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Single and cross-generation natural hedging of longevity and financial risk*

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Abstract

The paper provides natural hedging strategies among death benefits and annuities written on a single and on different generations. It obtains closed-form Delta and Gamma hedges, in the presence of both longevity and interest rate risk. We present an application to UK data on survivorship and bond dynamics. We first compare longevity and financial risk exposures: Deltas and Gammas for longevity risk are greater in absolute value than the corresponding sensitivities for interest rate risk. We then calculate the optimal hedges, both within and across generations. Our results apply to both asset and asset-liability management.

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1 Introduction

Longevity risk - i.e. the risk of unexpected changes in survivorship - is by now perceived as an important threat to the safety of insurance and re-insurance companies, as well as pension funds. Most actors in the financial market are long longevity risk. This stimulated the transformation of contracts subject to longevity risk into an asset class, as originally suggested by Blake and Burrows (2001). The creation of q-forwards, s-forwards and longevity bonds or swaps goes into this direction, but is still in its infancy. Waiting for the development of this market, insurance companies can benefit from the natural hedge between death benefits and life contracts, such as annuities. Exploiting the natural offsetting between death and life contracts - before using customized financial products such as longevity forwards - is by no doubt cheap and wise. The importance goes beyond theory, since Cox and Lin (2007) find empirical evidence that insurers whose liability portfolio benefits from natural hedging have a competitive advantage and charge lower premiums. In spite of being safe, sound and comparatively cheap, this task is not trivial, given the difficulties in assessing longevity risk on the one side, and given the subtle interactions between longevity and financial risk on the other. These interactions arise if one considers liability management only, since the value of the reserves is subject to interest rate risk, and - a fortiori - if one considers asset and liability management.

Natural hedging of longevity risk - as isolated from financial risk - has been recently addressed by Cox and Lin (2007), Wang et al. (2010) and Gatzert and Wesker (2010). Cox and Lin (2007), motivated by the empirical evidence mentioned above, propose to add mortality swaps between annuity providers and death assurance writers to make natural hedging feasible. Wang et al. (2010) are closer in spirit to our approach. They propose an immunization strategy by matching duration and convexity of life insurance and death benefits. They demonstrate that the strategy is effective in reducing longevity risk, once calibrated to the US mortality data. Gatzert and Wesker (2010) use simulations in order to select portfolios of policies which immunize the insurer's solvency against changes in mortality. Differently from Wang et al. (2010), they take into account the interaction between assets and liabilities and stress the importance of the asset side in order to improve hedging.
Natural hedging with financial risk has instead been studied by Stevens et al. (2011). They show that financial risk has a clear impact on the overall initial riskiness of the annuity-death benefit mix. It also affects its hedging possibilities, since ignoring financial risk may induce overestimates of the natural hedge of annuities provided by death benefits. The risk measure they adopt

is the minimal initial asset value required in order to end up with a positive and sufficiently large asset value. In their case, financial risk is due to the potential losses arising from assets only, while in our case it affects assets and the fair value of liabilities.

We extend the previous literature by providing a framework in which not only *longevity and financial risk* can be addressed together, with the latter referred to both assets and liabilities, but there is also the possibility of examining and exploiting the hedge *within a single generation and across generations or genders*. We provide all of these hedges in *closed form* and as Delta-Gamma hedges, which are notoriously less expensive than other strategies.

In a previous paper (Luciano et al. (2012)) we obtained closed form formulas for the fair price, Deltas and Gammas of the reserves of pure endowments. The risk factors against which to hedge were the differences between the mortality and interest rate intensities forecasted today and their actual realizations in the future. In this paper we provide the Deltas and Gammas in closed form for annuities and death assurances. The *main novelties* are two. We provide the natural hedges between life and death contracts including assets too (asset-liability management). Above all, we extend hedging between different generations. This requires splitting the longevity risk factor into a common and an idiosyncratic part.
The UK-calibrated application which concludes the paper confirms first *how relevant longevity risk is*, with respect to financial risk, in homogeneous (single generation) and heterogeneous (several generations or genders) portfolios. It permits to compare the magnitude of first and second order effects, i.e. Deltas and Gammas, within a single type of risk (longevity or financial) and across risks and generations. Second, the application shows how *straightforward the computation* of sensitivities and hedges is, given the presence of closed form solutions. It permits to appreciate its cost and feasibility. This happens for first order as well as for second order hedges. Third, it permits a wide range of *intra and cross-generation hedging strategies* which show how effective the Delta-Gamma hedge can be. Natural hedging involving only issued policies can be achieved exclusively when Delta or Delta–Gamma hedging the portfolio against longevity risk. When we consider also financial risk or self-financing strategies, reinsurance transactions are needed.

The paper is structured as follows: Section 2 reviews the mortality and interest rate model, recalls the corresponding survival probabilities and bond prices, as well as the Greeks of fairly-priced reserves for a pure endowment.

Section 3 extends pricing and hedging to annuities and death assurances. Section 4 focuses on the natural hedging opportunities provided by a portfolio mix of annuities and death assurances. It does that within a single generation and between cohorts. Section 5 presents the general framework for Delta-Gamma hedging longevity risk of life insurance liabilities. Section 6 presents an application calibrated to UK data, computes single-generation portfolios of annuities and death assurances which immunize the portfolio up to the first and second order and studies cross-generation immunization. Section 7 concludes.

2 Longevity and interest rate risk

We place ourselves in a continuous-time framework as suggested by Cairns et al. (2006a) and Cairns et al. (2008). In this framework we use a parsimonious, continuous-time model for mortality intensity - which extends the classical Gompertz law - and a benchmark model for interest rate risk - the Hull and White model. This section introduces the models for longevity and interest rate risk and the Delta-Gamma hedging for a pure endowment obtained in Luciano et al. (2012).

2.1 Model for longevity risk

We follow a well-established stream of literature (as summarized for instance in Cairns et al. (2006a)) and consider the time of death as the first jump time of a Poisson process with stochastic intensity, i.e. a Cox process. Let us introduce a filtered probability space $(\Omega, \mathbf{F}, \mathbb{P})$, equipped with a filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ which satisfies the usual properties of right-continuity and completeness.[1] Our approach is generation-based. We denote with $\lambda_x(t)$ the *spot mortality intensity* at calendar time t of a head belonging to a cohort (generation/gender) of individuals whose age is x at 0, the evaluation time. We assume that - under the $\mathbb{P}$ measure - the dynamics of the stochastic mortality intensity follows an Ornstein-Uhlenbeck process without mean reversion (OU):

$$d\lambda_x(t) = a_x \lambda_x(t)dt + \sigma_x dW_x(t),$$

where $a_x > 0$, $\sigma_x \geq 0$, W_x is a standard one-dimensional Brownian motion. Our choice of the OU process is motivated by its parsimony - very few parameters for calibration - and its appropriateness to fit human cohort-based

[1] This filtration reflects both the mortality and the financial information. For a discussion of its relationship with the natural filtration of the mortality-intensity and interest rate processes, as well as for the relevant change of measure, see Luciano et al. (2012).

life tables, because of its lack of mean reversion. It is an affine process - for which we can find closed-form expressions for the survival probability. Above all, it is a natural stochastic extension of the Gompertz model for the force of mortality, easy to interpret in the light of the traditional actuarial practice. Its major drawback is that λ_x can turn negative with positive probability, with the survival probability increasing in time. However, in practical applications we verify that this probability is negligible and that the survival probability is decreasing over the duration of human life.[2] Together with the spot intensity, we consider the *forward instantaneous intensity*, denoted as $f_x(t,T)$. This is the best forecast at time t of the spot intensity at T, since it converges to it when the horizon of the forecast goes to zero, or $T \to t$:

$$f_x(t,t) = \lambda_x(t).$$

Standard properties of affine processes allow us to represent the survival probability from time t to T as:

$$S_x(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda_x(s)ds\right) \mid \mathcal{F}_t\right] = e^{\alpha_x(T-t)+\beta_x(T-t)\lambda_x(t)}. \tag{1}$$

where $\alpha_x(\cdot)$ and $\beta_x(\cdot)$ are

$$\alpha_x(t) = \frac{\sigma^2}{2a_x^2}t - \frac{\sigma^2}{a_x^3}e^{a_x t} + \frac{\sigma^2}{4a_x^3}e^{2a_x t} + \frac{3\sigma^2}{4a_x^3}, \tag{2}$$

$$\beta_x(t) = \frac{1}{a_x}(1 - e^{a_x t}). \tag{3}$$

However, following Jarrow and Turnbull (1994), we can write the *survival probability* for the OU case in a more useful way:

$$S_x(t,T) = \frac{S_x(0,T)}{S_x(0,t)} \exp\left[-X_x(t,T)I(t) - Y_x(t,T)\right],$$

with

$$X_x(t,T) = \frac{\exp(a_x(T-t)) - 1}{a_x},$$

[2]See Luciano and Vigna (2008). In that paper, the authors argue that the OU model - together with other non-mean reverting affine processes - meets all the criteria - stated by Cairns et al. (2006a) - that a good mortality model should meet, apart from the strict positivity of the intensity. Indeed, it fits well historical data; its long-term future dynamics is biologically reasonable; it is convenient for pricing, valuation and hedging; its long-term mortality improvements are not mean-reverting. Most importantly for the case at hand, mortality-linked products can be priced using analytical methods.

$$Y_x(t,T) = -\sigma_x^2[1-e^{2a_x t}]X_x(t,T)^2/(4a_x),$$
$$I(t) := \lambda_x(t) - f_x(0,t).$$

$I(t)$ - the difference between the instantaneous mortality intensity at t and its forecast at time 0 - is what we interpret as the *longevity risk factor*: the error in forecast which makes insurance companies and pension funds exposed to longevity. At time 0, this is the only entry of future survival probabilities $S_x(t,T)$ which is random. A crucial feature which our hedging technique exploits is the dependence of I on t only (not on T). The same factor affects all survivals, independently of their horizon T.

2.2 Model for interest rate risk

Before introducing financial risk, let us clarify how it affects our strategies. Later on we compute the fair value of the reserves and assume that insurance companies hedge it: by so doing, we provide *liability hedging.* We also assume that insurance companies can set up the hedged portfolio under a self-financing constraint. If this constraint applies the (unique) premium received for death or life benefits is used in order to build the hedge. When self-financing is required, we are implementing at the same time *asset and liability management.* The assets are made by bonds only and have no stock component. While in the longevity domain we have modeled first spot intensities, then forward ones, for the financial domain we adopt the standard HJM framework (Heath et al. (1992)) and model directly the *instantaneous forward rate* $F(t,T)$, which is the time-t rate applying at instant T.
Also, we assume that no arbitrages exist and we start modelling directly under a risk-neutral measure equivalent to $\mathbb{P}$, which we call $\mathbb{Q}$. We assume that the process for the forward interest rate $F(t,T)$, defined on the probability space $(\Omega, \mathbf{F}, \mathbb{Q})$, is the well-known Hull and White (1990) model, with constant parameters:

$$dF(t,T) = -gF(t,T)dt + \Sigma e^{-g(T-t)}dW_F(t), \tag{4}$$

where $g > 0, \Sigma > 0$ and W_F is a univariate Brownian motion independent of W_x for all x. Financial and mortality risks are independent.
We recall that the limit of the forward rate when $T \to t$ is the short rate which applies instantaneously at t, $r(t)$:

$$F(t,t) = r(t). \tag{5}$$

Under the Hull and White choice, the *discount factor* from T to t is

$$B(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r(s)ds\right) \mid \mathcal{F}_t\right].$$

It can be written either in a form similar to (1) or, more effectively, as

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T)\right],$$

with

$$\bar{X}(t,T) := \frac{1-\exp(-g(T-t))}{g},$$

$$\bar{Y}(t,T) := \frac{\Sigma^2}{4g}\left[1-\exp(-2gt)\right]\bar{X}^2(t,T),$$

where K is the *financial risk factor*, measured by the difference between the time-t spot and forward rate:

$$K(t) := r(t) - F(0,t).$$

As in the longevity case, the financial risk factor is the difference between actual and forecasted rates and is the only source of randomness which affects bonds.[3] It is one ($K(t)$) across all bond maturities T.

2.3 Reserves and hedges for pure endowments

In the presence of both longevity and interest rate risk, the fairly-priced reserves of every insurance product become stochastic too. This generates the need for liability hedging and opens the way to asset and liability management, as recalled above. This section computes reserves and hedge ratios for pure endowments.

In order to compute the fair value of an insurance liability, a change of survival probability measure is still needed. Given the absence of arbitrage in the financial market, we choose a measure $\mathbb{Q}$ which allows the mortality intensity to remain affine under the changed measure. We also want the risk premium for longevity to be constant. This - quite standard - choice is equivalent to fixing a risk premium[4] $\theta_x(t) = \frac{q\lambda_x(t)}{\sigma_x}$, $q \in \mathbb{R}, q > -a_x$.

Since the assumption of independence between longevity and financial risk is preserved also after the change of measure, we can provide expressions for the fair value of insurance liabilities. Consider a pure endowment contract starting at time 0 and paying one unit of account if the head x is alive at

[3] Using this device, Jarrow and Turnbull (1994) set up the Delta-Gamma hedge of interest rate risk which we extend to longevity.

[4] Notice that, given the absence of a rich market for longevity bonds, there are no standard choices to apply in the choice of $\theta_x(t)$: see for instance the extensive discussion in Cairns et al. (2006b).

time T. Let us compute the *fair value* of such an insurance policy at time $t \geq 0$, $Z_{Ex}(t,T)$. We have:

$$\begin{aligned} Z_{Ex}(t,T) &= S_x(t,T)B(t,T) = \\ &= E_{\mathbb{Q}}\left[\exp\left(-\int_t^T \lambda_x(s)ds\right)|\mathcal{F}_t\right] E_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right)|\mathcal{F}_t\right] = \\ &= \frac{S_x(0,T)}{S_x(0,t)}\exp\left[-X_x(t,T)I(t) - Y_x(t,T)\right]\frac{B(0,T)}{B(0,t)}\exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T)\right], \end{aligned} \tag{6}$$

where the parameter a_x in X_x, Y_x has been turned into $a'_x = a_x + q > 0$, to account for the measure-change. From now on, and until Section 4 in which we introduce multiple cohorts, we suppress the subscript x which denotes the dependence on the cohort. Assuming a single premium paid at the policy issue, Z_E is also the time-t *reserve* for the policy, that needs to be hedged by the life office.
Using Ito's lemma, for given t we obtain the dynamics of the reserve Z_E as a function of the changes in the risk factors and the first and second-order sensitivities of survival probabilities and bond prices with respect to the risk factors:

$$dZ_E = B\left(\Delta^M \Delta I + \frac{1}{2}\Gamma^M \Delta I^2\right) + S\left(\Delta^F \Delta K + \frac{1}{2}\Gamma^F \Delta K^2\right),$$

where

$$\Delta^M(t,T) \doteq \frac{\partial S}{\partial I} = -S(t,T)X(t,T) < 0, \tag{7}$$

$$\Gamma^M(t,T) \doteq \frac{\partial^2 S}{\partial I^2} = S(t,T)X^2(t,T) > 0, \tag{8}$$

$$\Delta^F(t,T) \doteq \frac{\partial B}{\partial K} = -B(t,T)\bar{X}(t,T) < 0, \tag{9}$$

$$\Gamma^F(t,T) \doteq \frac{\partial^2 B}{\partial K^2} = B(t,T)\bar{X}^2(t,T) > 0. \tag{10}$$

We re-define the exposures to the risk factors for a pure endowment, so that

$$dZ_E = \Delta_E^M \Delta I + \frac{1}{2}\Gamma_E^M \Delta I^2 + \Delta_E^F \Delta K + \frac{1}{2}\Gamma_E^F \Delta K^2,$$

with

$$\begin{cases} \Delta_E^M(t,T) & \doteq & B(t,T)\Delta^M(t,T) < 0, \\ \Gamma_E^M(t,T) & \doteq & B(t,T)\Gamma^M(t,T) > 0, \\ \Delta_E^F(t,T) & \doteq & S(t,T)\Delta^F(t,T) < 0, \\ \Gamma_E^F(t,T) & \doteq & S(t,T)\Gamma^F(t,T) > 0. \end{cases} \tag{11}$$

In the next section we extend our approach by considering more complex insurance products and show how to compute their Deltas and Gammas.

3 Reserves and Delta-Gamma sensitivities of annuities and death assurances

In Sections 3.1 and 3.2 we compute the reserves and the Greeks for annuities and death assurances. These products present offsetting exposure to longevity risk and can provide natural hedging against it.

3.1 Annuities

Let us consider an annuity - with annual installments R - issued at time 0 to an individual belonging to generation x. We omit the subscript x to simplify the notation. Assuming the payment of a single premium at policy inception, the prospective reserve Z_A from $t > 0$ up to horizon T is:

$$Z_A(t,T) = R \sum_{u=1}^{T-t} B_{t,u} S_{t,u},$$

where we use the short notation $B_{t,u}$ for $B(t, t+u)$. We use the same shortcut for S and the Greeks below. The horizon T depends on the type of annuity issued, i.e. it is T with an annuity payable for T years and $\omega - x$ with a whole-life annuity. The change in the reserve for given t is straightforward to compute:

$$dZ_A = R \left[\Delta_A^M \Delta I + \frac{1}{2} \Gamma_A^M \Delta I^2 + \Delta_A^F \Delta K + \frac{1}{2} \Gamma_A^F \Delta K^2 \right],$$

where the Deltas and Gammas are

$$\Delta_A^M(t,T) = -\sum_{u=1}^{T-t} B_{t,u} S_{t,u} X_{t,u} = \sum_{u=1}^{T-t} \Delta_E^M(t, t+u) < 0,$$

$$\Gamma_A^M(t,T) = \sum_{u=1}^{T-t} B_{t,u} S_{t,u} [X_{t,u}]^2 = \sum_{u=1}^{T-t} \Gamma_E^M(t, t+u) > 0,$$

$$\Delta_A^F(t,T) = -\sum_{u=1}^{T-t} B_{t,u} S_{t,u} \bar{X}_{t,u} = \sum_{u=1}^{T-t} \Delta_E^F(t, t+u) < 0,$$

$$\Gamma_A^F(t,T) = \sum_{u=1}^{T-t} B_{t,u} S_{t,u} [\bar{X}_{t,u}]^2 = \sum_{u=1}^{T-t} \Gamma_E^F(t, t+u) > 0.$$

3.2 Death assurances

Let us consider a term assurance issued at time 0 to an individual belonging to generation x. Let it have maturity T and sum assured C. For simplicity, we assume that the benefit is paid at the end of the year of death, if it occurs before time T. Assuming the payment of a single premium at policy inception, the prospective reserve Z_D from $t > 0$ up to horizon T is:

$$Z_D(t,T) = C \sum_{u=1}^{T-t} B_{t,u}(S_{t,u-1} - S_{t,u}).$$

The change in the reserve Z_D at time t is

$$dZ_D = C \left[\Delta_D^M \Delta I + \frac{1}{2} \Gamma_D^M \Delta I^2 + \Delta_D^F \Delta K + \frac{1}{2} \Gamma_D^F \Delta K^2 \right],$$

where the following Deltas and Gammas with respect to longevity and financial risk appear:

$$\Delta_D^M(t,T) = \sum_{u=1}^{T-t} B_{t,u}(\Delta_{t,u-1}^M - \Delta_{t,u}^M) > 0, \tag{12}$$

$$\Gamma_D^M(t,T) = \sum_{u=1}^{T-t} B_{t,u}(\Gamma_{t,u-1}^M - \Gamma_{t,u}^M) < 0, \tag{13}$$

$$\Delta_D^F(t,T) = \sum_{u=1}^{T-t} (S_{t,u-1} - S_{t,u})\, \Delta_{t,u}^F < 0,$$

$$\Gamma_D^F(t,T) = \sum_{u=1}^{T-t} (S_{t,u-1} - S_{t,u})\, \Gamma_{t,u}^F > 0.$$

Notice that $\Delta_D^M(t,T)$ and $\Gamma_D^M(t,T)$, the Greeks for longevity risk of death assurances, have opposite signs with respect to those of annuities, since the bond prices which appear in the Deltas and Gammas of pure endowments are decreasing in the maturity u. Consistently across maturities first order sensitivities to longevity risk of death assurances are positive and second order sensitivities negative.
The sensitivities of each generations' reserves with respect to the risk factors provide closed-form exposure to mortality and interest rates forecast errors. They do more: they open the possibility of hedging.

4 Natural hedging within and across generations

4.1 Single generation natural hedging

Consistently with intuition and actuarial practice, a natural offsetting is possible between the change in the reserve of an annuity and a death assurance. Even in this stochastic context, where risk comes from both interest rate and longevity, the exposure of an annuity provider can be reduced through positions on standard term death assurances. It is possible to compute the number of offsetting contracts in closed form. Consider for simplicity an insurer who has issued m annuities with annuity rate R and maturity T_1 and n death assurances with sum assured C and maturity T_2 written on the same generation.[5] The change in the value of its liabilities $Z_H(t)$, for $t < \min\{T_1, T_2\}$, can be written as

$$\begin{aligned}
dZ_H(t) &= mdZ_A(t,T_1) + ndZ_D(t,T_2) = \\
&= (Rm\Delta_A^M + Cn\Delta_D^M)\Delta I(t) + \frac{1}{2}(Rm\Gamma_A^M + Cn\Gamma_D^M)\Delta I^2(t) + \\
&+ (Rm\Delta_A^F + Cn\Delta_D^F)\Delta K(t) + \frac{1}{2}(Rm\Gamma_A^F + Cn\Gamma_D^F)\Delta K^2(t).
\end{aligned}$$

where the coefficients multiplying the risk factor changes are simply the weighted sums of the sensitivities of each single liability. This follows from the fact that the risk factors considered are the same across products and maturities. As remarked above, they depend on time t only. What differs from product to product is the first and second order exposure - i.e. the Greeks - to the changes of the risk factors.

The crucial point for the natural hedging technique is that the sign of the first and second order sensitivities with respect to the mortality risk factor– $Rm\Delta_A^M + Cn\Delta_D^M$ and $Rm\Gamma_A^M + Cn\Gamma_D^M$ – is not uniquely determined. Indeed, it is possible to select m and n in such a way as to have the coefficients of ΔI and ΔI^2 equal to 0.[6] In this way longevity risk is hedged. In Section 5 we extend this technique so as to incorporate financial risk hedging. This requires more than two contracts.

[5] Everything still works when products are written on different maturities T_i. The calibrated example in Section 6 has indeed different maturities.

[6] For the meaning of positive and negative m and n we refer the reader to Section 5.

4.2 Natural hedge across generations

Imagine that we have to hedge the liability of a pure endowment written on generation x with the one written on another generation or gender y. We refer to generations for simplicity and use an index to denote the generation, in intensities, survival probabilities and Greeks. Imagine also that the two generations have instantaneously correlated Brownian intensities (after the change of measure)

$$\begin{aligned} d\lambda_x &= a'_x\lambda_x dt + \sigma_x dW_x(t), & (14) \\ d\lambda_y &= a'_y\lambda_y dt + \sigma_y dW_y(t). & (15) \end{aligned}$$

Let ρ be the correlation coefficient between the two Brownians. Appendix A shows that - by re-parametrizing the two sources of risk through independent Wieners, one can isolate the mortality risk which affects generations x and y (the common risk) from the one which affects y only (the idiosyncratic risk). The first one is the risk factor $I(t)$ we defined above for generation x, while the latter - which we denote as $I'(t)$ - is a factor which is instantaneously uncorrelated with $I(t)$.

The Greeks for pure endowments, annuities and death assurances we derived in the previous section apply for the products written on generation x. We now provide the analogous Greeks for generation y. We start with the sensitivities of the survival probabilities with respect to the factors. Appendix A shows that

$$\begin{aligned} dS_y(t,T) &= \frac{\partial S_y(t,T)}{\partial t}dt + \Delta_y^{M,x}(t,T)dI + \Delta_y^{M,y}(t,T)dI' + \\ &+ \frac{1}{2}\Gamma_y^{M,x}(t,T)dI^2 + \frac{1}{2}\Gamma_y^{M,y}(t,T)dI'^2. \qquad (16) \end{aligned}$$

where we use the superscripts M, x and M, y to denote the mortality hedging coefficients with respect to the common and idiosyncratic risk factor, and we define the Greeks as:

$$\begin{aligned} \Delta_y^{M,x}(t,T) &\doteq \frac{\partial S_y(t,T)}{\partial \lambda_y}\rho\frac{\sigma_y}{\sigma_x} = -\rho\frac{\sigma_y}{\sigma_x}X_y(t,T)S_y(t,T) = \rho\frac{\sigma_y}{\sigma_x}\Delta_y^M(t,T), & (17) \\ \Delta_y^{M,y}(t,T) &\doteq \frac{\partial S_y(t,T)}{\partial \lambda_y} = -X_y(t,T)S_y(t,T) = \Delta_y^M(t,T), & (18) \\ \Gamma_y^{M,x}(t,T) &\doteq \left(\rho\frac{\sigma_y}{\sigma_x}\right)^2\frac{\partial^2 S_y(t,T)}{\partial^2 \lambda_y} = \left(\rho\frac{\sigma_y}{\sigma_x}\right)^2 X_y^2(t,T)S_y(t,T) = \left(\rho\frac{\sigma_y}{\sigma_x}\right)^2\Gamma_y^M(t,T), & (19) \\ \Gamma_y^{M,y}(t,T) &\doteq \frac{\partial^2 S_y(t,T)}{\partial^2 \lambda_y} = X_y^2(t,T)S_y(t,T) = \Gamma_y^M(t,T). & (20) \end{aligned}$$

The first derivative with respect to common risk $\Delta_y^{M,x}$ has sign opposite to that of the correlation coefficient ρ, while the one with respect to idiosyncratic

risk $\Delta_y^{M,y}$ is negative. Notice that both Gamma coefficients are non negative, as usual. The Gamma with respect to the common risk $\Gamma_y^{M,x}$ is positive, unless there is no correlation, while the one with respect to idiosyncratic risk, $\Gamma_y^{M,y}$, is strictly positive.
For given t, for pure endowments written on generation y we have

$$dZ_{Ey} = B\left(\Delta_y^{M,x}\Delta I + \frac{1}{2}\Gamma_y^{M,x}\Delta I^2 + \Delta_y^{M,y}\Delta I' + \frac{1}{2}\Gamma_y^{M,y}\Delta I'^2\right) + S\left(\Delta_y^F \Delta K + \frac{1}{2}\Gamma_y^F \Delta K^2\right),$$

which, exploiting the notation introduced in (11),we rewrite as

$$dZ_{Ey} = \Delta_{Ey}^{M,x}\Delta I + \frac{1}{2}\Gamma_{Ey}^{M,x}\Delta I^2 + \Delta_{Ey}^{M,y}\Delta I' + \frac{1}{2}\Gamma_{Ey}^{M,y}\Delta I'^2 + \Delta_{Ey}^F \Delta K + \frac{1}{2}\Gamma_{Ey}^F \Delta K^2.$$

The change in the reserve of an annuity for given t has the expression

$$dZ_{Ay} = R\left[\Delta_{Ay}^{M,x}\Delta I + \frac{1}{2}\Gamma_{Ay}^{M,x}\Delta I^2 + \Delta_{Ay}^{M,y}\Delta I' + \frac{1}{2}\Gamma_{Ay}^{M,y}\Delta I'^2 + \Delta_{Ay}^F \Delta K + \frac{1}{2}\Gamma_{Ay}^F \Delta K^2\right],$$

where

$$\Delta_{Ay}^{M,j}(t,T) = \sum_{u=1}^{T-t} \Delta_{Ey}^{M,j}(t,t+u),$$

$$\Gamma_{Ay}^{M,j}(t,T) = \sum_{u=1}^{T-t} \Gamma_{Ey}^{M,j}(t,t+u) \geq 0,$$

with $j = x, y$. Again, while the sign of $\Delta_{Ay}^{M,y}$ is negative, as usual, the sign of $\Delta_{Ay}^{M,x}$ is opposite to the one of ρ. As for the Gammas, they are positive, with the exception of$\Gamma_{Ay}^{M,x}(t,T)$, which is null when $\rho = 0$.

We can write the change in the reserve for a death assurance as

$$dZ_{Dy} = C\left[\Delta_{Dy}^{M,x}\Delta I + \frac{1}{2}\Gamma_{Dy}^{M,x}\Delta I^2 + \Delta_{Dy}^{M,y}\Delta I' + \frac{1}{2}\Gamma_{Dy}^{M,y}\Delta I'^2 + \Delta_{Dy}^F \Delta K + \frac{1}{2}\Gamma_{Dy}^F \Delta K^2\right],$$

where

$$\begin{aligned}
\Delta_{Dy}^{M,j}(t,T) &= \sum_{u=1}^{T-t} B_{t,u}(\Delta_y^{M,j}(t,t+u-1) - \Delta_y^{M,j}(t,t+u)), \\
\Gamma_{Dy}^{M,j}(t,T) &= \sum_{u=1}^{T-t} B_{t,u}(\Gamma_y^{M,j}(t,t+u-1) - \Gamma_y^{M,j}(t,t+u)),
\end{aligned}$$

with $j = x, y$. For positive ρ, the comments on the sign of $\Delta_{Dy}^{M,x}(t,T)$ are the same as in (12). The opposite comments apply for negative correlation.

Provided that $\rho \neq 0$, the comments on the sign of $\Gamma^{M,x}_{Dy}(t,T)$ are the same as in (13). The same comments as in (12), (13) hold for the Delta and Gamma of generation y with respect to its factor, $\Delta^{M,y}_{Dy}(t,T), \Gamma^{M,y}_{Dy}(t,T)$.

The two-generations' problem has an important feature. If we have not enough insurance products on generation x available, we can still hedge against its longevity risk using some contract written on generation y. In this sense, generation's y products complete the insurance market for x. In Section 6.3 we provide a calibrated example of such a situation.

Using the properties outlined above, the next section describes the general framework for the Delta-Gamma hedging of a portfolio of insurance liabilities, which reduces to solving a system of linear equations.

5 Hedging the longevity risk of life insurance liabilities

Imagine an insurer who has issued n_H products with fair value Z_H. He is subject to both longevity and interest rate risk. He can Delta-Gamma hedge his position by assuming positions in n_i units of some other N instruments with fair value Z_i, $i = 1, ...N$, by creating portfolios whose exposures with respect to the risk factors are neutralized. In order to simplify the notation, the index i of each product denotes both the *type* of product (i.e. E, A, D) and the maturity of the product. We interpret negative positions $n_i < 0$ as short positions on the corresponding product, i.e. as a need for policy selling, positive solutions $n_i > 0$ as reinsurance purchases for instruments of that type and maturity.[7] The products available for hedging can be written either on the same cohort on which Z_H is written or on different ones. Hedging portfolios are obtained equating to zero $\Delta^{M,j}_{\Pi}$,Δ^{F}_{Π},$\Gamma^{M,j}_{\Pi}$,Γ^{F}_{Π}, where the subscript Π refers to the portfolio itself, the superscript M, j refers to to the j-th longevity risk factor, $j = 1, ..., J$ is the cohort on which the product is written.[8] F refers to the financial risk factor, which is unique across cohorts since, as explained in Section 2.2, all discount factors and bond prices - i.e., the whole term structure - is affected by the same risk factor, independently of the maturity. Bonds from the interest rate market can be used as hedging instruments for financial risk.

[7] Alternatively, we can interpret positive positions as need for mortality-linked contracts, such as survivor bonds and other derivatives.

[8] Notice that each time that the products written on a generation j are used to hedge those written on a generation x, the framework described in Section 4.2 is used. This means also that the number of longevity risk factors against which to hedge is the same as the number of generations of the portfolio.

The quantities $n_i, i = 1, ...N$ therefore solve the following system (for given n_H):

$$\begin{cases} n_H \Delta_H^{M,j}(t, T_H) + \sum_{i=1}^{N} n_i \Delta_i^{M,j}(t, T_i) = 0, \; j = 1, ..., J. & (21) \\ n_H \Delta_H^{F}(t, T_H) + \sum_{i=1}^{N} n_i \Delta_i^{F}(t, T_i) = 0, & (22) \\ n_H \Gamma_H^{M,j}(t, T_H) + \sum_{i=1}^{N} n_i \Gamma_i^{M,j}(t, T_i) = 0, \; j = 1, ..., J. & (23) \\ n_H \Gamma_H^{F}(t, T_H) + \sum_{i=1}^{N} n_i \Gamma_i^{F}(t, T_i) = 0. & (24) \end{cases}$$

The expressions for the Delta and Gamma coefficients take the forms we derived in the previous sections, depending on the type of the i-th product (pure endowment, annuity, death assurance). Notice the following. The system means that - in principle - one can hedge the liabilities of a single generation (say $j = 1$) using contracts on that generation only, on that and other generations ($j = 1, ..., J$) or on different generations only ($j \neq 1$). When we solve simultaneously

- all $2 + 2J$ equations (21),(22),(23),(24) we Delta-Gamma hedge the portfolio reserves against both longevity and interest rate risk;
- the $2J$ equations (21) and (23) we Delta-Gamma hedge longevity risk only;
- the $J+1$ equations (21),(22) we Delta-hedge longevity and interest rate risk;
- when solving (21) only we are Delta-hedging the longevity risk of the reserves.

In all cases we perform liability hedging. A further equation must be added to the system if one requires the portfolio to be self-financed, i .e. if one wants to focus on asset-liability management. If a single fair premium is paid at policy issue, self-financing strategies are characterized by the self-financing constraint:

$$n_H Z_H + \sum_{i=1}^{N} n_i Z_i = 0. \qquad (25)$$

Indeed, since positive values of n_i are interpreted as purchases of reinsurance contracts (assets), while negative values are short positions or sales of the corresponding insurance contracts (liabilities), the self-financing constraint is imposing nothing else than the use of the inflows from sales in order to finance asset purchases. The self-financing constraint means also that we are considering asset-liability management strategies, since the premium is used in order to buy hedging instruments. In this sense our results cover both liability management and ALM.
The reader can object that we do not consider among the assets and liabilities purely financial contracts, i.e., bonds. Actually, it is sufficient to expand our notation in order to include a contract Z_{N+1} whose value is the discount factor (or to imagine a fake generation which has survival probability constantly equal to one) to include in our equations also a zero-coupon bond.[9]

The quantities n_i to hold in order to implement a hedging strategy are the solutions of the systems described above, if they exist. We obtain a unique solution to the system of equations that solves the Delta or Delta-Gamma hedging problem if the matrix of the n_i coefficients is full-rank and the number of hedging instruments equals this rank. This imposes a restriction on how many life-insurance liabilities (and bonds, when they are admitted) are used for coverage:

- $N = 2 + 2J$ hedging instruments for Delta-Gamma hedging longevity and interest rate risk;
- $N = 2J$ hedging instruments for Delta-Gamma hedging longevity risk only;
- $N = J+1$ hedging instruments for Delta-hedging longevity and interest rate-risk;
- $N = J$ instruments in order to Delta-hedge longevity risk.

If one accepts multiple solutions, this restriction can be relaxed. In each of the above cases we require a further instrument if we want the portfolio to be self-financing.

6 UK-calibrated application

In this section we present a calibrated application which concerns two generations. In Section 6.1 we explain how we calibrate the model to UK data and

[9]Since any bond can be stripped into zero-coupon bonds, with some additional notation we can include coupon bonds too.

present the hedge ratios for the case at hand. In Section 6.2 we compute the portfolio mix of annuities and death assurances which - within each single generation - immunizes the portfolio up to the first and second order, comparing with the related literature. In Section 6.3 we study cross-generation immunization, considering two generations.

6.1 Calibration

In calibrating the model to UK data we do the further - methodologically irrelevant - assumption that, for each generation, $q = 0$, so that $a' = a$, and the risk premium on longevity risk is null.[10] We calibrate the parameters of the mortality intensity processes using projected tables for English annuitants (IML tables). We consider contracts written on the lives of male individuals who were 65 years old on 31/12/2010 (generation 1945, to which we refer as y), and on the lives of males who were 75 years old on 31/12/2010 (generation 1935, to which we refer as x). We calibrate the OU model to those two generations. The values of the parameters, considering $t = 0$, are: $a_y = 10.94\%$, $\sigma_y = 0.07\%$, $\lambda_y(0) = -\ln p_y = 0.885\%$ for the first and $a_x = 9.95\%$, $\sigma_x = 0.03\%$, $\lambda_x(0) = -\ln p_x = 1.14\%$.[11] In the application that follows, we consider both longevity risk only and the case in which interest rate risk is present too. In order to be ready to introduce interest rate risk, we calibrate a constant-parameter Hull-White model to the UK government bond markets at 31/12/2010. The calibrated parameters for the forward-rate dynamics are $g = 2.72\%$ and $\Sigma = 0.65\%$. We first analyse each generation separately. For each one, we compute the Deltas and Gammas for annuities and death assurances with different maturities. Table 1 summarizes the results for the two generations in terms of reserves and Greeks for three different products: a whole-life annuity with unit benefit and two death assurance (to which we refer from now on as DA) contracts with different maturities and insured sum $C = 100$.

It is evident from the table that - within each single generation - the Deltas and Gammas with respect to longevity are greater (in absolute value) than the Greeks for financial risk. This happens consistently across products and maturities. When, instead of looking at each single generation, we compare across them, the magnitude of the Greeks depends on the product. For annuities, both the longevity and financial Greeks are greater (in absolute

[10]This assumption could be easily removed by calibrating the model parameters to actual insurance products, as soon as a liquid market for mortality derivatives will exist.

[11]We refer the reader to Luciano and Vigna (2008) for a full description of the data set and the calibration procedure. Notice that the calibrated parameters satisfy the sufficient condition for biological reasonableness for the OU model, see footnote 3 above.

Table 1: Reserves and Greeks for annuities and death assurances

Maturity	Whole-life annuity	10-year DA	20-year DA
Generation 1935 (x)			
Value	12.66	15.53	33.59
Δ_i^M	-269.54	1240.69	2181.05
Γ_i^M	16164.35	-20053.31	-107139.46
Δ_i^F	-95.82	-83.22	-308.34
Γ_i^F	1007.17	537.53	3406.96
Generation 1945 (y)			
Value	13.09	12.94	30.05
Δ_i^M	-323.48	1355.29	2619.28
Γ_i^M	24847.66	-23225.97	-146827.81
Δ_i^F	-100.92	-70.48	-285.16
Γ_i^F	1075.37	459.63	3211.46

value) for the younger generation, y. For death assurances, the same result applies to the sensitivities to longevity risk, while the opposite happens for the Greeks with respect to interest rate risk. Notice that, while the Greeks with respect to longevity risk of annuities and DAs have opposite signs, all the sensitivities with respect to financial risk match in sign. This fact ensures us that, as we highlight in Section 6.2, there exists no portfolio mix without long positions on insurance contracts able to neutralize the exposure against both longevity and interest rate risk. Reinsurance - i.e. being long on some death or life contract - is needed.

6.2 Intra-generational natural hedge

Consider now an insurer who has issued a whole-life annuity with unit benefit on an individual belonging to generation y. We have $n_H = -1$ and we set the terminal life-table age ω =110. Using insurance contracts and bonds, he aims at achieving instantaneous neutrality to first and/or second order shocks to longevity and interest rates. We compute the hedging coefficients, i.e. the positions the insurer has to hold, when he wants to cover his position using DAs on the same cohort y of the annuitant. The DAs have benefit $C = 100$ and different maturities. We assume the existence of enough instruments to provide a unique solution to the system in Section 5 and compute the hedging strategies when the insurer aims at different objectives. The objectives are specified in the first column. The rest of Table 2 reports the quantities n_i needed for hedging. There is a column for each DA maturity. The last

Table 2: Hedging strategies for an annuity on generation y

Strategy/Instrument	5-y DA	10-y DA	15-y DA	20-y DA	30-y DA	Liquidity
D M				-0.23		30.45
DG M		0.66		-0.49		33.26
D MF				0.84	-0.73	17.29
DG MF		-10.34	13.82	-7.02	1.17	12.07
D M SF		-7.17		2.68		-
DG M SF		126.57	-112.63	28.75		-
D MF SF		4.78		-5.24	2.71	-
DG MF SF	-60.98	81.13	-47.53	8.08	2.05	-

In the table D stands for Delta-hedging, DG for Delta-Gamma hedging, M for longevity only, MF for both longevity and financial risk neutralization, SF for self-financing. Empty cells refer to instruments which are not used to set up the strategy and their optimal holding is set to zero.

column contains the initial value of the hedged portfolio, i.e. the left-hand side of (25). Self-financing strategies have obviously zero value. Empty cells refer to redundant hedging instruments, which are not used in the hedging strategy. Their optimal holdings are set to zero. The first row of the table shows that the only case in which a portfolio of policies issued by the insurer is naturally hedged without resorting to reinsurance is when we consider the Delta-hedging of longevity risk only, as in Wang et al. (2010). When we set up all the other hedging strategies, i.e. we want to neutralize the exposure to both sources of risk or to set up self-financing strategies, we are not able to find unique solutions with negative coefficients n_i only. These are the other rows. In other words, except for the first line, a portfolio made by liabilities only does not provide a perfect hedging strategy. It is sufficient to add financial risk to the picture or require the strategy to be self-financing to make it impossible to immunize the liabilities with the use of standard life insurance contracts (annuities and DAs) only.[12] This fact, as already mentioned, depends on the signs of the coefficients in system (21)-(25). Our comment therefore holds in general, not only in the example.

Notice that it is also possible to substitute some hedging instruments with bonds from the interest rate market.[13]

[12]Notice that in theory - but not in our application - it is possible to set up a Delta-Gamma hedging strategy of longevity risk only without resorting to reinsurance.

[13]For example, a self-financing Delta-Gamma hedging strategy involving bonds consists in issuing 5.02 15-year term DAs, buying reinsurance for 1.50 20-year DAs and 4.68 10-year DAs and by taking positions -9.14 and 32.73 on UK-government zero-coupon bonds with 5 and 10 years maturity respectively.

6.3 Cross-generational natural hedges

We now consider the case in which the insurer has a portfolio made by products issued on the two generations x and y and wants to hedge the longevity risk of the first generation. The financial risk factor obviously does not change across cohorts: hence, we can simply compute the Greeks regarding interest rate risk by summing up the Deltas and Gammas of each liability. As concerns the longevity risk factor, we know from Section 4.2 that in principle one can hedge a liability written on cohort x using either products on x or products on x and y. We can then construct Delta-Gamma neutral portfolios by using products from both cohorts.
As an example, imagine an annuity provider who has issued an annuity on a head of cohort x ($n_H = -1$ and Z_H is the fair value of A on generation x). There exists another generation y. Consider a market in which at least one of two DAs on x (with maturities 10 and 15 years) is available, while DAs with different maturities (10, 15, 20, 25, 30 years) written on the younger generation y are present. Table 3 reports the optimal hedging strategies achieving different goals. As in Section 6.2, we assume that there are enough products to guarantee the existence and uniqueness of the hedging strategy. As usual, the goals are specified in the first column, while the others contain the number of hedging instruments n_i. The last column contains again the portfolio value. Empty cells refer to available products which are not used as hedging instruments. Their optimal holding is set to zero. Notice that in this particular example the optimal hedging strategies do not depend on ρ. This is due to the fact that the product to be hedged is written on one cohort (x) only. If we hedge a portfolio of products written on both generations, the hedge coefficients do depend on ρ.[14]

In line 1 and line 3 of Table 3 the insurer has access to the 10-year DA

[14] Let us show how ρ enters the problem. In the presence of generations x and y, focus on the $J = 2$ equations (21) (for the 2 equations (23) a similar argument holds):

$$\begin{cases} -n_H \Delta_H^{M,x} = n_1 \Delta_1^{M,x} + ... + n_{N'} \Delta_{N'}^{M,x} + \rho \frac{\sigma_y}{\sigma_x} \left(n_{N'+1} \Delta_{N'+1}^{M,y} + ... + n_{N'+N''} \Delta_{N'+N''}^{M,y} \right) \\ -n_H \Delta_H^{M,y} = n_1 \cdot 0 + ... + n_{N'} \cdot 0 + n_{N'+1} \Delta_{N'+1}^{M,y} + ... + n_{N'+N''} \Delta_{N'+N''}^{M,y} \end{cases}$$

where N' is the number of hedging products written on x, N'' is the number of hedging products written on y and $N = N' + N''$. Notice that the right hand side of the second equation is equal to the term in parenthesis of the first equation. When the annuity is on generation x, its Delta with respect to longevity risk of generation y is zero ($\Delta_H^{M,y} = 0$). This implies that in the right hand side of the first equation the term multiplying ρ is zero. Hence, ρ does not affect the solution. If the initial liability is a product or a portfolio of products written also on y, $\Delta_H^{M,y} \neq 0$. This implies that the solution does depend on ρ. A deep analysis of how ρ affects the hedging strategies in general would require a detailed investigation of all possible cases.

on x only, while in the other lines he can use also the 15-year DA on that generation. Notice that in all examples products on generation y are needed to hedge generation's x liabilities. For instance, finding a unique solution to the Delta hedging of longevity and financial risk (line 1) is possible only when considering also the products on cohort y. Similarly, a Delta-Gamma self-financing hedge against the exposure to both longevity and interest rate risk (line 6) is possible, by using all the products available on both generations. In these - and all the intermediate - cases, the second cohort "completes" the market for the first generation. This can be of practical relevant importance when hedging annuities written on the lives of old generations, for which very few life insurance contracts are available. DAs written on a younger generation - as in our example - can be used to "complete" the market.

Table 3: Hedging strategies for an annuity on generation x

Instrument	10-y DA_x	15-y DA_x	5-y DA_y	10-y DA_y	15-y DA_y	20-y DA_y	30-y DA_y	Value
Strategy								
D MF	-0.22	N/A		-3.15	2.03			12.92
DG MF	0.45	-0.45	-22.93	19.22	-8.09	1.92		34.92
D M SF	2.58	N/A		-3.83	1.04			-
DG M SF	1.90	-1.45		121.43	-108.61	28.2		-
D MF SF	-62.14	41.88		70.75	-45.55			-
DG MF SF	0.45	-0.45	-0.42	2.38	-3.77	2.73	-0.49	-

The Table reports the hedging strategies when at least one product written on generation x is available on the market. Cells with "N/A" refer to products which are not available in the market. Empty cells refer to available products which are not used as hedging instruments.

7 Conclusions

In this paper we studied natural hedging of financial and longevity risk with the Delta-Gamma hedging technique, which is notoriously simple, comparatively inexpensive and easy to extend to more complex financial or insurance contracts. We assumed a continuous-time cohort-based model for longevity risk which generalizes the classical Gompertz law and a standard stochastic interest rate model (Hull-White).

We extended the previous literature by analyzing *financial and longevity risk* at the same time, by providing *closed form hedges* and by giving *intra-generation as well as cross-generation* hedges. We obtained portfolios which are immunized to longevity and financial risk up to the second order and we clarified the role of natural hedging between annuities and death benefits. The reinsurance needed for coverage - or the amount of additional premiums obtained by issuing natural hedging policies - depends crucially on the existence of longevity-only or longevity-and-financial risk, on the hedging requirements (Delta, Delta-Gamma, liabilities only or ALM), as well as on the number of generations in the portfolio.

Our numerical application to a UK sample achieves *three goals*. *First*, it permits to compare financial and longevity sensitivities (the Greeks), within and across generations. *Second*, it shows how to perform static natural hedging up to the second order in closed form and to appreciate its cost and feasibility. *Last but not least*, it allows to discuss intra and cross-generational hedge and its effectiveness. In particular, the application shows that when there are not enough products written on one generation, products written on other cohorts help completing the longevity market and make feasible natural hedging. This seems particularly important when hedging annuities written on very old people, such as pensioners.

Appendix A

This Appendix obtains the risk factors against which to hedge in the presence of two correlated Brownian motions which affect the intensities of two generations/genders.

We can write the dynamics of the generations x and y's intensities, (14) and (15)

$$\begin{aligned} d\lambda_x &= a'_x \lambda_x(t)dt + \sigma_x dW_x(t), \\ d\lambda_y &= a'_y \lambda_y(t)dt + \sigma_y dW_y(t). \end{aligned}$$

with $< dW_x(t), dW_y(t) >= \rho dt$, in terms of two independent Brownian motions, $\tilde{W}_x$ and $\tilde{W}_y$

$$d\lambda_x = a'_x\lambda_x(t)dt + \sigma_x d\tilde{W}_x(t), \tag{26}$$

$$d\lambda_y = a'_y\lambda_y(t)dt + \sigma_y(\rho d\tilde{W}_x(t) + \sqrt{1-\rho^2}d\tilde{W}_y(t)). \tag{27}$$

Recall that, since the processes (14) and (15) are two affine processes, the following expression for the survival probabilities holds true:

$$S_j(t,T) = e^{\alpha_j(t,T)+\beta_j(t,T)\lambda_j(t)},$$

with $j = x, y$ and

$$\beta_j(t,T) = \frac{1-e^{a_j(T-t)}}{a_j} = -X_j(t,T).$$

We can write dS_j using Ito's lemma:

$$dS_j = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial \lambda_j}d\lambda_j + \frac{1}{2}\frac{\partial^2 S}{\partial \lambda_j^2} < d\lambda_j, d\lambda_j > . \tag{28}$$

We highlight that

$$\frac{\partial S_j}{\partial \lambda}(t,T) = \beta_j(t,T)S_j(t,T) = -X_j(t,T)S_j(t,T),$$

and identify a common longevity risk factor as follows. Since from (26)

$$d\tilde{W}_x = \frac{d\lambda_x - a'_x\lambda_x(t)dt}{\sigma_x}, \tag{29}$$

we can rewrite the dynamics of the intensity of generation y as

$$d\lambda_y(t) = \rho\frac{\sigma_y}{\sigma_x}d\lambda_x(t) + (a'_y\lambda_y(t) - \rho\frac{\sigma_y}{\sigma_x}a'_x\lambda_x(t))dt + \sqrt{1-\rho^2}\sigma_y d\tilde{W}_y(t), \tag{30}$$

in which we show dependence on $d\lambda_x = dI(t)$ and on another factor, which we define:

$$d\lambda'_y(t) = (a'_y\lambda_y(t) - \rho\frac{\sigma_y}{\sigma_x}a'_x\lambda_x(t))dt + \sqrt{1-\rho^2}\sigma_y d\tilde{W}_y(t).$$

Expanding expression (28) we have:

$$dS_x(t,T) = \frac{\partial S_x(t,T)}{\partial t}dt + \frac{\partial S_x(t,T)}{\partial \lambda_x}d\lambda_x + \frac{1}{2}\frac{\partial^2 S_x(t,T)}{\partial \lambda_x^2}d\lambda_x d\lambda_x, \tag{31}$$

$$dS_y(t,T) = \frac{\partial S_y(t,T)}{\partial t}dt + \frac{\partial S_y(t,T)}{\partial \lambda_y}d\lambda_y + \frac{1}{2}\frac{\partial^2 S_y(t,T)}{\partial \lambda_y^2}d\lambda_y d\lambda_y. \tag{32}$$

Rearranging some terms, we have the following expression for the dynamics of the reserves of a pure endowment written on cohort y:

$$
\begin{aligned}
dS_y(t,T) &= \frac{\partial S_y(t,T)}{\partial t}dt + \frac{\partial S_y(t,T)}{\partial \lambda_y}\rho\frac{\sigma_y}{\sigma_x}d\lambda_x + \frac{\partial S_y(t,T)}{\partial \lambda_y}d\lambda'_y + \\
&+ \frac{1}{2}\left(\rho\frac{\sigma_y}{\sigma_x}\right)^2\frac{\partial^2 S_y(t,T)}{\partial \lambda_y^2}d\lambda_x d\lambda_x + \frac{1}{2}\frac{\partial^2 S_y(t,T)}{\partial \lambda_y^2}d\lambda'_y d\lambda'_y. \quad (33)
\end{aligned}
$$

We have thus decomposed the dynamics of S_y and highlighted its dependence on the two risk factors we identified above: the change in the mortality intensity of cohort x and the risk factor $d\lambda'_y$, which represents the residual uncertainty, the part of the unexpected change in the mortality intensity of λ_y which is uncorrelated with the dynamics of the mortality of cohort x. We further notice that the dynamics of the risk factor $I(t)$ we defined in Section 2.1 coincides with that of $\lambda_x(t)$. Comparing the above expressions (31) and (33) it is clear that the possibility of hedging across generations exists. Defining as Greeks the coefficients of the first and second order changes in the risk factors - according to (17), (18), (19), (20) - and relabeling the changes in the intensities as dI and dI' we obtain $dS_y(t,T)$ as reported in (16) in the text.

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