New York Journal of Mathematics

New York J. Math. **14** (2008) 305–323.

Spectral theory of Toeplitz and Hankel operators on the Bergman space A^1

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Abstract. The Fredholm properties of Toeplitz operators on the Bergman space A^2 have been well-known for continuous symbols since the 1970s. We investigate the case $p = 1$ with continuous symbols under a mild additional condition, namely that of the logarithmic vanishing mean oscillation in the Bergman metric. Most differences are related to boundedness properties of Toeplitz operators acting on A^p that arise when we no longer have $1 < p < \infty$; in particular bounded Toeplitz operators on A^1 were characterized completely very recently but only for bounded symbols. We also consider compactness of Hankel operators on A^1.

Contents

1. Introduction

Spectral theory of Toeplitz and Hankel operators has been extensively studied in the Hilbert space setting, most prolifically in the case of the Hardy space H^2 and of the Bergman space A^2, but also a very extensive

Received January 22, 2008.

Mathematics Subject Classification. 47B35, 47A53, 32A36, 32A37.

Key words and phrases. Toeplitz operators, Hankel operators, compactness, Fredholmness, Bergman space.

The first author was partially supported by the Väisälä foundation of the Finnish Academy of Science and Letters. The second author was supported by Academy of Finland Project 207048.

ISSN 1076-9803/08

theory exists for H^p and A^p when $1 < p < \infty$, see [4]. The endpoint cases have received less attention in the past, see [15, 18] and references therein for these operators acting on H^1 (and on its dual BMOA). In the present paper we deal with Toeplitz and Hankel operators on the Bergman space A^1.

Denote the unit disk of the complex plane $\mathbb{C}$ by $\mathbb{D}$ and the unit circle by $\mathbb{T}$. For $1 \le p < \infty$, the Bergman space A^p consists of all analytic functions that belong to $L^p = L^p(\mathbb{D})$; the space of all bounded analytic functions in $\mathbb{D}$ is denoted by H^∞. The standard Bergman projection is defined by

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2} dA(w),$$

where $dA(w) = dxdy/\pi$ is the normalized Lebesgue measure on $\mathbb{D}$. The Toeplitz operator with symbol $a \in L^1$ is defined by $T_a f = P(af)$ and the Hankel operator by $H_a f = Q(af) = af - P(af)$, where $Q = I - P$ is the complementary projection of P.

The boundedness of the Bergman projection on L^p with $1 < p < \infty$ has been known since the 1960s, from which it directly follows that Hankel and Toeplitz operators with bounded symbols are bounded on A^p when $1 < p < \infty$. The question of boundedness for unbounded symbols is still an open problem and only known for some special classes of symbols, such as positive, harmonic and radial symbols—see [8, 10, 22]. It is well-known that the Bergman projection fails to be bounded on L^1 (there are bounded projections from L^1 onto A^1, however, unlike in the case of H^1), and so boundedness of Toeplitz and Hankel operators needs further considerations. Indeed, K. Zhu [21] was the first one to study this question and found a sufficient condition providing a large class of bounded functions that generate bounded Toeplitz operators on A^1. Further conditions can be found in [1] and [19]. However, as in the case of $1 < p < \infty$ these results have been inconclusive in the sense that boundedness is completely characterized only for bounded symbols. We discuss this in some more detail in Section 3.

The Fredholm properties of Toeplitz operators acting on A^2 have been studied for several classes of symbols—see, e.g., results in Venugopalkrishna [17], Coburn [5], McDonald [12], McDonald and Sundberg [13], Luecking [11], and Böttcher [3]. Much of the recent progress is due to Grudsky, Karapetyants, and Vasilevski [6, 7, 9, 16]. The case $p = 1$ has not been exploited previously and so we aim to establish Fredholm theory first for symbols that are continuous up to the boundary of $\mathbb{D}$ and belong to $\mathrm{VMO}_{\partial \log}$; note that the fact that T_a is unbounded for some continuous symbols causes great difficulties in dealing with most symbol classes familiar from A^2 Fredholm theory.

We also deal with compactness of Hankel operators to a certain extent as needed in connection with Fredholm theory. Regarding Hankel operators acting on the Bergman space A^2, Stroethoff [14] gave a characterization for compactness when the symbol is bounded in $\mathbb{D}$, and Zhu [20] found a connection between compactness and the mean oscillation of a general symbol in the Bergman metric; recall also Axler's result [2], which is concerned with analytic symbols and shows that $H_{\overline{a}}$ is compact if and only if a is in the little Bloch space. Here we prove a useful sufficient condition for compactness of Hankel operators on A^1. Part of our approach is based on certain estimates in connection with the mean oscillation similar to those of Zhu [21] but now with the logarithmic weight.

We would like to thank Kehe Zhu for a useful discussion that took place when JAV visited the University at Albany in December 2006.

2. Logarithmic BMO in the Bergman metric

In this section we recall some results on bounded mean oscillation in the Bergman metric (see [22] for further details and proofs) and develop analogous theory for the logarithmic BMO_∂ and VMO_∂.

Denote the Bergman metric on $\mathbb{D}$ by $\beta(z,w)$ and the Bergman disk by $D(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}$. A function $f \in L^1(\mathbb{D})$ is said to be of bounded mean oscillation, $f \in \mathrm{BMO}_\partial$, if

$$\begin{aligned}
\mathrm{MO}_r(f)(z) :&= \left[\frac{1}{|D(z,r)|}\int_{D(z,r)} \left|f(w) - \hat{f}_r(z)\right|^2 dA(w)\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{2\,|D(z,r)|^2}\int_{D(z,r)}\int_{D(z,r)} |f(u) - f(v)|^2\, dA(u)dA(v)\right]^{\frac{1}{2}} \\
&= \left[\widehat{|f|^2_r}(z) - \left|\hat{f}_r(z)\right|^2\right]^{\frac{1}{2}}
\end{aligned}$$

is bounded, where

$$\hat{f}_r(z) = \frac{1}{|D(z,r)|}\int_{D(z,r)} f(w)dA(w), \quad z \in \mathbb{D};$$

note that this condition is independent of r as for each $r > 0$

$$\|f\|_{\mathrm{BMO}_r} := \sup_{z\in\mathbb{D}} \mathrm{MO}_r(f)(z)$$

is equivalent to

$$\|f\|_{\mathrm{BMO}_\partial} := \sup_{z\in\mathbb{D}} \mathrm{MO}(f)(z) := \left[\widetilde{|f|^2}(z) - |\widetilde{f}(z)|^2\right]^{1/2},$$

where $\widetilde{f}$ is the Berezin transform. Its closed subspace that consists of all functions f with vanishing mean oscillation,

$$\lim_{|z|\to 1} \mathrm{MO}(f)(z) = 0,$$

is denoted by VMO_∂. Note that for any $r > 0$, $f \in \mathrm{VMO}_\partial$ if and only if

$$\lim_{|z|\to 1} \mathrm{MO}_r(f)(z) = 0.$$

Let us next consider the spaces above with the logarithmic weight, denoted by $\mathrm{BMO}_{\partial\log}$ and $\mathrm{VMO}_{\partial\log}$, which are equipped with the following norm

$$\|f\|_{\mathrm{BMO}_{\partial\log}} = \sup_{z\in\mathbb{D}} \log \frac{1}{1-|z|^2} \mathrm{MO}_r(f)(z).$$

Proposition 1. *The spaces $L^\infty \cap \mathrm{BMO}_{\partial\log}$ and $C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial\log}$ are both Banach algebras.*

Proof. It suffices to note that

$$\left| fg - \hat{f}_r \hat{g}_r \right| \le \left| (f - \hat{f}_r) g \right| + \left| (g - \hat{g}_r) \hat{f}_r \right|$$

for all $f, g \in \mathrm{BMO}_{\partial\log}$. □

In order to describe the image of P on (weighted) BMO_∂ and VMO_∂, we state the definitions of the (logarithmic) Bloch and little Bloch spaces. Let f be analytic in $\mathbb{D}$. Then we say that f is a Bloch function and write $f \in \mathcal{B}$ if

$$\sup_{z\in\mathbb{D}} \left| f'(z) \right| (1 - |z|)^2 < \infty;$$

if in addition $(1-|z|^2)f'(z) \to 0$, then f is said to belong to the little Bloch space $\mathcal{B}_0$. The logarithmic versions of these spaces, denoted by $L\mathcal{B}$ and $L\mathcal{B}_0$, are defined simply by adding the factor $\log(1-|z|^2)^{-1}$ to the two conditions above.

Recall that $P(\mathrm{BMO}_\partial) = \mathcal{B}$ and $P(\mathrm{VMO}_\partial) = \mathcal{B}_0$; an analogous result, whose proof we omit here, holds in the case of weighted spaces:

Theorem 2. *We have $P(\mathrm{BMO}_{\partial\log}) = L\mathcal{B}$ and $P(\mathrm{VMO}_{\partial\log}) = L\mathcal{B}_0$.*

When dealing with Toeplitz operators on A^1, we restrict our study to the symbols in the Banach algebra $L^\infty \cap \mathrm{BMO}_{\partial\log}$, which is equipped with the following norm

$$\|f\| = \|f\|_\infty + \|f\|_{\mathrm{BMO}_{\partial\log}}.$$

Our next aim is to show that continuous functions can be approximated by C^∞ functions in the $L^\infty \cap \mathrm{BMO}_{\partial\log}$ norm. We start with a preliminary lemma.

Lemma 3. *Let $f \in C(\overline{\mathbb{D}})$. For every $\epsilon > 0$, there is a $g \in C^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that*

$$|f(z) - g(z)| < \epsilon(1 - |z|^2) \tag{2.1}$$

for all $z \in \mathbb{D}$.

Proof. We use mollification. Define the usual compactly supported C^∞ function $\varphi(z) := \exp(-1/(1-|z|^2))$, if $|z| < 1$, and $\varphi(z) = 0$ otherwise. Let $C := \int_{\mathbb{C}} \varphi dxdy$ and $J := \varphi/C$. For all $\delta > 0$ and $z := x + iy \in \mathbb{C}$ let $J_\delta(z) := \delta^{-2} J(z/\delta)$. The support of J_δ is the disc with center 0 and radius δ, and moreover,

$$\int_{\mathbb{C}} J_\delta \, dxdy = 1. \tag{2.2}$$

Let us define a positive valued auxiliary function δ on $\mathbb{D}$ as follows. Since f is uniformly continuous on $\overline{\mathbb{D}}$, for all $0 \le r < 1$ it is possible to find $\widetilde{\delta}(r) > 0$ such that $\widetilde{\delta}(r) \le (1-r)/2$ and

$$\sup_{|w-z|\le\widetilde{\delta}(|z|)} |f(z) - f(w)| \le \epsilon(1 - |z|^2). \tag{2.3}$$

Again by the uniform continuity of f, one can require that $\widetilde{\delta}$ is bounded from below by a strictly positive constant on every compact interval $[0, R]$, $0 < R < 1$. Hence, it is possible to find a C^∞ function $\delta : [0, 1[\to \mathbb{R}$ such that $0 < \delta(r) \le \widetilde{\delta}(r)$ for all $0 \le r < 1$. Finally, set $\delta(z) := \delta(|z|)$ for all $z \in \mathbb{D}$.

We define the approximating function g by $g(z) = f(z)$, if $|z| = 1$, and

$$g(z) := J_{\delta(z)} * f(z) := \int_{\mathbb{C}} J_{\delta(z)}(z - w) f(w) dw \quad \text{for } |z| < 1,$$

where $dw = dxdy$ and f is extended as 0 outside the closed unit disc; in view of the support of $J_{\delta(z)}$, it actually does not matter how f is extended there. Differentiating under the integral sign one verifies that $g \in C^\infty(\mathbb{D})$.

In view of (2.2), (2.3), the remark on the support of the function $J_{\delta(z)}$ and the fact that $\delta(z) \le \widetilde{\delta}(z)$, we have the following estimate for all $z \in \mathbb{D}$

$$\begin{aligned}
|f(z) - g(z)| &= \left| \int_{\mathbb{C}} J_{\delta(z)}(z - w)(f(w) - f(z)) dw \right| \\
&\le \int_{\mathbb{C}} J_{\delta(z)}(z - w) |f(w) - f(z)| dw \\
&\le \int_{\mathbb{C}} J_{\delta(z)}(z - w) dw \sup_{|w-z|\le\widetilde{\delta}(|z|)} |f(z) - f(w)| \\
&\le \epsilon(1 - |z|^2).
\end{aligned}$$

This proves the required approximation. It remains to prove that g is continuous on the boundary of the unit disc. However, this obviously follows from the continuity of f on $\overline{\mathbb{D}}$ and from (2.1). $\square$

Theorem 4. *The space* $C^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ *is dense in* $C(\overline{\mathbb{D}}) \cap \mathrm{BMO}_{\partial\log}$.

Proof. Let $f \in C(\overline{\mathbb{D}}) \cap \mathrm{BMO}_{\partial \log}$ and $\epsilon > 0$. According to the previous lemma, there is a function $g \in C^\infty \cap C(\overline{\mathbb{D}})$ so that $|f(z) - g(z)| \le \epsilon(1-|z|)$. It is clear that $\|f-g\|_\infty < \epsilon$, so it remains to estimate the difference in the $\mathrm{BMO}_{\partial \log}$ norm. Indeed, we have

$$\begin{aligned} \mathrm{MO}_r(f-g)(z) &\le \left[\frac{1}{|D(z,r)|} \int_{D(z,r)} |f(u) - g(u)|^2 \, dA(u) \right]^{1/2} \\ &\le \sup_{w \in D(z,r)} |f(w) - g(w)| , \end{aligned}$$

and thus

$$\log \frac{1}{1-|z|^2} \mathrm{MO}_r(f-g)(z) \le \epsilon \log \frac{1}{1-|z|^2} \sup_{w \in D(z,r)} (1-|w|^2).$$

Since Euclidean center and radius of $D(z,r)$ are given by

$$C_z = \frac{1-s^2}{1-s^2|z|^2} z \quad \text{and} \quad R_z = \frac{1-|z|^2}{1-s^2|z|^2} s,$$

respectively, where $s = \tanh r \in (0,1)$, we see that

$$\sup_{w \in D(z,r)} (1-|w|^2) = 1 - (C_z - R_z)^2$$

is asymptotically comparable to $1-|z|^2$ as $|z| \to 1$. Therefore,

$$\log \frac{1}{1-|z|^2} \sup_{w \in D(z,r)} (1-|w|^2)$$

is bounded for all $z \in \overline{D}$, and it follows that

$$\|f-g\| < \epsilon + C\epsilon$$

for some absolute constant C. □

3. Boundedness

Suppose that $a \in L^\infty$. Then according to [19], the Toeplitz operator $T_{\overline{a}}$ is bounded on A^1 if and only if $P(a)$ belongs to the logarithmic Bloch space $L\mathcal{B}$. This condition is unsuitable for our purposes as we need a symbol algebra to deal with the Fredholm theory of Toeplitz operators and it is far from clear whether the condition above forms an algebra. However, the following sufficient condition enables us to have a large class that is indeed a Banach algebra. According to [21], T_a is bounded on A^1 if

$$a \in L^\infty \cap \mathrm{BMO}_{\partial \log}.$$

We concentrate on the case in which a is continuous up to the boundary of $\mathbb{D}$ and belongs to $\mathrm{VMO}_{\partial \log}$. Note that $C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$ contains Hölder continuous functions.

Observe also that, for bounded symbols a, the boundedness of the Toeplitz operator T_a is both sufficient and necessary for the Hankel operator H_a to be bounded. Indeed, sufficiency follows from the following estimate

$$\|H_a f\| \leq \|af\| + \|T_a f\| \leq (\|a\|_\infty + \|T_a\|)\,\|f\|\,; \tag{3.1}$$

and, conversely, if H_a is bounded,

$$\|T_a f\| \leq \|af\| + \|H_a f\|$$

implies that T_a is bounded and thus $P(\overline{a}) \in L\mathcal{B}$.

In order to give a bound for the norm of these operators in terms of their symbols, we state a lemma whose proof is contained in the proof of [21, Theorem 4]. We include the proof for completeness.

Lemma 5. *Suppose that $a \in L^\infty \cap \mathrm{BMO}_{\partial\log}$. If $f \in \mathrm{BMO}_\partial$,*

$$\mathrm{MO}_r\big(af\big)(z) \leq 2\,\|a\|_\infty\,\|f\|_{\mathrm{BMO}_\partial} + |\hat{f}_r(z)|\mathrm{MO}_r(a)(z) \tag{3.2}$$

for all $z \in \mathbb{D}$.

Proof. Write

$$\begin{aligned} a(z)f(z) - (\widehat{af})_r(w) &= a(z)\big(f(z) - \hat{f}_r(w)\big) + \hat{f}_r(w)\big(a(z) - \hat{a}_r(w)\big) \\ &\quad + \hat{f}_r(w)\hat{a}_r(w) - (\widehat{af})_r(w), \end{aligned}$$

and so

$$\begin{aligned} |a(z)f(z) - (\widehat{af})_r(w)| &\leq \|a\|_\infty |f(z) - \hat{f}_r(w)| + |\hat{f}_r(w)||a(w) - \hat{a}_r(w)| \\ &\quad + |\hat{f}_r(w)\hat{a}_r(w) - (\widehat{af})_r(w)|. \end{aligned}$$

Let $D = D(w,r)$. Since

$$\begin{aligned} |\hat{a}_r(w)\hat{f}_r(w) - (\widehat{af})_r(w)| &= \left|\frac{1}{|D|}\int_D a(z)\big(f(z) - \hat{f}_r(w)\big)dA(z)\right| \\ &\leq \|a\|_\infty \left(\frac{1}{|D|}\int_D |f(z) - \hat{f}_r(w)|^2 dA(z)\right)^{1/2} \\ &\leq \|a\|_\infty \|f\|_{\mathrm{BMO}_\partial}, \end{aligned}$$

we have

$$\begin{aligned} |a(z)f(z) - (\widehat{af})_r(w)| &\leq \|a\|_\infty \|f\|_{\mathrm{BMO}_\partial} + \|a\|_\infty |f(z) - \hat{f}_r(w)| \\ &\quad + |\hat{f}_r(w)||a(z) - \hat{a}_r(w)|. \end{aligned}$$

Therefore,

$$\mathrm{MO}_r\big(af\big)(z) \leq 2\,\|a\|_\infty\,\|f\|_{\mathrm{BMO}_\partial} + |\hat{f}_r(z)|\mathrm{MO}_r(a)(z). \qquad \square$$

Theorem 6. *Let $a \in L^\infty(\mathbb{D}) \cap \mathrm{BMO}_{\log}$. Then there are constants C_1 and C_2 such that*

$$\|T_a\|_{\mathcal{L}(A^1)} \leq C_1\,\|a\|\,, \quad \|H_a\|_{\mathcal{L}(A^1,L^1)} \leq C_2\,\|a\|\,,$$

where $\|a\| = \|a\|_\infty + \|a\|_{\mathrm{BMO}_{\partial\log}}$.

Proof. Let $f \in \mathcal{B}$. Since P is bounded from BMO_∂ to the Bloch space $\mathcal{B}$, we have

$$\begin{aligned}\|T_a f\|_{\mathcal{B}} = \|P(af)\|_{\mathcal{B}} &\le \mathrm{const}\, \|af\|_{\mathrm{BMO}_\partial} \\ &\le \mathrm{const}\, \|a\|_\infty \, \|f\|_{\mathrm{BMO}_\partial} + \mathrm{const} \sup_{z\in\mathbb{D}} \hat{f}_r(z)\mathrm{MO}_r(a)(z),\end{aligned}$$

where the last inequality follows from the previous lemma. According to [21, Theorem 1] and its proof,

$$\sup\{|\hat{f}_r(z)| : \|f\|_{\mathrm{BMO}_\partial} \le 1,\ \hat{f}(0) = 0\} \le \mathrm{const}\, \beta(0,z)$$

for all $z \in \mathbb{D}$. Therefore, since T_a is a bounded operator and $\beta(0,z)$ is comparable to $\log(1/|D(z,r)|)$, it follows that

$$\|T_a\| \le \mathrm{const} \left(\|a\|_\infty + \|a\|_{\mathrm{BMO}_{\log}} \right).$$

Since $T_{\overline{a}} : \mathcal{B} \to \mathcal{B}$ is the adjoint of $T_a : A^1 \to A^1$, we have the desired inequality. The claim regarding the Hankel operator H_a now follows from (3.1). $\square$

4. Compactness

As usual, in the study of the Fredholm properties of Toeplitz operators, Hankel operators play an important role, especially their compactness. Also, we need to pay attention to the compactness of Toeplitz operators (for comparison, recall however that nontrivial Toeplitz operators on H^p are never compact).

In the definition of the mean oscillation MO_r choose $r > 0$ so small that always

$$D(z,r) \subset \{w \in \mathbb{D} \ :\ |w - z| < (1 - |z|)/2\}. \tag{4.1}$$

We also denote $W(z) := \log(e/(1 - |z|))$ for all $z \in \mathbb{D}$.

We start with a preliminary lemma that provides a key approximation result.

Lemma 7. *Let* $f \in \mathrm{VMO}_{\partial\log}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$.

(a) *Given* $\epsilon > 0$*, it is possible to find* $h \in \mathrm{VMO}_{\partial\log}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ *such that* h *is* C^1 *in an annular neighborhood that contains* $\mathbb{T}$ *and such that*

$$\|f - h\|_{\mathrm{BMO}_{\partial\log}} \le \epsilon \quad \text{and} \quad \|f - h\|_\infty \le \epsilon.$$

(b) *If in addition* $f(t) = 0$ *for all* $t \in \mathbb{T}$*, then it is possible to choose* h *to be compactly supported in* $\mathbb{D}$.

Proof. Suppose that $\|f\|_{\mathrm{BMO}_{\partial\log}} \le 1$ and $\|f\|_\infty \le 1$. Let $\epsilon > 0$ and choose $0 < R < 1$ such that

$$W(z)\mathrm{MO}_r(f)(z) \le \epsilon/2 \tag{4.2}$$

for $|z| \ge R$. In case (b), we fix R further so that

$$\sup\{|f(z)| : R < |z| < 1\} < \epsilon/2.$$

Let us define P in a neighborhood of $\overline{\mathbb{D}}$ such that it is C^1 and

$$\sup_{R<|z|<1} |P(z) - f(z)| \leq \epsilon/2\,.$$

(To find such a P, one can for example extend f as a uniformly continuous function on $2\,\overline{\mathbb{D}}$ by setting $f(z) = f(z/|z|)$ for $|z| > 1$, and then define P, say, on $\frac{3}{2}\,\mathbb{D}$ by mollifying the extended f.) In case (b), we simply set $P \equiv 0$. As a C^1 function, P definitely belongs to $\mathrm{VMO}_{\partial\log}(\mathbb{D})$ and thus there exists $R' \in (R, 1)$ such that

$$W(z)\mathrm{MO}_r(P)(z) \leq \epsilon/2 \tag{4.3}$$

for $|z| \geq R'$.

In order to define the desired function h, we construct a continuous radial function $\psi : \overline{\mathbb{D}} \to [0, 1]$ such that

$$\psi(z) = 0 \quad \text{for} \quad |z| \leq 1 - \delta/2, \tag{4.4}$$

where $\delta = 1 - R'$, $\psi(z) = 1$ for z sufficiently close to $\mathbb{T}$, and, in addition,

$$W(z)\mathrm{MO}_r(\psi)(z) \leq \epsilon \tag{4.5}$$

for all $z \in \mathbb{D}$. Indeed, let $N \in \mathbb{N}$ be such that $2^{-N} = \epsilon'$, where $\epsilon' > 0$ is so small that $\epsilon' < \delta/4$ and $10\epsilon'|\log \epsilon'| < \epsilon$. Let $\nu \in \mathbb{N}$ be the largest positive integer such that $\sum_{k=1}^{\nu} 1/k \leq 2^N$. For all z with $|z| \leq 1 - 2^{-N}$, define $\psi(z) = 0$. Then (4.4) holds, by the choice of ϵ'. For all n, $1 \leq n \leq \nu$, define

$$\psi(z) = \sum_{k=1}^{n} \frac{\epsilon'}{k},$$

where $z = 1 - 2^{-N-n}$. For all $n > \nu$, for $z = 1 - 2^{-N-n}$ we simply set

$$\psi(z) = 1.$$

We extend ψ affinely for other positive z and then radially all over the disc.

To prove (4.5) it is enough to consider the weighted mean oscillation for $D(z, r)$ with $1 - 2^{-N+2} \leq |z| \leq 1 - 2^{-N-\nu-2}$; for other values of z, the mean oscillation on $D(z, r)$ is 0, since ψ is constant there, see (4.1). If $n \leq \nu + 1$ is such that $1 - 2^{-N-n} \leq |z| < 1 - 2^{-N-n-1}$, we have

$$1 - 2^{-N-n+1} \leq |w| < 1 - 2^{-N-n-2}$$

for all $w \in D(z, r)$. Hence,

$$\epsilon' \sum_{k=1}^{n-1} \frac{1}{k} \leq \psi(w) \leq \epsilon' \sum_{k=1}^{n+2} \frac{1}{k}$$

and moreover

$$W(z) \leq 3\log(2^{N+n}) \leq 3n|\log \epsilon'|$$

for all $w \in D(z,r)$. Hence,

$$
\begin{aligned}
(4.6) \qquad & W(z)\mathrm{MO}_r(\psi)(z) \\
&\le 3n|\log\epsilon'|\Bigg(\frac{1}{|D(z,r)|}\int\limits_{D(z,r)}\Bigg|\epsilon'\sum_{k=1}^{n+2}\frac{1}{k}-\epsilon'\sum_{k=1}^{n-1}\frac{1}{k}\Bigg|^2 dA(w)\Bigg)^{\frac{1}{2}} \\
&\le 3n|\log\epsilon'|\Bigg(\frac{1}{|D(z,r)|}\int\limits_{D(z,r)}\Big(\frac{3\epsilon'}{n}\Big)^2 dA(w)\Bigg)^{\frac{1}{2}} \\
&\le 9|\log\epsilon'|\epsilon' < \epsilon
\end{aligned}
$$

for such z, and by the remark above, this estimate holds true for all $z \in \mathbb{D}$.

Define $h = (1-\psi)f + \psi P$. Let us show that h is the desired approximation. Note first that in case (b), the support of h is trivially compact since $P \equiv 0$. Clearly, since $\psi(z) = 0$ for all $|z| < R$,

$$\|f-h\|_\infty = \|\psi(f-P)\|_\infty < \epsilon$$

according to the choice of P. So it remains to estimate the other norm. Writing

$$\|f-h\|_{\mathrm{BMO}_{\partial\log}} \le \|\psi f\|_{\mathrm{BMO}_{\partial\log}} + \|\psi P\|_{\mathrm{BMO}_{\partial\log}} \tag{4.7}$$

we start with the first term on the right-hand side. By (4.4), $\psi(w) = 0$, if $w \in D(z,r)$ with $|z| \le 1-\delta$, hence,

$$\|\psi f\|_{\mathrm{BMO}_{\partial\log}} = \sup_{|z|\ge 1-\delta} W(z)\mathrm{MO}_r(\psi f)(z).$$

Let $|z| \ge 1-\delta$ and $w \in D(z,r)$. We write $M_r(f) = \hat{f}_r$ and

$$
\begin{aligned}
&\psi(w)f(w) - M_r(\psi f)(z) \\
&\qquad = \psi(w)\Big(f(w) - M_r(f)(z)\Big) + M_r(f)(z)\Big(\psi(w) - M_r(\psi)(z)\Big) \\
&\qquad\quad + M_r(\psi)(z)M_r(f)(z) - M_r(\psi f)(z).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&W(z)\mathrm{MO}_r(\psi f)(z)\\
&\quad\le C\left(\frac{1}{|D(z,r)|}\int\limits_{D(z,r)} W(z)^2\left|\psi(w)f(w)-M_r(\psi f)(z)\right|^2 dA(w)\right)^{\frac12}\\
&\quad\le C\left(\frac{1}{|D(z,r)|}\int\limits_{D(z,r)} \Big(W(z)^2\|\psi\|_\infty\left|f(w)-M_r(f)(z)\right|^2\right.\\
&\qquad + W(z)^2\|f\|_\infty\left|\psi(w)-M_r(\psi)(z)\right|^2\\
&\qquad \left.+ W(z)^2\left|M_r(\psi)(z)M_r(f)(z)-M_r(\psi f)(z)\right|^2\Big)dA(w)\right)^{\frac12}\\
&\quad\le C'W(z)\mathrm{MO}_r(f)(z)+C'W(z)\mathrm{MO}_r(\psi)(z)\\
&\qquad + W(z)\left|M_r(\psi)(z)M_r(f)(z)-M_r(\psi f)(z)\right|.
\end{aligned}
$$

The second and third but last terms are comparable to ϵ, by (4.2) and (4.5). The same is true for the last one, since

$$
\begin{aligned}
&W(z)\left|M_r(\psi)(z)M_r(f)(z)-M_r(\psi f)(z)\right| \\
&\quad\le \left|\frac{W(z)}{|D(z,r)|}\int\limits_{D(z,r)} f(w)\Big(\psi(w)-M_r(\psi)(z)\Big)dA(w)\right|\\
&\quad\le \|f\|_\infty W(z)\left(\frac{1}{|D(z,r)|}\int\limits_{D(z,r)} |\psi(w)-M_r(\psi)(z)|^2 dA(w)\right)^{1/2}\\
&\quad= \|f\|_\infty W(z)\left(\frac{1}{|D(z,r)|}\int\limits_{D(z,r)} |\psi(w)-M_r(\psi)(z)|^2 dA(w)\right)^{1/2}\\
&\quad= \|f\|_\infty W(z)\mathrm{MO}_r(\psi)(z)\le\epsilon
\end{aligned}
\tag{4.8}
$$

So we obtained the estimate

$$\|\psi f\|_{\mathrm{BMO}_{\partial\log}} < C\epsilon.$$

The other term $\|\psi P\|_{\mathrm{BMO}_{\partial\log}}$ in (4.7) is treated in the same way, using (4.3) instead of (4.2). □

We can now deal with compactness of Hankel and Toeplitz operators; in particular, we obtain sufficient conditions for compactness of these operators that are of great importance in the next section.

Corollary 8. *If $a \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial\log}$ and $a(t) = 0$ for all $t \in \mathbb{T}$, then $T_a : A^1 \to A^1$ is compact.*

Proof. Suppose that $a = 0$ on $\mathbb{T}$. If $K := \operatorname{supp} a$ is compact and $T : A^1 \to L^1$ is defined by $Tf(z) = \chi_K(z)f(z)$, then T is compact, and so $T_a = PM_aT$ must also be compact. Otherwise, use case (b) of Lemma 7 to approximate the given symbol a by a sequence of a_k with $\operatorname{supp} a_k$ compact, so that $\|T_a - T_{a_k}\| \to 0$ as $k \to \infty$ according to the norm estimate in Theorem 6. Thus T_a is compact. □

Lemma 9. *If $h \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$ is C^1 in an annular neighborhood Ω that contains $\mathbb{T}$, then $H_h : A^1 \to L^1$ is compact.*

Proof. One can easily check that the transpose of the operator

$$PH_{\bar{h}} : C(\overline{\mathbb{D}}) \to \mathcal{B}_0$$

is the operator $H_h : A^1 \to C(\overline{\mathbb{D}})^*$ (or, $H_h : A^1 \to L^1$, since L^1 is a closed subspace of $C(\overline{\mathbb{D}})^*$ and H_h is known to map A^1 into L^1). We prove that $H_{\bar{h}} : C(\overline{\mathbb{D}}) \to C(\overline{\mathbb{D}})$ is compact. Since $P : C(\overline{\mathbb{D}}) \to \mathcal{B}_0$ is bounded, the compactness of $H_h : A^1 \to L^1$ will follow.

Let $\epsilon > 0$ be so small that $\{|z| \geq 1 - \epsilon\}$ is contained in Ω. Define a C^∞ cutoff function $\chi(z) : \overline{\mathbb{D}} \to \mathbb{R}$ such that $0 \leq \chi(z) \leq 1$ for all z and such that $\chi(z) = 1$ for $|z| \leq 1 - 2\epsilon$ and $\chi(z) = 0$ for $|z| \geq 1 - \epsilon$.

The integral operator $\widetilde{H} : C(\overline{\mathbb{D}}) \to C(\overline{\mathbb{D}})$,

$$\widetilde{H}f(z) := \int_{\mathbb{D}} \frac{(\bar{h}(z) - \bar{h}(w))}{(1 - z\overline{w})^2} \chi(w) f(w) dA(w),$$

is compact (as a consequence of the Arzela–Ascoli theorem), since its kernel is a continuous function on $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Hence, there exist finitely many functions $f_j \in C(\overline{\mathbb{D}})$ such that

$$\widetilde{H}(B(0,1)) \subset \bigcup_j B(f_j, \epsilon); \tag{4.9}$$

here $B(g, r)$ denotes the open ball in $C(\overline{\mathbb{D}})$ with center g and radius r.

On the other hand, since $\bar{h} \in C^1$ on Ω, it is also Lipschitz on Ω, and so we have

$$\begin{aligned} |(H_{\bar{h}} - \widetilde{H})f(z)| &\leq \int_{\mathbb{D}} \frac{|\bar{h}(z) - \bar{h}(w)|}{|1 - z\overline{w}|^2} (1 - \chi(w)) |f(w)| dA(w) \\ &\leq C \int_{|w| \geq 1 - 2\epsilon} \frac{|z - w|}{|1 - z\overline{w}|^2} dA(w) \end{aligned}$$

for all $f \in B(0, 1)$. Since $|z - w|/|1 - z\overline{w}| \leq 1$, the above integral can be bounded by a constant times

$$\int_{|w| \geq 1 - 2\epsilon} \frac{1}{|1 - z\overline{w}|} dA(w)$$

By radial symmetry, we may assume that z is positive and obtain, with $w = \varrho e^{i\theta}$, a further bound

$$
\begin{aligned}
&C \int\limits_{1-2\epsilon}^{1} \int\limits_{-\pi}^{\pi} \frac{1}{|1 - z\varrho\cos\theta| + \varrho|\sin\theta|}\varrho d\theta d\varrho \\
&\leq C' \int\limits_{1-2\epsilon}^{1} \int\limits_{-3\pi/4}^{3\pi/4} \frac{1}{1-\varrho+\varrho|\theta|}\varrho d\theta d\varrho + C' \int\limits_{1-2\epsilon}^{1} \int\limits_{\frac{3}{4}\pi<|\theta|<\pi} \frac{1}{|1+z\varrho/2|}\varrho d\theta d\varrho \\
&\leq C'' \int\limits_{1-2\epsilon}^{1} |\log(1-\varrho)| d\varrho + C'' \int\limits_{1-2\epsilon}^{1} d\varrho \\
&\leq C'''\epsilon|\log\epsilon| \leq c\epsilon^{1/2}.
\end{aligned}
$$

Combining this with (4.9) we conclude that

$$H_{\bar{h}}(B(0,1)) \subset \bigcup_j B(f_j, C\epsilon^{1/2}),$$

i.e., $H_{\bar{h}} : C(\overline{\mathbb{D}}) \to C(\overline{\mathbb{D}})$ is compact. □

Theorem 10. *Let $a \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial\log}$. Then $H_a : A^1 \to L^1$ is compact.*

Proof. Apply Lemmas 7 and 9. □

5. Fredholmness and index

A bounded linear operator A on a Banach space X is said to be Fredholm if both its kernel and cokernel are finite-dimensional; the index of a Fredholm operator is defined by

$$\operatorname{Ind} A = \dim\ker A - \dim\operatorname{coker} A.$$

We also define the index (or the winding number) of a nonvanishing continuous function a by

$$\operatorname{ind} a = \frac{[\arg a]_{\mathbb{T}}}{2\pi},$$

where $[\arg a]_{\mathbb{T}}$ denotes the total increment of $\arg a(t)$ when t ranges over $\mathbb{T}$.

Using the results of the previous section we can easily obtain a sufficient condition for Fredholmness of T_a on A^1.

Theorem 11. *Suppose that $a \in C(\overline{D}) \cap \mathrm{VMO}_{\partial\log}$ with $a(t) \neq 0$ for any $t \in \mathbb{T}$. Then $T_a : A^1 \to A^1$ is Fredholm.*

Proof. There are $\epsilon > 0$ and $0 < \delta < 1$ such that $|a(z)| > \epsilon$ whenever $\delta < |z| \leq 1$. Define $b = 1/a$ on $\{\delta < |z| \leq 1\}$ and extend it continuously to the whole $\overline{\mathbb{D}}$. Then it is clear that $b \in C(\overline{\mathbb{D}}) \cap \mathrm{BMO}_{\partial \log}$ and also, for sufficiently small r, we have $W(z)\mathrm{MO}_r b(z) \to 0$ as $|z| \to 1$. Thus, $b \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$.

Since

$$\begin{aligned} T_a T_b &= I - P(I - M_{ab}) - PM_a(I-P)M_b \\ &= I - T_{1-ab} - PM_a H_b, \end{aligned}$$

Corollary 8 and Theorem 10 imply that $T_a T_b = I + K$ for some compact operator K; similarly $T_b T_a = I + K'$ with K' compact. Therefore T_b is a regularizer of T_a and hence T_a is Fredholm (see, e.g., [4, Theorem 1.12]). □

In order to deal with sufficiency we first consider the index of Fredholm Toeplitz operators with continuous symbols.

Theorem 12. *Suppose that $a \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$ and $a(t) \neq 0$ for any $t \in \mathbb{T}$. Then*

$$\operatorname{Ind} T_a = -\operatorname{ind} a|_{\mathbb{T}},$$

where $a|_{\mathbb{T}}$ denotes the restriction of a to $\mathbb{T}$.

Proof. Suppose first that the index of a is nonnegative. According to Lemma 9, there is a function $b \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$ that has no zeros on $\mathbb{T}$ and is in C^1 in an annulus containing $\mathbb{T}$ and approximates a so that $\operatorname{Ind} T_a = \operatorname{Ind} T_b$ and

$$\operatorname{ind} a|_{\mathbb{T}} = \operatorname{ind} b|_{\mathbb{T}} =: \kappa.$$

For $\tau \in [0,1]$, we define

$$F_\tau(t) = t^\kappa \exp\big(\tau \log g(t)\big) \quad (t \in \mathbb{T}),$$

where $g(t) = t^{-\kappa} b(t)$. Since $b \in C^1(\mathbb{T})$ and $\operatorname{ind} g = 0$, F_τ is a homotopy in $C^1(\mathbb{T})$ and has no zeros on $\mathbb{T}$. We can now extend $F_\tau : \mathbb{T} \to \mathbb{C}$ to a mapping that belongs to $C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$. Then T_{F_τ} is Fredholm for each $\tau \in [0,1]$ by Theorem 11, and since the index of a Fredholm operator is continuous (see, e.g., [4, Theorem 1.12]), we have

$$\operatorname{Ind} T_{z^\kappa} = \operatorname{Ind} T_{F_0} = \operatorname{Ind} T_{F_1} = \operatorname{Ind} T_b = \operatorname{Ind} T_a.$$

As $\kappa \geq 0$, it is not difficult to show that

$$-\kappa = \operatorname{Ind} T_{z^\kappa} = \operatorname{Ind} T_a.$$

The case $\kappa < 0$ can be reduced to the preceding one via duality. Indeed recall that the dual of A^1 is the Bloch space under the usual integral pairing

$$(f,g) = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z)$$

and then in particular $(T_a)^* = T_{\overline{a}}$. Therefore,

$$-\operatorname{ind} a|_{\mathbb{T}} = \operatorname{ind} \overline{a}|_{\mathbb{T}} = -\operatorname{Ind} T_{\overline{a}} = \operatorname{Ind} T_a$$

when $\kappa < 0$. □

We use the index formula and Lemma 7 to prove that the condition $0 \notin a(\mathbb{T})$ is also necessary for Fredholmness of T_a with $a \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$. It is worth noting that our approach is different from that of Coburn [5], where the compactness of Toeplitz operators and C^*-algebra techniques are applied, which, in our view, is not applicable to the setting at hand. In particular, we assume that T_a is Fredholm but the symbol a has a zero, and show that this leads to a contradiction. Before proceeding to the proof, we consider a result that deals with the distribution of the zeros of approximating functions and hence allows us to deal with the other possible zeros of a.

Lemma 13. *Assume that the function $f \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$ is also defined and C^1 in a neighborhood of $\mathbb{T}$ and has finitely many zeros $t_1, \ldots, t_N \in \mathbb{T}$. Given $\eta > 0$, one can find $b \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$ with*

$$\|f - b\| < \eta$$

as follows: b is also defined and C^1 in a neighborhood Ω of $\mathbb{T}$, $b(t_j) = 0$ for all j, and b has no other zeros in Ω.

Proof. Apply the proof of Lemma 7, where P should now be replaced with

$$P(re^{i\theta}) := rf(e^{i\theta})$$

for r in a sufficiently small open interval containing 1. For a sufficiently large $R < 1$ this is the desired approximation of f in the set $\{z \in \overline{\mathbb{D}} : R \le |z| \le 1\}$ with respect to sup-norm, and the rest of the proof of Lemma 7 applies word for word. $\square$

Theorem 14. *Let $a \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$. If T_a is Fredholm on A^1, then $a(t) \neq 0$ for any $t \in \mathbb{T}$.*

Proof. Assume that T_a is Fredholm. Let $\epsilon > 0$ be so small that $\operatorname{Ind} T_a = \operatorname{Ind} T_f$ for every symbol $f \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$ with

$$\|a - f\| \le \epsilon \tag{5.1}$$

(see Theorem 6). Supposing that a has a zero on the boundary we construct two symbols with (5.1) for which the indices are different.

First, using Lemma 7, we approximate a by a function $\widetilde{f}$ which is C^1 in a neighborhood of $\mathbb{T}$ such that $\|a - \widetilde{f}\| \le \epsilon/10$. There exists a constant $\alpha \in \mathbb{C}$ such that $|\alpha| \le \epsilon/10$ and such that the function $\widetilde{f} + \alpha$ has only a finitely many zeros on $\mathbb{T}$. (If this were not true, one would pick countably many different numbers z_n, $n \in \mathbb{N}$, with $|z_n| \le \epsilon/10$. The set $\mathbb{T}_n := \{t \in \mathbb{T} \ : \ \widetilde{f}(t) = z_n\}$ would be infinite for every $n \in \mathbb{N}$, so each set $\mathbb{T}_n$ would have an accumulation point $w_n \in \mathbb{T}$. The set $\{w_n : n \in \mathbb{N}\}$ would still have an accumulation point τ; however, this would lead to a contradictory behavior of $\widetilde{f}$ at τ.)

Set $f = \widetilde{f} + \alpha$. Approximate f by a function b as in Lemma 13 with $\eta := \epsilon/10$. Summing up, we then have

$$\|a - b\| \leq \frac{3\epsilon}{10}, \tag{5.2}$$

and for some $r' > 1$, the function b is defined in a set $\mathbb{D}' := \{|z| \leq r'\}$ and C^1 in the set $\mathbb{T}' := \{1/r' \leq |z| \leq r'\}$. The numbers $t_j \in \mathbb{T}$, $j = 1, \ldots, N$, denote the zeros of b on $\mathbb{T}$, and b does not have other zeros on $\mathbb{T}'$.

Let us select for each j an open disc $B_j := B(t_j, \delta_j) \subset \mathbb{T}'$ such that $B(t_j, 2\delta_j) \cap B(t_k, 2\delta_k) = \emptyset$ for $j \neq k$.

Since $|\nabla b|$ is bounded on $\mathbb{T}'$, we can find, using the mean value theorem, a number $0 < \delta < (1 - 1/r')/10$ with the following property

$$|b(z) - b(w)| \leq \frac{\epsilon}{10N}, \tag{5.3}$$

for all $z, w \in \mathbb{D}'$ with $|z - w| \leq 100\delta$. Moreover, we assume δ is so small that $W(z)\mathrm{MO}(b)(z) < \epsilon/(10N)$ for z with $|z| \geq 1 - 4\delta$. If necessary, we diminish the numbers δ_j so that each of them satisfies $\delta_j \leq \delta$.

We next modify the function b on each of the sets $B(t_j, \delta_j)$ as follows. We start with $j = 1$, and without loss of generality we may assume $t_1 = 1$. Let $z = x + iy \in \overline{\mathbb{D}}$. We define

$$f_1(z) = b(z)$$

if $x \leq 1 - \delta_1^2/100$, and

$$f_1(z) = b(1 - \delta_1^2/100 + iy),$$

if $x \geq 1 - \delta_1^2/100$ (notice that then $z = x + iy \in B_1$).

We also define

$$f_2(z) = b(z)$$

if $x \leq 1 - \delta_1^2/100$, and

$$f_2(z) = b(1 - \delta_1^2/100 + 2(x - (1 - \delta_1^2/100)) + iy)$$

if $x \geq 1 - \delta_1^2/100$.

Since the functions f_k coincide with b except in the set B_1, we have

$$\sup_{z \in \mathbb{D} \setminus B_1} |f_k(z) - b(z)| = 0.$$

If $z \in B_1$, then, by the construction of the functions f_k, we have

$$f_k(z) = b(w) \tag{5.4}$$

for a number w such that $|z - w| \leq \delta_1$. Hence, we still obtain by (5.3)

$$|f_k(z) - b(z)| = |b(w) - b(z)| < \frac{\epsilon}{10N}.$$

Therefore, $\|f_k - b\|_\infty \leq \epsilon/10N$.

Concerning $\|f_k - b\|_{\mathrm{BMO}_{\partial \log}}$, we show that

$$W(z)\mathrm{MO}(f_k)(z) \leq \frac{4\epsilon}{10N}$$

for the numbers z with $|z| \geq 1-4\delta$. Indeed, if $|z| \geq 1-4\delta$, then the Euclidean radius of the set $D(z)$ is smaller than 50δ. Moreover, (5.4) still holds. (If $z \notin B_1$, then $f_k(z) = b(z)$). Hence, by (5.3) and (5.4), $|f_k(w) - f_k(\zeta)| \leq 4\epsilon/(10N)$ for all $w, \zeta \in D(z)$, which implies $W(z)\mathrm{MO}(f_k)(z) \leq 4\epsilon/(10N)$.

On the other hand, for such z, we have $W(z)\mathrm{MO}(b)(z) \leq \epsilon/(10N)$ by assumption, so we get for $|z| \geq 1 - 4\delta$ the bound $W(z)\mathrm{MO}(b - f_k)(z) < 5\epsilon/(10N)$.

For $|z| \leq 1 - 4\delta$ we have $W(z)\mathrm{MO}(b - f_k)(z) = 0$, since the functions coincide on $D(z)$. As a conclusion,

$$\|b - f_k\| \leq \frac{\epsilon}{2N} \tag{5.5}$$

for $k = 1, 2$.

As for the other discs B_j, $j = 1, \ldots, N$, we modify b on them analogously to f_1 above. Eventually we thus get two modifications of b, namely b_2, which has exactly one zero in $\mathbb{T}'$ (the one sitting in B_1), and b_1, which has no zeros in $\mathbb{T}'$. Moreover, both functions satisfy,

$$\|b_k - b\| < \frac{4\epsilon}{10},$$

hence, by (5.2),

$$\|b_k - a\| < \epsilon,$$

and by construction, the indices of $b'_k := b_k \restriction \mathbb{T}$ $(k = 1, 2)$ are different since the increments of $\arg b'_1(t)$ and $\arg b'_2(t)$ are the same when t ranges over $\mathbb{T} \setminus B_1$ but differ when t ranges over $\mathbb{T} \cap B_1$. This contradicts the index theorem (see Theorem 12). $\square$

As a consequence, we see that the condition $a \restriction \mathbb{T} \equiv 0$ is also necessary for the compactness of T_a acting on A^1:

Proposition 15. *Let $a \in C(\overline{\mathbb{D}}) \cap \mathrm{VMO}_{\partial \log}$. Then T_a is compact on A^1 if and only if $a(t) = 0$ for all $t \in \mathbb{T}$.*

Proof. For sufficiency, see Corollary 8. Assume that T_a is compact but $a(t_0) \neq 0$ for some $t_0 \in \mathbb{T}$. It is well-known that $K - \lambda$ is Fredholm for all $\lambda \neq 0$ whenever K is compact (the Riesz's theorem for compact operators). In particular, $T_{a-a(t_0)}$ must then be Fredholm, which contradicts the previous theorem. $\square$

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