, $I(\mu_0) \neq -\infty$ for some μ_0. Hence $e^{I(\mu_0)} > 0$ (by the properties of the exponential function) and by Definition 2.3.2 it follows that $c(E) \neq 0$. Thus, one can almost see polar sets in potential theory as though they are negligible, just as the sets of measure zero are in measure theory.

For K a compact subset of $\mathbb{C}$ it is always true that $0 \leq c(K) \leq \delta(K)$, with $\delta(K)$ the diameter of K. Also if $E_1 \subset E_2$, then $c(E_1) \leq C(E_2)$ ([AUP1]; p. 178 and [RAN]; p. 128). Amongst all the properties of capacity the following theorem is the most important for us. It will be used in the proof of the Scarcity Theorem (cf. Chapter 3, Theorem 3.26).

Theorem 2.3.3
([RAN]; Theorem 5.1.3, p. 128)

If $B_1 \subset B_2 \subset B_3 \subset \ldots$ are Borel sets of $\mathbb{C}$ and $B = \bigcup_{n=1}^{\infty} B_n$, then

$$c(B) = \lim_{n\to\infty} c(B_n).$$

Theorem 2.3.4, due to H. Cartan, plays a significant role in the applications of subharmonic functions to spectral theory.

Theorem 2.3.4 Cartan's Theorem
([RAN]; Theorem 3.5.1, p. 65)

Let φ be a subharmonic function on a domain D of the complex plane $\mathbb{C}$ with φ not identical to $-\infty$. Then the set $E = \{z \in D : \varphi(z) = -\infty\}$ is a G_δ-polar set.

Thus, the set E in Cartan's Theorem is not only a G_δ-set but also has zero capacity.

The following Theorem 2.3.5 states that a closed disk always has non-zero capacity.

Theorem 2.3.5
([AUP1]; Corollary A.1.26, p. 179)

Let K be a closed disk of radius $r > 0$. Then $c(K) = r$.

Notes on Chapter 2

(**1**) The definition of a convex function ([RAN]; Definition 2.6.1, p. 43):

Let $-\infty \le a < b \le \infty$. *A function* $f : (a, b) \to \mathbb{R}$ *is called* **convex** *if whenever* $t_1, t_2 \in (a, b)$ *then,*

$$f\big((1-\alpha)t_1 + \alpha t_2\big) \le (1-\alpha)f(t_1) + \alpha f(t_2)$$

for $\alpha \in [0, 1]$.

(**2**) As stated in [RAN]; p. 83:

> *"Thinness can be characterized in terms of the so-called fine topology, namely the weakest topology on* $\mathbb{C}$ *with respect to which all subharmonic functions are continuous. In fact* S *is non-thin at* ζ *precisely when* ζ *is a fine limit of the point* S*. Several aspects of potential theory are mostly treated in the context of the fine topology."*

Chapter 3

Applications of Subharmonic Functions in General Banach Algebras

In general the spectrum function $x \mapsto \sigma(x)$, is not continuous, although the set of its points of continuity is a dense G_δ-set ([AUP]; Theorem 3.4.3, p. 50). The latter is actually an old result due to K. Kuratowski. We refer the reader to [JAM]; p. 35 – 36, for further reading, and the proof, of this classical result.
To discuss the continuity properties of the spectrum, we consider the concept of **Hausdorff distance**. The Hausdorff distance is a metric which measures the distance between two compact subsets K_1 and K_2 of $\mathbb{C}$. Its definition:

$$\triangle(K_1, K_2) = \max\Big(\sup_{z \epsilon K_2} \operatorname{dist}(z, K_1),\ \sup_{z \epsilon K_1} \operatorname{dist}(z, K_2)\Big).$$

Since the spectrum is compact, the Hausdorff distance is well-defined for our purpose. Thus, $x \mapsto \sigma(x)$ is said to be continuous at a point $a \in A$, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $||x - a|| < \delta$, then $\Delta(\sigma(x), \sigma(a)) < \varepsilon$.
Further, $x \mapsto \sigma(x)$ is continuous on $E \subset A$ if it is continuous at every point of E. Also, if the number $\delta > 0$ is independent of $a \in E$, then $x \mapsto \sigma(x)$ is said to be uniformly continuous on E ([AUP1]; p. 48).

Definition 3.1
([AUP1]; p. 48)

If K is a compact set of $\mathbb{C}$ and $r > 0$, then $K + r$ is defined:

$$K + r = \{z \in \mathbb{C} : \operatorname{dist}(z, K) \leq r\}.$$

Theorem 3.2
([AUP1]; Theorem 3.4.1, p. 48)

Let A be a Banach algebra. Suppose that $x, y \in A$ commute. Then

$$\sigma(y) \subset \sigma(x) + \rho(x - y),$$

and consequently we have

$$\Delta(\sigma(x), \sigma(y)) \leq \rho(x - y) \leq ||x - y||.$$

Furthermore, if A is commutative then the spectrum function is uniformly continuous on A.

Proof:

Suppose that the inclusion $\sigma(y) \subset \sigma(x) + \rho(x - y)$ is false. Then, there exists $\alpha \in \sigma(y)$ such that $\alpha \notin \sigma(x) + \rho(x - y)$. Thus, by Definition 3.1,

$$\rho(x - y) < \text{dist}(\alpha, \sigma(x)). \tag{3.2.1}$$

In particular, (3.2.1) implies that $\alpha \notin \sigma(x)$. Hence, by Theorem 1.3.3,

$$\text{dist}(\alpha, \sigma(x)) = \frac{1}{\rho((\alpha\mathbf{1} - x)^{-1})}.$$

By substitution of the latter into (3.2.1) we have

$$\rho((\alpha\mathbf{1} - x)^{-1})\rho(x - y) < 1.$$

Since $(\alpha\mathbf{1} - x)^{-1}$ and $x - y$ commute, it follows by Theorem 1.2.4 (e) that

$$\rho\big((\alpha\mathbf{1} - x)^{-1}(x - y)\big) < 1.$$

Consequently, by the latter $-1 \notin \sigma\big((\alpha\mathbf{1} - x)^{-1}(x - y)\big)$. But, $\alpha\mathbf{1} - y = (\alpha\mathbf{1} - x) + (x - y)$ whence,

$$\alpha\mathbf{1} - y = (\alpha\mathbf{1} - x)\big(\mathbf{1} + (\alpha\mathbf{1} - x)^{-1}(x - y)\big).$$

Since $G(A)$ is a group and $\alpha\mathbf{1} - x \in G(A)$, it must hold true that $\alpha\mathbf{1} - y$ is also invertible. Clearly this is a contradiction and the result follows.
Q.E.D.

Theorem 3.3 Upper Semi-Continuity of the Spectrum Function
([AUP1]; Theorem 3.4.2, p. 50)

Let A be a Banach algebra. Then the spectrum function $x \mapsto \sigma(x)$ is upper semi-continuous on A, that is, for every open set U containing $\sigma(x)$ there exists $\delta > 0$ such that $||x - y|| < \delta$ implies $\sigma(y) \subset U$.

Proof:

Suppose the inclusion is false for some open set U containing $\sigma(x)$. Then there exists a sequence (y_n) in A and a sequence (α_n) in $\mathbb{C}$ such that

$$x = \lim y_n,$$

but

$$\alpha_n \in \sigma(y_n) \cap (\mathbb{C}\backslash U).$$

Consequently, since $\alpha_n \in \sigma(y_n)$, it follows by the inequality $\rho(y_n) \leq ||y_n||$ that

$$|\alpha_n| \leq ||y_n|| \leq K,$$

since (y_n) is a convergent sequence, with K a positive constant. Hence, by the Bolzano-Weierstrass Theorem we may suppose without loss of generality that (α_n) converges to α. But, since $\mathbb{C}\backslash U$ is closed, we have $\alpha \notin U$. Thus, since $\sigma(x) \subset U$, $\alpha\mathbf{1} - x$ is invertible. Hence, by Theorem 1.2.1 (c), $\alpha_n\mathbf{1} - y_n$ is invertible for n large enough. Clearly this is a contradiction and the result follows.
Q.E.D.

Theorem 3.4 (J.D. Newburgh)
([AUP1]; Theorem 3.4.4, p. 51)

Let A a Banach algebra and $x \in A$. Suppose that U, V are two disjoint open sets such that

$$\sigma(x) \subset U \cup V,$$

and

$$\sigma(x) \cap U \neq \emptyset.$$

Then there exists $r > 0$ such that $||x - y|| < r$ implies $\sigma(y) \cap U \neq \emptyset$.

Proof:

Assume $\sigma(x) \subset U \cup V$ and $\sigma(x) \cap U \neq \emptyset$, with U, V two disjoint open sets. By the upper semi-continuity of the spectrum (Theorem 3.3) there exists $\delta > 0$ such that $||x - y|| < \delta$ implies $\sigma(y) \subset U \cup V$.

Suppose the conclusion is false. Then, there exists a sequence (y_n) converging to x such that for n large enough

$$\sigma(y_n) \cap U = \emptyset,$$

and

$$\sigma(y_n) \subset V.$$

Now, let f be the holomorphic function defined on $U \cup V$ by:

$$f(\lambda) = 1, \text{ for } \lambda \in U, \quad \text{and} \quad f(\lambda) = 0, \text{ for } \lambda \in V.$$

By the Holomorphic Functional Calculus[(1)]

$$\lim_{n\to\infty} f(y_n) = f(x) \quad \text{and} \quad f(y_n) = 0, \quad \text{for } n \text{ large enough.} \tag{3.4.1}$$

Consequently, by the Spectral Mapping Theorem (Theorem 1.3.2), it follows that $1 \in \sigma(f(x)) = f(\sigma(x))$. Hence, since $1 \notin \sigma(0)$, we have $f(x) \neq 0$. Clearly this is in

contradiction with (3.4.1), and the result follows.
Q.E.D.

Corollary 3.5 (J. D. Newburgh)
([AUP1]; Corollary 3.4.5, p. 51)

Suppose that the spectrum of $a \in A$ is totally disconnected. Then $x \mapsto \sigma(x)$ is continuous at a.

Proof:

Let $\varepsilon > 0$. Since $\sigma(a)$ is totally disconnected (and compact), it follows by Theorem B.1 that $\sigma(a)$ is included in the union U of a finite number of disjoint open sets, each intersecting $\sigma(a)$ and having diameters less than ε.

Further, by the upper semi-continuity of $x \mapsto \sigma(x)$ (Theorem 3.3), there exists a $r_1 > 0$ such that $||x - a|| < r_1$ implies $\sigma(x) \subset U$. Since U consists of a finite union of disjoint sets, it follows by Theorem 3.4 that there exists $r_2 > 0$ such that

$$||x - a|| < r_2$$

implies

$$\sup_{z \in \sigma(a)} \operatorname{dist}(z, \sigma(x)) < \varepsilon.$$

Choosing $r = \min(r_1, r_2)$ it follows that

$$||x - a|| < r$$

implies

$$\Delta(\sigma(a), \sigma(x)) < \varepsilon$$

proving the desired result.
Q.E.D.

Corollary 3.5 is useful because it implies that the spectral function is continuous at all elements with a finite or countable spectrum.

The following Lemma 3.6 and Vesentini's Theorem (Theorem 3.7) are the most frequently applied results in the work to come. The proofs of the latter depend strongly on the properties and characteristics of subharmonic functions.

Lemma 3.6
([AUP1]; Lemma 3.4.6, p. 52)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach space X. Then

$$\lambda \mapsto \log ||f(\lambda)||$$

is subharmonic on D.

Proof:

By application of the Beckenbach-Saks Theorem (Theorem 2.2.5) we show that $\lambda \mapsto |e^{p(\lambda)}| \cdot ||f(\lambda)||$ is subharmonic on D for each complex polynomial p, from which the desired result then follows.

First note that $||f(\lambda)|| \geq 0$ on D and that the function $\lambda \mapsto |e^{p(\lambda)}| \cdot ||f(\lambda)||$ is continuous, hence upper semi-continuous (cf. §2.2). We now prove that this function also satisfies the MVI.

Consider a closed disk $\overline{B}(\lambda_0, r)$ contained in D. Clearly for each $\lambda \in \partial\overline{B}(\lambda_0, r)$ we have $\lambda = \lambda_0 + re^{i\theta}$, where $\theta \in [0, 2\pi]$. We claim that for f analytic on $\overline{B}(\lambda_0, r)$,

$$f(\lambda_0) = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda_0 + re^{i\theta})\ d\theta.$$

Let $\phi \in X'$. Then, by Cauchy's Integral Theorem

$$(\phi \circ f)(\lambda_0) = \phi(f(\lambda_0)) = \frac{1}{2\pi i} \int_{\partial\overline{B}} \phi(f(\lambda))(\lambda - \lambda_0)^{-1} d\lambda.$$

By substitution it follows that

$$\phi(f(\lambda_0)) = \frac{1}{2\pi i} \int_0^{2\pi} \phi\big(f(\lambda_0 + re^{i\theta})\big)(re^{i\theta})^{-1}(ire^{i\theta})\ d\theta,$$

and thus by §1.3

$$\phi(f(\lambda_0)) = \frac{1}{2\pi} \int_0^{2\pi} \phi\big(f(\lambda_0 + re^{i\theta})\big)\ d\theta = \phi\Big(\frac{1}{2\pi} \int_0^{2\pi} f(\lambda_0 + re^{i\theta})\ d\theta\Big).$$

Hence, by a consequence of the Hahn-Banach Theorem,

$$f(\lambda_0) = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda_0 + re^{i\theta})\ d\theta,$$

and consequently

$$||f(\lambda_0)|| \leq \frac{1}{2\pi}\int_0^{2\pi} ||f(\lambda_0 + re^{i\theta})||\ d\theta. \tag{3.6.1}$$

Let p denote an arbitrary complex polynomial on $\mathbb{C}$. Then, the function $\lambda \mapsto e^{p(\lambda)}f(\lambda)$ is analytic from D into X and

$$||e^{p(\lambda)}f(\lambda)|| = |e^{p(\lambda)}| \cdot ||f(\lambda)||. \tag{3.6.2}$$

By replacing $f(\lambda)$ by $e^{p(\lambda)}f(\lambda)$ in (3.6.1), it follows that

$$||e^{p(\lambda_0)}f(\lambda_0)|| \leq \frac{1}{2\pi}\int_0^{2\pi} ||e^{p(\lambda_0)}f(\lambda_0 + re^{i\theta})||\ d\theta,$$

and consequently by (3.6.2)

$$|e^{p(\lambda_0)}| \cdot ||f(\lambda_0)|| \leq \frac{1}{2\pi}\int_0^{2\pi} |e^{p(\lambda_0)}| \cdot ||f(\lambda_0 + re^{i\theta})||\ d\theta.$$

Hence, $\lambda \mapsto |e^{p(\lambda)}| \cdot ||f(\lambda)||$ is subharmonic on D and the desired result follows.
Q.E.D.

Theorem 3.7 (E. Vesentini)
([AUP1]; Theorem 3.4.7, p. 52)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Then

$$\lambda \mapsto \rho(f(\lambda)) \quad \textit{and} \quad \lambda \mapsto \log\rho(f(\lambda))$$

are subharmonic on D.

Proof:

From Theorem 1.2.4 (a) we have $\rho(x) = \lim_{n\to\infty} ||x^n||^{1/n}$. Now consider the subsequence $||x^{2^n}||^{1/2^n}$, which is decreasing**(2)** and also with limit $\rho(x)$.

Focusing on the analytic function f, with $f(\lambda) \in A$, it then follows for the same reasons that $||f(\lambda)^{2^n}||^{1/2^n}$ is a deceasing sequence with limit $\rho(f(\lambda))$.
Consequently, since $\log t$ ($t \in \mathbb{R}^+$) is increasing, it follows that

$$\log||f(\lambda)^{2^n}||^{1/2^n} = \frac{1}{2^n}\log||f(\lambda)^{2^n}|| \tag{3.7.1}$$

is also a decreasing sequence. Further, by the continuity of the log-function and by taking into account (3.7.1), we have that

$$\frac{1}{2^n}\log||f(\lambda)^{2^n}|| \to \ \log\rho(f(\lambda)) \ \text{ as } n \to \infty. \tag{3.7.2}$$

Since $\lambda \mapsto f(\lambda)^{2^n}$ is analytic on D, $\lambda \mapsto \log ||f(\lambda)^{2^n}||$ is subharmonic on D by Lemma 3.6. Consequently, since $1/\ 2^n > 0$ and by Theorem 2.2.2 (a),

$$\lambda \mapsto \frac{1}{2^n} \log ||f(\lambda)^{2^n}||$$

is subharmonic on D for each n.

Hence, by (3.7.2), since $\log \rho(f(\lambda))$ is the limit of a decreasing sequence of subharmonic functions, it follows by Theorem 2.2.2 (b) that

$$\lambda \mapsto \log \rho(f(\lambda)) \tag{3.7.3}$$

is subharmonic on D.

Now, consider the convex (cf. §2.2), increasing function $t \mapsto e^t$, with $t \in \mathbb{R}$. By taking the composition of the latter with (3.7.3), it follows by the identity $e^{\log t} = t$ and Theorem 2.2.2 (c) that $\lambda \mapsto \rho(f(\lambda))$ is subharmonic on D.
Q.E.D.

Corollary 3.8
([AUP1]; Corollary 3.4.8, p. 53)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Suppose that $\alpha \notin \sigma(f(\lambda))$ for all $\lambda \in D$. Then

$$\lambda \mapsto 1/\mathrm{dist}(\alpha, \sigma(f(\lambda))) \quad \textit{and} \quad \lambda \mapsto -\log \mathrm{dist}(\alpha, \sigma(f(\lambda)))$$

are subharmonic on D.

Proof:

By Theorem 1.3.3

$$\rho\big((\alpha\mathbf{1} - f(\lambda))^{-1}\big) = 1/\mathrm{dist}\big(\alpha, \sigma(f(\lambda))\big), \text{ for } \alpha \notin \sigma(f(\lambda)).$$

Since $\lambda \mapsto (\alpha\mathbf{1} - f(\lambda))^{-1}$ is analytic on D, $\lambda \mapsto \rho((\alpha\mathbf{1} - f(\lambda))^{-1})$ is subharmonic on D (cf. Theorem 3.7). Thus,

$$\lambda \mapsto 1/\mathrm{dist}(\alpha, \sigma(f(\lambda)) \quad \text{and} \quad \lambda \mapsto -\log \mathrm{dist}(\alpha, \sigma(f(\lambda))$$

are subharmonic on D, where the latter holds true by Theorem 3.7.
Q.E.D.

Corollary 3.9
([AUP1]; Corollary 3.4.9, p. 53)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Define

$$u(\lambda) = \max\{\text{Re } u : u \in \sigma(f(\lambda))\}$$

and

$$v(\lambda) = \min\{\text{Re } v : v \in \sigma(f(\lambda))\}.$$

Then $u(\lambda) = \log \rho(e^{f(\lambda)})$, $v(\lambda) = -\log \rho(e^{-f(\lambda)})$, and u and $-v$ are subharmonic on D.

Proof:

It is obvious that $v(\lambda) = -\max\{\text{Re } v : v \in \sigma(-f(\lambda))\}$. Thus it is sufficient to prove that u is subharmonic on D.

We first prove that $\max\{\text{Re } u : u \in \sigma(x)\} = \log \rho(e^x)$ for $x \in A$.

Consider the function $f(x) = e^x$, $x \in A$. By the Spectral Mapping Theorem and by the definition of the spectral radius

$$\rho(e^x) = \max\{|e^u| : u \in \sigma(x)\}.$$

Since $\log t$ ($t \in \mathbb{R}^+$) is increasing, it then follows that

$$\log \rho(e^x) = \max\{\log |e^u| : u \in \sigma(x)\}.$$

Further, taking into account that $|e^u| = e^{\text{Re u}}$, we then have

$$\log \rho(e^x) = \max\{\log e^{\text{Re u}} : u \in \sigma(x)\},$$

and hence

$$\log \rho(e^x) = \max\{\text{Re } u : u \in \sigma(x)\} \text{ for } x \in A.$$

Thus for $f(\lambda) \in A$

$$u(\lambda) = \log \rho(e^{f(\lambda)}),$$

and consequently, by Theorem 3.7, u is subharmonic on D.
Q.E.D.

Corollary 3.10
([AUP1]; Corollary 3.4.10, p. 53)

Let $0 \leq r \leq s$, $0 < \theta_2 - \theta_1 < 2\pi$, $\Omega = \{z : |z| > s, \ \theta_1 < \arg z < \theta_2\}$, *and* f *an analytic function from a domain* D *of* $\mathbb{C}$ *into a Banach algebra* A *such that* $\sigma(f(\lambda)) \subset \Omega \cup B(0,r)$ *and* $\sigma(f(\lambda)) \cap \Omega \neq \emptyset$ *for all* $\lambda \in D$. *Define*

$$u(\lambda) = \max\{\arg z : z \in \sigma(f(\lambda)) \cap \Omega\},$$

$$v(\lambda) = \min\{\arg z : z \in \sigma(f(\lambda)) \cap \Omega\}.$$

Then u *and* $-v$ *are subharmonic on* D.

Proof:

First consider figure 3.1 below, illustrating Ω, $B(0,r)$, and their union, such that $\sigma(f(\lambda)) \subset \Omega \cup B(0,r)$:

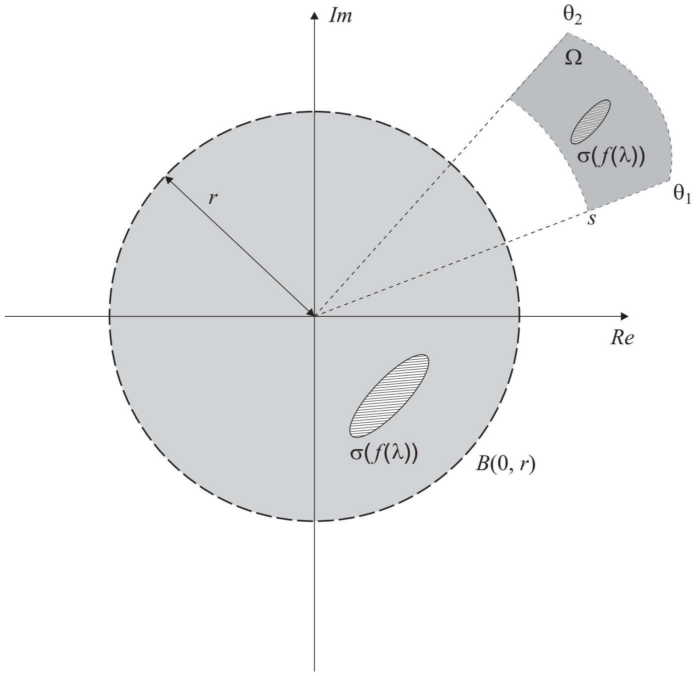

Figure 3.1: The spectrum of $\sigma(f(\lambda))$ contained in the union of $\Omega \cup B(0,r)$.

On Ω, consider an analytic branch of the logarithm $\log z = \log|z| + i \arg z$, which exists since Ω doesn't separate 0 from ∞. Define

$$h(z) = -i \log z, \quad \text{for } z \in \Omega, \tag{3.10.1}$$

and

$$h(z) = \alpha, \quad \text{for } z \in B(0,r), \tag{3.10.2}$$

where $\alpha < \theta_1$ is a fixed real number.
Since Ω and $B(0,r)$ are disjoint open sets, it follows that h is holomorphic on $\Omega \cup B(0,r)$. Thus, by the Spectral Mapping Theorem applied to $h(f(\lambda)) \in A$,

$$\sigma\big(h(f(\lambda))\big) = h\big(\sigma(f(\lambda))\big) = \{h(z) : z \in \sigma(f(\lambda))\}.$$

Consequently, since $\sigma(f(\lambda)) \subset \Omega \cup B(0,r)$, and by (3.10.1) and (3.10.2),

$$\sigma\big(h(f(\lambda))\big) = \{-i\log z : z \in \sigma(f(\lambda)) \cap \Omega\} \cup \{\alpha\}, \text{ for } \sigma(f(\lambda)) \cap B(0,r) \neq \emptyset,$$

and

$$\sigma\big(h(f(\lambda))\big) = \{-i\log z : z \in \sigma(f(\lambda)) \cap \Omega\}, \text{ for } \sigma(f(\lambda)) \cap B(0,r) = \emptyset.$$

Since $-i\log z = -i\log|z| + \arg z$, we then have that

$$\sigma\big(h(f(\lambda))\big) \subset \{-i\log|z| + \arg z : z \in \sigma(f(\lambda)) \cap \Omega\} \cup \{\alpha\}.$$

Let $u \in \sigma\big(h(f(\lambda))\big)$. If $u \in \{-i\log|z| + \arg z : z \in \sigma(f(\lambda)) \cap \Omega\}$, then Re $u = \arg z'$ for some $z' \in \sigma(f(\lambda)) \cap \Omega$. If $u \notin \{-i\log|z| + \arg z : z \in \sigma(f(\lambda)) \cap \Omega\}$, then $u = \alpha$. Consequently, defining

$$u(\lambda) = \max\{\text{Re } u : u \in \sigma\big(h(f(\lambda))\big)\},$$

as in Corollary 3.9, it then follows that

$$u(\lambda) = \max\{\arg z : z \in \sigma(f(\lambda)) \cap \Omega\}.$$

Thus, applying Corollary 3.9 to $h \circ f$, where $h(f(\lambda)) \in A$, it follows from

$$u(\lambda) = \max\{\arg z : z \in \sigma(f(\lambda)) \cap \Omega\} = \max\{\text{Re } u : u \in \sigma\big(h(f(\lambda))\big)\},$$

that u is subharmonic on D.
The proof of $-v$ is similar, where $-v = \max\{\text{Re } v : v \in \sigma(-h(f(\lambda)))\}$.
Q.E.D.

If x is in a Banach algebra, then the **peripherical spectrum** of x, denoted by $\sigma_p(x)$, is

$$\sigma_p(x) = \{\xi \in \sigma(x) : |\xi| = \rho(x)\}.$$

For example if $\sigma(x) = \{\xi \in \mathbb{C} : |\xi| \leq 1\}$, then $\sigma_p(x) = \{\xi \in \sigma(x) : |\xi| = 1\}$.

Theorem 3.11
([AUP1]; Theorem 3.4.11, p. 54)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra. Suppose that there exists $\lambda_0 \in D$ such that

$$\rho(f(\lambda)) \leq \rho(f(\lambda_0)) \ \ \textit{for all } \lambda \in D.$$

Then the peripherical spectrum of $f(\lambda)$ is constant on D.

Proof:

Consider the mapping $\lambda \mapsto \rho(f(\lambda))$ which is subharmonic on D by Theorem 3.7. Moreover, by our hypothesis, it follows by the Maximum Principle of Subharmonic Functions (Theorem 2.2.3), that there exists a constant c such that $\rho(f(\lambda)) = c$ for each $\lambda \in D$. Thus, we need to show that the set

$$\sigma_p(f(\lambda)) = \{\xi_\lambda \in \sigma(f(\lambda)) : |\xi_\lambda| = c\}$$

is constant on D.

For $c = 0$ the result is obvious. So suppose that $c > 0$, but $\sigma_p(f(\lambda))$ is not constant on D. That is, there exists $\lambda_1, \lambda_2 \in D$ and $z \in \mathbb{C}$ such that

$$z \in \sigma(f(\lambda_1)) \ \text{ and } \ |z| = c,$$

but

$$z \notin \sigma(f(\lambda_2)).$$

To obtain a contradiction we consider the analytic function $g(\lambda) = f(\lambda) + az\mathbf{1}$, with $a > 0$ fixed, on D. Further, we also consider the two disks $\overline{B}(az, c)$ and $\overline{B}(0, (a+1)c)$ in $\mathbb{C}$. Since g is analytic, it follows by Theorem 3.7 that

$$\lambda \mapsto \rho(g(\lambda)) = \rho(f(\lambda) + az\mathbf{1})$$

is subharmonic on D. We now show that $g(\lambda) \in A$ satisfies

$$\sigma(g(\lambda)) \subset \overline{B}(az, c) \subset \overline{B}(0, (a+1)c), \tag{3.11.1}$$

for each $\lambda \in D$.

To prove (3.11.1), consider the first inclusion let $\alpha \in \sigma(g(\lambda))$, that is, $\alpha \in \sigma((f(\lambda)) + az\mathbf{1})$. Then by the Spectral Mapping Theorem, $\alpha \in \sigma(f(\lambda)) + az$, that is, $\alpha - az \in \sigma(f(\lambda))$. Since $\rho(f(\lambda)) = c$ for all $\lambda \in D$, we have $|\alpha - az| \leq c$. But, by definition $\overline{B}(az, c) = \{\gamma \in \mathbb{C} : |\gamma - az| \leq c\}$, and hence $\alpha \in \overline{B}(az, c)$, that is, $\sigma(g(\lambda)) \subset \overline{B}(az, c)$. To prove the second inclusion let $\alpha \in \overline{B}(az, c)$. Then $|\alpha - az| \leq c$, from which it follows that $|\alpha| - |az| \leq c$, and $|\alpha| \leq c + a|z|$ since $a > 0$. Consequently, since $|z| = c$, $|\alpha - 0| \leq (1 + a)c$. Hence, by definition, $\alpha \in \overline{B}(0, (a + 1)c)$ and (3.11.1) holds true.

Further observe that the disks $\overline{B}(az, c)$ and $\overline{B}(0, (a+1)c)$ are tangent to each other at the point $(a+1)z$. That is, $(a+1)z$ is a boundary point of $\overline{B}(az, c)$ and $\overline{B}(0, (a+1)c)$. This holds true, since $|(a + 1)z - az| = |z| = c$ implies $(a + 1)z \in \partial\overline{B}(az, c)$, and $|(a + 1)z - 0| = (a + 1)|z| = (a + 1)c$ implies $(a + 1)z \in \partial\overline{B}(0, (a + 1)c)$.
This is illustrated in the following figure 3.2:

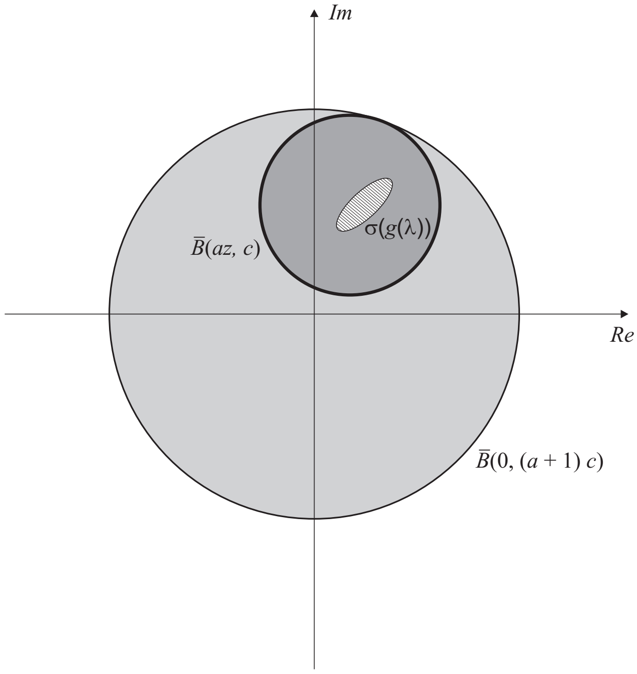

Figure 3.2: The tangent disks $\overline{B}(az, c)$ and $\overline{B}(0, (a + 1)c)$.

Consequently, by the latter and (3.11.1) we have

$$\rho(g(\lambda_2)) < (a + 1)c. \tag{3.11.2}$$

If this is not the case, then by (3.11.1), in particular for λ_2, it must hold that

$$\rho(g(\lambda_2)) = (a + 1)c,$$

since $\sigma(g(\lambda_2)) \subset \overline{B}(az, c) \subset \overline{B}(0, (a+1)c)$. Consequently, since the disks are tangent at $(a + 1)z$,

$$(a + 1)z \in \sigma(g(\lambda_2)),$$

that is,
$$(a+1)z \in \sigma((f(\lambda_2)) + az\mathbf{1}).$$
Thus, by the Spectral Mapping Theorem $(a+1)z \in \sigma(f(\lambda_2))+az\mathbf{1}$, and $z \in \sigma(f(\lambda_2))$, which contradicts our assumption. Hence (3.11.2) holds true.

Since $|z| = c$, we have that $\rho(g(\lambda_1)) = \rho(f(\lambda_1)) + az\mathbf{1}) = (a+1)c$. Hence by (3.11.1), it follows for $\lambda_1, \lambda \in D$, that
$$\rho(g(\lambda)) \leq (a+1)c = \rho(g(\lambda_1)). \tag{3.11.3}$$
Consequently, from (3.11.3) and by the Maximum Principle of Subharmonic Functions (Theorem 2.2.3), it follows that
$$\rho(g(\lambda)) = (a+1)c$$
for all $\lambda \in D$. But this contradicts (3.11.2) for λ_2, and the result follows.
Q.E.D.

Corollary 3.12
([AUP1]; Corollary 3.4.12, p. 54)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Suppose that
$$\sigma(f(\lambda)) \subset \mathbb{R} \quad \textit{for all } \lambda \in D.$$
Then $\sigma(f(\lambda))$ is constant on D.

Proof:

We first show that the spectrum is constant in a neighborhood of $\lambda_0 \in D$. Note that we may replace $f(\lambda)$ by $\alpha f(\lambda) + \beta\mathbf{1}$ for some suitable $\alpha, \beta > 0$ to ensure that
$$\sigma(f(\lambda_0)) \subset (0, 2\pi).$$
Since the spectrum is upper semi-continuous and $\sigma(f(\lambda)) \subset \mathbb{R}$ for all $\lambda \in D$, there exists $\delta > 0$ such that $|\lambda - \lambda_0| < \delta$ implies $\sigma(f(\lambda)) \subset (0, 2\pi)$. By the Holomorphic Functional Calculus we define the Banach algebra elements
$$g(\lambda) = e^{if(\lambda)},$$
for $|\lambda - \lambda_0| < \delta$. It then follows by the Spectral Mapping Theorem that
$$\sigma(g(\lambda)) = e^{i\sigma(f(\lambda))} \subset e^{i(0,2\pi)}.$$

Thus, by the latter $\sigma(g(\lambda))$ is contained in the unit circle and consequently

$$\rho(g(\lambda)) = \rho(g(\lambda_0)) \text{ for all } \lambda \text{ satisfying } |\lambda - \lambda_0| < \delta.$$

Hence, by Theorem 3.11, $\sigma_p(g(\lambda)) = \sigma(g(\lambda))$ is constant for $|\lambda - \lambda_0| < \delta$.

Using the fact that $\sigma(g(\lambda))$ is constant in a neighborhood of λ_0, we now show that the same holds true for $\sigma(f(\lambda))$. Let us say this is not the case. Then there exists λ_1 and $\lambda_2 \in B(\lambda_0, \delta)$ such that

$$\sigma(f(\lambda_1)) \neq \sigma(f(\lambda_2)).$$

So, we may suppose that there exists $\alpha \in \mathbb{C}$ such that

$$\alpha \in \sigma(f(\lambda_1)) \text{ but } \alpha \notin \sigma(f(\lambda_2)),$$

for $|\lambda_1 - \lambda_0| < \delta$ and $|\lambda_2 - \lambda_0| < \delta$.

Since $\sigma(g(\lambda)) = e^{i\,\sigma(f(\lambda))}$ is constant, it must hold true that

$$e^{i\,\alpha} \in \sigma(g(\lambda_1)) = \sigma(g(\lambda_2)),$$

or equivalently

$$e^{i\,\alpha} \in e^{i\,\sigma(f(\lambda_1))} = e^{i\,\sigma(f(\lambda_2))}.$$

Since $\sigma(f(\lambda_1) \subset (0, 2\pi)$, it follows that $\alpha \in (0, 2\pi)$. Also, since the mapping $t : (0, 2\pi) \mapsto e^{it}$ is one-to-one and $\sigma(f(\lambda_2) \subset (0, 2\pi)$, we obtain a contradiction for $\alpha \notin \sigma(f(\lambda_2))$. Hence, $\sigma(f(\lambda))$ is locally constant.

To prove $\sigma(f(\lambda))$ is constant on the whole of D, consider the set

$$E = \{\lambda \in D : \sigma(f(\lambda)) = \sigma(f(\lambda_0))\}.$$

If $\lambda_1 \in E$, then as shown in the previous argument, there exists $\delta_1 > 0$ such that if $|\lambda - \lambda_1| < \delta_1$, then

$$\sigma(f(\lambda)) = \sigma(f(\lambda_1)) = \sigma(f(\lambda_0)).$$

Hence E is the union of all open disks $B(\lambda_1, \delta_1)$, and consequently E is an open subset of D.

On the other hand, consider a sequence $(\lambda_n) \subset E$ such that $\lambda_n \to \lambda' \in D$. Then for n sufficiently large it follows by the first part of the theorem (where we have shown that $\sigma(f(\lambda))$ is locally constant on D) that

$$\sigma(f(\lambda_n)) = \sigma(f(\lambda')).$$

But by definition of E, $\sigma(f(\lambda_n)) = \sigma(f(\lambda_0))$ for each n, and hence $\sigma(f(\lambda')) = \sigma(f(\lambda_0))$ proving that $\lambda' \in E$. Thus E is a closed subset of D as well.
Since D is a domain of $\mathbb{C}$ (open and connected), we must have $D = E$ and consequently $\sigma(f(\lambda))$ is constant for all $\lambda \in D$.
Q.E.D.

The following example illustrates the usefulness of Corollary 3.12:

Let $x, y \in A$ fixed. Suppose $\sigma(x + \lambda y) \subset \mathbb{R}$ for all $\lambda \in \mathbb{C}$. Then y is quasinilpotent.

Consider the analytic function $f(\lambda) = x + \lambda y$ from $\mathbb{C}$ into A. Then, by our assumption and by Corollary 3.12 $\sigma(f(\lambda))$ constant. Thus $\sigma(x + \lambda y) = \sigma(x)$ for all $\lambda \in \mathbb{C}$. Multiplying throughout by $\mu = 1/\lambda$ we obtain

$$\sigma(\mu x + y) = \sigma(\mu x), \text{ for all } \mu = 1/\lambda \neq 0.$$

Hence

$$\rho(\mu x + y) = \rho(\mu x) = |\mu|\rho(x).$$

But, by Theorem 3.7

$$\mu \mapsto \rho(\mu x + y) \text{ is subharmonic on all of } \mathbb{C}.$$

Hence, by Theorem 2.2.2 (g)

$$\rho(y) = \limsup_{\mu \to 0,\ \mu \neq 0} \rho(\mu x + y) = \limsup_{\mu \to 0,\ \mu \neq 0} |\mu|\rho(x) = 0,$$

and y is quasinilpotent.

Theorem 3.13 Spectral Maximum Principle
([AUP1]; Theorem 3.4.13, p. 55)

Let f be analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Suppose that there exists $\lambda_0 \in D$ such that

$$\sigma(f(\lambda)) \subset \sigma(f(\lambda_0))$$

for all $\lambda \in D$. Then

$$\partial\sigma(f(\lambda_0)) \subset \partial\sigma(f(\lambda)) \quad \textit{and} \quad \sigma(f(\lambda_0))^\wedge = \sigma(f(\lambda))^\wedge$$

for all $\lambda \in D$. In particular if $\sigma(f(\lambda_0))$ has no interior points or if $\sigma(f(\lambda))$ does not separate the plane for all $\lambda \in D$, then $\sigma(f(\lambda))$ is constant on D.

Proof:

Assume $\sigma(f(\lambda)) \subset \sigma(f(\lambda_0))$ for all $\lambda \in D$. First we show that $\partial\sigma(f(\lambda_0)) \subset \partial\sigma(f(\lambda))$ for all $\lambda \in D$. If this is not the case, then we can find $\lambda_1 \in D$ such that

$$z_0 \in \partial\sigma(f(\lambda_0)) \;\text{ but }\; z_0 \notin \partial\sigma(f(\lambda_1)).$$

Note that z_0 is not an interior point $\sigma(f(\lambda_1))$, since it will imply by our hypothesis that z_0 is an interior point of $\sigma(f(\lambda_0))$, which is not the case. So, in conclusion $z_0 \notin \sigma(f(\lambda_1))$. Consequently, by the compactness of $\sigma(f(\lambda_1))$ and the normality of $\mathbb{C}$, it follows that we can find $r > 0$ such that

$$\overline{B}(z_0, r) \cap \sigma(f(\lambda_1)) = \emptyset.$$

Since $z_0 \in \partial\sigma(f(\lambda_0))$, it follows by definition that every neighborhood of z_0 contains points not belonging to $\sigma(f(\lambda_0))$. In particular we can find

$$z_1 \notin \sigma(f(\lambda_0)) \;\text{ with }\; |z_1 - z_0| < r/3.$$

Consider figure 3.3:

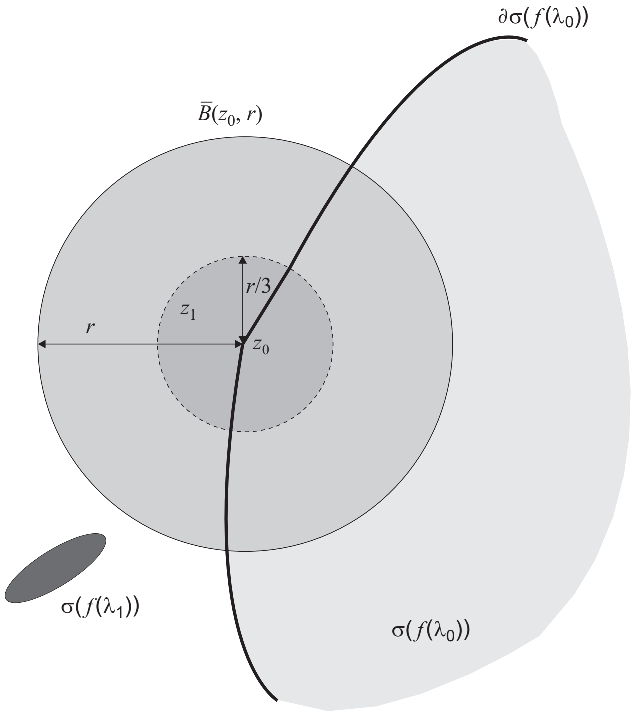

Figure 3.3: The relationship between $\overline{B}(z_0, r)$, $\sigma(f(\lambda_0))$ and $\sigma(f(\lambda_1))$.

From figure 3.3 it is clear that

$$\operatorname{dist}(z_1, \sigma(f(\lambda_0))) < r/3 \quad \text{and} \quad \operatorname{dist}(z_1, \sigma(f(\lambda_1))) > 2r/3. \tag{3.13.1}$$

Further, since $\sigma(f(\lambda)) \subset \sigma(f(\lambda_0))$ for all $\lambda \in D$, it follows that

$$\operatorname{dist}(z_1, \sigma(f(\lambda))) \geq \operatorname{dist}(z_1, \sigma(f(\lambda_0))).$$

Recall from Corollary 3.8 that $\lambda \mapsto 1/\operatorname{dist}(z_1, \sigma(f(\lambda)))$ is subharmonic on D. Since

$$1/\operatorname{dist}(z_1, \sigma(f(\lambda))) \leq 1/\operatorname{dist}(z_1, \sigma(f(\lambda_0)))$$

for all $\lambda \in D$, it then follows by the Maximum Principle for Subharmonic Functions (Theorem 2.2.3) that

$$\operatorname{dist}(z_1, \sigma(f(\lambda))) = \operatorname{dist}(z_1, \sigma(f(\lambda_0))).$$

But this is a contradiction with (3.13.1) for $\lambda = \lambda_1 \in D$. Thus

$$\partial\sigma(f(\lambda_0)) \subset \partial\sigma(f(\lambda)) \quad \text{for all } \lambda \in D.$$

For the proof of $\sigma(f(\lambda_0))^\wedge = \sigma(f(\lambda))^\wedge$, denote by $U(\lambda)$ the unbounded component of $\mathbb{C}\backslash\sigma(f(\lambda))$ for each $\lambda \in D$ (notation $\sigma(f(\lambda))^\wedge$ is discussed in Appendix B, after Theorem B.1). Since $\sigma(f(\lambda)) \subset \sigma(f(\lambda_0))$ it follows by De Morgan's Laws that

$$U(\lambda_0) \subset U(\lambda) \quad \text{for each } \lambda \in D.$$

Consequently to prove $\sigma(f(\lambda_0))^\wedge = \sigma(f(\lambda))^\wedge$, we need only to prove $U(\lambda_0) = U(\lambda)$, hence $U(\lambda) \subset U(\lambda_0))$ for all $\lambda \in D$.

Suppose this is not the case. Then we can find a $\lambda_1 \in D$ such that $z \in U(\lambda_1)$, but $z \notin U(\lambda_0)$, that is, $z \in \sigma(f(\lambda_0))^\wedge$. Now consider an arc Γ, connecting z to infinity and such that Γ is strictly contained within $U(\lambda_1)$. By definition, Γ is a continuous function $\phi : [0, \infty) \to \mathbb{C}$, such that $\phi(0) = z$ and $|\phi(t)| \to \infty$ as $t \to \infty$, and moreover, for each $t \in [0, \infty)$ we have

$$\phi(t) \in U(\lambda_1).$$

Further, consider the set:

$$R = \{\phi(t) : t \in [0, \infty)\} \cap \sigma(f(\lambda_0)).$$

We now show that R is not empty.
If $z \in \sigma(f(\lambda_0))$, then since $z = \phi(0)$ we have that $R \neq \emptyset$.
If $z \notin \sigma(f(\lambda_0))$, then z belongs to some hole of $\sigma(f(\lambda_0))$, or equivalently z belongs

to a bounded component of $\mathbb{C}\backslash\sigma(f(\lambda_0))$. Thus, if $R = \emptyset$, then ϕ connects a hole of $\sigma(f(\lambda_0))$ to infinity, which is a contradiction since a hole of $\mathbb{C}\backslash\sigma(f(\lambda))$ is disconnected from the unbounded component of $\mathbb{C}\backslash\sigma(f(\lambda))$. Hence $R \neq \emptyset$.

Now let

$$z_0 = \phi(t'), \quad \text{where } t' = \max\{t \in [0, \infty) : \phi(t) \in \sigma(f(\lambda_0))\}.$$

By the construction of z_0, it follows that $z_0 \in \partial\sigma(f(\lambda_0))$. Consequently by the first part of the theorem

$$z_0 \in \partial\sigma(f(\lambda)) \text{ for all } \lambda \in D.$$

In particular for $\lambda_1 \in D$, we have $z_0 \in \sigma(f(\lambda_1))$. But $z_0 = \phi(t') \in U(\lambda_1)$, hence contradicting the fact that $U(\lambda_1) \cap \sigma(f(\lambda_1)) = \emptyset$. Thus in conclusion, $U(\lambda_0) = U(\lambda)$, that is,

$$\sigma(f(\lambda_0))^\wedge = \sigma(f(\lambda))^\wedge \text{ for all } \lambda \in D.$$

If $\sigma(f(\lambda_0))$ contains no interior points then

$$\sigma(f(\lambda_0)) = \partial\sigma(f(\lambda_0)) \subset \partial\sigma(f(\lambda)) \subset \sigma(f(\lambda)) \subset \sigma(f(\lambda_0)).$$

That is,

$$\sigma(f(\lambda_0)) \subset \sigma(f(\lambda)) \subset \sigma(f(\lambda_0)),$$

and $\sigma(f(\lambda))$ is constant on D.

Also, if $\sigma(f(\lambda))$ does not separate the plane, then

$$\sigma(f(\lambda)) = \sigma(f(\lambda))^\wedge = \sigma(f(\lambda_0))^\wedge,$$

and $\sigma(f(\lambda))$ is constant for $\lambda_0 \in D$ fixed.
Q.E.D.

The following Liouville's Spectral Theorem (Theorem 3.14) is one of the most important and frequently applied theorems in our work – especially in Chapter 5, in the characterization of the McCrimmon radical. Theorem 3.14 also investigates the full spectrum, $\sigma(f(\lambda))^\wedge$, as in Theorem 3.13. It is interesting to note that in Theorem 3.14, we consider f to be analytic on the whole of $\mathbb{C}$, which is almost similar to Liouville's Theorem for entire functions in complex analysis, and Liouville's Theorem for Subharmonic Functions (Theorem 2.2.4). Moreover, the notion of boundedness also plays a role, as is the case with the latter two theorems. Further, Liouville's Theorem for Subharmonic Functions is applied in the proof of Theorem 3.14, in conjunction with the properties of polynomial convex sets, as discussed in Appendix B.

Theorem 3.14 Liouville's Spectral Theorem
([AUP1]; Theorem 3.4.14, p. 56)

Let f be an analytic function from $\mathbb{C}$ into a Banach algebra A. Suppose there exists a bounded set G such that

$$\sigma(f(\lambda)) \subset G \text{ for all } \lambda \in \mathbb{C}.$$

Then

$$\sigma(f(\lambda))^{\wedge} \text{ is constant on } \mathbb{C}.$$

<u>Proof:</u>

Consider the set $E = \overline{\cup\, \sigma(f(\lambda))}$ with $\lambda \in \mathbb{C}$. Clearly, it follows by our hypothesis that E is bounded. Also, since E is closed in $\mathbb{C}$ and $\mathbb{C}$ is finite-dimensional, we have that E is compact in $\mathbb{C}$. To obtain the desired result we prove that $E^{\wedge} = \sigma(f(\lambda))^{\wedge}$, for each $\lambda \in \mathbb{C}$.
The proof consists of two parts. First we show that $\text{dist}(z_0, \sigma(f(\lambda))$ is constant on $\mathbb{C}$ for $z_0 \notin E$. This result is then used in the second part to obtain a contradiction.

Let $z_0 \notin E$ and $\varepsilon > 0$ such that

$$B(z_0, \varepsilon) \cap E = \emptyset.$$

By definition of E it follows that $z_0 \notin \sigma(f(\lambda))$. Consequently, by Corollary 3.8

$$\lambda \mapsto -\log \text{dist}(z_0, \sigma(f(\lambda))$$

is subharmonic on $\mathbb{C}$. Moreover, we have that $-\log \text{dist}(z_0, \sigma(f(\lambda))$ is smaller than $-\log \varepsilon$. To prove the latter consider the following figure 3.4:

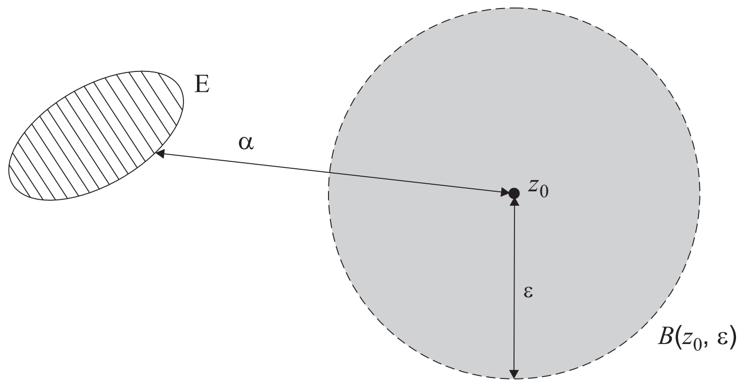

Figure 3.4: The relationship between the set E and the disk $B(z_0, \varepsilon)$.

Setting $\alpha = \text{dist}(z_0, \sigma(f(\lambda))$, and since $B(z_0, \varepsilon) \cap E = \emptyset$, it is clear that $\alpha > \varepsilon$. Hence

$$\log \text{dist}(z_0, \sigma(f(\lambda)) \geq \log \alpha > \log \varepsilon,$$

and

$$-\log \varepsilon > -\log \text{dist}(z_0, \sigma(f(\lambda)).$$

Consequently, by Liouville's Theorem for Subharmonic Functions (Theorem 2.2.4) we have that

$$\text{dist}(z_0, \sigma(f(\lambda)) \text{ is constant on } \mathbb{C}. \tag{3.14.1}$$

By using (3.14.1) we now show that $\partial E \subset \partial\sigma(f(\lambda))$ for all $\lambda \in \mathbb{C}$. Suppose that the latter inclusion is false. Then, there exists $z_1 \in \partial E$ and $\lambda_1 \in \mathbb{C}$ such that $z_1 \notin \partial\sigma(f(\lambda_1))$. Note that $z_1 \notin \sigma(f(\lambda_1))$, since $\sigma(f(\lambda_1)) \subset E$ and $z_1 \in \partial E$. Thus, by the normality of $\mathbb{C}$, it follows that there exists $r > 0$ such that $B(z_1, r) \cap \sigma(f(\lambda_1)) = \emptyset$. Let $\lambda_2 \in \mathbb{C}$ such that

$$B(z_1, r/5) \cap \sigma(f(\lambda_2)) \neq \emptyset. \tag{3.14.2}$$

Since $z_1 \in \partial E$ and E is compact, we can find a $z_2 \in \mathbb{C}$ such that

$$z_2 \notin E \text{ but } z_2 \in B(z_1, r/5). \tag{3.14.3}$$

Consider the following figure 3.5:

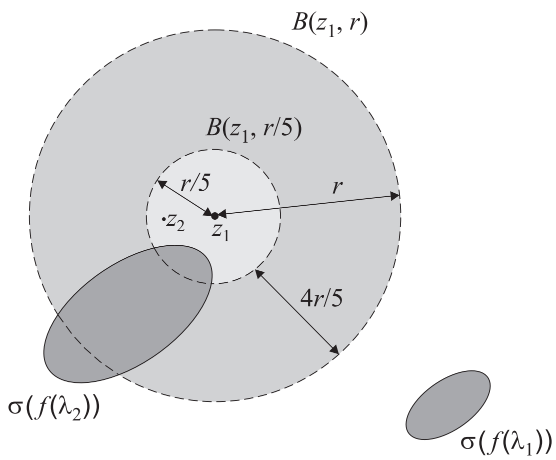

Figure 3.5: The relationship of $B(z_0, \varepsilon)$ with respect to $\sigma(f(\lambda_1))$ and $\sigma(f(\lambda_2))$.

It then follows by figure 3.5, (3.14.2), and (3.14.3) that

$$\text{dist}(z_2, \sigma(f(\lambda_2))) \leq 2r/5,$$

but

$$\text{dist}(z_2, \sigma(f(\lambda_1))) \geq 4r/5.$$

Clearly this is a contradiction if we apply (3.14.1) to $z_0 = z_2$. Thus $\partial E \subset \partial\sigma(f(\lambda))$ and consequently $E^\wedge \subset \sigma(f(\lambda))^\wedge$ (cf. Appendix B and [CON1]; p. 211). But $\sigma(f(\lambda)) \subset E$, so $\sigma(f(\lambda))^\wedge \subset E^\wedge$, and hence $\sigma(f(\lambda))^\wedge = E^\wedge$ for all $\lambda \in \mathbb{C}$.
Q.E.D.

To discuss the proof of Theorem 3.15, we first need to consider the action of the complex mapping $z \mapsto z^{-1}$ on polynomially convex sets in $\mathbb{C}$. Let $z = re^{i\theta}$. Then

$$z^{-1} = \frac{1}{re^{i\theta}} = \frac{1}{r}\, e^{-i\theta}.$$

For $z \neq 0$ arbitrary it is clear that $z \mapsto z^{-1}$ is continuous. Further, z^{-1} is obtained by measuring the distance $1/r$ on the line segment through 0 and z, and then reflecting this point in the real axis. This is shown in figure 3.6 below:

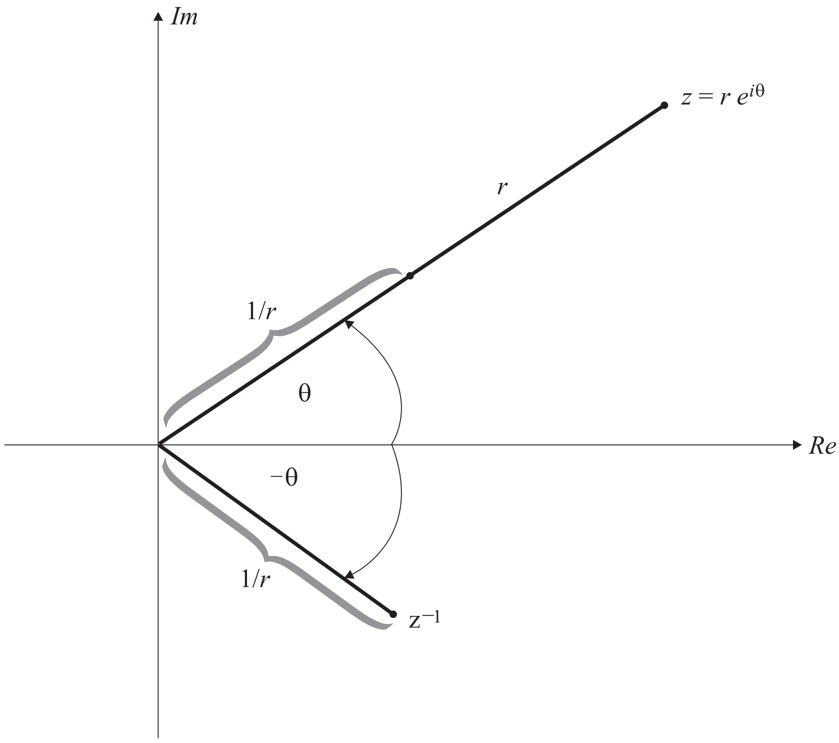

Figure 3.6: The mapping $z \mapsto z^{-1}$.

Consider the following fig 3.7, which will be used to show that $z \mapsto z^{-1}$ maps a polynomially convex set onto a polynomially convex set in $\mathbb{C}$:

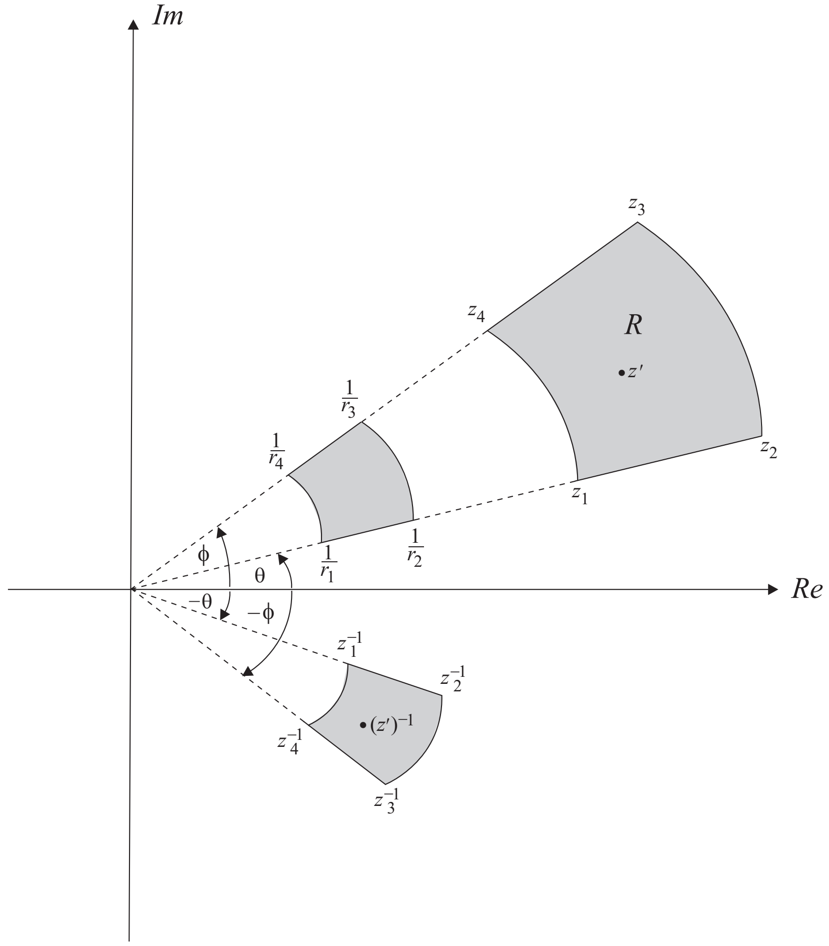

Figure 3.7: The action of $z \mapsto z^{-1}$ on a polynomial convex set in $\mathbb{C}$.

From figure 3.7 it is clear that the set R is polynomially convex. Let z_1, z_2, z_3, and z_4 be the points on the vertices of R, with $z_j = r_j e^{i\theta}$ for $j = 1, 2$ and $z_j = r_j e^{i\phi}$ for $j = 3, 4$. Further, let z' be arbitrary in R, and consider R under $z \mapsto z^{-1}$. Then, as in the previous argument, for each $z' \in R$, $|z'| = r'$ becomes $1/r'$ and z' is reflected in the real axis. Consequently, $z \mapsto z^{-1}$ maps R onto a set, say R', which contains no hole, and hence $z \mapsto z^{-1}$ maps polynomially convex sets onto polynomially convex sets.

Note that for the mapping $z \mapsto (z - z_0)^{-1}$ the same arguments holds true, since this mapping is merely a translation followed by inversion.

In the proof of Theorem 3.15, we use the fact that $z \mapsto (z - z_0)^{-1}$ maps polynomially convex sets onto polynomially convex sets and that this mapping is also injective.

Theorem 3.15
([AUP1]; Theorem 3.4.15, p. 57)

Let f be an analytic function from $\mathbb{C}$ into a Banach algebra A. Then either

$$\sigma(f(\lambda))^\wedge \textit{ is constant on } \mathbb{C},$$

or

$$\bigcup_{\lambda \in \mathbb{C}} \sigma(f(\lambda))^\wedge \textit{ is dense in } \mathbb{C}.$$

<u>Proof:</u>

Suppose $\bigcup_{\lambda \in \mathbb{C}} \sigma(f(\lambda))^\wedge$ is not dense in $\mathbb{C}$. Then we can find $z_0 \in \mathbb{C}$ and $r > 0$ such that

$$B(z_0, r) \cap \sigma(f(\lambda))^\wedge = \emptyset \text{ for all } \lambda \in \mathbb{C}. \tag{3.15.1}$$

Consider the function $u(z) = (z - z_0)^{-1}$ for $z \in \mathbb{C}$. Clearly u is holomorphic on a neighborhood, say Ω, of $\sigma(f(\lambda))^\wedge$. Further, by the argument preceding the theorem, we have that u maps any polynomially convex set of $G = \mathbb{C}\backslash\overline{B}(z_0, r)$ onto a polynomially convex set. Consequently by the Holomorphic Functional Calculus

$$\sigma(u(f(\lambda)))^\wedge = u(\sigma(f(\lambda))^\wedge).$$

Further, it also holds true that

$$\sigma(u(f(\lambda))^\wedge) \subset B(0, 1/r) \text{ for all } \lambda \in \mathbb{C}. \tag{3.15.2}$$

To prove (3.15.2) let $\alpha \in u(\sigma(f(\lambda))^\wedge)$. Then

$$\alpha \in \left\{(z - z_0)^{-1} : \; z \in \sigma(f(\lambda))^\wedge\right\}.$$

This implies $|\alpha| = |(z' - z_0)|^{-1}$ for some $z' \in \sigma(f(\lambda))^\wedge$. But by (3.15.1) we have that $|z - z_0| > r$ for all $z \in \sigma(f(\lambda))^\wedge$. Hence

$$\frac{1}{|z' - z_0|} < \frac{1}{r},$$

from which it follows that $|\alpha| < 1/r$, and hence $\alpha \in B(0, 1/r)$.

Thus, by (3.15.2) and Theorem 3.14, $\sigma(u(f(\lambda)))^\wedge$ is constant. Since u is injective and $\sigma(u(f(\lambda)))^\wedge = u(\sigma(f(\lambda))^\wedge)$, it follows for $\lambda_1, \lambda_2 \in \mathbb{C}$ arbitrary that

$$u(\sigma(f(\lambda_1))^\wedge) = \; u(\sigma(f(\lambda_2))^\wedge)$$

implies

$$u^{-1}u(\sigma(f(\lambda_1))^\wedge) = u^{-1}u(\sigma(f(\lambda_2))^\wedge),$$

and thus

$$\sigma(f(\lambda_1))^\wedge = \sigma(f(\lambda_2))^\wedge.$$

Hence $\sigma(f(\lambda))^\wedge$ is constant on $\mathbb{C}$.
Q.E.D.

Corollary 3.16
([AUP1]; Corollary 3.4.16, p. 57)

Let f be an analytic function from $\mathbb{C}$ into a Banach algebra A. Suppose there exists a constant C such that

$$\max\left\{|\mathrm{Re}\ u - \mathrm{Re}\ v| : u, v \in \sigma(f(\lambda))\right\} \leq C,$$

for all $\lambda \in \mathbb{C}$. Then there exists an entire function h such that

$$\sigma(f(\lambda))^\wedge = h(\lambda) - h(0) + \sigma(f(0))^\wedge.$$

Proof:

Recall from Corollary 3.9 that for $u, v \in \sigma(f(\lambda))$ we defined

$$u(\lambda) = \max\left\{\mathrm{Re}\ u : u \in \sigma(f(\lambda))\right\} \text{ and } v(\lambda) = \min\left\{\mathrm{Re}\ v : v \in \sigma(f(\lambda))\right\},$$

and obtained that $u(\lambda)$ and $-v(\lambda)$ are both subharmonic on $\mathbb{C}$. Further, by Theorem 2.2.2 (a), $u(\lambda) + (-v(\lambda)) = u(\lambda) - v(\lambda)$ is subharmonic on $\mathbb{C}$ where

$$u(\lambda) - v(\lambda) = \max\left\{\mathrm{Re}\ u : u \in \sigma(f(\lambda))\right\} - \min\left\{\mathrm{Re}\ v : v \in \sigma(f(\lambda))\right\},$$

and hence for all $\lambda \in \mathbb{C}$

$$u(\lambda) - v(\lambda) \leq \max\left\{|\mathrm{Re}\ u - \mathrm{Re}\ v| : u, v \in \sigma(f(\lambda))\right\} \leq C.$$

Thus, since $u(\lambda) - v(\lambda)$ is bounded it follows by the Maximum Principle of Subharmonic Functions (Theorem 2.2.3) that $u(\lambda) - v(\lambda)$ is constant, say $u(\lambda) - v(\lambda) = \alpha$, for all $\lambda \in \mathbb{C}$ and some constant α.
Since $r(\lambda) = \alpha$ is a constant function it is obviously upper semi-continuous. Further, we have that

$$\alpha = r(\lambda_0) = \frac{1}{2\pi}\int_0^{2\pi} r(\lambda_0 + \rho e^{it})\ dt = \frac{1}{2\pi}\ \alpha \int_0^{2\pi} dt = \alpha.$$

Thus, $r(\lambda) = \alpha$ satisfies the MVI for any closed disk $\overline{B}(\lambda_0, \rho)$ in $\mathbb{C}$, and it follows that $r(\lambda) = \alpha$ is subharmonic on $\mathbb{C}$. For similar reasons the same holds true for $t(\lambda) = -\alpha$. Consequently, since $-v(\lambda)$ and $-\alpha = v(\lambda) - u(\lambda)$ are subharmonic on $\mathbb{C}$, it follows by Theorem 2.2.2 (a), that $-u(\lambda) = -v(\lambda) + v(\lambda) - u(\lambda)$ is subharmonic on $\mathbb{C}$. Thus, since u and $-u$ are subharmonic on $\mathbb{C}$, we have that u is harmonic on $\mathbb{C}$, and hence there exists an entire function h such that $u(\lambda) = \text{Re } h(\lambda)$ for all $\lambda \in \mathbb{C}$ (cf. §2.1).

Further, since f is analytic and h is holomorphic on $\mathbb{C}$, $g(\lambda) = f(\lambda) - h(\lambda)\mathbf{1}$ is analytic from $\mathbb{C}$ into A where $h(\lambda) \in \mathbb{C}$. Hence

$$\sigma(g(\lambda)) = \sigma(f(\lambda) - h(\lambda)\mathbf{1}) = \sigma(f(\lambda)) - h(\lambda).$$

Now, define $\text{Re } \sigma(g(\lambda)) = \{\text{Re } v : v \in \sigma(g(\lambda))\}$. It then follows that $\text{Re } \sigma(g(\lambda)) = \text{Re } \sigma(f(\lambda)) - \text{Re } h(\lambda)$. But, since $u(\lambda) = \text{Re } h(\lambda)$ it follows that

$$\text{Re } \sigma(g(\lambda)) = \text{Re } \sigma(f(\lambda)) - u(\lambda),$$

for all $\lambda \in \mathbb{C}$. Also, since $u(\lambda) = \max\big\{\text{Re } u : u \in \sigma(f(\lambda))\big\}$,

$$\text{Re } \sigma(g(\lambda)) = \text{Re } \sigma(f(\lambda)) - \max\big\{\text{Re } u : u \in \sigma(f(\lambda))\big\},$$

thus

$$\sigma(g(\lambda)) \subset \{z \in \mathbb{C} : \text{Re } z \leq 0\} = B = B^\wedge.$$

Consequently

$$\sigma(g(\lambda))^\wedge \subset B^\wedge,$$

and hence

$$\bigcup_{\lambda \in \mathbb{C}} \sigma(g(\lambda))^\wedge \subset B^\wedge.$$

Now, by definition of B, it follows that $\cup_{\lambda \in \mathbb{C}} \sigma(g(\lambda))^\wedge$ is only contained in the left half-plane of the complex plane, hence $\cup_{\lambda \in \mathbb{C}} \sigma(g(\lambda))^\wedge$ isn't dense in $\mathbb{C}$. Thus by Theorem 3.15 (applied to g) it follows that $\sigma(g(\lambda))^\wedge$ is constant for all $\lambda \in \mathbb{C}$. Further, since $\sigma(g(\lambda)) = \sigma(f(\lambda)) - h(\lambda)$, we have

$$\sigma(g(\lambda))^\wedge = \big[\sigma(f(\lambda)) - h(\lambda)\big]^\wedge = \sigma(f(\lambda))^\wedge - h(\lambda) \text{ for all } \lambda \in \mathbb{C}.$$

But since $\sigma(g(\lambda))^\wedge$ is constant, it follows that

$$\sigma(g(\lambda))^\wedge = \sigma(g(0))^\wedge, \text{ for all } \lambda \in \mathbb{C}.$$

Thus

$$\sigma(f(\lambda))^\wedge - h(\lambda) = \sigma(f(0))^\wedge - h(0),$$

and hence

$$\sigma(f(\lambda))^\wedge = h(\lambda) - h(0) + \sigma(f(0))^\wedge,$$

for all $\lambda \in \mathbb{C}$.
Q.E.D.

In the proof of Theorem 3.17 we will be using many of the notations of Corollary 3.10 in conjunction with Corollary 3.12 to obtain the desired result.

Theorem 3.17
([AUP1]; Theorem 3.4.17, p. 58)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra. Suppose that $\sigma(f(\lambda)) = \{0, \alpha(\lambda)\}$ for all $\lambda \in D$, where α is a mapping from D into $\mathbb{C}$. Then α is holomorphic on D.

Proof:

Since $\sigma(f(\lambda))$ is finite it follows from Corollary 3.5 that α is continuous on D. Further let D' be the subset of D where $D' = \{\lambda \in D : \alpha(\lambda) \neq 0\}$. Note that if $D' = \emptyset$ we have the trivial case, since $\alpha = 0$ for all $\lambda \in D$ and α is then holomorphic on D. Thus we may assume $D' \neq \emptyset$.

We first show that D' is open by considering its complement $(D')^c = D \backslash D'$. Let (λ_n) be a sequence in $(D')^c$ such that $\lambda_n \to \lambda \in D$. By definition, $\alpha(\lambda_n) = 0$ for all n. Since α is continuous, $\alpha(\lambda_n) \to \alpha(\lambda)$ and it follows by the uniqueness of the limit that $\alpha(\lambda) = 0$. Hence $(D')^c$ is closed and D' is open.

Note, that by Radó's Extension Theorem (cf. Appendix C) it is enough to prove that α is locally holomorphic on D', so we only need to consider an arbitrary neighborhood of $\lambda_0 \in D'$.

Consider figure 3.8 as obtained from Corollary 3.10.
By definition $\Omega = \{z : |z| > s,\ \theta_1 < \arg z < \theta_2\}$, $0 \leq r \leq s$, and $0 < \theta_2 - \theta_1 < 2\pi$ as in Corollary 3.10.
Since α is continuous, it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|\lambda - \lambda_0| < \delta$ implies $|\alpha(\lambda) - \alpha(\lambda_0)| < \varepsilon$. So, $\alpha(\lambda)$ is in an ε-neighborhood of $\alpha(\lambda_0)$. Consequently, since $\sigma(f(\lambda)) = \{0, \alpha(\lambda)\}$, it follows that $\sigma(f(\lambda)) \cap \Omega = \{\alpha(\lambda)\}$ for $|\lambda - \lambda_0| < \delta$. This is illustrated in figure 3.8.

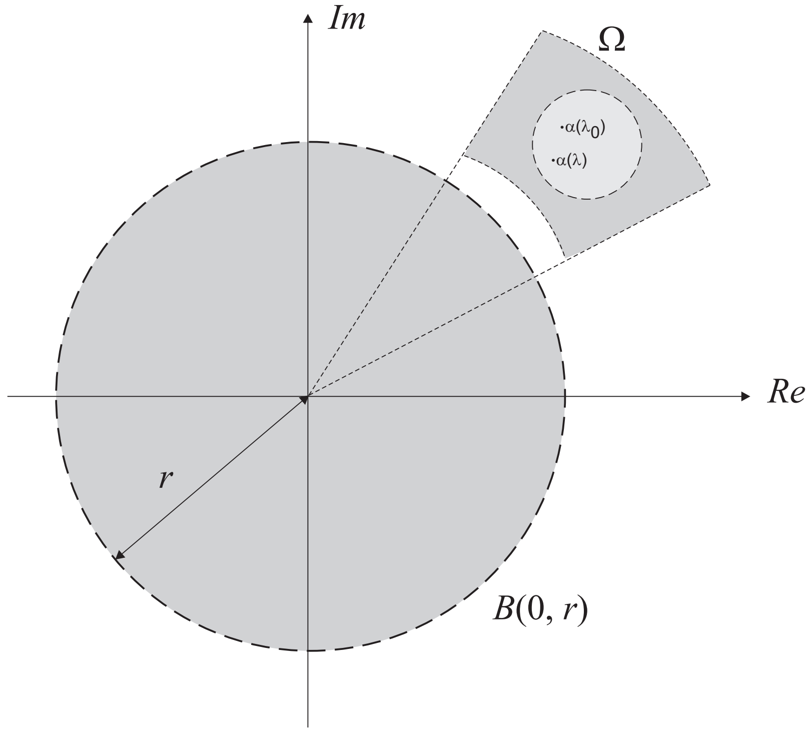

Figure 3.8: $\alpha(\lambda)$ in a neighborhood of $\alpha(\lambda_0)$ contained in Ω.

Further, as in Corollary 3.10, define

$$u(\lambda) = \max\{\arg z : z \in \sigma(f(\lambda)) \cap \Omega\ \},$$

and

$$v(\lambda) = \min\{\arg z : z \in \sigma(f(\lambda)) \cap \Omega\},$$

where by the same corollary u and $-v$ are subharmonic for $|\lambda - \lambda_0| < \delta$.

Since $\sigma(f(\lambda)) \cap \Omega = \{\alpha(\lambda)\}$, we have $u(\lambda) = v(\lambda)$, and hence

$$u(\lambda) = \arg\,\alpha(\lambda). \tag{3.17.1}$$

Further, $u = v$ implies $-u = -v$, thus $-u$ is also subharmonic. So for u and $-u$ both subharmonic, it follows that u is harmonic on $B(\lambda_0, \delta)$ (cf. §2.1). Thus, by the latter (similar to Corollary 3.16 and also by §2.2), there exists a holomorphic function k on $B(\lambda_0, \delta)$ such that $u(\lambda) = \operatorname{Im}\, k(\lambda)$. Consequently, by (3.17.1)

$$\operatorname{Im}\, k(\lambda) = \arg\,\alpha(\lambda). \tag{3.17.2}$$

Now, consider the analytic function $g : B(\lambda_0, \delta) \mapsto A$ with $g(\lambda) = e^{-k(\lambda)} f(\lambda)$, $e^{-k(\lambda)} \in \mathbb{C}$, and $f(\lambda) \in A$. Then, by the Spectral Mapping Theorem, it follows from

$$\sigma(g(\lambda)) = \sigma(e^{-k(\lambda)} f(\lambda)),$$

that

$$\sigma(g(\lambda)) = \left\{ e^{-k(\lambda)}\alpha(\lambda) : \alpha(\lambda) \in \sigma(f(\lambda)),\ \lambda \in B(\lambda_0, \delta) \right\} \cup \{0\}, \qquad (3.17.3)$$

since $\sigma(f(\lambda)) = \{0, \alpha(\lambda)\}$. Further, for $\alpha(\lambda) = e^{\log|\alpha(\lambda)|}e^{i\arg\alpha(\lambda)}$ in polar form, we have

$$e^{-k(\lambda)}\alpha(\lambda) = e^{-\operatorname{Re}k(\lambda)}e^{-i\operatorname{Im}k(\lambda)}e^{\log|\alpha(\lambda)|}e^{i\arg\ \alpha(\lambda)}.$$

By substitution of (3.17.2), it follows that

$$e^{-k(\lambda)}\alpha(\lambda) = e^{-\operatorname{Re}k(\lambda)}e^{\log|\alpha(\lambda)|}. \qquad (3.17.4)$$

The latter equation implies $e^{-k(\lambda)}\alpha(\lambda) \in \mathbb{R}$ is for all $\lambda \in B(\lambda_0, \delta)$. Hence, taking into account (3.17.3) and (3.17.4), we have that $\sigma(g(\lambda)) \subset \mathbb{R}$. This implies by Corollary 3.12 that $\sigma(g(\lambda))$ is a constant set, say $C = \{0, \beta\}$. Thus $\sigma(g(\lambda)) = \{0, \beta\} = \sigma(e^{-k(\lambda)}f(\lambda))$. Consequently, by the Spectral Mapping Theorem, $\beta = e^{-k(\lambda)}\alpha(\lambda)$ and

$$\alpha(\lambda) = \beta e^{k(\lambda)},$$

which is holomorphic on $B(\lambda_0, \delta)$.
Q.E.D.

The next corollary is an extension of the latter theorem. We omit the proof since it is similar to Theorem 3.17.

Corollary 3.18
([AUP1]; Corollary 3.4.18, p. 58)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Suppose that $\sigma(f(\lambda)) = \{\alpha(\lambda)\}$ for all $\lambda \in D$, where α is mapping from D into $\mathbb{C}$. Then α holomorphic on D.

In Theorem 3.19 we consider the spectrum of $f(\lambda)$ which is contained in a vertical line segment, say L_λ. Note that as λ varies $\sigma(f(\lambda))$ is contained in a different line segment L_λ.

Theorem 3.19
([AUP1]; Theorem 3.4.19, p. 59)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Suppose that $\sigma(f(\lambda))$ lies on a vertical line segment for each $\lambda \in D$. Then there exist a holomorphic function h on D and a fixed compact subset K of $\mathbb{R}$ such that

$$\sigma(f(\lambda)) = h(\lambda) + iK \textit{ for all } \lambda \in D.$$

Proof:

Assume that for each $\lambda \in D$, $\sigma(f(\lambda))$ is contained in a vertical line segment say, L_λ as shown in the following figure 3.9:

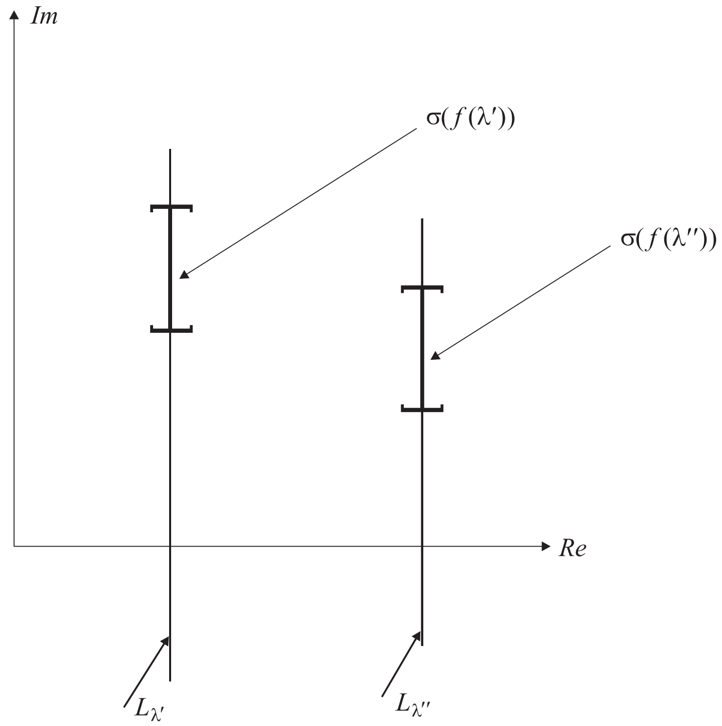

Figure 3.9: $\sigma(f(\lambda')) \subset L_{\lambda'}$ and $\sigma(f(\lambda'')) \subset L_{\lambda''}$, for λ' and $\lambda'' \in D$ arbitrary.

By Corollary 3.9, we have

$$u(\lambda) = \max\{\text{Re } u : u \in \sigma(f(\lambda))\},$$

and

$$v(\lambda) = \min\{\text{Re } v : v \in \sigma(f(\lambda))\},$$

with u and $-v$ subharmonic on D.

Since $\sigma(f(\lambda)) \subset L_\lambda$, it follows that $u(\lambda) = v(\lambda)$, for all $\lambda \in D$. Thus Re $u =$ Re u' for all $u, u' \in \sigma(f(\lambda))$ arbitrary. Moreover, from $-u(\lambda) = -v(\lambda)$ it follows that both u and $-u$ are subharmonic on D, and hence u is harmonic on D.

Now, denote by $h(\lambda)$ the element of $\sigma(f(\lambda))$ such that $h(\lambda)$ has the the smallest imaginary part with respect to all $z \in \sigma(f(\lambda))$, as shown figure 3.10:

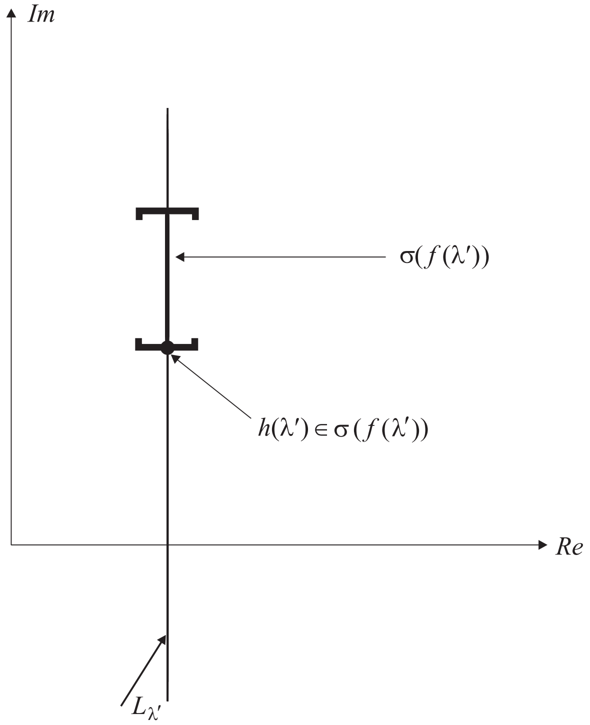

Figure 3.10: $\sigma(f(\lambda'))$ containing the element $h(\lambda')$ with smallest imaginary part.

Fix $\lambda_0 \in D$ and let $\delta > 0$ such that $\overline{B}(\lambda_0, \delta) \subset D$. We first show that h is holomorphic on $B(\lambda_0, \delta)$. Since u is harmonic on D there exists a holomorphic function k on $B(\lambda_0, \delta)$ such that $u(\lambda) = \text{Re } k(\lambda)$, and consequently by the definition of $u(\lambda)$

$$\text{Re } k(\lambda) = \text{Re } u = K,$$

for all $u \in \sigma(f(\lambda))$, where K is a constant.
Consider the analytic function $g_1(\lambda) = -i(f(\lambda) - k(\lambda)\mathbf{1})$ from $B(\lambda_0, \delta)$ into A. By the Spectral Mapping Theorem applied to g_1

$$\sigma(g_1(\lambda)) = -i\sigma(f(\lambda) - k(\lambda)\mathbf{1}) = -i\sigma(f(\lambda)) + ik(\lambda), \tag{3.19.1}$$

or equivalently

$$\sigma(g_1(\lambda)) = -i\big\{\text{Re } u + i \text{ Im } u : u \in \sigma(f(\lambda))\big\} + i\big(\text{Re } k(\lambda) + i \text{ Im } k(\lambda)\big).$$

Since Re $u = \text{Re } k(\lambda)$, we then have that

$$\sigma(g_1(\lambda)) = \big\{\text{Im } u - \text{Im } k(\lambda) : u \in \sigma(f(\lambda))\big\}.$$

But $\text{Im } u - \text{Im } k(\lambda) \in \mathbb{R}$, thus $\sigma(g_1(\lambda)) \subset \mathbb{R}$ and it follows by Corollary 3.12 that $\sigma(g_1(\lambda))$ is a constant set on $B(\lambda_0, \delta)$, say $\sigma(g_1(\lambda)) = C$.

Thus, by (3.19.1)

$$\sigma(g_1(\lambda)) = C = -i\big(\sigma(f(\lambda)) - k(\lambda)\big),$$

and the set $\sigma(f(\lambda)) - k(\lambda)$ is constant for all $\lambda \in B(\lambda_0, \delta)$.

Moreover, since Re $k(\lambda)$ = Re $u = K$ (K a constant) for $u \in \sigma(f(\lambda))$ and $\lambda \in B(\lambda_0, \delta)$, it follows that $\sigma(f(\lambda)) - k(\lambda)$ is a fixed set lying on the imaginary axis, as shown in the figure 3.11:

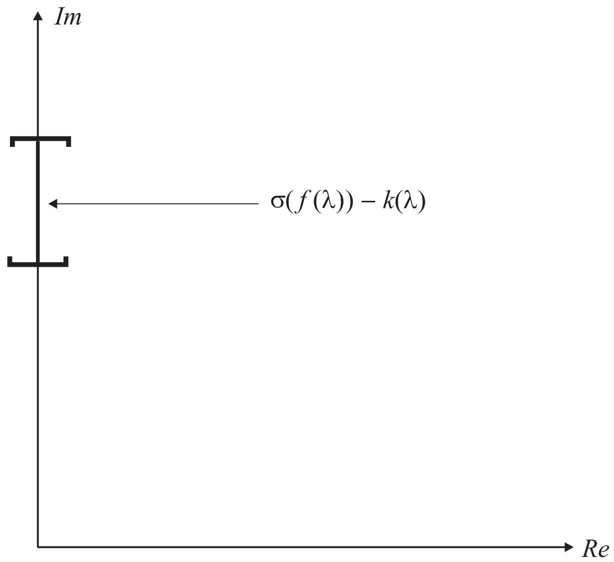

Figure 3.11: $\sigma(f(\lambda)) - k(\lambda)$, a constant set for all $\lambda \in B(\lambda_0, \delta)$.

Consequently, we have that for each $\lambda' \in B(\lambda_0, \delta)$, the spectrum of $f(\lambda')$ is mapped from L'_λ onto the fixed set $\sigma(f(\lambda)) - k(\lambda)$. Thus, $\sigma(f(\lambda)) - k(\lambda)$ can be seen as a translation of $\sigma(f(\lambda'))$ for each $\lambda' \in B(\lambda_0, \delta)$. This is illustrated in figure 3.12 for $\lambda_1 \in B(\lambda_0, \delta)$.

Since $h(\lambda)$ is the element of $\sigma(f(\lambda))$ with smallest imaginary part, it then follows (as $\lambda \in B(\lambda_0, \delta)$ varies) that $h(\lambda)$ is always mapped onto the same fixed element $z_0 \in \sigma(f(\lambda)) - k(\lambda)$, where z_0 has the smallest imaginary part of all $z \in \sigma(f(\lambda)) - k(\lambda)$.

Thus $h(\lambda) - k(\lambda) = z_0$, with Re $z_0 = 0$, and hence

$$h(\lambda) = k(\lambda) + z_0,$$

for all $\lambda \in B(\lambda_0, \delta)$.

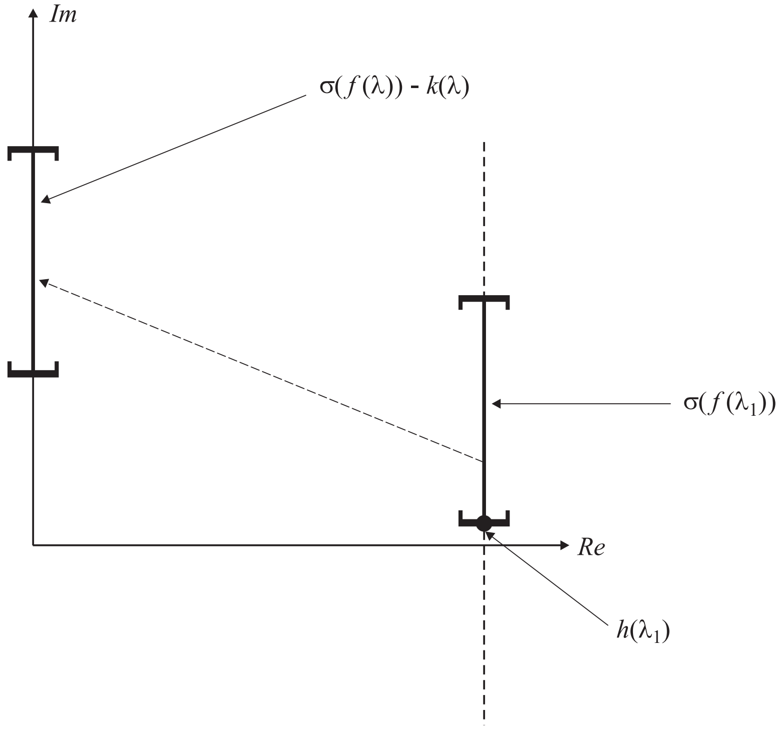

Figure 3.12: The mapping of $\sigma(f(\lambda_1))$ onto the constant set $\sigma(f(\lambda)) - k(\lambda)$.

Since $k \in H(B(\lambda_0, \delta))$, we have that the same holds true for h, and consequently h is holomorphic on the whole of D.

Now consider the analytic function $g_2(\lambda) = -i(f(\lambda) - h(\lambda)\mathbf{1})$ on D. Applying the same argument as in the previous part and Corollary 3.12 to g_2 (taking into account that Re $u =$ Re $h(\lambda)$ for this case), we obtain

$$\sigma(g_2(\lambda)) = \left\{\text{Im } u - \text{Im } h(\lambda) : u \in \sigma(f(\lambda)),\ h \in H(D)\right\}.$$

As before, we now have $\sigma(g_2(\lambda)) \subset \mathbb{R}$. Thus, $\sigma(g_2(\lambda)) = K$, where K is a constant compact set contained in $\mathbb{R}$.
Further, from $g_2(\lambda) = -i(f(\lambda) - h(\lambda)\mathbf{1})$ we have $f(\lambda) = ig_2(\lambda) + h(\lambda)\mathbf{1}$, and consequently by the Spectral Mapping Theorem

$$\sigma(f(\lambda)) = \sigma(ig_2(\lambda) + h(\lambda)\mathbf{1}) = i\sigma(g_2(\lambda)) + h(\lambda).$$

Hence, from the latter,

$$\sigma(f(\lambda)) = iK + h(\lambda), \quad \text{for all } \lambda \in D,$$

with $K = \sigma(g_2(\lambda))$ a fixed compact set in $\mathbb{R}$, and $h \in H(D)$.
Q.E.D.

Theorem 3.20 Holomorphic Variation of Isolated Spectral Values
([AUP1]; Theorem 3.4.20, p. 59)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Suppose there exists $\lambda_0 \in D$, $\alpha_0 \in \sigma(f(\lambda_0))$, and $r, \delta > 0$ such that $|\lambda - \lambda_0| < \delta$ implies that $\lambda \in D$ and $\sigma(f(\lambda)) \cap B(\alpha_0, r)$ contains only one point $\alpha(\lambda)$. Then α is holomorphic on a neighborhood of λ_0.

Proof:

First suppose that $\sigma(f(\lambda_0)) = \{\alpha_0\}$. Then $\sigma(f(\lambda_0))$ is contained in the open ball $B(\alpha_0, r)$. It then follows that $|\lambda - \lambda_0| \to 0$ implies $||f(\lambda) - f(\lambda_0)|| \to 0$ (since f is analytic) and consequently by the upper semi-continuity of the spectrum (Theorem 3.3) we have for λ in a neighborhood of λ_0 that $\sigma(f(\lambda)) \subset B(\alpha_0, r)$. Moreover, $\sigma(f(\lambda)) = \{\alpha(\lambda)\}$. Say this is not true, that is $\#\sigma(f(\lambda)) > 1$. Then, it would imply that at least two points of $\sigma(f(\lambda))$ is contained in $B(\alpha_0, r)$, which is contradictive with our hypothesis, since $\#(\sigma(f(\lambda)) \cap B(\alpha_0, r)) = 1$. Hence $\sigma(f(\lambda)) = \{\alpha(\lambda)\}$, and it follows by Corollary 3.18 that α is holomorphic for $|\lambda - \lambda_0| < \delta$.

Suppose now that $\sigma(f(\lambda_0))$ is larger than $\{\alpha_0\}$, that is, $\{\alpha_0\}$ is a proper subset of $\sigma(f(\lambda_0))$ and $\#\sigma(f(\lambda_0)) > 1$. Then, for similar reasons (as in the previous argument) we may suppose that for $|\lambda_0 - \lambda| < \delta$,

$$\sigma(f(\lambda)) \cap \partial B(\alpha_0, r) = \emptyset.$$

Further, it is clear that

$$\sigma(f(\lambda_0)) \cap \{z : |z - \alpha_0| > r\} \neq \emptyset.$$

Consequently, $\sigma(f(\lambda_0))$ is contained in the union of the two disjoint open sets; $B(\alpha_0, r)$ and $(\overline{B}(\alpha_0, r))^c$. Hence, since $\sigma(f(\lambda)) \cap B(\alpha_0, r) = \{\alpha(\lambda)\}$, it follows by Theorem 3.4 for $|\lambda - \lambda_0| < \delta$, that $\sigma(f(\lambda))$ is included in the union of $\alpha(\lambda) \in B(\alpha_0, r)$ and a non-empty set included in $(\overline{B}(\alpha_0, r))^c$.

Now, let h be the holomorphic function defined by

$$h(z) = z, \quad \text{for } z \in B(\alpha_0, r),$$

and

$$h(z) = 0, \quad \text{for } z \in (\overline{B}(\alpha_0, r))^c.$$

Then, by the Holomorphic Functional Calculus, we can define the Banach algebra element $h(f(\lambda))$ for $|\lambda - \lambda_0| < \delta$, where

$$h(f(\lambda)) = \frac{1}{2\pi i}\int_\Gamma h(z)(z\mathbf{1} - f(\lambda))^{-1}\,dz,$$

with Γ is a smooth closed contour surrounding $\sigma(f(\lambda))$. Further, by the Spectral Mapping Theorem $\sigma(h(f(\lambda))) = h(\sigma(f(\lambda)))$. Consequently, since h maps the point $\alpha(\lambda) \in B(\alpha_0, r) \cap \sigma(f(\lambda))$ onto itself, and all spectral points of $\sigma(f(\lambda))$ contained in $(\overline{B}(\alpha_0, r))^c$ onto 0, we have

$$\sigma(h(f(\lambda))) = \{0, \alpha(\lambda)\}.$$

Hence, by applying Theorem 3.17 to h, it follows that α is holomorphic on a neighborhood of λ_0.
Q.E.D.

The following lemma is a result of complex function theory. It is used in the proof of Theorem 3.22.

Lemma 3.21
([AUP1]; Lemma 3.4.21, p. 60)

Let $\phi_1, \ldots, \phi_n$ be upper semi-continuous functions on an open subset D of $\mathbb{C}$. Let $E \subset D$ and $\lambda_0 \in D \cap \overline{E}$. If

$$\phi_1(\lambda_0) + \ldots + \phi_n(\lambda_0) = \limsup\{\phi_1(\lambda) + \ldots + \phi_n(\lambda) : \lambda \to \lambda_0,\ \lambda \neq \lambda_0,\ \lambda \in E\},$$

then there exists a sequence (λ_k) converging to λ_0 such that $\lambda_k \neq \lambda_0$, $(\lambda_k) \subset E$ and $\phi_i(\lambda_0) = \lim_{k\to\infty} \phi_i(\lambda_k)$ for $i = 1, \ldots, n$.

In the following Theorem 3.22 we use the definition of non-thin sets as discussed in §2.2, after Theorem 2.2.2.

Theorem 3.22 Weak Lower Semi-continuity of the Boundary of the Spectrum
([AUP1]; Theorem 3.4.22, p. 60)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Suppose that $E \subset D\backslash\{\lambda_0\}$ is non-thin at $\lambda_0 \in D \cap \overline{E}$. Then there exists a sequence (μ_k) converging to λ_0, such that $\mu_k \in E$ and

$$\partial\sigma(f(\lambda_0)) \subset \partial\sigma(f(\mu_k)) + B(0, 1/k),\ \textit{for } k \geq 1.$$

Proof:

Let $\varepsilon > 0$ and $B(\xi_1, \varepsilon/2), \ldots, B(\xi_n, \varepsilon/2)$ be a finite covering of $\partial\sigma(f(\lambda_0))$ with $\xi_1, \ldots, \xi_n \in \partial\sigma(f(\lambda_0))$. Note that such a covering exists since the boundary of the spectrum is compact. Also, choose $\eta_1, \ldots, \eta_n \notin \sigma(f(\lambda_0))$ such that $|\xi_i - \eta_i| < \varepsilon/8$ for $i = 1, \ldots, n$. So, we have that $\eta_i \in B(\xi_i, \varepsilon/2)$ for each $i = 1, \ldots, n$. Consequently by the normality of $\mathbb{C}$ we can find an open subset U of $\mathbb{C}$ such that $\sigma(f(\lambda_0)) \subset U$, but $\eta_i \notin U$ for each $i = 1, \ldots, n$. Further, let $r > 0$ such that $\overline{B}(\lambda_0, r) \subset D$. Since f is analytic, it follows for $|\lambda - \lambda_0| < r$ and by the upper semi-continuity of the spectrum (Theorem 3.3) that

$$\sigma(f(\lambda)) \subset U \text{ and } \eta_i \notin U.$$

Hence, by the latter, the function

$$u_i(z) = (z - \eta_i)^{-1}, \; z \in \mathbb{C}$$

is holomorphic on U for $|\lambda - \lambda_0| < r$ and $i = 1, \ldots, n$. Thus, by the Holomorphic Functional Calculus $u_i(f(\lambda)) = (f(\lambda) - \mathbf{1}\eta_i)^{-1}$ is well-defined in A, and by Theorem 3.7 the functions

$$\phi_i(\lambda) = \rho(u_i(f(\lambda))) = \rho\big((f(\lambda) - \mathbf{1}\eta_i)^{-1}\big)$$

are subharmonic for $|\lambda - \lambda_0| < r$. Since $\eta_i \notin \sigma(f(\lambda))$, it follows by Theorem 1.3.3 that

$$\phi_i(\lambda) = \rho\big((f(\lambda) - \mathbf{1}\eta_i)^{-1}\big) = 1/\mathrm{dist}\big(\eta_i, \sigma(f(\lambda))\big),$$

where clearly $\phi_i > 0$ for each $i = 1, 2, \ldots, n$ (which is crucial to our proof). Further, by Theorem 2.2.2 (a), $\sum_{i=1}^n \phi_i(\lambda)$ is subharmonic for $|\lambda - \lambda_0| < r$.

Since E is non-thin at $\lambda_0 \in D \cap \overline{E}$, it follows by definition (cf. §2.2) that

$$\sum_{i=1}^n \phi_i(\lambda_0) = \limsup \Big\{ \sum_{i=1}^n \phi_i(\lambda) : \lambda \to \lambda_0, \lambda \neq \lambda_0, \lambda \in E \Big\}.$$

Further, since $\sum_{i=1}^n \phi_i$ subharmonic it is then upper semi-continuous (by definition). Hence, by Lemma 3.21, there exists a sequence $(\lambda_k) \in E$ such that $\lambda_k \to \lambda_0$, $\lambda_k \neq \lambda_0$, and

$$\phi_i(\lambda_0) = \lim_{k\to\infty} \phi_i(\lambda_k),$$

for $i = 1, \ldots, n$.

We now claim the latter implies that there exists $\mu(\varepsilon) \in E$ such that $|\mu(\varepsilon) - \lambda_0| < r$ and

$$\phi_i(\mu(\varepsilon)) \geq \phi_i(\lambda_0)/2, \quad \text{for } i = 1, \ldots, n. \tag{3.22.1}$$

Let us say this is false. That is, for all $\mu(\epsilon) \in E$ with $|\mu(\varepsilon) - \lambda_0| < r$, we can find an $i \in \{1, 2, \ldots, n\}$ such that

$$\phi_i(\mu(\varepsilon)) < \phi_i(\lambda_0)/2.$$

In particular, the latter inequality must also hold true for the sequence (λ_k) obtained from Lemma 3.21. Hence, for each k sufficiently large, we can find an i such that

$$\phi_i(\lambda_k) < \phi_i(\lambda_0)/2.$$

Since there are only finitely many i, we can find a $j \in \{1, \ldots, n\}$ and a subsequence (λ_{k_n}) of (λ_k), such that

$$\phi_j(\lambda_{k_n}) < \phi_j(\lambda_0)/2 \text{ for all } k_n.$$

Consequently

$$\lim_{n\to\infty} \phi_j(\lambda_{k_n}) \le \phi_j(\lambda_0)/2.$$

But, since $\phi_j(\lambda_0) > 0$, and by Lemma 3.21 we had $\lim_{n\to\infty} \phi_j(\lambda_{k_n}) = \phi_j(\lambda_0)$, it follows that

$$\lim_{n\to\infty} \phi_j(\lambda_{k_n}) = \phi_j(\lambda_0) > \phi_j(\lambda_0)/2$$

which is a contradiction. Hence (3.22.1) holds indeed true.

Now consider the expression

$$\operatorname{dist}\Big(\eta_i, \sigma\big(f(\mu(\varepsilon))\big)\Big) = 1/\rho\Big(u_i\big(f(\mu(\varepsilon))\big)\Big) = 1/\phi_i(\mu(\varepsilon)),$$

for $i = 1, \ldots, n$, $|\mu(\varepsilon) - \lambda_0| < r$, and $\eta_i \notin \sigma(f(\mu(\varepsilon)))$. It then follows by the above and (3.22.1) that

$$\operatorname{dist}\Big(\eta_i, \sigma\big(f(\mu(\varepsilon))\big)\Big) = 1/\phi_i(\mu(\varepsilon)) \le 2/\phi_i(\lambda_0) = 2\Big(\operatorname{dist}\big(\eta_i, \sigma(f(\lambda_0))\big)\Big).$$

But for $\xi_i \in \partial\sigma(f(\lambda_0))$,

$$\Big(\operatorname{dist}\big(\eta_i, \sigma(f(\lambda_0))\big)\Big) \le |\eta_i - \xi_i|.$$

Consequently, since $|\eta_i - \xi_i| < \varepsilon/8$, we have that

$$\operatorname{dist}\Big(\eta_i, \sigma\big(f(\mu(\varepsilon))\big)\Big) < \varepsilon/4, \tag{3.22.2}$$

for $i = 1, \ldots, n$ and $|\mu(\varepsilon) - \lambda_0| < r$.

Thus by (3.22.2), $\overline{B}(\eta_i, \epsilon/4)$ intersects $\partial\sigma(f(\mu(\varepsilon)))$, for $i = 1, \ldots, n$. But $\overline{B}(\eta_i, \varepsilon/4) \subset \overline{B}(\xi_i, \varepsilon/2)$, which is easily verified by the triangle inequality. Hence $\overline{B}(\xi_i, \varepsilon/2)$ intersects $\partial\sigma(f(\mu(\varepsilon)))$ for $i = 1, \ldots, n$ as illustrated in the following figure 3.13:

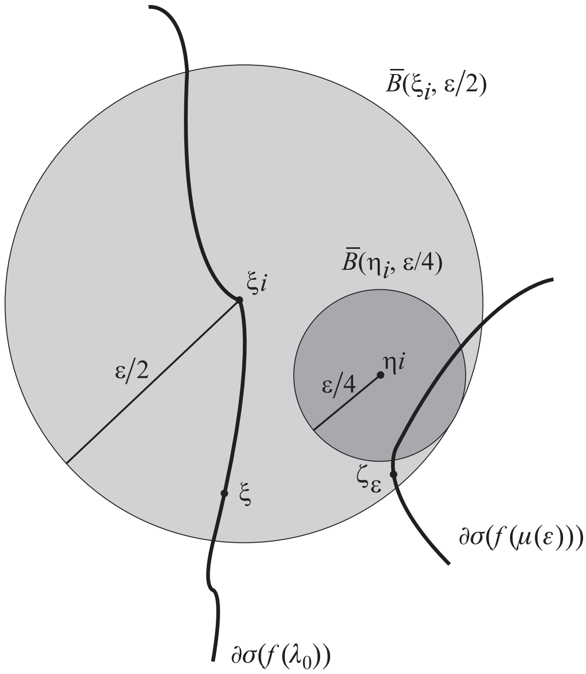

Figure 3.13: The intersection of $B(\xi_i, \varepsilon/2)$ with $\partial\sigma(f(\mu(\varepsilon)))$.

Now if $\xi \in \partial\sigma(f(\lambda_0))$, then ξ must belong to some $B(\xi_i, \varepsilon/2)$, because the sets $B(\xi_i, \varepsilon/2)$ are a finite covering of $\partial\sigma(f(\lambda_0))$. Thus, for some ξ_i we have $|\xi - \xi_i| < \varepsilon/2$, and hence

$$\operatorname{dist}\Big(\xi, \sigma\big(f(\mu(\varepsilon))\big)\Big) < \varepsilon,$$

which should be clear by figure 3.13. Consequently, from the latter inequality there exists $\zeta_\varepsilon \in \partial\sigma\big(f(\mu(\varepsilon))\big)$, such that $|\xi - \zeta_\varepsilon| < \varepsilon$, hence implying that $\xi - \zeta_\varepsilon \in B(0, \varepsilon)$. But $\xi = \zeta_\varepsilon + (\xi - \zeta_\varepsilon)$. Thus, since $\xi \in \partial\sigma(f(\lambda_0))$ was arbitrary, it follows that

$$\partial\sigma(f(\lambda_0)) \subset \partial\sigma(f(\mu(\varepsilon))) + B(0, \varepsilon).$$

Further, since $\varepsilon > 0$ is arbitrary, we let $\varepsilon = 1/k$. Also, we could find $\mu(\varepsilon) \in E$, such that $|\mu(\varepsilon) - \lambda_0| < r$. But, since we can choose $r > 0$ arbitrary small, say without loss of generality $r = 1/k$, it follows that

$$\partial\sigma(f(\lambda_0)) \subset \partial\sigma(f(\mu_k)) + B(0, 1/k), \text{ for } k \geq 1.$$

Q.E.D.

In the following Corollary 3.23 we will use the same method of proof used in Theorem 3.13.

Corollary 3.23
([AUP1]; Corollary 3.4.23, p. 61)

With the hypothesis of Theorem 3.22, *suppose that there exists a closed subset F of $\mathbb{C}$ such that*

$$\partial\sigma(f(\lambda)) \subset F, \quad \textit{for all } \lambda \in E.$$

Then

$$\partial\sigma(f(\lambda_0)) \subset F.$$

If, moreover, F has no interior points and does not separate the plane, then

$$\sigma(f(\lambda_0)) \subset F.$$

Proof:

Firstly, by Theorem 3.22 we have that for all $\lambda \in E$ that

$$\partial\sigma(f(\lambda_0)) \subset \partial\sigma(f(\lambda)) + B(0, 1/k),\ k \geq 1.$$

Since $\partial\sigma(f(\lambda)) \subset F$ for all $\lambda \in E$, it follows that

$$\partial\sigma(f(\lambda_0)) \subset F + B(0, 1/k),$$

and consequently as $k \to \infty$,

$$\partial\sigma(f(\lambda_0)) \subset F.$$

Now, assume F has no interior points and does not separate the plane. If $\sigma(f(\lambda_0))$ has an empty interior, then $\sigma(f(\lambda_0)) = \partial\sigma(f(\lambda_0))$ and the result holds trivially. So, we may assume $\operatorname{int}\sigma(f(\lambda_0)) \neq \emptyset$. Since F has no interior points, it follows that $\sigma(f(\lambda_0))$ can not be contained in F. Thus there exists $z_0 \in \operatorname{int}\sigma(f(\lambda_0))$ with $z_0 \in \mathbb{C}\backslash F$. Using the same argument as in the proof of Theorem 3.13, we can connect z_0 to infinity by an arc Γ, where Γ is included in $\mathbb{C}\backslash F$. Consequently (as in the proof of Theorem 3.13) Γ will cross $\partial\sigma(f(\lambda_0)) \subset F$, which is a contradiction. Hence $\sigma(f(\lambda_0)) \subset F$.
Q.E.D.

In Appendix D we discuss n'th spectral diameter and one of its properties which plays a crucial role in the proof Theorem 3.26. For A a Banach algebra and $x \in A$ we define the **n'th spectral diameter** of x, for $n \geq 1$, by the δ_n**-formula**:

$$\delta_n(x) = \max\left(\prod_{1\leq i<j\leq n+1} |z_i - z_j|\right)^{2/n(n+1)}.$$

This formula is applied to $n+1$ spectral points of $\sigma(x)$, with $z_k \in \{\lambda_1, \lambda_2, \ldots, \lambda_{n+1}\}$. For $n = 1$ it is the classical diameter of the spectrum of x, denoted by $\delta(x)$.
In Theorem 3.24 we show that we can construct two new subharmonic functions by using the n'th spectral diameter for $n = 1$. In the case of $n \geq 2$ the proof is much more complicated and uses tensor products and the concept of the joint spectrum. The first proof was given in 1982 by Z. Słodkowski. We refer the reader to [AUP1]; Theorem 7.1.3 and Theorem 7.1.13, p. 145 and p. 150 for further reading on this case.

Theorem 3.24
([AUP1]; Theorem 3.4.24, p. 62)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Then the functions

$$\lambda \mapsto \delta(f(\lambda)) \text{ and } \lambda \mapsto \log \delta(f(\lambda))$$

are subharmonic on D, where $\delta(x)$ is the diameter of $\sigma(x)$.

Proof:

Let $\alpha \in \mathbb{C}$ such that $|\overline{\alpha}| = 1$, and let $x \in A$. Denote by L_α the length of the projection of $\sigma(x)$ on the line $\overline{\alpha}t$, $t \in \mathbb{R}$ as illustrated in figure 3.14. We first show that L_α is given by

$$\log \rho(e^{\alpha x}) + \log \rho(e^{-\alpha x}),$$

by considering the behavior of $\sigma(\alpha x) = \alpha\sigma(x)$ with respect to the line $\overline{\alpha}t$.

Since $|\overline{\alpha}| = |\alpha| = 1$, $\alpha\sigma(x)$ is a rotation of $\sigma(x)$ with respect to the origin of the complex plane. That is, $\sigma(x)$ isn't stretched or compressed in any way, and hence the length L_α stays constant. Thus, as $\sigma(x)$ is rotated by multiplying $\sigma(x)$ by α, the line $\overline{\alpha}t$ is also rotated by multiplication of α with $\overline{\alpha}t$. Consequently, by this rotation, $\alpha\overline{\alpha}t$ coincides with the real axis as shown in the figure 3.15.
Clearly by figure 3.15, for $u, v \in \alpha\sigma(x)$, we have

$$L_\alpha = \max(\operatorname{Re} u) - \min(\operatorname{Re} v).$$

By applying Corollary 3.9 to $\sigma(\alpha x) = \alpha\sigma(x)$, it follows that

$$\max\{\operatorname{Re} u : u \in \alpha\sigma(x)\} = \log \rho(e^{\alpha x}) \text{ and } \min\{\operatorname{Re} v : v \in \alpha\sigma(x)\} = -\log \rho(e^{-\alpha x}).$$

Consequently we have that

$$L_\alpha = \log \rho(e^{\alpha x}) + \log \rho(e^{-\alpha x}).$$

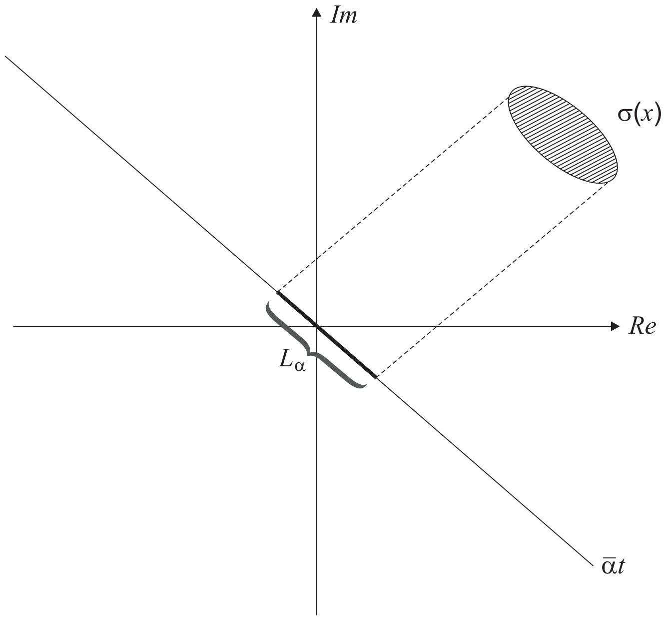

Figure 3.14: The length L_α of the projection of $\sigma(x)$ on the line $\overline{\alpha}t$.

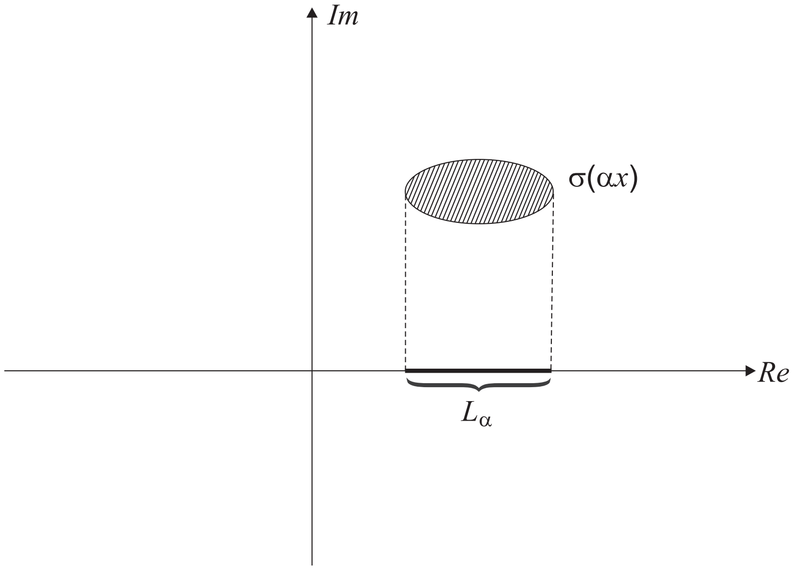

Figure 3.15: The projection length L_α coinciding with the line $\alpha\overline{\alpha}t$.

Further, it should be clear from the figure 3.15 that $\delta(x)$ and $\max_{|\alpha|=1} L_\alpha$ coincides, that is,

$$\delta(x) = \max_{|\alpha|=1} \Bigg(\log \rho(e^{\alpha x}) + \log \rho(e^{-\alpha x}) \Bigg).$$

Consequently, for $f(\lambda) \in A$, the same arguments holds true:

$$L_\alpha = \log \rho(e^{\alpha f(\lambda)}) + \log \rho(e^{-\alpha f(\lambda)}),$$

where L_α denotes the length of the projection of $\alpha\sigma(f(\lambda))$ on the line $\overline{\alpha}t$. Also, the spectral diameter of $f(\lambda)$ is given by

$$\delta(f(\lambda)) = \max_{|\alpha|=1}\Bigg(\log \rho\big(e^{\alpha f(\lambda)}\big) + \log \rho\big(e^{-\alpha f(\lambda)}\big)\Bigg).$$

Since f is analytic, it is clear that the functions $\lambda \mapsto e^{\alpha f(\lambda)}$ and $\lambda \mapsto e^{-\alpha f(\lambda)}$ are analytic on D. Hence by Theorem 3.7 and Theorem 2.2.2 (a)

$$\phi(\lambda) = \log \rho\big(e^{\alpha f(\lambda)}\big) + \log \rho\big(e^{-\alpha f(\lambda)}\big)$$

is subharmonic on D. So, by definition, ϕ satisfies the MVI. Thus, for $\overline{B}(\lambda_0, r)$ an arbitrary closed disk in D,

$$\phi(\lambda_0) \leq \frac{1}{2\pi}\int_0^{2\pi} \log \rho(e^{\alpha f(\lambda_0 + re^{i\theta})}) + \log \rho(e^{-\alpha f(\lambda_0 + re^{i\theta})})\ d\theta.$$

Now, considering the maximum of all $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$, we have that there exists at least one such an α, say $\alpha_0 = \max_{|\alpha|=1}\{\alpha\}$. Thus, by definition,

$$\max_{|\alpha|=1} \phi(\lambda) = \log \rho\big(e^{\alpha_0 f(\lambda)}\big) + \log \rho\big(e^{-\alpha_0 f(\lambda)}\big).$$

Consequently, since ϕ satisfies the MVI, it follows that

$$\max_{|\alpha|=1} \phi(\lambda_0) \leq \frac{1}{2\pi}\int_0^{2\pi} \log \rho(e^{\alpha_0 f(\lambda_0 + re^{i\theta})}) + \log \rho(e^{-\alpha_0 f(\lambda_0 + re^{i\theta})})\ d\theta,$$

that is,

$$\log \rho\big(e^{\alpha_0 f(\lambda)}\big) + \log \rho\big(e^{-\alpha_0 f(\lambda)}\big) \leq \frac{1}{2\pi}\int_0^{2\pi} \log \rho(e^{\alpha_0 f(\lambda_0 + re^{i\theta})}) + \log \rho(e^{-\alpha_0 f(\lambda_0 + re^{i\theta})})\ d\theta.$$

By using the latter, we now show that $\lambda \mapsto \delta(f(\lambda)) = \max_{|\alpha|=1} \phi(\lambda)$ also satisfies the MVI. Suppose the contrary, that is,

$$\max_{|\alpha|=1} \phi(\lambda_0) > \frac{1}{2\pi}\int_0^{2\pi} \max_{|\alpha|=1}\Bigg(\log \rho(e^{\alpha f(\lambda_0 + re^{i\theta})}) + \log \rho(e^{-\alpha f(\lambda_0 + re^{i\theta})})\Bigg)\ d\theta,$$

for all α such that $|\alpha| = 1$. In particular this must hold true for α_0, which is a contradiction. Hence, $\lambda \mapsto \delta(f(\lambda))$ satisfies the MVI.

In order to show that $\lambda \mapsto \delta(f(\lambda))$ is subharmonic on D, we still need to prove that this mapping is upper semi-continuous.
Since $\log \rho\big(e^{\alpha f(\lambda)}\big) + \log \rho\big(e^{-\alpha f(\lambda)}\big)$ is continuous it is also upper semi-continuous. Thus, by definition (cf. §2.2)

$$\limsup_{\lambda\to\lambda_0} \left(\log \rho\big(e^{\alpha f(\lambda)}\big) + \log \rho\big(e^{-\alpha f(\lambda)}\big) \right) \leq \log \rho\big(e^{\alpha f(\lambda_0)}\big) + \log \rho\big(e^{-\alpha f(\lambda_0)}\big).$$

Again, as in the previous argument, there exists α_0, with $|\alpha_0| = 1$, such that

$$\max_{|\alpha|=1} \left(\log \rho\big(e^{\alpha f(\lambda)}\big) + \log \rho\big(e^{-\alpha f(\lambda)}\big) \right) = \log \rho\big(e^{\alpha_0 f(\lambda)}\big) + \log \rho\big(e^{-\alpha_0 f(\lambda)}\big).$$

Consequently,

$$\limsup_{\lambda\to\lambda_0} \left\{ \max_{|\alpha|=1} \left(\log \rho\big(e^{\alpha f(\lambda)}\big) + \log \rho\big(e^{-\alpha f(\lambda)}\big) \right) \right\} = \limsup_{\lambda\to\lambda_0} \left(\log \rho\big(e^{\alpha_0 f(\lambda)}\big) + \log \rho\big(e^{-\alpha_0 f(\lambda)}\big) \right),$$

and hence,

$$\limsup_{\lambda\to\lambda_0} \left(\log \rho\big(e^{\alpha_0 f(\lambda)}\big) + \log \rho\big(e^{-\alpha_0 f(\lambda)}\big) \right) \leq \log \rho\big(e^{\alpha_0 f(\lambda_0)}\big) + \log \rho\big(e^{-\alpha_0 f(\lambda_0)}\big).$$

Now, suppose that $\lambda \mapsto \delta(f(\lambda))$ is not upper semi-continuous, that is,

$$\limsup_{\lambda\to\lambda_0} \left\{ \max_{|\alpha|=1} \left(\log \rho\big(e^{\alpha f(\lambda)}\big) + \log \rho\big(e^{-\alpha f(\lambda)}\big) \right) \right\} > \max_{|\alpha|=1} \left(\log \rho\big(e^{\alpha f(\lambda)}\big) + \log \rho\big(e^{-\alpha f(\lambda)}\big) \right),$$

for all α, with $|\alpha| = 1$. In particular, for $\alpha_0 = \max_{|\alpha|=1}\{\alpha\}$, we have

$$\limsup_{\lambda\to\lambda_0} \left(\log \rho\big(e^{\alpha_0 f(\lambda)}\big) + \log \rho\big(e^{-\alpha_0 f(\lambda)}\big) \right) > \log \rho\big(e^{\alpha_0 f(\lambda_0)}\big) + \log \rho\big(e^{-\alpha_0 f(\lambda_0)}\big),$$

which is a contradiction. Hence, $\lambda \mapsto \delta(f(\lambda))$ is upper semi-continuous, and thus subharmonic on D.

To prove that $\lambda \mapsto \log \delta(f(\lambda))$ is subharmonic, consider the analytic function $\lambda \mapsto e^{p(\lambda)} f(\lambda)$ from D into A, where p is a complex polynomial on D. By application of the Beckenbach-Saks Theorem (Theorem 2.2.5) we show that $\lambda \mapsto |e^{p(\lambda)}| \cdot \delta(f(\lambda))$ is subharmonic, hence implying that $\lambda \mapsto \log \delta(f(\lambda))$ is subharmonic on D.

Since $\delta(e^{p(\lambda)} f(\lambda)) = |e^{p(\lambda)}| \cdot \delta(f(\lambda))$ and $\lambda \mapsto \delta(e^{p(\lambda)} f(\lambda))$ is subharmonic by the first part of the theorem, it follows that $\lambda \mapsto |e^{p(\lambda)}| \cdot \delta(f(\lambda))$ is subharmonic on D. Thus, $\lambda \mapsto \log \delta(f(\lambda))$ is subharmonic on D.
Q.E.D.

Lemma 3.25
([CON2]; Theorem 3.7, p. 78)

Let D be a domain in $\mathbb{C}$ and $f : D \to \mathbb{C}$ be a holomorphic function. Then the following are equivalent statements:

(a) $f \equiv 0$,
(b) *the set* $\{z \in D : f(z) = 0\}$ *has a limit point in* D.

Theorem 3.26 Scarcity of Elements with Finite Spectrum
([AUP1]; Theorem 3.4.25, p. 63)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Then either the set of $\lambda \in D$ such that $\sigma(f(\lambda))$ is finite is:

(i) a Borel set having zero capacity, or

(ii) there exists an integer $n \geq 1$ and a closed discrete subset E of D such that $\#\sigma(f(\lambda)) = n$ for $\lambda \in D\backslash E$ and $\#\sigma(f(\lambda)) < n$ for $\lambda \in E$. Further, in the case of $\#\sigma(f(\lambda)) = n$, the n points are locally holomorphic functions.

Proof:

Let

$$F = \{\lambda \in D : \#\sigma(f(\lambda)) < \infty\}, \quad \text{and}$$
$$F_k = \{\lambda \in D : \#\sigma(f(\lambda)) \leq k\}.$$

Further, consider Appendix D on the n'th spectral diameter. It was shown that if $x \in A$ with $\#\sigma(x) \leq n$, then $\delta_n(x) = 0$. By defining $\log 0 \equiv -\infty$, we first show that

$$\lambda \in F_k \text{ if and only if } \log \delta_k(f(\lambda)) = -\infty.$$

Assume $\lambda \in F_k$. Then, by definition of F_k and Appendix D, $\delta_k(f(\lambda)) = 0$, and hence $\log \delta_k(f(\lambda)) = -\infty$.
If $\log \delta_k(f(\lambda)) = -\infty$, then it must hold true that $\delta_k(f(\lambda)) = 0$, and thus by Appendix D, $\#\sigma(f(\lambda)) \leq k$, that is, $\lambda \in F_k$.

Now, by Theorem 3.24, it follows that the mapping $\lambda \mapsto \phi_k(\lambda) = \log \delta_k(f(\lambda))$ ($k \geq 1$), is subharmonic on D. It is then clear from the latter, that for each $k \geq 1$ we have the equivalent characterization

$$F_k = \{\lambda \in D : \phi_k(\lambda) = -\infty\}.$$

By applying Cartan's Theorem (Theorem 2.3.4) to ϕ_k, we have the following two cases:

Case 1:

If ϕ_k is not identical to $-\infty$, then $F_k = \{\lambda \in D : \phi_k(\lambda) = -\infty\}$ is a G_δ-polar set, for all $k \geq 1$ (that is, each F_k is a G_δ-set with zero capacity).

Case 2:

If there exists a smallest integer $n \geq 1$ such that $c(F_n) > 0$, then follows for all $\lambda \in D$ that $\phi_n(\lambda) \equiv -\infty$, and hence $F_n = D$, by definition of F_n and Cartan's Theorem.

The first case implies that F is a Borel set with $c(F) = 0$. To prove the latter note that it is clear by the definitions of F and F_k that $F = \cup_{k\geq 1} F_k$. Since each F_k is a G_δ-set, hence a Borel set, it follows that the union F is a Borel set as well. Further, since $F_1 \subset F_2 \subset \ldots \subset F_k \subset \ldots$, it follows by Theorem 2.3.3 that $c(F) = \lim_{k\to\infty} c(F_k) = 0$.

For the second case we obtained $F_n = D$. Hence $\#\sigma(f(\lambda)) \leq n$ for all $\lambda \in D$. Now, define $E \subset D$ as follows:

$$E = \{\lambda \in D : \#\sigma(f(\lambda)) < n\}.$$

We now show that E is closed and discrete in D to obtain the desired end result.

• E closed:

Let (λ_l) be an arbitrary sequence in E such that $\lambda_l \to \lambda_0 \in D$, but $\lambda_0 \notin E$. By definition of E we have $\#\sigma(f(\lambda_l)) < n$, and $\#\sigma(f(\lambda_0)) = n$, say

$$\sigma(f(\lambda_0)) = \{\alpha_1, \ldots, \alpha_n\},$$

with $\alpha_i \neq \alpha_j$ for $i \neq j$.
Since $\sigma(f(\lambda_0))$ is finite, we can choose $\varepsilon > 0$ such that the disks $B(\alpha_i, \varepsilon)$ are disjoint for $i = 1, \ldots, n$. Hence $\sigma(f(\lambda_0)) \subset \cup_{i=1}^n B(\alpha_i, \varepsilon)$. Since f is analytic, $|\lambda_l - \lambda_0| \to 0$ implies $||f(\lambda_l) - f(\lambda_0)|| \to 0$. Thus, by Theorem 3.4, for l large enough it follows that

$$\sigma(f(\lambda_l)) \cap B(\alpha_i, \varepsilon) \neq \emptyset$$

for each $i = 1, \ldots, n$. Consequently, $\#\sigma(f(\lambda_l)) = n$. Clearly this is a contradiction and E is closed in D.

• E discrete:

Again let $\lambda_0 \notin E$. Then, as before, $\sigma(f(\lambda_0)) = \{\alpha_1, \ldots, \alpha_n\}$ with $\alpha_i \neq \alpha_j$ for $i \neq j$. Further, we choose $\varepsilon > 0$, such that the disks $B(\alpha_i, \varepsilon)$ are disjoint and hence $\sigma(f(\lambda_0)) \subset \cup_{i=1}^{n} B(\alpha_i, \varepsilon)$. Consequently, $|\lambda - \lambda_0| \to 0$ implies $||f(\lambda) - f(\lambda_0)|| \to 0$, and it follows by Theorem 3.4 for $i = 1, \ldots, n$ and λ in neighborhood of λ_0 that

$$\#\Big(\sigma(f(\lambda)) \cap B(\alpha_i, \varepsilon)\Big) = 1.$$

Thus, $\sigma(f(\lambda)) = \{\alpha_1(\lambda), \ldots, \alpha_n(\lambda)\}$, for $|\lambda - \lambda_0|$ sufficiently small. Moreover, by Theorem 3.20, it holds true that each α_i is holomorphic on a neighborhood of $\lambda_0 \notin E$. Now for $\lambda \in D \backslash E$, define the function

$$\varphi(\lambda) = \prod_{1 \leq i < j \leq n} \big(\alpha_i(\lambda) - \alpha_j(\lambda)\big)^2.$$

Note that φ is well-defined on $D \backslash E$. This holds true since φ is holomorphic on $D \backslash E$, since each of the α_i can be holomorphically extended onto the whole of $D \backslash E$. To prove the latter let $\lambda_0, \lambda_1 \in D \backslash E$ arbitrary and $\lambda' \in B(\lambda_0, r_0) \cap B(\lambda_1, r_1)$ as shown in the following figure 3.16:

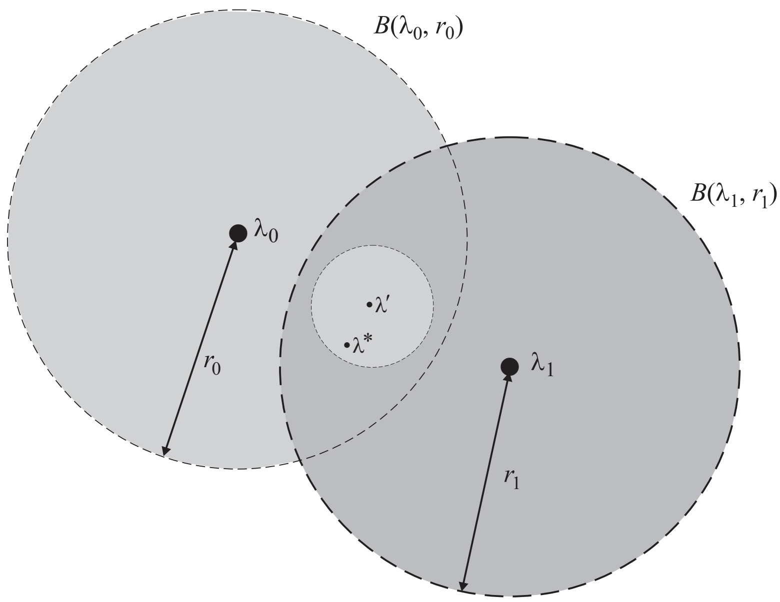

Figure 3.16: The intersection of the disks $B(\lambda_0, r_0)$ and $B(\lambda_1, r_1)$.

For $|\lambda - \lambda_0| < r_0$ it follows, as in the previous arguments, that

$$\sigma(f(\lambda)) \cap B(\alpha_i, \varepsilon) = \{\alpha_i^{(0)}(\lambda)\},$$

and for $|\lambda - \lambda_1| < r_1$,

$$\sigma(f(\lambda)) \cap B(\alpha_i, \varepsilon) = \{\alpha_i^{(1)}(\lambda)\},$$

for $i = 1, \ldots, n$.
We claim for λ^* arbitrary in a neighborhood of $\lambda' \in B(\lambda_0, r_0) \cap B(\lambda_1, r_1)$, it holds true that

$$\sigma(f(\lambda^*)) \cap B(\alpha_i, \varepsilon) = \{\alpha_i^{(0)}(\lambda^*)\} = \{\alpha_i^{(1)}(\lambda^*)\}.$$

That is, the holomorphic functions $\alpha_i^{(0)}$ and $\alpha_i^{(1)}$ coincide for all $\lambda' \in B(\lambda_0, r_0) \cap B(\lambda_1, r_1)$.

Suppose this is false. Then choose $\lambda' \in B(\lambda_0, r_0) \cap B(\lambda_1, r_1)$ fixed, such that $B(\lambda', r') \subsetneq B(\lambda_0, r_0) \cap B(\lambda_1, r_1)$.

Now, for $B(\lambda', r') \subsetneq B(\lambda_0, r_0)$ it follows for $|\lambda^* - \lambda'| < r'$ that

$$\sigma(f(\lambda^*)) \cap B(\alpha_i, \varepsilon) = \{\alpha_i^{(0)}(\lambda^*)\} \text{ for } i = 1, \ldots, n. \tag{3.26.1}$$

Also, for $B(\lambda', r') \subsetneq B(\lambda_1, r_1)$, we have that if $|\lambda^* - \lambda'| < r'$, then

$$\sigma(f(\lambda^*)) \cap B(\alpha_i, \varepsilon) = \{\alpha_i^{(1)}(\lambda^*)\} \text{ for } i = 1, \ldots, n. \tag{3.26.2}$$

Thus, (3.26.1) and (3.26.2) implies that for $|\lambda^* - \lambda_0| < r_0$ and $|\lambda^* - \lambda_1| < r_1$,

$$\#\Big(\sigma(f(\lambda^*)) \cap B(\alpha_i, \varepsilon)\Big) > 1,$$

which is a contradiction. Hence for all $\lambda' \in B(\lambda_0, r_0) \cap B(\lambda_1, r_1)$, we have that $\alpha_i^{(0)}(\lambda') = \alpha_i^{(1)}(\lambda')$, $i = 1, \ldots, n$. Thus, each α_i can be holomorphically extended over $D\backslash E$, and it follows by definition of φ, that φ is holomorphic and hence well-defined on $D\backslash E$.

Further, note that φ is nowhere zero on $D\backslash E$. To prove this let λ_p be in $D\backslash E$ with $p \in \{1, 2, \ldots\}$. Then for $|\lambda - \lambda_p|$ small enough, it follows as in previous arguments that

$$\sigma(f(\lambda)) = \{\alpha_1^{(p)}, \alpha_2^{(p)}, \ldots, \alpha_n^{(p)}\},$$

where

$$\alpha_i^{(p)} \neq \alpha_j^{(p)},$$

for $i \neq j$. Hence for all open balls B_{λ_p} center λ_p in $D\backslash E$, it follows by the definition of φ, that φ is nowhere zero on $D\backslash E$.

We now show that E has no interior points. To prove this, let us recall that for

$\lambda \in D \backslash E$, $\# \sigma(f(\lambda)) = n$, and for $\lambda \in E$, $\# \sigma(f(\lambda)) < n$. Consequently, by Appendix D

$$\delta_{n-1}(f(\lambda)) \neq 0 \text{ for } \lambda \in D \backslash E,$$

and

$$\delta_{n-1}(f(\lambda)) = 0 \text{ for } \lambda \in E.$$

Since f is analytic on the whole of D, it follows by Theorem 3.24 that $\lambda \mapsto \log \delta_{n-1}(f(\lambda))$ is subharmonic on D. Moreover, it is clear that for $\lambda \in D \backslash E$, $\log \delta_{n-1}(f(\lambda)) \neq -\infty$, and for $\lambda \in E$, $\log \delta_{n-1}(f(\lambda)) = -\infty$. Thus, the mapping $\lambda \mapsto \log \delta_{n-1}(f(\lambda))$ isn't identical to $-\infty$ on D. Hence, by Cartan's Theorem (Theorem 2.3.4), E has zero capacity.
Let us assume that E does indeed contain interior points. Then we can find an open set $U \subset E$, and a closed disk $\overline{B} \subset U$. It follows by Theorem 2.3.5, that $c(E) > 0$, since $c(\overline{B}) > 0$. But this is contradictive with the fact that $c(E) = 0$ and hence E has an empty interior.

If we define φ on E by $\varphi(\lambda) = 0$ it follows that φ is continuous on the whole of D (since φ is holomorphic on $D \backslash E$). To prove this, first note that since φ is nowhere zero on $D \backslash E$, we have that

$$\varphi(\lambda) = 0 \quad \text{if and only if} \quad \lambda \in E.$$

Thus, the set of zero's of φ is exactly E.
To prove the continuity of φ on D let $\lambda_1 \notin E$. Since φ is continuous at λ_1, we may assume that $\lambda_1 \in E$. So, we choose a sequence $(\lambda_m) \subset D \backslash E$ such that if $\lambda_m \to \lambda_1$, then we need to show that $\varphi(\lambda_m) \to \varphi(\lambda_1) = 0$.

Since $\lambda_1 \in E$, we have $\# \sigma(f(\lambda_1)) < n$. Say

$$\sigma(f(\lambda_1)) = \{\beta_1, \beta_2, \ldots, \beta_k\}, \text{ with } k < n.$$

Then, we can find $\varepsilon > 0$ and consider the disjoint disks $B(\beta_i, \varepsilon)$ such that

$$\sigma(f(\lambda_1)) \subset \bigcup_{i=1}^{k} B(\beta_i, \varepsilon).$$

Further, for $(\lambda_m) \notin E$ we have that $\# \sigma(f(\lambda_m)) = n$, say

$$\sigma(f(\lambda_m)) = \{\alpha_1^{(m)}, \alpha_2^{(m)}, \ldots, \alpha_n^{(m)}\}.$$

Consequently, by Theorem 3.4, it follows for m sufficiently large that

$$\sigma(f(\lambda_m)) \subset \bigcup_{i=1}^{k} B(\beta_i, \varepsilon).$$

Thus, since $\#\sigma(f(\lambda_m)) = n$ and $\#\sigma(f(\lambda_1)) < n$, we have that at at least two sequences, say $\alpha_i^{(m)}$ and $\alpha_j^{(m)}$ $(i \neq j)$ of $\sigma(f(\lambda_m))$, will have a common limit, say $\beta_1 \in \sigma(f(\lambda_1))$. Thus, as $\lambda_m \to \lambda_1$,

$$\left(\alpha_i^{(m)}(\lambda_m) - \alpha_j^{(m)}(\lambda_m)\right)^2 \to 0.$$

Hence

$$\varphi(\lambda_m) = \prod_{1 \leq i < j \leq n} \left(\alpha_i(\lambda_m) - \alpha_j(\lambda_m)\right)^2 \to 0,$$

since one of the factors in the above product tends to zero as $\lambda_m \to \lambda_1$. Thus, $\varphi(\lambda_m) \to 0$, and φ is continuous on E, and therefore on D. Consequently, by Radó's Extension Theorem (cf. Appendix C), we have that φ is holomorphic on the whole of D, since φ is holomorphic on E. Hence, since φ is not identical to zero on D, it follows by Lemma 3.25 that the set of zero's of φ, which is precisely E, is discrete in D.
Q.E.D.

Lemma 3.27
([CON2]; Theorem 1.2, p. 55)

If a complex function f has an isolated singularity at λ_0, then the point λ_0 is a removable singularity if and only if

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0) f(\lambda) = 0.$$

Theorem 3.28
([AUP1]; Theorem 3.4.26, p. 64)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Suppose for all $\lambda \in D$ that the spectrum of $f(\lambda)$ has at most 0 as a limit point. Let $z \neq 0$. Then either the set of $\lambda \in D$ such that $z \in \sigma(f(\lambda))$ is closed and discrete in D, or $z \in \sigma(f(\lambda))$ for all $\lambda \in D$.

<u>Proof:</u>

Suppose $z \neq 0$ and $z \in \sigma(f(\lambda_0))$ for some $\lambda_0 \in D$. Let E be the subset of D where

$$E = \{\lambda \in D : z \in \sigma(f(\lambda))\}.$$

We first show that either λ_0 is isolated in E, or that λ_0 is in the interior of E.

Since $z \neq 0$, it follows by our hypothesis that z is isolated in $\sigma(f(\lambda_0))$. If this was not the case, then it would follow that $z \neq 0$ is a limit point of $\sigma(f(\lambda_0))$, contradicting our hypothesis. Thus, for z isolated, we can find an open disk Δ, centered at z such that Δ contains no other points of $\sigma(f(\lambda_0))$. Also, if $z_1 \in \overline{\Delta}$, with $z_1 \neq z$ and $z_1 \in \sigma(f(\lambda_0))$, then $z_1 \in \partial\Delta$. Hence, we can always find an open disk $\Delta' \subset \Delta$, with the radius of Δ' smaller than that of Δ, such that

$$\overline{\Delta'} \cap \sigma(f(\lambda_0)) = \{z\}.$$

Thus, without loss of generality, we may assume that

$$\overline{\Delta} \cap \sigma(f(\lambda_0)) = \{z\}.$$

Note that although $\#\big(\overline{\Delta} \cap \sigma(f(\lambda_0))\big) = 1$, this does not necessarily imply that $\#\sigma(f(\lambda_0)) = 1$. We can have that $\#\sigma(f(\lambda_0)) > 1$.

Since $\mathbb{C}$ is a normal space, we can always find an open set $(\overline{\Delta})^c = D \backslash \overline{\Delta}$ such that $\sigma(f(\lambda_0)) \subset (\overline{\Delta})^c \cup \Delta$, but $(\overline{\Delta})^c \cap \Delta = \emptyset$. Consequently, since $\sigma(f(\lambda_0)) \cap \Delta \neq \emptyset$, it follows by Theorem 3.4, that there exists $r_1 > 0$ such that if $|\lambda - \lambda_0| < r_1$, then

$$\sigma(f(\lambda)) \cap \Delta \neq \emptyset. \tag{3.27.1}$$

Further, since $\sigma(f(\lambda_0)) \subset (\overline{\Delta})^c \cup \Delta$, it follows by the upper semi-continuity of the spectrum (Theorem 3.3), that there exists $r_2 > 0$ such that if $|\lambda - \lambda_0| < r_2$, then

$$\sigma(f(\lambda)) \subset (\overline{\Delta})^c \cup \Delta, \text{ and hence } \sigma(f(\lambda)) \cap \partial\Delta = \emptyset. \tag{3.27.2}$$

To ensure that both (3.27.1) and (3.27.2) hold true, we choose $r = \min\{r_1, r_2\}$. Hence it follows by (3.27.1) and (3.27.2) that

$$\sigma(f(\lambda)) \subset \Delta \cup (\overline{\Delta})^c,$$

for $|\lambda - \lambda_0| < r$, with Δ and $(\overline{\Delta})^c$ open in $\mathbb{C}$.

Let h be the complex function defined as follows:

$$h(z) = z, \text{ for } z \in \Delta,$$

and

$$h(z) = 0, \text{ for } z \in (\overline{\Delta})^c.$$

Clearly h is holomorphic on $\Delta \cup (\overline{\Delta})^c$, and by the Holomorphic Functional Calculus and the Spectral Mapping Theorem, for $|\lambda - \lambda_0| < r$

$$\sigma(h(f(\lambda))) = h(\sigma(f(\lambda))) = \{h(z) : z \in \sigma(f(\lambda))\}.$$

So, by the definition of h, $\sigma(h(f(\lambda)))$ consists of all the points of $\sigma(f(\lambda))$ that lie inside Δ, and possibly the point $0 \notin \Delta$, since $h(\alpha) = 0$ for $\alpha \in \sigma(f(\lambda)) \cap (\overline{\Delta})^c$.
Note that if $\sigma(f(\lambda))$ has infinitely many points inside Δ, then since Δ is bounded, it follows by the Bolzano-Weierstrass Theorem that $\sigma(f(\lambda))$ will have a limit point in $\overline{\Delta}$. Since $0 \notin \overline{\Delta}$ we obtain a contradiction with our hypothesis. Thus $\sigma(f(\lambda))$ has only finitely many points in Δ, and hence

$$\#\big(\sigma(h(f(\lambda)))\big) < \infty \quad \text{for } |\lambda - \lambda_0| < r.$$

Further, since $\sigma(h(f(\lambda)))\backslash\{0\} = \sigma(f(\lambda)) \cap \Delta$, it follows that

$$\#\big(\sigma(h(f(\lambda)))\backslash\{0\}\big) = \#\big(\sigma(f(\lambda)) \cap \Delta\big).$$

In conclusion, it follows without loss of generality by Theorem 3.4 that

$$\sigma(h(f(\lambda))) = \sigma(f(\lambda)) \cap \Delta \quad \text{for all } \lambda \in B(\lambda_0, r),$$

or

$$\sigma(h(f(\lambda))) = \big(\sigma(f(\lambda)) \cap \Delta\big) \cup \{0\} \quad \text{for all } \lambda \in B(\lambda_0, r),$$

where the latter holds when $\sigma(f(\lambda))$ contains points in $(\overline{\Delta})^c$.

Since $B(\lambda_0, r)$ has non-zero capacity (cf. Theorem 2.3.5), it follows by the Scarcity Theorem (Theorem 3.26), that there exists an integer $n \geq 1$ and a closed discrete subset F of $B(\lambda_0, r)$ such that for $\lambda \in B(\lambda_0, r)\backslash F$

$$\sigma(f(\lambda)) \cap \Delta = \{\alpha_1(\lambda), \alpha_2(\lambda), \ldots, \alpha_n(\lambda)\},$$

where $\alpha_1(\lambda), \alpha_2(\lambda), \ldots, \alpha_n(\lambda)$ are holomorphic on $B(\lambda_0, r)\backslash F$.

Further, it follows that there exists s such that $0 < s \leq r$ and

$$B(\lambda_0, s) \cap F \subset \{\lambda_0\}.$$

To prove the above, let us say this is not the case. That is, for all $s \leq r$ we have that $B(\lambda_0, s) \cap F$ contains other points than λ_0. Choosing $s_n \leq r$ such that $s_n \to 0$, it follows that λ_0 will be an accumulation point of F. Since F is closed we have $\lambda_0 \in F$. But this contradicts the fact that F is discrete – F can not contain any accumulation points. Hence $B(\lambda_0, s) \cap F \subset \{\lambda_0\}$.
Note that the reason why $B(\lambda_0, s) \cap F$ is a subset of $\{\lambda_0\}$, and does not coincide with $\{\lambda_0\}$, is since $B(\lambda_0, s) \cap F = \emptyset$ is a possibility.
Hence, from $B(\lambda_0, s) \cap F \subset \{\lambda_0\}$ and since $\{\alpha_1(\lambda), \alpha_2(\lambda), \ldots, \alpha_n(\lambda)\}$ are only holomorphic on $B(\lambda_0, r)\backslash F$, it follows that for each $i = 1, \ldots, n$, we have that α_i is

holomorphic on $B(\lambda_0, s)$ except perhaps at λ_0. Thus, λ_0 is at most an isolated singularity of each α_i.

Moreover, by the upper semi-continuity of the spectrum (Theorem 3.3) we have

$$\lim_{\lambda \to \lambda_0} \alpha_i(\lambda) = z \quad \text{for all } i \in \{1, \dots, n\}.$$

Suppose this is false. Then there exists at least one i such that $\lim_{\lambda \to \lambda_0} \alpha_i(\lambda) \neq z$. That is, there exists $\delta > 0$ and a sequence $(\lambda_j) \subset B(\lambda_0, s)$ such that $\lambda_j \to \lambda_0$, but $|\alpha_i(\lambda_j) - z| > \delta$ for all j. So, we can find a neighborhood V_δ of z, such that $\alpha_i(\lambda_j) \notin V_\delta$ for all j. But, as discussed in the beginning of the proof, we can choose the radius of Δ small enough such that $\Delta \subsetneq V_\delta$. Hence, for $|\lambda_j - \lambda_0|$ small enough we have

$$\sigma(f(\lambda_j)) \cap \Delta = \{\alpha_1(\lambda_j), \alpha_2(\lambda_j), \dots, \alpha_n(\lambda_j)\}.$$

Clearly this is a contradiction, and it follows that $\lim_{\lambda \to \lambda_0} \alpha_i(\lambda) = z$ holds indeed true.

The existence of the latter limit implies that each α_i can be holomorphically extended to the whole disk $B(\lambda_0, s)$. That is, λ_0 is a removable singularity of each α_i, for $i = 1, \dots, n$. This holds true by Lemma 3.27 since

$$\lim_{\lambda \to \lambda_0} [(\lambda - \lambda_0)\alpha_i(\lambda)] = 0(z) = 0.$$

That each of the α_i can be holomorphically extended to $B(\lambda_0, s)$ implies that either

$$\alpha_{i_0} \equiv z \quad \text{for some } i_0 \in \{1, \dots, n\},$$

or there exists t with $0 < t \leq s$ such that

$$\alpha_{i(t)}(\lambda) \neq z,$$

for all $\lambda \in B(\lambda_0, t) \backslash \{\lambda_0\}$ and $i = 1, \dots, n$, where i depends on t.

To prove the above suppose the second case is false. Then, for every t with $0 < t \leq s$, we can find $i(t)$ such that $\alpha_{i(t)} = z$ for some $\lambda \in B(\lambda_0, t) \backslash \{\lambda_0\}$.

It then follows that we can find a sequence $(t_m) \to 0$, such that for each m we can find a corresponding i_m and $\lambda_m \in B(\lambda_0, t_m) \backslash \{\lambda_0\}$ such that $\alpha_{i_m}(\lambda_m) = z$. But, since there are only finitely many i_m, it implies that some i_m will occur infinitely many times. If we denote the latter by i_0, then it follows that there exists a sequence $(\lambda_m^{(0)}) \in B(\lambda_0, t_m) \backslash \{\lambda_0\}$ such that $\alpha_{i_m}(\lambda_m^{(0)}) = z$. Now consider $g(\lambda) = \alpha_{i_0}(\lambda) - z$ which is holomorphic on $B(\lambda_0, s)$. Since $g(\lambda_m^{(0)}) \to 0$ as $\lambda_m^{(0)} \to \lambda_0$ and $t_m \to 0$, it

follows that λ_0 is a limit point of the set $\{\lambda \in B(\lambda_0, s) : f(\lambda) = 0\}$.
Hence, by Lemma 3.25, $g \equiv 0$ on $B(\lambda_0, s)$. Thus $g(\lambda) = \alpha_{i_0}(\lambda) - z = 0$, that is, $\alpha_{i_0}(\lambda) = z$ for all $\lambda \in B(\lambda_0, s)$.

Since $\alpha_{i_0}(\lambda) = z \in \sigma(f(\lambda))$ for all $\lambda \in B(\lambda_0, s)$ (by the first case), it follows by the definition of E that $B(\lambda_0, s) \subset E$, and hence λ_0 is an interior point of E. The second case implies that $\alpha_i(\lambda_0) = z$, thus $\lambda_0 \in E$. Hence, $B(\lambda_0, t)\backslash\{\lambda_0\}$ is not contained in E and λ_0 is isolated in E.

We now prove that E is closed in D. Let E' denote the set of limit points of E. To obtain the desired result we need only show that $E' \subset E$. Let (λ_n) be a sequence in E such that $\lambda_n \to \lambda' \in E'$. Then, by definition of E, $z \in \sigma(f(\lambda_n))$. We claim that $\lambda' \in E$. If this is not the case, then by definition of E, $z \notin \sigma(f(\lambda'))$. It then follows by the normality of the complex plane that there exist disjoint sets U and V such that

$$\sigma(f(\lambda')) \subset U \text{ and } \{z\} \subset V.$$

Consequently, as $|\lambda_n - \lambda'| \to 0$, $||f(\lambda_n) - f(\lambda')|| \to 0$, and it follows by the upper semi-continuity of the spectrum (Theorem 3.3) that $z \in \sigma(f(\lambda_n)) \subset U$ for n large enough. But this is a contradiction. Hence, $\lambda' \in E$ and E is closed in D, implying that E' is closed in D as well.

We also show that E' is open in D. Let $\mu' \in E'$. Thus, since μ' is a limit point, it follows that μ' is not isolated in E, implying that μ is an interior point of E and hence of E' as well. Thus, E' is open in D.

Since D is a domain and E' is both open and closed in D, it follows that either one of the following is true:

$$E' = \emptyset \quad \text{or} \quad E' = D.$$

For $E' = D$ we have that $E = D$, thus $z \in \sigma(f(\lambda))$ for all $\lambda \in D$. If $E' = \emptyset$, it follows that we have the case of E discrete in D.
Q.E.D.

Notes on Chapter 3

(**1**) *With the hypothesis and proof of* Theorem 3.4.4, *we have that as* $y_n \to x$, *it then follows by the Holomorphic Functional Calculus that* $\lim_{n\to\infty} f(y_n) = f(x)$.

Proof:

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be smooth curve in U and V, such that Γ_1 surrounds $\sigma(x)$ and $\sigma(y_n)$ (for n large), and Γ_2 surrounds the the remaining spectrum of x. Then by the Holomorphic Functional Calculus the following elements in A are well-defined:

$$f(y_n) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - y_n)^{-1}\, d\lambda$$

and

$$f(x) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - x)^{-1}\, d\lambda.$$

So, we need to show that

$$\lim_{n\to\infty}\left(\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - y_n)^{-1}\, d\lambda\right) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - x)^{-1}\, d\lambda.$$

Now $y_n \to x$ implies $\lambda\mathbf{1} - y_n \to \lambda\mathbf{1} - x$ as $n \to \infty$. Further, since the mapping $x \mapsto x^{-1}$ is continuous on $G(A)$ (Theorem 3.2.3), it follows that $(\lambda\mathbf{1} - y_n)^{-1} \to (\lambda\mathbf{1} - x)^{-1}$. Thus

$$\begin{aligned}
&\left|\left|\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - y_n)^{-1}d\lambda - \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - x)^{-1}d\lambda\right|\right| \\
&= \left|\left|\frac{1}{2\pi i}\int_\Gamma f(\lambda)\cdot\Big((\lambda\mathbf{1} - y_n)^{-1} - (\lambda\mathbf{1} - x)^{-1}\Big)d\lambda\right|\right| \\
&\le \frac{1}{2\pi}\int_\Gamma |f(\lambda)|\cdot||(\lambda\mathbf{1} - y_n)^{-1} - (\lambda\mathbf{1} - x)^{-1}||d\lambda.
\end{aligned}$$

Now, consider the compact set $\Gamma \times F$, with $F = \Big\{\{y_n\}_{n=1}^\infty \cup \{x\}\Big\}$. Also, consider the function $g(\lambda, y_j) = ||(\lambda\mathbf{1} - y_j)^{-1} - (\lambda\mathbf{1} - x)^{-1}||$ from $\Gamma \times F$ into $\mathbb{R}^+$. Clearly g is continuous. Since $\Gamma \times F$ is compact, it follows that there exists $K \in \mathbb{R}^+$ such that

$$g(\lambda, y_j) = ||(\lambda\mathbf{1} - y_j)^{-1} - (\lambda\mathbf{1} - x)^{-1}|| \le K. \tag{i}$$

Let $h(\lambda) = K$. Clearly h is integrable. Consequently, by (i) and since $g(\lambda, y_j) \to 0$ as $j \to \infty$, it follows by the Lebesgue Dominated Convergence Theorem that

$$\left|\left|\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - y_n)^{-1}d\lambda - \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - x)^{-1}d\lambda\right|\right| \to 0.$$

Hence $\lim_{n\to\infty} f(y_n) = f(x)$.
Q.E.D.

(**2**) *For A a Banach algebra and $x \in A$, the sequence $||x^{2^n}||^{1/2^n}$ is decreasing.*

Proof:

We prove that the following statement $S(n)$ holds true for each n by means of induction:

$$S(n): \ ||x^{2^{n+1}}||^{1/2^{n+1}} \leq \ ||x^{2^n}||^{1/2^n}$$

Note, that by the multiplicative inequality $||a^2|| \leq ||a||^2$ for each $a \in A$ (this plays a role in our proof).

For $n = 1$:

$$||x^{2^2}||^{1/2^2} = ||x^4||^{1/4} = ||x^2x^2||^{1/4} \leq \left(||x^2|| \cdot ||x^2||\right)^{1/4}.$$

Thus, we have

$$||x^{2^2}||^{1/2^2} \leq \left(||x^2||^2\right)^{1/4} = ||x^2||^{1/2}.$$

Assume that $S(n)$ is true for $n = k$, that is:

$$||x^{2^{k+1}}||^{1/2^{k+1}} \leq ||x^{2^k}||^{1/2^k}.$$

For $n = k + 1$ we need to prove:

$$||x^{2^{k+2}}||^{1/2^{k+2}} \leq ||x^{2^{k+1}}||^{1/2^{k+1}}.$$

Since $2^{k+2} = 2^{k+1}2$, it follows that

$$||x^{2^{k+2}}||^{1/2^{k+2}} = ||x^{2^{k+1}2}||^{1/2^{k+1}2} = ||(x^{2^{k+1}})^2||^{1/2^{k+1}2}.$$

But

$$||(x^{2^{k+1}})^2||^{1/2^{k+1}2} \leq \left(|| \ x^{2^{k+1}}||^2\right)^{1/2^{k+1}2}.$$

Hence we have

$$||x^{2^{k+2}}||^{1/2^{k+2}} \leq \left(|| \ x^{2^{k+1}}||^2\right)^{1/2^{k+1}2} = ||x^{2^{k+1}}||^{1/2^{k+1}}.$$

Q.E.D.

Chapter 4

Spectral Characterizations of the Jacobson Radical

In this chapter we apply our subharmonic function theory of chapters 2 and 3 to obtain spectral characterizations of the radical.
Our two main results regarding the latter are Theorems 4.4 and 4.6. In the proof of Theorem 4.4 we see the usefulness of the much applied Vesentini's Theorem (Theorem 3.7), and the Maximum Principle of Subharmonic Functions (Theorem 2.3.3). The concept of capacity (§2.3) and its relationship with polar sets, now plays a prominent role when applying Cartan's Theorem (Theorem 2.3.4) and Theorem 2.3.5. On the other hand, Theorem 4.6 uses the results of our newly obtained Theorem 4.4 in conjunction with Vesentini's and Cartan's Theorem, and Liouville's Theorem for Subharmonic Functions (Theorem 2.2.4).

Although the proof of Theorem 4.4 strongly depends on subharmonic theory, Theorems 4.1, 4.2, and 4.3; all of which are characterizations of center modulo the radical of A, denoted by $Z(A)$ (definition below). Hence, we first turn our attention to the latter three theorems, with representation theory (§1.5) as their basis. Important results such as Schur's, Jacobson's, and Kaplansky's Theorem are used in conjunction with two definitions of the radical – Definition 1.5.6 (due to Jacobson) and Theorem 1.5.7.

Now for the definition of $Z(A)$:
Given a Banach algebra A, we denote by $Z(A)$ the **center modulo the radical** of A, which is the set of all $a \in A$ such that $ax - xa \in \operatorname{Rad} A$ for all $x \in A$ ([AUP1]; p. 92).

To obtain the desired end result of the following Theorem 4.1, due to Le Page, we use the fact that the radical is the intersection of the kernels of all continuous irreducible representations of A (Definition 1.5.6) .

Theorem 4.1
([AUP1]; Theorem 5.2.1, p. 92)

Let A be a Banach algebra with $a \in A$ such that $\#\sigma(ax - xa) = 1$ for all $x \in A$. Then $a \in Z(A)$.

Proof:

Let π be an arbitrary continuous irreducible representation of A on a Banach space X. We first prove that $\pi(a)$ is of algebraic degree less than or equal to two. The proof is by contradiction and by application of Kaplansky's Theorem (Theorem 1.5.5). We claim that for $\xi \in X$ arbitrary, $\xi, \pi(a)\xi = \eta$, and $\pi(a)^2\xi = \pi(a)\eta$ are linearly dependent in X. Suppose this is not the case. Then, by Theorem 1.5.3, there exists $x \in A$ such that

$$\pi(x)\xi = 0, \quad \pi(x)\eta = -\xi, \quad \text{and} \quad \pi(x)\pi(a)\eta = -\eta.$$

Consequently

$$\pi(ax - xa)\xi = \xi,$$

since $\pi(ax - xa)\xi = \pi(a)\pi(x)\xi - \pi(x)\pi(a)\xi = 0 - \pi(x)\eta = -(-\xi) = \xi$.
Also

$$\pi(ax - xa)\eta = 0,$$

since $\pi(ax - xa)\eta = \pi(a)\pi(x)\eta - \pi(x)\pi(a)\eta = \pi(a)(-\xi) - (-\eta) = -\eta + \eta = 0$.
Thus, from the above it follows that

$$\{0, 1\} \subset \sigma_{\pi(A)}\big(\pi(ax - xa)\big).$$

The above inclusion holds true since the operators $I - \pi(ax - xa)$ and $\pi(ax - xa)$ are not one-to-one. Further, by Theorem 1.5.1

$$\sigma_{\pi(A)}\big(\pi(ax - xa)\big) \subset \sigma_A(ax - xa),$$

which is a contradiction with our hypothesis. Hence $\xi, \pi(a)\xi = \eta$, and $\pi(a)^2\xi = \pi(a)\eta$ are linearly independent for all $\xi \in X$, and by Kaplansky's Theorem it follows that $\pi(a)$ is of algebraic degree less than or equal to two.

Without loss of generality we may assume that $\pi(a)^2 = \gamma I$ for some $\gamma \in \mathbb{C}$. Say this is not the case and let $\pi(a)^2 = \alpha\pi(a) + \beta I$, $\alpha, \beta \in \mathbb{C}$. Further, by taking $a' = a - (\alpha\mathbf{1})/2$ it then follows trivially that for all $x \in A$

$$\sigma_A(a'x - xa') = \sigma_A(ax - xa).$$

Moreover

$$\pi(a')^2 = (\beta + \alpha^2/4)I.$$

The above holds true since

$$\pi(a')^2 = \pi(a')\pi(a') = \pi(a - \frac{\alpha}{2}\mathbf{1})\pi(a - \frac{\alpha}{2}\mathbf{1}),$$

and consequently

$$\pi(a')^2 = \pi(a)^2 - \alpha\pi(a) + (\alpha^2/4)I.$$

But we assumed $\pi(a)^2 = \alpha\pi(a) + \beta I$, so

$$\pi(a')^2 = \alpha\pi(a) + \beta\mathrm{I} - \alpha\pi(a) + (\alpha^2/4)I,$$

and hence

$$\pi(a')^2 = (\beta + \alpha^2/4)I.$$

Consequently, since $\sigma_A(a'x - xa') = \sigma_A(ax - xa)$, we may assume that $\pi(a)^2 = \gamma I$. Clearly then $\pi(a)^2\xi = \gamma\xi$.

We now show that $\pi(a)$ is in actual fact of algebraic degree one. Suppose this is not the case. Then, by the contrapositive of Theorem 1.5.5, there exists $\xi \in X$ such that ξ and $\pi(a)\xi = \eta$ are linearly independent. It then follows, by Theorem 1.5.3, that there exists $x \in A$ such that

$$\pi(x)\xi = \xi \quad \text{and} \quad \pi(x)\eta = \xi + \eta.$$

Consequently

$$\pi(ax - xa)\xi = -\xi,$$

since $\pi(ax - xa)\xi = \pi(a)\pi(x)\xi - \pi(x)\pi(a)\xi = \pi(a)\xi - \pi(x)\eta = \eta - (\xi + \eta) = -\xi$. Also

$$\pi(ax - xa)\eta = \eta,$$

since $\pi(ax-xa)\eta = \pi(a)\pi(x)\eta - \pi(x)\pi(a)\eta = \pi(a)(\xi+\eta) - \pi(x)\pi(a)^2\xi = \eta + \pi(a)^2\xi - \pi(x)\pi(a)^2\xi$. But $\pi(a)^2\xi = \gamma\xi$. So $\pi(ax-xa)\eta = \eta + \gamma\xi - \pi(x)(\gamma\xi) = \eta + \gamma\xi - \gamma\xi = \eta$. Thus, by the latter it follows that

$$\{-1, 1\} \subset \sigma_{\pi(A)}\big(\pi(ax - xa)\big)$$

which holds true by similar reasons as in the previous arguments. Hence, as in the first part of the proof,

$$\{-1, 1\} \subset \sigma_{\pi(A)}\big(\pi(ax - xa)\big) \subset \sigma_A(ax - xa),$$

which is a contradiction with our hypothesis, and thus $\pi(a)$ is of algebraic degree one. So, by definition

$$\alpha_0\mathbf{1} + \alpha_1\pi(a) = 0,$$

that is

$$\pi(a) = -\alpha_0/\alpha_1\mathbf{1} = \alpha\mathbf{1}$$

with $-\alpha_0/\alpha_1 = \alpha \in \mathbb{C}$. Hence $\pi(ax - xa) = 0$ and by Definition 1.5.6 we have that $ax - xa \in \mathrm{Rad}\, A$, that is $a \in Z(A)$.

Q.E.D.

By Theorem 1.2.4 (e) we know that if A is commutative then the spectral radius is subadditive and submultiplicative. In Theorem 4.2, if we assume that $a \in Z(A)$, it follows that the result is reminiscent to that of Theorem 1.2.4 (e), but we also have the following equivalences:

Theorem 4.2
([AUP1]; Theorem 5.2.2, p. 93)

Let A be a Banach algebra with $a \in A$. Then the following properties are equivalent:

(a) $a \in Z(A)$,
(b) *there exists $M > 0$ such that*

$$\rho(a + x) \leq M(1 + \rho(x))$$

for every $x \in A$,
(c) *there exists $N > 0$ such that*

$$\rho((a - \lambda\mathbf{1})^{-1}x) \leq N\rho((a - \lambda\mathbf{1})^{-1})\rho(x)$$

for every $x \in A$ and $\lambda \notin \sigma(a)$.

Proof:

Considering $Z(A) = \{a \in A : a + \text{Rad } A \in Z(A/\text{Rad } A)\}$ and changing A for $A/\text{Rad}\,A$ if necessary, we may suppose without loss of generality that A is semi-simple.

(a) $\Rightarrow$ (b): Let $a \in Z(A)$ be fixed. Since A is semi-simple it follows by the definition of $Z(A)$ that $ax = xa$ for all $x \in A$. We consider the following two possibilities:
If $0 \leq \rho(a) < 1$, then by Theorem 1.2.4 (e)

$$\rho(a + x) \leq \rho(a) + \rho(x) \leq 1 + \rho(x).$$

Suppose $\rho(a) > 1$. Since $\sigma(a)$ is bounded there exists $M > 0$ such that $1 < \rho(a) \leq M$. Thus $\rho(a + x) \leq \rho(a) + \rho(x) \leq M + M\rho(x)$, that is,

$$\rho(a + x) \leq M(1 + \rho(x)).$$

Choosing $M \geq \max(1, \rho(a))$, it follows that $\rho(a + x) \leq M(1 + \rho(x))$.

(a) $\Rightarrow$ (c): Again let $a \in Z(A)$. Then $ax = xa$ for all $x \in A$. In particular,

$(a-\lambda\mathbf{1})x = x(a-\lambda\mathbf{1})$ and consequently, for $\lambda \notin \sigma(a)$, $(a-\lambda\mathbf{1})^{-1}x = x(a-\lambda\mathbf{1})^{-1}$. Thus, by Theorem 1.2.4 (e)

$$\rho((a-\lambda\mathbf{1})^{-1}x) \leq \rho((a-\lambda\mathbf{1})^{-1})\rho(x).$$

Hence, for $N \geq 1$ we have

$$\rho\big((a-\lambda\mathbf{1})^{-1}x\big) \leq N\rho\big((a-\lambda\mathbf{1})^{-1}\big)\rho(x).$$

(b) $\Rightarrow$ (a): Let $u \in A$ fixed and $\lambda \in \mathbb{C}$. Consider the function f from $\mathbb{C}$ into A defined as

$$f(\lambda) = \frac{a - e^{\lambda u}ae^{-\lambda u}}{\lambda}, \quad \text{for } \lambda \neq 0,$$

and

$$f(\lambda) = [a,u], \quad \text{for } \lambda = 0,$$

where $[a,u] = au - ua$.
We first show that f is continuous at $\lambda = 0$. Considering

$$\lim_{\lambda\to 0} f(\lambda) = \lim_{\lambda \to 0}\left(\frac{a - e^{\lambda u}ae^{-\lambda u}}{\lambda}\right),$$

it follows by L'Hôpital's Rule for Banach algebra elements that

$$\lim_{\lambda\to 0} f(\lambda) = \lim_{\lambda\to 0}\bigg(-\frac{d}{d\lambda}(e^{\lambda u}ae^{-\lambda u})\bigg).$$

But $-\frac{d}{d\lambda}(e^{\lambda u}ae^{-\lambda u}) = -e^{\lambda u}(u)ae^{-\lambda u} - e^{\lambda u}ae^{-\lambda u}(-u)$. Hence

$$\lim_{\lambda\to 0}\left(e^{\lambda u}ae^{-\lambda u}(u) - e^{\lambda u}(u)ae^{-\lambda u}\right) = au - ua = [a,u].$$

Thus $\lim_{\lambda\to 0} f(\lambda) = f(0)$ and f is continuous at $\lambda = 0$.

Further let φ be a bounded linear functional on A. Clearly, from the continuity of f at 0,

$$\lim_{\lambda\to 0}\ (\lambda)\varphi(f(\lambda)) = 0,$$

and hence $\lambda = 0$ is a removable singularity of the function $\varphi \circ f : \mathbb{C} \to \mathbb{C}$ (by Lemma 3.27). Thus, the latter implies $\varphi \circ f$ is holomorphic on $\mathbb{C}$. Consequently, we have that f is analytic on $\mathbb{C}$ (cf. Chapter 1), and hence by Theorem 3.7, $\lambda \mapsto \rho(f(\lambda))$ is subharmonic on $\mathbb{C}$.
Now for $\lambda \neq 0$

$$\rho(f(\lambda)) = \rho\left(\frac{a - e^{\lambda u}ae^{-\lambda u}}{\lambda}\right) = \frac{1}{|\lambda|}\Big(\rho(a - e^{\lambda u}ae^{-\lambda u})\Big),$$

and it follows by our hypothesis that

$$\rho(f(\lambda)) \leq \frac{M}{|\lambda|}\Big(1 + \rho(e^{\lambda u} a e^{-\lambda u})\Big),$$

for some $M > 0$. Further, by Theorem 1.2.3 (a), $\rho(a) = \rho(ae^{-\lambda u}e^{\lambda u}) = \rho(e^{\lambda u}ae^{-\lambda u})$, and hence

$$\rho(f(\lambda)) \leq \frac{M}{|\lambda|}\Big(1 + \rho(a)\Big).$$

Now, consider the ball $\overline{B}(0,r)$ in $\mathbb{C}$. Clearly $\lambda \mapsto ||f(\lambda)||$ is continuous on $\overline{B}(0,r)$ and hence it assumes its maximum on $\overline{B}(0,r)$. It then follows by the above inequality and since $\rho(a) \leq ||a||$, that $\lambda \mapsto \rho(f(\lambda))$ is bounded on $\overline{B}(0,r)$, and consequently on the whole of $\mathbb{C}$. Hence by the Maximum Principle of Subharmonic Functions (Theorem 2.2.3) it follows that $\rho(f(\lambda))$ is constant on $\mathbb{C}$. Thus, since $\rho(f(\lambda)) \to 0$ as $|\lambda| \to \infty$, we have that $\rho(f(\lambda)) = 0$ for all $\lambda \in \mathbb{C}$. In particular for $\lambda = 0$, $\rho([a,u]) = 0$. Hence $\#\sigma(au - ua) = 1$ and $a \in Z(A)$ by Theorem 4.1.

(c) $\Rightarrow$ (b): Assume there exists $N > 0$ such that

$$\rho\big((a - \lambda\mathbf{1})^{-1}x\big) \leq N\rho\big((a - \lambda\mathbf{1})^{-1}\big)\rho(x)$$

for every $x \in A$ and $\lambda \notin \sigma(a)$. We claim that the above implies

$$\sigma(y) \subset \sigma(a) + N\rho(y - a)$$

for all $y \in A$. If this is not the case, then by Definition 3.1, there exists $y \in A$ and $\alpha \in \sigma(y)$ such that

$$N\rho(y - a) < \operatorname{dist}(\alpha, \sigma(a)).$$

Since, by Theorem 1.3.3

$$\operatorname{dist}(\alpha, \sigma(a)) = \frac{1}{\rho((a - \alpha\mathbf{1})^{-1})},$$

it follows that

$$N\rho\big(a - \lambda\mathbf{1}\big)^{-1})\rho(y - a) < 1.$$

Thus, by our hypothesis, it must hold that

$$\rho\bigg((a - \lambda\mathbf{1})^{-1}(y - a)\bigg) < 1$$

for all $y \in A$ and $\lambda \notin \sigma(a)$. In particular, choosing $y = 2a - \lambda\mathbf{1}$ we have that $\rho(\mathbf{1}) < 1$. But this is contradictive, and hence $\sigma(y) \subset \sigma(a) + N\rho(y - a)$ for all $y \in A$. Clearly then

$$\rho(y) \leq \rho(a) + N\rho(y - a).$$

Letting $y = a + x$ (for $x \in A$ arbitrary) we have that $\rho(a+x) \leq \rho(a) + N\rho(x)$. Choosing $M \geq \max(\rho(a), N)$ it follows for all $x \in A$ that

$$\rho(a+x) \leq M(1+\rho(x))$$

which is the desired result.
Q.E.D.

In Theorem 3.24 we showed that $\lambda \mapsto \delta(f(\lambda))$ is subharmonic on $\mathbb{C}$, which is used in the proof of the following Theorem 4.3:

Theorem 4.3
([AUP1]; Theorem 5.2.4, p. 94)

Let A be a Banach algebra. Then

$$a \in Z(A) \quad \textit{if and only if} \quad \sup_{x \in A} \delta(e^x a e^{-x} - a) < +\infty.$$

Proof:

Considering $Z(A) = \{a \in A : a + \text{Rad } A \in Z(A/\text{Rad } A)\}$ we may suppose without loss of generality that A is semi-simple. Let $a \in Z(A)$. Then $ab = ba$ for all $b \in A$. Now, it is easy to show that for $x \in A$ arbitrary

$$\rho(x) \leq \delta(x) \leq 2\rho(x).$$

Also, if $xy = xy$, then it is well-known that

$$\delta(x+y) \leq \delta(x) + \delta(y),$$

(cf. [AUP1]; p. 66).
Thus, since $a \in Z(A)$,

$$\delta(e^x a e^{-x} - a) \leq \delta(e^x a e^{-x}) + \delta(a),$$

and hence

$$\delta(e^x a e^{-x} - a) \leq 2\big(\rho(e^x a e^{-x}) + \rho(a)\big).$$

But, by Theorem 1.2.3 (a) $\rho(e^x a e^{-x}) = \rho(e^{-x} e^x a)$, thus

$$\delta(e^x a e^{-x} - a) \leq 4\rho(a)$$

for all $x \in A$, and consequently

$$\sup_{x \in A} \delta(e^x a e^{-x} - a) < +\infty.$$

For the converse let $u \in A$ be fixed and $\lambda \in \mathbb{C}$. Further, consider the analytic function $f : \mathbb{C} \mapsto A$, as in Theorem 4.2, where

$$f(\lambda) = \frac{a - e^{\lambda u} a e^{-\lambda u}}{\lambda}, \quad \text{for } \lambda \neq 0,$$

and

$$f(\lambda) = [a, u] = au - ua, \quad \text{for } \lambda = 0.$$

Now by Theorem 3.24, we know that $\lambda \mapsto \delta(f(\lambda))$ is subharmonic on $\mathbb{C}$, where for $\lambda \neq 0$,

$$\delta(f(\lambda)) = \frac{1}{|\lambda|}\delta(a - e^{\lambda u} a e^{-\lambda u}).$$

But, by our hypothesis, there exists $0 < k < +\infty$ such that

$$\delta(a - e^{\lambda u} a e^{-\lambda u}) \leq k.$$

Thus

$$\delta(f(\lambda)) \leq \frac{k}{|\lambda|}.$$

Consequently, by similar reasons as in Theorem 4.2, we have that $\lambda \mapsto \delta(f(\lambda))$ is bounded on $\mathbb{C}$. Hence, it follows by the Maximum Principle for Subharmonic Functions (Theorem 2.2.3) that the latter mapping is constant on $\mathbb{C}$. In particular, for $\lambda = 0$, we have $\delta(au - ua) = 0$. Thus $\#\sigma(au - ua) = 1$, and by Theorem 4.1 $a \in Z(A)$.
Q.E.D.

Theorem 4.4 Spectral Characterization of the Radical (J. Zemánek) ([AUP1]; Theorem 5.3.1, p. 95)

Let A be a Banach algebra. Then the following properties are equivalent:

(a) *a is in the Jacobson radical of A,*
(b) *$\sigma(a + x) = \sigma(x)$, for all $x \in A$,*
(c) *$\rho(a + x) = 0$, for all quasi-nilpotent elements x in A,*
(d) *$\rho(a + x) = 0$, for all quasi-nilpotent elements x in a neighborhood of $0 \in A$,*
(e) *there exists $C > 0$ such that $\rho(x) \leq C||x - a||$, for all x in a neighborhood of $a \in A$.*

Proof:

(a) $\Rightarrow$ (b): Denote by $\mathcal{I}$ the radical of A. By the discussion following Corollary 1.5.8, Chapter 1, we had that $\sigma_A(x) = \sigma_{A/\mathcal{I}}(\widehat{x})$ for all $x \in A$. Let $a \in \mathcal{I}$. Then it follows by repeated use of the latter equality that

$$\sigma_A(a+x) = \sigma_{A/\mathcal{I}}(a+x+\mathcal{I}) = \sigma_{A/\mathcal{I}}(x+a+\mathcal{I}) = \sigma_{A/\mathcal{I}}(x+\mathcal{I}) = \sigma_A(x)$$

for each $x \in A$ and $a \in \operatorname{Rad} A = \mathcal{I}$.

(b) $\Rightarrow$ (e) and consequently (a) $\Rightarrow$ (e): Clearly, if $a \in \mathcal{I}$ then $-a \in \mathcal{I}$, and hence from (a) $\Rightarrow$ (b) we have that $\sigma(x-a) = \sigma(x)$ for each $x \in A$. In particular, for the maximum spectral values, we have that $\rho(x-a) = \rho(x)$ for each $x \in A$. Since $\rho(x-a) \leq ||x-a||$ it follows that $\rho(x) \leq ||x-a||$ for each $x \in A$ and hence in particular for all x in a neighborhood of $a \in A$.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are both obvious and consequently (b) $\Rightarrow$ (d) holds as well.

(c) $\Rightarrow$ (a): Assume $\rho(a+x) = 0$ for all $x \in Q_A$ where Q_A denotes the set of all quasi-nilpotent elements in A (cf. §1.5). Letting $x = 0$ we have $\rho(a) = 0$. Thus, it follows that $\rho(a) = \rho(ae^{-x}e^x) = 0$ and hence by Theorem 1.2.3 (a)

$$\rho(e^x a e^{-x}) = 0 \text{ for each } x \in A.$$

The latter implies that $e^x a e^{-x} \in Q_A$ and hence, by our assumption,

$$\rho(a - e^x a e^{-x}) = 0 \text{ for each } x \in A.$$

Since $\delta(x) \leq 2\rho(x)$ and $\sup_{x \in A} \rho(a - e^x a e^{-x}) < +\infty$, we have that

$$\sup_{x \in A} \delta(a - e^x a e^{-x}) < +\infty.$$

Thus by Theorem 4.3 $a \in Z(A)$, that is, $ax - xa \in \mathcal{I} = \operatorname{Rad} A$.
Now, let π be an arbitrary continuous irreducible representation of A. Then it follows by Definition 1.5.6 that $\pi(ax - xa) = 0$, that is, $\pi(a)\pi(x) = \pi(x)\pi(a)$ for all $x \in A$. Consequently by Schur's Theorem (Theorem 1.5.2)

$$\pi(a) = \alpha I, \quad \text{for } \alpha \in \mathbb{C}.$$

Moreover, since $\rho(a) = 0$ and $\sigma(\pi(a)) \subset \sigma(a)$, it follows by the continuity of π that $\rho(\pi(a)) = 0$, that is, $\rho(\alpha I) = 0$. Thus $\alpha = 0$, implying $\pi(a) = 0$, and it follows by Definition 1.5.6 that $a \in \operatorname{Rad} A$.

(d) $\Rightarrow$ (c): Assume $\rho(a+x) = 0$ for each $x \in Q_A$ with $||x-0|| = ||x|| < r$. Let $q \neq 0$ be an arbitrary element of Q_A. Clearly $\lambda q \in Q_A$ for $\lambda \in \mathbb{C}$. If $|\lambda| < r/||q||$, then

$$||\lambda q|| = |\lambda| \cdot ||q|| < r.$$

Thus by our hypothesis

$$\rho(a + \lambda q) = 0, \quad \text{for } |\lambda| < r/||\lambda||.$$

Since $\lambda \mapsto a + \lambda q$ is analytic for all $\lambda \in \mathbb{C}$, it follows by Theorem 3.7 that $\varphi(\lambda) = \log(\rho(a + \lambda q))$ is subharmonic on $\mathbb{C}$.

Now $\rho(a + \lambda q) = 0$ implies that $\varphi(\lambda) = \log(\rho(a + \lambda q)) = -\infty$ for $|\lambda| < r/||\lambda||$. Hence by Cartan's Theorem (Theorem 2.3.4) it follows that $\varphi(\lambda) \equiv -\infty$ for all $\lambda \in \mathbb{C}$. If this was not the case, then by the latter theorem, the set

$$F = \{\lambda \in \mathbb{C} : |\lambda| < r/||q||\}$$

has capacity zero. Clearly this is a contradiction by application of Theorem 2.3.5 to F. Thus $\varphi(\lambda) \equiv -\infty$ and it follows that $\rho(a + \lambda q) = 0$ for all $\lambda \in \mathbb{C}$ and $q \in Q_A$ arbitrary. In particular, for $\lambda = 1$, we then obtain $\rho(a + q) = 0$.

(e) $\Rightarrow$ (d): Assume there exists a $C > 0$ such that $\rho(x) \leq C||x - a||$ for all x in a neighborhood of $a \in A$ and let $q \in Q_A$.

Consider the analytic function $\lambda \mapsto \lambda a + q$ from $\mathbb{C}$ into A. Then, by Theorem 3.7, $\lambda \mapsto \rho(f(\lambda))$ is subharmonic on $\mathbb{C}$. Further consider the element $a + q/\lambda$ which is in a neighborhood of a for $|\lambda|$ sufficiently large. This holds true since

$$||(a + q/\lambda) - a|| = ||q/\lambda|| = ||q||/|\lambda| \to 0,$$

as $|\lambda| \to \infty$. Hence by our hypothesis, there exists $C > 0$ such that

$$\rho(\lambda a + q) = |\lambda| \cdot \rho(a + q/\lambda) \leq C||q||,$$

for $|\lambda|$ large enough.

Thus, $\lambda \to \rho(a+\lambda q)$ is bounded on $\mathbb{C}$ and hence constant by the Maximum Principle for Subharmonic Functions (Theorem 2.2.3). Consequently $\rho(a + q) = \rho(q) = 0$.
Q.E.D.

Theorem 4.5
([AUP1]; Theorem 5.3.2, p. 96)

Let A be a semi-simple Banach algebra and $a \in A$. Then there exist $\alpha \in \mathbb{C}$ such that

$$a = \alpha \mathbf{1} \quad \textit{if and only if} \quad \#\sigma(a+q) = 1 \textit{ for all } q \in Q_A.$$

Proof:

If $a = \alpha\mathbf{1}$, then

$$\sigma(a+q) = \sigma(\alpha\mathbf{1}+q) = \{\alpha\} + \sigma(q).$$

Since $\sigma(q) = \{0\}$ we have

$$\sigma(a+q) = \{\alpha\} + \{0\} = \{\alpha\},$$

for all $q \in Q_A$.

For the converse assume that $\#\sigma(a+q) = 1$ for all $q \in Q_A$. Also consider the analytic function f from $\mathbb{C}$ into A where $f(\lambda) = a + \lambda q$. Then, by our assumption for each $\lambda \in \mathbb{C}$, we have that

$$\sigma(f(\lambda)) = \sigma(a+\lambda q) = \{h(\lambda)\},$$

where h is an entire function by Corollary 3.18.

Since $h(\lambda) \in \sigma(a+\lambda q)$, it follows for $\lambda \neq 0$ that

$$\left|\frac{h(\lambda)}{\lambda}\right| \;\leq\; \rho(a/\lambda + q),$$

and thus

$$\limsup_{|\lambda|\to\infty} \left|\frac{h(\lambda)}{\lambda}\right| \;\leq\; \limsup_{|\lambda|\to\infty} \rho(a/\lambda + q).$$

Letting $1/\lambda = \gamma$ we have

$$\limsup_{|\lambda|\to\infty} \rho(a/\lambda + q) = \limsup_{|\gamma|\to 0} \rho(\gamma a + q),$$

and by application of Theorem 2.2.2 (g) it follows that

$$\limsup_{|\gamma|\to 0} \rho(\gamma a + q) = \rho(q).$$

Thus

$$\limsup_{|\lambda|\to\infty} \left|\frac{h(\lambda)}{\lambda}\right| \leq \rho(q) = 0, \quad \text{for } q \in Q_A.$$

Further, since h is an entire function, it follows that

$$h(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n,$$

which converges for all $\lambda \in \mathbb{C}$, and

$$h(\lambda)/\lambda = \alpha_0/\lambda + \sum_{n=1}^{\infty} \alpha_n \lambda^{n-1}.$$

Now, let

$$g(\lambda) = \sum_{n=1}^{\infty} \alpha_n \lambda^{n-1}.$$

Since $\limsup_{|\lambda|\to\infty} |h(\lambda)/\lambda| = 0$, it follows that

$$\limsup_{|\lambda|\to\infty} |g(\lambda)| = 0,$$

and hence g is bounded. So, by Liouville's Theorem for entire functions, $g(\lambda)$ is a constant, and consequently g is precisely zero. Thus $\alpha_n = 0$ for each $n \in \{1, 2, 3, \ldots\}$. Hence $h(\lambda) = \alpha_0 = \alpha$ (say), and h is constant.

Thus $\sigma(f(\lambda)) = \sigma(a + \lambda q) = \{h(\lambda)\}$ is a constant set for all $\lambda \in \mathbb{C}$, and consequently

$$\sigma(a + \lambda q) = \sigma(a), \quad \text{for all } \lambda \in \mathbb{C}.$$

Since $\#\sigma(a + q) = 1$ it follows from the above equality that $\sigma(a) = \{\alpha\}$, implying that $a = \alpha\mathbf{1} + q_0$ for some $q_0 \in Q_A$ and $\alpha \in \mathbb{C}$. Thus

$$\sigma(q_0 + q) = \sigma(a - \alpha\mathbf{1} + q),$$

from which it follows by application of the Spectral Mapping Theorem that

$$\sigma(q_0 + q) = \sigma(a + q) - \{\alpha\} = \{0\}.$$

Since $q \in Q_A$ arbitrary, it follows by Theorem 4.4 (c) that $q_0 \in$ Rad$A$. Hence, since A is semi-simple, $q_0 = 0$ and thus $a = \alpha\mathbf{1}$.
Q.E.D.

Theorem 4.6 is another spectral characterization of the radical and it uses the fact that if a is in the Jacobson radical, then $\sigma(a + x) = \sigma(x)$ for all $x \in A$ (cf. Theorem 4.4).

Theorem 4.6 Spectral Characterization of the Radical
([AUP2]; p. 31)

Let a be an element of a Banach algebra A. Then a is in the Jacobson radical of A if and only if for every $x \in A$

$$\sup\{\rho(x+ta) : t \in \mathbb{C}\} < +\infty.$$

<u>Proof:</u>

Let a be in the Jacobson radical of A. Then, by Theorem 4.4 (b), $\rho(x+ta) = \rho(x)$ for all $t \in \mathbb{C}$ and $x \in A$ arbitrary. Since $\rho(x) < +\infty$, it follows that $\sup\{\rho(x+ta) : t \in \mathbb{C}\} < +\infty$ for all $x \in A$.

For the converse, consider for each $x \in A$ the functions $t \mapsto \rho(x+ta)$ on $\mathbb{C}$. Clearly, by Theorem 3.7 each of these functions are subharmonic. Moreover, since $t \mapsto \rho(x+ta)$ is bounded, it is then constant on $\mathbb{C}$ by Liouville's Theorem for Subharmonic Functions (Theorem 2.2.4). In particular, $\rho(x+ta) = \rho(x)$ for all $x \in A$ and $t \in \mathbb{C}$.

Let $|\mu| > \rho(x)$. Then $\mu\mathbf{1} - x \in G(A)$. Since $\rho(x+ta) = \rho(x)$, it follows that $\mu\mathbf{1} - (x+ta) \in G(A)$, for $t \in \mathbb{C}$. For $t \in \mathbb{C}$ consider the relation

$$\begin{aligned} \mu\mathbf{1} - (x+ta) &= (\mu\mathbf{1} - x)\Big(\mathbf{1} - (\mu\mathbf{1} - x)^{-1}ta\Big) \\ &= (\mu\mathbf{1} - x)t\left(\frac{\mathbf{1}}{t} - (\mu\mathbf{1} - x)^{-1}a\right). \end{aligned}$$

Since $G(A)$ is a group it follows that $1/t \notin \sigma\big((\mu\mathbf{1} - x)^{-1}a\big)$, and hence, since the spectrum is non-empty,

$$\rho\big((\mu\mathbf{1} - x)^{-1}a\big) = 0.$$

Let $y \in A$ be arbitrary and $\lambda \in \mathbb{C}$ and such that $2\rho(y) < |\lambda|$, thus $\rho(y) < |\lambda|$. We now show that for each $y \in A$ there exists a corresponding $x \in A$ such that if $2\rho(y) < |\lambda|$, then $y - \lambda\mathbf{1} = (\mu\mathbf{1} - x)^{-1}$, or equivalently $(y - \lambda\mathbf{1})^{-1} = \mu\mathbf{1} - x$, where $|\mu| > \rho(x)$.

Since $\rho(y) < |\lambda|$, we can write

$$(y - \lambda\mathbf{1})^{-1} = -\frac{1}{\lambda}\sum_{j=0}^{\infty}\left(\frac{y}{\lambda}\right)^j.$$

To show that this series expansion exists we consider the following series:

$$\sum_{j=0}^{\infty} \left\| \left(\frac{y}{\lambda}\right)^j \right\|.$$

Using the root test, and the fact that $\rho(y) < |\lambda|$ we have

$$\lim_{j\to\infty} \left\| \left(\frac{y}{\lambda}\right)^j \right\|^{1/j} = \frac{1}{|\lambda|} \lim_{j\to\infty} ||y^j||^{1/j} = \frac{\rho(y)}{|\lambda|} < 1.$$

Hence,

$$\sum_{j=0}^{\infty} \left\| \left(\frac{y}{\lambda}\right)^j \right\|$$

converges. Consequently, since absolute convergence implies convergence in a Banach algebra,

$$(y - \lambda \mathbf{1})^{-1} = -\frac{1}{\lambda} \sum_{j=0}^{\infty} \left(\frac{y}{\lambda}\right)^j,$$

is convergent in A ([KRE]; p. 68), and

$$(y - \lambda \mathbf{1})^{-1} = -\frac{\mathbf{1}}{\lambda} - \frac{y}{\lambda^2} - \frac{y^2}{\lambda^3} - \frac{y^3}{\lambda^4} - \dots$$

Letting $\mu = -1/\lambda$ and

$$x = \frac{y}{\lambda^2} + \frac{y^2}{\lambda^3} + \frac{y^3}{\lambda^4} + \dots$$

it follows that $(y - \lambda \mathbf{1})^{-1} = \mu \mathbf{1} - x$, and hence

$$y - \lambda \mathbf{1} = (\mu \mathbf{1} - x)^{-1} = \left(-\frac{\mathbf{1}}{\lambda} - \left(\frac{y}{\lambda^2} + \frac{y^2}{\lambda^3} + \frac{y^3}{\lambda^4} + \dots \right) \right)^{-1}.$$

We show that $\rho(x) < 1/|\lambda|$. Note that $x = \frac{y}{\lambda^2} + \frac{y^2}{\lambda^3} + \dots$ can be written in closed form as $x = \frac{y}{\lambda}(\lambda \mathbf{1} - y)^{-1}$. Hence it follows by Theorem 1.3.3 that

$$\rho(x) \le \rho\left(\frac{y}{\lambda}\right) \rho((\lambda \mathbf{1} - y)^{-1}) = \frac{\rho(y)}{|\lambda| \text{dist}(\lambda, \sigma(y))}.$$

But since $2\rho(y) < |\lambda|$ we have that $\rho(y) < \text{dist}(\lambda, \sigma(y))$ from which the desired result follows immediately.
Consequently we have $\rho(x) < \frac{1}{|\lambda|} = |\mu|$ and moreover that $\rho((y - \lambda \mathbf{1})a) = 0$ for all λ with $2\rho(y) < |\lambda|$. Let $f(\lambda) = (y - \lambda \mathbf{1})a$ with $y \in A$ arbitrary. Then it follows that

there is a neighborhood U in $\mathbb{C}$ such that $\sigma(f(\lambda)) = \{0\}$ for all $\lambda \in U$. But, since f is analytic on $\mathbb{C}$, the Scarcity Theorem implies that $\#\sigma(f(\lambda)) = 1$ for all $\lambda \in \mathbb{C}$. In particular, if $\lambda = 0$ we obtain $\#\sigma(ya) = 1$ for all $y \in A$. Further, one can now easily show that $a \notin G(A)$, from which it follows that $\sigma(ya) = \{0\}$ for all $y \in A$, and hence $a \in \text{Rad } A$.
Q.E.D.

Chapter 5
Jordan-Banach Algebras

A Jordan-Banach algebra is a Jordan algebra equipped with a norm such that the Jordan algebra is also a Banach space. To understand what is meant by this definition we first need to consider the structure of Jordan algebras. This includes a study in quadratic operators and quadratic Jordan algebras, invertibility and quasi-invertibility, and the McCrimmon radical of a Jordan algebra.

Jordan Algebras

Jordan algebras were introduced in 1934 by P. Jordan, J von Neumann and E. Wigner in order to generalize the quantum mechanics formalism. The outcome of all this experimentation was a distillation of the algebraic essence of quantum mechanics into an axiomatically defined algebraic system (cf. [MCR3]; p. 3) and which led to the following definition:

Definition 5.1
([JAC]; p. 6)

A Jordan algebra J is a non-associative algebra over $\mathbb{C}$ with product composition $x \bullet y$, called a Jordan product, satisfying

$$\text{(a)}\quad x \bullet y = y \bullet x \quad \textit{and} \quad \text{(b)}\quad (x^2 \bullet y) \bullet x = x^2 \bullet (y \bullet x)$$

where $x^2 = x \bullet x$.

The identity (b) in the above definition is referred to as the **Jordan identity**, and can be seen as weak form of associativity.
We shall always assume that J is **unital**, that is, there exists an identity element (also called unit) $\mathbf{1} \in J$ such that $x \bullet \mathbf{1} = x$ for all $x \in J$. If no such identity element exists we can always imbed J into a unital Jordan algebra, called its **unital hull** $J' = \Phi\mathbf{1} + J$ with Φ the complex field ([HOG]; p. 156).

Since Jordan algebras are non-associative it follows for $i \neq j$, with $i, j \in \{1, 2, 3\}$, that $a_1 \bullet a_2 \bullet a_3$ will stand for $(a_1 \bullet a_2) \bullet a_3$. In general we have

$$a_1 \bullet a_2 \bullet \ldots \bullet a_n = (\ldots((a_1 \bullet a_2) \bullet a_3) \ldots \bullet a_n).$$

Although Jordan algebras are non-associative, they still have the property of power associativity. A Jordan algebra is **power associative** if the subalgebras generated by the single elements of the algebra are associative. This is equivalent to the identities $a^k \bullet a^l = a^{k+l}$ with $k, l = 1, 2, \ldots, n$, if a^k is defined by $a^1 = a$ and $a^k = a^{k-1} \bullet a$ ([JAC]; p. 33, 36).

One of the most important Jordan product compositions in a Jordan algebra J is the **Jordan triple product** $\{abc\}$ where

$$\{abc\} = (a \bullet b) \bullet c + (b \bullet c) \bullet a - (a \bullet c) \bullet b$$

for $a, b, c \in J$ ([JAC]; p. 36).
The Jordan triple product is of great importance when defining quadratic operators on Jordan algebras. A **quadratic operator** (or just U-operator) $U_{a,b}$ is a linear mapping $x \mapsto \{axb\}$ where $U_{a,b}(x) = \{axb\}$. A standard notation which is also used is $U_{a,b} = a \circ b$.
It is also convenient to define $U_{a,b}$ in terms of the linear mapping R_a on J, where $R_a(x) = a \bullet x$ $(= x \bullet a)$ for $x, a \in J$, and from which we have the following identity

$$U_{a,b} = R_a R_b + R_b R_a - R_{a \bullet b} \quad (a, b \in J).$$

Using the abbreviation $U_{a,a} = U_a$ we obtain the important **Identity 5.2**:

$$U_a = 2R_a^2 - R_{a^2}$$

([JAC]; p. 36).

Most authors refer to a Jordan algebra defined by quadratic operators as a quadratic Jordan algebra. K. McCrimmon introduced the method of using these operators to obtain new and previously unknown results on general Jordan algebras. This resulted into a more uniform method in the investigation of Jordan algebras, making the studies of these algebras more manageable.

Anyone who undertakes an in depth study of Jordan algebras will realize there are a number of important identities ([MCR3]; p. 202). One identity that is of great importance to us is the so-called Fundamental Formula:

Theorem 5.3
([MCR3]; p. 202)

Every Jordan algebra J satisfies the **Fundamental Formula**:

$$U_{U_x(y)} = U_x U_y U_x$$

for all $x, y \in J$.

Quadratic operators are also used in the characterizations of invertible elements in Jordan algebras, but first a definition.

Definition 5.4
([MCR3]; p. 70)

Let J be a Jordan algebra with identity $\mathbf{1}$. Then an element x of J is invertible, with inverse y (denoted by x^{-1}) if and only if

$$\text{(a) } x \bullet y = \mathbf{1} \quad \textit{and} \quad \text{(b) } x^2 \bullet y = x.$$

Theorem 5.5
([MCR3]; p. 216)

Let J be a Jordan algebra with identity $\mathbf{1}$ and $x \in J$. Then x^n, $n \geq 1$, is invertible if and only if x is invertible, in which case

$$(x^n)^{-1} = (x^{-1})^n.$$

For the above theorem we shall use the notation x^{-n} instead of $(x^n)^{-1} = (x^{-1})^n$. Moreover it follows by Definition 5.4 and Theorem 5.5 that power associativity also holds true for negative powers of elements in J ([MCR3]; p. 216).

Invertibility can also be defined by means of U-operators, as given by the following Definition 5.6, which is equivalent to Definition 5.4 (cf. [MCR3]; p. 215).

Definition 5.6
([MCR3]; p. 211, 212)

An element x of a unital Jordan Algebra is invertible if it has inverse y (denoted by x^{-1}) satisfying both the conditions

$$\text{(a) } U_x(y) = x \quad \textit{and} \quad \text{(b) } U_x(y^2) = \mathbf{1}.$$

The following theorem shows the usefulness of U-operators with respect to characterizations of invertible elements.

Theorem 5.7
([MCR3]; p. 212, 363)

For x an element of a unital Jordan algebra J the following conditions are equivalent:

(a) *the element x is invertible in J,*

(b) *the operator* U_x *is invertible on* J,
(c) *the operator* U_x *is surjective on* J, *that is,* $U_x(J) = J$,
(d) *the identity* $\mathbf{1}$ *lies in the range of* U_x, *that is,* $\mathbf{1} \in U_x(J)$,
(e) $U_x(J)$ *contains an invertible element.*

Properties of U-operators with respect to inverses are as follows:

Theorem 5.8
([MCR3]; p. 212, 216)

Let J *be a unital Jordan algebra. If* x *is invertible, then write* $U_x^{-1} = (U_x)^{-1}$ *and* $U_x^{-n} = (U_x)^{-n}$ *for* $n \geq 1$.

(a) *If* x *is invertible in* J *its inverse is uniquely determined by*

$$x^{-1} = U_x^{-1}(x).$$

(b) *The* U*-operator of the inverse is the inverse of the original* U*-operator, that is,*

$$U_{x^{-1}} = U_x^{-1}.$$

(c) $U_x(y)$ *is invertible if and only if* x *and* y *are invertible, where*

$$\Big(U_x(y)\Big)^{-1} = U_{x^{-1}}(y^{-1}).$$

(d) U_x *and* $U_{x^{-1}}$ *are each others inverses.*

The McCrimmon Radical of a Jordan Algebra

Now that we have the definition of invertibility we can define the Jacobson radical of a Jordan algebra, which is formulated in terms of quasi-invertibility.

If J is an arbitrary unital Jordan algebra and $x \in J$, then we say that x is **quasi-invertible** if $\mathbf{1} - x$ is invertible in J. That is, there exists $y \in J$ such that $\mathbf{1} - y = (\mathbf{1} - x)^{-1}$ is the inverse of $\mathbf{1} - x$, where y is called the **quasi-inverse** of x. Further, it is clear that quasi-nilpotency implies quasi-invertibility. The reader should note that both equations (a) and (b) of Definition 5.4 are satisfied by $\mathbf{1} - x$ and $\mathbf{1} - y$ ([JAC]; p. 55 and [MCR3]; p. 366). We shall say that an ideal is quasi-invertible if each of its elements is quasi-invertible ([MCR3]; p. 367).

In [AUP2], the author defines the **McCrimmon radical** of a Jordan algebra J as the unique maximal quasi-invertible ideal of J. Clearly, by definition, the McCrimmon radical contains all quasi-invertible ideals. Further, a Jordan algebra is semi-simple if the McCrimmon radical is identical to the set only containing zero.

The following theorem is used in the spectral characterization of the McCrimmon radical.

Theorem 5.9
([MCR4]; Proposition 2, p. 672)

Let J be an arbitrary unital Jordan algebra. If z belongs to a quasi-invertible ideal of J and u is invertible, then $u - z$ is invertible.

Proof:

We first prove an auxiliary result:

$$u - z \textit{ is invertible if and only if } U_u^{-1}(u-z)^2 \textit{ is invertible.}$$

Let $u - z$ be invertible. Then, by Theorem 5.5, $(u-z)^2$ is invertible. Since u is invertible, it then follows by Theorem 5.8 (c), that $U_u(u-z)^2$ is invertible, and consequently $U_u^{-1}(u-z)^2$ is also invertible by Theorem 5.8 (b) and (c).
For the converse let $U_u^{-1}(u-z)^2$ be invertible. Since u is invertible it must then hold true, by Theorem 5.8 (c), that $(u-z)^2$ is invertible.

By using our auxiliary result we now show that if z belongs to a quasi-invertible ideal and u is invertible, then $U_u^{-1}(u-z)^2$ is invertible.
By Theorem 5.8 (b) we have

$$U_u^{-1}(u-z)^2 = U_{u^{-1}}(u-z)^2,$$

that is,

$$U_u^{-1}(u-z)^2 = U_{u^{-1}}\big((u-z) \bullet (u-z)\big) = U_{u^{-1}}\big(u^2 - 2u \bullet z + z^2\big).$$

So

$$U_u^{-1}(u-z)^2 = U_{u^{-1}}(u^2) - U_{u^{-1}}\big(2u \bullet z - z^2\big).$$

By Definition 5.6 (b), we have $U_{u^{-1}}(u^2) = \mathbf{1}$. Thus, letting $z' = U_{u^{-1}}\big(2u \bullet z - z^2\big)$, it follows that $U_u^{-1}(u-z)^2 = \mathbf{1} - z'$.
Now, since $z' = U_{u^{-1}}\big(2u \bullet z - z^2\big)$, and by Identity 5.2, $U_a = 2R_a^2 - R_{a^2}$,

$$z' = 2R_{u^{-1}}^2\big(2u \bullet z - z^2\big) - R_{(u^{-1})^2}\big(2u \bullet z - z^2\big),$$

and hence

$$z' = 2u^{-1} \bullet \big(u^{-1} \bullet \big(2u \bullet z - z^2\big)\big) - (u^{-1})^2 \bullet \big(2u \bullet z - z^2\big).$$

Since z belongs to a quasi-invertible ideal, it follows that z' also belongs to the same quasi-invertible ideal. Hence $\mathbf{1} - z' = U_u^{-1}(u - z)^2$ is invertible and it follows by our auxiliary result that $u - z$ is invertible.
Q.E.D.

Jordan-Banach Algebras

A unital **Jordan-Banach algebra** A is a complex Jordan algebra with product $a \bullet b$, having identity $\mathbf{1}$, and norm $||\cdot||$, such that A is a Banach space, $||\mathbf{1}|| = 1$, and for $a, b \in A$ we have

$$||a \bullet b|| \leq ||a|| \cdot ||b||$$

([YOU]; p. 261).

Just as in the associative case, the spectrum of an element a, denoted by $\sigma(a)$, is defined by

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \text{ is not invertible}\},$$

where $\sigma(a)$ is non-empty and compact ([AUP2]; p. 33, [AUP3]; p. 481, and [YOU]; p. 262).

Further, the mapping $x \mapsto \sigma(x)$ from A into a compact subset of $\mathbb{C}$, is upper semi-continuous, and $\lambda \mapsto (\lambda\mathbf{1} - x)^{-1}$ is analytic on the complement of $\sigma(x)$ ([AUP2]; p. 33, [MAO]; p. 3187, and [YOU]; p. 262).

The spectral radius of $a \in A$, denoted by $\rho(a)$, is defined by

$$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\},$$

where it also holds true that $\rho(a) = \lim_{n\to\infty} ||a^n||^{1/n} \leq ||a||$ ([AUP2]; p. 33, and [YOU]; p. 262).

The standard Holomorphic Functional Calculus in Banach algebras can also be extended to Jordan-Banach algebras. This comes from the fact that the closed sub-algebra generated by $\mathbf{1}$ and x is associative, and consequently a Banach algebra ([AUP3]; p. 481).

Theorem 5.10 Holomorphic Functional Calculus for Jordan-Banach Algebras
([AUP3]; Theorem 2.3, p. 481)

Let A be a Jordan-Banach algebra, x an element of A and Ω a neighborhood of $\sigma(x)$. If h is holomorphic on Ω, then we set

$$h(x) = \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda\mathbf{1} - x)^{-1}\, d\lambda,$$

where Γ is a positively orientated curve included in Ω and surrounding $\sigma(x)$. Further, we then have the following properties:

(a) *$h(x)$ is independent of the choice of Γ surrounding $\sigma(x)$ in Ω,*
(b) *$\phi : h \to h(x)$ is an algebraic homomorphism from $H(\Omega)$ into the smallest closed associative subalgebra containing $\mathbf{1}$ and x. Moreover, if $h(\lambda) = 1$, then $\phi(h) = \mathbf{1}$, and if $h(\lambda) = \lambda$, then $\phi(h) = x$,*
(c) *$\sigma(h(x)) = h(\sigma(x))$.*

Theorem 5.10 (c) shall also be referred to as the Spectral Mapping Theorem.

Since the standard Holomorphic Functional Calculus can be extended to Jordan-Banach algebras, it gives a sufficient condition under which we can define the roots of Jordan-Banach algebra elements. This result is similar to Theorem 1.4.1 (cf. Chapter 1, §1.4), and it states that if the spectrum of $x \in J$ does not separate 0 from ∞, then for every $n \geq 1$ there exists $z \in J$ such that $z^n = x$, that is, x has roots of all order in J. This result is is applied in Corollary 5.12.

The following identity is used in Theorem 5.11:

$$U_{x-\lambda\mathbf{1}} = U_x - 2\lambda R_x + \lambda^2 I.$$

To prove the above we use Identity 5.2, $U_x = 2R_x^2 - R_{x^2}$. Now

$$\begin{aligned}
U_{x-\lambda\mathbf{1}}(a) &= 2R^2_{x-\lambda\mathbf{1}}(a) - R_{(x-\lambda\mathbf{1})^2}(a) \\
&= 2(x-\lambda\mathbf{1}) \bullet \big((x-\lambda\mathbf{1}) \bullet a\big) - (x-\lambda\mathbf{1})^2 \bullet a \\
&= 2(x-\lambda\mathbf{1}) \bullet \big(x \bullet a - \lambda a\big) - \big((x-\lambda\mathbf{1}) \bullet (x-\lambda\mathbf{1})\big) \bullet a \\
&= 2\big((x-\lambda\mathbf{1}) \bullet (x \bullet a) - (x-\lambda\mathbf{1}) \bullet \lambda a\big) - \big((x-\lambda\mathbf{1}) \bullet x - \lambda\mathbf{1} \bullet (x-\lambda\mathbf{1})\big) \bullet a \\
&= 2\big(x \bullet (x \bullet a) - 2\lambda x \bullet a + \lambda^2 a\big) - \big(x \bullet x - 2\lambda x + \lambda^2\mathbf{1}\big) \bullet a \\
&= 2x \bullet (x \bullet a) - 4\lambda x \bullet a + 2\lambda^2 a - x^2 \bullet a + 2\lambda x \bullet a - \lambda^2 a \\
&= 2x \bullet (x \bullet a) - x^2 \bullet a - 2\lambda x \bullet a + \lambda^2 a.
\end{aligned}$$

Hence, by applying Identity 5.2 once again, we then obtain the desired result

$$U_{x-\lambda\mathbf{1}}(a) = U_x(a) - 2\lambda R_x(a) + \lambda^2 I(a).$$

Theorem 5.11 Characterization of the McCrimmon Radical of a Jordan-Banach Algebra
([AUP2]; Theorem 2, p. 33)

Let a be an element of a Jordan-Banach algebra A. Then a is in the McCrimmon radical of A if and only if

$$\sup\{\rho(x + ta) : t \in \mathbb{C}\} < +\infty$$

for every $x \in A$.

Proof:

Throughout the proof let $\sigma(x)^\wedge$ denote the full spectrum, that is, the spectrum with the holes filled, called the polynomially convex hull of $\sigma(x)$ (cf. Appendix B).

Assume that a is the McCrimmon radical of A, that is, a is an element of a quasi-invertible ideal of A. To obtain the desired result we show that $\sigma(x+ta) = \sigma(x)$ for $x \in A$ and $t \in \mathbb{C}$ arbitrary. This is equivalent to showing that $\mathbb{C}\backslash\sigma(x+ta) = \mathbb{C}\backslash\sigma(x)$. Let $\lambda \notin \sigma(x)$. Then $\lambda\mathbf{1} - x$ is invertible. So, by Theorem 5.9, $\lambda\mathbf{1} - x - ta = \lambda\mathbf{1} - (x + ta)$ is invertible. Thus $\lambda \notin \sigma(x + ta)$, and $\sigma(x + ta) \subset \sigma(x)$. Now, let $\lambda \notin \sigma(x + ta)$. So, $\lambda\mathbf{1} - x - ta$ is invertible, and consequently, by Theorem 5.9, $\lambda\mathbf{1} - x - ta - (-ta)$ is invertible. So, $\lambda\mathbf{1} - x$ is invertible and $\lambda \notin \sigma(x)$. Hence $\sigma(x) \subset \sigma(x + ta)$. Thus, $\sigma(x + ta) = \sigma(x)$ and hence $\rho(x + ta) = \rho(x)$ for all $x \in A$ and $t \in \mathbb{C}$. Consequently, for all $x \in A$,

$$\sup\{\rho(x + ta) : t \in \mathbb{C}\} < +\infty.$$

For the converse, consider for each $x \in A$ the functions $t \mapsto \rho(x + ta)$ on $\mathbb{C}$. It is clear, by Vesentini's Theorem (Theorem 3.7), that each of these functions are subharmonic on $\mathbb{C}$. Since $\sup\{\rho(x + ta) : t \in \mathbb{C}\} < +\infty$ for all $x \in A$, it follows by Liouville's Spectral Theorem (Theorem 3.14) that $\sigma(x + ta)^\wedge$ is constant on $\mathbb{C}$, and consequently $\sigma(x + ta)^\wedge = \sigma(x)^\wedge$ for all $x \in A$.

Let $\mathcal{I}$ be defined by

$$\mathcal{I} = \{b \in A : \sigma(x + b)^\wedge = \sigma(x)^\wedge \text{ for all } x \in A\}.$$

We now show that $\mathcal{I}$ is a linear subspace of A. Clearly $0 \in \mathcal{I}$. Let $b, c \in \mathcal{I}$. Then, by repeated use of the definition of $\mathcal{I}$, it follows that

$$\sigma(x + (b + c))^\wedge = \sigma((x + b) + c)^\wedge = \sigma(x + b)^\wedge = \sigma(x)^\wedge,$$

and thus $b + c \in \mathcal{I}$.
To prove that $\alpha b \in \mathcal{I}$, let $b \in \mathcal{I}$ and $\alpha \in \mathbb{C}$. Without loss of generality we may suppose $\alpha \neq 0$. Since $\sigma(\alpha x)^\wedge = \alpha\sigma(x)^\wedge$ for each $x \in A$, it follows that

$$\sigma(x + \alpha b)^\wedge = \alpha\sigma\left(\frac{x}{\alpha} + b\right)^\wedge = \alpha\sigma\left(\frac{x}{\alpha}\right)^\wedge = \sigma(x)^\wedge.$$

Hence, $\alpha b \in \mathcal{I}$, and $\mathcal{I}$ is a linear subspace of A.
Moreover, in particular for $x = 0$, it follows by the definition of $\mathcal{I}$, that $\sigma(b)^\wedge = 0$, so $\rho(b) = 0$, and thus is b quasi-nilpotent. Hence, $\mathcal{I}$ consists of quasi-nilpotent elements.

The key to establishing the converse now lies in the proof of the following:

If $a \in \mathcal{I}$, then $U_x(a) \in \mathcal{I}$ for every $x \in A$.

The above-mentioned is then used to show that $\mathcal{I}$ is also an ideal, from which we obtain the desired end result.

Assume that $0 \notin \operatorname{cohull} \sigma(x)$. Then there exists a straight line Δ containing 0, with Δ extending to infinity in both directions, such that Δ does not intersect $\sigma(x)$**(1)**. Consider the complex function $f(z) = z^{-2}$ on a neighborhood Ω containing $\sigma(x)$ with $0 \notin \Omega$. By the Holomorphic Functional Calculus we have that $f(x) = x^{-2}$ is well-defined in A. Further, let Γ be one of the half-lines, defined by Δ, with origin zero. Then, $|f(\Gamma)| =: \{|f(z)| = |z^{-2}| : z \in \Gamma, z \neq 0\} = (0, \infty)$, and $f(\Gamma)$ is a continuous path joining 0 and infinity. Moreover, $f(\Gamma)$ does not intersect $\sigma(x^{-2})$, and hence $0 \notin \sigma(x^{-2})^\wedge$ **(2)**.

Let $y \in A$ arbitrary and $\mu \notin \sigma(y)^\wedge$. Then $\mu\mathbf{1} - y$ is invertible, and it follows by Theorem 5.8 (c) that $U_{x^{-1}}(\mu\mathbf{1} - y)$ is invertible in A. Using Identity 5.2, $U_a = 2R_a^2 - R_{a^2}$,

$$\begin{aligned} U_{x^{-1}}(\mu\mathbf{1} - y) &= 2R_{x^{-1}}^2(\mu\mathbf{1} - y) - R_{(x^{-1})^2}(\mu\mathbf{1} - y) \\ &= 2x^{-1} \bullet \big(x^{-1} \bullet (\mu\mathbf{1} - y)\big) - x^{-2} \bullet (\mu\mathbf{1} - y) \\ &= 2x^{-1} \bullet \big(\mu x^{-1} - x^{-1} \bullet y\big) - \mu x^{-2} + x^{-2} \bullet y \\ &= 2\mu x^{-2} - 2x^{-1} \bullet (x^{-1} \bullet y) - \mu x^{-2} + x^{-2} \bullet y \\ &= \mu x^{-2} - \big(2x^{-1} \bullet (x^{-1} \bullet y) - x^{-2} \bullet y\big). \end{aligned}$$

Thus, by $U_a = 2R_a^2 - R_{a^2}$,

$$U_{x^{-1}}(\mu\mathbf{1} - y) = \mu x^{-2} - U_{x^{-1}}(y) = \mu\left(x^{-2} - \frac{\mathbf{1}}{\mu}U_{x^{-1}}(y)\right).$$

By using the above and the upper semi-continuity of the spectrum, we now show that $0 \notin \sigma(U_{x^{-1}}(\mathbf{1}\mu - y))^\wedge$.
Since $0 \notin \sigma(x^{-2})^\wedge$, it follows by the normality of the complex plane that there exists an open set W containing $\sigma(x^{-2})^\wedge$, such that W does not separate 0 from infinity. Now, if $|\mu| \to \infty$, then

$$\left(x^{-2} - \frac{\mathbf{1}}{\mu} U_{x^{-1}}(y)\right) \to x^{-2}.$$

Hence, by the upper semi-continuity of the spectrum

$$\sigma\left(x^{-2} - \frac{\mathbf{1}}{\mu} U_{x^{-1}}(y)\right) \subset W, \text{ for } |\mu| \text{ sufficiently large.}$$

Since $0 \notin W$, it follows that 0 can not belong to some hole of $\sigma(x^{-2} - \frac{\mathbf{1}}{\mu} U_{x^{-1}}(y))$. Moreover,

$$\mu\sigma\left(x^{-2} - \frac{\mathbf{1}}{\mu} U_{x^{-1}}(y)\right) = \sigma\left(\mu\left(x^{-2} - \frac{\mathbf{1}}{\mu} U_{x^{-1}}(y)\right)\right) \subset \mu W.$$

Note that, since W does not separate 0 from infinity the same holds true for μW, for $|\mu|$ sufficiently large. Hence

$$0 \notin \sigma\left(\mu\big(x^{-2} - \frac{\mathbf{1}}{\mu} U_{x^{-1}}(y)\big)\right)^\wedge.$$

But $U_{x^{-1}}(\mu\mathbf{1} - y) = \mu(x^{-2} - \frac{\mathbf{1}}{\mu} U_{x^{-1}}(y))$, thus for $\mu \notin \sigma(y)^\wedge$ and $|\mu|$ sufficiently large,

$$0 \notin \sigma(U_{x^{-1}}(\mu\mathbf{1} - y))^\wedge.$$

Using the fact that $\sigma(x + ta)^\wedge = \sigma(x)^\wedge$, for all $x \in A$ and $t \in \mathbb{C}$, it follows that

$$0 \notin \sigma(U_{x^{-1}}(\mu\mathbf{1} - y) - ta)^\wedge = \sigma(U_{x^{-1}}(\mu\mathbf{1} - y))^\wedge,$$

and thus $U_{x^{-1}}(\mu\mathbf{1} - y) - ta$ is invertible.
We assumed $0 \notin \operatorname{cohull} \sigma(x)$. Thus it follows by Theorem 5.8 (c) that

$$U_x\big(U_{x^{-1}}(\mu\mathbf{1} - y) - ta\big)$$

is invertible. That is

$$\mu\mathbf{1} - y - tU_x(a)$$

is invertible, and thus $\mu \notin \sigma(y + tU_x(a))$ for $|\mu|$ sufficiently large.
Hence we can find a bounded set C such that $\sigma(y + tU_x(a)) \subset C$ for all $t \in \mathbb{C}$. Thus, by Liouville's Spectral Theorem (Theorem 3.14) we have that $\sigma(y + tU_x(a))^\wedge$ is constant for all $t \in \mathbb{C}$. Consequently, $\sigma(y + tU_x(a))^\wedge = \sigma(y)^\wedge$, for all $t \in \mathbb{C}$, $y \in A$

arbitrary, and $0 \notin \operatorname{cohull} \sigma(x)$.

This result is now used to finally show that if $a \in \mathcal{I}$, then $U_x(a) \in \mathcal{I}$ for all $x \in A$. Let $x \in A$ arbitrary, $y \in A$ be fixed, and $t \in \mathbb{C}$. Also, consider $\mu \in \mathbb{C}$, with $|\mu| > \rho(x)$. Clearly $\mu \notin \operatorname{cohull} \sigma(x)$. Since $\operatorname{cohull} \sigma(x) - \{\mu\} = \operatorname{cohull} \sigma(x - \mu\mathbf{1})$ and $|\mu| > \rho(x)$, we then have that $0 \notin \operatorname{cohull} \sigma(x - \mu\mathbf{1})$. Thus, by our previous argument

$$\sigma(y + tU_{x-\mu\mathbf{1}}(a))^{\wedge} = \sigma(y)^{\wedge},$$

for $t \in \mathbb{C}$ arbitrary and y fixed, but arbitrary. That is, $\sigma(y + tU_{x-\mu\mathbf{1}}(a))^{\wedge}$ is constant for $|\mu| > \rho(x)$. Further, by the definition of $\mathcal{I}$, we also have that $U_{x-\mu\mathbf{1}}(a) \in \mathcal{I}$. Moreover, $\sigma(y + tU_{x-\mu\mathbf{1}}(a))^{\wedge}$ is constant for all $\mu \in \mathbb{C}$. To prove this let $|\mu| \leq \rho(x)$. Then, for $t_0 \in \mathbb{C}$ fixed and since $U_{x-\mu\mathbf{1}}(a) = U_x(a) + 2R_x(a) + \mu^2 a$,

$$\begin{aligned} \rho(y + t_0 U_{x-\mu\mathbf{1}}(a)) &\leq ||y|| + |t_0| \cdot ||U_{x-\mu\mathbf{1}}(a)|| \\ &\leq ||y|| + |t_0| \cdot \Big(||U_x(a)|| + 2||R_x(a)|| + |\mu|^2 ||a|| \Big) \\ &\leq ||y|| + |t_0| \cdot \Big(||U_x(a)|| + 2||R_x(a)|| + \rho(x)^2 ||a|| \Big). \end{aligned}$$

Thus, from the latter we have that $\sigma(y + t_0 U_{x-\mu\mathbf{1}}(a)) \subset K$, with K a bounded set in $\mathbb{C}$, depending on t_0. Consequently, for $t_0 \in \mathbb{C}$ arbitrary, we can find a bounded set M_{t_0} in $\mathbb{C}$ such that for all $\mu \in \mathbb{C}$,

$$\sigma(y + t_0 U_{x-\mu\mathbf{1}}(a)) \subset M_{t_0}.$$

Hence, by Liouville's Spectral Theorem (Theorem 3.14) we have that $\sigma(y+t_0U_{x-\mu\mathbf{1}}(a))^{\wedge}$ is constant for all μ. But we know that $\sigma(y + t_0U_{x-\mu\mathbf{1}}(a))^{\wedge} = \sigma(y)^{\wedge}$ for $|\mu| > \rho(x)$. Thus, in particular with $\mu = 0$ and $t_0 = 1$ we have that $U_x(a) \in \mathcal{I}$ by definition of $\mathcal{I}$.

Using the identity $U_{x-\lambda\mathbf{1}} = U_x - 2\lambda R_x + \lambda^2 I$, and the fact that $\mathcal{I}$ is linear subspace, we have

$$R_x(a) = \frac{\mathbf{1}}{2\lambda}\Big(U_{x-\lambda\mathbf{1}}(a) - U_x(a) - \lambda^2 a \Big) \in \mathcal{I}.$$

Thus $ax = xa \in \mathcal{I}$ for all $x \in \mathcal{I}$, so that $\mathcal{I}$ is an ideal in A and $a \in \mathcal{I}$. But $\mathcal{I}$ consists of quasi-nilpotent elements. Thus $\mathcal{I}$ is a quasi-invertible ideal in A, and hence, by definition $\mathcal{I}$ is included in the McCrimmon radical of A.
Q.E.D.

The following two corollaries are both spectral characterizations of the McCrimmon radical, where we use Theorem 5.11 to obtain the desired result.

Corollary 5.12
([AUP2]; p. 34)

Let A be a Jordan-Banach algebra and $a \in A$. *Then a is in the McCrimmon radical of A if and only if $\rho(U_x(a)) = 0$ for every x in A.*

Proof:

Let a be in the McCrimmon radical of A. Now, by Identity 2.2 we have that $U_x(a) = 2R_x^2(a) - R_{x^2}(a) = 2x \bullet (x \bullet a) - x^2 \bullet a$, for every $x \in A$. Since the McCrimmon radical is an ideal, $U_x(a)$ then belongs to the McCrimmon radical. Hence, by definition $\rho(U_x(a)) = 0$, for all $x \in A$.

For the converse assume $\rho(U_x(a)) = 0$ for every $x \in A$. We show that $\sup\{\rho(x+ta) : t \in \mathbb{C}\} < +\infty$ for all $x \in A$ to obtain the desired result.

Let $y \in A$ be arbitrary such that $|\mu| > \rho(y)$. Then $\sigma(\mu\mathbf{1} - y)$ does not separate zero from infinity. Thus, there exists $b \in A$ (with b dependant on μ), such that $b^2 = \mu\mathbf{1} - y$. Thus by Theorem 5.5, b^{-2} exists where

$$b^{-2} = (b^{-1})^2 = (\mu\mathbf{1} - y)^{-1}.$$

Now, denote $x = b^{-1} = \sqrt{(\mu\mathbf{1} - y)^{-1}}$ and consider $U_x(\mu\mathbf{1} - y - ta)$, with $t \in \mathbb{C}$. Then

$$U_x(\mu\mathbf{1} - y - ta) = U_x(\mu\mathbf{1} - y) - tU_x(a) = 2R_x^2(\mu\mathbf{1} - y) - R_{x^2}(\mu\mathbf{1} - y) - tU_x(a),$$

that is,

$$U_x(\mu\mathbf{1} - y - ta) = 2x \bullet (x \bullet (\mu\mathbf{1} - y)) - x^2 \bullet (\mu\mathbf{1} - y) - tU_x(a).$$

By substituting $x = b^{-1}$ and $b^2 = \mu\mathbf{1} - y$, we then have

$$U_x(\mu\mathbf{1} - y - ta) = 2b^{-1} \bullet (b^{-1} \bullet b^2) - b^{-2} \bullet b^2 - tU_x(a).$$

But, by Definition 5.4 (b) $b^{-1} \bullet b^2 = b$, and since $b^{-2} \bullet b^2 = \mathbf{1}$, it follows that

$$U_x(\mu\mathbf{1} - y - ta) = 2b^{-1} \bullet b - \mathbf{1} - tU_x(a) = 2(\mathbf{1}) - \mathbf{1} - tU_x(a),$$

and thus

$$U_x(\mu\mathbf{1} - y - ta) = \mathbf{1} - tU_x(a).$$

Since $\rho(U_x(a)) = 0$ it follows that $\mathbf{1} - tU_x(a) = U_x(\mu\mathbf{1} - y - ta)$ is invertible for all $t \in \mathbb{C}$.

Considering the Fundamental Formula, as given in Theorem 5.3 we have the following:

$$U_{U_x(\mu\mathbf{1}-y-ta)} = U_x U_{(\mu\mathbf{1}-y-ta)} U_x.$$

By Theorem 5.7 (b), since $U_x(\mu\mathbf{1}-y-ta)$ is invertible, it then follows that $U_{U_x(\mu\mathbf{1}-y-ta)}$ is invertible. But $x = b^{-1}$, so $U_{x^{-1}} = (U_x)^{-1}$ exists by Theorem 5.8 (b), and consequently

$$U_{(\mu\mathbf{1}-y-ta)} = U_{x^{-1}} U_{U_x(\mu\mathbf{1}-y-ta)} U_{x^{-1}}$$

is invertible. So, again by Theorem 5.8 (b) we have

$$(U_{(\mu\mathbf{1}-y-ta)})^{-1} = U_{(\mu\mathbf{1}-y-ta)^{-1}},$$

thus $\mu\mathbf{1} - y - ta$ is invertible. Hence $\mu \notin \sigma(y - ta)$, for $|\mu| > \rho(y)$ and $t \in \mathbb{C}$.

This result is now used to show that $\rho(y + ta) \leq \rho(y)$ for all $t \in \mathbb{C}$ and $y \in A$. Suppose this is not the case. Then, we can find a $t' \in \mathbb{C}$ such that $\rho(y + t'a) > \rho(y)$. Since $\sigma(y + t'a)$ is closed, we can find $\beta \in \sigma(y + t'a)$ such that $\rho(y + t'a) = |\beta|$, and hence $|\beta| > \rho(y)$. But from the result in the preceding paragraph, it follows that $\beta \notin \sigma(y + t'a)$. Clearly this a contradiction and thus $\rho(y + ta) \leq \rho(y)$. Consequently $\sup\{\rho(y + ta) : t \in \mathbb{C}\} < +\infty$, and the result follows by Theorem 5.11.
Q.E.D.

Corollary 5.13
([AUP2]; p. 34)

Let A be a Jordan-Banach algebra and $a \in A$. Then a is in the McCrimmon radical of A if and only if there exists $C \geq 0$ such that $\rho(x) \leq C||x - a||$ for all x in a neighborhood of a.

<u>Proof:</u>

Let a be in the McCrimmon radical of A and $x \in A$ arbitrary. Then, by Theorem 5.11, it follows that the subharmonic function $t \mapsto \rho(x + ta)$ is bounded for all $t \in \mathbb{C}$. So, by Liouville's Theorem for Subharmonic functions, $t \mapsto \rho(x + ta)$ is then constant on $\mathbb{C}$. Consequently, for all $x \in A$, we have $\rho(x) = \rho(x - a) \leq ||x - a||$, which is the desired result with $C = 1$.

Assume now that there exists $C \geq 0$ such that $\rho(x) \leq C||x - a||$ for all x in a neighborhood of a. Again we consider the subharmonic function $t \mapsto \rho(x + ta)$ on $\mathbb{C}$.

Consider the element $a + x/t$ in A. Since

$$||(a + x/t) - a|| = \frac{1}{|t|}||x|| \to 0, \quad \text{as } |t| \to \infty,$$

it follows that $a+x/t$ is in a neighborhood of a for t sufficiently large. Thus, by our hypothesis

$$\rho(a + x/t) \leq \frac{C}{|t|}||x||.$$

So there exists a $K \in \mathbb{R}^+$, such that if $|t| > K$, then $\rho(x + ta) \leq M||x||$ for t sufficiently large (where $M = C/K$).
If $|t| \leq K$, then

$$\rho(x + ta) \leq ||x + ta|| \leq ||x|| + |t| \cdot ||a|| \leq ||x|| + K||a||,$$

for t sufficiently large. Hence, choosing $N = \max\{M||x||, ||x|| + K||a||\}$, we have

$$\rho(x + ta) \leq N, \quad \text{for all } t \in \mathbb{C}.$$

Thus, by Liouville's Theorem for Subharmonic Functions it follows that $t \mapsto \rho(x+ta)$ is bounded, and consequently constant on $\mathbb{C}$. In particular $\rho(x + ta) = \rho(x)$ for all $t \in \mathbb{C}$, and the desired result follows by Theorem 5.11.
Q.E.D.

Notes on Chapter 5

(**1**) *If* $0 \notin \text{cohull}\, \sigma(x)$, *then there exists a straight line* Δ *containing* 0, *such that* Δ *extends to infinity in both directions and* $\sigma(x) \cap \Delta = \emptyset$.

Proof:

Suppose $0 \notin \text{cohull}\, \sigma(x)$. Clearly then $0 \notin \sigma(x)$. Further, suppose that the contrary is true, that is, for all straight lines through 0, Δ does intersect $\sigma(x)$.

We may assume that the real axis contains a spectral point of x, otherwise Δ can be chosen to be the real axis. Without loss of generality we may also assume there is a spectral point of x, say z_1, on the positive real axis (thus the negative real axis has an empty intersection with $\sigma(x)$, otherwise $0 \in \text{cohull}\, \sigma(x)$). A similar argument holds true for the imaginary axis with $z_2 \in \sigma(x)$ on the imaginary axis. Consequently, we have the following situation as illustrated in figure 5.1 below, where L' denotes the line segment connecting z_1 and z_2. By definition $L' \subset \text{cohull}\, \sigma(x)$:

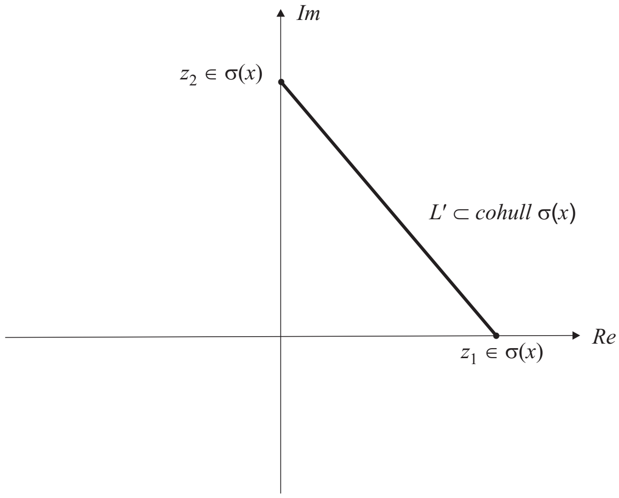

Figure 5.1: The line segment L' contained in $\text{cohull}\, \sigma(x)$.

If there are no spectral points of x in the second and fourth quadrants, then, obviously we can find a required line Δ. Without loss of generality we may assume that there exists a spectral point in the second quadrant. Now choose a spectral point of x in the second quadrant, say z_3, such that this point has the largest possible principle argument (denoted by Arg). Note that such a point exists since $\sigma(x)$ is closed. Moreover, observe that $\text{Arg}\, z \neq \pi$, because otherwise $0 \in \text{cohull}\, \sigma(x)$. Further, let l be the straight line through z_3 and 0 as depicted in Figure 5.2:

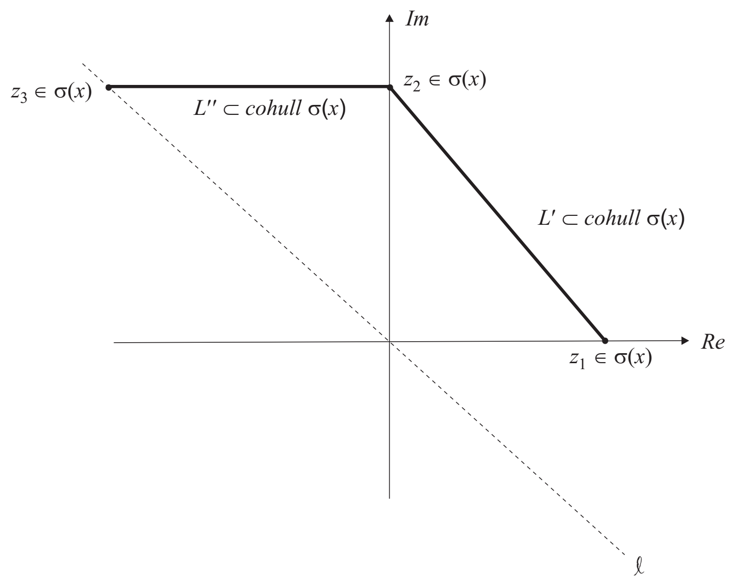

Figure 5.2: The line segments L' and L'' contained in cohull $\sigma(x)$.

Now denote by M the max Arg. Then each line, through the origin, in the second quadrant with Arg $\in (M, \pi)$ has an empty intersection with $\sigma(x)$. If every line in the fourth quadrant with Arg $\in (M - \pi, 0)$ contains a spectral point of x, then, since $\sigma(x)$ is compact, l must contain a spectral point of x in the fourth quadrant. But this contradicts the fact that $0 \notin \operatorname{cohull} \sigma(x)$. Hence, there exists a line, with negative gradient, through the origin not intersecting $\sigma(x)$.
Q.E.D.

(**2**) *Let* Γ *be a half-line defined by* Δ, *through the origin such that* $\Delta \cap \sigma(x) = \emptyset$. *Also, define* $|f(\Gamma)| =: \{|f(z)| : z \in \Gamma, z \neq 0\} = (0, \infty)$ *where* $|f(z)| = |z^{-2}|$. *Then*

$$f(\Gamma) \cap \sigma(x^{-2}) = \emptyset.$$

Proof:

Consider the function $g(\lambda) = 1/f(\lambda) = \lambda^2$. Let Γ be a half-line through the origin such that $f(\Gamma) = 1/g(\Gamma)$ and $\Gamma \cap \sigma(x) = \emptyset$. We first show that $g(\Gamma) \cap \sigma(x^2) = \emptyset$. Clearly $g(\Gamma) = \{\lambda^2 : \lambda \in \Gamma\}$ and $\sigma(x^2) = g(\sigma(x))$. Let us say that $g(\Gamma) \cap \sigma(x^2) \neq \emptyset$. Then there exists λ_1 such that $\lambda_1 = \lambda_0^2$ for $\lambda_0 \in \Gamma$ and $\lambda_1 = \alpha_0^2$ for $\alpha_0 \in \sigma(x)$. Hence $\alpha_0 = \pm\lambda_0$. But this contradicts the fact that $\Delta \cap \sigma(x) = \emptyset$. Thus $g(\Gamma) \cap \sigma(x^2) = \emptyset$. So, in conclusion, since $f(\Gamma) = 1/g(\Gamma)$ and $\sigma(x^{-2}) = 1/\sigma(x^2)$, it follows that $\sigma(x^{-2}) \cap f(\Gamma) = \emptyset$. Further, since $f(\Gamma) = (0, \infty)$, it follows that $0 \notin \sigma(x^{-2})^\wedge$.
Q.E.D.

Appendix A
Borel Measures

When studying capacities of sets in the complex plane the term Borel measure occurs frequently. It is in one of our main references for subharmonic function theory ([RAN]; p. 53 – 57, 127, 209) that the latter concept is used to define capacity. In [RAN]; p. 209, the author states the following:

> *"By the term Borel measure is meant a positive measure on the Borel σ-algebra of a topological space."*

This can be quite a mouthful for the reader who isn't familiar with measure theory. Thus, the author felt that a short appendix should be next to hand if any questions should arise.

Throughout this appendix Ω will denote a nonempty set and $\mathcal{F}$ a collection of subsets of Ω such that $\Omega \in \mathcal{F}$.
The standard definition of a topological space, as in [RUD2]; p. 8, is to be used. That is, a topology τ on a set Ω is a collection of subsets of Ω such that the following three properties hold:

(1) $\emptyset \in \tau$ and $\Omega \in \tau$
(2) If $V_1, \ldots, V_n \in \tau$, then $V_1 \cap V_2 \ldots \cap V_n \in \tau$
(3) If $\{V_\alpha\}$ is any arbitrary collection of members of τ,
then $\bigcup_\alpha V_\alpha \in \tau$.

For τ a topology on Ω, (Ω, τ) will be called a topological space, where the members of τ are called the open sets in Ω. We now define what is meant by a σ-algebra.

Definition A.1
([ASH]; p. 4)

Let $\mathcal{F}$ be a collection of subsets of a nonempty set Ω. Then $\mathcal{F}$ is called an **algebra** *if and only if the following conditions hold*:

(1) $\Omega \in \mathcal{F}$.
(2) *If* $A \in \mathcal{F}$, *then* $A^c \in \mathcal{F}$.
(3) *If* $A_1, \ldots, A_n \in \mathcal{F}$, *then* $\bigcup_{i=1}^n \in \mathcal{F}$.

If (3) *is replaced by closure under countable unions, that is,* *If* $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}$, *then* $\bigcup_{i=1}^{\infty} \in \mathcal{F}$,

we call $\mathcal{F}$ *a* **σ-algebra**.

To define a Borel σ-algebra we first have to note what is meant by a σ-algebra generated by $\mathcal{F}$. It is an important fact that every collection of subsets $\mathcal{F}$ of Ω is contained in a smallest σ-algebra, with respect to the inclusion relation of sets ([ALI]; p. 97 and [RUD2]; p. 12). This σ-algebra is the intersection of all σ-algebras that contain $\mathcal{F}$. It is called the σ-algebra **generated** by $\mathcal{F}$.

The above concept now leads to the definition of Borel sets and Borel σ-algebras as given in [ALI]; p. 97 and [RUD2]; p. 12:
Consider a topological space (Ω, τ). The **Borel sets** of (Ω, τ) are the members of the σ-algebra generated by the open sets of (Ω, τ). This σ-algebra of all Borel sets is denoted by $\mathcal{B}$. In particular, by Definition A.1, closed sets are also Borel sets, and so are countable unions of closed sets and countable intersections of open sets. Thus, a **Borel σ-algebra $\mathcal{B}$**, is a σ-algebra such that its members are the Borel sets.

We shall need only one more main definition which enables us to understand Ransford's quoted words. This is the definition of a measure on a σ-algebra.

Definition A.2
([ASH]; p. 16 and [RUD2]; p. 16)

A positive measure (or henceforth measure) μ, *defined on a* σ*-algebra* $\mathcal{F}$, *is a real-valued function with range* $[0, \infty]$ *such that if* $A_1, A_2, \ldots$ *forms a countable infinite collection of disjoints sets in* $\mathcal{F}$, *then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

It can be shown from the above definition that $\mu(\emptyset) = 0$ is always true.
Also, if $\mu(\Omega) = 1$ then μ is called a **probability measure** ([ASH]; p. 16). This measure plays an important role when defining the capacity of a set in the complex plane.

Finally we are at the point of defining a Borel measure.
A **Borel measure** is a measure μ on the Borel σ-algebra of a topological space (Ω, τ) such that $\mu(K) < \infty$ for each compact subset K of Ω ([ALI]; p. 136).

Appendix B

Topological Concepts

This appendix contains the most important topological concepts used in our work. The theory we discuss was obtained from the following references: [ALI], [CON2], [JAM], [KRE], [WIL], and [ZEL].

The standard definition of a topology τ on a set X applies throughout our work (cf. Appendix A). By convention, a topological space is denoted by (X, τ). Should no confusion arise about which topology we are referring to, it will be abbreviated simply by X. A metric space is denoted by (X, d), with d the metric on X. The definition, as given in [KRE]; p. 3, is used. Again, should no confusion arise as to which metric is concerned, the abbreviation X is used. The interior and closure of a set A, in X, are denoted by $\operatorname{int} A$ and $\overline{A}$ respectively, and the boundary of A by ∂A.

If X is a topological space, then X is said to be **Hausdorff** if any two distinct points in X have disjoint neighborhoods contained in X. Further, X is **normal**, if given any two disjoint closed sets in X, there exists disjoint open sets in X, containing each of the closed sets respectively. The complex plane is an example of a normal space ([JAM]; p. 39, 41).

Two of the most important topological concepts used in our work are compactness and connectedness.

To discuss compactness, we first consider the definition of a covering of a set. A family $\mathcal{G}$ of sets is a **covering** of a set S if its union contains S, that is, $S \subset \bigcup_{G \in \mathcal{G}} G$. An **open covering** is a covering whose members are all open sets. A topological space X is then said to be **compact** if every open covering of X has a finite subcovering.

Some important properties of compactness that features in our work:

- *Every compact subset of a topological space is closed.*
- *Every compact Hausdorff space is normal.*
- *A closed subset of a compact space is compact.*
- *The continuous image of a compact set is compact.*

If X is a metric space, then X is said to be **sequentially compact** if every sequence in X has a subsequence converging to a point in X. It is interesting to note the following:

In a metric space (X, d) the following are equivalent:

(a) *X is a compact metric space,*
(b) *every infinite set in X has a limit point,*
(c) *X is sequentially compact.*

Moreover, for any finite dimensional normed space X, a subset M of X is compact if and only if it is closed and bounded in X ([ALI]; Corollary 4.7, p. 25). In connection with finite dimensional normed spaces, we also recall the well-known **Bolzano-Weierstrass Theorem**:

> *Every bounded sequence in X has a convergent subsequence in X, with the limit belonging to X.*

The second topological property is connectedness. A topological space X is said to be **connected** if the following equivalent assertions hold:

(a) *X cannot be partitioned into two disjoint non-empty closed sets,*
(b) *there are no subsets, other than X and $\emptyset$, which are simultaneously open and closed.*

If X is not connected then X is said to be **disconnected**.

A **path-connected** space X, is a space X with the property that each pair of points in X can be joined by a curve which lies entirely in X. That is, for each $a, b \in X$, there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = a$, $f(1) = b$, and $f(I) \subset X$, with $I = [a, b]$.

Consider a topological space X and $\xi \in X$. A **component** M, containing ξ, is a maximal connected set in X containing ξ. That is, if $\mathcal{U}$ is the collection of all connected subsets of X containing ξ, then $M = \bigcup_{T \in \mathcal{U}} T$. If one thinks of X as country surrounded by water, then the country is an archipelago of which the constituent islands are the components of X.
In particular, X is said to be **totally disconnected** if and only if the components in A are the one-point sets ([WIL]; p. 210).

The following theorem pertaining to the latter property will be used in our subharmonic applications:

Theorem B.1
([WIL]; Theorem 29.15, p. 213)

Let X be a totally disconnected compact metric space. Then for each $n = 0, 1, 2, \ldots$ there exists a finite open cover U_n of X of disjoint open sets of diameter $< 1/2^n$ such that $U_{n+1} \subset U_n$ for each $n \geq 0$.

In chapters 3 and 5 we consider the polynomially convex hull of the spectrum of $x \in A$, denoted by $\sigma(x)^\wedge$. It is also referred to as the full spectrum of x. If K is a compact subset of $\mathbb{C}$ we define the **polynomial convex hull** $K^\wedge$ of K as the union of K with the bounded components of its complement (the latter is also referred to as the holes of K). Further, we say that K is **polynomially convex** if $K = K^\wedge$ ([AUP1]; p. 77 and [CON1]; p. 211).

Appendix C
Radó's Extension Theorem

Radó's Extension Theorem is used in the applications of subharmonic functions to spectral theory in conjunction with proofs regarding holomorphic functions (cf. Chapter 3; Theorem 3.17 and Theorem 3.26). In this appendix we discuss the proof of Radó's Extension Theorem, in which we also apply the well-known Maximum-Modulus Theorem. We include two forms of the latter:

Maximum-Modulus Theorem
([PRI]; p. 76)

Form 1:

Let G be a bounded domain in $\mathbb{C}$ with f holomorphic on G and continuous on $\overline{G}$. Then $|f|$ obtains a maximum on the boundary of G.

Form 2:

Let K be the closure of Ω, where Ω is a bounded region in $\mathbb{C}$. Further let $f \in H(\Omega)$ and continuous on K and define $||f||_{\partial K}$ as $||f||_{\partial K} = \sup\{|f(z)| : z \in \partial K\}$. Then $|f(z)| \leq ||f||_{\partial K}$ for each $z \in \Omega$.

Throughout this appendix, the disk with center 0 and radius 1 will be referred to as the open unit disk and denoted by U. T will denote the unit circle, that is, $T = \{z \in \mathbb{C} : |z| = 1\}$, and thus $\partial U = T$. Further we only mention the following theorem which will be used in the proof of Radó's Extension Theorem.

Theorem C.1
([RUD2]; Theorem 12.12, p. 280)

Suppose M is the vector space of continuous complex functions on the closed unit disc $\overline{U}$, with the following properties:
(a) $1 \in M$.
(b) *If $f \in M$, then also $If \in M$ with $I(z) = z$.*
(c) *If $f \in M$, then $||f||_U = ||f||_T$.*

Then every $f \in M$ is holomorphic on U.

Radó's Extension Theorem
([RUD2]; Theorem 12.13, p. 280)

Assume $f \in C(\overline{U})$, where $C(\overline{U})$ is the class of all complex continuous functions on the closed unit disk $\overline{U}$. Further let Ω be the set of $z \in U$ such that $f(z) \neq 0$, that is, $\Omega = \{z \in U : f(z) \neq 0\}$, and let $f \in H(\Omega)$. Then f is holomorphic on U.

Proof:

The proof consists of two parts. In the first part we show that Ω is dense in U. For the second part of the proof, the density of Ω is used in conjunction with Theorem C.1 to obtain the desired end result.

Part 1:
Assume $\Omega \neq \emptyset$. We first prove that Ω is open in U by showing that $U \backslash \Omega$ is closed. Let (z_n) be a sequence in $U \backslash \Omega$ such that $z_n \to z \in U$. We prove that $z \in U \backslash \Omega$. Now $z \in U \backslash \Omega$ if an only if $f(z) = 0$. But for each n we have $f(z_n) = 0$. Since f is continuous it follows that $f(z_n) \to f(z)$. Thus we must have $f(z) = 0$ and hence Ω is open in U, and consequently in $\overline{U}$.

Suppose Ω is not dense in U. Then there exists $\alpha \in \Omega$ and $\beta \in U \backslash \Omega$ such that

$$2|\beta - \alpha| < 1 - |\beta|.$$

We now prove the above inequality.
If Ω is not dense in U then, by definition, there exists $\beta_0 \in U$ and $\varepsilon > 0$ such that $B(\beta_0, \varepsilon) \cap \Omega = \emptyset$. Thus, without loss of generality

$$\overline{B}(\beta_0, \varepsilon) \subset \{z \in U : f(z) = 0\}.$$

If for each $\varepsilon > 0$ and $\beta_0 \in U$ we have that

$$\overline{B}(\beta_0, \varepsilon) \cap U \subset \{z \in U : f(z) = 0\},$$

then f is zero on all of U and theorem holds trivially. So, we may assume that there exists an $\varepsilon > 0$ such that

$$\overline{B}(\beta_0, \varepsilon) \cap U \not\subseteq \{z \in U : f(z) = 0.\}$$

Let

$$\varepsilon_0 = \sup \left\{ \varepsilon > 0 : \overline{B}(\beta_0, \varepsilon) \subset \{z \in U : f(z) = 0\} \right\}.$$

The latter implies that we let the radius of the ball $\overline{B}(\beta_0, \varepsilon)$ increase until it meets Ω. Thus we can find a $\beta_1 \in \partial\overline{B}(\beta_0, \varepsilon)$ such that $\beta_1 \in \partial\Omega$. Hence β_1 is the limit of some sequence, say (α_n), in Ω.

Since $\beta_1 \in U$ we have that $|\beta_1|/2 < 1/2$. Consequently for N_1 sufficiently large,

$$|\alpha_{N_1} - \beta_1| + |\beta_1|/2 < 1/2.$$

On the other hand we can find a sequence (β_n) in $\overline{B}(\beta_0, \varepsilon)$ such that $\beta_n \to \beta_1$. So for N_2 sufficiently large

$$|\alpha_{N_1} - \beta_{N_2}| + |\beta_{N_2}|/2 < 1/2.$$

Hence, by the above two inequalities, we have the desired result that $2|\beta - \alpha| < 1 - |\beta|$.

Further, since $\alpha \in \Omega$, $f(\alpha) \neq 0$, and thus $|f(\alpha)| \neq 0$. Moreover,

$$2^n|f(\alpha)| \to \infty, \quad \text{as} \quad n \to \infty.$$

Thus, for $||f||_T = \sup\{|f(z)| : z \in T\}$ and n sufficiently large we have that

$$2^n|f(\alpha)| > ||f||_T.$$

Define h to be the function

$$h(z) = (z - \beta)^{-n} f(z), \quad \text{for } z \in \overline{\Omega}.$$

If $z \in U \cap \partial\Omega$, then $z \notin \Omega$ since Ω is open. Hence $f(z) = 0$, and thus $h(z) = 0$ as well. Now, let $z \in T \cap \partial\Omega$, and consider

$$|h(z)| = \frac{|f(z)|}{|z - \beta\ |^n}.$$

Since $\big||z| - |\beta|\big| \leq \big|z - \beta\big|$ and $|z| = 1$ for $z \in T \cap \partial\Omega$, we have

$$(1 - |\beta|)^n \leq |z - \beta|^n,$$

and thus

$$\frac{1}{|z - \beta|^n} \leq \frac{1}{(1 - |\beta|)^n}.$$

So, for $|h(z)|$,

$$|h(z)| = \frac{|f(z)|}{|z - \beta|^n} \ \leq \ \frac{|f(z)|}{(1 - |\beta|)^n}.$$

Consequently, since $2|\beta-\alpha| < 1-|\beta|$, it follows that

$$|h(z)| = \frac{|f(z)|}{|z-\beta|^n} \;\leq\; \frac{||f||_T}{(1-|\beta|)^n} \;<\; \frac{||f||_T}{2^n|\alpha-\beta|^n}.$$

Thus, from $2^n|f(\alpha)| > ||f||_T$ and for n sufficiently large, we obtain

$$|h(z)| \;<\; \frac{||f||_T}{2^n|\alpha-\beta|^n} \;<\; \frac{|f(\alpha)|}{|\alpha-\beta|^n} \;=\; |h(\alpha)|.$$

So, $|h(z)| < |h(\alpha)|$ for $z \in (T \cap \partial\Omega) \cup (U \cap \partial\Omega)$. Clearly this is a contradiction with the Maximum-Modulus Theorem and thus Ω is indeed dense in U.

Part 2:
Denote by M the vector space of all $g \in C(\overline{U})$ with $g \in H(\Omega)$. Clearly by our hypothesis $f \in M$. We now show that M satisfies the properties (a) – (c) of Theorem C.1. Clearly (a) and (b) are satisfied, thus it remains to verify (c). Fix $g \in M$. It then follows by definition of Ω that $fg^n = 0$ on $U \cap \partial\Omega$ for $n \in \{1, 2, 3, \ldots\}$. Since $fg \in H(\Omega)$ and $fg \in C(\overline{U})$ it follows by an application of the Maximum-Modulus Theorem that for all $\alpha \in \Omega$

$$|(fg^n)(\alpha)| = |f(\alpha)||g(\alpha)|^n \leq ||fg^n||_{\partial\Omega}.$$

Since

$$\sup_{z\in\partial\Omega} |(fg)^n(z)| = \sup_{z\in T\cap\partial\Omega} |(fg)^n(z)| \leq \sup_{z\in T} |(fg)^n(z)|,$$

we have

$$||fg^n||_{\partial\Omega} \leq ||fg^n||_T.$$

Hence, for each $\alpha \in \Omega$

$$|f(\alpha)| \cdot |g(\alpha)|^n \leq ||fg^n||_T \leq ||f||_T \cdot ||g||_T^n,$$

where the latter inequality holds by the definition of $||f||_T$.
So, we obtained $|f(\alpha)| \cdot |g(\alpha)|^n \leq ||f||_T \cdot ||g||_T^n$. By taking the n'th roots both sides and the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} |f(\alpha)|^{1/n} \cdot |g(\alpha)| \leq \lim_{n\to\infty} ||f||_T^{1/n} \cdot ||g||_T.$$

Since $\lim_{n\to\infty} |f(\alpha)|^{1/n} = \lim_{n\to\infty} ||f||_T^{1/n} = 1$, it follows that for each $\alpha \in \Omega$,

$$|g(\alpha)| \leq ||g||_T.$$

From the latter we now show that $||g||_U = ||g||_T$. We first prove that $||g||_U \leq ||g||_T$. Let $z' \in U$ arbitrary. Since Ω is dense in U there exists for $z' \in U$ a sequence (z_n) in Ω such that $z_n \to z'$. Since g is continuous, $g(z_n) \to g(z')$ and consequently

$$\lim_{n\to\infty} |g(z_n)| = |g(z')| \leq ||g||_T.$$

This being true for each $z' \in U$, it follows that

$$\sup_{z' \in U} |g(z')| = ||g||_U \leq ||g||_T.$$

To prove that $||g||_T \leq ||g||_U$, let us suppose the contrary. So $||g||_T > ||g||_U$, that is, $\sup_{|z|=1} |g(z)| > \sup_{|z|<1} |g(z)|$. Since ∂U is compact we can find z_0 such that $\sup_{|z|=1} |g(z)| = |g(z_0)|$. Thus, for all z with $|z| < 1$, we have that $|g(z_0)| > |g(z)|$. Hence, there exists $l > 0$, such that $|g(z_0)| > l > |g(z)|$ for all z with $|z| < 1$. Let (z_n) be a sequence in U with $z_n \to z_0$. Then, since $g \in C(\overline{U})$, it follows that $|g(z_n)| \to |g(z_0)|$. But $\lim_{n\to\infty} |g(z_n)| \leq l < |g(z_0)|$ which is a contradiction, and we have that $||g||_T \leq ||g||_U$. Hence $||g||_U = ||g||_T$. Since M satisfies the hypothesis Theorem C.1 and $f \in H(U)$ it then follows that $f \in H(\Omega)$.
Q.E.D.

Appendix D
The n'th Spectral Diameter

In this appendix we discuss an important property of the n'th Spectral Diameter which is used in the proof of the Scarcity Theorem (cf. Chapter 3, Theorem 3.26).
We shall use the definition of the n'th spectral diameter, as given by [AUP1]; p. 62.

For A a Banach algebra and x an element of A we define the **n'th spectral diameter** of x, for $n \geq 1$, by the **δ_n-formula**:

$$\delta_n(x) = \max \left(\prod_{1 \leq i < j \leq n+1} |z_i - z_j| \right)^{2/n(n+1)}.$$

The above formula is applied to $n+1$ spectral points of $\sigma(x)$ with $z_k \in \{\lambda_1, \lambda_2, \ldots, \lambda_{n+1}\}$.
The reader should also note that it doesn't necessarily mean that $\sigma(x)$ has precisely $n+1$ points – it can have more, or less than $n+1$.

The notation we used in the δ_n-formula is not precisely the same as that of [AUP1]. The letters z_i and z_j are used as "placeholders" to avoid any ambiguity. That is, z_i and z_j do not denote spectral points of $\sigma(x)$ – we shall substitute are chosen spectral points of $\sigma(x)$ into z_i and z_j respectively.
There is just one important condition that needs to be satisfied with the substitution: the relation $i < j$ must hold at all times. For example if $j \leq 3$, it follows that the only values i can assume are $i = 1$ and $i = 2$.

Further, we may have that the spectral points need not be distinct. We are allowed to make use of repeated values. However, if a value in $\{\lambda_1, \lambda_2, \ldots, \lambda_{n+1}\}$ is to be repeated, then $\prod_{1 \leq i < j \leq n+1} |z_i - z_j|$ equals zero. The latter holds true since there will be a factor equal to zero in our product, that is $|z_i - z_j| = 0$ for some $i \neq j$, but $z_i = z_j$.

Let us consider the first three cases of the δ_n-formula: $n = 1, 2,$ and 3.

<u>For $n = 1$:</u>

$$\delta_1(x) = \max \left(\prod_{1 \leq i < j \leq 2} |z_i - z_j| \right),$$

becomes

$$\delta_1(x) = \max\left\{\ |z_1 - z_2|\ \right\}.$$

Hence, choosing any two spectral points, say $\alpha,\ \beta$ of $\sigma(x)$, we substitute them respectively in z_1 and z_2 above. The result is $\delta_1(x) = \max\left\{|\alpha - \beta|\right\}$ for all pairs $\alpha,\ \beta \in \sigma(x)$, and where δ_1 is nothing else than the classical **spectral diameter**, denoted by the standard symbol $\delta(x)$. Further, if $\#\sigma(x) = 1$, then the only possibility is that $z_1 = z_2 = \alpha$ (say) and hence $\delta_1(x) = 0$.

<u>For $n = 2$:</u>

$$\delta_2(x) = \max\left(\prod_{1\leq i<j\leq 3} |z_i - z_j|\right)^{1/3},$$

reduces to

$$\delta_2(x) = \max\left\{\left(|z_1 - z_2| \cdot |z_1 - z_3| \cdot |z_2 - z_3|\right)^{1/3}\ :\ z_1,\ z_2,\ z_3 \in \sigma(x)\right\}.$$

<u>For $n = 3$:</u>

$$\delta_3(x) = \max\left(\prod_{1\leq i<j\leq 4} |z_i - z_j|\right)^{1/6},$$

becomes

$$\delta_3(x) = \max\left\{\left(|z_1-z_2|\cdot|z_1-z_3|\cdot|z_1-z_4|\cdot|z_2-z_3|\cdot|z_2-z_4|\cdot|z_3-z_4|\right)^{1/6}\ :\ z_1,\ z_2,\ z_3\ z_4 \in \sigma(x)\right\}.$$

Again we want to bring it under the reader's attention that repetition is allowed. We mentioned that an important property of the n'th spectral diameter is to be considered, and which plays an important role in the Scarcity Theorem. This property can be stated as follows:

> *If $\#\sigma(x) \leq n$, then $\delta_n(x) = 0$. If this is not the case, that is, $\#\sigma(x) > n$, then $\delta_n(x) > 0$.*

The above holds true since if $\#\sigma(x) \leq n$, then repetition of values will occur. Hence one of the factors $|z_i - z_j| = 0$ for some $i \neq j$, but $z_i = z_j$. If $\#\sigma(x) > n$ there is no repeated values and all the factors $|z_i - z_j| > 0$ for all i, j. Thus $\delta_n(x) > 0$.

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