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The relation between monotonicity and strategy-proofness

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Abstract The Muller–Satterthwaite Theorem (J Econ Theory 14:412–418, 1977) establishes the equivalence between Maskin monotonicity and strategy-proofness, two cornerstone conditions for the decentralization of social choice rules. We consider a general model that covers public goods economies as in Muller–Satterthwaite (J Econ Theory 14:412–418, 1977) as well as private goods economies. For private goods economies, we use a weaker condition than Maskin monotonicity that we call unilateral monotonicity. We introduce two *easy-to-check* preference domain conditions which separately guarantee that (i) unilateral/Maskin monotonicity implies strategy-proofness (Theorem 1) and (ii) strategy-proofness implies unilateral/Maskin monotonicity (Theorem 2). We introduce and discuss various classical single-peaked preference domains and show which of the domain conditions they satisfy (see Propositions 1 and 2 and an overview in Table 1). As a by-product of our analysis, we obtain some extensions of the Muller–Satterthwaite Theorem as summarized in Theorem 3. We also discuss some new "Muller–Satterthwaite preference domains" (e.g., Proposition 3).

1 Introduction

The Muller–Satterthwaite Theorem (Muller and Satterthwaite 1977) states the equivalence between strategy-proofness and Maskin monotonicity, two cornerstone con-

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ditions for the decentralization of (social choice) rules.[1] As a consequence of the Muller–Satterthwaite Theorem, the class of Maskin monotonic rules is fairly small: only dictatorial rules are strategy-proof.[2] However, it is by now well understood that the aforementioned theorem strongly relies on the assumption of an unrestricted domain of strict preferences—what we refer to as the Arrovian preference domain. In many situations though, it is natural to work with more structured preference domains. For instance, consider a group of agents who have to choose the location of a public facility on their street. A natural preference domain restriction is to assume that agents have single-peaked preferences over the possible locations (Black 1948). We know that the class of strategy-proof rules for this type of economies is large (Moulin 1980); and a natural question is whether the same conclusion holds for the class of Maskin monotonic rules. So, despite the equivalence provided by the Muller–Satterthwaite Theorem, it seems that for many preference domains and models of interest, the logical relation between Maskin monotonicity and strategy-proofness is not fully understood. In addition, notice that in public goods models, a rule selects an alternative at each preference profile, whereas in private goods models, an allocation will be selected, i.e., a bundle for each agent. An allocation is an object whose nature is different from an alternative in several aspects. For instance, the bundle that an agent (or a group of agents) receives at some preference profile may be conditional on the shape of preferences of some other agents. Rules that have this feature violate the well-known non-bossiness condition (Satterthwaite and Sonnenschein 1981). Because of this difference between the two models, it is not clear whether there is a "direct" logical relation between Maskin monotonicity and strategy-proofness in private goods models.

Our contribution: Our goal is to provide a better understanding of the logical relation between monotonicity conditions and strategy-proofness. We consider a model that covers public goods as well as private goods economies.[3] In addition to Maskin monotonicity, we introduce a weaker condition called unilateral monotonicity which pertains to unilateral changes in preferences.[4] The use of this condition is pertinent when we refer to private goods models.

We introduce two easy-to-check preference domain conditions. Condition R1 is a preference domain richness condition, whereas Condition R2 is a preference domain restriction condition. A rule defined on a preference domain satisfying Condition R1 is unilaterally monotonic/Maskin monotonic only if it is strategy-proof (Theorem 1 and Corollary 1). Examples of rich preference domains include the Arrovian preference domain as well as various single-peaked (single-plateaued) preference domains,

[1] Both conditions are central in the mechanism design literature. Strategy-proofness is a necessary condition for implementation in dominant strategies, whereas Maskin monotonicity is a necessary condition for implementation in Nash equilibrium.

[2] A version of the Muller–Satterthwaite Theorem has as well-known corollary the Gibbard–Satterthwaite Theorem (see Reny 2001): any onto and strategy-proof rule defined on a domain of unrestricted linear orderings must be dictatorial.

[3] For private goods economies, our model covers both the infinitely divisible goods case as well as the indivisible goods case.

[4] As far as we know, unilateral monotonicity was first introduced by Takamiya (2001).

but exclude preference domains in which preferences are both single-peaked and symmetric. More generally, the domain of convex star-shaped preferences satisfies Condition R1 (Proposition 1). Next, for public goods models, Condition R2 entails that strategy-proofness implies Maskin monotonicity; and for private goods models, strategy-proofness implies unilateral monotonicity (Theorem 2). Indeed, in private goods models, there exist rules that are strategy-proof but not Maskin monotonic. As argued above, an important difference between public goods and private goods models turns out to be the existence of rules that violate non-bossiness in the latter.[5] As a consequence, for several preference domains, the "set-inclusion connection" between the class of Maskin monotonic rules and the class of strategy-proof rules may be lost for private goods models.[6] However, when Condition R2 is satisfied, a logical relation between strategy-proofness and Maskin monotonicity can be recovered thanks to non-bossiness: strategy-proofness and non-bossiness together imply Maskin monotonicity (Corollary 2). Examples of preference domains satisfying Condition R2 include the Arrovian preference domain as well as some (symmetric) single-peaked (single-plateaued) preference domains but exclude larger preference domains like the single-peaked preference domain.[7] More generally, any convex norm induced preference domain satisfies Condition R2 (Proposition 2).

Next, we come to the Muller–Satterthwaite Theorem and its extensions. As a by-product of our results, we obtain an extended version of the Muller–Satterthwaite Theorem that applies to the model at hand (Theorem 3). A straightforward corollary is the standard version of the theorem (Muller and Satterthwaite 1977) for the public goods case, along with a new and direct proof. We then discuss some new "Muller–Satterthwaite preference domains" of interest (Proposition 3). This shows that the conclusion of the Muller–Satterthwaite Theorem can also spread to restricted preference domains.

Relation to the literature: The investigation of the relation between monotonicity conditions and strategy-proofness is not new. A seminal paper dealing with the relation between Maskin monotonicity and strategy-proofness is Dasgupta et al. (1979). They introduce a preference domain richness condition and prove that any Maskin monotonic rule defined on a rich preference domain is strategy-proof. More recently, Takamiya (2001, 2003) studies the relation between coalition strategy-proofness and Maskin monotonicity for a broad class of economies with indivisible goods. Takamiya (2007) generalizes the results obtained in his former two papers. Finally, in an article independent of ours, Berga and Moreno (2009) study the relation between strategy-proofness, Maskin monotonicity, and non-bossiness for the single-peaked and single-plateaued preference domain for the provision of a pure public good.

[5] For preference domains satisfying Condition R2, unilateral monotonicity and non-bossiness imply Maskin monotonicity (Lemma 1).

[6] For example, in private goods models, the symmetric single-peaked preference domain admits rules that are strategy-proof but not Maskin monotonic, as well as rules that are Maskin monotonic but not strategy-proof.

[7] However, the domain of strict single-peaked preferences satisfies Condition R2. In fact, any preference domain composed only of strict preference relations satisfies Condition R2.

In addition to Dasgupta et al. (1979), preference domain richness conditions are used in articles close to ours: Fleurbaey and Maniquet (1997) and Le Breton and Zaporozhets (2009). Note that the preference domain richness condition (Condition R1) that we introduce differs from the conditions uncovered in the aforementioned articles, and it does not imply any "cross-profile" requirements. We discuss in the Appendix the logical relations between the latter conditions and our Condition R1.

The plan of the article is the following. In Sect. 2, we introduce a general model that encompasses public goods as well as private goods economies, and we present the definitions and preference domains necessary for the article. In Sect. 3, we define our two preference domain conditions and we prove our main results. In Sect. 4, we check both these conditions for well-known preference domains. We also provide an extended version of the Muller–Satterthwaite Theorem that applies to the model at hand. Finally, in the Appendix, we compare our preference domain richness condition (Condition R1) to the ones introduced in related articles.

2 The model, key properties, and preference domains

2.1 The model

Let $N = \{1, \ldots, n\}$ be a *set of agents*. Let $A = A_1 \times \ldots \times A_n$ be a *set of alternatives*. For $i \in N$, we call A_i *agent i's individual set of alternatives*. We assume that for all $i, j \in N$, $A_i = A_j$. Furthermore, we assume that if $A_i \subseteq \mathbb{R}^m$ and $|A_i| = \infty$, then A_i is convex. Let $x = (x_1, \ldots, x_n) \in A$ be an alternative and $\mathbb{1} \equiv (1, \ldots, 1) \in \mathbb{R}^n$. If alternative x is such that for all $i, j \in N$, $x_i = x_j = \alpha$, then we denote $x = \alpha\mathbb{1}$. Next, let $F \subseteq A$ be the *set of feasible alternatives*. If for all $x \in F$ there exists α such that $x = \alpha\mathbb{1}$, then the set of feasible alternatives F *determines a public goods economy*. Otherwise, the set of feasible alternatives F *determines an economy with at least one private goods component*. Hence, our model encompasses public and private goods economies.

To fix ideas, let us give two examples. It will be clear from these examples that given the set A of alternatives, the set F of feasible alternatives fully determines whether we are working with a public or a private goods model. Note that the Cartesian product notation we use for the set of alternatives is for notational convenience only; none of our results require it.

Example 1 Let $A = \{a_1, \ldots, a_n\} \times \ldots \times \{a_1, \ldots, a_n\}$.

Public goods model: Suppose that the agents have to choose one candidate out of the set $\{a_1, \ldots, a_n\}$ of possible candidates. Then, $F = \{x \in A : \text{for all } i, j \in N,\ x_i = x_j\}$.

Private goods model: On the other hand, if agents have to allocate the set of indivisible objects or tasks $\{a_1, \ldots, a_n\}$ among themselves, then $F = \{x \in A : \text{for all } i, j \in N,\ x_i \neq x_j\}$. ◇

Example 2 Let $A = [0, 1] \times \ldots \times [0, 1]$.
Public goods model: Suppose that the agents have to choose a single point in the interval [0,1] that everyone will consume without rivalry, e.g., a public facility on a street (see Moulin 1980). Then, $F = \{x \in A : \text{for all } i, j \in N,\ x_i = x_j\}$.

Private goods model: On the other hand, if agents have to choose a division of one unit of an infinitely divisible good among themselves (see Sprumont 1991), then feasibility is determined by the size of the resource and $F = \{x \in A : \text{for all } i \in N,\ x_i \geq 0 \text{ and } \sum_{i \in N} x_i = 1\}$. ◇

For all $i \in N$, *preferences* are represented by a complete, reflexive, and transitive binary relation R_i over A_i.[8] As usual, for all $x, y \in A$, $x_i\ R_i\ y_i$ is interpreted as "i weakly prefers x to y", $x_i\ P_i\ y_i$ as "i strictly prefers x to y", and $x_i\ I_i\ y_i$ as "i is indifferent between x and y". Preferences R_i over A_i are *strict* if for all $x_i, y_i \in A_i$, $x_i\ R_i\ y_i$ implies $x_i\ P_i\ y_i$ or $x_i = y_i$.

For public goods models, preferences R_i over the individual set of alternatives A_i can easily be extended to preferences over the set of alternatives A (since each agent consumes the same public alternative). Whenever our model captures a private goods component, we assume that agents only care about their own consumption. Then, for both public and private goods models, we can easily extend preferences R_i over the individual set of alternatives A_i to preferences over the set of alternatives A (both preference relations only depend on agent i's consumption in A_i). Therefore, from now on, we use R_i to describe agent i's preferences over A_i as well as over A, i.e., we use both notations $x\ R_i\ y$ and $x_i\ R_i\ y_i$. Note that for private goods models, strict preferences over A_i do not need to be strict over A.

For all $i \in N$, we call a set of preferences over A_i, denoted by $\mathcal{R}_i$, a *preference domain*. We assume that for all $i, j \in N$, $\mathcal{R}_i = \mathcal{R}_j$ and denote this *common preference domain* by $\mathcal{R}$. Let $\mathcal{R}^N$ denote the set of *preference profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$.

For all $i \in N$, all preference relations $R_i \in \mathcal{R}$, and all alternatives $x \in A$, the *lower contour set of* R_i *at* x is $L(R_i, x) \equiv \{y \in A : x\ R_i\ y\}$; the *strict lower contour set of* R_i *at* x is $\mathrm{SL}(R_i, x) \equiv \{y \in A : x\ P_i\ y\}$; the *upper contour set of* R_i *at* x is $U(R_i, x) \equiv \{y \in A : y\ R_i\ x\}$; and the *strict upper contour set of* R_i *at* x is $\mathrm{SU}(R_i, x) \equiv \{y \in A : y\ P_i\ x\}$.

Let the set of alternatives A, the set of feasible alternatives F, and the common preference domain $\mathcal{R}$ be given. Then, a *rule* φ is a function that assigns to every preference profile $R \in \mathcal{R}^N$ a feasible alternative $\varphi(R) \in F$.

2.2 Properties of rules

We discuss in turn two central properties of the mechanism design literature. First, strategy-proofness is an incentive property that requires that no agent ever benefits from misrepresenting his preference relation. In game theoretical terms, a rule is strategy-

[8] Note that we do not impose continuity on preferences whenever individual sets of alternatives $A_i \subseteq \mathbb{R}^m$. However, various preference domains that we consider later contain only continuous preferences.

proof if in its associated direct revelation game form, it is a weakly dominant strategy for each agent to announce his true preference relation. By the revelation principle, strategy-proofness is a necessary condition for dominant strategy implementability.

For agent $i \in N$, preference profile $R \in \mathcal{R}^N$, and preference relation $R'_i \in \mathcal{R}$, we obtain preference profile (R'_i, R_{-i}) by replacing R_i at R by R'_i.

Strategy-proofness: A rule φ is *strategy-proof* if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi(R)\ R_i\ \varphi(R'_i, R_{-i})$.

Next, Maskin monotonicity is a property that requires the robustness (or invariance) of a rule with respect to specific preference changes. A rule φ is Maskin monotonic if an alternative x that is chosen at preference profile R remains chosen at a preference profile R' at which x is considered (weakly) better by all agents. An important result of the mechanism design literature is that Maskin monotonicity is a necessary condition for Nash implementability of a rule (see Maskin 1977, 1999). Apart from its importance for Nash implementability, we consider Maskin monotonicity to be an appealing property in itself.

In order to introduce Maskin monotonicity, we first define monotonic transformations. Loosely speaking, for any alternative x and any preference profile R, if at a preference profile R' all agents $i \in N$ consider alternative x to be (weakly) better, then R' is a monotonic transformation of R at x. For preferences R_i, $R'_i \in \mathcal{R}$ and alternative $x \in A$, R'_i is a *monotonic transformation of* R_i *at* x if $L(R_i, x) \subseteq L(R'_i, x)$. By $\mathrm{MT}(R_i, x)$ we denote the *set of all monotonic transformations of* R_i *at* x and by $\mathrm{MT}(R, x)$ we denote the *set of all monotonic transformations of* R *at* x, i.e., $R' \in \mathrm{MT}(R, x)$ if for all $i \in N$, $R'_i \in \mathrm{MT}(R_i, x)$.

Maskin monotonicity: A rule φ is *Maskin monotonic* if for all R, $R' \in \mathcal{R}^N$, $x \equiv \varphi(R)$ and $R' \in \mathrm{MT}(R, x)$ imply $\varphi(R') = x$.

For one of our "private goods results", we use the following weaker monotonicity property: a rule φ is unilaterally Maskin monotonic if given that alternative x is chosen at preference profile R, agent i's component x_i remains chosen at a unilateral deviation profile $R' = (R'_i, R_{-i})$ at which agent i considers x_i to be (weakly) better.

Unilateral monotonicity: A rule φ is *unilaterally monotonic* if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $x \equiv \varphi(R)$ and $R'_i \in \mathrm{MT}(R_i, x)$ imply $\varphi_i(R'_i, R_{-i}) = x_i$.

Note that Maskin monotonicity implies unilateral monotonicity. To be more precise, for public goods economies, Maskin monotonicity and unilateral monotonicity are equivalent and for private goods economies Maskin monotonicity implies unilateral monotonicity. However, the converse does not hold as shown in Example 7 in Sect. 3.

We close this section by introducing non-bossiness (in allocations) (see Satterthwaite and Sonnenschein 1981), an auxiliary property that we use for some of our "private goods results". The property states that, by changing his preference relation, an agent cannot change components of the allocation for the other agents without affecting his own. Obviously, this property is vacuous in a public goods model.

Non-bossiness: A rule φ is *non-bossy* if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R_i' \in \mathcal{R}$, $\varphi_i(R) = \varphi_i(R_i', R_{-i})$ implies that $\varphi(R) = \varphi(R_i', R_{-i})$.

For private goods models, Maskin monotonicity implies non-bossiness under our rich preference domain condition (Condition R1), while the converse is not true. On the other hand, the conjunction of strategy-proofness and non-bossiness is equivalent to Maskin monotonicity under our preference domain restriction condition (Condition R2). These relations will be made clear in Sect. 3.

2.3 Well-known preference domains

2.3.1 The Arrovian preference domain

By $\mathcal{R}_A$ we denote the preference domain that contains *all* strict preferences over A_i. We call this "unrestricted strict" domain of preferences $\mathcal{R}_{\mathcal{A}}$ the *Arrovian preference domain.*

2.3.2 One-dimensional single-peaked and single-plateaued preferences

Here, we introduce the general single-peaked preference domain and several of its well-known preference subdomains. We start by defining the smallest preference domain we consider, the symmetric single-peaked preference domain (e.g., Border and Jordan 1983).

Symmetric single-peaked preferences over $A_i \subseteq \mathbb{R}$: Preferences R_i over $A_i \subseteq \mathbb{R}$ are *symmetrically single-peaked* if there exists a point $p(R_i) \in A_i$ such that for all $x_i, y_i \in A_i$, $x_i \; R_i \; y_i$ if and only if $|p(R_i) - x_i| \leq |p(R_i) - y_i|$. We call $p(R_i)$ the *peak (alternative) of* R_i. We refer to the set of *all* symmetric single-peaked preferences (over A_i) as the *domain of symmetric single-peaked preferences (over* A_i*).*

Note that a symmetric single-peaked preference relation R_i is completely determined by it's peak. By relaxing the symmetry assumption, one obtains the domain of single-peaked preferences (e.g., Black 1948; Moulin 1980).

Single-peaked preferences over $A_i \subseteq \mathbb{R}$: Preferences R_i over $A_i \subseteq \mathbb{R}$ are *single-peaked* if there exists a point $p(R_i) \in A_i$ such that for all $x_i, y_i \in A_i$ satisfying either $y_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < y_i$, $x_i \; P_i \; y_i$. We call $p(R_i)$ the *peak (alternative) of* R_i. We refer to the set of *all* single-peaked preferences (over A_i) as the *domain of single-peaked preferences (over* A_i*).*

We now introduce two superdomains of the single-peaked preference domain (i.e., each of the following two preference domains contains the set of all single-peaked preferences). First, consider again the location of a public facility on a street. As in Example 2, we assume that agents' preferences are single-peaked, but that in addition they have an outside option so that if the public facility is too far away, they will not use it. This class of preferences is introduced and analyzed by Cantala (2004).

Single-peaked preferences over $A_i \subseteq \mathbb{R}$ reflecting an outside option: Preferences R_i over $A_i \subseteq \mathbb{R}$ are *single-peaked and reflect an outside option* if there exists an interval $[a, b] \subseteq A_i$ and a point $p(R_i) \in (a, b)$ such that (i) R_i is single-peaked on $[a, b]$ with peak $p(R_i)$; (ii) for all $x_i \in (a, b)$ and $y_i \in A_i \setminus [a, b]$, $x_i \; P_i \; y_i$; and (iii) for all $x_i, y_i \in A_i \setminus (a, b)$, $x_i \; I_i \; y_i$. We refer to the set of *all* single-peaked preferences (over A_i) reflecting an outside option as the *domain of single-peaked preferences (over A_i) reflecting an outside option* or the *Cantala preference domain (over A_i)* for short.

The second superdomain of the single-peaked preference domain frequently encountered in the literature (see Moulin 1984) is the so-called single-plateaued preference domain. For such a preference domain, we allow agents to have an interval of best points, so that instead of the peak we have a plateau.

Single-plateaued preferences over $A_i \subseteq \mathbb{R}$: Preferences R_i over $A_i \subseteq \mathbb{R}$ are *single-plateaued* if there exists an interval $[\underline{p}(R_i), \overline{p}(R_i)] \subseteq A_i$ such that (i) for all $x_i, y_i \in [\underline{p}(R_i), \overline{p}(R_i)]$, $x_i \, I_i \, y_i$; (ii) for all $x_i \in [\underline{p}(R_i), \overline{p}(R_i)]$ and all $y_i \in A_i \setminus [\underline{p}(R_i), \overline{p}(R_i)]$, $x_i \; P_i \; y_i$; and (iii) for all $x_i, y_i \in A_i \setminus [\underline{p}(R_i), \overline{p}(R_i)]$ satisfying either $y_i < x_i \leq \underline{p}(R_i)$ or $\overline{p}(R_i) \leq x_i < y_i$, $x_i \; P_i \; y_i$. We call $[\underline{p}(R_i), \overline{p}(R_i)]$ the *plateau (of alternatives) of* R_i. We refer to the set of *all* single-plateaued preferences (over A_i) as the *domain of single-plateaued preferences (over A_i)*.

Note that the definition above only allows for a unique closed interval of indifferent alternatives, namely the plateau of best alternatives. Dasgupta et al. (1979), DHM for short, consider an even more general preference domain (which they call the single-peaked preference domain) by allowing for additional closed intervals of indifferent alternatives left and right from the "top-plateau".

DHM preferences over $A_i \subseteq \mathbb{R}$: Preferences R_i over $A_i \subseteq \mathbb{R}$ are *DHM preferences* if there exists an interval $[\underline{p}(R_i), \overline{p}(R_i)] \subseteq A_i$ such that (i) for all $x_i, y_i \in [\underline{p}(R_i), \overline{p}(R_i)]$, $x_i \, I_i \, y_i$; (ii) for all $x_i \in [\underline{p}(R_i), \overline{p}(R_i)]$ and all $y_i \in A_i \setminus [\underline{p}(R_i), \overline{p}(R_i)]$, $x_i \; P_i \; y_i$; and (iii) for all $x_i, y_i \in A_i \setminus [\underline{p}(R_i), \overline{p}(R_i)]$ satisfying either $y_i < x_i \leq \underline{p}(R_i)$ or $\overline{p}(R_i) \leq x_i < y_i$, then $x_i \; R_i \; y_i$. We refer to the set of *all* DHM preferences (over A_i) as the *DHM preference domain (over A_i)*.

Set-relationships between one-dimensional single-peaked and single-plateaued preferences are depicted in Fig. 1.

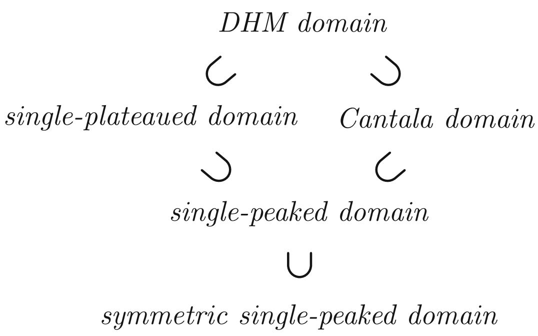

Fig. 1 Set-relationships between one-dimensional single-peaked preference domains

2.3.3 Higher-dimensional single-peaked preferences

There are various extensions of the one-dimensional single-peaked preference domains to higher dimensions. We start again by defining the smallest preference domains first. The first two preference domains are extensions of the one-dimensional symmetric single-peaked preferences introduced before (see Border and Jordan 1983). The domain of symmetric single-peaked preferences over $A_i \subseteq \mathbb{R}^m$ is induced by the Euclidean norm $\|\cdot\|_E$.

Symmetric single-peaked (Euclidean) preferences over $A_i \subseteq \mathbb{R}^m$: Preferences R_i over $A_i \subseteq \mathbb{R}^m$ are *symmetrically single-peaked* (or Euclidean) if there exists a point $p(R_i) \in A_i$ such that for all $x_i, y_i \in A_i$, $x_i \; R_i \; y_i$ if and only if $\|p(R_i) - x_i\|_E \leq \|p(R_i) - y_i\|_E$. We call $p(R_i)$ the *peak (alternative) of* R_i. We refer to the set of *all* symmetric single-peaked (Euclidean) preferences (over A_i) as the *domain of symmetric single-peaked (Euclidean) preferences (over* A_i*)*.

Note that for symmetric single-peaked preferences, upper contour sets are spheres. The following preference domain loosely speaking extends the symmetric preference domain to also allow for ellipsoids as upper contour sets (with axes that are parallel to the coordinate axes).

Separable quadratic preferences over $A_i \subseteq \mathbb{R}^m$: Preferences R_i over $A_i \subseteq \mathbb{R}^m$ are *separable quadratic* if there exists a point $p(R_i) \in A_i$, scalars $\alpha_1(R_i), \ldots, \alpha_m(R_i) > 0$, and a utility representation u_i of R_i such that for all $x_i \in A_i$, $u_i(x_i) = -\sum_{k=1}^{m} \left(\alpha_k(R_i)(x_{i,k} - p_k(R_i)\right)^2$. Note that if for all g, h such that $1 \leq g, h \leq m$, $\alpha_g(R_i) = \alpha_h(R_i)$, then preferences R_i are symmetrically single-peaked over $A_i \subseteq \mathbb{R}^m$. We call $p(R_i)$ the *peak (alternative) of* R_i. We refer to the set of *all* separable quadratic preferences (over A_i) as the *domain of separable quadratic preferences (over* A_i*)*.

In order to introduce the next preference domain, we need some definitions and notation. We define the convex hull of two points $x_i, y_i \in \mathbb{R}^m$ by $\text{conv}(x_i, y_i) = \{z_i \in \mathbb{R}^m$: there exists $t \in [0, 1]$ such that $z_i = tx_i + (1 - t)y_i\}$. Let $\|\cdot\|$ be a strictly convex norm, i.e.,

(i)	for all $x_i \in \mathbb{R}^m$, $\|x_i\| \geq 0$,	(positivity)
(ii)	for all $x_i \in \mathbb{R}^m$, $\|x_i\| = 0$ if and only if $x_i = 0$,	(positive definiteness)
(iii)	for all $x_i \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, $\|\alpha x_i\| = \lvert\alpha\rvert \, \|x_i\|$,	(positive homogeneity)
(iv)	for all $x_i, y_i \in \mathbb{R}^m$, $\|x_i\| + \|y_i\| \geq \|x_i + y_i\|$,	(triangular inequality)
(v)	for all $x_i, y_i, z_i \in \mathbb{R}^m$, $\|x_i - y_i\| + \|y_i - z_i\| = \|x_i - z_i\|$ if and only if $y_i \in \text{conv}(x_i, z_i)$.	(strict convexity)

Note that the requirement of strict convexity means that any sphere of positive radius does not contain any line segment that is not reduced to a point. Our definition of strict convexity for norms is based on Papadopoulos (2005, Proposition 7.2.1), which also lists various equivalent conditions for the strict convexity of a norm. For instance, the so-called ℓ^p norm $\|\cdot\|_p$ over $A_i \subseteq \mathbb{R}^m$ is strictly convex for any $p > 1$ (see Papadopoulos 2005, Proposition 7.3.2).[9]

The following preference domain includes the two previously introduced preference domains.

Preferences over $A_i \subseteq \mathbb{R}^m$ that are induced by a strictly convex norm $\|\cdot\|$: Preferences R_i over $A_i \subseteq \mathbb{R}^m$ are *induced by a strictly convex norm* $\|\cdot\|$ if there exists a point $p(R_i) \in A_i$ such that for all $x_i, y_i \in A_i$, $x_i \; R_i \; y_i$ if and only if $\|p(R_i) - x_i\| \leq \|p(R_i) - y_i\|$. We call $p(R_i)$ the *peak (alternative) of* R_i. We refer to the set of *all* preferences (over A_i) that are induced by a strictly convex norm as the *domain of preferences (over A_i) that are induced by a strictly convex norm.*

Finally, we introduce the most general higher-dimensional single-peaked preference domain that we are aware of [see Border and Jordan (1983)]. These preferences are called *star-shaped* by Border and Jordan (1983) because upper contour sets can have the shape of a star and look as follows $\star$.

Star-shaped preferences over $A_i \subseteq \mathbb{R}^m$: Preferences R_i over $A_i \subseteq \mathbb{R}^m$ are *star-shaped* if there exists a point $p(R_i) \in A_i$ such that for all $x_i \in A_i \setminus \{p(R_i)\}$ and all $\lambda \in (0, 1)$, $p(R_i) \; P_i \; [\lambda x_i + (1-\lambda)p(R_i)] \; P_i \; x_i$. We call $p(R_i)$ the *peak (alternative) of* R_i. We refer to the set of *all* star-shaped preferences (over A_i) as the *domain of star-shaped preferences (over A_i).*

Note that star-shaped preferences R_i are characterized by the fact that the restriction of R_i to each line passing through the peak $p(R_i)$ is one-dimensionally single-peaked.

If in addition to star-shapedness we require convexity of preferences, we obtain the following class of preferences.

Convex star-shaped preferences over $A_i \subseteq \mathbb{R}^m$: Preferences R_i over $A_i \subseteq \mathbb{R}^m$ are *convex star-shaped* if they are star-shaped and for all $x \in A$, $U(R_i, x)$ is a convex set.[10] We refer to the set of *all* convex star-shaped preferences (over A_i) as the *domain of convex star-shaped preferences (over A_i).*

Set-relationships between higher-dimensional single-peaked preference domains are as follows: *symmetric single-peaked domain* $\subset$ *separable quadratic domain* $\subset$ *convex norm induced domain* $\subset$ *convex star-shaped domain* $\subset$ *star-shaped domain.*

3 Monotonicity and strategy-proofness

3.1 Rich preference domains: monotonicity implies strategy-proofness

For $i \in N$ and $R_i \in \mathcal{R}$, by $b(R_i)$, we denote agent i's best alternatives in A, i.e., $b(R_i) \equiv \{x \in A : \text{for all } y \in A, \; x \; R_i \; y\}$. To establish our first result, we introduce the following preference domain "richness" condition.

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[9] For $p > 1$ and $x \in \mathbb{R}^m$, $\|x\|_p = \left(\sum_{j=1}^m |x_j|\right)^{\frac{1}{p}}$. For $p = 1$ and $x \in \mathbb{R}^m$, $\|x\|_1 = \sum_{j=1}^m |x_j|$; the associated ℓ^1 (or taxicab) norm is not strictly convex.

[10] Alternatively, preferences R_i over A_i are convex star-shaped if they are star-shaped and for all $x_i, y_i \in A_i$ and $\lambda \in [0, 1]$, $x_i \; R_i \; y_i$ implies $\lambda x_i + (1-\lambda) y_i \; R_i \; y_i$.

Condition R1: Let $i \in N$, $R_i \in \mathcal{R}$, and $x, y \in A$ be such that $y \; P_i \; x$. Then, there exists $R_i' \in \mathcal{R}$ such that $y \in b(R_i')$ and $L(R_i, x) \subseteq L(R_i', x)$.[11]

Remark 1 Note that Condition R1 is different from Dasgupta et al.'s (1979) or Fleurbaey and Maniquet's (1997) preference domain richness conditions. Our condition involves one preference relation R_i while the other two richness conditions are based on conditions involving two preference relations R_i and R_i'. The preference domain richness condition closest to ours seems to be the one introduced by Le Breton and Zaporozhets (2009). We briefly state and discuss the relation between these richness conditions in more detail in Appendix A. △

Examples of rich preference domains satisfying Condition R1 are the Arrovian preference domain, the single-peaked preference domain over $A_i \subseteq \mathbb{R}$, and more generally the convex star-shaped preference domain over $A_i \subseteq \mathbb{R}^m$ (see Proposition 1). We will check if the preference domains introduced above satisfy Condition R1 in Sect. 4 and give a short survey in Table 1.

Theorem 1 *Let A and F be given. Let $\mathcal{R}$ satisfy Condition R1 and let rule φ be defined on $\mathcal{R}^N$. If φ is unilaterally monotonic, then it is strategy-proof.*

Proof Suppose φ is unilaterally monotonic, but not strategy-proof. Then, there exist $R \in \mathcal{R}^N$, $i \in N$, and $\bar{R}_i \in \mathcal{R}$ such that $\varphi(\bar{R}_i, R_{-i}) \; P_i \; \varphi(R)$. Denote $\varphi(R) = x$ and $\varphi(\bar{R}_i, R_{-i}) = y$. Hence, $y_i \; P_i \; x_i$ and by Condition R1 there exists $R_i' \in \mathcal{R}$ such that $y \in b(R_i')$ and $L(R_i, x) \subseteq L(R_i', x)$. Thus, $R_i' \in \mathrm{MT}(\bar{R}_i, y)$ and $R_i' \in \mathrm{MT}(R_i, x)$. By unilateral monotonicity, $\varphi_i(R_i', R_{-i}) = y_i$ and $\varphi_i(R_i', R_{-i}) = x_i$. Hence, $x_i = y_i$; contradicting our assumption that $y_i \; P_i \; x_i$. □

Corollary 1 *Let A and F be given. Let $\mathcal{R}$ satisfy Condition R1 and let rule φ be defined on $\mathcal{R}^N$. If φ is Maskin monotonic, then it is strategy-proof.*

We demonstrate for the public as well as for the private goods case that strategy-proofness does not necessarily imply unilateral/Maskin monotonicity; the notation "unilateral/Maskin monotonicity" indicates that the corresponding statement holds for "unilateral monotonicity as well as for Maskin monotonicity". For both examples, we use the domain of single-peaked preferences (over $A_i = [0, 1]$), which satisfies Condition R1 (this follows from Proposition 1).

Example 3 We consider Moulin (1980) model as introduced in Example 2. Thus, for all $i \in N$, $A_i = [0, 1]$, $F = \{x \in A : \text{for all } i, j \in N, \; x_i = x_j\}$, and $\mathcal{R}$ is the domain of single-peaked preferences over A_i. Let $c_1, c_2 \in [0, 1]$, $c_1 < c_2$, and $k \in N$. Then, for all $R \in \mathcal{R}^N$,

$$\varphi(R) \equiv \begin{cases} c_1 \mathbb{1} \text{ if } c_1 \; P_k \; c_2 \text{ or if } c_1 \; I_k \; c_2 \text{ and } p(R_k) \in \mathbb{Q}; \\ c_2 \mathbb{1} \text{ if } c_2 \; P_k \; c_1 \text{ or if } c_1 \; I_k \; c_2 \text{ and } p(R_k) \notin \mathbb{Q}. \end{cases}$$

It is easy to see that φ is strategy-proof, but not unilateral/Maskin monotonic. ◇

[11] Note that in the proof of Theorem 1 and in all results concerning single-peaked preference domains, we could strengthen Condition R1 by requiring $L(R_i, x) = L(R_i', x)$ instead of $L(R_i, x) \subseteq L(R_i', x)$.

Example 4 We consider Sprumont (1991) model as introduced in Example 2. Thus, for all $i \in N$, $A_i = [0,1]$, $F = \{x \in A : \text{for all } i \in N,\ x_i \geq 0 \text{ and } \sum_{i \in N} x_i = 1\}$, and $\mathcal{R}$ is the domain of single-peaked preferences over A_i. Note that in this model, a two agents division problem corresponds to a two agents location problem in Moulin (1980) model. Hence, by adapting the rule of Example 3, we can construct a strategy-proof rule φ' that is not unilaterally/Maskin monotonic for Sprumont (1991) model as follows. Let $c_1, c_2 \in [0,1]$, $c_1 < c_2$, $k \in N$, and φ be the rule defined in Example 3. Let $j \in N \setminus \{k\}$. Then for all $R \in \mathcal{R}^N$, $\varphi'_k(R) = \varphi_k(R)$, $\varphi'_j(R) = 1 - \varphi_k(R)$, and for all $i \in N \setminus \{j,k\}$, $\varphi'_i(R) = 0$. ◇

3.2 Restricted preference domains: strategy-proofness implies monotonicity

To establish our second result, we introduce the following preference domain "restriction" condition.

Condition R2: Let $i \in N$, $R_i, R'_i \in \mathcal{R}$, and $x \in A$ be such that $R'_i \in \mathrm{MT}(R_i, x)$ and $R'_i \neq R_i$. Then, for all $y \in L(R_i, x) \cap U(R'_i, x)$, $y_i = x_i$.

Examples of restricted preference domains satisfying Condition R2 are the Arrovian preference domain (and any domain containing only strict preference relations), the symmetric single-peaked preference domain over $A_i \subseteq \mathbb{R}$, the separable quadratic preference domain over $A_i \subseteq \mathbb{R}^m$, and more generally any strictly convex norm induced preference domain (see Proposition 2). We state two straightforward consequences concerning Condition R2 in the following remark.

Remark 2 If a preference domain $\mathcal{R}$ satisfies Condition R2, then any preference domain $\tilde{\mathcal{R}} \subset \mathcal{R}$ satisfies Condition R2. If a preference domain $\mathcal{R}$ only contains strict preferences, then $\mathcal{R}$ satisfies Condition R2. △

We will check if the preference domains introduced above satisfy Condition R2 in Sect. 4 and give a short survey in Table 1.

Theorem 2 *Let A and F be given. Let $\mathcal{R}$ satisfy Condition R2 and let rule φ be defined on $\mathcal{R}^N$.*

(a) *If φ is strategy-proof, then it is unilaterally monotonic.*
(b) *Let F determine a public goods economy. If φ is strategy-proof, then it is Maskin monotonic.*

Proof (a) Suppose φ is strategy-proof, but not unilaterally monotonic. Then, there exist $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$ such that $\varphi(R) = x$, $R'_i \in \mathrm{MT}(R_i, x)$, and $\varphi_i(R'_i, R_{-i}) = y_i \neq x_i$. By strategy-proofness, $x\ R_i\ y$ and $y\ R'_i\ x$. Thus, $y \in L(R_i, x)$ and $y \in U(R'_i, x)$. Hence, $y \in L(R_i, x) \cap U(R'_i, x)$ and $y_i \neq x_i$; a contradiction with Condition R2.

(b) Because unilateral monotonicity and Maskin monotonicity coincide for public goods economies, this implication follows from (a). □

We demonstrate for the public as well as for the private goods case that unilateral/Maskin monotonicity does not necessarily imply strategy-proofness. For both examples, we use the domain of symmetric single-peaked preferences, which satisfies Condition R2 (this follows from Proposition 2).

Example 5 We consider Moulin (1980) model as described in Examples 2 and 3, but with symmetric single-peaked preferences. Thus, for all $i \in N$, $A_i = [0, 1]$, $F = \{x \in A : \text{for all } i, j \in N,\ x_i = x_j\}$, and $\mathcal{R}$ is the domain of symmetric single-peaked preferences over A_i. Let $c_1, c_2 \in [0, 1]$, $c_1 < c_2$, and $k \in N$. Then, for all $R \in \mathcal{R}^N$,

$$\varphi(R) \equiv \begin{cases} p(R_k)\mathbb{1} & \text{if } p(R_k) \leq c_1; \\ c_2\mathbb{1} & \text{otherwise.} \end{cases}$$

It is easy to see that φ is unilaterally/Maskin monotonic, but not strategy-proof. ◇

Example 6 We consider Sprumont (1991) model discussed in Examples 2 and 4, but with symmetric single-peaked preferences. Thus, for all $i \in N$, $A_i = [0, 1]$, $F = \{x \in A : \text{for all } i \in N,\ x_i \geq 0 \text{ and } \sum_{i \in N} x_i = 1\}$, and $\mathcal{R}$ is the domain of symmetric single-peaked preferences over A_i. Similarly as in Example 4, we can adapt the rule of Example 5 to construct a unilaterally/Maskin monotonic rule φ' that is not strategy-proof. Let $c_1, c_2 \in [0, 1]$, $c_1 < c_2$, $k \in N$, and φ be the rule defined in Example 5. Let $j \in N \setminus \{k\}$. Then for all $R \in \mathcal{R}^N$, $\varphi'_k(R) = \varphi_k(R)$, $\varphi'_j(R) = 1 - \varphi_k(R)$, and for all $i \in N \setminus \{j, k\}$, $\varphi'_i(R) = 0$. ◇

The following example demonstrates that for private goods economies Condition R2 and strategy-proofness do not necessarily imply Maskin monotonicity (hence, Theorem 2 (b) cannot be extended to private goods economies). We use the domain of separable quadratic preferences, which satisfies Condition R2 (this follows from Proposition 2).

Example 7 We consider a two-dimensional extension of Sprumont (1991) model with separable quadratic preferences. Thus, for all $i \in N$, $A_i = [0, 1]^2$, $F = \{x \in A : \text{for all } i \in N,\ x_i \geq 0\mathbb{1} \text{ and } \sum_{i \in N} x_i = 1\mathbb{1}\}$, and $\mathcal{R}$ is the domain of separable quadratic preferences over A_i. Let $c \in [0, 1)^2$. We define φ as follows. First, for all $R \in \mathcal{R}^N$, $\varphi_1(R) = c$. Second, if R_1 is symmetric, then $\varphi_2(R) = 1\mathbb{1} - c$ and $\varphi_3(R) = 0$, and otherwise, $\varphi_2(R) = 0$ and $\varphi_3(R) = 1\mathbb{1} - c$. It is easy to see that φ is strategy-proof, unilaterally monotonic, but not Maskin monotonic. ◇

Theorem 2 as well as Examples 6 and 7 show an important difference between public goods and private goods models. For the former, and for almost all the preference domains we cover[12], the class of Maskin monotonic rules is either a subset, a superset, or coincides with the class of strategy-proof rules (see Table 1). In the private goods case, this "set-inclusion connection" between the class of Maskin monotonic rules and the class of strategy-proof rules is lost for some preference domains, e.g., the symmetric single-peaked preference domain for which there exist rules that are Maskin monotonic but not strategy-proof, as well as rules that are strategy-proof but not Maskin monotonic.

[12] An exception is the star-shaped preference domain.

A key feature of Example 7 is that φ violates non-bossiness. With the next lemma we can show easily that Theorem 2 (b) can be extended to private goods economies if non-bossiness is added.

Lemma 1 *Let A and F be given. Let rule φ be defined on $\mathcal{R}^N$. If φ is unilaterally monotonic and non-bossy, then it is Maskin monotonic.*

Proof Suppose that φ is unilaterally monotonic and non-bossy. Let $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$ be such that $\varphi(R) = x$ and $R'_i \in \text{MT}(R_i, x)$. Then, by unilateral monotonicity, $\varphi_i(R'_i, R_{-i}) = x_i$. Hence, by non-bossiness, $\varphi(R'_i, R_{-i}) = x$. The proof that for all $R, R' \in \mathcal{R}^N$ such that $R' \in \text{MT}(R, x)$, $\varphi(R) = \varphi(R') = x$ follows from an iteration of the previous arguments (by switching agents one by one from R_i to R'_i). Hence, φ is Maskin monotonic. □

Corollary 2 *Let A and F be given. Let $\mathcal{R}$ satisfy Condition R2 and let rule φ be defined on $\mathcal{R}^N$. If φ is strategy-proof and non-bossy, then it is Maskin monotonic.*

4 Rich preference domains, restricted preference domains, and the Muller–Satterthwaite Theorem

We now analyze which of our preference domains are rich and which are restricted.

4.1 Condition R1: rich preference domains

It is clear from Examples 5 and 6 that the domain of symmetric single-peaked preferences violates Condition R1. We show below that the domain of convex star-shaped preference is rich.

Proposition 1 *The domain of convex star-shaped preferences satisfies Condition R1.*

The following notation for star-shaped preferences is useful in the proof of Proposition 1. Let R_i be a star-shaped preference relation and assume that $x_i \in A_i \setminus \{p(R_i)\}$. Then, for all $z_i \in A_i$ such that $z_i \; R_i \; x_i$ there exists $x'_i \in A_i$, $x'_i \; I_i \; x_i$ and $\lambda(R_i; x_i, z_i) \in [0, 1]$ such that $z_i = \lambda(R_i; x_i, z_i) p(R_i) + (1 - \lambda(R_i; x_i, z_i)) x'_i$. Note that if $\lambda(R_i; x_i, z_i) = 0$, then $z_i \; I_i \; x_i$ and if $\lambda(R_i; x_i, z_i) = 1$, then $z_i = p(R_i) \; P_i \; x_i$.

Proof Let R_i be a convex star-shaped preference relation and assume that $x, y \in A$ such that $y \; P_i \; x$. In order to verify Condition R1, we construct convex star-shaped preferences R'_i such that $y \in b(R'_i)$ and $L(R_i, x) \subseteq L(R'_i, x)$. If $y_i = p(R_i)$, then we are done by choosing $R'_i = R_i$. Thus, we assume that $y_i \neq p(R_i)$.

Loosely speaking, we construct R'_i by "lifting y_i up" to become the peak of a new preference relation R'_i such that preferences over $L(R_i, x)$ do not change. To be more precise, we construct preferences R'_i as follows:

(i) $y_i = p(R'_i)$, i.e., y_i is the peak of R'_i;
(ii) for all $z, z' \in L(R_i, x)$, $z \; R_i \; z'$ if and only if $z \; R'_i \; z'$, i.e., preferences on $L(R_i, x)$ do not change;

(iii) for all $z \in U(R_i, x)$ and $z' \in \text{SL}(R_i, x)$, $z\ P_i'\ z'$, i.e., preferences between $U(R_i, x)$ and $\text{SL}(R_i, x)$ do not change;
(iv) for all $z, z' \in U(R_i, x)$, $z\ R_i'\ z'$ if and only if $\lambda(R_i'; x_i, z_i) \geq \lambda(R_i'; x_i, z_i')$, i.e., we parameterize all $z, z' \in U(R_i, x)$ using line segments from the indifference set $I_i(R_i, x_i) = \{x_i' \in A_i : x_i'\ I_i\ x_i\}$ to the peak $p(R_i') = y_i$ and $\lambda(R_i'; x_i, \cdot)$.

Note that by (i) and (iv), $b(R_i') = \{y\}$ and by (ii), $L(R_i, x) = L(R_i', x)$ (in particular, $I_i(R_i, x_i) = I_i(R_i', x_i)$). Next, we prove that convex star-shapedness is preserved by our construction of R_i' from R_i.

First, we show that star-shapedness is preserved when going from R_i to R_i'. Let $w, z \in A$, and $\lambda \in (0, 1)$ be such that $w_i, z_i \in A_i \setminus \{p(R_i')\}$ and $z_i = \lambda y_i + (1-\lambda) w_i$. We prove that $z_i\ P_i'\ w_i$. We have two cases to consider:

Case 1. $w \in \text{SL}(R_i, x)$

Hence, (a) $w, z \in \text{SL}(R_i, x)$ or (b) $[w \in \text{SL}(R_i, x)$ and $z \in U(R_i, x)]$. For (a), since $y_i\ P_i\ w_i$, by convexity, $z_i\ R_i\ w_i$. Suppose, by contradiction, that $z_i\ I_i\ w_i$. Since, $y_i\ P_i\ w_i$ there exists $w' \in A$ with $w_i'\ I_i\ w_i$ and such that $y_i = \bar{\lambda} p(R_i) + (1-\bar{\lambda}) w_i'$ for some $\bar{\lambda} \in (0, 1)$. If $w_i' = w_i$ or $w_i' = z_i$, then we are done.[13] So suppose that w_i' is distinct from w_i and z_i. Since, $z_i = \lambda w_i + (1-\lambda) y_i$ and $y_i = \bar{\lambda} p(R_i) + (1-\bar{\lambda}) w_i'$, we obtain

$$\begin{aligned} z_i &= \lambda w_i + (1-\lambda)[\bar{\lambda} p(R_i) + (1-\bar{\lambda}) w_i'] \\ &= \lambda w_i + (1-\lambda)(1-\bar{\lambda}) w_i' + (1-\lambda)\bar{\lambda} p(R_i). \end{aligned}$$

Let $v \in A$ be such that v_i is the following convex combination of w_i, w_i', and z_i:

$$v_i = \lambda w_i + (1-\lambda)(1-\bar{\lambda}) w_i' + (1-\lambda)\bar{\lambda} z_i.$$

By convexity, $v_i\ R_i\ z_i\ I_i\ w_i\ I_i\ w_i'$. Notice that $z_i - (1-\lambda)\bar{\lambda} p(R_i) = v_i - (1-\lambda)\bar{\lambda} z_i$. Therefore,

$$z_i = \frac{1}{(1+(1-\lambda)\bar{\lambda})} v_i + \frac{(1-\lambda)\bar{\lambda}}{(1+(1-\lambda)\bar{\lambda})} p(R_i) = \tilde{\lambda} v_i + (1-\tilde{\lambda}) p(R_i)$$

with $\tilde{\lambda} = \frac{1}{(1+(1-\lambda)\bar{\lambda})} \in (0, 1)$. Hence, by star-shapedness of R_i, $z_i\ P_i\ v_i$, contradicting $v_i\ R_i\ z_i$. Therefore, $z_i\ P_i\ w_i$ and by the construction of R_i' [see (ii)], it follows that $z_i\ P_i'\ w_i$.

For (b), $w \in \text{SL}(R_i, x)$, $z \in U(R_i, x)$, and the construction of R_i' [see (iii)] imply $z_i\ P_i'\ w_i$.

[13] If $w_i' = w_i$ or $w_i' = z_i$, then for some $\lambda^* \in (0, 1)$, $z_i = \lambda^* w_i + (1-\lambda^*) p(R_i)$ and by star-shapedness, $z_i\ P_i\ w_i$.

Case 2. $w \in U(R_i, x_i)$

Hence, by the convex star-shapedness of R_i, $z_i\ P_i\ w_i$ and $z \in \mathrm{SU}(R_i, x_i)$. Thus, $\lambda(R_i'; x_i, z_i) > \lambda(R_i'; x_i, w_i)$. Hence, by construction of R_i' [see (iv)], this implies $z_i\ P_i'\ w_i$.

Second, we show that convexity is preserved when going from R_i to R_i'. Instead of the standard definition of convex preferences given in Footnote 10, it is well known that convexity of preferences can be defined via the convexity of upper contour sets. Recall that we do not change preferences on $L(R_i, x)$. An immediate implication is that for each $y' \in L(R_i', x)$, $U(R_i', y')$ is a convex set. Therefore, to show our claim, we only need to consider upper contour sets for points that are in $\mathrm{SU}(R_i, x)$. Hence, let $v, w \in \mathrm{SU}(R_i', x)$, $v \neq w$, $v\ I_i'\ w$, and $\alpha \in (0, 1)$ such that $z_i = \alpha v_i + (1 - \alpha) w_i$. We have to show that $z\ R_i'\ v\ I_i'\ w$. By construction of R_i' (see (iv)), this implies that we have to prove

$$\lambda(R_i'; x_i, z_i) \geq \lambda(R_i'; x_i, v_i) = \lambda(R_i'; x_i, w_i). \tag{1}$$

Note that $v, w \in \mathrm{SU}(R_i', x)$ and $v \neq w$ imply that $1 > \lambda(R_i'; x_i, v_i) = \lambda(R_i'; x_i, w_i) > 0$.

Let $\hat{z}_i = \alpha v_i' + (1 - \alpha) w_i'$. There exist $v', w', z' \in A$ such that $v_i'\ I_i\ w_i'\ I_i\ z_i'\ I_i\ x_i$ (recall that $I_i(R_i, x_i) = I_i(R_i', x_i)$), $v_i = \lambda(R_i'; x_i, v_i) y_i + (1 - \lambda(R_i'; x_i, v_i)) v_i'$, $w_i = \lambda(R_i'; x_i, w_i) y_i + (1 - \lambda(R_i'; x_i, w_i)) w_i'$, and $\hat{z}_i = \lambda(R_i'; x_i, \hat{z}_i) y_i + (1 - \lambda(R_i'; x_i, \hat{z}_i)) z_i'$. By convexity, $\hat{z}\ R_i'\ v'\ I_i'\ w'$. By construction of R_i' [see (iv)], $\lambda(R_i'; x_i, \hat{z}_i) \geq \lambda(R_i'; x_i, v_i') = \lambda(R_i'; x_i, w_i') = 0$.

Next, we can derive $z_i = [\lambda(R_i'; x_i, v_i) + \lambda(R_i'; x_i, \hat{z}_i) - \lambda(R_i'; x_i, v_i)\lambda(R_i'; x_i, \hat{z}_i)] y_i + [(1 - \lambda(R_i'; x_i, v_i))(1 - \lambda(R_i'; x_i, \hat{z}_i))] z_i'$.[14] Since, $z_i = \lambda(R_i'; x_i, z_i) y_i + (1 - \lambda(R_i'; x_i, z_i)) z_i'$, it follows that $\lambda(R_i'; x_i, z_i) = \lambda(R_i'; x_i, v_i) + \lambda(R_i'; x_i, \hat{z}_i) - \lambda(R_i'; x_i, v_i)\lambda(R_i'; x_i, \hat{z}_i)$. Hence, $1 > \lambda(R_i'; x_i, v_i) > 0$ and $\lambda(R_i'; x_i, \hat{z}_i) \geq 0$ imply

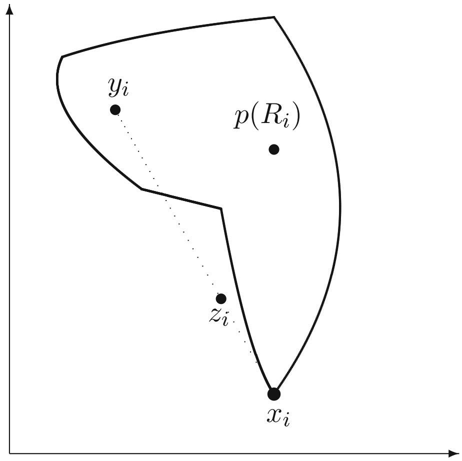
$$\begin{aligned}
&\Leftrightarrow \lambda(R_i'; x_i, \hat{z}_i) && \geq \lambda(R_i'; x_i, v_i)\lambda(R_i'; x_i, \hat{z}_i) \\
&\Leftrightarrow \lambda(R_i'; x_i, \hat{z}_i) - \lambda(R_i'; x_i, v_i)\lambda(R_i'; x_i, \hat{z}_i) && \geq 0 \\
&\Leftrightarrow \lambda(R_i'; x_i, v_i) + \lambda(R_i'; x_i, \hat{z}_i) - \lambda(R_i'; x_i, v_i) \\
&\quad\ \lambda(R_i'; x_i, \hat{z}_i) && \geq \lambda(R_i'; x_i, v_i) \\
&\Leftrightarrow \lambda(R_i'; x_i, z_i) && \geq \lambda(R_i'; x_i, v_i).
\end{aligned}$$

Hence, the desired inequality (1) holds and we have proven convexity of the preference relation R_i'. □

Corollary 3 *Let A such that for all $i \in N$, $A_i \subseteq \mathbb{R}^m$ and F be given. Let $\mathcal{R}$ be the domain of all convex star-shaped preferences and let rule φ be defined on $\mathcal{R}^N$. If φ is Maskin monotonic, then it is strategy-proof.*

[14] For completeness, $z_i = \alpha v_i + (1 - \alpha) w_i = \alpha[\lambda(R_i'; x_i, v_i) y_i + (1 - \lambda(R_i'; x_i, v_i)) v_i'] + (1 - \alpha)[\lambda(R_i'; x_i, v_i) y_i + (1 - \lambda(R_i'; x_i, v_i)) w_i'] = \lambda(R_i'; x_i, v_i) y_i + (1 - \lambda(R_i'; x_i, v_i))[\alpha v_i' + (1 - \alpha) w_i'] = \lambda(R_i'; x_i, v_i) y_i + (1 - \lambda(R_i'; x_i, v_i)) \hat{z}_i = \lambda(R_i'; x_i, v_i) y_i + (1 - \lambda(R_i'; x_i, v_i))[\lambda(R_i'; x_i, \hat{z}_i) y_i + (1 - \lambda(R_i'; x_i, \hat{z}_i)) z_i'] = [\lambda(R_i'; x_i, v_i) + \lambda(R_i'; x_i, \hat{z}_i) - \lambda(R_i'; x_i, v_i)\lambda(R_i'; x_i, \hat{z}_i)] y_i + [(1 - \lambda(R_i'; x_i, v_i))(1 - \lambda(R_i'; x_i, \hat{z}_i))] z_i'$.

Fig. 2 Star-shaped non-convex preferences that do not satisfy Condition R1



The following example demonstrates that convexity of preferences is a necessary assumption for star-shaped preferences to satisfy Condition R1.

Example 8 Let $A = [0, 1]^2 \times \ldots \times [0, 1]^2$ and let $\mathcal{R}$ be the domain of star-shaped preferences. In Fig. 2, we depict a preference relation R_i over $A_i = \mathbb{R}^2_+$ with peak $p(R_i)$ and with a non-convex upper contour set at $x_i \in A_i$ (marked by the indifference curve through x_i). It is easy to see that there does not exist $R'_i \in \mathcal{R}$ with $y_i = p(R'_i)$ and $L(R_i, x) \subseteq L(R'_i, x)$. Indeed, for any such R'_i, star-shapedness implies that for all $z' \in A$ with $z'_i = z_i$, $z \in SU(R'_i, x)$ while Condition R1 implies that $z \in L(R'_i, x)$; a contradiction. Thus Condition R1 is violated. ◇

Since, the single-peaked preference domain (over $A_i \subseteq \mathbb{R}$) is the one-dimensional equivalent of convex star-shaped preferences, Proposition 1 implies that also the one-dimensional single-peaked preference domain satisfies Condition R1. More generally, the following holds.

Remark 3 Let $\mathcal{R}$ be a preference domain formed by convex preferences that contains the domain of convex star-shaped preferences. Note that for such a preference domain $\mathcal{R}$, we can simply replicate the proof of Proposition 1 by noting that the preference relation R'_i defined in the proof is convex star-shaped. Hence, preference domain $\mathcal{R}$ satisfies Condition R1. As demonstrated by Example 8, the convexity assumption cannot be dropped.

In particular, the domain of single-peaked preferences (over $A_i \subseteq \mathbb{R}$), the domain of single-peaked preferences (over $A_i \subseteq \mathbb{R}$) reflecting an outside option, the domain of single-plateaued preferences (over $A_i \subseteq \mathbb{R}$), and the domain of DHM preferences (over $A_i \subseteq \mathbb{R}$) all satisfy Condition R1. △

4.2 Condition R2: restricted preference domains

It is clear from Examples 3 and 4 that general single-peaked preferences violate Condition R2. We show below that all preferences that are induced by a strictly convex norm satisfy R2.

Proposition 2 *The domain of preferences that are induced by a strictly convex norm satisfies Condition R2.*

Proof Let $\|\cdot\|$ be a strictly convex norm and R_i, R'_i be preferences over $A_i \subseteq R^m$ induced by $\|\cdot\|$. Furthermore, let $x \in A$ be such that $R'_i \in \mathrm{MT}(R_i, x)$ and $R'_i \neq R_i$. Note that then $p(R_i) \neq p(R'_i)$. Let $y \in L(R_i, x) \cap U(R'_i, x)$. Since $R'_i \in \mathrm{MT}(R_i, x)$, $y \in L(R'_i, x) \cap U(R'_i, x)$. Hence, $y_i \; I'_i \; x_i$ and

$$\|p(R'_i) - y_i\| = \|p(R'_i) - x_i\|. \tag{2}$$

Furthermore, $y \in L(R_i, x)$ implies

$$\|p(R_i) - y_i\| \geq \|p(R_i) - x_i\|. \tag{3}$$

Consider $\mathrm{line}(p(R_i), p(R'_i)) = \{z_i \in \mathbb{R}^m : \text{there exists } t \in \mathbb{R} \text{ such that } z_i = tx_i + (1-t)y_i\}$. Then, there exist two distinct points $\hat{z}_i, \tilde{z}_i \in \mathrm{line}(p(R_i), p(R'_i))$ such that $\hat{z}_i \; I_i \; x_i$ and $\tilde{z}_i \; I_i \; x_i$ (possibly $\hat{z}_i = x_i$ or $\tilde{z}_i = x_i$). Note that we can give an orientation to the line such that one of these points is to the left of $p(R_i)$ and the other is to the right of $p(R_i)$. Without loss of generality, assume that $p(R'_i)$ and $\tilde{z}_i$ are to the right of $p(R_i)$. Since, $R'_i \in \mathrm{MT}(R_i, x)$, $\tilde{z}_i \; I_i \; x_i$ implies $\tilde{z}_i \in L(R'_i, x)$ and

$$\|p(R'_i) - \tilde{z}_i\| \geq \|p(R'_i) - x_i\|. \tag{4}$$

Case 1. $p(R'_i) \notin \mathrm{conv}(p(R_i), \tilde{z}_i)$

Then, $\|p(R_i) - p(R'_i)\| > \|p(R_i) - \tilde{z}_i\| = \|p(R_i) - x_i\|$. Hence, $p(R'_i) \in L(R_i, x)$ and by $R'_i \in \mathrm{MT}(R_i, x)$, $p(R'_i) \in L(R'_i, x)$. Hence, $x_i = p(R'_i)$ and by (2), $x_i = y_i$.

Case 2. $p(R'_i) \in \mathrm{conv}(p(R_i), \tilde{z}_i)$

Then, by strict convexity, $\|p(R_i) - \tilde{z}_i\| = \|p(R_i) - p(R'_i)\| + \|p(R'_i) - \tilde{z}_i\| \overset{(4)}{\geq} \|p(R_i) - p(R'_i)\| + \|p(R'_i) - x_i\| \overset{(*)}{\geq} \|p(R_i) - x_i\|$, where $(*)$ follows from the triangular inequality. However, since $\|p(R_i) - \tilde{z}_i\| = \|p(R_i) - x_i\|$, $(*)$ is an equality and by strict convexity, $p(R'_i) \in \mathrm{conv}(p(R_i), x_i)$. Hence, $x_i = \tilde{z}_i$.

If $p(R'_i) \notin \mathrm{conv}(p(R_i), y_i)$, then, by strict convexity, $\|p(R_i) - y_i\| < \|p(R_i) - p(R'_i)\| + \|p(R'_i) - y_i\| \overset{(4)}{=} \|p(R_i) - p(R'_i)\| + \|p(R'_i) - x_i\| = \|p(R_i) - x_i\|$. Thus, $\|p(R_i) - y_i\| < \|p(R_i) - x_i\|$; contradicting (3). Hence, $p(R'_i) \in \mathrm{conv}(p(R_i), y_i)$. But then, (2) and (3) together imply, $x_i = y_i$.

To summarize, we have proven that for any $y \in L(R_i, x) \cap U(R'_i, x)$, it follows that $y_i = x_i$. Hence, preferences that are induced by a strictly convex norm satisfy Condition R2. □

Corollary 4 *Let A such that for all $i \in N$, $A_i \subseteq \mathbb{R}^m$ and F be given. Let $\mathcal{R}$ be a domain of preferences that are induced by a strictly convex norm and let rule φ be defined on $\mathcal{R}^N$.*

(a) *If φ is strategy-proof, then it is unilaterally monotonic.*
(b) *Let F determine a public goods economy. If φ is strategy-proof, then it is Maskin monotonic.*

Remark 4 Examples of preferences induced by a strictly convex norm are symmetric single-peaked preferences (over $A_i \subseteq \mathbb{R}^m$) and separable quadratic preferences (over $A_i \subseteq \mathbb{R}^m$).

As mentioned previously (see also Footnote 9), the ℓ^p norm over $A_i \subseteq \mathbb{R}^m$ is strictly convex for $p > 1$. A natural question to ask is whether the strictness of the convex norm is really needed for Proposition 2. Some evidence that this is the case can be obtained from the ℓ^1 or "taxicab" norm, which is convex, but not strictly so: the preference domain induced by the taxicab norm violates Condition R2 (incidentally, this preference domain also violates Condition R1). An example is available upon request. △

We now turn our attention to the Muller–Satterthwaite Theorem and its extensions.

4.3 An extended Muller–Satterthwaite Theorem

To conclude the section, we now state some immediate consequences of Theorems 1 and 2, and Corollaries 1 and 2.

Theorem 3 *An Extension of the Muller–Satterthwaite Theorem*
Let A and F be given. Let $\mathcal{R}$ satisfy Conditions R1 and R2 and let rule φ be defined on $\mathcal{R}^N$.

(a) *Then, φ is unilaterally monotonic if and only if it is strategy-proof.*
(b) *Let F determine a public goods economy. Then, φ is Maskin monotonic if and only if it is strategy-proof.*
(c) *Then, φ is Maskin monotonic if and only if it is strategy-proof and non-bossy.*

Theorem 3 states an extension of the Muller–Satterthwaite Theorem that covers both the public goods and the private goods case. Items (a) and (c) establish that the only monotonicity condition equivalent to strategy-proofness in a private goods model is the unilateral monotonicity condition. As Corollary 2 made clear, for preference domains satisfying both R1 and R2, only a subset of the set of strategy-proof rules coincide with the set of Maskin monotonic rules, namely the set of strategy-proof rules that satisfy non-bossiness. Because non-bossiness is vacuous in public goods models, item (c) directly implies item (b). The equivalence between Maskin monotonicity and strategy-proofness as stated in the original version of the Muller–Satterthwaite Theorem can thus be obtained only for public goods models.

Corollary 5 *The Muller–Satterthwaite Theorem*
Let A and F be given such that F determines a public goods economy. Let rule φ be defined on the Arrovian preference domain $\mathcal{R}_A$. Then, φ is Maskin monotonic if and only if it is strategy-proof.

Next, we show that the conclusion of the Muller–Satterthwaite Theorem is not only limited to the Arrovian preference domain; Theorem 3 has bite for various single-peaked preference domains. A first example is the domain of strict single-peaked preferences over $A_i \subseteq \mathbb{R}$ or the domain of strict single-peaked preferences defined on a finite set of alternatives. Indeed, preferences being single-peaked implies Condition R1 and preferences being strict implies Condition R2.

Finally, we introduce a new "Muller–Satterthwaite preference domain". Suppose that a public facility, e.g., a phone booth is to be located on a street that is very safe on one end of the street and becomes more and more dangerous when moving toward the other end of the street. Then, it is natural to assume that agents' preferences are single-peaked (the phone booth in front of one's house would be best) and prefer any location in the safer part of the street to a location in the more dangerous part of the street. The following preference domain describes the situation when the street is very safe on its "left side" and becomes more dangerous toward its "right side".[15]

Left-right single-peaked preferences over $A_i \subseteq \mathbb{R}$: Preferences R_i over $A_i \subseteq \mathbb{R}$ are *left-right single-peaked* if R_i is single-peaked over $A_i \subseteq \mathbb{R}$ with peak $p(R_i) \in A_i$ and such that for all $x_i, y_i \in A_i$ satisfying $x_i \leq p(R_i) < y_i$, $x_i \; P_i \; y_i$.

Note that any left-right single-peaked preference relation is uniquely defined by its peak.

Proposition 3 *Left-right single-peaked preferences satisfy Conditions R1 and R2.*

Proof Note that the domain of left-right single-peaked preferences only contains strict preferences and therefore satisfies Condition R2. In order to verify Condition R1, let R_i be a left-right single-peaked preference relation and assume that $x, y \in A$ such that $y \; P_i \; x$. Consider the left-right single-peaked preference relation R_i' with $p(R_i') = y_i$. By the definition of left-right single-peaked preferences:

(i) if $x_i > p(R_i)$, then $L(R_i, x) = A \cap [x, \infty) = L(R_i', x)$;
(ii) if $x_i \leq p(R_i)$, then $x_i < y_i \leq p(R_i)$ and
$L(R_i, x) = A \cap ((-\infty, x] \cup [p(R_i)\mathbb{1}, \infty)) \subseteq A \cap ((-\infty, x] \cup [y, \infty)) = L(R_i', x)$.

Hence, $y \in b(R_i')$ and $L(R_i, x) \subseteq L(R_i', x)$. Thus, the domain of left-right single-peaked preferences also satisfies Condition R1. □

Similarly, we can define the domain of right-left single-peaked preferences over $A_i \subseteq \mathbb{R}$ by assuming that the street is very safe on its "right side" and becomes more dangerous toward its "left side".

4.4 Conclusion

We conclude by summarizing which conditions our single-peaked preference domains satisfy (or not) in Table 1 below. We briefly explain how the cells in the table are filled.

[15] We thank Bernardo Moreno for suggesting this type of preference domain.

Table 1 Preference Domains and Conditions R1 and R2

Preference Domain(s) of	Condition R1	Condition R2
Arrovian preferences	Yes	Yes
Strict single-peaked preferences over $A_i \subseteq \mathbb{R}$	Yes	Yes
Left-right single-peaked preferences over $A_i \subseteq \mathbb{R}$	Yes	Yes
Right-left single-peaked preferences over $A_i \subseteq \mathbb{R}$	Yes	Yes
Symmetric single-peaked (Euclidean) preferences over $A_i \subseteq \mathbb{R}^m$	No	Yes
Separable quadratic preferences over $A_i \subseteq \mathbb{R}^m$	No	Yes
Strict convex norm induced preferences over $A_i \subseteq \mathbb{R}^m$	No	Yes
Single-peaked preferences over $A_i \subseteq \mathbb{R}$	Yes	No
Single-peaked preferences over $A_i \subseteq \mathbb{R}$ reflecting an outside option	Yes	No
Single-plateaued preferences over $A_i \subseteq \mathbb{R}$	Yes	No
DHM preferences over $A_i \subseteq \mathbb{R}$	Yes	No
Convex star-shaped preferences over $A_i \subseteq \mathbb{R}^m$	Yes	No
Star-shaped preferences over $A_i \subseteq \mathbb{R}^m$	No	No

Consider the first block formed by preference domains satisfying both Conditions R1 and R2. For the first two preference domains condition R2 is immediate since preferences are strict (Remark 2). The implication for Condition R1 is also immediate for the Arrovian preference domain while the conclusion for the strict single-peaked preference domain can be derived similarly as in Proposition 1 (i.e., the construction of preferences R'_i in the proof is similar, but for (iv) instead of star-shapedness one ensures strictness of R'_i). Proposition 3 establishes that both conditions are satisfied for the left-right and right-left single-peaked preference domains.

Next, take the second block formed by preference domains violating Condition R1 but satisfying Condition R2. By Proposition 2, the three preference domains in this block satisfy Condition R2. By Theorem 1 and Examples 5 and 6, the domain of symmetric single-peaked preferences violates Condition R1. It is easy to see that both examples can be extended to the separable quadratic preference domains and, more generally, to the strict convex norm induced preference domains (by choosing two points $c_1, c_2 \in A_i \subseteq \mathbb{R}^m$ such that $\|c_1\| < \|c_2\|$). Hence, these domains also violate Condition R1.

Consider now the third block formed by preference domains satisfying Condition R1 but violating Condition R2. By Remark 3 all these preference domains satisfy Condition R1. By Theorem 2 and Examples 3 and 4, the domain of single-peaked preferences over $A_i \subseteq \mathbb{R}$ violates Condition R2. Note that the next three preference domains are supersets of the single-peaked preference domain over $A_i \subseteq \mathbb{R}$. Hence, by Remark 2, Condition R2 is violated. Finally, it is easy to see that Examples 3 and 4 can be extended to the domain of strict convex star-shaped preferences (by choosing two points $c_1, c_2 \in A_i \subseteq \mathbb{R}^m$ such that $\|c_1\| < \|c_2\|$ and choosing one component of $p(R_k)$). Hence, this preference domain also violates Condition R2.

The last argument can also be used to show that the domain of star-shaped preferences violates Condition R2. Finally, Example 8 shows that the star-shaped preference domain violates Condition R1.

As a final remark, we would like to mention that our analysis and results are for single-valued rules. It might be interesting to see in how far our results can be extended to multi-valued rules/correspondences.

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A Appendix: Richness conditions

First, we introduce Dasgupta et al.'s (1979) richness condition. A preference domain is (Dasgupta, Hammond, and Maskin) rich if it satisfies the following condition.

Condition DHM: Let $R_i, R_i' \in \mathcal{R}$ and $x, y \in A$ such that (a) $x\ R_i\ y \Rightarrow x\ R_i'\ y$ and (b) $x\ P_i\ y \Rightarrow x\ P_i'\ y$. Then, there exists $R_i'' \in \mathcal{R}$ such that (i) $R_i'' \in \mathrm{MT}(R_i, x)$ and (ii) $R_i'' \in \mathrm{MT}(R_i', y)$.

Maskin (1985) called the Dasgupta et al. (1979) rich preference domain monotonically closed. Note that Condition DHM does not imply Condition R1. For instance, *strictly monotonic preference domains* satisfying Condition DHM do not satisfy Condition R1.[16] On the other hand, all the preference domains satisfying Condition R1 that we look at in the paper satisfy Condition DHM, but in general Condition R1 does not imply Condition DHM.[17]

Fleurbaey and Maniquet (1997) also use a preference domain richness condition under the name of strict monotonic closedness. Their rich preference domain satisfies the following condition.

Condition FM: Let $R_i, R_i' \in \mathcal{R}$ and $x, y \in A$ such that (a) $x\ P_i\ y$. Then, there exists $R_i'' \in \mathcal{R}$ such that for all $z \in A$, $z \neq x, y$, (i) $x\ R_i'\ z$ implies $x\ P_i''\ z$, (ii) $y\ R_i\ z$ implies $y\ P_i''\ z$, and (iii) [not $x\ I_i''\ y$].

Note that Conditions $R1$ and FM are logically independent. The preference domain of single-plateaued preferences over $A_i \subseteq \mathbb{R}$ is rich according to Condition $R1$, but not according to Condition FM [on the single-plateaued preference domain it might not be possible to satisfy Condition FM (iii)]. On the other hand, strictly monotonic domains satisfying Condition FM do not satisfy Condition R1.

Finally, we consider Le Breton and Zaporozhets's (2009) rich preference domain condition.

Condition LBZ: Let $R_i \in \mathcal{R}$ and $x, y \in A$ such that $y\ P_i\ x$ and $y \in b(\bar{R}_i)$ for some $\bar{R}_i \in \mathcal{R}$, there exists $R_i' \in \mathcal{R}$ such that $y \in b(R_i')$ and for all z with $z_i \neq x_i$ such that $x\ R_i\ z$, $x\ P_i'\ z$.

[16] A preference domain $\mathcal{R}$ is strictly monotonic with respect to A_i, $|A_i| = \infty$, if for each $R_i \in \mathcal{R}$, and each $x_i, y_i \in A_i$ with $y_i > x_i$, $y_i\ P_i\ x_i$.

[17] Let $A_i = \{x, y, a, b\}$ and $\mathcal{R} = \{R, R', R^1, R^2, R^3, R^4\}$ with $a\ P_i\ x\ I_i\ y\ P_i\ b$, $b\ P_i'\ x\ I_i'\ y\ P_i'\ a$, $a\ I_i^1\ x\ P_i^1\ y\ P_i^1\ b$, $a\ I_i^2\ y\ P_i^2\ x\ P_i^2\ b$, $b\ I_i^3\ x\ P_i^3\ y\ P_i^3\ a$, $b\ I_i^4\ y\ P_i^4\ x\ P_i^4\ a$. Then, $\mathcal{R}$ satisfies Condition R1 (R^1, R^2, R^3, R^4 serve to "complete" preferences R, R' to establish Condition R1), but violates Condition DHM (e.g., for R, R' and x, y).

While, Condition LBZ implies Condition R1, the converse is not true. Observe that Condition LBZ requires that $L(R_i, x) \setminus \{x\} \subseteq \mathrm{SL}(R'_i, x)$; a stronger requirement than $L(R_i, x) \subseteq L(R'_i, x)$ imposed by Condition R1. Condition LBZ requires sufficient degrees of freedom to undo at R'_i the possible indifferences with respect to x present at R_i.[18] On the other hand, all the preference domains satisfying Condition R1 that we look at in the article satisfy Condition LBZ.

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[18] Let $A_i = \{x, y, z\}$ and $\mathcal{R} = \{R, R', R'', R'''\}$ with $x\ P_i\ y\ I_i\ z$, $y\ P'_i\ x\ I'_i\ z$, $z\ P''_i\ x\ I''_i\ y$, and $x\ I'''_i\ y\ I'''_i\ z$. Then, $\mathcal{R}$ satisfies Condition R1 but violates Condition LBZ.