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A GAME THEORETICAL APPROACH TO THE ALGEBRAIC COUNTERPART OF THE WAGNER HIERARCHY

THESE

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Jérémie CABESSA

Titulaire d'un diplôme d'ingénieur mathématicien de
l'Ecole Polytechnique Fédérale de Lausanne (EPFL)

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Thesis Committee

- Jacques DUPARC, Professor at the University of Lausanne, Co-supervisor.
- Jean-Eric PIN, Professor at the University Paris Diderot - Paris 7 and member of the CNRS, Co-supervisor.
- Marco TOMASSINI, Professor at the University of Lausanne, Internal member.
- Olivier CARTON, Professor at the University Paris Diderot - Paris 7, External member.
- Thomas HENZINGER, Professor at the Ecole Polytechnique Fédérale de Lausanne (EPFL), External member.
- Victor SELIVANOV, Professor at the Novosibirsk Pedagogical University, External member and referee.
- Pascal WEIL, Professor at the University Bordeaux I and member of the CNRS, External member and referee.

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Contents

Résumé

Ce travail traite de la classification topologique des langages ω-réguliers, question qui a déjà été abordée sous de multiples facettes que sont la théorie des automates, théorie descriptive des ensembles, ou encore théorie des semigroupes, en algèbre.

En effet, d'une part, l'approche automatique de la théorie des langages formels révèle l'équivalence entre les langages ω-réguliers et ceux reconnus par automates de Büchi, Muller ou Rabin. Dans ce contexte, Klaus Wagner décrivit alors une fine et pertinente hiérarchisation topologique des langages ω-réguliers – la hiérarchie de Wagner –, et ce en classifiant les automates de Muller sous-jacents par rapport à une notion de complexité graphique. Cette hiérarchie possède une hauteur de ω^ω, est décidable, et s'avère coïncider avec la restriction de la hiérarchie de Wadge aux ensembles ω-réguliers.

D'autre part, en 1998, Victor Selivanov proposa quant à lui une description complète de cette hiérarchie d'un point de vue purement ensembliste.

Au cours des mêmes années, l'approche algébrique de la théorie des langages formels introduisit la structure d'ω-semigroupe fini comme contrepartie pertinente des langages ω-réguliers. Cette considération algébrique possède un intérêt bien spécifique dans le fait qu'il existe, pour tout langage ω-régulier, une structure minimale – dite syntaxique – qui le caractérise ; une propriété qui ne trouve pas de contrepartie convaincante du point de vue ensembliste ou automatique.

Ce travail de thèse vise à renforcer ce point de vue algébrique, en présentant une description détaillée de la contrepartie algébrique de la hiérarchie de Wagner, et ce par le biais de la théorie descriptive des jeux.

Les chapitres 1 à 3 présentent l'étroite correspondance entre les considérations automatique et algébrique des langages ω-réguliers. On y introduit la notion d'ω-semigroupe, qui, dans le cas fini, apparaît comme contrepartie algébrique pertinente des automates de Büchi. On montre ensuite que tout langage ω-régulier possède un ω-semigroupe syntaxique correspondant qui vérifie les propriétés de minimalité requises.

Dans les chapitres 4 et 5, on présente, par le biais de la théorie des jeux, la hiérarchie de Wadge des ω-ensembles Boréliens, ainsi que la hiérarchie de Wagner, vue comme trace de la hiérarchie de Wadge sur les ensembles ω-réguliers.

Les chapitre 6, 7 et 8 fournissent une description détaillée de la contrepartie algébrique de la hiérarchie de Wagner. Ces résultats reposent principalement sur une transposition de la théorie des jeux de Wadge dans le cadre des ω-semigroupes. Ainsi, on définit d'abord une réduction de type Wadge sur les

ω-semigroupes finis pointés. On prouve que la hiérarchie algébrique qui en résulte est effectivement isomorphe à la hiérarchie de Wagner, correspondant alors à un ordre partiel décidable de hauteur ω^ω et de largeur 2. On décrit ensuite une procédure de décision efficace de cette hiérarchie. Pour ce faire, on introduit une représentation graphique des ω-semigroupes finis pointés, révélant des invariants de Wagner algébriques *a priori* sensiblement différents des invariants automatiques. Une reformulation de la procédure de Wagner en termes d'ordinaux permet alors de calculer le degré de Wagner de tout ω-semigroupe fini pointé à partir de sa représentation graphique, et ce en un temps polynomial. Il en résulte que le degré de Wagner de tout langage ω-rationnel peut être calculé directement sur son image syntaxique. Par la suite, on décrit également deux méthodes constructives, l'une directe et l'autre inductive, permettant d'exhiber un ω-semigroupe fini pointé de degré de Wagner quelconque. On introduit finalement un invariant topologique caractérisant chaque classe de Wagner de cette hiérarchie algébrique.

Le chapitre 9 présente quelques propriétés additionnelles concernant la contrepartie algébrique de la hiérarchie de Wagner, et par là même conclut ce travail. En particulier, à équivalence près, on montre que les structures algébrique non auto-duales de cette hiérarchie sont exactement les ω-monoïdes finis pointés. De plus, les ω-semigroupes finis simplifiables à gauche, ω-groupes finis, et ω-semigroupes cycliques finis, lorsque pointés, se trouvent être tous de degré trivial dans cette hiérarchie.

Introduction

Automata theory arose in the thirties, before being more deeply investigated from the middle of the fifties. More precisely, in 1936, Alan Turing introduced the concept of a Turing machine as an abstract model of a computer [38], a notion which happens to already capture the entire concept of a finite automaton. In 1943, the two neuroscientists Warren S. McCulloch and Walter Pitts presented a mathematical formalization of the neural network in terms of finite automata. Later, in 1956, Stephen Kleene proved the equivalence between languages recognized by finite automata and regular languages [18], creating a significant bridge between abstract machines and formal languages [11, 12]. Automata theory kept on developing during the following years, providing many practical applications in lexical analysis, text processing, software verification, etc.

In the eighties, an algebraic approach to automata theory emerged, introducing finite semigroups as a relevant algebraic counterpart to finite automata, and revealing a succeeding correspondence between pseudo-varieties of semigroups and varieties of formal languages [28, 29]. Nowadays, automata theory stands at the crossroad of finite state machine, formal language, and semigroup theories.

In a parallel development, Richard Büchi's seminal work leading to the decidability of the monadic second order logic brought him to consider an extension of automata reading finite words to automata reading infinite words [2], thus opening the study of non-terminating processes. Thomas Wilke generalized Kleene's theorem in this context [42], stating the equivalence between languages recognized by infinite words reading automata and so-called ω-rational languages, and hence strengthening the link between automata and formal languages. In 1979, Klaus Wagner proposed an efficient classification of ω-rational languages by focusing on graph theoretical properties of their underlying automata, the Wagner hierarchy [41, 43]. This hierarchy was further proved to correspond to the restriction of the Wadge hierarchy – the most refined hierarchy in descriptive set theory – to ω-rational languages [39, 40, 34].

In the nineties, the algebraic approach to automata theory was extended from finite to infinite words. Jean-Eric Pin introduced the notion of an ω-semigroup as the algebraic counterpart to automata reading infinite words [26, 30]. In this framework, Olivier Carton and Dominique Perrin went into the algebraic reformulation of the Wagner hierarchy [4, 5, 6], a work carried on by Jacques Duparc and Mariane Riss in [10]. The present work follows this perspective, and hopes to provide a complete description of the algebraic counterpart of the Wagner hierarchy by means of a game theoretical approach.

Hence, this writing lies at the crossroad of two mathematical fields: the algebraic theory of automata working on infinite words, and hierarchical games, in descriptive set theory. Each of these two components enriches the strict mechanical aspect of automata theory.

The algebraic approach draws the equivalence between Büchi automata and ω-semigroups [27], providing several interesting properties. Firstly, given a finite Büchi automaton, one can effectively compute a finite ω-semigroup recognizing the same ω-language, and vice versa. Secondly, there exists a minimal finite ω-semigroup among all the ones recognizing a given ω-language – called the syntactic ω-semigroup –, whereas there is no convincing notion of Büchi (or Muller) minimal automaton. Thirdly, ω-semigroups appear as a powerful classification tool: for instance, an ω-language is first-order definable if and only if it is recognized by an aperiodic ω-semigroup [20, 37, 25], a generalization to infinite words of Schützenberger and McNaughton's famous result. Also, topological properties (being open, closed, clopen, $\mathbf{\Sigma^0_2}$, $\mathbf{\Pi^0_2}$, $\mathbf{\Delta^0_2}$) can be characterized by algebraic properties on ω-semigroups (see [31] or [27, Chap. 3]).

Hierarchical games aim to classify subsets of topological spaces, in particular by means of the following Wadge reduction: given two topological spaces E and F, and two subsets $X \subseteq E$ and $Y \subseteq F$, one says that X Wadge reduces to Y if there exists a continuous function from E into F such that $X = f^{-1}(Y)$, or equivalently, if there exists a winning strategy for Player II in the Wadge game $\mathbb{W}(X, Y)$. The resulting Wadge hierarchy appeared of a special interest for computer scientists, for it enlightens the study of classifying ω-rational languages. In this context, two main questions arise when X Wadge reduces to Y:

- *Effectivity*: if X and Y are given effectively, is it then possible to effectively compute a continuous function f such that $X = f^{-1}(Y)$?
- *Automaticity*: if X and Y are recognized by finite ω-automata, is there also an *automatic*[1] continuous function f such that $X = f^{-1}(Y)$?

An extended literature exists on both questions. In particular, Klaus Wagner answered positively to the second problem [41], and the restriction of the Wadge hierarchy to ω-rational sets is in fact entirely known. It corresponds precisely to the original Wagner hierarchy – an ordered set of width 2 and height ω^ω –, and the Wagner degree of any ω-rational set is efficiently computable [43]. Wagner's original proofs rely on a careful analysis of Muller automata, away from the algebraic framework. Olivier Carton and Dominique Perrin [4, 5, 6] investigated the algebraic reformulation of the Wagner hierarchy, a work carried on by Jacques Duparc and Mariane Riss [10]. However, this new approach is not yet entirely satisfactory, for it fails to provide a complete algorithm computing the Wagner degree of any ω-rational set directly on its syntactic ω-semigroup. Our work fills this gap, and provides a complete description of the algebraic counterpart of the Wagner hierarchy by means of hierarchical games.

In Chapter 1, we introduce the preliminary definitions and results involved in this work. We particularly focus on ordinals below ω^ω, and ordinal arithmetic.

Chapter 2 is a reminder of the classical definitions of a Büchi, Muller, and Rabin automaton. We conclude by mentioning the generalization of Kleene's theorem in the case of infinite words, stating the equivalence between languages recognized by automata reading infinite words, and ω-rational languages.

[1] i.e. computed by some finite automaton.

Chapter 3 describes the basis of the algebraic approach to automata theory, in both cases of finite and infinite words. First of all, we describe the equivalence between finite automata reading finite words and finite semigroups. We then define and prove the minimality properties of the syntactic semigroup of a rational language. We finally show that the morphism reduction between rational languages precisely coincides with the division relation on their syntactic structures. Thereafter, as a generalization of these results, we prove the equivalence between finite automata reading infinite words and finite ω-semigroups. We explore factorization properties of infinite words in finite semigroups, and prove that every finite ω-semigroup is entirely defined by only a finite amount of data. We finally define and state the expected minimality properties of syntactic ω-semigroups.

Chapter 4 is devoted to the description of the Wadge hierarchy. We define the continuous reduction via Wadge games, and introduce the resulting Wadge hierarchy. We then prove the determinacy of Wadge games with Borel winning sets, a key result providing a detailed description of the Borel Wadge hierarchy.

In Chapter 5, we describe the Wagner hierarchy as the trace of the Wadge hierarchy on ω-rational languages. We show that this hierarchy is decidable, and has height ω^ω. We prove that the Wagner degree of an ω-rational language is given by the length of the maximal chains contained in a complete underlying Muller automata. We finally show by a direct argument that the Wagner degree is indeed a syntactic invariant.

In Chapter 6, we translate the Wadge theory from the ω-rational languages to the ω-semigroups context. We define a reduction on pointed ω-semigroups by means of games, without any direct reference to the Wagner hierarchy. The resulting hierarchy, called the $\mathbb{SG}$-hierarchy, happens to be a generalization of the Wadge hierarchy. Many results concerning Wadge games are proved to also hold in this framework.

In Chapter 7, we first state that the restriction of the $\mathbb{SG}$-hierarchy to finite pointed ω-semigroups is the precise algebraic counterpart of the Wagner hierarchy, and hence corresponds to a refinement of the hierarchies of chains and superchains introduced by Olivier Carton and Dominique Perrin. We then provide a complete description of this hierarchy. We present a graph representation of finite pointed ω-semigroups, and deduce an algorithm on graphs that computes the precise Wagner degree of any such structure. We then show how to build a finite pointed ω-semigroup of any given Wagner degree. Finally, we introduce the normal form of any finite pointed ω-semigroup, which is a topological invariant for its Wagner class.

Chapter 8 explores the computational complexity of the decidability of the $\mathbb{FSG}$-hierarchy. We prove that the Wagner degree of any finite pointed ω-semigroup is efficiently computable in time $\mathcal{O}(n^3)$, where n is the cardinality of the finite semigroup given in input.

Chapter 9 provides additional results concerning the algebraic counterpart of the Wagner hierarchy, and concludes this work. Among other properties, we prove that finite ω-semigroups build on left-cancelable semigroups, groups, and cyclic semigroups only contain subsets of trivial Wagner degrees.

Chapter 1

Preliminaries

1.1 Ordinals

1.1.1 Classical presentation

We present some basic definitions and facts about ordinals, focusing particularly on the ordinal arithmetic. A more detailed and complete presentation can be found in [32, 19, 22, 23, 17].

Let E and F be two sets. A *binary relation* on E and F is a subset $R \subseteq E \times F$. Such a relation is called *left-total* if for all $x \in E$, there exists $y \in F$ such that $(x, y) \in R$. It is called *right-total* if for all $y \in F$, there exists $x \in E$ such that $(x, y) \in R$. It is *functional* if for all $x \in E$ and $y, z \in F$, the two relations $(x, y) \in R$ and $(x, z) \in R$ imply $y = z$.

A *relation* on E is a subset R of $E \times E$. The expression $(x, y) \in R$ is usually denoted by xRy. The relation R is *reflexive* if xRx, for all $x \in E$. It is *irreflexive* if xRx holds for no $x \in E$. It is *symmetric* if xRy implies yRx, for all $x, y \in E$. It is *antisymmetric* if xRy and yRx imply $x = y$, for all $x, y \in E$. It is *transitive* if xRy and yRz imply xRz, for all $x, y, z \in E$. Finally, it is *trichotomic* if either $x = y$, or xRy, or yRx holds, for all $x, y \in E$.

An *equivalence relation* is a reflexive, transitive, and symmetric relation. A *preorder* is a reflexive and transitive relation. An *order* (or *partial order*) is a reflexive, transitive, and antisymmetric relation. A *total order* is an irreflexive, transitive, and trichotomic relation. A *well-ordering* on E is a total order R on E such that every nonempty subset of E has an R-least element. In this case, one also says that the relation R well-orders E. Finally, a set E is called *transitive* if every element of E is also a subset of E.

An *ordinal* is a transitive set well-ordered by the membership relation $\in$. From now on, ordinals always will be denoted by greek letters. Given a set of ordinals X, the expression $\sup(X)$ denotes $\bigcup X$, and in case $X \neq \emptyset$, $\inf(X)$ denotes $\bigcap X$. Both $\sup(X)$ and $\inf(X)$ are ordinals. For any ordinal α, the set $S(\alpha) = \alpha \cup \{\alpha\}$ is also an ordinal called the *successor* of α. An ordinal α is said to be *successor* if there exists an ordinal β such that $\alpha = S(\beta)$; it is called *limit* otherwise.

The *natural numbers* are the finite ordinals defined by induction as follows: $0 = \emptyset$, and $n+1 = S(n)$, for every integer $n \geq 0$. This way, every ordinal number is defined as the set of its predecessors. The element 0 has no predecessor, it is the empty set. Then $1 = S(0) = \{0\} = \{\emptyset\}$, $2 = S(1) = \{0,1\} = \{\emptyset, \{\emptyset\}\}$, and so on, and so forth. The least infinite ordinal, denoted by ω, is defined by $\omega = \sup\{0, 1, 2, \ldots\}$, and thus corresponds to the *set of all natural numbers*. Afterwards, a succession of larger ordinals can be defined by induction on α as follows: for $\alpha = 0$, one sets $\omega_0 = \omega$; then for every ordinal $\alpha > 0$, the set $\omega_{\alpha+1}$ is defined as the least ordinal such that there is no injection from ω_α into ω_α; for α limit, one has $\omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\}$.

We now introduce the arithmetical operations on ordinals with some of their properties. A formal definition of the *ordinal sum* can be found in many textbooks, for instance [19]. We present an equivalent definition by transfinite induction. The intuitive interpretation of the expression "$\alpha + \beta$" is exactly the same as with integers: the number of items that we get when we lay α items on a table followed by β other items. Given two ordinals α, β, then

- $\alpha + 0 = \alpha$,
- $\alpha + S(\beta) = S(\alpha + \beta)$,
- $\alpha + \beta = \sup\{\alpha + \xi \mid \xi < \beta\}$, if β is a limit ordinal.

LEMMA 1.1. *Let α, β, γ be ordinals.*

(1) *If $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*
(2) *If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.*
(3) *If $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$.*
(4) *$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.*
(5) *$\alpha \leq \beta$ if and only if there exists an ordinal δ such that $\alpha + \delta = \beta$.*
(6) *$\alpha < \beta$ if and only if there exists an ordinal $\delta > 0$ such that $\alpha + \delta = \beta$.*
(7) *If B is a nonempty set of ordinals, then $\alpha + \sup\{\beta \mid \beta \in B\} = \sup\{\alpha + \beta \mid \beta \in B\}$.*

The ordinal sum over the natural numbers coincides with the usual addition. It is associative and commutative in this context. However, the ordinal sum is generally not commutative: for instance,

$$1 + \omega = 1 + \sup\{n \mid n < \omega\} = \sup\{1 + n \mid n < \omega\} = \omega < \omega + 1.$$

In addition, for every $k < \omega$, one has the following absorption property: $k + \omega = \sup\{k + n \mid n < \omega\} = \sup\{n \mid n < \omega\} = \omega$. For example, $3 + 15 + \omega + 7 + 2 = \omega + 7 + 2 = \omega + 9$.

Here again, we do not present the formal definition of the *ordinal multiplication*, but an equivalent definition by transfinite induction. The intuitive interpretation of the expression "$\alpha \cdot \beta$" is the number of items that we get when we count α items β times. Given two ordinals α, β, then

- $\alpha \cdot 0 = 0$,
- $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$,
- $\alpha \cdot \beta = \sup\{\alpha \cdot \xi \mid \xi < \beta\}$, if β is a limit ordinal.

LEMMA 1.2. *Let α, β, γ be ordinals.*

(1) *If $\alpha \neq 0$ and $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.*
(2) *If $\alpha \leq \beta$, then $\alpha \cdot \gamma \leq \beta \cdot \gamma$.*

(3) *If* $\alpha \neq 0$ *and* $\alpha \cdot \beta = \alpha \cdot \gamma$, *then* $\beta = \gamma$.
(4) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
(5) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
(6) *If* B *is a nonempty set of ordinals, then* $\alpha \cdot \sup\{\beta \mid \beta \in B\} = \sup\{\alpha \cdot \beta \mid \beta \in B\}$.

The ordinal multiplication over the natural numbers coincides with the usual multiplication. However, the ordinal multiplication is not commutative in general: for instance,

$$2 \cdot \omega = \sup\{2 \cdot n \mid n < \omega\} = \omega < \omega + \omega = \omega \cdot 2.$$

As for the ordinal sum, for every $k < \omega$, one has $k \cdot \omega = \sup\{k \cdot n \mid n < \omega\} = \sup\{n \mid n < \omega\} = \omega$. Hence, one has $15 + 3 \cdot \omega \cdot 4 + 2 = \omega \cdot 4 + 2$.

Apart from its combinatorial definition, the *ordinal exponentiation* can be defined by transfinite induction via the ordinal multiplication as follows: given any two ordinals α, β, one sets

- $\alpha^0 = 1$,
- $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$,
- $\alpha^\beta = \sup\{\alpha^\xi \mid \xi < \beta\}$, if β is a limit ordinal.

Lemma 1.3. *Let* α, β, γ *be ordinals.*
(1) *If* $\alpha > 1$, *then* $\beta < \gamma$ *if and only if* $\alpha^\beta < \alpha^\gamma$.
(2) *If* $\alpha > 1$, *then* $\alpha^\beta = \alpha^\gamma$ *implies* $\beta = \gamma$.
(3) *If* $\alpha \leq \beta$, *then* $\alpha^\gamma \leq \beta^\gamma$.
(4) *If* $\alpha > 1$, *then* $\beta \leq \alpha^\beta$.
(5) $\alpha^{(\beta+\gamma)} = \alpha^\beta \cdot \alpha^\gamma$.
(6) $(\alpha^\beta)^\gamma = \alpha^{(\beta \cdot \gamma)}$.
(7) *If* B *is a nonempty set of ordinals, then* $\alpha^{\sup\{\beta \mid \beta \in B\}} = \sup\{\alpha^\beta \mid \beta \in B\}$.

By combining the properties of the ordinal sum, multiplication and addition, one can show that $\omega^p \cdot \omega^q = \omega^q$, whenever $p < q$. This property will be particularly important throughout this work, since we will mostly deal with ordinal expressions of this form. For instance, one has $\omega^9 \cdot 3 + \omega^9 \cdot 2 + \omega^4 + \omega^6 \cdot 2 + 7 = \omega^9 \cdot 5 + \omega^6 \cdot 2 + 7$.

Finally, every ordinal can be uniquely written in a peculiar form called the *Cantor normal form (CNF) of base* ω, which is kind of a generalization of the Euclidian division of integers.

Theorem 1.4 (Cantor). *Given an ordinal* $\alpha \geq 1$, *there exists a unique integer* $k \geq 0$, *and two unique sequences of ordinals* $\beta_0 > \beta_1 > \ldots > \beta_k \geq 0$, *and* $0 < n_i < \omega$, *such that*

$$\alpha = \sum_{i=0}^{k} \omega^{\beta_i} \cdot n_i.$$

1.1.2 Alternative presentation

This work only involves ordinals strictly below ω^ω and we choose to present an alternative characterization of those ones. The set of ordinals strictly below ω^ω (that is ω^ω itself) is isomorphic to the set

$$Ord_{<\omega^\omega} = \{0\} \cup \bigcup_{k\in\mathbb{N}} \left(\mathbb{N}\backslash\{0\} \times \mathbb{N}^k\right)$$

– that is the set containing the integer 0 plus all finite nonempty sequences of integers whose left most component is strictly positive – equipped with the following ordering: 0 is the least element and given any two sequences $\alpha = (a_0, \dots, a_m)$, $\beta = (b_0, \dots, b_n) \in Ord_{<\omega^\omega}$, then

$$\alpha < \beta \text{ if and only if } \begin{cases} \text{either } m < n, \\ \text{or } m = n \text{ and } \alpha <_{lex} \beta, \end{cases}$$

where $<_{lex}$ denote the lexicographic order. This relation is clearly a well-ordering. For instance, one has $(7,3,0,0,1) < (1,0,0,0,0,0)$ and $(7,3,0,0,1) < (7,3,1,0,1)$. As usual, given such a sequence α, the i^{th} element of α is denoted by $\alpha(i)$. For example, if $\alpha = (3,0,0,2,1)$, then $\alpha(0) = 3$ and $\alpha(3) = 2$.

Every ordinal $\xi < \omega^\omega$ can then be associated in a unique way with an element of $Ord_{<\omega^\omega}$ as described hereafter: the ordinal 0 is associated with 0, and every ordinal $0 < \xi < \omega^\omega$ with Cantor normal form $\omega^{n_k} \cdot p_k + \cdots + \omega^{n_0} \cdot p_0$ is associated with the sequence of integers $\bar{\xi}$ of length $n_k + 1$ defined by $\bar{\xi}(n_k - i)$ being the multiplicative coefficient of the term ω^i in this Cantor normal form. The sequence $\bar{\xi}$ is thence an encoding of the Cantor normal form of ξ. For instance, the ordinal $\omega^4 \cdot 3 + \omega^3 \cdot 5 + \omega^0 \cdot 1$ corresponds to the sequence $(3,5,0,0,1)$. The ordinal ω^n corresponds the sequence $(1,0,0,\dots,0)$ containing n 0's. This correspondence is an isomorphism from ω^ω into $Ord_{<\omega^\omega}$, and from this point onward, we will make no more distinction between non-zero ordinals strictly below ω^ω and their corresponding sequences of integers.

In this framework, the ordinal sum on sequences of integers is defined as follows: given $\alpha = (a_0, \dots, a_m)$, $\beta = (b_0, \dots, b_n) \in Ord_{<\omega^\omega}$, then

$$\alpha + \beta = \begin{cases} \beta & \text{if } m < n, \\ (\alpha(0), \dots, \alpha(n-m-1), \alpha(n-m) + \beta(0), \beta(1), \dots, \beta(n)) & \text{if } m \geq n. \end{cases}$$

For instance, one has $(7,3,1,2,3) + (1,0,0,0,0,0) = (1,0,0,0,0,0)$,

$$(7,3,1,2,5) + (4,0,3) = \begin{array}{rccccc} & (7, & 3, & 1, & 2, & 5) \\ + & (& & 4, & 0, & 3) \\ \hline & (7, & 3, & 5, & 0, & 3) \end{array},$$

and $(7,3,1,2,5) + (5,0,0,0,1) = (12,0,0,0,1)$. As usual, the multiplication by an integer is defined by induction via the ordinal sum.

1.1.3 Some new definitions

A *signed ordinal* is a pair (ε, ξ), where ξ is an ordinal strictly below ω^ω and $\varepsilon \in \{+, -, \pm\}$. It will be denoted by $[\varepsilon]\xi$ instead. Signed ordinal are equipped with the following partial ordering: $[\varepsilon]\xi < [\varepsilon']\xi'$ if and only if $\xi < \xi'$. Therefore the signed ordinals $[+]\xi$, $[-]\xi$, and $[\pm]\xi$ are all three incomparable.

Given an ordinal $0 < \xi < \omega^\omega$ with Cantor normal form $\omega^{n_k} \cdot p_k + \cdots + \omega^{n_0} \cdot p_0$, the *playground* of ξ, denoted by $pg(\xi)$, is simply defined as the integer n_0. When regarded as a sequence of integers, the playground of ξ is the number of successive 0's from the right end of ξ. For instance, if $pg((2, 4, 0, 5, 0, 0)) = 2$.

Finally, given a signed ordinal $[\varepsilon]\xi$ with $\varepsilon \in \{+, -\}$ and Cantor normal form $\xi = \omega^{n_k} \cdot p_k + \cdots + \omega^{n_0} \cdot p_0$, a *cut* of $[\varepsilon]\xi$ is a signed ordinal $[\varepsilon']\xi' < [\varepsilon]\xi$ satisfying the following properties:

(1) $\xi' = \omega^{n_k} \cdot p_k + \cdots + \omega^{n_i} \cdot q_i$, for some $0 \leq i \leq k$ and $q_i \leq p_i$;

(2) if $n_i = n_0$, then $\varepsilon' = \varepsilon$ if and only if p_i and q_i have the same parity; whereas if $n_i > n_0$, then $\varepsilon' \in \{+, -\}$ with no restriction.

If ξ is regarded as the sequence of integers $(a_0, \ldots, a_n)$, a cut of $[\varepsilon]\xi$ is a signed ordinal $[\varepsilon'](b_0, \ldots, b_n) < [\varepsilon](a_0, \ldots, a_n)$ satisfying the following properties:

(1) there exists an index i such that: firstly, $b_j = a_j$, for each $0 \leq j < i$; secondly, $b_i < a_i$; thirdly, $b_j = 0$, for each $i < j \leq n$;

(2) if $pg(a_0, \ldots, a_n) = pg(b_0, \ldots, b_n) = p$, then $\varepsilon' = \varepsilon$ if and only if a_{n-p} and b_{n-p} have the same parity; whereas if $pg(a_0, \ldots, a_n) \neq pg(b_0, \ldots, b_n)$, then $\varepsilon' \in \{+, -\}$ with no restriction.

For instance, the successive cuts of the signed ordinal $[+](2, 0, 3, 0)$ are $[-](2, 0, 2, 0)$, $[+](2, 0, 1, 0)$, $[+](2, 0, 0, 0)$, $[-](2, 0, 0, 0)$, $[+](1, 0, 0, 0)$, and $[-](1, 0, 0, 0)$. As another example, the cuts of the signed ordinal $[-](4, 2, 0, 3, 0)$ are all listed below by decreasing order (i.e. $[\varepsilon]\xi$ can access $[\varepsilon']\xi'$ iff $[\varepsilon]\xi > [\varepsilon']\xi'$).

$$
\begin{tikzcd}
 & {[-](4,2,0,3,0)} \arrow[dl] \\
{[+](4,2,0,2,0)} \arrow[dr] & \\
 & {[-](4,2,0,1,0)} \arrow[dl] \arrow[d] \\
{[+](4,2,0,0,0)} \arrow[d] \arrow[dr] & {[-](4,2,0,0,0)} \arrow[d] \arrow[dl] \\
{[+](4,1,0,0,0)} \arrow[d] \arrow[dr] & {[-](4,1,0,0,0)} \arrow[d] \arrow[dl] \\
{[+](4,0,0,0,0)} \arrow[d] \arrow[dr] & {[-](4,0,0,0,0)} \arrow[d] \arrow[dl] \\
{[+](3,0,0,0,0)} \arrow[d] \arrow[dr] & {[-](3,0,0,0,0)} \arrow[d] \arrow[dl] \\
{[+](2,0,0,0,0)} \arrow[d] \arrow[dr] & {[-](2,0,0,0,0)} \arrow[d] \arrow[dl] \\
{[+](1,0,0,0,0)} & {[-](1,0,0,0,0)}
\end{tikzcd}
$$

1.2 Topology

First of all, we recall that T is a *tree* over the alphabet A if T is a subset of A^* closed under prefixes, that is if $v \in T$ and $u \subseteq v$, then $u \in T$. A *finite branch* of T is a finite word $u \in T$ such that there is no $v \in T$ satisfying $u \subseteq v$. An *infinite branch* of T is an infinite word $\alpha \in A^\omega$ such that $\alpha[0, n] \in T$, for all $n \geq 0$. The set of infinite branches of T is denoted by $[T]$.

Given any alphabet A, the set A^ω will always be equipped with the product topology of the discrete topology on A. The basic open sets of A^ω are thus of the form uA^ω, where $u \in A^*$. Hence, a set $X \subseteq A^\omega$ is open if and only if it is of the form $X = UA^\omega$, where $U \subseteq A^*$. A set $X \subseteq A^\omega$ is closed if and only if there exists a tree $T \subseteq A^*$ such that $X = [T]$. We will say that the finite word u *entered* the open set $X = UA^\omega$ if any infinite extension of u belongs to X, or equivalently, if $u \in UA^*$. Conversely, we say that the finite word u *left* the closed set $X = [T]$ if $u \notin T$, that is if sooner or later, any extension of u will exit T.

We recall that given any topological space E, the class of *Borel* subsets – or the *Borel σ-algebra* – of E is the smallest class $\mathcal{B}$ containing the open sets, and closed under countable union and complementation. The *Borel hierarchy* consists of a collection of classes of Borel subsets which stratifies the whole Borel algebra with respect to these operations of countable union and complementation. More precisely, for any countable ordinal α, the Borel classes are defined by induction as follows:

- $\boldsymbol{\Sigma}^0_1$ is the class of all open sets,
- $\boldsymbol{\Pi}^0_\alpha = \left\{X^c \mid X \in \boldsymbol{\Sigma}^0_\alpha\right\}$,
- $\boldsymbol{\Delta}^0_\alpha = \boldsymbol{\Sigma}^0_\alpha \cap \boldsymbol{\Pi}^0_\alpha$,
- $\boldsymbol{\Sigma}^0_\alpha = \left\{X = \bigcup_{i\in\mathbb{N}} X_i \mid X_i \in \bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta\right\}$.

One can show that

$$\mathcal{B} = \bigcup_{\alpha<\omega_1} \boldsymbol{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \boldsymbol{\Pi}^0_\alpha = \bigcup_{\alpha<\omega_1} \boldsymbol{\Delta}^0_\alpha.$$

This hierarchy is partially illustrated in Figure 1.1, where arrows represent the inclusion relation between classes.

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
\boldsymbol{\Sigma}^0_3 & & \boldsymbol{\Pi}^0_3 \\
& \boldsymbol{\Delta}^0_3 \arrow[ul] \arrow[ur] & \\
\boldsymbol{\Sigma}^0_2 \arrow[ur] & & \boldsymbol{\Pi}^0_2 \arrow[ul] \\
& \boldsymbol{\Delta}^0_2 \arrow[ul] \arrow[ur] & \\
\boldsymbol{\Sigma}^0_1 \arrow[ur] & & \boldsymbol{\Pi}^0_1 \arrow[ul] \\
& \boldsymbol{\Delta}^0_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 1.1: The Borel hierarchy.

Another preliminary notion: a subset $F \subseteq 2^\omega$ is called a *flip set* [1] if changing one single bit of any infinite word $x \in 2^\omega$ shifts it from F to its complement, or *vice versa*. In other words, for all $\alpha, \beta \in 2^\omega$, if there exists a unique k such that $\alpha(k) \neq \beta(k)$, then $\alpha \in F$ if and only if $\beta \notin F$. Without proving it, we will use the fact that no flip set is Borel, since Borel sets satisfy the Baire property, whereas flip sets do not [1].

Finally, given a subset X of E, the complement of X in E is denoted by X^c. Given an integer i, we also set

$$X^{c(i)} \begin{cases} X & \text{if } i \text{ is even,} \\ X^c & \text{if } i \text{ is odd.} \end{cases}$$

1.3 Languages

An *alphabet* is a simply set whose elements are called *letters*. In this work, alphabets are always finite. A *finite words* is a finite sequence of letters. An *infinite words* is a infinite sequence of letters of length ω. The *empty word* is denoted by ε. For a better comprehension, letters of the alphabets will be denoted by latin letters like a, b, or c, finite words by latin letters like u, v, w, and infinite words by greek letters like α, β, or γ. The *length* of a finite word is the number of letters that it contains, and the length of an infinite word is ω. Given a finite word u and a finite or infinite word v, then u_i denotes the i-th letter of u, $u[i,j]$ denotes the finite word $u_i u_{i+1} \ldots u_j$, with $i < j$, and in particular, $u[0,n]$ denotes the restriction of u to its n first letters; the expressions uv denotes the *concatenation* of u and v, and $u \subseteq v$ expresses that u is a *prefix* of v. Besides, an infinite word $\alpha = a_0 a_1 a_2 \ldots$ is *periodic* if there exists a positive integer p such that $a_k = a_{k+p}$, for all $k \in \mathbb{N}$. It is *ultimately periodic* if there exist two positive integers k_0 and p such that $a_k = a_{k+p}$, for every $k \geq k_0$. Finally, a *factorization* of the infinite word α is an infinite sequence $(u_n)_{n\in\omega}$ of finite words such that $\alpha = u_0 u_1 u_2 \cdots$. A factorization $(u'_n)_{n\in\omega}$ of α is a *superfactorization* of $(u_n)_{n\in\omega}$ if there exists a strictly increasing sequence of strictly positive integers $(k_n)_{n\in\omega}$, such that $u'_0 = u_0 u_1 \cdots u_{k_0 - 1}$ and $u'_{n+1} = u_{k_n} u_{k_n+1} \cdots u_{k_{n+1}-1}$, for every $n \geq 0$.

A set of finite words is called a *language*. A set of infinite words is called an *ω-language*. Given an alphabet A, then A^n, A^*, A^+, A^ω, and A^∞ respectively denote the sets of finite words with n letters, finite words, nonempty finite words, infinite words, and finite or infinite words, each of them over the alphabet A. Given $X \subseteq A^*$ and $Y \subseteq A^\infty$, the *concatenation* of X and Y is defined by $XY = \{xy \mid x \in X \text{ and } y \in Y\}$, the *star* or *finite iteration* of X by $X^* = \{x_1 \cdots x_n \mid n \geq 0 \text{ and } x_1, \ldots, x_n \in X\}$, and the *infinite iteration* of X by $X^\omega = \{x_0 x_1 x_2 \cdots \mid x_i \in X, \text{ for all } i \in \mathbb{N}\}$. One also sets $uY = \{uv \mid v \in Y\}$, and $u^{-1}Y = \{v \mid uv \in Y\}$. Finally, given a language $L \subseteq A^*$, the set $\overrightarrow{L}$ denotes the ω-language consisting of all infinite words which have infinitely many prefixes in L.

The class of *rational languages* of A^*, denoted $Rat(A^*)$, is the smallest class of subsets of A^* containing the empty word ε, the singletons $\{a\}$, for all $a \in A$, and closed under finite union, finite concatenation, and finite iteration. The class of *ω-rational languages* of A^∞, denoted $Rat(A^\infty)$, is the smallest class of subsets of A^∞ containing ε, the singletons $\{a\}$, for all $a \in A$, and closed under finite union, finite concatenation, finite iteration, and infinite iteration.

The class of ω-rational languages of A^ω, denoted by $Rat(A^\omega)$, is simply the restriction of $Rat(A^\infty)$ to the sets of infinite words. This class of languages is particularly important in automata theory. The following proposition gives a precise characterization of $Rat(A^\omega)$ [27].

Proposition 1.5. *An ω-language $L \subseteq A^\omega$ is ω-rational if and only if it can be written as $L = \bigcup_{i=1}^{n} XY^\omega$, with X and Y being rational languages of A^*, and $n \in \mathbb{N}$.*

1.4 The Gale-Stewart game

Game theory plays a crucial role in this work. We will particularly focus on infinite two-player games with perfect information. The first player, denoted by *Player I*, will always be assumed masculine, and the second player, denoted by *Player II*, will be assumed feminine. We recall that a two-player game is said to be *determined* if either Player I or Player II has a winning strategy in this game. The following Gale-Stewart game is of particular importance.

Let A be an alphabet, and $X \subseteq A^\omega$. The *Gale-Stewart game* $\mathbb{G}(X)$ [15] is an infinite two-player game with perfect information, where the players alternately play letters from A. Player I begins. After ω turns, their successive moves produce an infinite word $\alpha \in A^\omega$. Player I wins $\mathbb{G}(X)$ if and only if $\alpha \in X$. A play in this game is illustrated below.

$$
\begin{array}{lccccccccccc}
\text{I}: & x_0 & & & & x_2 & & \cdots\cdots\cdots & & & x_{n+1} & & \cdots\cdots\cdots \\
 & & \searrow & & \nearrow & & \searrow & & & \nearrow & & \searrow & \\
\text{II}: & & & x_1 & & & & \cdots\cdots & x_n & & & & \cdots\cdots
\end{array}
$$

In 1975, D. A. Martin proved *Borel determinacy*, a major result stating that Gale-Stewart games were determined for any Borel winning set [21]. The Borel determinacies of the Wadge and $\mathbb{SG}$-games that we will introduce further on directly follow from this result.

Theorem 1.6 (Martin). *Let A be an alphabet, and $X \subseteq A^\omega$ be Borel. The Gale-Stewart game $\mathbb{G}(X)$ is determined.*

Chapter 2

Automata

Summary

This chapter provides a survey of basic definitions and results in automata theory. An extended presentation of automata reading finite and infinite words can be found respectively in [35] and [27].

First, we define the general notion of an automaton, which is a mathematical model of a very simplistic computer. Such a machine jumps from one computational state to another, depending on its current state, and on the last information bit it receives in input. The automaton is said to be deterministic, if for each input, the resulting computation is uniquely determined. An automaton is commonly represented as a graph, and every one of its computations is described by a path in the induced graph.

We then focus on the conventional finite automata reading finite inputs. In this context, the final state reached by the computation determines whether the input is accepted or rejected by the machine. Kleene's theorem states that the input sets accepted by these machines are exactly the rational languages.

When extended to infinite inputs, every computation induces an infinite path in the graph of the automaton. Several other acceptance conditions can be defined, leading in particular to the definitions of a Büchi, Muller, and Rabin automaton. A generalization of Kleene's theorem states that the expressive powers of these three kinds of machines are equivalent, and correspond precisely to the ω-rational languages.

2.1 General concept

We introduce the general notion of an automaton which will be further extended on the one hand to automata reading finite words, and on the other hand to automata reading infinite words.

DEFINITION 2.1. *A finite automaton is a 4-tuple* $\mathcal{A} = (Q, A, E, I)$*, where*

- Q *is a finite set called the set of states,*
- A *is an alphabet,*
- E *is a subset of* $Q \times A \times Q$ *called the set of transitions,*
- I *is a subset of* Q *called the set of initial states.*

An automaton can be regarded as an abstract machine moving from state to state depending on the input it receives. The set of transitions describes the rules for going from one state to another according to the input symbol that is read. The transition $(p, a, q) \in E$ means that if the automaton is in state p and receives the input letter a, then it will move to state q. A finite automaton is generally represented by a directed labeled graph. The nodes of the graph correspond to the states of the automaton. The nodes corresponding to the initial states are marked by an incoming arrow. The labeled edges of the graph represent the transitions of the automaton, meaning that there is an edge labeled by a from p to q if and only if (p, a, q) is a transition of the automaton. Therefore, the behavior of the automaton on a given input is represented by a path in this graph.

EXAMPLE 2.2. The finite automaton $\mathcal{A} = (Q, A, E, I)$ defined by $Q = \{q_0, q_1\}$, $A = \{0, 1\}$, $E = \{(q_0, 0, q_0), (q_0, 1, q_1), (q_1, 0, q_0), (q_1, 1, q_1)\}$, and $I = \{q_0\}$, is illustrated in Figure 2.1. On the sample input string 0110, the automaton $\mathcal{A}$ starts in the initial state q_0, and then proceeds to the successive states q_0 again, q_1, q_1 again, and back to q_0.

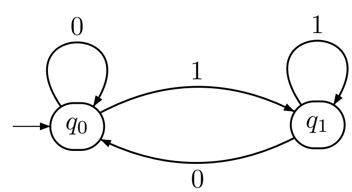


Figure 2.1: A finite automaton.

Given an automaton $\mathcal{A} = (Q, A, E, I)$ and a state $p \in Q$, the expression $\mathcal{A}_p$ denotes the automaton $\mathcal{A}$ whose set of initial state has been changed into the singleton p, namely $\mathcal{A}_p = (Q, A, E, \{p\})$. In addition, for any transition $(p, a, q) \in E$, the state p is called the *origin*, q is the *end*, and a is the *label* of this transition. Two transitions (p, a, q) et (p', a', q') are called *consecutive* if $q = p'$. A *finite path* in $\mathcal{A}$ is a finite sequence of consecutive transitions

$$c = ((q_0, a_1, q_1), \ldots, (q_{n-1}, a_n, q_n)),$$

also denoted by $c : q_0 \xrightarrow{a_1} q_1 \cdots q_{n-1} \xrightarrow{a_n} q_n$, or simply $c : q_0 \xrightarrow{a_1 \cdots a_n} q_n$. The states q_0 and q_n are respectively called the *origin* and the *end* of the path c. A state q_n is said to be *accessible* from q_0 if there exists a path $q_0 \xrightarrow{a_1 \cdots a_n} q_n$. One says that such a path c *passes through* or *visits* the states $q_0, q_1, \ldots, q_n$, and the finite word $a_1 \cdots a_n$ is called the *label* of c. More generally, an *infinite path* in $\mathcal{A}$ is an infinite sequence of consecutive transitions

$$x = ((q_0, a_1, q_1), (q_1, a_2, q_2), \ldots)$$

also denoted by $x : q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \cdots$. The state q_0 is the *origin* of x and the infinite word $a_1 a_2 \cdots$ is its *label*. The path x *passes infinitely often through* or *visits infinitely often* the state q if there exist infinitely many integers n such that $q_n = q$. The set of states visited infinitely often by x is denoted by $Inf(x)$.

An automaton $\mathcal{A} = (Q, A, E, I)$ is called *complete* if for every state $q \in Q$ and every letter $a \in A$, there exists at least one transition $(q, a, q') \in E$. It is called *deterministic* if it has only one initial state q_0, and if for every state $q \in Q$ and every letter $a \in A$, there exists at most one transition $(q, a, q') \in E$. In this case, the set of transitions E is the graph of a partial function $\delta : Q \times A \longrightarrow A$, called the *transition function* of the automaton, which maps every state q and every letter a on the unique state $q \cdot a$ – if it exists – such that $(q, a, q \cdot a) \in E$. The transition function can be naturally extended to a partial function from $Q \times A^*$ into Q by setting $q \cdot 1 = q$ and $q \cdot (ua) = (q \cdot u) \cdot a$, for every word $u \in A^+$ and every letter $a \in A$ such that $(q \cdot u)$ and $(q \cdot u) \cdot a$ are defined. An automaton as defined in Definition 2.1 is said to be *nondeterministic*. Therefore, a deterministic automaton is simply a specific nondeterministic one.

2.2 Automata over finite words

Finite automata on finite words describe abstract computers with an extremely limited amount of memory, those reacting according to their current state and the unique forthcoming bit to be read. As simple as it may be, this computational model stands for the cornerstone of any other more sophisticated model, like counter automata or pushdown automata, or even Turing machines.

An *automaton* reading finite words is a 5-tuple $\mathcal{A} = (Q, A, E, I, F)$, where (Q, A, E, I) is an automaton, and F is a subset of Q called the set of final states. A finite path of $\mathcal{A}$ is called *successful* if its origin is in I and its end is in F. A finite word is said to be *recognized* by $\mathcal{A}$ if it is the label of a successful finite path of $\mathcal{A}$, and the *language recognized* by $\mathcal{A}$, denoted by $L(\mathcal{A})$, is the set of finite words recognized by $\mathcal{A}$. Two automata are said to be *equivalent* if they recognize the same language. A language is called *recognizable* if there exists an automaton that recognizes it.

Example 2.3. Consider the automaton $\mathcal{A} = (Q, A, E, I, F)$ defined by $Q = \{q_0, q_1\}$, $A = \{0, 1\}$, $E = \{(q_0, 0, q_0), (q_0, 1, q_1), (q_1, 0, q_0), (q_1, 1, q_1)\}$, $I = \{q_0\}$, and $F = \{q_1\}$. This automaton is illustrated in Figure 2.2, where final states are double-circled. One easily notices that the language recognized by $\mathcal{A}$ corresponds to the set of finite words ending by 1, that is $L(\mathcal{A}) = A^*1$.

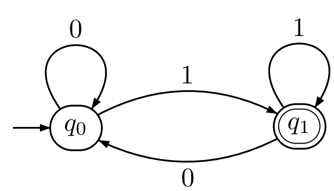


Figure 2.2: An automaton reading finite words.

One can show that the languages recognized by nondeterministic and deterministic automata are the same [35]. Kleene's theorem precisely characterizes these languages.

THEOREM 2.4 (KLEENE). *A language is recognizable if and only if it is rational.*

2.3 Automata over infinite words

In the sixties, the work of Julius Richard Büchi on the decidability of the monadic second order logic led up to an extension of the classical notion of an automaton to the case of infinite words. In this context, we present three main different kinds of automata reading infinite words, namely the Büchi, Muller, and Rabin automata, and mention that their respective expressive powers, which may appear different at first sight, are in fact the very same. A detailed presentation of these concepts can be found in [27].

2.3.1 Büchi automata.

Büchi automata over infinite words are the ones with the weakest mode of recognition [2]. A *Büchi automaton* reading infinite words is a 5-tuple $\mathcal{A} = (Q, A, E, I, F)$, where (Q, A, E, I) is an automaton and F is a subset of Q. An infinite path x of $\mathcal{A}$ is called *successful* if its origin is in I and if it visits F infinitely often, that is if $Inf(x) \cap F \neq \emptyset$. An infinite word is said to be *recognized* by $\mathcal{A}$ if it is the label of a successful infinite path in $\mathcal{A}$, and the ω-language *recognized* by $\mathcal{A}$, denoted by $L^\omega(\mathcal{A})$, is the set of infinite words recognized by $\mathcal{A}$. An ω-language is said to be *recognizable* if there exists a finite Büchi automaton that recognizes it.

EXAMPLE 2.5. Consider the Büchi automaton $\mathcal{A} = (Q, A, E, I, F)$, defined by $Q = \{q_0, q_1\}$, $A = \{0, 1\}$, $E = \{(q_0, 0, q_0), (q_0, 1, q_1), (q_1, 0, q_0), (q_1, 1, q_1)\}$, $I = \{q_0\}$, and $F = \{q_1\}$, illustrated in Figure 2.3. The ω-language recognized by this automaton is the set of infinite words containing infinitely many 1's, i.e. $L^\omega(\mathcal{A}) = (A^*1)^\omega$.

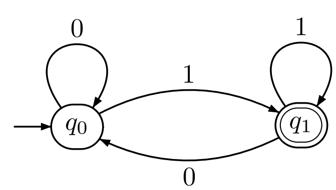


Figure 2.3: A Büchi automaton reading infinite words.

Nondeterministic Büchi automata are strictly more powerful than deterministic ones. In order to establish this statement, consider the ω-language $L = (0 \cup 1)^*1^\omega$. This language is recognized by the nondeterministic Büchi automata of Figure 2.4. However, it cannot by recognized by a deterministic Büchi automaton. Indeed, towards a contradiction, assume that there exists a deterministic Büchi automaton $\mathcal{A}$ recognizing L. Since the infinite word 1^ω belongs to L, it is recognized by $\mathcal{A}$. Then there necessarily exists an integer n_0 such that $\mathcal{A}$ ends up in a final state after reading the prefix $u_0 = 1^{n_0}$. Furthermore, since $1^{n_0}01^\omega$ belongs to L, it is recognized by $\mathcal{A}$, and there necessarily exists an integer n_1 such that $\mathcal{A}$ ends up in a final state after reading the prefix $u_1 = 1^{n_0}01^{n_1}$. By iterating this construction, one produces an infinite word $\alpha = 1^{n_0}01^{n_1}01^{n_2}0\cdots$ such that $\mathcal{A}$ ends up in a final state after reading every prefix u_i of α. Therefore, the automaton $\mathcal{A}$ visits a final state infinitely often when reading the infinite word α, meaning that α is recognized by $\mathcal{A}$. A contradiction, since $\alpha \notin L$.

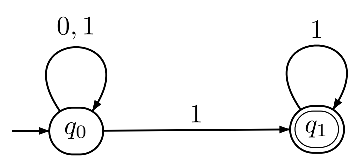


Figure 2.4: A nondeterministic Büchi automaton recognizing the ω-language $L = (0 \cup 1)^*1^\omega$.

2.3.2 Muller automata.

Muller automata are deterministic automata with a more powerful acceptance mode than the Büchi automata have [24]. A *Muller automaton* is a 5-tuple $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$, where $(Q, A, \delta, \{i\})$ is a deterministic automaton, and $\mathcal{T}$ is a subset of $\mathcal{P}(Q)$ called the *table* of $\mathcal{A}$. An infinite path x of $\mathcal{A}$ is said to be *successful* if it starts in the state i and if $Inf(x)$ belongs to $\mathcal{T}$. An infinite word is *recognized* by $\mathcal{A}$ if it is the label of a successful infinite path in $\mathcal{A}$. The ω-language *recognized* by $\mathcal{A}$, denoted $L^\omega(\mathcal{A})$, is the set of infinite words recognized by $\mathcal{A}$. Besides, a subset T of Q is called *admissible* if there exists an infinite initial path x such that $Inf(x) = T$. The table $\mathcal{T}$ is said to be *full* if for every admissible set T of $\mathcal{T}$ and for every admissible set T' containing T, then T' also belongs to $\mathcal{T}$.

Example 2.6. Consider the Muller automaton $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$ defined by $Q = \{q_0, q_1\}$, $A = \{0, 1\}$, the graph of δ being given by

$$\{(q_0, 0, q_0), (q_0, 1, q_1), (q_1, 0, q_0), (q_1, 1, q_1)\},$$

$i = q_0$, and $\mathcal{T} = \{\{q_1\}\}$. This automaton is illustrated in Figure 2.5. The ω-language recognized by this automaton is the set of infinite words ending with infinitely many successive 1's, that is $L^\omega(\mathcal{A}) = A^*1^\omega$.

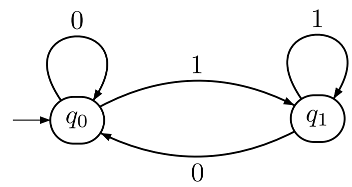


Figure 2.5: A Muller automaton.

2.3.3 Rabin automata

Rabin automata propose another mode of recognition. A *Rabin automaton* is a 5-tuple $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{R})$, where $(Q, A, \delta, \{i\})$ is a deterministic automaton, and $\mathcal{R} = \{(U_j, V_j) \mid j \in J\}$ is a family of pairs of sets of states. An infinite path x of $\mathcal{A}$ is said to be *successful* if it starts in the state i and if there exists an index $j \in J$ such that x visits U_j finitely often and V_j infinitely often. An infinite word is said to be *recognized* by $\mathcal{A}$ if it is the label of a successful infinite path in $\mathcal{A}$, and the ω-language *recognized* $\mathcal{A}$, denoted by $L^\omega(\mathcal{A})$, is the set of infinite words recognized by $\mathcal{A}$.

Example 2.7. Consider the Rabin automaton $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$ defined by $Q = \{q_0, q_1\}$, $A = \{0, 1\}$, the graph of δ being given by

$$\{(q_0, 0, q_0), (q_0, 1, q_1), (q_1, 0, q_0), (q_1, 1, q_1)\},$$

$i = q_0$, and $\mathcal{R} = \{(\{q_0, q_1\}, \{q_1\})\}$. This automaton is illustrated in Figure 2.6. The ω-language recognized by this automaton is the set of infinite words ending with infinitely many successive 1's, that is $L^\omega(\mathcal{A}) = A^*1^\omega$.

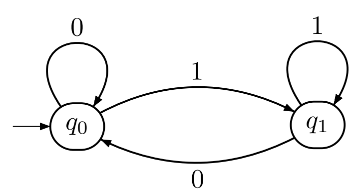


Figure 2.6: A Rabin automaton.

2.3.4 Recognizable ω-languages

Finally, as a generalization of Kleene's theorem, we mention two results describing the expressive powers of these different kinds of automata. The proofs of these results can be found in [27].

THEOREM 2.8. *Let L be an ω-language. The following conditions are equivalent:*

(1) *L is ω-rational.*
(2) *L is recognizable (by a finite Büchi automaton).*
(3) *L is recognizable by a finite Muller automaton.*
(4) *L is recognizable by a finite Rabin automaton.*

THEOREM 2.9. *Let L be an ω-language. The following conditions are equivalent:*

(1) *L is recognizable by a finite deterministic Büchi automaton.*
(2) *L is recognizable by a finite Muller automaton with full table.*
(3) *L can be written of the form $L = \overrightarrow{X}$, with X recognizable subset of A^+.*

Chapter 3

Algebra and automata

Summary

The algebraic approach to automata theory draws a tight correspondence between automata and specific algebraic structures. Semigroups are proved to be a convincing algebraic counterpart to automata reading finite words, and many properties on rational languages benefit from a purely algebraic characterization. Furthermore, this relation can be extended to the case of infinite words, in which finite ω-semigroups appear as the algebraic representatives of Büchi automata [27]. The present chapter describes these equivalences for both finite and infinite word cases.

First of all, we introduce basic notions and results of semigroup theory. We particularly focus on properties characterizing the structure of finite semigroups. In this context, the study of infinite words in finite semigroups plays an important role. A key result from Ramsey shows that every such infinite sequence can be factorized into a specific form $seee\cdots$, called a linked pair, and usually written as (s, e). As a consequence, every infinite word can be uniquely associated with an equivalence class of linked pairs. Linked pairs thus provide a finite classification of infinite words according to their factorizations.

Thereafter, we establish the equivalence between semigroups and automata reading finite words. We define an algebraic notion of recognizability, and prove that one can effectively shift from an automaton to a semigroup recognizing the same language, and conversely, from a semigroup to a corresponding automaton. In addition, we prove that there exists a unique minimal semigroup among all those recognizing a given rational language, called the syntactic semigroup. Finally, we present an algebraic characterization of the morphism reduction on rational languages.

We further extend these results to the case of infinite words. For this purpose, we introduce the concept of an ω-semigroup as a generalization of a semigroup. We then exclusively focus on finite ω-semigroups. Based on the study of infinite words in finite semigroups, we show that finite ω-semigroups are equivalent to other algebraic structures, called Wilke algebras, hence determined by only a finite amount of data.

Finally, we prove the expected equivalence between finite ω-semigroups and Büchi automata. We conclude by introducing the notion of a syntactic ω-

semigroup, and prove the same minimality properties as in the semigroup context. This feature is particularly interesting, since there is no convincing notion of minimal Büchi or Muller automaton.

3.1 Semigroups

3.1.1 Generalities

In this section, we expose basic definitions and facts of semigroup theory, focusing particularly on finite semigroups.

Let S be a set. A *binary operation* on S is a mapping from $S \times S$ into S. The image of the couple (x, y) under this mapping is usually denoted by xy and is called the *product* of x and y. A binary operation on S is *associative* if the equality $(xy)z = x(yz)$ holds, for all $x, y, z \in S$. It is *commutative* if $xy = yx$ holds, for all $x, y \in S$. A *semigroup* is a pair consisting of a set S and an associative binary operation on S. When the binary operation is clear from the context, a semigroup is simply denoted by its set. A semigroup S is called *finite* if S is a finite set and *infinite* otherwise. Once equipped with an identity element, a semigroup becomes a *monoid*. If S is a semigroup, S^1 denotes S if S is a monoid, and $S \cup \{1\}$ otherwise, with the operation of S completed by the relations $1 \cdot x = x \cdot 1 = x$, for all $x \in S^1$. Finally, a *pointed semigroup* [33] is a pair (S, X), where S is a semigroup and X is a subset of S.

Example 3.1. The *trivial monoid*, denoted by $\{1\}$, consists of a single identity element.

Example 3.2. The semigroup U_1 is the set $\{1, 0\}$ equipped with the usual multiplication. This semigroup is a commutative monoid.

Example 3.3. The set $\{0, 1, 2, \ldots, n\}$ equipped with the max operation is a commutative monoid whose identity is the element 0.

Example 3.4. The set $\{a, b, c, d\}$ is a semigroup for the operation defined by the relations:

$$\begin{array}{llll} a^2 = a & ab = a & ac = a & ba = a \\ b^2 = b & bc = c & ca = d & cb = c \\ c^2 = c & & & \end{array}$$

Given a semigroup S, an *idempotent* of S is an element e such that $e^2 = e$. The set of idempotents of S is denoted by $E(S)$, or simply by E when the semigroup involved is clear from the context. The following result will be frequently used throughout this work. It shows that any element of a finite semigroup has an idempotent power.

Lemma 3.5 ([27, pp. 441–442]). *Let S be a finite semigroup. There exists an integer π such that, for each $s \in S$, s^π is idempotent.* The least integer satisfying this property is called the *exponent* of S.

Proof. Since S is finite, for all $x \in S$, there exist two strictly positive integers i and p such that $x^{i+p} = x^i$ (otherwise S would be infinite). The minimal i and p satisfying this property are respectively called the *index* and the *period* of x. For any $k \geq i$, one has

$$x^k = x^{i+n} = x^i x^n = x^{i+p} x^n = x^i x^p x^n = x^{i+n} x^p = x^{k+p}.$$

In particular, if k is a multiple of p, that is $k = qp$, then

$$(x^k)^2 = x^{2k} = x^{k+qp} = x^k,$$

showing that x^k is idempotent. Therefore, every element s of S has a least idempotent power s^{n_s}. The conclusion is drawn by setting π as the least common multiple of all the n_s. □

A *semigroup morphism* is a mapping φ from a semigroups S into a semigroup T such that $\varphi(xy) = \varphi(x)\varphi(y)$ holds, for every $x, y \in S$. A morphism $\varphi : S \longrightarrow T$ is an *isomorphism* if there exists another morphism $\psi : T \longrightarrow S$ such that $\varphi \circ \psi = id_T$ and $\psi \circ \varphi = id_S$. A morphism is an isomorphism if and only if it is a bijection. As a general rule, isomorphic semigroups shall be identified. From this perspective, a semigroup S is a *subsemigroup* of a semigroup T if there exists an injective morphism from S into T. A semigroup S is a *quotient* of a semigroup T if there exists a surjective morphism from T onto S. Finally, a semigroup S *divides* a semigroup T if S is quotient of a subsemigroup of T. The two following results prove that the division relation is transitive and is a partial order on finite semigroups, up to isomorphism.

Proposition 3.6. *The division relation is transitive.*

Proof. Assume that S_1 divides S_2 and S_2 divides S_3. Then there exist a subsemigroup T_1 of S_2, a subsemigroup T_2 of S_3, and surjective morphisms $\pi_1 : T_1 \longrightarrow S_1$ and $\pi_2 : T_2 \longrightarrow S_2$. Let us set $T = \pi_2^{-1}(T_1)$. Then T is a subsemigroup of S_3 and S_1 is a quotient of T, since $\pi_1(\pi_2(T)) = \pi_1(T_1) = S_1$. Therefore, S_1 divides S_3. □

Proposition 3.7. *Two finite semigroups that divide each other are isomorphic.*

Proof. This proof uses the notations of the proof of Proposition 3.6 with $S_3 = S_1$. Since T_1 is a subsemigroup of S_2 and T_2 is a subsemigroup of S_1, then $|T_1| \leq |S_2|$ and $|T_2| \leq |S_1|$. Furthermore, since π_1 and π_2 are surjective, then $|S_1| \leq |T_1|$ and $|S_2| \leq |T_2|$. It follows that $|S_1| = |T_1| = |S_2| = |T_2|$, and thus $T_1 = S_2$ and $T_2 = S_1$, up to isomorphism. Finally, π_1 and π_2 are bijections and therefore, S_1 and S_2 are isomorphic. □

Finally, a *morphism of pointed semigroups* from a pointed semigroup (S, X) into a pointed semigroup (T, Y) is a semigroup morphism $\varphi : S \longrightarrow T$ such that $\varphi^{-1}(Y) = X$. The notions of subsemigroups, quotient, and division can be easily adapted in the context of pointed semigroups, and the two previous results also hold in this case.

A *congruence* on a semigroup S is an equivalence relation $\sim$ such that $s \sim t$ implies $xsy \sim xty$, for each $s, t \in S$ and each $x, y \in S^1$. The quotient set $S/{\sim}$ of equivalence classes of S is naturally equipped with a structure of semigroup. The function which maps every element onto its equivalence class is the *canonical* morphism from S onto $S/{\sim}$. When two congruences are comparable, their associated quotient structures can also be compared.

PROPOSITION 3.8. *Let $\sim_1$ and $\sim_2$ be two congruences on a semigroup S. If $\sim_2$ is coarser than $\sim_1$, then $S/{\sim_2}$ is a quotient of $S/{\sim_1}$.*

PROOF. Let $\pi_1 : S \longrightarrow S/{\sim_1}$ and $\pi_2 : S \longrightarrow S/{\sim_2}$ be the corresponding canonical morphisms. Since $\sim_2$ is coarser than $\sim_1$ and since π_1 is surjective, the relation $\pi_2 \circ \pi_1^{-1}$ is a function from $S/{\sim_1}$ into $S/{\sim_2}$. Since π_1 and π_2 are morphisms and π_2 is onto, this function is a surjective morphism. Therefore, $S/{\sim_2}$ is a quotient of $S/{\sim_1}$. $\square$

For any semigroup morphism φ from S into T, the equivalence $\sim_\varphi$ on S defined by $s \sim_\varphi t$ if and only if $\varphi(s) = \varphi(t)$ is a congruence called the *nuclear congruence* of φ. In addition, given a subset P of a semigroup S, the *syntactic congruence* of P is the congruence $\sim_P$ over S defined by $s \sim_P t$ if and only if $xsy \in P \Leftrightarrow xty \in P$, for every $x, y \in S^1$. The quotient semigroup $S/{\sim_P}$ is called the *syntactic semigroup* of P and the canonical morphism from S onto $S/{\sim_P}$ is the *syntactic morphism* of P.

Let A be an alphabet. The set A^+ equipped with the concatenation is a semigroup called the *free semigroup* on A. The set A^* is called the *free monoid* on A. The following universal property justifies this terminology.

PROPOSITION 3.9. *Let φ be a function from A into a semigroup S. Then there exists a unique semigroup morphism $\bar{\varphi} : A^+ \longrightarrow S$ such that $\bar{\varphi}(a) = \varphi(a)$, for each $a \in A$.*

PROOF. The mapping $\bar{\varphi}$ from A^+ into S defined by

$$\bar{\varphi}(a_0 a_1 \cdots a_n) = \varphi(a_0)\varphi(a_1)\cdots\varphi(a_n),$$

for every finite word $a_0 a_1 \cdots a_n$, is the required morphism. In addition, any other morphism $\bar{\varphi}'$ such that $\bar{\varphi}'(a) = \varphi(a)$ for each $a \in A$ must satisfy the previous equalities. Therefore, $\bar{\varphi}$ is unique. $\square$

In particular, the morphism from S^+ onto S induced by the identity over S is called the *natural* morphism associated with S. This leads to the mention of the following useful corollary.

COROLLARY 3.10. *Let $\mu : A^+ \longrightarrow S$ be a morphism and $\sigma : T \longrightarrow S$ be a surjective morphism. Then there exists a morphism $\varphi : A^+ \longrightarrow T$ such that $\mu = \sigma \circ \varphi$.*

PROOF. Let us associate with each letter $a \in A$ an element $\varphi(a)$ of $\sigma^{-1}(\mu(a))$. This defines a function $\varphi : A \longrightarrow T$, which, by Proposition 3.9, can be extended to a morphism $\varphi : A^+ \longrightarrow T$ such that $\mu = \sigma \circ \varphi$. $\square$

$$\begin{tikzcd} A^+ \arrow[rr, dashed, "\varphi"] \arrow[dr, "\mu"'] & & T \arrow[dl, "\sigma"] \\ & S & \end{tikzcd}$$

We now introduce the different *ideals* of a semigroup. Let S be a semigroup. A *left ideal* L of S if a subset L of S such that $S^1L \subseteq L$. Symmetrically, a *right ideal* R of S if a subset R of S such that $RS^1 \subseteq R$. An *ideal* I of S is a subset which is simultaneously a left and right ideal of S. If K is a subset of S, the ideal (resp. left ideal, right ideal) *generated* by K is the set S^1KS^1 (resp. S^1K, KS^1). In addition, a nonempty ideal I of S is called *minimal* if, for every nonempty ideal J of S, the relation $J \subseteq I$ implies $J = I$. Minimal left and right ideals are defined similarly. Finally, a subset I of S is a *universal minimal ideal* of S if it is a minimal ideal contained in every other ideal of S. Universal minimal left and right ideals are defined similarly. We conclude by the following result.

Proposition 3.11. *Every finite nonempty semigroup has a unique minimal ideal.*

Proof. Let S be a nonempty finite semigroup. Since S itself is a nonempty ideal, there exists at least a minimal ideal in S. In addition, let I_1 and I_2 be two minimal ideals of S. One has $S^1I_1I_2S^1 = (S^1I_11)(1I_2S^1) \subseteq I_1I_2$, and therefore, I_1I_2 is an ideal contained in $I_1 \cap I_2$. The minimality of I_1 and I_2 implies $I_1 = I_2$. □

Green's relations are fundamental equivalence relations introduced by James Alexander Green in 1951 [16]. They characterize the elements of a semigroup in terms of the principal ideals they generate.

Given a semigroup S, we first introduce the following four preorder relations:

- $x \leq_{\mathcal{L}} y$ if and only if there exists $u \in S^1$ such that $x = uy$,
- $x \leq_{\mathcal{R}} y$ if and only if there exists $u \in S^1$ such that $x = yu$,
- $x \leq_{\mathcal{J}} y$ if and only if there exist $u, v \in S^1$ such that $x = uyv$,
- $x \leq_{\mathcal{H}} y$ if and only if $x \leq_{\mathcal{L}} y$ and $x \leq_{\mathcal{R}} y$.

These preorders can be reformulated in terms of ideals as follows: $x \leq_{\mathcal{L}} y$ if and only if $S^1x \subseteq S^1y$; $x \leq_{\mathcal{R}} y$ if and only if $xS^1 \subseteq yS^1$; $x \leq_{\mathcal{J}} y$ if and only if $S^1xS^1 \subseteq S^1yS^1$; $x \leq_{\mathcal{H}} y$ if and only if $S^1x \subseteq S^1y$ and $xS^1 \subseteq yS^1$.

Green's relations are the equivalence relations induced by these four preorders, namely:

- $x \mathrel{\mathcal{L}} y$ if and only if $x \leq_{\mathcal{L}} y$ and $y \leq_{\mathcal{L}} x$, or equivalently $S^1x = S^1y$,
- $x \mathrel{\mathcal{R}} y$ if and only if $x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} x$, or equivalently $xS^1 = yS^1$,
- $x \mathrel{\mathcal{J}} y$ if and only if $x \leq_{\mathcal{J}} y$ and $y \leq_{\mathcal{J}} x$, or equivalently $S^1xS^1 = S^1yS^1$,
- $x \mathrel{\mathcal{H}} y$ if and only if $x \mathrel{\mathcal{L}} y$ and $x \mathrel{\mathcal{R}} y$.

Hence, two elements x and y are $\mathcal{L}$-equivalent (resp. $\mathcal{R}$-equivalent, $\mathcal{J}$-equivalent) if they generate the same left ideal (resp. right ideal, ideal). If $\mathcal{K}$ denotes one of the Green's relation, then both $x \leq_{\mathcal{K}} y$ and $x \not\mathrel{\mathcal{K}} y$ will be denoted by $x <_{\mathcal{K}} y$. The equivalence classes of the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, and $\mathcal{H}$ are respectively called the $\mathcal{L}$-classes, $\mathcal{R}$-classes, $\mathcal{J}$-classes and $\mathcal{H}$-classes of S. For any element $x \in S$, the $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{J}$-class, and $\mathcal{H}$-class of x are respectively denoted by $\mathcal{L}(x)$, $\mathcal{R}(x)$, $\mathcal{J}(x)$, and $\mathcal{H}(x)$. The following proposition presents the basic stability properties of the Green's relation.

Proposition 3.12. *In any semigroup S, the relations $\leq_{\mathcal{R}}$ and $\mathcal{R}$ are stable on the left and the relations $\leq_{\mathcal{L}}$ and $\mathcal{L}$ are stable on the right.*

Proof. If $x \leq_{\mathcal{R}} t$, then $xS^1 \subseteq yS^1$, and thus $uxS^1 \subseteq uyS^1$. Therefore, $ux \leq_{\mathcal{R}} uy$, for every $u \in S$. The other cases are proved analogously. □

We now introduce the fifth Green's relation denoted by $\mathcal{D}$. The relation $\mathcal{D}$ is the least equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$. The equivalence classes of $\mathcal{D}$ are called the $\mathcal{D}$-classes of S, and the $\mathcal{D}$-class of an element x is denoted by $\mathcal{D}(x)$. The two following results show that the relation $\mathcal{D}$ is actually equal to $\mathcal{L} \circ \mathcal{R}$ and to $\mathcal{R} \circ \mathcal{L}$.

Proposition 3.13. *Let S be a semigroup. The preorder relations $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ commute in S. The equivalence relations $\mathcal{R}$ and $\mathcal{L}$ commute in S.*

Proof. Suppose that $s \leq_{\mathcal{R}} r$ and $r \leq_{\mathcal{L}} t$. Then $s = ru$ and $r = vt$, for some $u, v \in S^1$. Hence $s = vtu \leq_{\mathcal{L}} tu \leq_{\mathcal{R}} t$, and therefore, $\leq_{\mathcal{R}} \circ \leq_{\mathcal{L}} \subseteq \leq_{\mathcal{L}} \circ \leq_{\mathcal{R}}$. The opposite inclusion holds by duality, and thus $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ commute. A similar proof shows that the relations $\mathcal{L}$ and $\mathcal{R}$ also commute. □

Corollary 3.14. $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

Proof. Let $\mathcal{C} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. We first show that $\mathcal{C}$ is an equivalence relation. It is clearly reflexive and symmetric. It is also transitive, since $\mathcal{C} \circ \mathcal{C} = (\mathcal{L} \circ \mathcal{R}) \circ (\mathcal{L} \circ \mathcal{R}) = \mathcal{L} \circ (\mathcal{R} \circ \mathcal{L}) \circ \mathcal{R} = \mathcal{L} \circ (\mathcal{L} \circ \mathcal{R}) \circ \mathcal{R} = (\mathcal{L} \circ \mathcal{L}) \circ (\mathcal{R} \circ \mathcal{R}) = \mathcal{L} \circ \mathcal{R} = \mathcal{C}$. We finally show that $\mathcal{C} = \mathcal{D}$. Since $\mathcal{C}$ is an equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$, it contains also $\mathcal{D}$. Conversely, one has $\mathcal{C} = \mathcal{L} \circ \mathcal{R} \subseteq \mathcal{D} \circ \mathcal{D} = \mathcal{D}$, which concludes the proof. □

Therefore, the relation $\mathcal{D}$ can be equivalently defined as follows: $x \mathrel{\mathcal{D}} y$ if and only if there exists $u \in S$ such that $x \mathrel{\mathcal{L}} u$ and $u \mathrel{\mathcal{R}} y$, or equivalently, if and only if there exists $u \in S$ such that $x \mathrel{\mathcal{R}} u$ and $u \mathrel{\mathcal{L}} y$. Corollary 3.14 ensures that the three relations $x \mathrel{\mathcal{D}} y$, $\mathcal{R}(x) \cap \mathcal{L}(y) \neq \emptyset$, and $\mathcal{L}(x) \cap \mathcal{R}(y) \neq \emptyset$ are equivalent. For this reason, a $\mathcal{D}$-class is commonly represented by an "egg-box picture", as illustrated in Figure 3.1, where each row describes an $\mathcal{R}$-class, each column an $\mathcal{L}$-class, and each cell an $\mathcal{H}$-class.

In addition, the relation $\mathcal{D}$ is thinner than $\mathcal{J}$. Indeed, if $x \mathrel{\mathcal{D}} y$, there exists $u \in S$ such that $x \mathrel{\mathcal{L}} u$ and $u \mathrel{\mathcal{R}} y$. In other terms, one has $S^1x = S^1u$ and $uS^1 = yS^1$, and hence $S^1xS^1 = S^1yS^1$, that is $x \mathrel{\mathcal{J}} y$. The following diagram summarizes the connections between the five Green's relations, where arrows stand for the usual implication.

$$\begin{tikzcd}
& x \mathrel{\mathcal{R}} y \arrow[dr, Rightarrow] & & \\
x \mathrel{\mathcal{H}} y \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & x \mathrel{\mathcal{D}} y \arrow[r, Rightarrow] & x \mathrel{\mathcal{J}} y \\
& x \mathrel{\mathcal{L}} y \arrow[ur, Rightarrow] & &
\end{tikzcd}$$

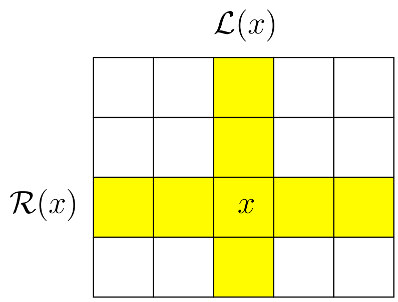


Figure 3.1: A $\mathcal{D}$-class

The two following result focus on specific properties of the $\mathcal{D}$-classes and $\mathcal{H}$-classes. The first proposition describes the structure of a $\mathcal{D}$-class. The second one gives a characterization of the $\mathcal{H}$-classes that are groups.

Proposition 3.15 (Green's lemma). *Let x and y be two $\mathcal{R}$-equivalent elements of a semigroup S. If $x = yu$ and $y = xv$ for some $u, v \in S^1$, the mappings $s \mapsto su$ and $s \mapsto sv$ define inverse bijections between $\mathcal{L}(x)$ and $\mathcal{L}(y)$, and these bijections preserve the $\mathcal{H}$-classes.*

Proof. Let $x' \in \mathcal{L}(x)$. Since $\mathcal{L}$ is stable on the right, then $x'v \in \mathcal{L}(xv) = \mathcal{L}(y)$. Furthermore, there exists $p \in S^1$ such that $x' = px$, and thus $x'vu = pxvu = pyu = px = x'$. Similarly, if $y' \in \mathcal{L}(y)$, then $y'uv = y'$. Therefore, the maps $s \mapsto su$ and $s \mapsto sv$ define inverse bijections between $\mathcal{L}(x)$ and $\mathcal{L}(y)$. Finally, Proposition 3.12 ensures that these maps preserve the $\mathcal{H}$-classes. □

Proposition 3.16. *Let H be an $\mathcal{H}$-class of a semigroup S. Then H contains an idempotent if and only if H is a group.*

Proof. If H is a group, then it contains an idempotent: its identity element. Conversely, let e be an idempotent of H. We first show that e is an identity for the elements of H. Let $x \in H$, then $x \mathrel{\mathcal{L}} e$ and $x \mathrel{\mathcal{R}} e$. Hence, there exist $u, v \in S^1$ such that $x = eu$ and $x = ve$, and thus $ex = eeu = eu = x$ and $xe = vee = ve = x$. We now show H is a semigroup. Let $x, y \in H$. The relation $y \mathrel{\mathcal{R}} e$ implies that $e = yy'$, for some $y' \in S^1$. Hence, by setting $\bar{y} = y'e$, one obtains $xy\bar{y} = xyy'e = xee = xe = x$. Thus $x = xy\bar{y} \leq_{\mathcal{R}} xy \leq_{\mathcal{R}} x$, meaning that $xy \in \mathcal{R}(x)$. We shall prove similarly that $xy \in \mathcal{L}(y)$. Therefore, $xy \in \mathcal{R}(x) \cap \mathcal{L}(y) = H$ and H is a semigroup. Finally, for each $x \in H$, the Green's lemma ensures that the map $s \mapsto sx$ is a permutation on H. In particular, there exists $\bar{x} \in H$ such that $\bar{x}x = e$. Therefore, the class H is a group with identity e. □

In the sequel, we will particularly focus on the restriction of the preorder $\leq_{\mathcal{H}}$ to the set of idempotents of a semigroup S. This preorder is called the *natural order* on $E(S)$ and is denoted by $\leq$. The next proposition shows that this preorder is actually a partial ordering.

Proposition 3.17. *Let S be a semigroup and let e and f be two idempotents of S. The following conditions are equivalent:*

(1) $e \leq f$,
(2) $ef = fe = e$,
(3) $fef = e$.

Proof. We first show the equivalence between (1) and (2). If $e \leq f$, then $e \leq_{\mathcal{L}} f$ and $e \leq_{\mathcal{R}} f$. Therefore, there exist $u, v \in S^1$ such that $e = fu = vf$. It follows that $ef = vff = vf = e$ and $fe = ffu = fu = e$. Conversely, if $ef = fe = e$, then $e \leq_{\mathcal{L}} f$ and $e \leq_{\mathcal{R}} f$, and thus $e \leq f$. We now prove the equivalence between (2) and (3). If $fef = e$, then $ef = feff = fef = e$ and $fe = ffef = fef = e$. Conversely, if $ef = fe = e$, then $fef = f(ef) = fe = e$. □

The following two propositions concern some properties of the Green's relations in finite semigroups.

Proposition 3.18. *In a finite semigroup S, the relations $\mathcal{J}$ and $\mathcal{D}$ are equal.*

Proof. If $x \,\mathcal{D}\, y$, there exists $z \in S$ such that $x \,\mathcal{L}\, z$ and $z \,\mathcal{R}\, y$. Therefore, $x \,\mathcal{J}\, z$ and $z \,\mathcal{J}\, y$, that is $x \,\mathcal{J}\, y$. Conversely, assume that $x \,\mathcal{J}\, y$. Then there exist $u, v, u', v' \in S^1$ such that $uxv = y$ and $u'yv' = x$. Hence $(u'u)x(vv') = x$ and thus $(u'u)^k x(vv')^k = x$, for every $k > 0$. In particular, since S is finite, one may choose k as the exponent π of S. Therefore, the elements $e = (u'u)^\pi$ and $f = (vv')^\pi$ are idempotents and one has $x = exf$. It follows that $(u'u)^\pi x = ex = eexf = exf = x$ and $x(vv')^\pi = xf = exff = exf = x$. Hence $ux \,\mathcal{L}\, x$ and $xv \,\mathcal{R}\, x$. Then the right stability of $\mathcal{L}$ implies $y = uxv \,\mathcal{L}\, xv$. Finally, the two relations $y \,\mathcal{L}\, xv$ and $xv \,\mathcal{R}\, x$ imply $y \,\mathcal{D}\, x$. □

Proposition 3.19. *Let S be a finite semigroup and let $x, y \in S$.*
(1) *$x \,\mathcal{J}\, y$ and $x \leq_{\mathcal{R}} y$ implies $x \,\mathcal{R}\, y$.*
(2) *$x \,\mathcal{J}\, y$ and $x \leq_{\mathcal{L}} y$ implies $x \,\mathcal{L}\, y$.*

Proof. By symmetry, it suffices to prove the first assertion. Since $x \,\mathcal{J}\, y$, there exists $u, v \in S^1$ such that $y = uxv$. Since $x \leq_{\mathcal{R}} y$, there also exists $t \in S^1$ such that $x = yt$. It follows that $y = uytv = u^k y(tv)^k$, for all $k > 0$. In particular, since S is finite, one may choose k as the exponent π of S. Therefore, the elements $e = u^\pi$ and $f = (tv)^\pi$ are idempotents, and $y = u^\pi y(tv)^\pi = eyf = eyff = yf = y(tv)^\pi$. This implies $y \,\mathcal{R}\, yt = x$. □

3.1.2 Infinite words in finite semigroups.

We now describe the specific behavior of infinite words in finite semigroups. For that purpose, we introduce the key notion of a linked pair. We show that every infinite word in a finite semigroup can be associated with a specific equivalence class of linked pairs which witnesses the way it can be factorized. This way, two infinite words are associated with a same class of linked pairs if and only if they can be factorized into the same form. Linked pairs will be of special importance, when considering infinite words of finite semigroups from a playful perspective. A more detailed analysis of this issue can be found in [27, Chapter II - 2].

Given a semigroup S, a pair $(s,e) \in S^2$ is called a *linked pair* if $se = s$ and e is idempotent. The elements s and e are respectively called the *prefix* and the *idempotent* of the linked pair. The set of all prefixes of linked pairs of S is denoted by $P(S)$, or simply by P if the semigroup involved is clear from the context. The set of idempotents associated with a given prefix s is defined by $E(s,S) = \{e \in E(S) \mid se = s\}$, and is simply denoted by $E(s)$ when the semigroup involved is clear from the context. In addition, two linked pairs (s,e) and (s',e') of S^2 are said to be *conjugate*, denoted by $(s,e) =_c (s',e')$, if there exist $x, y \in S^1$ such that $e = xy$, $e' = yx$, and $s' = sx$. The following lemma shows that the conjugacy relation between linked pairs is an equivalence relation. The conjugacy class of a linked pair (s,e) of S^2 will be denoted by

$$[s,e] = \left\{(s',e') \in S^2 \mid (s',e') =_c (s,e)\right\}.$$

LEMMA 3.20. *The conjugacy relation on linked pairs is an equivalence relation.*

PROOF. Reflexivity is trivial. For the symmetry, if $(s,e) =_c (s',e')$, there exist $x, y \in S^1$ such that $e = xy$, $e' = yx$, and $s' = sx$. Therefore, $s = se = sxy = s'y$, meaning that $(s',e') =_c (s,e)$. For the transitivity, if (s,e) and (s',e') are conjugate and (s',e') and (s'',e'') are conjugate, then there exist elements $x, y, x', y' \in S^1$ such that

$$s' = sx,\ e = xy,\ e' = yx,\ s'' = s'x',\ e' = x'y',\ e'' = y'x'.$$

Hence $x'y' = yx$, and thus $(xx')(y'y) = x(x'y')y = x(yx)y = (xy)(xy) = e$. Similarly $(y'y)(xx') = y'(x'y')x' = e''$, and also $s(xx') = s''$. Therefore, (s,e) and (s'',e'') are conjugate. □

Given a semigroup S, an infinite sequence of elements of S is called an *infinite word* of S. Infinite words will be generally written in the form $a_0a_1a_2\cdots$, instead of $(a_0, a_1, a_2, \cdots)$. If $\alpha = (x_n)_{n\in\mathbb{N}}$ and $\beta = (y_n)_{n\in\mathbb{N}}$ are two infinite words of S, then β is said to be a *factorization* of α if there exists a strictly increasing sequence of integers $(k_n)_{n\geq 0}$ such that $y_0 = x_0 \cdots x_{k_0-1}$ and $y_{n+1} = x_{k_n} \cdots x_{k_{n+1}-1}$, for each $n \geq 0$. Factorizations of such infinite words will be generally denoted by use of parentheses. For instance, if $ab = c$, then the infinite word $(ab)(ab)(ab)(ab)\cdots = cccc\cdots$ is a factorization of $abababab\cdots$. The two following propositions tightly bind infinite words over finite semigroups with conjugacy classes of linked pairs.

PROPOSITION 3.21. *Let S be a finite semigroup and let $\alpha = (x_n)_{n\in\mathbb{N}}$ be an infinite word of S. Then there exist a linked pair (s,e) of S^2, and a factorization $(k_n)_{n\in\mathbb{N}}$ of α such that both $x_0x_1\cdots x_{k_0-1} = s$ and $x_{k_n}x_{k_n+1}\cdots x_{k_{n+1}-1} = e$, for all $n \geq 0$.*

In this case, the infinite word α is said to be *associated* with the linked pair (s,e). One also says that α can be factorized into the form $seee\cdots$.

PROOF. We inductively define a sequence of pairs (U_i, n_i), where U_i is an infinite subset of $\mathbb{N}$ and $n_i = \min U_i$. First of all, let us set $U_0 = \mathbb{N}$ and thus $n_0 = 0$. Then, suppose that U_i and n_i have already been defined. Since S is finite, there exists at least one element $s_i \in S$ such that the set

$$T = \{n \in U_i \mid x_{n_i} \cdots x_{n-1} = s_i\}$$

is infinite. Therefore, one sets $U_{i+1} = T$ and $n_{i+1} = \min U_{i+1}$. The sequence on indices $(n_0, n_1, n_2, \ldots)$ is illustrated in Figure 3.2. By construction, one has $x_{n_i} \cdots x_{n_{i+j}-1} = s_i$, for all $i \geq 0$ and $j > 0$. Hence, since S is finite again, there exists an element e of S such that $s_i = e$ for infinitely many integers i. These integers define a subsequence of indices $(m_0, m_1, \ldots)$ such that $x_0 \cdots x_{m_0-1} = t$ and $x_{m_i} \cdots x_{m_{i+j}-1} = e$, for all $i \geq 0$ and $j > 0$, as illustrated in Figure 3.3. In addition, the element e is idempotent:

$$e = x_{m_0} \cdots x_{m_1-1} = x_{m_1} \cdots x_{m_2-1} = x_{m_0} \cdots x_{m_2-1} = ee.$$

Finally, the required linked pair and sequence of indices are obtained by setting $(s, e) = (te, e)$ and $k_i = m_{i+1}$, for all $i \geq 0$: indeed, one has

$$se = tee = te = s \text{ and } e^2 = e,$$
$$x_0 \cdots x_{k_0-1} = x_0 \cdots x_{m_0-1} x_{m_0} \cdots x_{m_1-1} = te = s,$$
$$x_{k_n} \cdots x_{k_{n+1}-1} = x_{m_{n+1}} \cdots x_{m_{n+2}-1} = e \text{ , for all } n > 0.$$

□

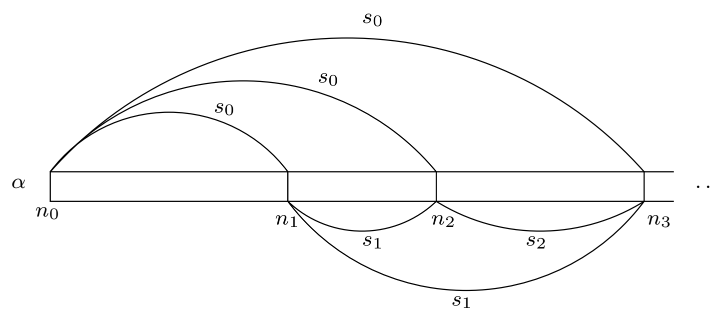


Figure 3.2: The sequence of indices.

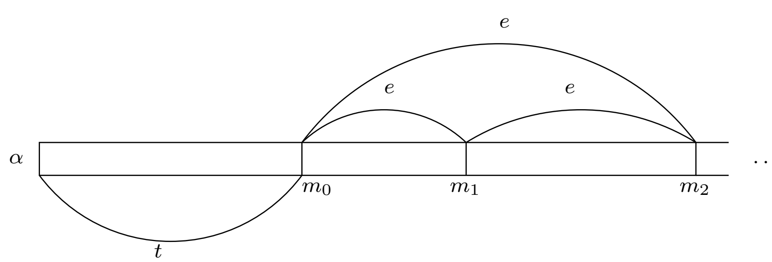


Figure 3.3: The subsequence of indices.

Proposition 3.22. *Let S be a finite semigroup. There exists an infinite word of S which can be associated with different linked pairs if and only if these linked pairs are conjugate.*

PROOF. Assume that there exists an infinite word $\alpha = (s_i)_{i\in\mathbb{N}}$ of S^ω associated with two different linked pairs (s, e) and (s', e'). Then there exist two factorizations $(k_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ of α, such that

$$s_0 \cdots s_{k_0-1} = s \quad \text{and} \quad s_{k_n} \cdots s_{k_{n+1}-1} = e, \text{ for all } n \geq 0,$$
$$s_0 \cdots s_{l_0-1} = s' \quad \text{and} \quad s_{l_n} \cdots s_{l_{n+1}-1} = e', \text{ for all } n \geq 0.$$

Notice that since (s, e) and (s', e') are linked pairs, any subsequence of $(k_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ still verifies the relations above, and therefore, up to considering subsequences of $(k_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$, we may assume that $k_i \leq l_i \leq k_{i+1}$, for all i. In addition, let us set $x_i = s_{k_i} \cdots s_{l_i-1}$ and $y_i = s_{l_i} \cdots s_{k_{i+1}-1}$, for all i. Since S is finite, there exists a pair (x, y) such that $(x_i, y_i) = (x, y)$ for infinitely many indices i. Let p and q be any two such indices such that $p < q$. Then

$$x s_{l_p} \cdots s_{k_q-1} = e \ , \ s_{l_p} \cdots s_{k_q-1} x = e' \ , \text{ and } sx = s',$$

showing that the linked pairs (s, e) and (s', e') are effectively conjugate. Conversely, let (s, e) and (s', e') be two conjugate linked pairs. There exist elements $x, y \in S^1$ such that $sx = s'$, $e = xy$, and $e' = yx$. Therefore, the infinite word $\alpha = s(xy)^\omega$ of S^ω is associated with the two linked pairs (s, e) and (s', e'). □

COROLLARY 3.23. *Let S be a finite semigroup, and let α and β be two infinite words of S such that β is a factorization of α. Then the linked pairs associated with α and β are conjugate.*

PROOF. Let (s, e) and (t, f) be two linked pairs associated with α and β, respectively. Since β is a factorization of α, the linked pair (t, f) is also associated with the infinite word α. Therefore, the two linked pairs (s, e) and (t, f) are associated with α, and Proposition 3.22 shows that these linked pairs are conjugate. □

Given a finite semigroup S, Propositions 3.21 and 3.22 ensure the existence of an onto mapping

$$\pi : S^\omega \longrightarrow \big\{[s, e] : (s, e) \text{ is a linked pair of } S^2\big\}$$

defined by $\pi\big((s_n)_{n\in\mathbb{N}}\big) = [s, e]$, where (s, e) is a linked pair associated with the infinite word $(s_n)_{n\in\mathbb{N}}$ in the sense of Proposition 3.21. Proposition 3.22 ensures that this mapping is consistently defined, since its definition is independent of the choice of the linked pair. This mapping will be of a particular importance in the description of finite ω-semigroups.

3.2 Semigroups and rational languages

3.2.1 Semigroups and automata

The algebraic approach to automata theory presents semigroups as the algebraic counterpart of automata reading finite words. This section provides the description of this tight correspondence. We first give the algebraic definition of recognizable sets. We then show that one can shift from an automaton to a semigroup recognizing the same language, and conversely, from a semigroup to an automaton recognizing the same language.

Let S and T by two semigroups. A surjective morphism of semigroups $\varphi : S \longrightarrow T$ *recognizes* a subset I of S if there exists a subset J of T such that $\varphi^{-1}(J) = I$. By extension, a semigroup T *recognizes* a subset I of S if there exist a surjective morphism $\varphi : S \longrightarrow T$ that recognizes I. Finally, a subset is said to be *recognizable* if it is recognized by a finite semigroup.

Example 3.24. Let $A = \{a, b\}$ be an alphabet, let U_1 be the semigroup $\{0, 1\}$ equipped with usual multiplication, and let φ be the surjective morphism from A^+ onto U_1 induced by the equalities $\varphi(a) = 0$ and $\varphi(b) = 1$. Then the languages of A^+ recognized by φ are the following: $\varphi^{-1}(\emptyset) = \emptyset$, $\varphi^{-1}(0) = A^*aA^*$, $\varphi^{-1}(1) = b^+$, and $\varphi^{-1}(\{0, 1\}) = A^+$.

The two following propositions describe the equivalence between automata and semigroups. We show that the working of an automaton can be simulated by a specific morphism of semigroups, and conversely, the behavior of a morphism from the free semigroup can be described by a specific automaton. These results provide two constructions: firstly, given an automaton on finite words, one can build a semigroup recognizing the same language; secondly, given a semigroup recognizing a certain language, one can build an automaton recognizing the same language. These constructions do not require any finiteness assumption on the semigroup or the automaton involved. But when applied to finite structures, these constructions are obviously effective. Therefore, the languages recognized by finite automata and by finite semigroups coincide and precisely correspond to the rational languages.

Proposition 3.25. *Given any automaton on finite words, there exists a semigroup recognizing the same language.*

Proof. Let $\mathcal{A} = (Q, A, E, I, F)$ be an automaton on finite words recognizing the language $L(\mathcal{A})$ of A^+. We built a surjective morphism of semigroups recognizing $L(\mathcal{A})$. Let $\mathcal{R}(Q)$ be the semigroup of binary relations on Q equipped with the usual concatenation on relations, and let $\varphi_{\mathcal{A}}$ be the mapping from A^+ into $\mathcal{R}(Q)$ which associates with each word u the set of couples of states related by u in $\mathcal{A}$, that is

$$\varphi_{\mathcal{A}}(u) = \Big\{(p, q) \in Q \times Q \mid \text{there exists a path } p \xrightarrow{u} q \text{ in } \mathcal{A}\Big\}.$$

Then $\varphi_{\mathcal{A}}$ is a morphism of semigroups. Therefore, the set $S_{\mathcal{A}} = \varphi_{\mathcal{A}}(A^+)$ is a semigroup and the restriction $\varphi_{\mathcal{A}} : A^+ \longrightarrow S_{\mathcal{A}}$ is a surjective morphism of semigroups. Moreover, the definition of $\varphi_{\mathcal{A}}$ implies

$$L(\mathcal{A}) = \varphi_{\mathcal{A}}^{-1}\{r \in S_{\mathcal{A}} \mid r \cap (I \times F) \neq \emptyset\}.$$

Consequently, the semigroup morphism $\varphi_{\mathcal{A}}$ recognizes the language $L(\mathcal{A})$. $\square$

Proposition 3.26. *Given a semigroup recognizing a certain language, there exists an automaton on finite words recognizing the same language.*

Proof. Let $\varphi : A^+ \longrightarrow S$ be a morphism of semigroups recognizing the language L. Then there exists a subset F of S such that $\varphi^{-1}(F) = L$. Now, let $\mathcal{A}_\varphi$ be the deterministic automaton defined as follows: the set of states is the monoid S^1; the initial state is the identity of the monoid; the set of final states is F; the transition function is given by $\delta(q,a) = q\varphi(a)$, and can be naturally extended to $\delta^* : Q \times A^* \longrightarrow Q$ defined by $\delta^*(q,u) = q\varphi(u)$. We prove that the language $L(\mathcal{A}_\varphi)$ recognized by $\mathcal{A}_\varphi$ corresponds precisely to L. One has

$$\begin{aligned} L(\mathcal{A}_\varphi) &= \{u \in A^+ : \delta^*(1,u) \in F\} = \{u \in A^+ : 1\varphi(u) \in F\} \\ &= \{u \in A^+ : \varphi(u) \in F\} = \varphi^{-1}(F) = L. \end{aligned}$$

□

The semigroup $S_{\mathcal{A}}$ defined in the first part of the proof is called the *transition semigroup* of the automaton $\mathcal{A}$. For practical computations, it can be represented as a semigroup of boolean matrices of order $|Q| \times |Q|$ by setting

$$\varphi_{\mathcal{A}}(u)_{p,q} = \begin{cases} 1 & \text{if there exists a path } p \xrightarrow{u} q \text{ in } \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

The following two examples illustrate the equivalence between automata and semigroups in the case of finite structures. Firstly, starting from a finite automaton, we build the transition semigroup morphism. Secondly, starting from a surjective semigroup morphism from the free structure, we build the corresponding automaton.

Example 3.27. Consider the finite automaton $\mathcal{A} = (Q, A, E, I, F)$ represented in Figure 3.4 and recognizing the language $L = (ab)^+$. Then the transition semigroup of $\mathcal{A}$ is given by

$$S_{\mathcal{A}} = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\}$$

equipped with the usual matrix multiplication, and the semigroup morphism $\varphi_{\mathcal{A}}$ from A^+ onto $S_{\mathcal{A}}$ is induced by the equalities $\varphi_{\mathcal{A}}(a) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\varphi_{\mathcal{A}}(b) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. One has $\varphi_{\mathcal{A}}^{-1}\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = (ab)^+$, and thus the semigroup $S_{\mathcal{A}}$ also recognizes the language L.

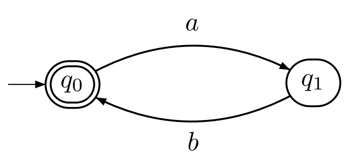


Figure 3.4: An automaton recognizing the language $L = (ab)^+$.

Example 3.28. Consider the finite semigroup $S = \{0, 1\}$ equipped with the usual multiplication, the surjective morphism of semigroups φ from $A^+ = \{a,b\}^+$ onto S defined by $\varphi(a) = 0$ and $\varphi(b) = 1$, and the language $L = A^+aA^+ = \varphi^{-1}\{0\}$ recognized by S. Then L is also recognized by the finite automaton illustrated in Figure 3.5.

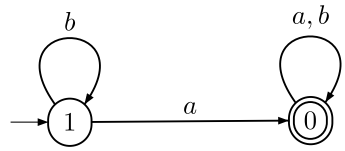


Figure 3.5: An automaton recognizing the langauge $L = A^+aA^+$.

3.2.2 Syntactic semigroups

The notion of a syntactic semigroup is central in the algebraic approach to formal languages. Many properties on rational languages can be characterized on their syntactic structures. For instance, Schützenberger's theorem state that a rational language is star-free if and only if its syntactic semigroup is aperiodic [28, 29]. In addition, the syntactic semigroup of a rational language is precisely the transition semigroup of the minimal automaton recognizing this language. In this section, we show that the syntactic semigroup of a language is the unique (up to isomorphism) minimal (for the division order) semigroup recognizing this language. We also prove that the morphism reduction between rational languages corresponds precisely to the division relation on their syntactic structures, a result that will be generalized to the case of infinite words.

Let L be a language of A^+. The *syntactic congruence* of L is the congruence $\sim_L$ over A^+ defined by $u \sim_L v$ if and only if $xuy \in L \Leftrightarrow xvy \in L$, for every $x, y \in A^*$. The quotient semigroup $S(L) = A^+/{\sim_L}$ is called the *syntactic semigroup* of L, and the canonical morphism μ from A^+ onto $S(L)$ is the *syntactic morphism* of L. The syntactic pointed semigroup $(S(L), \mu(L))$ is denoted by $Synt(L)$. The two following results show that the syntactic semigroup of a language is the smallest semigroup recognizing this language.

Proposition 3.29. *Let L be a language of A^+. Then $S(L)$ recognizes L.*

Proof. Let $\mu : A^+ \longrightarrow S(L)$ be the syntactic morphism of L. We show that $\mu^{-1}(\mu(L)) = L$. The inclusion $L \subseteq \mu^{-1}(\mu(L))$ is obvious. Conversely, let $u \in \mu^{-1}(\mu(L))$. Then $\mu(u) \in \mu(L)$, and there exists $v \in L$ such that $\mu(v) = \mu(u)$, meaning that $v \sim_L u$. Since both $v \sim_L u$ and $v \in L$, one obtains $u \in L$, by setting $x = y = \varepsilon$ in the definition of $\sim_L$. Thus $\mu^{-1}(\mu(L)) \subseteq L$. Finally, by setting $P = \mu(L) \subseteq S(L)$, one has $\mu^{-1}(P) = L$. Therefore, $S(L)$ recognizes L. □

Proposition 3.30. *Let L be a language of A^+ and let S be a semigroup. Then S recognizes L if and only if $S(L)$ divides S.*

Proof. Assume that $S(L)$ divides S, and let μ be the syntactic morphism of L. Then, there exist a semigroup T, an injective morphism ι from T into S, and a surjective morphism σ from T onto $S(L)$. By Corollary 3.10, there exists a morphism φ from A^+ into T such that $\sigma \circ \varphi = \mu$. Now, set $P = \iota(\sigma^{-1}(\mu(L)))$.

Since ι is injective and μ is the syntactic morphism of L, one obtains

$$(\iota \circ \varphi)^{-1}(P) = \varphi^{-1}(\iota^{-1}(\iota(\sigma^{-1}(\mu(L))))) = \varphi^{-1}(\sigma^{-1}(\mu(L))) = \mu^{-1}(\mu(L)) = L.$$

Therefore, S recognizes L. Conversely, assume that S recognizes L. There exist a morphism φ from A^+ onto S, and a subset P of S, such that $\varphi^{-1}(P) = L$. Let us set $T = \varphi(A^+)$. Then T is a subsemigroup of S. Now, assume that $u \sim_\varphi v$ and let $x, y \in A^*$. One has

$$\begin{aligned} xuy \in L &\Leftrightarrow \varphi(xuy) \in P \Leftrightarrow \varphi(x)\varphi(u)\varphi(y) \in P \\ &\Leftrightarrow \varphi(x)\varphi(v)\varphi(y) \in P \Leftrightarrow \varphi(xvy) \in P \Leftrightarrow xvy \in L. \end{aligned}$$

Hence, the relation $u \sim_\varphi v$ implies $u \sim_L v$, and Proposition 3.8 shows that $S(L)$ is a quotient of T. Therefore, $S(L)$ divides S. ☐

Let K and L be two languages of A^+ and B^+, respectively. One says that the language K *reduces by morphism* to L if there exists a semigroup morphism $\varphi : A^+ \longrightarrow B^+$ such that $\varphi^{-1}(L) = K$. We show that the morphism reduction between rational languages coincides with the division relation on their syntactic pointed semigroups.

PROPOSITION 3.31. *Let K and L be two rational languages of A^+ and B^+, respectively. Then K reduces by morphism to L if and only if $Synt(K)$ divides $Synt(L)$.*

PROOF. Let μ and ν be the syntactic morphisms of K and L, respectively. Assume that K reduces by morphism to L. Then there exists a semigroup morphism $\varphi : A^+ \longrightarrow B^+$ such that $\varphi^{-1}(L) = K$, as illustrated below. We

$$\begin{tikzcd} (A^+, K) \arrow[r, "\varphi"] \arrow[d, "\mu"'] \arrow[dr, dashed, "f"] & (B^+, L) \arrow[d, "\nu"] \\ Synt(K) & Synt(L) \end{tikzcd}$$

show that the mapping $f = \nu \circ \varphi$ is a morphism of pointed semigroups from (A^+, K) into $Synt(L)$. By composition, it is clearly a semigroup morphism. Moreover, since ν is the syntactic morphism of L, then

$$f^{-1}(\nu(L)) = \varphi^{-1}(\nu^{-1}(\nu(L))) = \varphi^{-1}(L) = K.$$

Hence $(f(A^+), \nu(L))$ is a pointed subsemigroup of $Synt(L)$. We now prove that the relation $u \sim_f v$ implies $u \sim_K v$. Let $u, v \in A^+$ such that $u \sim_f v$, and let $x, y \in A^*$. Then

$$\begin{aligned} xuy \in K &\Leftrightarrow \varphi(xuy) \in L \Leftrightarrow \nu(\varphi(xuy)) \in \nu(L) \Leftrightarrow f(xuy) \in \nu(L) \\ &\Leftrightarrow f(x)f(u)f(y) \in \nu(L) \Leftrightarrow f(x)f(v)f(y) \in \nu(L) \\ &\Leftrightarrow f(xvy) \in \nu(L) \Leftrightarrow xvy \in f^{-1}(\nu(L)) = K. \end{aligned}$$

Hence, Proposition 3.8 shows that $Synt(K)$ is a quotient of $(f(A^+), \nu(L))$. Therefore, $Synt(K)$ divides $Synt(L)$. Conversely, assume that $Synt(K)$ divides

$Synt(L)$. Then there exist a pointed semigroup (S, P), an injective morphism $\iota : (S, P) \longrightarrow Synt(L)$, and a surjective morphism $\sigma : (S, P) \longrightarrow Synt(K)$, as illustrated below. Since σ and ι are morphisms of pointed ω-semigroups, the

$$\begin{tikzcd}
(A^+, K) \arrow[rr, dashed, "g"] \arrow[d, "\mu"'] \arrow[dr, dashed, "f"] & & (B^+, L) \arrow[d, "\nu"] \\
Synt(K) & (S, P) \arrow[l, "\sigma"'] \arrow[r, "\iota"] & Synt(L)
\end{tikzcd}$$

equalities $\sigma^{-1}(\mu(K)) = P = \iota^{-1}(\nu(L))$ hold. Now, since A^+ is free and σ is surjective, Proposition 3.10 ensures that there exists a morphism of ω-semigroups $f : A^+ \to S$ such that $\sigma \circ f = \mu$. In addition, since μ is the syntactic morphism of K, then

$$f^{-1}(P) = f^{-1}(\sigma^{-1}(\mu(K))) = \mu^{-1}(\mu(K)) = K.$$

Thus $f : (A^+, K) \longrightarrow (S, P)$ is a morphism of pointed ω-semigroups. By composition, the mapping $\iota \circ f$ from (A^+, K) into $Synt(L)$ is also a morphism of pointed ω-semigroups. Once again, since A^+ is free and ν is surjective, there exists a morphism of free semigroups $g : A^+ \longrightarrow B^+$ such that $\nu \circ g = \iota \circ f$. Moreover, since ν is the syntactic morphism of L, one has

$$g^{-1}(L) = g^{-1}(\nu^{-1}(\nu(L))) = f^{-1}(\iota^{-1}(\nu(L))) = f^{-1}(P) = K.$$

Therefore, K reduces to L by morphism. □

3.3 ω-Semigroups

3.3.1 Generalities

The notion of an ω-semigroup was first introduced by Jean-Eric Pin [26, 30] as a generalization of semigroups. In case of finite structures, these objects represent a convincing algebraic counterpart to automata reading infinite words. In this section, we present general definitions and results concerning ω-semigroups. Some statements are straightforward generalizations of results presented in Section 3.1.1. Their proofs will be omitted in this case.

Definition 3.32 (see [27, p. 92]). *An* ω-semigroup *is an algebra consisting of two components, $S = (S_+, S_\omega)$, and equipped with the following operations:*

- *a binary operation on S_+, denoted multiplicatively, such that S_+ equipped with this operation is a semigroup;*
- *a mapping $S_+ \times S_\omega \longrightarrow S_\omega$, called mixed product, which associates with each pair $(s, t) \in S_+ \times S_\omega$ an element of S_ω, denoted by st, and such that for every $s, t \in S_+$ and for every $u \in S_\omega$, then $s(tu) = (st)u$;*
- *a surjective mapping $\pi_S : S_+^\omega \longrightarrow S_\omega$, called infinite product, which is compatible with the binary operation on S_+ and the mixed product in the*

following sense: for every strictly increasing sequence of integers $(k_n)_{n>0}$, for every sequence $(s_n)_{n\geq 0} \in S_+^\omega$, and for every $s \in S_+$, then

$$\pi_S(s_0 s_1 \cdots s_{k_1-1}, s_{k_1} \cdots s_{k_2-1}, \ldots) = \pi_S(s_0, s_1, s_2, \ldots),$$
$$s\pi_S(s_0, s_1, s_2, \ldots) = \pi_S(s, s_0, s_1, s_2, \ldots).$$

Intuitively, an ω-semigroup is a semigroup equipped with a suitable infinite product. The conditions on the infinite product ensure that one can replace the notation $\pi_S(s_0, s_1, s_2, \ldots)$ by the notation $s_0 s_1 s_2 \cdots$ without ambiguity. In addition, since an ω-semigroup is a pair (S_+, S_ω), it is convenient to call *+-subsets* and *ω-subsets* for the subsets of S_+ and S_ω, respectively. Then given an ω-subset $X \subseteq S_\omega$ and an element u of S_+, we set

$$uX = \{u\alpha \in S_\omega \mid \alpha \in X\},$$
$$u^{-1}X = \{\alpha \in S_\omega \mid u\alpha \in X\}.$$

An ω-semigroup is said to be *finite* if its first component is a finite semigroup. It is *infinite* otherwise. In the sequel, we will essentially focus on finite ω-semigroups. Finally, a *pointed ω-semigroup* is a pair (S, X), where S is an ω-semigroup and X is a subset of S.

Example 3.33. The *trivial ω-semigroup* is the finite ω-semigroup $1 = (\{1\}, \{a\})$, obtained by equipping the trivial semigroup $\{1\}$ with the infinite product π defined by $\pi(1, 1, 1, \ldots) = a$.

Example 3.34. The set $S = (\{0, 1\}, \{a\})$ is an ω-semigroup for the operations defined as follows: the set $\{0, 1\}$ is equipped with the usual multiplication and every infinite product is equal to a.

Given two ω-semigroups $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$, a *morphism of ω-semigroups* from S into T is a pair $\varphi = (\varphi_+, \varphi_\omega)$, where $\varphi_+ : S_+ \longrightarrow T_+$ is a semigroup morphism and $\varphi_\omega : S_\omega \longrightarrow T_\omega$ is a mapping preserving the infinite product in the following sense: for every infinite sequence $(s_n)_{n\in\mathbb{N}}$ of elements of S_+,

$$\varphi_\omega(s_0 s_1 s_2 \cdots) = \varphi_+(s_0)\varphi_+(s_1)\varphi_+(s_2) \cdots .$$

In this case, the morphism φ also preserves the mixed product, that is, for every $s \in S_+$ and for every $t \in S_\omega$, then $\varphi_+(s)\varphi_\omega(t) = \varphi_\omega(st)$. Indeed, given $t = t_0 t_1 t_2 \cdots$, one has

$$\begin{aligned}
\varphi_+(s)\varphi_\omega(t) &= \varphi_+(s)\varphi_\omega(t_0 t_1 t_2 \cdots) \\
&= \varphi_+(s)\varphi_+(t_0)\varphi_+(t_1)\varphi_+(t_2) \cdots \\
&= \varphi_\omega(s t_0 t_1 t_2 \cdots) = \varphi_\omega(st).
\end{aligned}$$

As in the semigroup context, a morphism of ω-semigroups $\varphi : S \longrightarrow T$ is an *isomorphism* if there exists another morphism $\psi : T \longrightarrow S$ such that $\varphi \circ \psi = id_T$ and $\psi \circ \varphi = id_S$. Isomorphic ω-semigroups are generally assumed to be identical. By this convention, one says that S is an *ω-subsemigroup* of T if there exists an injective morphism from S into T. The ω-semigroup S is a *quotient* of T if there exists a surjective morphism from T onto S. The ω-semigroup S *divides* T if S is quotient of an ω-subsemigroup of T. A straightforward generalization of propositions 3.6 and 3.7 shows that the division relation is transitive and that it is a partial order on finite ω-semigroups, up to isomorphism.

Finally, a *morphism of pointed ω-semigroups* from (S, X) into (T, Y) is morphism of ω-semigroups $\varphi : S \longrightarrow T$ such that $\varphi^{-1}(Y) = X$. The notions of subsemigroups, quotient, and division can be easily adapted in the context of pointed ω-semigroups.

A *congruence of an ω-semigroup* $S = (S_+, S_\omega)$ is a pair $(\sim_+, \sim_\omega)$, where the relation $\sim_+$ is a semigroup congruence on S_+, the relation $\sim_\omega$ is an equivalence relation on S_ω, and these relations are stable for the infinite and the mixed products: if $(s_0, s_1, \ldots)$ and $(t_0, t_1, \ldots)$ are sequences of elements of S_+ such that $s_i \sim_+ t_i$, for every $i \geq 0$, then $s_0 s_1 s_2 \cdots \sim_\omega t_0 t_1 t_2 \cdots$, and if $s, s' \in S_+$ and $x, x' \in S_\omega$ are such that $s \sim_+ s'$ and $x \sim_\omega x'$, then $sx \sim_\omega s'x'$. The quotient set $S/{\sim} = (S/{\sim_+}, S/{\sim_\omega})$ is naturally equipped with a structure of ω-semigroup. In addition, if $(\sim_i)_{i \in I}$ is a family of congruences on a an ω-semigroup, then the congruence $\sim$, defined by $s \sim t$ if and only if $s \sim_i t$ for all $i \in I$, is called the *lower bound* of the family $(\sim_i)_{i \in I}$. The *upper bound* of the family $(\sim_i)_{i \in I}$ is the lower bound of the congruences that are coarser than all the $\sim_i$. The following result is a straightforward generalization of Proposition 3.8.

Proposition 3.35. *Let $\sim_1$ and $\sim_2$ be two congruences on an ω-semigroup S. If $\sim_2$ is coarser than $\sim_1$, then $S/{\sim_2}$ is a quotient of $S/{\sim_1}$.*

As in the context of semigroups, given a morphism of ω-semigroups φ from S into T, the *nuclear congruence* of φ is the equivalence $\sim_\varphi$ on S defined by $s \sim_\varphi t$ if and only if $\varphi(s) = \varphi(t)$. However, the syntactic congruence cannot be defined in such a convenient way as for semigroups. It will be presented properly in a further section.

The notion of free structure can also be extended to the case of ω-semigroups. Let A be an alphabet. The ω-semigroup $A^\infty = (A^+, A^\omega)$ equipped with the usual concatenation is the *free ω-semigroup* over the alphabet A. Indeed, this ω-semigroup satisfies the following universal property characterizing free structures.

Proposition 3.36. *Let A be an alphabet, $S = (S_+, S_\omega)$ be an ω-semigroup, and φ be a function from A into S_+. Then there exists a unique morphism of ω-semigroups $\bar{\varphi} : A^\infty \longrightarrow S$ such that $\bar{\varphi}(a) = \varphi(a)$ holds, for each $a \in A$.*

Proof. The mapping $\bar{\varphi}$ from A^∞ into S defined for each finite word $a_0 a_1 \cdots a_n$ and for each infinite word $a_0 a_1 a_2 \cdots$ by the relations

$$\begin{aligned} \bar{\varphi}(a_0 a_1 \cdots a_n) &= \varphi(a_0)\varphi(a_1)\cdots\varphi(a_n) \\ \bar{\varphi}(a_0 a_1 a_2 \cdots) &= \varphi(a_0)\varphi(a_1)\varphi(a_2)\cdots \end{aligned}$$

is the required morphism. In addition, any other morphism $\bar{\varphi}$ such that $\bar{\varphi}(a) = \varphi(a)$ for every $a \in A$ must satisfy the previous equalities. Therefore, $\bar{\varphi}$ is unique. □

In particular, the morphism from S_+^∞ onto $S = (S_+, S_\omega)$ induced by the identity over S_+ is called the *natural* morphism associated with S. The following result is a generalization of Corollary 3.10.

COROLLARY 3.37. *Let $\mu : A^\infty \longrightarrow S$ be a morphism of ω-semigroups, and $\sigma : T \longrightarrow S$ be a surjective morphism of ω-semigroups. Then, there exists a morphism $\varphi : A^\infty \longrightarrow T$ such that $\mu = \sigma \circ \varphi$.*

$$\begin{tikzcd} A^\infty \arrow[rr, dashed, "\varphi"] \arrow[dr, "\mu"'] & & T \arrow[dl, "\sigma"] \\ & S & \end{tikzcd}$$

Finally, we define a topology over ω-subsets of ω-semigroups. Given an ω-semigroup $S = (S_+, S_\omega)$ and an ω-subset $X \subseteq S_\omega$, then X is a *basic open set* of S_ω if and only if $\pi_S^{-1}(X)$ is an open subset of S_+^ω, where S_+^ω is equipped with the product topology of the discrete topology on S_+. This topology is the final topology on S_ω defined by the infinite product π_S, and it makes π_S become a continuous function by definition.

REMARK 3.38. The topology defined by setting sS_ω as a basic open set for every $s \in S_+$ would be more natural at first sight. However, this topology is too weak for our purpose. For example, in case S_+ is a group, then Borel subsets of S_ω reduce to the empty set and the whole space: indeed, given any basic open set sS_ω, then $S_\omega = ss^{-1}S_\omega \subseteq sS_\omega$, and thus $sS_\omega = S_\omega$.

3.3.2 Finite ω-semigroups

We now particularly focus on finite ω-semigroups. First, we show that any finite semigroup can be extended to a maximal (for the division relation) finite ω-semigroup. Then, we prove that a finite ω-semigroup is entirely and uniquely determined by some specific values of its infinite product.

Any finite semigroup can be extended to a finite ω-semigroup. Given a finite semigroup S_+, we set $S = (S_+, S_\omega)$, where S_ω is defined by

$$S_\omega = \left\{ [s, e] \mid (s, e) \text{ is a linked pair of } S_+^2 \right\}.$$

We also set an infinite product $\pi_S : S_+^\omega \longrightarrow S_\omega$ defined by $\pi_S\big((s_n)_{n\geq 0}\big) = [s, e]$, where (s, e) is a linked pair associated with $(s_n)_{n\geq 0}$ as described in Proposition 3.21. Proposition 3.22 warrants that this mapping is consistently defined. Finally, we set a mixed product from $S_+ \times S_\omega$ into S_ω defined by $x[s, e] = \pi_S(x, s, e, e, e, \ldots) = [xs, e]$. We say that the structure $S = (S_+, S_\omega)$ is *induced* by the semigroup S_+. We prove that S is the maximal (for the division relation) ω-semigroup containing S_+ as first component.

LEMMA 3.39. *Let $S = (S_+, S_\omega)$ be induced by the finite semigroup S_+. Then S is a finite ω-semigroup.*

PROOF. We show that the mixed product defined above satisfies the required property. Let $x, y \in S_+$ and let $[s, e] \in S_\omega$, then

$$x(y[s, e]) = x[ys, e] = [x(ys), e] = [(xy)s, e] = (xy)[s, e].$$

We now show that the infinite product defined above also satisfies the required properties. First, propositions 3.21 and 3.22 ensure that this product is a surjective mapping from S_+^ω onto S_ω. Now, let $x \in S_+$, and let $(s_n)_{n\in\mathbb{N}}$ be an infinite word of S_+^ω associated with the linked pair (s,e). Then, by definitions of both the infinite and the mixed products, and by Corollary 3.23, the following equalities hold:

$$x\pi_S(s_0,s_1,s_2,\ldots) = x[s,e] = \pi_S(x,s_0,s_1,s_2,\ldots),$$
$$\pi_S(s_0\cdots{}_{k_0-1}, s_{k_0}\cdots{}_{k_1-1},\ldots) = [s,e] = \pi_S(s_0,s_1,s_1,\ldots).$$

Therefore, $S=(S_+,S_\omega)$ is a finite ω-semigroup. □

Proposition 3.40. *Let $S=(S_+,S_\omega)$ be induced by the finite semigroup S_+. Then every ω-semigroup containing S_+ as first component is a quotient of S.*

Proof. Lemma 3.39 shows that $S=(S_+,S_\omega)$ is a finite ω-semigroup. Now, let $T=(S_+,T_\omega)$ be another ω-semigroup containing S_+ as first component. Let also $\varphi=(\varphi_+,\varphi_\omega)$ be the mapping from S into T defined as follows: $\varphi_+ = id_{S_+}$, and $\varphi_\omega([s,e]) = \pi_T(s,e,e,e,\ldots)$. We prove that the mapping φ_ω is consistently defined. Let (s,e) and (s',e') be two conjugate linked pairs. Then there exist $x,y\in S_+^1$ such that $s=s'y$, $e=xy$, and $e'=yx$. Therefore, the properties of the infinite product ensure that

$$\begin{aligned}
\varphi_\omega([s,e]) &= \pi_T(s,e,e,e,\ldots)\\
&= \pi_T(s'y,xy,xy,xy,\ldots)\\
&= \pi_T(s',yx,yx,yx,\ldots)\\
&= \pi_T(s',e',e',e',\ldots)\\
&= \varphi_\omega([s',e']).
\end{aligned}$$

We now show that φ is a morphism of ω-semigroups. Let $(s_n)_{n\in\mathbb{N}}$ be an infinite word of S_+^ω associated with the linked pair (s,e). Then there exists a factorization $(k_n)_{n\in\mathbb{N}}$ of α such that $s_0\cdots s_{k_0-1}=s$ and $s_{k_{i-1}}\cdots s_{k_i-1}=e$, for every $i>0$. Therefore, the properties of the infinite product implies

$$\begin{aligned}
\varphi\left(\pi_S(s_0,s_1,s_2,\ldots)\right) &= \varphi\left([s,e]\right) = \pi_T(s,e,e,e,\ldots)\\
&= \pi_T(s_0\cdots s_{k_0-1}, s_{k_0}\cdots s_{k_1-1}, s_{k_1}\cdots s_{k_2-1},\ldots)\\
&= \pi_T(s_0,s_1,s_2,\ldots)\\
&= \pi_T\left(\varphi_+(s_0),\varphi_+(s_1),\varphi_+(s_2),\ldots\right).
\end{aligned}$$

By the same argument, we finally prove that φ is surjective. Let $y\in T_\omega$. Since π_T is surjective by definition, there exists $\alpha\in S_+^\omega$ such that $y=\pi_T(\alpha)$. By setting $x=\pi_S(\alpha)\in S_\omega$, the equalities above show that $\varphi(x)=\varphi(\pi_S(\alpha)) = \pi_T(\alpha)=y$. Therefore, T is a quotient of S. □

Example 3.41. Consider the finite semigroup $U_1=\{0,1\}$ equipped with the usual multiplication. The elements 0 and 1 are idempotents and the pairs $(0,0)$, $(0,1)$, and $(1,1)$ are linked and pairwise non-conjugate. Therefore, the finite semigroup U_1 induces the finite ω-semigroup $U=(U_1,U_{1\omega})$ given by

$$U_{1\omega} = \{[0,0],[0,1],[1,1]\}.$$

In addition, the infinite product π_U is defined by the relations

$$\begin{aligned}
\pi_U^{-1}([0,0]) = (\{0,1\}^*0)^\omega &: \text{ set of infinite words containing infinitely many 0's,}\\
\pi_U^{-1}([0,1]) = \{0,1\}^*01^\omega &: \text{ set of infinite words containing finitely many 0's,}\\
\pi_U^{-1}([1,1]) = 1^\omega &: \text{ set of infinite words containing no 0 at all.}
\end{aligned}$$

EXAMPLE 3.42. Given an integer $n \geq 0$, consider the finite semigroup $S_+ = \{0, \ldots, n\}$ equipped with the max operation. The linked pairs are of the form (i, j), for every $i \geq j$. The semigroup S_+ induces the finite ω-semigroup $S = (S_+, S_\omega)$ given by

$$S_\omega = \{[i, j] \mid j \leq i \text{ and } 0 \leq i, j \leq n\}.$$

If we set $i + 1 = \{0 \ldots, i\}$ for every $i \geq 0$, then for every $0 \leq j \leq i \leq n$, the infinite product π_S is defined by the relations

$$\pi_S^{-1}([i, j]) = (i + 1)^* i (j + 1)^\omega.$$

Now, we prove that a finite ω-semigroup is entirely and uniquely defined by a finite amount of data, namely its underlying finite semigroup, its mixed product, and some specific values of its infinite products. For that purpose, we introduce another algebraic structure, the Wilke algebra, precisely defined by these features. We then show that every finite Wilke algebra can be equipped, in a unique way, with a structure of ω-semigroup.

DEFINITION 3.43. *A Wilke algebra is an algebra consisting of two components, $S = (S_+, S_\omega)$, and equipped with the following operations:*

- *a binary operation on S_+, denoted multiplicatively, such that S_+ equipped with this operation is a semigroup;*
- *a mapping $S_+ \times S_\omega \longrightarrow S_\omega$, called mixed product, which associates with each pair $(s, t) \in S_+ \times S_\omega$ an element of S_ω, denoted by st, and such that for every $s, t \in S_+$ and for every $u \in S_\omega$, then $s(tu) = (st)u$;*
- *a mapping $S_+ \longrightarrow S_\omega$, called ω-operation and denoted by $s \longmapsto s^\omega$, such that every element of S_ω can be written as st^ω, for some $s, t \in S_+$, and such that for every $s, t \in S_\omega$ and for every integer n, then*

$$s(ts)^\omega = (st)^\omega,$$
$$(s^n)^\omega = s^\omega.$$

First of all, we show that any finite semigroup can also be extended to a finite Wilke algebra. Given a finite semigroup S_+, we set $S = (S_+, S_\omega)$, where S_ω is defined by

$$S_\omega = \{[s, e] \mid (s, e) \text{ is a linked pair of } S_+^2\}.$$

Then, for every $[s, e] \in S_\omega$ and every $t \in S_+$, we define an ω-operation and a mixed product by $t^\omega = [t^\pi, t^\pi]$ and $t[s, e] = [ts, e]$, where π is the exponent of S_+. The definition of the mixed product is consistent, since if $[s, e] = [s', e']$, then $[ts, e] = [ts', e']$. We prove that S is a Wilke algebra.

LEMMA 3.44. *Let $S = (S_+, S_\omega)$ be defined by the the finite semigroup S_+ as described above. Then S is a Wilke algebra.*

PROOF. Let $u, v \in S_+$ and let $[s, e] \in S_\omega$. By definitions of the ω-operation and the mixed product, one has

$$u(v[s, e]) = u[vs, e] = [u(vs), e] = [(uv)s, e] = (uv)[s, e],$$
$$u(vu)^\omega = u[(vu)^\pi, (vu)^\pi] = [u(vu)^\pi, (vu)^\pi] = [(uv)^\pi, (uv)^\pi] = (uv)^\omega,$$
$$(u^n)^\omega = [(u^n)^\pi, (u^n)^\pi] = [u^\pi, u^\pi] = u^\omega. \qquad \square$$

Now, we prove that every finite ω-semigroup is equivalent to a finite Wilke algebra. As a consequence, a finite ω-semigroup is entirely and uniquely defined by the mixed product and the infinite products of the form $sss\cdots = s^\omega$, and can thus be described by only a finite amount of data. This result will be specially useful in the sequel, for we will no more distinguish between finite Wilke algebras and finite ω-semigroups.

Proposition 3.45. *For any finite Wilke algebra $S = (S_+, S_\omega)$, there is a unique infinite product π_S from S_+^ω into S_ω making S a finite ω-semigroup, and such that $s^\omega = \pi_S(s, s, s, \ldots)$, for all $s \in S_+$.*

Proof. By Proposition 3.21, any infinite word $(s_n)_{n\in\mathbb{N}}$ of S_+^ω can be associated with a linked pair (s, e). Therefore, if S can be equipped with a structure of ω-semigroup satisfying the properties of a Wilke algebra, the infinite product $s_0s_1s_2\cdots$ is forced to be equal to se^ω. Now, let $S = (S_+, S_\omega)$ be a finite Wilke algebra. We equip S with a structure of ω-semigroup that satisfies the required properties. If $(s_i)_{i\in\mathbb{N}}$ is an infinite word of S_+^ω, we define the infinite product π_S by $\pi_S(s_0, s_1, s_2, \ldots) = se^\omega$, where (s, e) is a linked pair associated with the infinite word $s_0s_1s_2\cdots$. We show that this definition is independent of the choice of the linked pair (s, e). Let (s, e) and (s', e') be two linked pairs associated with the infinite word $s_0s_1s_2\cdots$. Then Proposition 3.22 shows that these linked pairs are conjugate, and thus there exist $x, y \in S_+^1$ such that $s = s'y$, $e = xy$, and $e' = yx$. Therefore, the properties of the ω-operation imply that

$$se^\omega = s'y(xy)^\omega = s'(yx)^\omega = s'e'^\omega.$$

In addition, we show that this definition satisfies the properties of an infinite product. Let $(s_n)_{n\in\mathbb{N}}$ be an infinite word of S_+^ω associated with the linked pair (s, e), let $t \in S_+$, and let $(k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of integers. By definition of the infinite product, by the properties of the ω-operation, and by Corollary 3.23, one has

$$\begin{gathered} t\pi_S(s_0, s_1, s_2, \ldots) = t(se^\omega) = (ts)e^\omega = \pi_S(t, s_0, s_1, s_2, \ldots), \\ \pi_S(s_0\cdots s_{k_0-1}, s_{k_0}\cdots s_{k_1-1}, \ldots) = \pi_S(s_0, s_1, s_2, \ldots). \end{gathered}$$

Finally, it remains to show that $\pi_S(s, s, s, \ldots) = s^\omega$, for every $s \in S_+$. Let π be the exponent of S_+. Then, by definition of π_S and by the properties of the ω-operation, one has $\pi_S(s, s, s, \ldots) = (s^\pi)^\omega = s^\omega$. □

Example 3.46. The set $S = (\{0, 1\}, \{0^\omega, 1^\omega\})$ is an ω-semigroup for the operations defined as follows:

$$\begin{array}{cccc} 0\cdot 0 = 0 & 0\cdot 1 = 0 & 1\cdot 0 = 0 & 1\cdot 1 = 1 \\ 00^\omega = 0^\omega & 10^\omega = 0^\omega & 01^\omega = 1^\omega & 11^\omega = 1^\omega \end{array}$$

In this ω-semigroup, the infinite product π_S is defined by $\pi_S^{-1}(0^\omega) = (\{0,1\}^*0)^\omega$ and $\pi_S^{-1}(1^\omega) = \{0,1\}^*01^\omega$.

Example 3.47. The set $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$ is an ω-semigroup for the operations defined as follows:

$$\begin{array}{cccc}
a^2 = a & ab = a & ac = a & ba = a \\
b^2 = b & bc = c & cb = c & c^2 = c \\
b^\omega = a^\omega & c^\omega = 0 & aa^\omega = a^\omega & a(ca)^\omega = a^\omega \\
ba^\omega = a^\omega & b(ca)^\omega = (ca)^\omega & ca^\omega = (ca)^\omega & c(ca)^\omega = (ca)^\omega
\end{array}$$

3.4 ω-Semigroups and ω-rational languages

3.4.1 ω-Semigroups and automata

In this section, we explore the generalization of the correspondence between finite semigroups and finite automata. We prove that finite ω-semigroups stand for the algebraic counterpart of finite Büchi automata. In order to state this result, we first introduce the notions of recognizability for finite ω-semigroups. We then show a construction to go from a finite Büchi automaton to a finite ω-semigroup recognizing the same ω-language, and conversely, we prove that any ω-language recognized by a finite ω-semigroup is ω-rational, so that there also exists a construction to go from a finite ω-semigroup to a finite Büchi automaton recognizing the same ω-language [27].

Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ by two ω-semigroups. A surjective morphism of semigroups $\varphi : S \longrightarrow T$ *recognizes* a subset $I = (I_+, I_\omega)$ of S if there exists a subset $J = (J_+, J_\omega)$ of T such that $\varphi^{-1}(J) = I$, that is $\varphi^{-1}(J_+) = I_+$ and $\varphi^{-1}(J_\omega) = I_\omega$. By extension, an ω-semigroup T *recognizes* a subset I of an ω-semigroup S if there exist a surjective morphism $\varphi : S \longrightarrow T$ that recognizes I. In addition, a congruence $\sim$ on the ω-semigroup S *recognizes* a subset I of S if the natural morphism $\varphi : S \longrightarrow S/\!\sim$ recognizes I. Finally, a subset is said to be *recognizable* if it is recognized by a finite ω-semigroup.

Example 3.48. Let $A = \{a, b\}$ be an alphabet, let U be the finite ω-semigroup defined in Example 3.41, and let φ be the surjective morphism of ω-semigroups from A^∞ onto U induced by the relations $\varphi(a) = 0$ and $\varphi(b) = 1$. Then the ω-languages of A^ω recognized by φ are the following: $\varphi^{-1}([0,0]) = (A^*a)^\omega$, $\varphi^{-1}([0,1]) = A^*ab^\omega$, and $\varphi^{-1}([1,1]) = b^\omega$.

Both following results show that the ω-languages recognized by finite automata and by finite ω-semigroups coincide, hence correspond precisely to the ω-rational languages. These results provides two effective constructions: firstly, given a Büchi automaton, one can build a Wilke algebra recognizing the same ω-language; secondly, given an ω-semigroup recognizing a certain ω-language, one can build an ω-rational expression describing this language, and thus, one can also build a Büchi automaton recognizing this ω-language (see [27]).

Proposition 3.49. *Given any finite Büchi automaton recognizing an ω-language, there exists a finite ω-semigroup recognizing the same ω-language.*

Proof. Let $\mathcal{A} = (Q, A, E, I, F)$ be some finite Büchi automaton recognizing the language $L(\mathcal{A}) \subseteq A^\omega$. We build a finite Wilke algebra $S_\mathcal{A} = (S_+, S_\omega)$ recognizing it. First of all, we associate with every finite word u of A^+ a $(Q \times Q)$-matrix

$\mu(u)$ which expresses, for all $p, q \in Q$, whether there exists a path $p \xrightarrow{u} q$ in $\mathcal{A}$, and in case such a path exists, whether it visits a final state of $\mathcal{A}$. For that purpose, let $k = \{-\infty, 0, 1\}$ be the semiring whose addition is the maximum for the ordering $-\infty < 0 < 1$, and whose multiplication is described in the table below:

$\cdot \nearrow$	$-\infty$	0	1
$-\infty$	$-\infty$	$-\infty$	$-\infty$
0	$-\infty$	0	1
1	$-\infty$	1	1

Then the set $k^{Q\times Q}$ of $(Q \times Q)$-matrices over k is a semigroup. Therefore, the mapping μ from A into $k^{Q\times Q}$ defined for all $a \in A$ by

$$\mu(a)_{p,q} = \begin{cases} -\infty & \text{if } (p,a,q) \notin E, \\ 0 & \text{if } (p,a,q) \in E,\ p \notin F \text{ and } q \notin F, \\ 1 & \text{if } (p,a,q) \in E,\ p \in F \text{ or } q \in F, \end{cases}$$

can be naturally extended to a semigroup morphism from A^+ into $k^{Q\times Q}$. Now, we set $S_{\mathcal{A}} = (S_+, S_\omega)$, where $S_+ = k^{Q\times Q}$, and $S_\omega = k^Q$ is the set of vectors over k indexed by Q. It remains to equip $S_{\mathcal{A}}$ with a structure of Wilke algebra. Hence, we define an ω-operation from $k^{Q\times Q}$ into k^Q which describes whether a given transition matrix $\mu(u)$ associated with the finite word u may yield an infinite successful path in $\mathcal{A}$. For that purpose, given any matrix $s \in S_+$, we call infinite s-path starting at p an infinite sequence $(p_0 = p, p_1, p_2, \ldots)$ of states of $\mathcal{A}$ such that $s_{p_i,p_{i+1}} \neq \infty$, for all i; such an s-path is called successful if $s_{p_i,p_{i+1}} = 1$ for infinitely many integers i. Then, for any $s \in S_+$, we define $s^\omega \in S_\omega$ by: for all $p \in Q$,

$$s^\omega_p = \begin{cases} 1 & \text{if there exists a successful } s\text{-path of origin } p, \\ -\infty & \text{otherwise,} \end{cases}$$

Furthermore, the mixed product is the usual product between matrices and vectors. These operations provide $S_{\mathcal{A}}$ with a structure of finite Wilke algebra, see [27, Proposition 7.1, p. 107] for more details. Therefore, Proposition 3.45 shows that $S_{\mathcal{A}}$ can be equipped in a unique way with a structure of finite ω-semigroup. Furthermore, proposition 3.36 and 3.45 show that the morphism μ can be extended in a unique way to a morphism of ω-semigroups $\mu_{\mathcal{A}}$ from A^∞ into $S_{\mathcal{A}}$. In order to conclude, we prove that the morphism $\mu_{\mathcal{A}}$ recognizes the ω-language $L(\mathcal{A})$. Let us set

$$P = \{s \in S_\omega \mid s_i = 1 \text{ for some } i \in I\}.$$

By construction, one has $\mu_{\mathcal{A}}(\alpha) \in P$ if and only if there exists an infinite initial and successful path labeled by α. In other words, $L(\mathcal{A}) = {\mu_{\mathcal{A}}}^{-1}(P)$, thus $\mu_{\mathcal{A}}$ recognizes $L(\mathcal{A})$. $\square$

Proposition 3.50. *Any ω-language recognized by a finite ω-semigroup is ω-rational.*

Proof. Let $S = (S_+, S_\omega)$ be a finite ω-semigroup and let $\varphi : A^\infty \longrightarrow S$ be a surjective morphism of ω-semigroups recognizing the ω-language L of A^ω. Then there exists an ω-subset X of S_ω such that $\varphi^{-1}(X) = L$. We prove that

$$L = \bigcup_{(s,e)\in Y} \left(\varphi^{-1}(s)\right)\left(\varphi^{-1}(e)\right)^{\omega}, \tag{3.1}$$

where Y is the set of liked pairs (s,e) of S_+^2 such that $se^\omega \in X$. Let $\alpha = a_0a_1a_2\cdots \in L$. Since $L = \varphi^{-1}(X)$, there exists $x \in X$ such that

$$\varphi(\alpha) = \pi_S(\varphi(a_0), \varphi(a_1), \varphi(a_2), \ldots) = x.$$

By definition of the infinite product π_S, one has $x = se^\omega$, for some linked pair (s,e) of S_+^2 associated with the infinite word $(\varphi(a_i))_{i\in\mathbb{N}}$. Thus $(s,e) \in Y$. Moreover, there exists a factorization of the infinite word $(\varphi(a_i))_{i\in\mathbb{N}}$ such that

$$\begin{aligned} \varphi(a_0)\cdots\varphi(a_{k_0-1}) &= \varphi(a_0\cdots a_{k_0-1}) = s,\\ \varphi(a_{k_i})\cdots\varphi(a_{k_{i+1}-1}) &= \varphi(a_{k_i}\cdots a_{k_{i+1}-1}) = e, \end{aligned}$$

for all $i \geq 0$. It follows that $a_0\cdots a_{k_0-1} \in \varphi^{-1}(s)$ and $a_{k_i}\cdots a_{k_{i+1}-1} \in \varphi^{-1}(e)$, for all $i \geq 0$. Therefore, $\alpha = a_0a_1a_2\cdots \in \left(\varphi^{-1}(s)\right)\left(\varphi^{-1}(e)\right)^\omega$ for some $(s,e) \in Y$. Conversely, let $\alpha \in \bigcup_{(s,e)\in Y}\left(\varphi^{-1}(s)\right)\left(\varphi^{-1}(e)\right)^\omega$. There exists a specific linked pair $(s,e) \in Y$ such that $\alpha \in \left(\varphi^{-1}(s)\right)\left(\varphi^{-1}(e)\right)^\omega$. Since φ is a surjective morphism of ω-semigroups and since (s,e) is a linked pair of Y, one has

$$\begin{aligned} \varphi(\alpha) &\in \varphi\left(\left(\varphi^{-1}(s)\right)\left(\varphi^{-1}(e)\right)^\omega\right)\\ &= \pi_S\left(\varphi\circ\varphi^{-1}(s), \varphi\circ\varphi^{-1}(e), \varphi\circ\varphi^{-1}(e), \ldots\right)\\ &= \pi_S\left(s,e,e,\ldots\right) = se^\omega \in X. \end{aligned}$$

Therefore, $\alpha \in \varphi^{-1}(X) = L$. Finally, Proposition 3.26 shows that every term $\varphi^{-1}(s)$ and $\varphi^{-1}(e)$ of (1.1) are rational languages. Therefore, Proposition 1.5 ensures that L is ω-rational. □

The following two examples illustrate the equivalence between the recognition by finite Büchi automata and by finite ω-semigroups. Firstly, given a finite Büchi automaton, we build a morphism ω-semigroups recognizing the same ω-language. Secondly, given a surjective morphism of ω-semigroups from the free structure, we exhibit a Büchi automaton recognizing the same ω-language.

Example 3.51. Let $A = \{a,b\}$ and consider the finite Büchi automaton $\mathcal{A} = (Q,A,E,I,F)$ illustrated in Figure 3.6. This automaton recognizes the ω-language $L(\mathcal{A}) = A^*ba^\omega$. The construction described in Proposition 3.49 gives the finite ω-semigroup $S_\mathcal{A} = (\{x,y\},\{x^\omega,y^\omega,yx^\omega\})$ defined by

$$x = \begin{pmatrix} 0 & -\infty \\ -\infty & 1\end{pmatrix},\ y = \begin{pmatrix} 0 & 1 \\ -\infty & -\infty\end{pmatrix},\ x^\omega = \begin{pmatrix} -\infty \\ 1\end{pmatrix},\ y^\omega = \begin{pmatrix} -\infty \\ -\infty\end{pmatrix},\ yx^\omega = \begin{pmatrix} 1 \\ -\infty\end{pmatrix},$$

and by the relations $x^2 = x$, $xy = yx = y^2 = y$, $xx^\omega = x^\omega$, and $xy^\omega = xx^\omega = x^\omega$. It also gives the morphism of ω-semigroups $\mu_\mathcal{A} : A^\infty \longrightarrow S_\mathcal{A}$ defined by $\mu_\mathcal{A}(a) = x$ and $\mu_\mathcal{A}(b) = y$. Then the ω-language $L(\mathcal{A})$ is recognized by the morphism $\mu_\mathcal{A}$ since $L(\mathcal{A}) = A^*ba^\omega = \mu_\mathcal{A}^{-1}(yx^\omega)$.

Example 3.52. Let $A = \{a,b\}$ be an alphabet and let A^∞ be the free ω-semigroup over A. Consider the finite ω-semigroup $U = (U_1, U_{1\omega})$ of Example 3.41, the surjective morphism of ω-semigroups $\varphi : A^\infty \longrightarrow U$ defined by $\varphi(a) = 0$ and $\varphi(b) = 1$, and the ω-language $L = (A^*a)^\omega = \varphi^{-1}([0,0])$ recognized by φ. Then L also is recognized by the finite Büchi automaton of Figure 3.7.

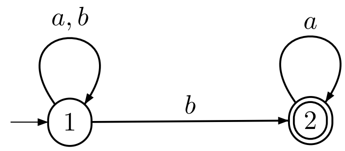


Figure 3.6: A Büchi automaton

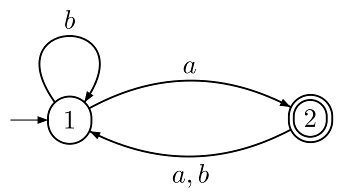


Figure 3.7: A Büchi automaton

3.4.2 Syntactic ω-semigroups

In this section, we focus on the notion of syntactic congruence for ω-semigroups. Unlike in the semigroup framework, there is no simple constructive definition of this congruence in the case of ω-semigroups, except for some particular cases. Nevertheless, we show that every recognizable ω-language admits a syntactic congruence, and as in the semigroup context, the syntactic ω-semigroup of an ω-rational language is the unique (up to isomorphism) minimal (for the division ordering) ω-semigroup recognizing this language. This feature is specially interesting since there is no convincing notion of Büchi or Muller minimal automaton.

Let $S = (S_+, S_\omega)$ be an ω-semigroup and let X be a subset of S. The *syntactic congruence* of X, denoted by $\sim_X$, is the upper bound of the family of congruences recognizing X, if this upper bound still recognizes X. It is undefined otherwise. Whenever defined, the quotient $S(X) = S/{\sim_X}$ is called the *syntactic ω-semigroup* of X, the corresponding morphism $\mu : S \longrightarrow S(X)$ is the *syntactic morphism* of X, and the set $\mu(X)$ is the *syntactic image* of X. The pointed ω-semigroup $(S(X), \mu(X))$ will be denoted by $Synt(X)$.

We now define an equivalence relation associated with every recognizable ω-subset. We will further prove that this relation is precisely the required syntactic congruence. Let $S = (S_+, S_\omega)$ be an ω-semigroup, and let X be a recognizable ω-subset of S. We set the equivalence relation $\sim_X$ on S, defined on S_+ by

$s \sim_X t$ if and only if

$$xsyz^\omega \in X \Leftrightarrow xtyz^\omega \in X,$$
$$x(sy)^\omega \in X \Leftrightarrow x(ty)^\omega \in X,$$

hold, for every $x, y \in S_+^1$ and every $z \in S_+$, and defined on S_ω by $u \sim_X v$ if and only if

$$xu \in X \Leftrightarrow xv \in X,$$

holds, for every $x \in S_+^1$. The following two results show that this relation is precisely the syntactic congruence of the recognizable subset X.

PROPOSITION 3.53. *Let X be a recognizable subset of an ω-semigroup $S = (S_+, S_\omega)$. Then the relation $\sim_X$ is a congruence of ω-semigroup.*

PROOF. First of all, we prove that the relation $\sim_X$ is a congruence of semigroup on S_+. Let s, t be two elements of S_+ such that $s \sim_X t$, and let x, y be two elements of S_+^1. We show that $xsy \sim_X xty$. Let $x', y' \in S_+^1$, and let $z \in S_+$. By definition of $\sim_X$ and by the properties of the ω-operation, one has

$$\begin{aligned}
x'(xsy)y'z^\omega \in X &\Leftrightarrow (x'x)s(yy')z^\omega \in X \Leftrightarrow (x'x)t(yy')z^\omega \in X \\
&\Leftrightarrow x'(xty)y'z^\omega \in X, \text{ and} \\
x'((xsy)y')^\omega \in X &\Leftrightarrow x'x(syy'x)^\omega \in X \Leftrightarrow x'x(tyy'x)^\omega \in X \\
&\Leftrightarrow x'((xty)y')^\omega \in X.
\end{aligned}$$

Furthermore, the relation $\sim_X$ is an equivalence relation on S_ω. We now prove that $\sim_X$ is stable for the mixed product. For that purpose, we use our key hypothesis. Since X is recognizable, there exist a finite ω-semigroup $T = (T_+, T_\omega)$, a subset Y of T, and a surjective morphism of ω-semigroups φ from S onto T such that $\varphi^{-1}(Y) = X$. Let $s, t \in S_+$ and let $u, v \in S_\omega$ such that $s \sim_X t$ and $u \sim_X v$. Since T is finite and φ is surjective, there exist $y = \varphi(a)$ and $z = \varphi(b)$ in T_ω such that $\varphi(v) = yz^\omega = \varphi(ab^\omega)$. Let $x \in S_+^1$, one has

$$\begin{aligned}
x(su) = (xs)u \in X &\Leftrightarrow (xs)v \in X \\
&\Leftrightarrow \varphi(x)\varphi(s)\varphi(ab^\omega) \in Y \\
&\Leftrightarrow xsab^\omega \in X \Leftrightarrow xtab^\omega \in X \\
&\Leftrightarrow \varphi(x)\varphi(t)\varphi(v) \in Y \\
&\Leftrightarrow x(tv) \in X
\end{aligned}$$

Therefore, $su \sim_X tv$. We finally prove that $\sim_X$ is stable for the infinite product. Let $(s_0, s_1, s_2, \ldots)$ and $(s'_0, s'_1, s'_2, \ldots)$ be two infinite sequences of S_+^ω such that $s_i \sim_X s'_i$, for each $i \geq 0$. We show that $s_0 s_1 s_2 \cdots \sim_X s'_0 s'_1 s'_2 \cdots$ holds. We use our key hypothesis again. Since X is recognizable, there exist a finite ω-semigroup $T = (T_+, T_\omega)$, a subset Y of T, and a surjective morphism of ω-semigroups φ from S onto T such that $\varphi^{-1}(Y) = X$. Since the pair $(\varphi(s_0)\varphi(s_1)\varphi(s_2)\cdots, \varphi(s'_0)\varphi(s'_1)\varphi(s'_2)\cdots)$ is an infinite word of the finite semigroup $T_+ \times T_+$, Proposition 3.21 shows that there exist a strictly increasing sequence of integers $(k_n)_{n\geq 0}$ and a linked pair $((s, s'), (e, e'))$ of $(T_+ \times T_+)^2$ such that

$$\begin{aligned}
\varphi(s_0)\cdots\varphi(s_{k_0-1}) &= \varphi(s_0\cdots s_{k_0-1}) = s, \\
\varphi(s'_0)\cdots\varphi(s'_{k_0-1}) &= \varphi(s'_0\cdots s'_{k_0-1}) = s', \\
\varphi(s_{k_i})\cdots\varphi(s_{k_{i+1}-1}) &= \varphi(s_{k_i}\cdots s_{k_{i+1}-1}) = e \quad \text{, for each } i \geq 0, \\
\varphi(s'_{k_i})\cdots\varphi(s'_{k_{i+1}-1}) &= \varphi(s'_{k_i}\cdots s'_{k_{i+1}-1}) = e' \quad \text{, for each } i \geq 0.
\end{aligned}$$

Now, let $x \in S^1_+$, one obtains

$$\begin{aligned}
xs_0s_1s_2\cdots \in X &\Leftrightarrow \varphi(x)\varphi(s_0)\varphi(s_1)\cdots = \varphi(x)se^\omega \in Y \\
&\Leftrightarrow \varphi(x(s_0\cdots s_{k_0-1})(s_{k_0}\cdots s_{k_1-1})^\omega) \in Y \\
&\Leftrightarrow x(s_0\cdots s_{k_0-1})(s_{k_0}\cdots s_{k_1-1})^\omega \in X \\
&\Leftrightarrow x(s'_0\cdots s'_{k_0-1})(s_{k_0}\cdots s_{k_1-1})^\omega \in X \\
&\Leftrightarrow x(s'_0\cdots s'_{k_0-1})(s'_{k_0}\cdots s'_{k_1-1})^\omega \in X \\
&\Leftrightarrow \varphi(x(s'_0\cdots s'_{k_0-1})(s'_{k_0}\cdots s'_{k_1-1})^\omega) \in Y \\
&\Leftrightarrow \varphi(x)s'e'^\omega = \varphi(x)\varphi(s'_0)\varphi(s'_1)\cdots \in Y \Leftrightarrow xs'_0s'_1s'_2\cdots \in X.
\end{aligned}$$

Therefore, $s_0s_1s_2\cdots \sim_X s'_0s'_1s'_2\cdots$, which concludes the proof. □

PROPOSITION 3.54. *Let X be any recognizable subset of an ω-semigroup $S = (S_+, S_\omega)$. The relation $\sim_X$ is the syntactic congruence of X.*

PROOF. We first show that the congruence $\sim_X$ is coarser than all the congruences recognizing X. Let $\sim$ be a congruence recognizing X, and let $\varphi : S \longrightarrow S/\!\sim$ be the corresponding canonical morphism. Since φ recognizes X, the set X is a union of $\sim$-equivalence classes, and thus $\varphi^{-1}(\varphi(X)) = X$. Now, let $s, t \in S_+$, we show that $s \sim t$ implies $s \sim_X t$. For that purpose, let $x, y \in S^1_+$ and $z \in S_+$. By the properties of the morphism φ, and since $s \sim t$ precisely means that $\varphi(s) = \varphi(t)$, one has

$$\begin{aligned}
xsyz^\omega \in X &\Leftrightarrow \varphi(xsyz^\omega) \in \varphi(X) \\
&\Leftrightarrow \varphi(x)\varphi(s)\varphi(y)\varphi(z)^\omega \in \varphi(X) \\
&\Leftrightarrow \varphi(x)\varphi(t)\varphi(y)\varphi(z)^\omega \in \varphi(X) \\
&\Leftrightarrow \varphi(xtyz^\omega) \in \varphi(X) \\
&\Leftrightarrow xstyz^\omega \in X, \text{ and} \\
x(sy)^\omega \in X &\Leftrightarrow \varphi(x(sy)^\omega) \in \varphi(X) \\
&\Leftrightarrow \varphi(x)(\varphi(s)\varphi(y))^\omega \in \varphi(X) \\
&\Leftrightarrow \varphi(x)(\varphi(t)\varphi(y))^\omega \in \varphi(X) \\
&\Leftrightarrow \varphi(x(ty)^\omega) \in \varphi(X) \\
&\Leftrightarrow x(ty)^\omega \in X.
\end{aligned}$$

Similarly, let $u, v \in S_\omega$, we show that $u \sim v$ implies $u \sim_X v$. Let $x \in S^1_+$, then

$$\begin{aligned}
xu \in X &\Leftrightarrow \varphi(xu) \in \varphi(X) \\
&\Leftrightarrow \varphi(x)\varphi(u) \in \varphi(X) \\
&\Leftrightarrow \varphi(x)\varphi(v) \in \varphi(X) \\
&\Leftrightarrow \varphi(xv) \in \varphi(X) \\
&\Leftrightarrow xv \in X.
\end{aligned}$$

Finally, we prove that the congruence $\sim_X$ recognizes X itself, meaning that X is a union of $\sim_X$-equivalence classes. This is true on S_+, since $\sim_X$ is a congruence of semigroups. On S_ω, if $u \sim_X v$ and $u = 1u$ belongs X, then, by definition of the relation $\sim_X$, $1v = v$ also belongs to X. $\square$

We proved that the syntactic congruence of every recognizable subsets exists and is well defined. In particular, Proposition 3.49 shows that the syntactic congruence of any ω-rational language is well defined. We now prove that the syntactic ω-semigroup of an ω-rational language is the minimal structure – for the division relation – recognizing this ω-language. This result is a straightforward generalization of Proposition 3.30.

PROPOSITION 3.55. *Let L be an ω-language of A^∞. Then S recognizes L if and only if $S(L)$ divides S.*

PROOF. First, assume that $S(L)$ divides S, and let μ be the syntactic morphism of L. Then there exist a semigroup T, an injective morphism ι from T into S, and a surjective morphism σ from T onto $S(L)$. By Corollary 3.37, there exists a morphism φ from A^∞ into T such that $\sigma \circ \varphi = \mu$. Now, let us set $P = \iota(\sigma^{-1}(\mu(L)))$. Since ι is injective and μ is the syntactic morphism, one has

$$(\iota \circ \varphi)^{-1}(P) = \varphi^{-1}(\iota^{-1}(\iota(\sigma^{-1}(\mu(L))))) = \varphi^{-1}(\sigma^{-1}(\mu(L))) = \mu^{-1}(\mu(L)) = L.$$

Therefore, S recognizes L. Conversely, assume that S recognizes L. There exist both a morphism φ from A^∞ onto S, and a subset P of S such that $\varphi^{-1}(P) = L$. Set $T = \varphi(A^\infty)$. Then T is an ω-subsemigroup of S. Now, let $u, v \in A^+$ such that $u \sim_\varphi v$, let $\alpha, \beta \in A^\omega$ such that $\alpha \sim_\varphi \beta$, and let $x, y \in A^*$ and $z \in A^+$. One has

$$\begin{aligned}
xuyz^\omega \in L &\Leftrightarrow \varphi(xuyz^\omega) \in P \\
&\Leftrightarrow \varphi(x)\varphi(u)\varphi(y)\varphi(z)^\omega \in P \\
&\Leftrightarrow \varphi(x)\varphi(v)\varphi(y)\varphi(z)^\omega \in P \\
&\Leftrightarrow \varphi(xvyz^\omega) \in P \\
&\Leftrightarrow xvyz^\omega \in L, \\
x(uy)^\omega \in L &\Leftrightarrow \varphi(x(uy)^\omega) \in P \\
&\Leftrightarrow \varphi(x)(\varphi(u)\varphi(y))^\omega \in P \\
&\Leftrightarrow \varphi(x)(\varphi(v)\varphi(y))^\omega \in P \\
&\Leftrightarrow \varphi(x(vy)^\omega) \in P \\
&\Leftrightarrow x(vy)^\omega \in L, \text{ and} \\
x\alpha \in L &\Leftrightarrow \varphi(x\alpha) \in P \\
&\Leftrightarrow \varphi(x)\varphi(\alpha) \in P \\
&\Leftrightarrow \varphi(x)\varphi(\beta) \in P \\
&\Leftrightarrow \varphi(x\beta) \in P \\
&\Leftrightarrow x\beta \in L.
\end{aligned}$$

Hence, the relations $u \sim_\varphi v$ and $\alpha \sim_\varphi \beta$ respectively imply $u \sim_L v$ and $\alpha \sim_L \beta$, and Proposition 3.35 shows that $S(L)$ is a quotient of T. Therefore, $S(L)$ divides S. $\square$

EXAMPLE 3.56. Let $A = \{a, b\}$, and let $K = (A^* a)^\omega$ be an ω-language over A. The ω-semigroup S given in Example 3.46 is the syntactic ω-semigroup of K. The morphism φ from A^∞ into S defined by the relations $\varphi(a) = 0$ and $\varphi(b) = 1$ is the syntactic morphism of K. The ω-subset $X = \{0^\omega\}$ of S is the syntactic image of K.

EXAMPLE 3.57. Let $B = \{a, b, c\}$, and let $L = (a\{b, c\}^* \cup \{b\})^\omega$ be an ω-language over B. The ω-semigroup T given in Example 3.47 is the syntactic ω-semigroup of L. The morphism ψ from B^∞ into T defined by $\psi(a) = a$, $\psi(b) = b$, and $\psi(c) = c$ is the syntactic morphism of L . The ω-subset $Y = \{a^\omega\}$ of T is the syntactic image of L.

Chapter 4

The Wadge hierarchy

Summary

This chapter describes the theory of Wadge games – a crucial topic through this work [39, 40]. The infinite games over ω-semigroups presented in Chapter 6 are highly inspired by the Wadge games, and many results of the Wadge theory presented below will be translated to the ω-semigroup context.

First, we define the Wadge reduction on ω-languages by means of the Wadge game, an infinite two-player game with perfect information. This reduction was introduced by William W. Wadge in order to provide a game theoretical reformulation of the continuous reduction. Consequently, the Wadge reduction is a preorder, and hence gives rise to an equivalence relation on ω-languages. The collection of ω-languages ordered by the Wadge reduction is then called the Wadge hierarchy.

In addition, we prove that Borel determinacy of Gale-Stewart games induces determinacy of Wadge games for every Borel winning set. This key result implies strong consequences on the Wadge hierarchy of Borel ω-languages. Therefore, up to Wadge equivalence and complementation, the Borel Wadge hierarchy is proved to be a well-ordering.

4.1 The Wadge game

Let A and B be two alphabets, and let $X \subseteq A^\omega$ and $Y \subseteq B^\omega$. The *Wadge game* $\mathbb{W}((A, X), (B, Y))$ [39] is a two-player infinite game with perfect information, where Player I is in charge of the subset X and Player II is in charge of the subset Y. Players I and II alternately play letters from the alphabets A and B, respectively. Player I begins. Player II is allowed to skip her turn – which is formally denoted by the symbol "–" – provided she plays infinitely many letters, whereas Player I is not allowed to do so. After ω turns, players I and II have respectively produced two infinite words $\alpha \in A^\omega$ and $\beta \in B^\omega$. Player II wins $\mathbb{W}((A, X), (B, Y))$ if and only if $(\alpha \in X \Leftrightarrow \beta \in Y)$. From this point onward, the Wadge game $\mathbb{W}((A, X), (B, Y))$ will be denoted by $\mathbb{W}(X, Y)$ and the alphabets involved will always be assumed to be known from the context.

A play of this game is illustrated below.

$$\begin{array}{lccccccc}
(X)\ \mathrm{I}\ : & a_0 & & & a_1 & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & \alpha = a_0 a_1 a_2 \cdots \\
 & & \searrow & \nearrow & & & & \\
(Y)\ \mathrm{II}\ : & & b_0 & & & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & \beta = b_0 b_1 b_2 \cdots
\end{array}$$

Along the play, the finite sequence of the previous moves of a given player is called the *current position* of this player. A *strategy* for Player I is a mapping from $(B \cup \{-\})^*$ into A. A *strategy* for Player II is a mapping from A^+ into $B \cup \{-\}$. A strategy is *winning* if the player following it must win, no matter what his opponent plays.

The *Wadge reduction* is defined via the Wadge game. The set X is said to be *Wadge reducible* to Y, denoted by $X \leq_W Y$, if and only if Player II has a winning strategy in $\mathbb{W}(X, Y)$. One then sets $X \equiv_W Y$ if and only if $X \leq_W Y$ and $Y \leq_W X$, and $X <_W Y$ if and only if $X \leq_W Y$ and $X \not\equiv_W Y$. The sets X and Y are called *incomparable* when both $X \not\leq_W Y$ and $Y \not\leq_W X$ hold. In addition, a set X is called *self-dual* if $X \equiv_W X^c$, and *non-self-dual* if $X \not\equiv_W X^c$. The following lemma sates a basic property of the Wadge reduction.

Lemma 4.1. *The Wadge reduction is a preorder.*

Proof. We first prove the reflexivity. Let X be an ω-language, we describe a winning strategy for Player II in the game $\mathbb{W}(X, X)$. Player II simply copies Player I's moves and wins. For the transitivity, let X, Y, and Z be three ω-languages such that $X \leq_W Y$ and $Y \leq_W Z$. Then Player II has two winning strategies σ and τ in the respective games $\mathbb{W}(X, Y)$ and $\mathbb{W}(Y, Z)$. The composition of these two strategies is a winning strategy for Player II in the game $\mathbb{W}(X, Z)$. Therefore, $X \leq_W Z$. $\square$

Lemma 4.1 implies in particular that the relation $\equiv_W$ is an equivalence relation, called the *Wadge equivalence*. In addition, the following result shows that the empty set and the full space are incomparable one to the other, but Wadge reducible to any other set. Other basic properties follow.

Lemma 4.2. *Let A be an alphabet, and let $X \subseteq A^\omega$.*
(1) *If $X \neq A^\omega$, then $\emptyset \leq_W X$.*
(2) *If $X \neq \emptyset$, then $A^\omega \leq_W X$.*
(3) *$\emptyset$ and A^ω are incomparable.*

Proof.
(1) We describe a winning strategy for Player II in the game $\mathbb{W}(\emptyset, X)$. At the end of the play, the infinite sequence played by Player I does not belong to $\emptyset$. Hence, the winning strategy for Player II consists in playing an infinite sequence $(s_0, s_1, s_2, \ldots)$ which doesn't belong to X. This is possible, since $X \neq A^\omega$.
(2) Similarly, we describe a winning strategy for Player II in $\mathbb{W}(A^\omega, X)$. At the end of the play, the infinite sequence played by Player I obviously belongs to A^ω. Therefore, Player II wins the game by playing an infinite sequence which belongs to X. This is indeed possible, since $X \neq \emptyset$.

(3) We show that Player II has no winning strategy in the game $\mathbb{W}(\emptyset, A^\omega)$. At the end of the play, Player I's infinite sequence does not belong to $\emptyset$ whereas Player II's infinite sequence belongs to A^ω. Therefore, $\emptyset \not\leq_W A^\omega$. The same argument shows that $A^\omega \not\leq_W \emptyset$. □

LEMMA 4.3. *Let A and B be two alphabets, and let $X \subseteq A^\omega$ and $Y \subseteq B^\omega$.*

(1) *$X \leq_W Y$ if and only if $X^c \leq_W Y^c$.*
(2) *X and X^c are either equivalent or incomparable.*
(3) *If $X <_W Y$, then $Y \not\leq_W X$ and $Y^c \not\leq_W X$.*

PROOF.

(1) By definition of the winning conditions of the Wadge game, a strategy is winning for Player II in $\mathbb{W}(X, Y)$ if and only if it is also winning for Player II in $\mathbb{W}(X^c, Y^c)$. Indeed, since $\alpha \in X \Leftrightarrow \beta \in Y$ if and only if $\alpha \in X^c \Leftrightarrow \beta \in Y^c$, every infinite play is winning for II in $\mathbb{W}(X, Y)$ if and only if it is also winning for II in $\mathbb{W}(X^c, Y^c)$.
(2) Either $X \leq_W X^c$, or $X \not\leq_W X^c$. If $X \leq_W X^c$, then (1) implies $X^c \leq_W X$, thus $X \equiv_W X^c$. If $X \not\leq_W X^c$, then (1) implies $X^c \not\leq_W X$, hence X and X^c are incomparable.
(3) If $X <_W Y$, then $Y \not\leq_W X$, by definition. Now, assume that $Y^c \leq_W X$. Then $Y^c \leq_W X$ and $X <_W Y$ imply $Y^c <_W Y$, which contradicts (2). □

Finally, the following result recalls that the Wadge games were precisely introduced in order that the Wadge reduction coincides with the continuous reduction.

LEMMA 4.4 (WADGE). *Let A and B be two alphabets, and let $X \subseteq A^\omega$ and $Y \subseteq B^\omega$. Then X is Wadge reducible to Y if and only if X is continuously reducible to Y.*

PROOF. Assume that $X \leq_W Y$. Then there exists a winning strategy σ for Player II in the game $\mathbb{W}(X, Y)$. This strategy naturally induces the function $\bar{\sigma}$ from A^∞ into B^∞ which maps every finite word u and every infinite word α to Player II's respective answers to u and α via σ. We prove that the restriction of this mapping from A^ω into B^ω is continuous. Let VB^ω be an open set of B^ω, for some $V \subseteq B^*$. Then $\bar{\sigma}^{-1}(VB^\omega) = (\bar{\sigma}^{-1}V)A^\omega$, showing that the preimage of any open set is an open set. Moreover, the winning condition of the Wadge game states that $\alpha \in X$ if and only if $\bar{\sigma}(\alpha) \in Y$, and therefore, $\bar{\sigma}^{-1}(Y) = X$. Conversely, let $f : A^\omega \longrightarrow B^\omega$ be a continuous mapping such that $f^{-1}(Y) = X$. We describe a winning strategy for Player II in the game $\mathbb{W}(X, Y)$. For that purpose, consider an enumeration $\{b_0, \ldots, b_n\}$ of the elements of B. Since f is continuous, the sets $A_i = f^{-1}(b_i B^\omega)$ form a partition of A^ω in open sets. Therefore, as long as I's play does not enter any of the sets A_i, II skips her turn. As soon as I's play enters a set A_{i_0}, for some $i_0 \leq n$, II plays b_{i_0}. Notice that since the sets A_i form a partition of A^ω, I's play is forced to enter a set A_{i_0} after a finite amount of time. Then, consider the sets $A'_i = f^{-1}(b_{i_0} b_i B^\omega)$, for every $i \leq n$, and proceed the same way. As long as I's play doesn't enter any of the sets A'_i's, II skips her turn. As soon as I's play enters an A'_{i_1}, II plays b_{i_1}. And so on and so forth. At the end of the play, the infinite words α and β played respectively by players I and II satisfy $f(\alpha) = \beta$. Moreover, since $f^{-1}(Y) = X$, the relation $\alpha \in X$ if and only if $\beta \in Y$ holds. Therefore, this strategy is winning for Player II in the game $\mathbb{W}(X, Y)$, thus $X \leq_W Y$. □

4.2 The Wadge hierarchy

The *Wadge hierarchy* consists of the collection of all ω-languages ordered by the Wadge reduction, and the *Borel Wadge hierarchy* is the restriction of the Wadge hierarchy to the Borel ω-languages. In the sequel, we will particularly focus on the Borel Wadge hierarchy. For a start, we prove that Martin's Borel determinacy [21] implies the determinacy of Wadge games, for every pair of Borel winning sets. This result is the cornerstone of the description of the Borel Wadge hierarchy.

THEOREM 4.5 (BOREL WADGE DETERMINACY). *Let X and Y be two Borel ω-languages of A^ω and B^ω, respectively. Then $\mathbb{W}(X,Y)$ is determined.*

PROOF. We define a Borel subset $Z \subseteq (A \cup B)^\omega$ such that a given player has a winning strategy in $\mathbb{G}(Z)$ if and only if the same player has a winning strategy in $\mathbb{W}(X,Y)$. Theorem 1.6 then leads to the conclusion. Let p_1 and p_2 be the continuous projections from $(A \cup B)^\omega$ into $(A \cup B)^\omega$ defined by

$$\begin{aligned} p_1(u_0u_1u_2u_3\cdots) &= u_0u_2u_4\cdots, \\ p_2(u_0u_1u_2u_3\cdots) &= u_1u_3u_5\cdots. \end{aligned}$$

Let also $X' = p_1^{-1}(X)$, $X'' = p_1^{-1}(X^c)$, $Y' = p_2^{-1}(Y)$, and $Y'' = p_2^{-1}(Y^c)$. By continuity of the functions p_1 and p_2, all these sets are Borel. Now, set $Z = (X' \cap Y') \cup (X'' \cap Y'')$. This set is Borel and satisfies the required property. □

The Wadge Borel determinacy induces the following corollaries: the $\leq_W$-antichains have length at most two, and the only incomparable ω-languages are – up to Wadge equivalence – of the form X and X^c, for X non-self-dual. Furthermore, the Wadge reduction is wellfounded on Borel sets.

PROPOSITION 4.6. *Let X and Y be two Borel ω-languages of A^ω and B^ω, respectively. The following properties hold.*

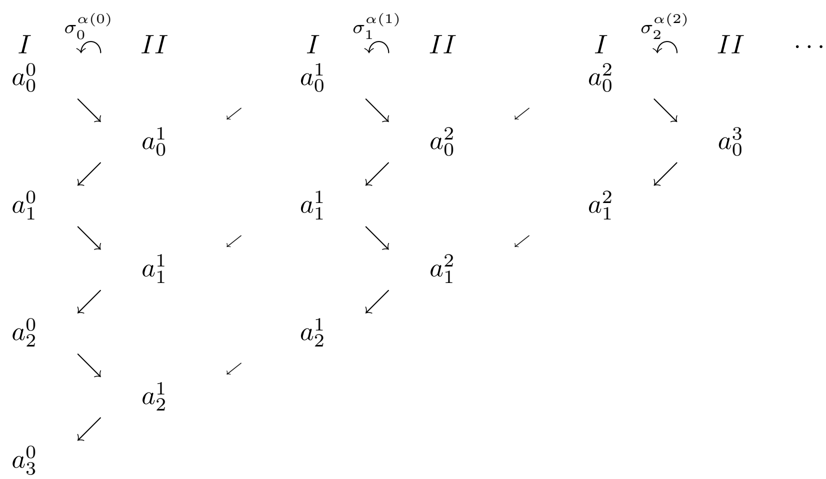
(1) *(Wadge's Lemma) Either $X \leq_W Y$, or $Y \leq_W X^c$.*
(2) *If X and Y are incomparable, then $X \equiv_W Y^c$.*
(3) *The $\leq_W$-antichains have length at most two.*

PROOF.

(1) Either $X \leq_W Y$, or $X \not\leq_W Y$. If $X \not\leq_W Y$, then Player II has no winning strategy in $\mathbb{W}(X,Y)$. Hence, by determinacy, Player I has a winning strategy σ in this game. Therefore, we describe a winning strategy for Player II in $\mathbb{W}(Y,X^c)$. On her first move, regardless Player I's move, Player II answers by $\sigma(\varepsilon)$. Then, Player II answers to every current position $(a_0,a_1,\ldots,a_n)$ of Player I by the move $\sigma(a_0,a_1,\ldots,a_{n-1})$. At the end of the play, the definition of σ ensures that $a_0a_1a_2\cdots$ belong to Y if and only if $\sigma(\varepsilon)\sigma(a_0)\sigma(a_0,a_1)\ldots$ doesn't belongs to X. Hence, Player II wins the game $\mathbb{W}(Y,X^c)$, and therefore $Y \leq_W X^c$.
(2) If $X \not\leq_W Y$ and $Y \not\leq_W X$, then (1) implies that $Y \leq_W X^c$ and $X \leq_W Y^c$. Therefore, $Y \leq_W X^c$ and $X^c \leq_W Y$, and thus $X^c \equiv_W Y$.
(3) Let X, Y and Z be such that $X \not\leq_W Y$ and $Y \not\leq_W Z$. Then (1) shows that $Y \leq_W X^c$ and $Z \leq_W Y^c$. Therefore, $Z \leq_W Y^c$ and $Y^c \leq_W X$, thus $Z \leq_W X$. □

PROPOSITION 4.7 (MARTIN, MONK). *The Wadge strict reduction is wellfounded on Borel ω-languages.*

PROOF. Towards a contradiction, assume that there exists an infinite strictly $<_W$-descending sequence of Borel ω-languages $(X_n)_{n\geq 0}$, where $X_i \subseteq A_i^\omega$ for all i. For all $n \geq 0$, the relation $X_n >_W X_{n+1}$ implies $X_n \not\leq_W X_{n+1}$ and $X_n^c \not\leq_W X_{n+1}$, meaning by determinacy that Player I has the winning strategies σ_n^0 and σ_n^1 in the respective games $\mathbb{W}(X_n, X_{n+1})$ and $\mathbb{W}(X_n^c, X_{n+1})$. Now, for any $\alpha \in \{0,1\}^\omega$, consider the infinite sequence of strategies $(\sigma_n^{\alpha(n)})_{n\geq 0}$ and the infinite sequence of games $(\mathbb{W}(X_n^{c(\alpha(n))}, X_{n+1}))_{n\geq 0}$ defined as follows: in the game $\mathbb{W}(X_k^{c(\alpha(k))}, X_{k+1})$, Player I applies his winning strategy $\sigma_k^{\alpha(k)}$, and Player II copies Player I's moves from the next game $\mathbb{W}(X_{k+1}^{c(\alpha(k+1))}, X_{k+2})$. Therefore, in the first game, Player I applies his winning strategy $\sigma_0^{\alpha(0)}$. Since it is a strategy for Player I, it gives the first letter a_0^0 before Player II has ever played anything. Then Player II copies Player I's first move a_0^1 in the second game, and Player I answers with his winning strategy. And so on and so forth for every move and every game. This infinite sequence of games are illustrated below. Big arrows stand for playing and little ones for copying.



Let x_α be the infinite word played by Player I in the first game, and let $\varphi : \{0,1\}^\omega \longrightarrow A_0^\omega$ be defined by $\varphi(\alpha) = x_\alpha$. We show that φ is continuous. By definition of these chained games, the k first letters of x_α only depend on the k first letters of α, since the games number $k+1, k+2, \ldots$ do not influence the construction the prefix $x_\alpha[0,k]$. Thus, for any $U \subseteq A_0^*$, one has $\varphi^{-1}(UA_0^\omega) = V\{0,1\}^\omega$, with $V \subseteq \{0,1\}^*$, meaning that the preimage by φ of any open set is an open set. Now, consider $F = \varphi^{-1}(X_0)$. By construction of these chained games, F is a flip set, because if α and α' only differ by one position (meaning if there exists a unique i such that $\alpha(i) \neq \alpha'(i)$), then $\alpha \in F$ if and only if $\alpha' \notin F$. On the other hand, the set F is also Borel, since φ is continuous, a contradiction. $\square$

Propositions 4.6 and 4.7 show that, up to complementation and Wadge equivalence, the Borel Wadge hierarchy is a well-ordering. Therefore, there exists a unique ordinal, called the *height* of the Borel Wadge hierarchy, and a mapping d_W from the Borel Wadge hierarchy onto its height, called the *Wadge* *degree*, such that $d_W(X) < d_W(Y)$ if and only if $X <_W Y$ and $d_W(X) = d_W(Y)$ if and only if $X \equiv_W Y$ or $X \equiv_W Y^c$, for every Borel ω-languages X and Y. The wellfoundness of the Borel Wadge hierarchy ensures that the Wadge degree can be defined by induction as follows:

$$d_W(X) = \begin{cases} 0 & \text{if } X = \emptyset \text{ or } X = \emptyset^c, \\ \sup\{d_W(B) + 1 : B <_W A\} & \text{otherwise.} \end{cases}$$

Another consequence of the Borel determinacy of Wadge games due to Martin and Wadge gives a precise characterization of non-self-duals sets. As a corollary, every self-dual set can be described by translations of strictly $\leq_W$-smaller non-self-dual sets.

Proposition 4.8 (Martin, Wadge). *Let X be a Borel subset of A^ω. Then X is non-self-dual if and only if there exists $\alpha \in A^\omega$ such that $(\alpha[0,n])^{-1}X \equiv_W X$, for all $n \in \mathbb{N}$.*

Proof. Notice that a player in charge of the set $u^{-1}X$ in a Wadge game is exactly as strong as a player in charge of X, but having already played the finite word u. Therefore, the relation $u^{-1}X \leq_W X$ always holds, for any $u \in A^*$. Indeed, the winning strategy for Player II in $\mathbb{W}(u^{-1}X, X)$ consists in: first, playing the finite word u, and then copying Player I's infinite sequence letter by letter. The proposition that we have to prove then reduces to the following: the set X is non-self-dual if and only if there exists $\alpha \in A^\omega$ such that $X \leq_W (\alpha[0,n])^{-1}X$, for all $n \in \mathbb{N}$.

($\Rightarrow$) Assume that X is non-self-dual. Then there exists a winning strategy σ for Player I in the game $\mathbb{W}(X, X^c)$. Let $\alpha = \sigma(-)\sigma(-,-)\sigma(-,-,-)\cdots$. We show that α satisfies $X \leq_W (\alpha[0,n])^{-1}X$, for all $n \in \mathbb{N}$. Let Player II apply the strategy σ in the game $\mathbb{W}(X, (\alpha[0,n])^{-1}X)$, and let β and β' be the infinite words respectively played by players I and II in this game. The definition of σ ensures that $\beta \in X$, if and only if $\alpha[0,n]\beta' \notin X^c$, if and only if $\alpha[0,n]\beta' \in X$, if and only if $\beta' \in (\alpha[0,n])^{-1}X$. Therefore, the strategy σ is winning for Player II in the game $\mathbb{W}(X, (\alpha[0,n])^{-1}X)$, for all $n \in \mathbb{N}$.

($\Leftarrow$) We prove that if X is self-dual, then for all $\alpha \in A^\omega$, there exists an integer $n \geq 0$ such that $X \not\leq_W (\alpha[0,n])^{-1}X$. Towards a contradiction, assume that there exists $\alpha \in A^\omega$ satisfying $X \leq_W (\alpha[0,n])^{-1}X$, for all $n \in \mathbb{N}$. Then for each integer n, on the one hand, there exists a winning strategy σ_n^0 for Player II in the game $\mathbb{W}(X, (\alpha[0,n])^{-1}X)$, and on the other hand, since X is self-dual, there also exists a winning strategy σ_n^1 for Player II in the game $\mathbb{W}(X, (\alpha[0,n])^{-1}X^c)$. Now, for any $\gamma \in \{0,1\}^\omega$, consider the infinite sequence of strategies $(\sigma_k^{\gamma(k)})_{k\geq 0}$, and the infinite sequence of games $(\mathbb{W}(X, (\alpha[0,n_k])^{-1}X^{c(\gamma(k))}))_{k\geq 0}$ defined as follows: in the game $\mathbb{W}(X, (\alpha[0,n_k])^{-1}X^{c(\gamma(k))})$, Player II applies his winning strategy $\sigma_k^{\gamma(k)}$, and Player I copies Player II's moves in the next game $\mathbb{W}(X, (\alpha[0,n_{k+1}])^{-1}X^{c(\gamma(k+1))})$, where n_k is the least integer larger than n_{k-1} such that the sequence of strategies $\sigma_0^{\gamma(0)} \circ \sigma_1^{\gamma(1)} \circ \ldots \circ \sigma_{k-1}^{\gamma(k-1)}$ applied to the finite word $\alpha[0,n_k]$ yields a word of length at least k. This infinite sequence of games is illustrated below. Big arrows denote the action of playing and little ones denote the action of copying.

$$\begin{array}{cccccccccccc}
II & \overset{\sigma_0^{\gamma(0)}}{\curvearrowleft} & I & & II & \overset{\sigma_1^{\gamma(1)}}{\curvearrowleft} & I & & II & \overset{\sigma_2^{\gamma(2)}}{\curvearrowleft} & I & \cdots \\
 & & a_0^1 & & & & a_0^2 & & & & a_0^3 & \\
 & \swarrow & & \nwarrow & & \swarrow & & \nwarrow & & \swarrow & & \\
a_0^0 & & & & a_0^1 & & & & a_0^2 & & & \\
 & \searrow & & & & \searrow & & & & & & \\
 & & a_1^1 & & & & a_1^2 & & & & & \\
 & \swarrow & & \nwarrow & & \swarrow & & & & & & \\
a_1^0 & & & & a_1^1 & & & & & & & \\
 & \searrow & & & & & & & & & & \\
 & & a_2^1 & & & & & & & & & \\
 & \swarrow & & & & & & & & & & \\
a_2^0 & & & & & & & & & & &
\end{array}$$

Let $a_0^0 a_1^0 a_2^0 \cdots$ be the infinite word played by Player II in the first of these chained game induced by γ. Then the mapping $f : \{0,1\}^\omega \longrightarrow A^\omega$ defined by $f(\gamma) = a_0^0 a_1^0 a_2^0 \cdots$ is continuous, and thus the set $Y = f^{-1}(X)$ is Borel. However, by construction of these chained game, the set Y is also a flip set, a contradiction. $\square$

Corollary 4.9. *If X is a self-dual Borel subset of A^ω, then there exist a set $I \subseteq A^+$ and a family of non-self-dual subsets $(X_i)_{i \in I}$ satisfying $X_i <_W X$, such that $X = \bigcup_{i \in I} iX_i$.*

Proof. The proof goes by induction on $d_W(X)$. If $d_W(X) = 0$, then X is the empty set or the full space. Since these sets are non-self-dual, there is nothing to prove in this case. For the induction step, assume that X is a self-dual set. Then Proposition 4.8 shows that, for all $\alpha \in A^\omega$, there exists a least integer n_α, such that $(\alpha[0, n_\alpha])^{-1} X <_W X$. Let us set $I = \{\alpha[0, n_\alpha] \mid \alpha \in A^\omega\}$. One obviously has $X = \bigcup_{u \in I} u\left(u^{-1}X\right)$. More precisely, consider the partition of I in I' and I'' given by $I' = \{u \in I \mid u^{-1}X \text{ is non-self-dual}\}$, and $I'' = \{u \in I \mid u^{-1}X \text{ is self-dual}\}$. Then

$$X = \bigcup_{u \in I'} u\left(u^{-1}X\right) \cup \bigcup_{u \in I''} u\left(u^{-1}X\right). \tag{4.1}$$

Now, by construction, every set $u^{-1}X$ satisfies $u^{-1}X <_W X$, for all $u \in I$. If $u^{-1}X$ is self-dual, the induction hypothesis guarantees that $u^{-1}X = \bigcup_{v \in J_u} vX_v$, for some subset J_u of A^+, and some non-self-dual sets X_v satisfying $X_v <_W u^{-1}X <_W X$. Therefore, one has

$$\bigcup_{u \in I''} u\left(u^{-1}X\right) = \bigcup_{u \in I''} u \bigcup_{v \in J_u} vX_v = \bigcup_{u \in I'', v \in J_u} u(vX_v) = \bigcup_{uv \in I''J_u} (uv)X_v.$$

Finally, by replacing the expression above in (4.1), one obtains the desired formula

$$X = \bigcup_{u \in I'} u\left(u^{-1}X\right) \cup \bigcup_{uv \in I''J_u} (uv)X_v.$$

$\square$

Corollary 4.9 shows that every self-dual ω-languages is a finite union of translations of strictly smaller non-self-dual sets. Hence, in order to exclusively concentrate on the non-self-dual sets, we consider another definition of the Wadge degree which sticks every self-dual set on a non-self-dual set lower at just one level in the hierarchy.

$$d_w(X) = \begin{cases} 1 & \text{if } X = \emptyset \text{ or } X = \emptyset^c, \\ \sup\{d_w(Y) + 1 \mid Y \text{ n.s.d. and } Y <_W X\} & \text{if } X \text{ is non-self-dual}, \\ \sup\{d_w(Y) \mid Y \text{ n.s.d. and } Y <_W X\} & \text{if } X \text{ is self-dual}. \end{cases}$$

Finally, one can show that the Borel Wadge hierarchy consists of an alternating succession of non-self-dual and self-dual sets, except for some levels of special cofinality that we will not discuss here. This hierarchy is partially illustrated in Figure 4.1, where circles denote the Wadge equivalence classes, and arrows represent the Wadge reduction. The Borel Wadge hierarchy drastically refines the Borel hierarchy: the Borel sets of only finite ranks provide Wadge degrees ranging from 1 to the first fixpoint of the exponentiation of base ω_1.

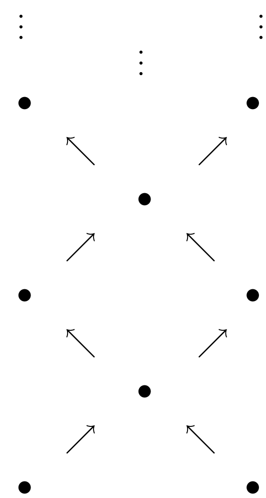


Figure 4.1: The Wadge hierarchy.

Chapter 5

The Wagner hierarchy

Summary

In 1979, Wagner defined a partial ordering on ω-rational languages by analyzing the graphs of their underlying Muller automata. The resulting hierarchy is a fine and effective classification of ω-rational sets, known as the Wagner hierarchy [41]. It was proved to be decidable, and has a height of ω^ω. In addition, this hierarchy happens to coincide with the restriction of the Wadge hierarchy to ω-rational languages, and therefore refines the lower levels of the Borel hierarchy. The Wagner reduction thus corresponds to the Wadge or the continuous reduction; but it also coincides with the sequential reduction – a reduction defined by means of automata – on the class of ω-rational languages [27, Thm 5.2, p. 209].

The Wagner hierarchy has been thoroughly investigated since then. Wilke and Yoo described an efficient algorithm computing the Wagner degree of any ω-rational language in polynomial time [43]. Carton, Perrin, Duparc, and Riss [4, 5, 6, 10] studied an algebraic description of this hierarchy in connection with the theory of ω-semigroup. Selivanov proposed a purely descriptive set theoretical description of the Wagner hierarchy in [34].

In this chapter, the Wagner hierarchy is described as the trace of the Wadge hierarchy on ω-rational sets [10]. This description relies mainly on the following observation: every Wadge play involving two ω-rational languages naturally induces two infinite paths in the respective underlying Muller automata. Therefore, the Wadge complexity of an ω-rational language obviously depends on the graph structure of its underlying Muller automaton.

Hence, we first introduce a simplified graph representation of Muller automata. We then define step by step three different notions of chains in the graph of Muller automata. We further show that the maximal chains contained in a Muller automaton is a signature of the Wagner equivalence class of the corresponding ω-rational language. Consequently, we prove that the Wagner hierarchy has height ω^ω, and is decidable. The Wagner degree of an ω-rational language is then precisely given by the length of a maximal chain contained in the corresponding underlying Muller automaton. Finally, we show that the Wagner degree is a *syntactic invariant*: if two ω-rational languages have the same syntactic ω-semigroup, then they share the same Wagner degree.

5.1 The DAG representation of Muller automata

The description of the Wagner hierarchy is rather technical but relies on the following main consideration. If $\mathcal{A}$ is a complete Muller automaton recognizing the language $L(\mathcal{A})$, then any Wadge play according to $L(\mathcal{A})$ traces an infinite path in the graph of the automaton $\mathcal{A}$. Such a play belongs to $L(\mathcal{A})$ if an only if the corresponding infinite path is accepted by $\mathcal{A}$. Similarly, any position u in the Wadge game leads to a state q in the graph of the automaton. The Wadge positions and plays can thus be visualized in the corresponding automaton graph. Therefore, a Wadge player in charge of $L(\mathcal{A})$ is constantly aware of the accepting and rejecting plays that he can still produce from his current position: it corresponds precisely to the accepting and rejecting loops in $\mathcal{A}$ that are reachable from his current state. For this reason, we introduce a pruned representation of the graph of $\mathcal{A}$ which precisely reveals these characteristics. We first need the following definitions.

DEFINITION 5.1. *Let $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$ be a complete Muller automaton, and let q and q' be two states of $\mathcal{A}$. We set $q \leq q'$ if either $q = q'$ or if there exists a path in $\mathcal{A}$ going from q to q'.*

This accessibility relation is reflexive and transitive. It induces an equivalence relation defined by $q \equiv q'$ if and only if $q \leq q'$ and $q' \leq q$. The equivalence classes of this relation are the strongly connected components of the graph of $\mathcal{A}$. Given two strictly connected components Q_i and Q_j of the graph of $\mathcal{A}$, we set $Q_i < Q_j$ if and only if there exists a path in $\mathcal{A}$ from some state in Q_i to some state in Q_j.

DEFINITION 5.2. *Let $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$ be a complete Muller automaton. A* loop *in $\mathcal{A}$ is a set of states $l = \{q_0, \ldots, q_n\}$ such that there exists a path is $\mathcal{A}$ which starts at q_0, visits successively every state of l, and comes back to q_0.*

In addition, a loop l of $\mathcal{A}$ is called *accepting* if it belongs to $\mathcal{T}$ and *rejecting* otherwise. A sequence of loops $(l_0, \ldots, l_n)$ is called *alternating* if l_i is accepting if and only if l_{i+1} is rejecting. It is called *increasing* if $l_i \subset l_{i+1}$. Finally, the *height* of a loop l, denoted by $ht(l)$, is the maximal length of an alternating increasing sequence of loops that are all strictly contained in l. For instance, if we consider the semi-lattice ordered by inclusion of all loops contained in a given strictly connected component, then every loop at the bottom has height zero.

The quotient of the graph of a Muller automaton $\mathcal{A}$ by the equivalence relation "$\equiv$" is a directed acyclic graph (DAG) whose nodes are the strictly connected components of the graph of $\mathcal{A}$. Furthermore, every node of this graph can be associated with the semi-lattice of loops (ordered by inclusion) contained in this strictly connected component. This enriched graph will be called the *DAG representation* of the automaton $\mathcal{A}$. It is illustrated in Figure 5.1, where edges stand for the accessibility relation between the strictly connected components. The DAG representation of $\mathcal{A}$ allows to follow the moves of a Wadge player in charge of the set $L(\mathcal{A})$. The successive positions of such a player induce a path in this graph which either remains indefinitely in a given semi-lattice, or climbs along a branch, with no chance of returning.

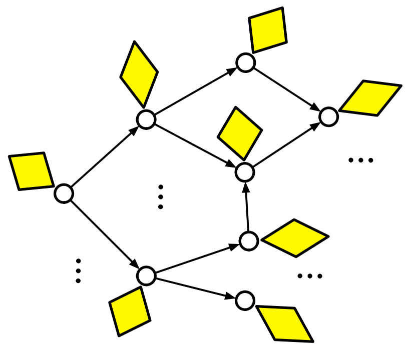


Figure 5.1: The DAG representation of a Muller automaton: every strictly connected component is associated with its corresponding semi-lattice of loops.

5.2 Chains in Muller automata

We introduce three successive definitions of chains in Muller automata. We will further prove that the maximal chains contained in a Muller automata describe entirely the Wagner class of the corresponding ω-rational language.

DEFINITION 5.3 (SEE [10]). *Let $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$ be a complete Muller automaton, and let n be a positive integer. An ω^n-$\mathcal{T}$-chain in $\mathcal{A}$ is a sequence $R_0 \subset R_1 \subset \ldots \subset R_n$ of admissible subsets of Q such that $R_k \in \mathcal{T}$ if and only if k is even.*

Hence, an ω^n-$\mathcal{T}^c$-chain in $\mathcal{A}$ is a sequence $R_0 \subset R_1 \subset \ldots \subset R_n$ of admissible subsets of Q such that $R_k \in \mathcal{T}$ if and only if k is odd.

EXAMPLE 5.4. Consider the Muller automaton represented in Figure 5.2 whose table is given by

$$\mathcal{T} = \{\{q_0\}, \{q_0, q_1, q_2\}\}.$$

Then $\{q_0\} \subset \{q_0, q_1\} \subset \{q_0, q_1, q_2\}$ is an ω^2-$\mathcal{T}$-chain, and $\{q_0, q_1\} \subset \{q_0, q_1, q_2\}$ is an ω^1-$\mathcal{T}^c$-chain.

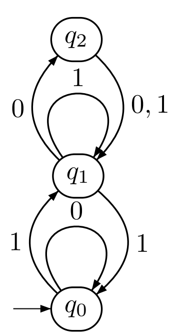


Figure 5.2: A Muller automaton.

DEFINITION 5.5 (SEE [10]). *Let $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$ be a complete Muller automaton, and let $n \geq 0$ and $l > 0$ be two integers. An $(\omega^n \cdot l)$-$\mathcal{T}$-chain in $\mathcal{A}$ is a sequence $(C_1, \ldots, C_l)$ such that*

- *each C_i is an ω^n-$\mathcal{T}^{c(i-1)}$-chain in $\mathcal{A}$,*
- *each C_{i+1} is accessible from C_i, meaning that there exists a path in $\mathcal{A}$ going from some state in C_i to some state in C_{i+1}.*

EXAMPLE 5.6. Consider the Muller automaton represented in Figure 5.3 whose table is given by

$$\mathcal{T} = \{\{q_0\}, \{q_2, q_3\}, \{q_4\}\} .$$

Then $(\{q_0\} \subset \{q_0, q_1\}, \{q_2\} \subset \{q_2, q_3\}, \{q_4\} \subset \{q_4, q_5\})$ is an $(\omega^1 \cdot 3)$-$\mathcal{T}$-chain, and $(\{q_2\} \subset \{q_2, q_3\}, \{q_4\} \subset \{q_4, q_5\})$ is an $(\omega^1 \cdot 2)$-$\mathcal{T}^c$-chain.

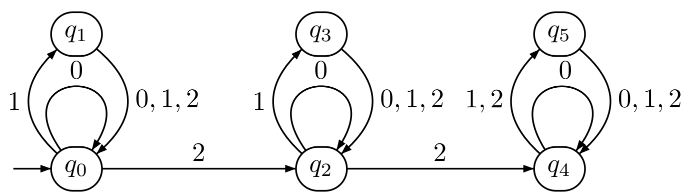
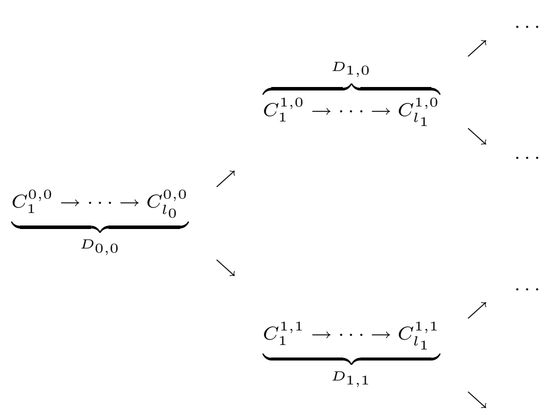


Figure 5.3: A Muller automaton.

DEFINITION 5.7 (SEE [10]). *Let $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$ be a complete Muller automaton, let $(n_i)_{i=k}^{0}$ be a strictly decreasing sequence of positive integers, and let $(l_i)_{i=k}^{0}$ be a sequence of strictly positive integers. An $(\omega^{n_k} \cdot l_k + \omega^{n_{k-1}} \cdot l_{k-1} + \ldots + \omega^{n_0} \cdot l_0)$-$\mathcal{T}$-chain in $\mathcal{A}$ is a sequence $(D_{i,j})_{i \leq k, j < 2^i}$ such that*

- *each $D_{i,j} = (C_1^{i,j}, C_2^{i,j}, \ldots, C_{l_i}^{i,j})$ is an $(\omega^{n_i} \cdot l_i)$-$\mathcal{T}^{c(j)}$-chain in $\mathcal{A}$,*
- *each $D_{i+1,2j}$ and $D_{i+1,2j+1}$ are both accessible from $D_{i,j}$, meaning that there exist two paths in $\mathcal{A}$, one going from some state in $C_{l_i}^{i,j}$ to some state in $C_1^{i+1,2j}$, and the other one going from some state in $C_{l_i}^{i,j}$ to some state in $C_1^{i+1,2j+1}$.*

The accessibility relation in a such a chain is illustrated below.



Example 5.8. Consider the Muller automaton represented in Figure 5.4 whose table is given by

$$\mathcal{T} = \{\{q_0\}, \{q_2\}, \{q_3\}, \{q_5, q_6\}, \{q_7\}, \{q_9, q_{10}\}, \{q_{11}\}, \{q_{13}, q_{14}\}\}.$$

Then $(D_{0,0}, D_{1,0}, D_{1,1})$ is an $(\omega^1 \cdot 3 + \omega^0 \cdot 3)$-$\mathcal{T}$-chain, where

- $D_{0,0} = (\{q_0\}, \{q_1\}, \{q_2\})$,
- $D_{1,0} = (\{q_3\} \subset \{q_3, q_4\}, \{q_5\} \subset \{q_5, q_6\}, \{q_7\} \subset \{q_7, q_8\})$,
- $D_{1,1} = (\{q_9\} \subset \{q_9, q_{10}\}, \{q_{11}\} \subset \{q_{11}, q_{12}\}, \{q_{13}\} \subset \{q_{13}, q_{14}\})$.

In addition $(D'_{0,0}, D'_{1,0}, D'_{1,1})$ is an $(\omega^1 \cdot 3 + \omega^0 \cdot 2)$-$\mathcal{T}^c$-chain, where

- $D'_{0,0} = (\{q_1\}, \{q_2\})$,
- $D'_{1,0} = (\{q_3\} \subset \{q_3, q_4\}, \{q_5\} \subset \{q_5, q_6\}, \{q_7\} \subset \{q_7, q_8\})$,
- $D'_{1,1} = (\{q_9\} \subset \{q_9, q_{10}\}, \{q_{11}\} \subset \{q_{11}, q_{12}\}, \{q_{13}\} \subset \{q_{13}, q_{14}\})$.

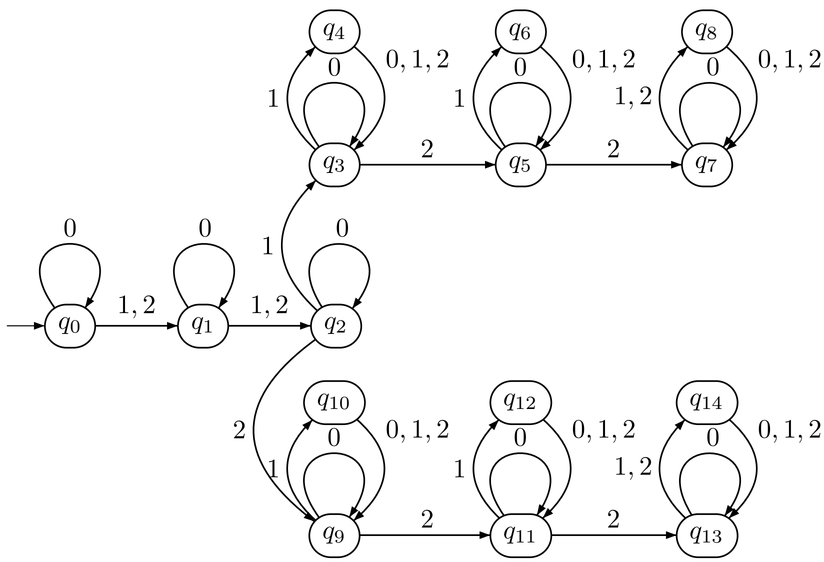


Figure 5.4: A Muller automaton.

In the sequel, an $(\omega^{n_k} \cdot l_k + \omega^{n_{k-1}} \cdot l_{k-1} + \ldots + \omega^{n_0} \cdot l_0)$-$\mathcal{T}$-chain will be called a ξ-$\mathcal{T}$-chain, where ξ denotes the ordinal whose Cantor normal form of base ω is

$$\xi = \omega^{n_k} \cdot l_k + \omega^{n_{k-1}} \cdot l_{k-1} + \ldots + \omega^{n_0} \cdot l_0.$$

A ξ-$\mathcal{T}^c$-chain is defined in a similar way. Definitions 5.3, 5.5, and 5.7 ensure that ξ-$\mathcal{T}$-chains and ξ-$\mathcal{T}^c$-chains are well defined for every ordinal ξ such that $0 < \xi < \omega^\omega$. When the automaton involved is supposed to be known, a ξ-$\mathcal{T}$-chain (a ξ-$\mathcal{T}^c$-chain) will be simply called a *ξ-chain* (a *co-ξ-chain*), or even a *chain* (a *co-chain*), when no information about the underlying ordinal is mentioned. Furthermore, given two complete Muller automata $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{A}$ is said to *contain more chains (co-chains)* than $\mathcal{B}$ if, for every ordinal ξ, the existence of an ξ-chain (co-ξ-chains) in $\mathcal{B}$ implies the existence of ξ-chain (co-ξ-chains) in $\mathcal{A}$. The automata $\mathcal{A}$ and $\mathcal{B}$ *contain the same chains (co-chains)* if, for every ordinal ξ, there exists a ξ-chain (co-ξ-chains) in $\mathcal{A}$ if and only if there exists an ξ-chain (co-ξ-chains) in $\mathcal{B}$. Finally, a chain is called *maximal* in $\mathcal{A}$ if it is a ξ-chain such that there is no other η-chain and co-η-chain satisfying $\eta > \xi$. A co-chain is called maximal in $\mathcal{A}$ if the same condition holds. We will further see that a complete Muller automaton always contains either a maximal chain, or a maximal co-chain, or both of them.

5.3 Chains as topological invariants

The following theorem shows that the chains and co-chains contained in a given Muller automaton are topological invariants for the Wagner class of the corresponding ω-rational language. In other words, any two ω-rational languages have the same Wagner degree if and only if their underlying Muller automata contain the same chains and co-chains.

THEOREM 5.9. *Let $\mathcal{A} = (Q_\mathcal{A}, A, \delta_\mathcal{A}, \{i_\mathcal{A}\}, \mathcal{T}_\mathcal{A})$ and $\mathcal{B} = (Q_\mathcal{B}, B, \delta_\mathcal{B}, \{i_\mathcal{B}\}, \mathcal{T}_\mathcal{B})$ be two complete Muller automata recognizing the respective ω-languages $L(\mathcal{A})$, and $L(\mathcal{B})$. Then $L(\mathcal{A}) \leq_W L(\mathcal{B})$ if and only if $\mathcal{B}$ contains more chains and co-chains than $\mathcal{A}$. In particular, $L(\mathcal{A}) \equiv_W L(\mathcal{B})$ if and only if $\mathcal{A}$ and $\mathcal{B}$ contain the same chains and co-chains.*

PROOF. The left-to-right and right-to-left directions are respectively given by the forthcoming propositions 5.14 and 5.15. The second part of the theorem is an immediate consequence of the first part. $\square$

The proof of Theorem 5.9 relies on the following preliminary results. First of all, if there exists a ξ-chain in a Muller automaton $\mathcal{A}$, then for every $0 < \eta < \xi$, there also exist an η-chain and a co-η-chain. Furthermore, a Muller automaton always contains either a maximal chain, or a maximal co-chain, or both of them. Therefore, an automaton $\mathcal{B}$ contains more chains (co-chains) than an automaton $\mathcal{A}$ if and only if the existence of a maximal chain (co-chain) in $\mathcal{A}$ implies the existence of a maximal chain (co-chain) in $\mathcal{B}$. Moreover, two automata contain the same chains (co-chains) if and only if they contain the same maximal chains (co-chains). This property could lead to an equivalent formulation of Theorem 5.9 in terms of existence of maximal chains and co-chains.

LEMMA 5.10. *Let $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$ be a complete Muller automaton. If there exists a ξ-chain or a co-ξ-chain in $\mathcal{A}$, then for each $0 < \eta < \xi$, there also exist both an η-chain and a co-η-chain.*

PROOF. We only consider the case of a ξ-chain, the case of a co-ξ-chain being symmetric. The proof goes by induction on ξ. It is obviously for $\xi = 1$, and the induction step relies mainly on the following argument. By its very definition, an ω^p-chain contains both an ω^q-chain and a co-ω^q-chain, for each $q < p$. In addition, since the accessibility relation always holds inside an ω^p-chain, then it can also give rise to both an $(\omega^q \cdot l)$-chain and a co-$(\omega^q \cdot l)$-chain, for each $q < p$ and every $l > 0$. More generally, every ω^p-chain contains both a θ-chain and a co-θ-chain, for each $0 < \theta < \omega^p$. Hence, let $(D_{i,j})_{i \leq k, j < 2^i}$ be a ξ-chain, where the Cantor normal form of ξ is given by

$$\xi = \omega^{n_k} \cdot l_k + \omega^{n_{k-1}} \cdot l_{k-1} + \ldots + \omega^{n_0} \cdot l_0.$$

(1) If $k = 0$, two cases occur.

   (i) If $l_k = 1$, then either $\xi = 1$, and there is nothing to prove, or $\xi = \omega^{p+1}$ for some $p \geq 0$, and the above argument leads to the conclusion.

   (ii) If $l_k > 1$, then $\xi = \omega^p \cdot (q+1)$ for some $p, q \geq 0$. By the main argument, the $(\omega^p \cdot (q+1))$-chain contains both an $(\omega^p \cdot q + \theta)$-chain and a co-$(\omega^p \cdot q + \theta)$-chain, for each $\theta < \omega^p$. The induction hypothesis leads to the conclusion.

(2) If $k \neq 0$, then again two cases occur.

   (i) If $n_0 = 0$, then the ordinal ξ is a successor, and either $l_0 = 1$ or $l_0 > 1$. If $l_0 = 1$, the sequence $(D_{i,j})_{0 < i \leq k, 0 \leq j < 2^{i-1}}$ is a $(\xi - 1)$-chain and $(D_{i,j})_{0 < i \leq k, 2^{i-1} \leq j < 2^i}$ is a co-$(\xi-1)$-chain. The induction hypothesis leads to the conclusion. If $l_0 > 1$, the sequence $D_{0,0} = (C_1^{0,0}, \ldots, C_{l_0}^{0,0})$ is an $(\omega^0 \cdot l_0)$-chain. By the main argument, this chain contains both an $(\omega^0 \cdot (l_0 - 1))$-chain and a co-$(\omega^0 \cdot (l_0-1))$-chain. Therefore, by replacing in the ξ-chain the sequence $D_{0,0}$ by each of these two chains, one obtains a $(\xi-1)$-chain and a co-$(\xi - 1)$-chain, respectively. The induction hypothesis leads to the conclusion.

   (ii) If $n_0 > 0$, then either $l_0 = 1$ or $l_0 > 1$. If $l_0 = 1$, then $D_{0,0} = (C_1^{0,0})$ where $C_1^{0,0}$ is an ω^{n_0}-chain. By the main argument, this ω^{n_0}-chain can give rise to both a θ-chain and a co-θ-chain, for each $0 < \theta < \omega^{n_0}$. Therefore, by replacing in the ξ-chain the sequence $D_{0,0}$ by each of these chains, one finds an $(\omega^{n_k} \cdot l_k + \ldots + \omega^{n_1} \cdot l_1 + \theta)$-chain and a co-$(\omega^{n_k} \cdot l_k + \ldots + \omega^{n_1} \cdot l_1 + \theta)$-chain, respectively, for each $0 < \theta < \omega^{n_0}$. The induction hypothesis leads to the conclusion. If $l_0 > 1$, then $D_{0,0} = (C_1^{0,0}, \ldots, C_{l_0}^{0,0})$ is an $(\omega^{n_0} \cdot l_0)$-chain and the root $C_1^{0,0}$ is an ω^{n_0}-chain. Once again, by the main argument, this chain can give rise to a θ-chain, for each $0 < \theta < \omega^{n_0}$. Since both $(C_1^{0,0}, \ldots, C_{l_0-1}^{0,0})$ and $(C_2^{0,0}, \ldots, C_{l_0}^{0,0})$ are accessible from this θ-chain, one can find an $(\omega^{n_k} \cdot l_k + \ldots + \omega^{n_0} \cdot (l_0 - 1) + \theta)$-chain, for each $0 < \theta < \omega^{n_0}$. Similarly, one may also find a co-$(\omega^{n_k} \cdot l_k + \ldots + \omega^{n_0} \cdot (l_0-1) + \theta)$-chain, for each $0 < \theta < \omega^{n_0}$. The induction hypothesis leads to the conclusion. □

Lemma 5.11. *Let $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$ be a complete Muller automaton. Then there exists either a maximal chain, or a maximal co-chain in $\mathcal{A}$. (There may exist both a maximal chain and a maximal co-chain.)*

Proof. First of all, we show that the ordinals associated with the chains and the co-chains of $\mathcal{A}$ are bounded. To this end, notice that an ω^n-chain or a co-ω^n-chain involves at least $n+1$ different states. Therefore, if we set $c = card(Q)$, then there is no α-chain and no co-α-chain in the automaton $\mathcal{A}$, for every $\alpha \geq \omega^c$. Now, consider the least ordinal α such that there is no α-chain and no co-α-chain in $\mathcal{A}$. If α is a successor, that is $\alpha = \xi + 1$, then the minimality of α ensures that

ξ is the largest ordinal such that there exists either a ξ-chain, or a co-ξ-chain, or even both. In addition, since α obviously stands strictly below ω^ω, so does ξ. This concludes the proof in this case. We now show by contradiction that α cannot be limit. Assume that α is a limit ordinal of Cantor normal form

$$\alpha = \omega^{n_k} \cdot l_k + \omega^{n_{k-1}} \cdot l_{k-1} + \ldots + \omega^{n_0} \cdot l_0,$$

where $n_0 > 0$, and consider the ordinal $\beta = \omega^{n_k} \cdot l_k + \omega^{n_{k-1}} \cdot l_{k-1} + \ldots + \omega^{n_0} \cdot (l_0 - 1)$. By minimality of α, there exists either a $(\beta + \omega^{n_0-1} \cdot (c+1))$-chain or a co-$(\beta + \omega^{n_0-1} \cdot (c+1))$-chain denoted by $(D_{i,j})_{i \leq k+1, j < 2^i}$. Let us assume, without loss of generality, that $(D_{i,j})_{i \leq k+1, j < 2^i}$ is a chain, the case of a co-chain is symmetric. Then the sequence $D_{0,0} = (C_1^{0,0}, \ldots, C_{c+1}^{0,0})$ is an $(\omega^{n_0-1} \cdot (c+1))$-chain, and since $card(Q) < c+1$, there exists a state q appearing in both $C_i^{0,0}$ and $C_j^{0,0}$, for some $i < j$. The strictly connected component of q thus contains the two chains $C_i^{0,0}$ and $C_j^{0,0}$. In addition, there exist in this strictly connected component two chains $C_k^{0,0} = R_0 \subset \ldots \subset R_{n_0-1}$ and $C_l^{0,0} = S_0 \subset \ldots \subset S_{n_0-1}$ of opposite acceptance for some $i \leq k < l \leq j$. In particular, the set $R_{n_0-1} \cup S_{n_0-1}$ is either an accepting or a rejecting loop. Therefore, one of the two chains $R_0 \subset \ldots \subset R_{n_0-1} \subset (R_{n_0-1} \cup S_{n_0-1})$ or $S_0 \subset \ldots \subset S_{n_0-1} \subset (R_{n_0-1} \cup S_{n_0-1})$ is either an ω^{n_0}-chain or a co-ω^{n_0}-chain related to both $D_{0,1}$ and $D_{1,0}$. Finally, by replacing in the $(\beta + \omega^{n_0-1} \cdot (c+1))$-chain the sequence $D_{0,0}$ by this new (co-)ω^{n_0}-chain, one obtains either an α-chain, or a co-α-chain. A contradiction. □

Corollary 5.12. *Let $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T})$ be a complete Muller automaton such that Q is a loop. Then, for some integer $n \geq 0$, there is a maximal ω^n-chain in $\mathcal{A}$ if and only if there is no maximal co-ω^n-chain in $\mathcal{A}$.*

Proof. Lemma 5.11 ensures the existence of a maximal chain or a maximal co-chain in $\mathcal{A}$. Consider the largest ordinal $\xi = \omega^n$ such that there exists either an ω^n-chain or a co-ω^n-chain in $\mathcal{A}$. Let us assume without loss of generality that it is an ω^n-chain $R_0 \subset R_1 \subset \ldots \subset R_n$ and that R_n is an accepting loop. We first show that there is no co-ω^n-chain in $\mathcal{A}$. We then show that n is indeed the length of the maximal chains of $\mathcal{A}$. Towards a first contradiction, assume that there also exists an co-ω^n-chain $S_0 \subset S_1 \subset \ldots \subset S_n$. Since Q is a loop, then so is $R_n \cup S_n$. In addition, if $R_n \cup S_n$ is accepting, then $S_0 \subset \ldots \subset S_n \subset R_n \cup S_n$ is a co-ω^{n+1}-chain, a contradiction with the maximality of n. If $R_n \cup S_n$ is rejecting, then $R_0 \subset \ldots \subset R_n \subset R_n \cup S_n$ is an ω^{n+1}-chain, a similar contradiction. Towards a second contradiction, assume that the ω^n-chain is not maximal. Then there exists an (co-)η-chain, for some $\eta > \omega^n$. Lemma 5.10 thus ensures the existence of both an ω^n-chain and a co-ω^n-chain, a contradiction. □

The two forthcoming propositions establish the proof of Theorem 5.9. The left-to-right direction is first proved in the specific case involving automata which contain a single strictly connected component. This result is then extended to the general case.

Lemma 5.13. *Let $\mathcal{A} = (Q_\mathcal{A}, A, \delta_\mathcal{A}, \{i_\mathcal{A}\}, \mathcal{T}_\mathcal{A})$ and $\mathcal{B} = (Q_\mathcal{B}, B, \delta_\mathcal{B}, \{i_\mathcal{B}\}, \mathcal{T}_\mathcal{B})$ be two complete Muller automata recognizing respectively the ω-languages $L(\mathcal{A})$ and $L(\mathcal{B})$, and such that $Q_\mathcal{A}$ and $Q_\mathcal{B}$ are loops. If $L(\mathcal{A}) \leq_W L(\mathcal{B})$, then $\mathcal{B}$ contains more chains and co-chains than $\mathcal{A}$.*

Proof. First of all, since $Q_{\mathcal{B}}$ is a loop, then $q \equiv i_{\mathcal{B}}$, for all $q \in Q_{\mathcal{B}}$. Therefore, the relation $u^{-1}L(\mathcal{B}) \equiv_W L(\mathcal{B})$ holds for all $u \in B^*$, and Proposition 4.8 shows that $L(\mathcal{B})$ is non-self-dual. In addition, Corollary 5.12 ensures the existence of a maximal ω^n-chain $R_0 \subset R_1 \subset \ldots \subset R_n$ in $\mathcal{A}$, for some integer n, and the non-existence of a co-ω^n-chain in $\mathcal{A}$, or vice versa. Let us assume without loss of generality that $R_0 \subset R_1 \subset \ldots \subset R_n$ is a chain, the case of a co-chain being symmetric. By Lemma 5.10, we only need to prove that there also exists an ω^n-chain in $\mathcal{B}$. Towards a contradiction, assume that this is not the case. We show that Player II has a winning strategy in $\mathbb{W}(L(\mathcal{B})^c, L(\mathcal{A}))$, which implies $L(\mathcal{B})^c \leq_W L(\mathcal{A}) \leq_W L(\mathcal{B})$, contradicting the fact that $L(\mathcal{B})$ is non-self-dual. By the hypothesis, since there is no ω^n-chain in $\mathcal{B}$, then either $ht(Q_{\mathcal{B}}) < n$ or $ht(Q_{\mathcal{B}}) = n$ and $Q_{\mathcal{B}} \in \mathcal{T}_{\mathcal{B}}$ if and only if $Q_{\mathcal{A}} \notin \mathcal{T}_{\mathcal{A}}$. If $ht(Q_{\mathcal{B}}) < n \leq ht(Q_{\mathcal{A}})$, then each time Player I runs through a loop $Q \subseteq Q_{\mathcal{B}}$, Player II is able to run through $R_{ht(Q)+1}$. If $Q_{\mathcal{B}} \in \mathcal{T}_{\mathcal{B}}$ if and only if $Q_{\mathcal{A}} \notin \mathcal{T}_{\mathcal{A}}$, then each time Player I runs through a loop $Q \subseteq Q_{\mathcal{B}}$, Player II runs through $R_{ht(Q)}$. In this manner, the plays produced by players I and II have an opposite acceptance. Player II wins $\mathbb{W}(L(\mathcal{B})^c, L(\mathcal{A}))$. □

Proposition 5.14. *Let $\mathcal{A} = (Q_{\mathcal{A}}, A, \delta_{\mathcal{A}}, \{i_{\mathcal{A}}\}, \mathcal{T}_{\mathcal{A}})$ and $\mathcal{B} = (Q_{\mathcal{B}}, B, \delta_{\mathcal{B}}, \{i_{\mathcal{B}}\}, \mathcal{T}_{\mathcal{B}})$ be two complete Muller automata recognizing respectively the ω-languages $L(\mathcal{A})$ and $L(\mathcal{B})$. If $L(\mathcal{A}) \leq_W L(\mathcal{B})$, then $\mathcal{B}$ contains more chains and co-chains than $\mathcal{A}$.*

Proof. We prove that the existence of a ξ-chain in $\mathcal{A}$ implies the existence of another ξ-chain in $\mathcal{B}$. The proof concerning the co-ξ-chains is dual. We proceed by induction on ξ. Let $(D_{i,j})_{i \leq k, j < 2^i}$ be a ξ-chain in $\mathcal{A}$, where $\xi = \omega^{n_k} \cdot l_k + \ldots + \omega^{n_0} \cdot l_0$ and $D_{0,0} = (C_1^{0,0}, \ldots, C_{l_0}^{0,0})$. If $\xi = \omega^0 = 1$, then a ξ-chain in $\mathcal{A}$ is simply an accepting loop. Since $L(\mathcal{A}) \leq_W L(\mathcal{B})$, there also exists an accepting loop in $\mathcal{B}$, which concludes the proof in this case. If $\xi > 1$, then let σ be a winning strategy for Player II in $\mathbb{W}(A, B)$. Now, consider Player II's response to Player I's run that remains forever in the root $C_1^{0,0}$, and let $\sigma(C_1^{0,0})$ denote the states of $Q_{\mathcal{B}}$ visited by Player II during this response. Let also S be the $<$-maximal strictly connected component of $\mathcal{B}$ verifying $S \cap \sigma(C_1^{0,0}) \neq \emptyset$. Since both $C_1^{0,0}$ is an ω^{n_0}-chain in $\mathcal{A}$ and $L(\mathcal{A}) \leq_W L(\mathcal{B})$, Lemma 5.13 ensures that there also exists an ω^{n_0}-chain $S_1^{0,0}$ in $S \subseteq Q_{\mathcal{B}}$. Moreover, for every finite word u and v leading respectively to some states $p \in C_1^{0,0}$ and $q \in S$ – that is $\delta_{\mathcal{A}}(i_{\mathcal{A}}, u) = p \in C_1^{0,0}$ and $\delta_{\mathcal{B}}(i_{\mathcal{B}}, v) = q \in S$ – then $u^{-1}L(\mathcal{A}) \leq_W v^{-1}L(\mathcal{B})$. Now, two cases occur:

(1) If $l_0 > 1$, then the ξ-chain $(D_{i,j})_{i \leq k, j < 2^i}$ with its root $C_1^{0,0}$ off is a co-$(\omega^{n_k} \cdot l_k + \ldots + \omega^{n_0} \cdot (l_0 - 1))$-chain of $\mathcal{A}_p$. By the induction hypothesis, since $\omega^{n_k} \cdot l_k + \ldots + \omega^{n_0} \cdot (l_0 - 1) < \xi$ and $u^{-1}L(\mathcal{A}) \leq_W v^{-1}L(\mathcal{B})$, there also exists a co-$(\omega^{n_k} \cdot l_k + \ldots + \omega^{n_0} \cdot (l_0 - 1))$-chain in $\mathcal{B}_q$. Therefore, by joining the ω^{n_0}-chain $S_1^{0,0}$ and this last co-chain, one finds a ξ-chain in $\mathcal{B}$.
(2) If $l_0 = 1$, then either $k = 0$ or $k > 0$. In the case $k = 0$, Lemma 5.13 leads to the conclusion. In the case $k > 0$, by a similar argument and by the induction hypothesis, there exist an $(\omega^{n_k} \cdot l_k + \ldots + \omega^{n_1} \cdot l_1)$-chain and a co-$(\omega^{n_k} \cdot l_k + \ldots + \omega^{n_0} \cdot l_1)$-chain in $\mathcal{B}$ both accessible from $S_1^{0,0}$. As before, this gives rise to a ξ-chain in $\mathcal{B}$. □

Proposition 5.15. *Let $\mathcal{A} = (Q_{\mathcal{A}}, A, \delta_{\mathcal{A}}, \{i_{\mathcal{A}}\}, \mathcal{T}_{\mathcal{A}})$ and $\mathcal{B} = (Q_{\mathcal{B}}, B, \delta_{\mathcal{B}}, \{i_{\mathcal{B}}\}, \mathcal{T}_{\mathcal{B}})$ be two complete Muller automata recognizing respectively the ω-languages $L(\mathcal{A})$*

and $L(\mathcal{B})$. If $\mathcal{B}$ contains more chains and co-chains than $\mathcal{A}$, then $L(\mathcal{A}) \leq_W L(\mathcal{B})$.

PROOF. The proof is a formal transcription of the following argument. As already mentioned, a play of the Wadge game $\mathbb{W}(L(\mathcal{A}), L(\mathcal{B}))$ can be followed in the DAG representations of $\mathcal{A}$ and $\mathcal{B}$. Hence if $\mathcal{B}$ contains more chains and co-chains than $\mathcal{A}$, then Player II disposes of a DAG representation with a richer alternation of accepting and rejecting loops. Therefore, Player II will always be able to follow Player I's play in order to produce a run of the same acceptance and win the corresponding Wadge game. In technical terms, the proof goes by induction on the ordinal ξ of a maximal ξ-chain or co-ξ-chain of $\mathcal{A}$. We prove that Player II has a winning strategy in $\mathbb{W}(L(\mathcal{A}), L(\mathcal{B}))$. Therefore, $L(\mathcal{A}) \leq_W L(\mathcal{B})$.

(1) If $\xi = \omega^0 = 1$, then all chains and co-chains of $\mathcal{A}$ are reduced to single loops, and by the maximality of ξ, all these loops are pairwise inaccessible. Two cases may occur. Firstly, the automaton $\mathcal{A}$ contains at least an ω^0-chain and no co-ω^0-chain (or vice-versa, the other case is symmetric). By the hypothesis, $\mathcal{B}$ also contains at least one ω^0-chain. Therefore, Player II runs indefinitely through this chain and wins $\mathbb{W}(L(\mathcal{A}), L(\mathcal{B}))$. Secondly, the automaton $\mathcal{A}$ contains at least an ω^0-chain and a co-ω^0-chain. By the hypothesis, $\mathcal{B}$ also contains both an ω^0-chain and a co-ω^0-chain. Hence Player II skips her turn until Player I runs through a loop of $\mathcal{A}$ (this will inevitably occur after a finite amount of time, otherwise there would be a sequence of two accessible loops). Then the inaccessibility between the loops of $\mathcal{A}$ ensures that Player I gets stuck indefinitely inside this loop. Therefore, Player II reaches a loop of the same acceptance and wins $\mathbb{W}(L(\mathcal{A}), L(\mathcal{B}))$.

(2) If $\xi = \omega^{n_k} \cdot l_k + \cdots + \omega^{n_0} \cdot l_0 > 1$, then two cases may occur. Firstly, the automaton $\mathcal{A}$ contains a ξ-chain and no co-ξ-chain (or vice versa). Then by the hypothesis, the automaton $\mathcal{B}$ also contains a ξ-chain. Hence, for every state p reached by Player I, Player II first computes the graph of the automaton $\mathcal{A}_p$. If $\mathcal{A}_p$ still contains a ξ-chain, the position p belongs to a strictly connected component C which contains or accesses an ω^{n_0}-chain $R_0 \subsetneq \ldots \subsetneq R_{n_0}$ (a root of a ξ-chain). By maximality of ξ, there is no ω^n-chain and no co-ω^n-chain in C, for any $n > n_0$ (this would lead to a (co-)η-chain in $\mathcal{A}$, for some $\eta > \xi$). As long as Player I's play persists in such a strictly connected component C, Player II plays in the root of a ξ-chain of $\mathcal{B}$, an ω^{n_0}-chain $S_0 \subsetneq \ldots \subsetneq S_{n_0}$. Every time Player I runs through a loop R_i, Player II marks the loop S_i of the same height – and thus also the same acceptance. If this situation persists indefinitely, the largest loops visited infinitely often by players I and II have the same height and thus the same acceptance. Therefore, Player II wins the game $\mathbb{W}(L(\mathcal{A}), L(\mathcal{B}))$. However, if Player I reaches a state p such that $\mathcal{A}_p$ contains no more ξ-chains, then Player II reaches a position q such that $\mathcal{B}_q$ contains the same chains and co-chains than $\mathcal{A}_p$. The induction hypothesis ensures that $p^{-1}L(\mathcal{A}) \leq_W q^{-1}L(\mathcal{B})$. Therefore, Player II has a winning strategy in $\mathbb{W}(p^{-1}L(\mathcal{A}), q^{-1}L(\mathcal{B}))$, and thus she also has a winning strategy in $\mathbb{W}(L(\mathcal{A}), L(\mathcal{B}))$. Secondly, the automaton $\mathcal{A}$ contains both a ξ-chain and a co-ξ-chain. By the hypothesis, $\mathcal{B}$ also contains both a ξ-chain and a co-ξ-chain. By the maximality of ξ, after a finite amount of time, Player

I's play is forced to go through a state p such that $\mathcal{A}_p$ contains a maximal ξ-chain and no co-ξ-chain, or vice versa (otherwise there would exist a loop accessing both a ξ-chain and a co-ξ-chain, which induces an η-chain or a co-η-chain for some $\eta > \xi$). Hence, Player II first skips her turn until Player I passes such a state p; then she reaches a state q such that $\mathcal{A}_p$ and $\mathcal{B}_q$ contain the same chains and co-chains; she plays as described in the first case and wins $\mathbb{W}(L(\mathcal{A}), L(\mathcal{B}))$. □

The two previous proposition complete the proof of Theorem 5.9. Finally, Proposition 5.16 presents a characterization of self-dual and non-self-dual ω-languages in terms of maximal chains and co-chains. A description of the relative position of some specific self-dual and non-self-dual sets follows.

PROPOSITION 5.16. *Let $\mathcal{A} = (Q, A, \delta, \{i\}, \mathcal{T}$ be a complete Muller automaton recognizing the ω-languages $L(\mathcal{A})$. Then $L(\mathcal{A})$ is self-dual if and only if $\mathcal{A}$ contains both a maximal chain and a maximal co-chain.*

PROOF. The left-to-right direction is a consequence of Proposition 5.14 and Lemma 5.11. Conversely, we prove that if $L(\mathcal{A})$ is non-self-dual, then $\mathcal{A}$ contains a maximal chain and no maximal co-chain, or vice versa. Since $L(\mathcal{A})$ is non-self-dual, Proposition 4.8 establishes the existence of an infinite word $\alpha \in A^\omega$ satisfying $(\alpha[0,n])^{-1}L(\mathcal{A}) \equiv_W L(\mathcal{A})$, for every integer $n \geq 0$. This infinite word induces an infinite path in the graph of $\mathcal{A}$. Since the graph of $\mathcal{A}$ is finite, this infinite path contains at least one loop. Consider then a state q belonging to a $<$-maximal such loop. Then the language $L(\mathcal{A}_q)$ recognized by the automaton $\mathcal{A}_q$ satisfies $L(\mathcal{A}_q) \equiv_W (\alpha[0,n])^{-1}L(\mathcal{A})$ for some integer $n \geq 0$, and therefore $L(\mathcal{A}_q) \equiv_W L(\mathcal{A})$. Hence, by proposition 5.14, the automata $\mathcal{A}$ and $\mathcal{A}_q$ contain the same chains and co-chains. Now, towards a contradiction, assume that the automaton $\mathcal{A}_q$ contains both a maximal ξ-chain and a maximal co-ξ-chain, for some $0 < \xi < \omega^\omega$. Then there exist in $\mathcal{A}_q$ a ξ-chain and a co-ξ-chain which are both accessible from the loop of q. But this gives rise to a (co-)$(\xi+1)$-chain in $\mathcal{A}$. A contradiction to the maximality of ξ. □

COROLLARY 5.17. *Let $\mathcal{A}$ and $\mathcal{B}$ be two complete Muller automata recognizing the ω-languages $L(\mathcal{A})$ and $L(\mathcal{B})$, respectively. The following conditions are equivalent:*

(1) *$\mathcal{A}$ contains a maximal ξ-chain and no maximal co-ξ-chain (or vice-versa), and $\mathcal{B}$ contains both a maximal ξ-chain and a maximal co-ξ-chain, for some $0 < \xi < \omega^\omega$;*
(2) *$L(\mathcal{A})$ is non-self-dual, $L(\mathcal{B})$ is self-dual, $L(\mathcal{A}) <_W L(\mathcal{B})$, and there is no ω-rational language Z such that $L(\mathcal{A}) <_W Z <_W L(\mathcal{B})$.*

PROOF. We first prove (1) implies (2). Proposition 5.16 shows that $L(\mathcal{A})$ is non-self-dual and $L(\mathcal{B})$ is self-dual. Assume that there exists an ω-rational language Z such that $L(\mathcal{A}) <_W Z <_W L(\mathcal{B})$. Then Z is the ω-language recognized by a complete Muller automaton $\mathcal{C}$. Since $Z <_W L(\mathcal{B})$, Theorem 5.9 shows that $\mathcal{B}$ contains more chains and co-chains than $\mathcal{C}$. If $\mathcal{B}$ and $\mathcal{C}$ contain the same chains and co-chains, then Theorem 5.9 ensures that $L(\mathcal{B}) \equiv_W Z$, a contradiction. If $\mathcal{B}$ and $\mathcal{C}$ do not contain the same chains and co-chains, then $\mathcal{C}$ contains a maximal η-chain or a maximal co-η-chain, for some $0 < \eta < \xi$. Therefore, Lemma 5.10 and Theorem 5.9 show that $Z \leq_W L(\mathcal{A})$, a contradiction. We now prove (2) implies (1). Since $L(\mathcal{A})$ is non-self-dual and $L(\mathcal{B})$ is self-dual, $\mathcal{A}$ contains a maximal ξ-chain and no maximal co-ξ-chain (or vice-versa), for some $0 < \xi < \omega^\omega$, and $\mathcal{B}$ contains both a maximal η-chain and a maximal co-η-chain, for some $0 < \eta < \omega^\omega$. By Lemma 5.10 and Theorem 5.9, the relation $L(\mathcal{A}) <_W L(\mathcal{B})$ implies $\xi \leq \eta$. We prove that $\xi < \eta$ leads to a contradiction, and therefore $\xi = \eta$. Consider a complete Muller automaton $\mathcal{C}$ which contains both a maximal ξ-chain and a maximal co-ξ-chain. Proposition 5.16 shows that $L(\mathcal{C})$ is self-dual. Since $L(\mathcal{A})$ is non-self-dual and $L(\mathcal{C})$ contains more chains and co-chains than $L(\mathcal{A})$, Theorem 5.9 ensures that $L(\mathcal{A}) <_W L(\mathcal{C})$. In addition, since $\xi < \eta$, Lemma 5.10 and Theorem 5.9 show that $L(\mathcal{C}) <_W L(\mathcal{B})$, a contradiction. Therefore, $\xi = \eta$. □

5.4 Description of the Wagner hierarchy

The Wagner hierarchy is the trace of the Wadge hierarchy on ω-rational sets. It consists of the collection of every ω-rational sets ordered by the Wadge reduction. Since every ω-rational language is Borel, the Wagner hierarchy appears as a restriction of the Borel Wadge hierarchy, so that up to complementation and Wadge equivalence, the Wagner hierarchy is a well ordering. Therefore, there exist a unique ordinal, called the *height* of the Wagner hierarchy, and a mapping d_{WAG} from the Wagner hierarchy onto its height, called the *Wagner degree*, such that $d_{WAG}(X) < d_{WAG}(Y)$ if and only if $X <_W Y$ and $d_{WAG}(X) = d_{WAG}(Y)$ if and only if $X \equiv_W Y$ or $X \equiv_W Y^c$, for every ω-rational languages X and Y. As in the Wadge framework, we consider a modified Wagner degree which attaches the self-dual sets to the non-self-dual ones located just one level below in the hierarchy.

$$d_{wag}(X) = \begin{cases} 1 & \text{if } X = \emptyset \text{ or } X = \emptyset^c, \\ \sup\{d_{wag}(Y) + 1 \mid Y \text{ n.s.d. and } Y <_W X\} & \text{if } X \text{ is non-self-dual,} \\ \sup\{d_{wag}(Y) \mid Y \text{ n.s.d. and } Y <_W X\} & \text{if } X \text{ is self-dual.} \end{cases}$$

The Wagner hierarchy also consists of an alternating succession of non-self-dual and self-dual sets, as illustrated in Figure 5.5, where circles denote the Wadge equivalence classes, and arrows stand for the Wadge reduction. This hierarchy refines the three lower levels of the Borel hierarchy, since every ω-rational set is a boolean combination of $\mathbf{\Sigma^0_2}$ or $\mathbf{\Pi^0_2}$ sets.

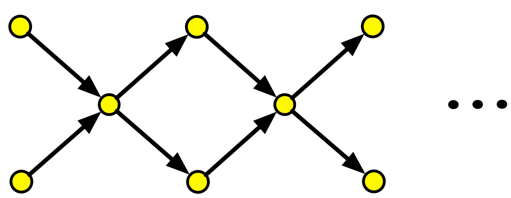


Figure 5.5: The Wagner hierarchy.

We now fulfill the description of the Wagner hierarchy. We prove that the Wagner degree of an ω-rational language is precisely the ordinal associated with the maximal chains and/or co-chains contained in the underlying Muller automaton. Consequently, the Wagner hierarchy has height ω^ω, and is decidable.

THEOREM 5.18. *Let $\mathcal{A}$ be a complete Muller automaton recognizing the language $L(\mathcal{A})$, and let ξ be an ordinal such that $0 < \xi < \omega^\omega$. Then $d_{wag}(L(\mathcal{A})) = \xi$ if and only if*

- *either $\mathcal{A}$ contains a maximal ξ-chain and no maximal co-ξ-chain,*
- *or $\mathcal{A}$ contains a maximal co-ξ-chain and no maximal ξ-chain,*
- *or $\mathcal{A}$ contains both a maximal ξ-chain and a maximal co-ξ-chain.*

PROOF. The Wagner degree is defined such that $d_{wag}(L(\mathcal{A})) = d_{wag}(L(\mathcal{B}))$ if and only if either $L(\mathcal{A}) \equiv_W L(\mathcal{B})$, or $L(\mathcal{A}) \equiv_W L(\mathcal{B})^c$, or $L(\mathcal{A})$ is non-self-dual and $L(\mathcal{B})$ is a self-dual set just $\leq_W$-above $L(\mathcal{A})$ (in the sense of Corollary 5.17), or symmetrically, $L(\mathcal{B})$ is non-self-dual and $L(\mathcal{A})$ is a self-dual set just $\leq_W$-above $L(\mathcal{B})$. In addition, Theorem 5.9 and Corollary 5.17 show that the disjunction of these four conditions holds if and only if each of the automata $\mathcal{A}$ and $\mathcal{B}$ contains either a maximal ξ-chain, or a maximal co-ξ-chain, or both of them, for some $0 < \xi < \omega^\omega$. Thus, $d_{wag}(L(\mathcal{A})) = d_{wag}(L(\mathcal{B}))$ if and only if each of the automata $\mathcal{A}$ and $\mathcal{B}$ contains either a maximal ξ-chain, or a maximal co-ξ-chain, or both of them. Moreover, for any $0 < \xi < \omega^\omega$, there exists an automaton $\mathcal{A}$ containing a maximal ξ-chain. Therefore, $d_{wag}(L(\mathcal{A})) = \xi$ if and only if $\mathcal{A}$ contains either a maximal ξ-chain, or a maximal co-ξ-chain, or both of them. □

COROLLARY 5.19. *The Wagner hierarchy has height ω^ω.*

PROOF. Theorem 5.18 ensures that the Wagner degree is bounded by ω^ω. In addition, for any ordinal $0 < \xi < \omega^\omega$, one can find an ω-rational language whose Wagner degree is equal to ξ. □

COROLLARY 5.20. *The Wagner hierarchy is decidable.*

PROOF. Since the graph of a Muller automaton is finite, the maximal chains and co-chains are effectively computable. Theorem 5.18 leads to the conclusion. □

The decidability of the Wagner hierarchy, obtained by computing the maximal chains and co-chains in Muller automata, is actually a reformulation of *Wagner's naming procedure*, described in [41, 43]. Wilke and Yoo described an efficient algorithm computing the name – or Wagner degree – of every ω-rational language [43]. We conclude with the following example.

EXAMPLE 5.21. Let $\mathcal{A}$ be the Muller automaton illustrated in Example 5.8. The language $L(\mathcal{A})$ that it recognizes satisfies $d_{wag}(L(\mathcal{A})) = \omega^1 \cdot 3 + \omega^0 \cdot 3$, and is non-self-dual.

5.5 The Wagner degree as a syntactic invariant

This section aims to prove that the Wagner degree is a *syntactic invariant*: if two ω-rational languages have the same syntactic pointed ω-semigroups, then they also have the same Wagner degree. This result is a partial generalization of

Proposition 3.31 to the case of ω-semigroups. It ensures that the Wagner degree of ω-rational languages can be characterized by an algebraic invariant on their syntactic pointed ω-semigroups. The description of this algebraic invariant will be presented in the sequel.

LEMMA 5.22. *Let $\varphi = (\varphi_+, \varphi_\omega) : A^\infty \longrightarrow B^\infty$ be a morphism of free ω-semigroups. Then $\varphi_\omega : A^\omega \longrightarrow B^\omega$ is continuous.*

PROOF. We prove that the inverse image by φ_ω of any open set of B^ω is an open set of A^ω. Let $V \subseteq B^*$, and let $\alpha = a_0 a_1 a_2 \cdots \in A^\omega$, one has

$$\begin{aligned}
\alpha \in \varphi^{-1}(VB^\omega) \quad &\Leftrightarrow \quad \varphi(\alpha) \in VB^\omega \\
&\Leftrightarrow \quad \varphi(a_0)\varphi(a_1)\varphi(a_2) \cdots \in VB^\omega \\
&\Leftrightarrow \quad \text{there exists } n \geq 0 \text{ such that } \varphi(a_0) \cdots \varphi(a_n) \in V \\
&\Leftrightarrow \quad \text{there exists } n \geq 0 \text{ such that } \varphi(a_0 \cdots a_n) \in V \\
&\Leftrightarrow \quad \text{there exists } n \geq 0 \text{ such that } a_0 \cdots a_n \in \varphi^{-1}(V) \\
&\Leftrightarrow \quad \alpha = a_0 a_1 a_2 \cdots \in \varphi^{-1}(V) A^\omega
\end{aligned}$$

Thus $\varphi_\omega^{-1}(VB^\omega) = \varphi_+^{-1}(V)A^\omega$, meaning that the inverse image by φ_ω of any open set of B^ω is an open set of A^ω. Therefore, φ_ω is continuous. $\square$

PROPOSITION 5.23. *Let K and L be two ω-rational languages of A^ω and B^ω, respectively. If $Synt(K)$ divides $Synt(L)$, then $K \leq_W L$.*

PROOF. Let μ and ν be the syntactic morphisms of K and L, respectively. If $Synt(K)$ divides $Synt(L)$, then there exist a pointed ω-semigroup (S, P), an injective morphism $\iota : (S, P) \longrightarrow Synt(L)$, and a surjective morphism $\sigma : (S, P) \longrightarrow Synt(K)$, as illustrated below. In particular, since σ and ι are mor-

$$\begin{tikzcd}
(A^\infty, K) \arrow[rr, dashed, "g"] \arrow[d, "\mu"] \arrow[dr, dashed, "f"] & & (B^\infty, L) \arrow[d, "\nu"] \\
Synt(K) & (S, P) \arrow[l, "\sigma"'] \arrow[r, "\iota"] & Synt(L)
\end{tikzcd}$$

phisms of pointed ω-semigroups, both equalities $\sigma^{-1}(\mu(K)) = P = \iota^{-1}(\nu(L))$ hold. Now, since A^∞ is free and σ is surjective, Corollary 3.37 ensures that there exists a morphism of ω-semigroups $f : A^\infty \to S$ such that $\sigma \circ f = \mu$. Moreover, since μ is the syntactic morphism of K, then

$$f^{-1}(P) = f^{-1}(\sigma^{-1}(\mu(K))) = \mu^{-1}(\mu(K)) = K.$$

Thus $f : (A^\infty, K) \longrightarrow (S, P)$ is a morphism of pointed ω-semigroups. By composition, the mapping $\iota \circ f$ from (A^∞, K) into $Synt(L)$ is a also morphism of pointed ω-semigroups. Once again, since A^∞ is free and ν is surjective, there exists a morphism of free ω-semigroups $g = (g_+, g_\omega) : A^\infty \longrightarrow B^\infty$ such that

$\nu \circ g = \iota \circ f$. Lemma 5.22 shows that g_ω is continuous. Moreover, since ν is the syntactic morphism of L, one has

$$g^{-1}(L) = g^{-1}(\nu^{-1}(\nu(L))) = f^{-1}(\iota^{-1}(\nu(L))) = f^{-1}(P) = K.$$

Therefore, $K \leq_W L$. □

COROLLARY 5.24. *If two ω-rational languages have the same syntactic pointed ω-semigroup, then they have the same Wagner degree.*

PROOF. A direct consequence of Proposition 5.23. □

Chapter 6

The $\mathbb{SG}$-hierarchy

Summary

We define a reduction relation on pointed ω-semigroups by means of an infinite two-player game inspired by the Wadge game. This reduction induces a hierarchy of Borel ω-subsets, called the $\mathbb{SG}$-hierarchy. Most of the results of the Wadge theory presented in Chapter 4 also apply in this framework, and provide a detailed description of the $\mathbb{SG}$-hierarchy.

6.1 The $\mathbb{SG}$-game

Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two ω-semigroups, and let $X \subseteq S_\omega$ and $Y \subseteq T_\omega$ be two ω-subsets. The game $\mathbb{SG}((S,X),(T,Y))$ [3] is an infinite two-player game with perfect information, where Player I is in charge of X, Player II is in charge of Y, and players I and II alternately play elements of S_+ and $T_+ \cup \{-\}$, respectively. Player I begins. Unlike Player I, Player II is allowed to skip her turn – denoted by the symbol "$-$" –, provided she plays infinitely many moves. After ω turns each, players I and II produced respectively two infinite sequences $(s_0, s_1, \ldots) \in S_+^\omega$ and $(t_0, t_1, \ldots) \in T_+^\omega$. The winning condition is given as follows: Player II wins $\mathbb{SG}((S,X),(T,Y))$ if and only if $\pi_S(s_0, s_1, \ldots) \in X \Leftrightarrow \pi_T(t_0, t_1, \ldots) \in Y$. From this point onward, the game $\mathbb{SG}((S,X),(T,Y))$ will be denoted by $\mathbb{SG}(X,Y)$ and the ω-semigroups involved will always be known from the context. A play in this game is illustrated below.

$$
\begin{array}{lccccccc}
(X)\ \mathrm{I}\ : & s_0 & & & s_1 & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & (s_0, s_1, s_2, \ldots) \\
 & & \searrow & \nearrow & & & & \\
(Y)\ \mathrm{II}\ : & & t_0 & & & \cdots\cdots & \xrightarrow{\text{after } \omega \text{ moves}} & (t_0, t_1, t_2, \ldots)
\end{array}
$$

A player is said to be *in position* s if the product of his/her previous moves $(s_1, \ldots, s_n)$ is equal to s. A *strategy* for Player I is a mapping $\sigma : (T_+ \cup \{\varepsilon\})^* \longrightarrow S_+$. A *strategy* for Player II is a mapping $\sigma : S_+^+ \longrightarrow T_+ \cup \{\varepsilon\}$. A *winning strategy* for a given player is a strategy such that this player always wins when using it. Notice finally that a player in charge of the set $s^{-1}X$ is exactly as strong as a player in charge of X, but having already reached position s.

Similarly to the Wadge framework, the $\mathbb{SG}$-*reduction* is defined by $X \leq_{SG} Y$ if and only if Player II has a winning strategy in $\mathbb{SG}(X, Y)$. As usual, we set $X \equiv_{SG} Y$ if and only if $X \leq_{SG} Y$ and $Y \leq_{SG} X$, and also $X <_{SG} Y$ if and only if $X \leq_{SG} Y$ and $X \not\equiv_{SG} Y$. An ω-subset X is called self-dual if $X \leq_{SG} X^c$, and non-self-dual otherwise. A straightforward generalization of Lemma 4.1 shows that the relation $\leq_{SG}$ is reflexive and transitive, and that $\equiv_{SG}$ is an equivalence relation. Lemmas 4.2 and 4.3 also apply in this context.

LEMMA 6.1. *Let $S = (S_+, S_\omega)$ be an ω-semigroup and let $X \subseteq S_\omega$.*

(1) *If $X \neq S_\omega$, then $\emptyset \leq_{SG} X$.*
(2) *If $X \neq \emptyset$, then $S_\omega \leq_{SG} X$.*
(3) *$\emptyset$ and S_ω are incomparable.*

PROOF.

(1) We describe a winning strategy for Player II in the game $\mathbb{SG}(\emptyset, X)$. At the end of the play, the infinite product of the infinite sequence played by I does obviously not belong to $\emptyset$. Hence, the winning strategy for II consists in playing an infinite sequence $(s_0, s_1, s_2, \ldots)$ such that $\pi_S(s_0, s_1, s_2, \ldots) \notin X$. This is possible, since $X \neq S_\omega$.
(2) Similarly, we describe a winning strategy for Player II in $\mathbb{SG}(S_\omega, X)$. At the end of the play, the infinite product of the infinite sequence played by Player I obviously belongs to S_ω. Therefore, Player II wins the game by playing an infinite sequence $(s_0, s_1, s_2, \ldots)$ such that $\pi_S(s_0, s_1, s_2, \ldots) \in X$. This is possible, since $X \neq \emptyset$.
(3) We first show that Player II has no winning strategy in the game $\mathbb{SG}(\emptyset, S_\omega)$. At the end of the play, the infinite product of Player I's infinite sequence does not belong to $\emptyset$, whereas the infinite product of Player II's infinite sequence belongs to S_ω. Thus $\emptyset \not\leq_{SG} S_\omega$. The same argument shows that $S_\omega \not\leq_{SG} \emptyset$. □

LEMMA 6.2. *Let $S = (S_+, S_\omega)$ and $T = (S_+, S_\omega)$ be two ω-semigroups, and let $X \subseteq S_\omega$ and $Y \subseteq T_\omega$.*

(1) *$X \leq_{SG} Y$ if and only if $X^c \leq_{SG} Y^c$.*
(2) *X and X^c are either equivalent, or incomparable.*
(3) *If $X <_{SG} Y$, then both $Y \not\leq_{SG} X$ and $Y^c \not\leq_{SG} X$.*

PROOF.

(1) By definition of the winning conditions of the $\mathbb{SG}$-game, a strategy is winning for Player II in $\mathbb{SG}(X, Y)$ if and only if it is also winning for Player II in $\mathbb{SG}(X^c, Y^c)$.
(2) Either $X \leq_{SG} X^c$, or $X \not\leq_{SG} X^c$. If $X \leq_{SG} X^c$, then (1) shows that $X^c \leq_{SG} X$, thus $X \equiv_{SG} X^c$. If $X \not\leq_{SG} X^c$, then (1) shows that $X^c \not\leq_{SG} X$, hence X and X^c are incomparable.
(3) If $X <_{SG} Y$, then $Y \not\leq_{SG} X$ by definition. Now, assume that $Y^c \leq_{SG} X$. Then $Y^c \leq_{SG} X$ and $X <_{SG} Y$ imply $Y^c <_{SG} Y$, a contradiction with (2). □

EXAMPLE 6.3. The ω-subsets $X = \{0^\omega\}$ and $Y = \{a^\omega\}$ given respectively in examples 3.56 and 3.57 satisfy $X \leq_{SG} Y$. Indeed, Player II has a winning strategy in the game $\mathbb{SG}(X, Y)$. First of all, regardless of Player I's first move, Player II answers with the element a. Afterwards, as long as Player I stays in position 1, Player II plays the element c. If this situation persists until the end of the play, then players I and II respectively produce the elements $1^\omega \notin X$ and $ac^\omega = 0 \notin Y$. Therefore, Player II wins the game. Now, if Player I reaches position 0, then Player II stays in position a, but answers as follows: when Player I plays 1, Player II plays c, and when Player I plays 0, Player II plays ca. Therefore, at the end of the play, two cases may occur: either players I and II respectively produce $01^\omega = 1^\omega \notin X$, and $ac^\omega = 0 \notin Y$, thus Player II wins the game, or they respectively produce $00^\omega = 0^\omega \in X$, and $a(ca)^\omega = a^\omega \in Y$, and Player II also wins the game.

6.2 The $\mathbb{SG}$-hierarchy

The collection of Borel ω-subsets ordered by the $\leq_{SG}$-relation is called *the* $\mathbb{SG}$-*hierarchy*, in order to underline the semigroup approach. Notice that the restriction of the $\mathbb{SG}$-hierarchy to Borel ω-subsets of free ω-semigroups is exactly the Borel Wadge hierarchy. The restriction of the $\mathbb{SG}$-hierarchy to ω-subsets of finite ω-semigroups will be called *the* $\mathbb{FSG}$-*hierarchy*, in order to underline the finiteness of the ω-semigroups involved. We will further on particularly focus on this hierarchy. Now, we prove that Borel Wadge determinacy implies the determinacy of $\mathbb{SG}$-games for every Borel winning sets. As in the Wadge framework, this result induces several consequences on the $\mathbb{SG}$-hierarchy.

THEOREM 6.4 ($\mathbb{SG}$-BOREL DETERMINACY). *Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two ω-semigroups, and let $X \subseteq S_\omega$ and $Y \subseteq T_\omega$ be two Borel ω-subsets. The game $\mathbb{SG}(X, Y)$ is determined.*

PROOF. By definition, since X and Y are Borel, the sets $\pi_S^{-1}(X)$ and $\pi_T^{-1}(Y)$ are respectively Borel subsets of S_+^ω and T_+^ω. In addition, a given player has a winning strategy in the game $\mathbb{SG}(X, Y)$ if and only if this same player has a winning strategy in the game $\mathbb{W}(\pi_S^{-1}(X), \pi_T^{-1}(Y))$. The Borel determinacy of Wadge games leads to the conclusion. $\square$

Propositions 4.6 and 4.7 can be easily reformulated in case of the $\mathbb{SG}$-reduction: the $\leq_{SG}$-antichains have length at most two, and the $\leq_{SG}$-relation is wellfounded on Borel ω-subsets.

PROPOSITION 6.5. *Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two ω-semigroups, and let $X \subseteq S_\omega$ and $Y \subseteq T_\omega$ be two Borel ω-subsets.*

(1) *(Wadge's Lemma) Either $X \leq_{SG} Y$, or $Y \leq_{SG} X^c$.*
(2) *If X and Y are incomparable, then $X \equiv_{SG} Y^c$.*
(3) *The $\leq_{SG}$-antichains have length at most two.*

PROOF.

(1) Either $X \leq_{SG} Y$, or $X \not\leq_{SG} Y$. If $X \not\leq_{SG} Y$, then Player II has no winning strategy in $\mathbb{SG}(X, Y)$. Hence, by determinacy, Player I has a winning strategy σ in this game. Therefore, Player II has the following winning strategy in $\mathbb{SG}(Y, X^c)$: she plays $\sigma(\varepsilon)$ on her first move, and then, she answers to every current position $(x_0, \ldots, x_n)$ of Player I by the move $\sigma(x_0 \cdots x_{n-1})$. Thus $Y \leq_{SG} X^c$.

(2) If $X \not\leq_{SG} Y$ and $Y \not\leq_{SG} X$, then (1) implies that $Y \leq_{SG} X^c$ and $X \leq_{SG} Y^c$. Therefore, $Y \leq_{SG} X^c$ and $X^c \leq_{SG} Y$, hence $X^c \equiv_{SG} Y$.

(3) Let X, Y, and Z be three ω-subsets such that $X \not\leq_{SG} Y$ and $Y \not\leq_{SG} Z$. Then point (1) shows that $Y \leq X^c$ and $Z \leq_{SG} Y^c$. Therefore, $Z \leq_{SG} Y^c$ and $Y^c \leq X$, thence $Z \leq_{SG} X$. □

Proposition 6.6 (Martin, Monk). *The partial ordering $\leq_{SG}$ is wellfounded on Borel ω-subsets.*

Proof. Towards a contradiction, assume that there exists an infinite sequence of ω-semigroups $(S_i = (S_{i,+}, S_{i,\omega}))_{i\geq 0}$, and an infinite strictly $<_{SG}$-descending sequence of Borel ω-subsets $X_0 >_{SG} X_1 >_{SG} X_2 \ldots$, where $X_i \subseteq S_{i,\omega}$, for all $i \geq 0$. By Lemma 6.2 (3), the relation $X_n >_{SG} X_{n+1}$ implies that $X_n \not\leq_{SG} X_{n+1}$ and $X_n^c \not\leq_{SG} X_{n+1}$, for all $n \geq 0$. Therefore, by determinacy, Player I has the winning strategies σ_n^0 and σ_n^1 in the respective games $\mathbb{SG}(X_n, X_{n+1})$ and $\mathbb{SG}(X_n^c, X_{n+1})$, for all $n \geq 0$. Now, for any $\alpha \in \{0,1\}^\omega$, consider the infinite sequence of strategies $(\sigma_n^{\alpha(n)})_{n\geq 0}$, and the infinite sequence of games $(\mathbb{SG}(X_n^{c(\alpha(n))}, X_{n+1}))_{n\geq 0}$ related as follows: in the game $\mathbb{SG}(X_k^{c(\alpha(k))}, X_{k+1})$, Player I applies his winning strategy $\sigma_k^{\alpha(k)}$, and Player II copies Player I's moves in the next game $\mathbb{SG}(X_{k+1}^{c(\alpha(k+1))}, X_{k+2})$. Hence, in the first game, Player I applies his winning strategy $\sigma_0^{\alpha(0)}$. Since it is a strategy for Player I, it gives the first letter a_0^0 before Player II has ever played anything. Then Player II copies Player I's first move a_0^1 of the second game, and Player I answers with his winning strategy. And so on and so forth, for every move and every game. This infinite sequence of games is illustrated below. Big arrows denote the action of playing, and little ones stand for copying.

$$
\begin{array}{ccccccccccccc}
I & \overset{\sigma_0^{\alpha(0)}}{\curvearrowleft} & II & & I & \overset{\sigma_1^{\alpha(1)}}{\curvearrowleft} & II & & I & \overset{\sigma_2^{\alpha(2)}}{\curvearrowleft} & II & & \cdots \\
a_0^0 & & & & a_0^1 & & & & a_0^2 & & & & \\
 & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & & \\
 & & a_0^1 & & & & a_0^2 & & & & a_0^3 & & \\
 & \swarrow & & & & \swarrow & & & & \swarrow & & & \\
a_1^0 & & & & a_1^1 & & & & a_1^2 & & & & \\
 & \searrow & & \swarrow & & \searrow & & \swarrow & & & & & \\
 & & a_1^1 & & & & a_1^2 & & & & & & \\
 & \swarrow & & & & \swarrow & & & & & & & \\
a_2^0 & & & & a_2^1 & & & & & & & & \\
 & \searrow & & \swarrow & & & & & & & & & \\
 & & a_2^1 & & & & & & & & & & \\
 & \swarrow & & & & & & & & & & & \\
a_3^0 & & & & & & & & & & & &
\end{array}
$$

Let $x_\alpha = a_0^0 a_1^0 a_2^0 \cdots$ be the infinite word played by Player I in the first game, let

$$\varphi : \{0,1\}^\omega \longrightarrow S_{0,+}^\omega$$

be defined by $\varphi(\alpha) = x_\alpha$, and let

$$\psi = \pi_{S_0} \circ \varphi : \{0,1\}^\omega \longrightarrow S_{0,\omega}$$

be defined by $\psi(\alpha) = \pi_{S_0}(x_\alpha) = \pi_{S_0}(a_0^0, a_1^0, a_2^0, \ldots)$. We show that φ is continuous. By definition of these chained games, the k first letters of x_α only depend on the k first letters of α, since we do not need games number $k+1, k+2, \ldots$ to determine $x_\alpha[0,k]$. Thus, for any $U \subseteq S_{0,+}^*$, one has $\varphi^{-1}(US_{0,+}^\omega) = V\{0,1\}^\omega$, with $V \subseteq \{0,1\}^*$, and hence the preimage by φ of any open set is an open set. Now, since φ and π_{S_0} are continuous, then so is ψ. Consider $F = \psi^{-1}(X_0)$. By construction of these chained games, F is a flip set, because if α and α' only differ by one position (meaning if there exists a unique i such that $\alpha(i) \neq \alpha'(i)$), then $\alpha \in F$ if and only if $\alpha' \notin F$. On the other hand, the set F is also Borel, since ψ is continuous. A contradiction. □

Propositions 6.5 and 6.6 show that, up to complementation and $\leq_{SG}$-equivalence, the $\mathbb{SG}$-hierarchy is a well ordering. Therefore, there exist a unique ordinal, called the *height* of the $\mathbb{SG}$-hierarchy, and a mapping d_{SG} from the $\mathbb{SG}$-hierarchy onto its height, called the $\mathbb{SG}$-*degree*, such that $d_{SG}(X) < d_{SG}(Y)$ if and only if $X <_{SG} Y$, and $d_{SG}(X) = d_{SG}(Y)$ if and only if $X \equiv_{SG} Y$ or $X \equiv_{SG} Y^c$, for every Borel ω-subsets X and Y. The wellfoundness of the $\mathbb{SG}$-hierarchy ensures that the $\mathbb{SG}$-degree can be defined by induction as follows:

$$d_{SG}(X) = \begin{cases} 0 & \text{if } X = \emptyset \text{ or } X = \emptyset^c, \\ \sup\{d_{SG}(B) + 1 : B <_{SG} A\} & \text{otherwise.} \end{cases}$$

Moreover, the $\mathbb{SG}$-hierarchy has the same familiar "scaling shape" as the Borel or Wadge hierarchies: an increasing sequence of non-self-dual sets with self-dual sets in between, as illustrated in Figure 6.1, where circles represent the $\equiv_{SG}$-equivalence classes of Borel ω-subsets, and arrows stand for the $<_{SG}$-relation.

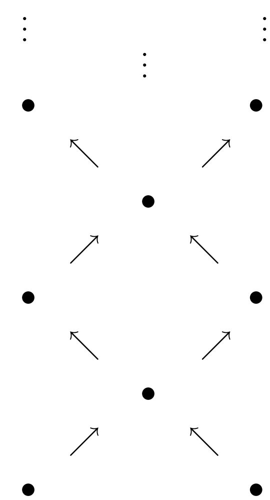


Figure 6.1: The $\mathbb{SG}$-hierarchy.

From this point onward, we will specially focus on the description of the $\mathbb{FSG}$-hierarchy. In this context, the following results present a game theoretical characterization of the self-dual and the non-self-dual ω-subsets of finite ω-semigroups. We first introduce the following notions.

Given a finite ω-semigroup $S = (S_+, S_\omega)$, an ω-subset $X \subseteq S_\omega$, and two elements $s, e \in S_+$: we say that s is a *prefix position* if s is a prefix of some linked pair of S_+^2; we say that e is a *waiting move for the prefix position* s if (s, e) is a linked pair; we say that s is a *critical position for* X if $s^{-1}X <_{SG} X$. We finally also introduce the imposed game $\overline{\mathbb{SG}}(\_,\_)$, very similar to $\mathbb{SG}(\_,\_)$, except that Player I is allowed to skip his turn, provided he plays infinitely often, whereas Player II is not allowed to do so, and is forced to play from one prefix position to another. This infinite game induces the reduction relation $\leq_{\overline{SG}}$ defined as usual by $X \leq_{\overline{SG}} Y$ if and only if Player II has a winning strategy in $\overline{\mathbb{SG}}(X, Y)$. The following results prove that an $\mathbb{SG}$-player is in charge of a self-dual ω-subset if and only if he his forced to reach some critical position for this set. Equivalently, an $\mathbb{SG}$-player is in charge of a non-self-dual ω-subset if and only if he has the possibility to remain indefinitely as strong as in his initial position. As a corollary, every self-dual set can be written as a finite union of $<_{SG}$-smaller non-self-duals sets.

Lemma 6.7. *Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two finite ω-semigroups, let $X \subseteq S_\omega$ and $Y \subseteq T_\omega$, and let s be a prefix of a linked pair of T_+^2. Then*

$$X \leq_{SG} s^{-1}Y \text{ if and only if } X \leq_{\overline{SG}} s^{-1}Y.$$

Proof.

($\Leftarrow$) Notice that Player II is more constrained in the $\overline{\mathbb{SG}}$-game than in the $\mathbb{SG}$-game. Hence, if Player II has a winning strategy in $\overline{\mathbb{SG}}(X, s^{-1}Y)$, then she also has a winning strategy in $\mathbb{SG}(X, s^{-1}Y)$.

($\Rightarrow$) In the game $\overline{\mathbb{SG}}(X, s^{-1}Y)$, we may assume that Player II is in charge of the subset Y, and is already in the prefix position s in the beginning of the play. Now, given a winning strategy σ for Player II in $\mathbb{SG}(X, s^{-1}Y)$, we describe a winning strategy for Player II in $\overline{\mathbb{SG}}(X, s^{-1}Y)$. For that purpose, let $a_0, a_1, a_2, \ldots$ denote the subsequence of non-skipping moves played by Player I in $\overline{\mathbb{SG}}(X, s^{-1}Y)$, and let $b_i = \sigma(a_0, \ldots, a_i)$ be the answers of Player II in the other game $\mathbb{SG}(X, s^{-1}Y)$, for all $i \geq 0$. Then, while I begins to play his very first successive moves, II first waits in her initial prefix position s by playing an idempotent e such that $se = s$. As soon as I's moves induce an answer $b_0 \cdots b_m$ such that $b_0 \cdots b_{k-1} = s'$, $b_k \cdots b_m = e'$, and (s', e') is a linked pair, then II either stays in or reaches position s'. She then waits in this position by playing the idempotent e' until I's moves induce another finite word $b_0 \cdots b_n$, with $n > m$, such that $b_0 \cdots b_{m+i} = s''$, $b_{m+i+1} \cdots b_n = e''$, $i \geq 0$, and (s'', e'') is a linked pair. As before, she either stays in or reaches position s'' by playing the element $(b_{m+1} \cdots b_{m+i})$, when it exists, and waits in this position for another similar situation by playing the idempotent e''. And so on and so forth. Proposition 3.21 shows that this configuration is forced to happen again and again along the play, so that this strategy is well defined. In the end, the infinite word played by Player II is a factorization of the infinite word $b_0 b_1 b_2 \ldots$. Corollary 3.23 shows that these two infinite words have the same image under the infinite product π_T. Therefore, since σ is winning for Player II in $\mathbb{SG}(X, s^{-1}Y)$, the strategy described above is aslo winning for II in $\overline{\mathbb{SG}}(X, s^{-1}Y)$. Hence $X \leq_{\overline{SG}} s^{-1}Y$. □

PROPOSITION 6.8. *Let $S = (S_+, S_\omega)$ be a finite ω-semigroup, and let $X \subseteq S_\omega$. The following conditions are equivalent:*

(1) *X is non-self-dual.*
(2) *$X \leq_{\overline{SG}} X$.*
(3) *There exists a prefix s of a linked pair of S_+^2 such that $X \equiv_{SG} s^{-1}X$.*

PROOF.

(2) $\Rightarrow$ (1) Given a winning strategy σ for Player II in $\overline{\mathbb{SG}}(X, X)$, we describe a winning strategy for Player I in $\mathbb{SG}(X, X^c)$: Player I first plays $\sigma(-)$, and then applies σ to Player II's moves. He wins.

(1) $\Rightarrow$ (2) Conversely, given a winning strategy σ for Player I in $\mathbb{SG}(X, X^c)$, we describe a winning strategy for Player II in $\overline{\mathbb{SG}}(X, X)$: she first computes the moves $\sigma(\varepsilon), \sigma(-), \sigma(-,-), \sigma(-,-,-)\ldots$, and plays the first of these elements which is a prefix position. Notice that such a move always exists, since S_+ is finite. From this prefix position, she then applies σ to Player I's moves, but restricts herself to playing from one prefix position to another, exactly as described in Lemma 6.7. She wins the game.

(3) $\Rightarrow$ (2) Given any element $s \in S_+$, the relation $s^{-1}X \leq_{SG} X$ always holds. Indeed, the winning strategy for Player II consists in first playing s, and then copying Player I's moves. The relation $X \equiv_{SG} s^{-1}X$ is thus equivalent to $X \leq_{SG} s^{-1}X$, and Lemma 6.7 ensures that $X \leq_{SG} s^{-1}X$ if and only if $X \leq_{\overline{SG}} s^{-1}X$, for any prefix s. Thus, given a prefix s and a winning strategy σ for II in $\overline{\mathbb{SG}}(X, s^{-1}X)$, we describe a winning strategy for II in $\overline{\mathbb{SG}}(X, X)$: she plays s and then applies σ.

(2) $\Rightarrow$ (3) Assume that $X \not\equiv_{SG} s^{-1}X$, for every prefix s of S_+. This means that, for every prefix s, Player I has a winning strategy σ_s in the game $\mathbb{SG}(X, s^{-1}X)$. We then describe a winning strategy for Player I in the game $\overline{\mathbb{SG}}(X, X)$: Player I skips his first move; Player II's answer is forced to be a prefix position s, by definition of the $\overline{\mathbb{SG}}$-game; then, Player I applies σ_s, and wins. $\square$

COROLLARY 6.9. *Let $S = (S_+, S_\omega)$ be a finite ω-semigroup, and let $X \subseteq S_\omega$. If X is self-dual, then $X = \bigcup_{s \in I} sY_s$, for some subset $I \subseteq S_+$, and some family of non-self-dual ω-subsets $(Y_s)_{s \in I}$ satisfying $Y_s <_{SG} X$.*

PROOF. Let $X \subseteq S_\omega$ be self-dual, and let I be the set of prefixes of linked pairs of S_+^2. We observe that

$$X = \bigcup_{s \in I} s\left(s^{-1}X\right).$$

Now, since X is self-dual, Proposition 6.8 ensures that $s^{-1}X <_{SG} X$, for every prefix $s \in I$. Moreover, for every prefix $s \in I$, there exists an idempotent e such that (s, e) is a liked pair. Since $se = s$, one has $s^{-1}X = (se)^{-1}X = e^{-1}(s^{-1}X)$, thus in particular $s^{-1}X \equiv_{SG} e^{-1}(s^{-1}X)$. Moreover, since e is a prefix of the linked pair (e, e), Proposition 6.8 shows that the set $s^{-1}X$ is non-self-dual, for all $s \in I$. This concludes the proof. $\square$

By the previous corollary, the self-dual ω-subsets of finite ω-semigroups can be expressed as finite unions of translations of strictly smaller non-self-dual sets. Hence, in order to exclusively concentrate on the non-self-dual sets, we consider another definition of the $\mathbb{SG}$-degree which sticks any self-dual set to the non-self-dual sets located just one level below it.

$$d_{sg}(X) = \begin{cases} 1 & \text{if } X = \emptyset \text{ or } X = \emptyset^c, \\ \sup\{d_{sg}(Y) + 1 \mid Y \text{ n.s.d. and } Y <_{SG} X\} & \text{if } X \text{ is non-self-dual,} \\ \sup\{d_{sg}(Y) \mid Y \text{ n.s.d. and } Y <_{SG} X\} & \text{if } X \text{ is self-dual.} \end{cases}$$

Example 6.10. Consider the finite ω-semigroup U given in Example 3.41. One can prove that its ω-subsets have the following $\mathbb{SG}$-degrees:

- $d_{sg}(\emptyset) = d_{sg}(S_\omega) = 1$.
- $d^o_{sg}(\{[1,1]\}) = d^o_{sg}(\{[0,0],[0,1]\}) = 2$.
- $d^o_{sg}(\{[0,1]\}) = d^o_{sg}(\{[0,0],[1,1]\}) = \omega$.
- $d^o_{sg}(\{[0,0]\}) = d^o_{sg}(\{[0,1],[1,1]\}) = \omega$.

Chapter 7

The $\mathbb{FSG}$-hierarchy

Summary

In this chapter, we prove that the $\mathbb{FSG}$-hierarchy is precisely the algebraic counterpart of the Wagner hierarchy. We then present a complete description of the $\mathbb{FSG}$-hierarchy.

First, we show that the $\mathbb{FSG}$ and Wagner hierarchies are isomorphic, so that the $\mathbb{FSG}$-hierarchy is decidable, and has a height of ω^ω. The given isomorphism associates every ω-rational language with its corresponding syntactic pointed ω-semigroup. Hence, an ω-rational language and its syntactic image always share the same Wagner degree. This result also induces the two following properties. First, the $\mathbb{SG}$-reduction on syntactic structures appears as the exact algebraic counterpart of the Wadge reduction on ω-rational languages. Second, the $\mathbb{SG}$-degree is invariant under surjective morphism, meaning that syntactic pointed ω-semigroups are minimal representatives of their $\mathbb{SG}$-equivalence classes, whereas there is no minimal Büchi or Muller automaton of a given Wagner degree.

Furthermore, we describe an algorithm that computes the $\mathbb{SG}$-degree of every finite pointed ω-semigroup, without any reference to the corresponding Muller automaton. This procedure may thus compute the Wagner degree of every ω-rational language directly on its syntactic structure.

To this end, we first notice that linked pairs provide relevant positions in and moves of the $\mathbb{SG}$-games. Indeed, prefixes of linked pairs correspond precisely to stable positions, since playing the corresponding idempotents makes the player stay in his current position. Idempotents then represent the specific waiting moves for these stable positions. Therefore, we introduce a graph representation of finite pointed ω-semigroups according to these particular positions and moves, and such that every play in the $\mathbb{SG}$-game induces two paths in the corresponding graphs. These arenas are represented as directed acyclic graphs whose nodes are flowers. Each heart of a flower corresponds to a class of pairwise accessible stable positions in the $\mathbb{SG}$-game. Each petal associated with a given stable position consists of all potential waiting moves in this position. The accessibility relation between flowers represents the accessibility between stable positions in the $\mathbb{SG}$-game.

Then, we prove that every $\mathbb{SG}$-player waiting in a given stable position by playing elements of the corresponding petal can restrict his waiting moves to

some specific idempotents, called the vein of the given petal. Similarly, every $\mathbb{SG}$-player waiting in a given reachability class of stable positions by playing elements of the whole corresponding flower can restrict his moves to some specific idempotents, called the main vein of the flower. Therefore, the graph representation of finite pointed ω-semigroups can be pruned by deleting all flowers, but only keeping the main veins of these. The resulting graph reveals the "automatic" structure of pointed ω-semigroups. Consequently, one can apply (a reformulation of) Wagner's naming procedure on these graphs, in order to decide the $\mathbb{SG}$-degree of every ω-subset in the $\mathbb{FSG}$-hierarchy.

Thereafter, we provide two methods for building an ω-subset of any given $\mathbb{SG}$-degree. The first construction is direct, and the second one is inductive. The latter construction describes the algebraic counterpart of the ordinal operations. That is, given two pointed structures of respective $\mathbb{SG}$-degrees d_1 and d_2, one can built a third structure with an $\mathbb{SG}$-degree of $d_1 + d_2$; one can also build two other structures of $\mathbb{SG}$-degrees $d_1 \cdot n$, for every integer n, and $d_1 \cdot \omega$. Therefore, starting from the empty set or the full space, one can inductively build an ω-subset of any given $\mathbb{SG}$-degree.

Aftherwards, we introduce the *normal form* of any finite pointed ω-semigroup, and prove that it is an algebraic invariant for the $\mathbb{SG}$-equivalence class of the given structure. More precisely, any two pointed structures have the same $\mathbb{SG}$-degree if and only if they have the same normal form. This invariant consists of a subgraph that encodes the $\mathbb{SG}$-degree of the finite pointed ω-semigroup.

We conclude with additional graphical and algebraic properties.

7.1 The $\mathbb{FSG}$ and the Wagner hierarchies

This section shows that the $\mathbb{FSG}$-hierarchy is precisely the algebraic counterpart of the Wagner hierarchy. Consequently, the $\mathbb{FSG}$-hierarchy has height ω^ω and is decidable.

These results rely mainly on the following observation. Let $S = (S_+, S_\omega)$ be a finite ω-semigroup, and let $\varphi : A^\infty \longrightarrow S$ be a surjective morphism of ω-semigroups, for some finite alphabet A. Then every ω-subset X of S_ω can be lifted to an ω-rational language $\varphi^{-1}(X)$ of A^ω. The next proposition proves that this lifting induces an embedding from the $\mathbb{FSG}$-hierarchy into the Wagner hierarchy.

Proposition 7.1. *Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two finite ω-semigroups, let $X \subseteq S_\omega$ and $Y \subseteq T_\omega$, and let $\varphi : A^\infty \longrightarrow S$ and $\psi : B^\infty \longrightarrow T$ be two surjective morphisms of ω-semigroups, where A and B are finite alphabets. Then*

$$X \leq_{SG} Y \text{ if and only if } \varphi^{-1}(X) \leq_W \psi^{-1}(Y).$$

Proof.

($\Rightarrow$) Given a winning strategy σ for Player II in $\mathbb{SG}(X, Y)$, we describe a winning strategy τ for this same player in the game $\mathbb{W}\left(\varphi^{-1}(X), \psi^{-1}(Y)\right)$. Assume Player I is in position $(a_0, \ldots, a_n)$. Then II computes the move $\sigma(\varphi(a_0), \ldots, \varphi(a_n))$. If it is not a skipping move, she chooses a finite word v_n such that $\psi(v_n) = \sigma(\varphi(a_0), \ldots, \varphi(a_n))$, keeps it in mind while she finishes to play letter by letter the finite words she had previously chosen, and then plays v_n letter by letter. If it is a skipping move, then either she finishes to play letter by letter the finite words she had previously chosen, or she skips her turn if it is already done. This strategy is illustrated below.

$$
\begin{array}{ccc}
\multicolumn{3}{c}{\mathbb{SG}\text{-game}}\\
I\ (X) & \overset{\sigma}{\curvearrowright} & II\ (Y)\\ \hline
\varphi(a_0) & & \sigma(\varphi(a_0))\\
\varphi(a_1) & & \sigma(\varphi(a_0),\varphi(a_1))\\
\vdots & & \vdots\\
\vdots & & \vdots\\
\vdots & & \vdots
\end{array}
\qquad \xleftarrow{\varphi,\psi} \qquad
\begin{array}{ccc}
\multicolumn{3}{c}{\text{Wadge game}}\\
I\ (\varphi^{-1}(X)) & \overset{\tau}{\curvearrowright} & II\ (\psi^{-1}(Y))\\ \hline
\begin{array}{c} a_0\\ a_1\\ \vdots \end{array} & & \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} v_0\\
\begin{array}{c} \vdots \end{array} & & \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} v_1\\
\vdots & & \vdots
\end{array}
$$

It remains to prove that this strategy is winning for Player II. Since φ and ψ are surjective morphisms of ω-semigroups, one obtains

$$
\begin{aligned}
a_0a_1a_2\cdots \in \varphi^{-1}(X) \quad &\Leftrightarrow \quad \varphi(a_0a_1a_2\cdots) \in X\\
&\Leftrightarrow \quad \varphi(a_0)\varphi(a_1)\varphi(a_2)\cdots \in X\\
&\Leftrightarrow \quad \sigma(\varphi(a_0))\sigma(\varphi(a_0),\varphi(a_1))\cdots \in Y\\
&\Leftrightarrow \quad \psi(v_0)\psi(v_1)\psi(v_2)\cdots \in Y\\
&\Leftrightarrow \quad v_0v_1v_2\cdots \in \psi^{-1}(Y).
\end{aligned}
$$

Consequently, $\varphi^{-1}(X) \leq_W \psi^{-1}(Y)$.

($\Leftarrow$) Given a winning strategy σ for Player II in $\mathbb{W}\left(\varphi^{-1}(X), \psi^{-1}(Y)\right)$, we describe a winning strategy τ for this same player in $\mathbb{SG}(X,Y)$. Assume Player I is in position $(s_0,\ldots,s_n)$. Then II chooses a finite word $u_n = u_{n,0}\ldots u_{n,k_n}$ of $\varphi^{-1}(s_n)$, and computes the successive elements $\sigma(u_0,\ldots,u_{n-1},u_{n,0})$, $\sigma(u_0,\ldots,u_{n-1},u_{n,0},u_{n,1})$, and so on, in the game $\mathbb{W}(\varphi^{-1}(X),\varphi^{-1}(Y))$. After that, she successively plays the images by ψ of these moves when they are non-skipping, and she skips her turn otherwise. This strategy is illustrated below.

$$
\begin{array}{ccc}
\multicolumn{3}{c}{\text{Wadge game}}\\
I\ (\varphi^{-1}(X)) & \overset{\sigma}{\curvearrowright} & II\ (\psi^{-1}(Y))\\ \hline
u_0\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & & \begin{array}{c} \sigma(u_{0,0})\\ \sigma(u_{0,0},u_{0,1})\\ \vdots \end{array}\\
u_1\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & & \vdots\\
\vdots & & \vdots
\end{array}
\qquad \xrightarrow{\varphi,\psi} \qquad
\begin{array}{ccc}
\multicolumn{3}{c}{\mathbb{SG}\text{-game}}\\
I\ (X) & \overset{\tau}{\curvearrowright} & II\ (Y)\\ \hline
s_0 & & \psi\circ\sigma(u_{0,0})\\
s_1 & & \psi\circ\sigma(u_{0,0},u_{0,1})\\
\vdots & & \vdots\\
\vdots & & \vdots\\
\vdots & & \vdots
\end{array}
$$

It remains to prove that the strategy τ is winning for Player II. Since φ

and ψ are surjective morphisms of ω-semigroups, one has

$$\begin{aligned} s_0 s_1 s_2 \cdots \in X &\Leftrightarrow \varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots \in X \\ &\Leftrightarrow u_0 u_1 u_2 \cdots \in \varphi^{-1}(X) \\ &\Leftrightarrow u_{0,0}\cdots u_{0,k_0} u_{1,0}\cdots u_{1,k_1}\cdots \in \varphi^{-1}(X) \\ &\Leftrightarrow \sigma(u_{0,0})\sigma(u_{0,0},u_{0,1})\cdots \in \psi^{-1}(Y) \\ &\Leftrightarrow \psi\left(\sigma(u_{0,0})\sigma(u_{0,0},u_{0,1})\cdots\right) \in Y \\ &\Leftrightarrow \psi(\sigma(u_{0,0}))\psi(\sigma(u_{0,0},u_{0,1}))\cdots \in Y. \end{aligned}$$

Therefore, $X \leq_{SG} Y$. □

The previous proposition shows that the Wadge reduction on ω-rational languages and the $\mathbb{SG}$-reduction on ω-subsets recognizing these languages coincide. This property holds in particular for ω-rational languages and their syntactic images, as mentioned in Corollary 7.2 below. In addition, Corollary 7.2 and Proposition 5.23 show that the $\mathbb{SG}$-relation on subsets of ω-semigroups is weaker than the division relation, and is the appropriate algebraic characterization of the Wadge reduction on ω-rational languages. Corollary 7.2 is the precise generalization of Proposition 3.31 for the case of ω-rational languages.

Corollary 7.2. *Let K and L be two ω-rational languages, and let $\mu(K)$ and $\nu(L)$ be their syntactic images. Then $K \leq_W L$ if and only if $\mu(K) \leq_{SG} \nu(L)$.*

Proof. Since μ and ν are syntactic morphisms, one has $\mu^{-1}(\mu(K)) = K$ and $\nu^{-1}(\nu(L)) = L$. Proposition 7.1 leads to the conclusion. □

Example 7.3. Consider the ω-subsets $X = \{0^\omega\}$ and $Y = \{a^\omega\}$, and the ω-rational languages $K = (A^*a)^\omega$ and $L = (a\{b,c\}^* \cup \{b\})^\omega$ given respectively in examples 3.56 and 3.57. Example 6.3 shows that $X \leq_{SG} Y$, and thus $K \leq_W L$.

As another consequence, the $\mathbb{SG}$-degree of an ω-subset is invariant under surjective morphism, and in particular under syntactic morphism. Therefore, syntactic finite pointed ω-semigroups are minimal representatives of their $\leq_{SG}$-equivalence class.

Corollary 7.4. *Let $\mu : S \longrightarrow T$ be a surjective morphism of finite ω-semigroups, let $Y \subseteq T_\omega$, and let $X = \mu^{-1}(Y)$. Then $X \equiv_{SG} Y$.*

Proof. Let $\varphi : S_+^\infty \longrightarrow S$ be the canonical morphism of ω-semigroups associated with S, and let $\psi = \mu \circ \varphi : S_+^\infty \longrightarrow T$. The mapping ψ is also a surjective morphism of ω-semigroups. It satisfies $\psi^{-1}(Y) = \varphi^{-1} \circ \mu^{-1}(Y) = \varphi^{-1}(X)$, thus in particular $\varphi^{-1}(X) \equiv_W \psi^{-1}(Y)$. Proposition 7.1 shows that $X \equiv_{SG} Y$. □

Finally, the following theorem proves that the Wagner hierarchy and the $\mathbb{FSG}$-hierarchy are isomorphic. The required isomorphism is the mapping which associates every ω-rational language with its syntactic image. Therefore, the Wagner degree of an ω-rational language and the $\mathbb{SG}$-degree of its syntactic image are the same. In order to establish this result, let us denote by $\mathbb{WAG}H$ the class of all ω-rational languages over finite alphabets, and by $\mathbb{FSG}H$ the class of all ω-subsets of finite ω-semigroups.

Theorem 7.5. *The partial orderings $(\mathbb{WAG}H, \leq_W)$ and $(\mathbb{FSG}H, \leq_{SG})$ are isomorphic.*

Proof. Consider the mapping from the Wagner hierarchy into the $\mathbb{FSG}$-hierarchy which associates every ω-rational language with its syntactic image. We prove that this mapping is an embedding. Let K and L be two ω-rational languages, and let $X = \mu(K)$ and $Y = \nu(L)$ be their syntactic images. Corollary 7.2 ensures that $K \leq_W L$ if and only if $X \leq_{SG} Y$. We now show that, up to $\equiv_{SG}$-equivalence, this mapping is onto. Let X be any ω-subset of a finite ω-semigroup $S = (S_+, S_\omega)$, let $\mu : S \longrightarrow S(X)$ be the syntactic morphism of X, and let $Y = \mu(X)$ be its syntactic image. Corollary 7.4 ensures that $X \equiv_{SG} Y$. Now, let also $\varphi : S_+^\infty \longrightarrow S$ be the canonical morphism associated with S_+, and let $L = \varphi^{-1}(X)$. The morphism of ω-semigroups $\psi = \mu \circ \varphi : S_+^\infty \longrightarrow S(X)$ is the syntactic morphism of L [27], and $\psi(L) = Y \equiv_{SG} X$. □

As a corollary, we prove that the $\mathbb{FSG}$-hierarchy is *decidable*: for every ω-subset X of the hierarchy, one can effectively compute the Cantor normal form of base ω of the ordinal $d_{sg}(X)$.

Corollary 7.6. *The $\mathbb{FSG}$-hierarchy has height ω^ω, and it is decidable.*

Proof. By the previous theorem, the $\mathbb{FSG}$ and Wagner hierarchies have the same height, namely ω^ω. In addition, given an ω-subset X of a finite ω-semigroup $S = (S_+, S_\omega)$, its $\mathbb{SG}$-degree can be computed as described hereafter. Let $\varphi : S_+^\infty \longrightarrow S$ be the canonical morphism associated with S_+, and let $L = \varphi^{-1}(X)$. Theorem 7.5 shows that the $\mathbb{SG}$-degree of X and the Wagner degree of L are the same. Furthermore, the Wagner degree of L can effectively be computed as follows. First, one can effectively compute an ω-rational expression describing $L = \varphi^{-1}(X)$ [27, Corollary 7.4, p. 110]. Next, one can shift from this rational expression to some finite Muller automaton recognizing L, see [27, Chapter I, sections 10.1, 10.3, and 10.4]. Finally, the Wagner degree of the ω-language recognized by a finite Muller automaton is effectively computable, as described in Chapter 5. □

Example 7.7. Consider the ω-subsets $X = \{0^\omega\}$ and $Y = \{a^\omega\}$ given respectively in examples 3.56 and 3.57. The algorithm presented in the following chapter shows that $d_{sg}(X) = \omega$ and $d_{sg}(Y) = \omega^2$. In addition, since these sets are the syntactic images of the ω-languages $K = (A^*a)^\omega$ and $L = (a\{b,c\}^* \cup \{b\})^\omega$, Theorem 7.5 shows that this result can also be obtained by computing the Wagner degrees of K and L. [27, 41, 10].

7.2 Describing finite pointed ω-semigroups

7.2.1 Finite semigroups as graphs

In this section, we describe a graph representation of finite semigroups by focusing on specific positions in, and moves of the $\mathbb{SG}$-game. The notion of a linked pair is essential to this description. As a consequence, every $\mathbb{SG}$-play induces a unique path in the graph inherited from the semigroup involved. From this point forward, the set S_+ denotes a fixed finite semigroup. We recall that P and E respectively denote the sets of prefixes and idempotents of S_+.

Linked pairs satisfy the following game theoretical properties. First of all, Proposition 6.8 shows that any $\mathbb{SG}$-player in charge of a non-self-dual ω-subset can restrict himself to only reaching *prefix positions*. Also, an $\mathbb{SG}$-player can stay indefinitely in a position s if and only if s is a prefix. He does so by playing idempotents in $E(s)$. Finally, for every $s \in P$, each idempotent e of $E(s)$ corresponds to a specific *waiting move* for the prefix position s. These specific positions and moves yield two preorders on the sets of prefixes and idempotents of linked pairs.

Firstly, we consider the restriction of the preorder $\leq_{\mathcal{R}}$ to the set of prefixes P, also denoted by $\leq_{\mathcal{R}}$ without ambiguity. By definition, this preorder satisfies the *accessibility relation* $s \geq_{\mathcal{R}} s'$ if and only if there exists $x \in S^1_+$ such that $sx = s'$, for all $s, s' \in P$. As usual, one has $s >_{\mathcal{R}} s'$ if and only if $s \geq_{\mathcal{R}} s'$ and $s' \not\geq_{\mathcal{R}} s$, and also $s \mathrel{\mathcal{R}} s'$ if and only if $s \geq_{\mathcal{R}} s'$ and $s' \geq_{\mathcal{R}} s$. This preorder can be naturally extended to the set of $\mathcal{R}$-classes of prefixes $P/\mathcal{R}$ by setting $\bar{s} \geq_{\mathcal{R}} \bar{t}$ if and only if there exist $s' \in \bar{s}$ and $t' \in \bar{t}$ such that $s' \geq_{\mathcal{R}} t'$, for all $\bar{s}, \bar{t} \in P/\mathcal{R}$. The pair $(P/\mathcal{R}, \geq_{\mathcal{R}})$ is therefore a partial ordering.

Secondly, we consider the *natural order* on idempotents, denoted by $\leq$, and defined as the restriction of the preorder $\leq_{\mathcal{H}}$ to the set E. It satisfies the *absorption relation* $e \geq e'$ if and only if $ee' = e'e = e'$ holds, for all $e, e' \in E$. As usual, one has $e > e'$ if and only if both $e \geq e'$ and $e' \not\geq e$ hold. Proposition 3.17 shows that the pair $(E, \geq)$ is also a partial ordering [27].

These two relations satisfy the following properties, central in the description of an $\mathbb{SG}$-play. Firstly, a player can move from the prefix position s to the prefix position s' if and only if $s \geq_{\mathcal{R}} s'$. He can go from s to s' and back to s if and only if $s \mathrel{\mathcal{R}} s'$. Secondly, a player which forever stays in the prefix position s by playing infinitely many e's and f's in $E(s)$ produces an infinite play α of the form $(s, e, f, f, e, f, e, e, \ldots)$. If $e \geq f$, since the f's absorb all the e's, the infinite word $(s, f, f, f, \ldots)$ is a factorization of α, and the properties of the infinite product ensure that $\pi_S(\alpha) = sf^\omega$. Therefore, only the $\leq$-least idempotents that are played infinitely often in a given prefix position are involved in the acceptance of the final play.

The graph of the preorder $(P, \geq_{\mathcal{R}})$ is a subgraph of the right Cayley graph of S_+, and its strongly connected components are the $\mathcal{R}$-classes of P. The graph of the partial order $(P/\mathcal{R}, \geq_{\mathcal{R}})$ is thus a directed acyclic graph (DAG) where vertices represent the $\mathcal{R}$-classes of prefixes and directed edges stand for the strict accessibility relation $>_{\mathcal{R}}$, as illustrated in Figure 7.1, where transitive arrows are not drawn, for reasons of clarity (that is every time there is an edge from i to j, and from j to k, the induced edge from i to k is not represented). The successive moves of an $\mathbb{SG}$-player should be traced inside this graph. Indeed, every $\mathbb{SG}$-play according to elements of S_+ induces a sequence of prefix positions which progresses deeper and deeper inside this structure: any prefix position $s = (a_0, a_1, \ldots, a_n)$ only extends to some prefix position of the form $s' = (a_0, \ldots, a_n, a_{n+1}, \ldots, a_k)$ such that $s \geq_{\mathcal{R}} s'$. Therefore, an infinite $\mathbb{SG}$-play induces a unique path in this DAG that either remains in an $\mathcal{R}$-class of prefixes, or climbs along the edges, with no chance of going back (this justifies the consideration of the partial order $(P/\mathcal{R}, \geq_{\mathcal{R}})$ instead of $(P/\mathcal{R}, \leq_{\mathcal{R}})$).

Furthermore, every prefix t can be associated with the partial ordered set $(E(t), \geq)$, called the *petal* associated with t, and denoted by *petal*(t). The graph of this set is also a DAG, and given $e, f \in \mathit{petal}(t)$, there is an edge from e to f if and only if $e \geq f$. Hence, this set consists of all the possible waiting moves for

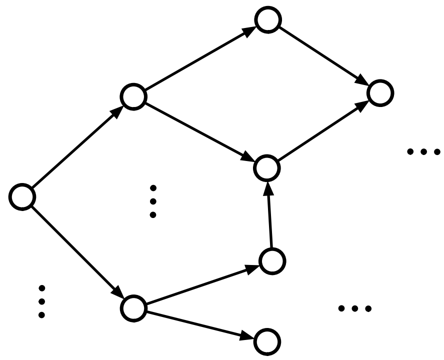


Figure 7.1: The directed acyclic graph representation of the partial order $(P/\mathcal{R}, \geq_{\mathcal{R}})$. A play of an $\mathbb{SG}$-player induces a unique path in this DAG.

the prefix position t ordered by their absorption capacity. Up to making copies of idempotents, we assume all petals to be disjoint. Then, for every $\mathcal{R}$-class of prefixes $\bar{s}$, the set $\bigcup_{t\in\bar{s}} petal(t)$ is called the *flower* associated with $\bar{s}$, denoted by $flower(\bar{s})$. This set contains all the possible waiting moves for some prefix position in $\bar{s}$. Figure 7.2 illustrates a flower in detail.

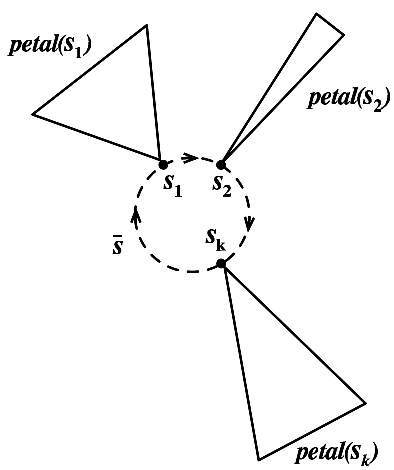


Figure 7.2: The set $flower(\bar{s})$ associated with the $\mathcal{R}$-class of prefixes $\bar{s}$. Every prefix s_i in $\bar{s}$ is associated with its corresponding petal. The circle describes the $\geq_{\mathcal{R}}$-accessibility relation between the prefixes s_i of $\bar{s}$.

Finally, the enriched graph representation of $(P/\mathcal{R}, \geq_{\mathcal{R}})$ where each $\mathcal{R}$-class of prefixes is associated with its corresponding flower will be called the *DAG representation* of the finite semigroup S_+. It can be drawn like a bunch of flowers, as illustrated in Figure 7.3. This graph represents an arena for an $\mathbb{SG}$-player moving in S_+. It allows to follow the successive prefix positions reached along the play, and for every prefix position, it describes all the possible waiting moves ordered by their absorption capacity.

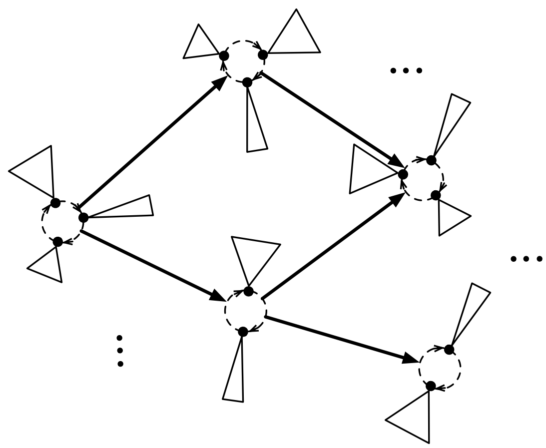


Figure 7.3: The DAG representation of a finite semigroup S_+: every $\mathcal{R}$-class of prefixes is associated with its corresponding flower. This DAG is an arena for every $\mathbb{SG}$-player whose moves are inside the semigroup S_+.

EXAMPLE 7.8. Let $U_2 = \{1, a, b\}$ be the finite monoid defined by the relations $aa = ba = a$ and $ab = bb = b$. The DAG representation of U_2 is illustrated in Figure 7.4.

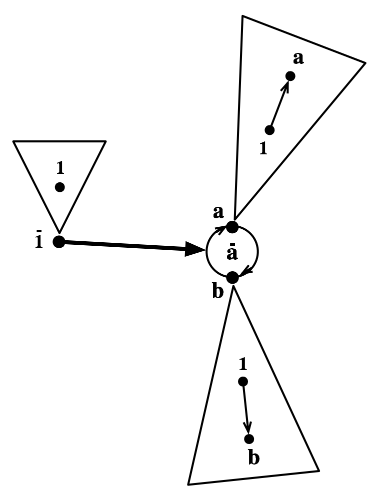


Figure 7.4: The DAG representation of the finite semigroup U_2.

7.2.2 Finite pointed ω-semigroups as graphs

The DAG representation of finite semigroups can easily be extended to some graph representation of finite pointed ω-semigroups. For that purpose, we introduce the *signature* of a petal. From this point onward, the pair (S, X) denotes a fixed finite pointed ω-semigroup, where $S = (S_+, S_\omega)$ is a finite ω-semigroup and X is a subset of S_ω.

DEFINITION 7.9. *Let $s \in P$. The* signature *of the set $petal(s)$ according to X is the mapping $sign_X : petal(s) \longrightarrow \{+, -\}$ defined by*

$$sign_X(e) = \begin{cases} + & \text{if } se^\omega \in X, \\ - & \text{if } se^\omega \notin X. \end{cases}$$

The pair $(petal(s), sign_X)$ is called the *signed petal* associated with s, denoted by $petal_X(s)$. The union for t running in $\bar{s}$ of the sets $petal_X(t)$ is called the *signed flower* associated with $\bar{s}$, and is denoted by $flower_X(\bar{s})$.

The graph of the partial order $(P/\mathcal{R}, \geq_\mathcal{R})$ where each $\mathcal{R}$-class of prefixes $\bar{s}$ is associated with its corresponding signed flower $flower_X(\bar{s})$ is called the *signed DAG representation* of the finite pointed ω-semigroup (S, X), and is illustrated in Figure 7.5. This graph is an arena for an $\mathbb{SG}$-player in charge of X: the successive prefix positions reached along the play can be traced inside this graph, just as described in section 7.2.1. But in addition, the signs associated with the idempotents provide information about the acceptance of an $\mathbb{SG}$-play according to X: an infinite play belongs to X if and only if it can be factorized into the form se^ω, for some positive $e \in petal_X(s)$. Finally, by finiteness of this DAG, every infinite play will eventually remain forever in a signed flower, and hit at least one of the corresponding signed petals infinitely often.

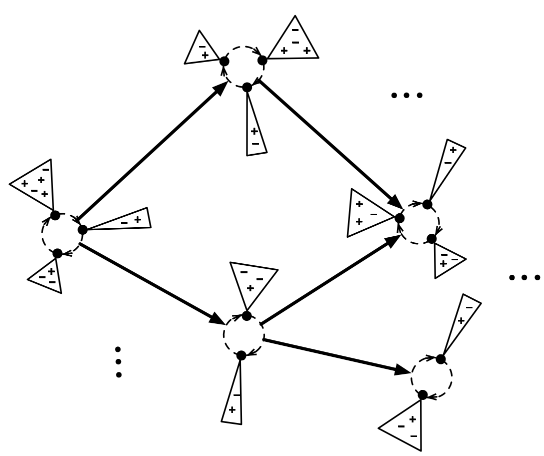


Figure 7.5: The signed DAG representation of a finite pointed ω-semigroup (S, X): an enriched arena for an $\mathbb{SG}$-player in charge of X.

EXAMPLE 7.10. Let $S = (\{0,1\}, \{0^\omega, 1^\omega\})$ be the finite ω-semigroup defined by the following relations:

$$\begin{array}{cccc} 0 \cdot 0 = 0 & 0 \cdot 1 = 0 & 1 \cdot 0 = 0 & 1 \cdot 1 = 1 \\ 00^\omega = 0^\omega & 10^\omega = 0^\omega & 01^\omega = 1^\omega & 11^\omega = 1^\omega \end{array}$$

Let $X = \{0^\omega\} \subseteq S$. The signed DAG representation of (S, X) is illustrated in Figure 7.6

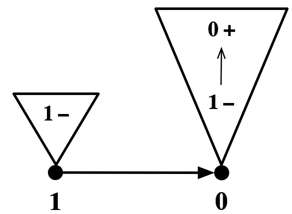


Figure 7.6: The signed DAG representation of (S, X).

EXAMPLE 7.11. Let $T = (\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\})$ be the finite ω-semigroup defined by the following relations:

$$\begin{array}{cccc} a^2 = a & ab = a & ac = a & ba = a \\ b^2 = b & bc = c & cb = c & c^2 = c \\ b^\omega = a^\omega & c^\omega = 0 & aa^\omega = a^\omega & a(ca)^\omega = a^\omega \\ ba^\omega = a^\omega & b(ca)^\omega = (ca)^\omega & ca^\omega = (ca)^\omega & c(ca)^\omega = (ca)^\omega \end{array}$$

Let $Y = \{a^\omega\} \subseteq T$. The signed DAG representation of (T, Y) is illustrated in Figure 7.7.

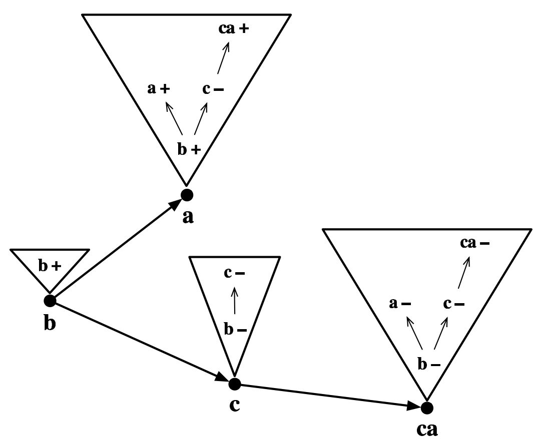


Figure 7.7: The signed DAG representation of (T, Y).

7.2.3 Alternating chains

The following sections describe step by step the game theoretical characteristics of the signed DAG representation of a finite pointed ω-semigroup. For that purpose, we introduce the notion of an *alternating chain* of idempotents in a signed petal. This definition refines the notion of a *chain* in finite ω-semigroups, introduced in [5, Theorem 6].

DEFINITION 7.12. *Let $s \in P$. An* alternating chain *in $petal_X(s)$ is a strictly descending sequence of idempotents of $petal_X(s)$ $e_0 > e_1 > \ldots > e_n$ satisfying the following properties:*

(1) *signs alternation: one has $sign_X(e_k) \neq sign_X(e_{k+1})$, for all $k < n$;*
(2) *each e_k is minimal for its sign: if $e_k > e$ and $sign_X(e_k) = sign_X(e)$, then there exists f such that $e_k > f > e$ and $sign_X(e_k) \neq sign_X(f)$.*

An alternating chain in a signed flower is simply an alternating chain in a signed petal of this signed flower.

Let $C : e_0 > e_1 > \ldots > e_n$ be an alternating chain in $petal_X(s)$. The *length* of C, denoted by $l(C)$, is n (number of its elements minus one, or equivalently, the number of signs alternations). The chain C is said to be *maximal* in $petal_X(s)$ if there is no other alternating chain of strictly larger length in $petal_X(s)$. Maximal alternating chains in signed petals and flowers will play a central role in the sequel. In addition, the chain C is called *positive* if $sign_X(e_0) = +$, and *negative* otherwise. Two alternating chains $e_0 > \ldots > e_n$ and $e'_0 > \ldots > e'_n$ of the same length are said to have *the same signs* if $sign_X(e_n) = sign_X(e'_n)$, and *opposite signs* otherwise. Condition (1) of Definition 7.12 implies that these chains have the same signs if and only if $sign_X(e_i) = sign_X(e'_i)$, for all i. Finally, we say that an alternating chain C *captures* the idempotent e if $e \geq e_0$, or if there exist e_i and e_{i+1} such that $e_i > e \geq e_{i+1}$. If $e \geq e_0$, the rank of e in C is defined as $rank_C(e) = 0$, and if $e_i > e \geq e_{i+1}$, then $rank_C(e) = i+1$. An alternating chain of length 3 capturing the elements e and e' is illustrated below. Every idempotent is associated with its sign, and arrows represent the $>$-relation.

$$(e_0, +) \longrightarrow (e_1, -) \rightarrow {\scriptstyle (e,+)} \rightarrow (e_2, +) \rightarrow {\scriptstyle (e',-)} \rightarrow (e_3, -)$$

EXAMPLE 7.13. Consider the finite pointed ω-semigroup (T, Y) given in Example 7.11. The sequence $b > c > ca$ is a positive alternating chain of length 2 in the signed petal $petal_Y(a)$. Inside the signed petal $petal_Y(ca)$, the element ca is a negative alternating chain of length 0 capturing the idempotents b and c.

Alternating chains satisfy the following property.

LEMMA 7.14. *Let $x \in petal_X(s)$. Among all the alternating chains capturing x, any longest two have the same signs, and induce the same rank for x.*

Consequently, we simply denote by $rank(e)$ the rank of e in any longest alternating chain capturing e.

PROOF. Let $C_1 : e_0 > \ldots > e_n$ and $C_2 : f_0 > \ldots > f_n$ be any two of the longest alternating chains capturing x. We prove that their $\leq$-minimal elements e_n and f_n have the same sign. Consider $e = (e_n f_n e_n)^\pi$ and $f = (f_n e_n f_n)^\pi$, where π is the exponent of S_+. Then e and f are idempotent and $se = sf = s$, hence e and f both belong to $petal_X(s)$. Moreover, $e_n e = e e_n = e$, thus $e_n \geq e$. Since C_1 is a longest alternating chain capturing x, and e_n is minimal in this chain, the elements e and e_n have the same sign. Condition (2) of Definition 7.12 then implies that $e_n = e$. Similarly, $f_n = f$. Hence, the properties of the ω-operation imply

$$se^\omega = s(e_n f_n e_n)^\omega = s(e_n f_n f_n e_n)^\omega = se_n f_n (f_n e_n e_n f_n)^\omega = s(f_n e_n f_n)^\omega = sf^\omega.$$

Therefore, the idempotents $e = e_n$ and $f = f_n$ have the same sign, hence C_1 and C_2 also have the same signs. We now prove that x has the same rank in C_1 and C_2. Let k and l be the respective ranks of x in C_1 and C_2. We may assume, without loss of generality, that $k \leq l$. Therefore,

$$\begin{gathered} e_0 > e_1 > \ldots > e_{k-1} > f_\ell > \ldots > f_n, \\ f_0 > f_1 > \ldots > f_{\ell-1} > e_k > \ldots > e_n \end{gathered}$$

are two alternating chains of respective lengths $(k-1)+(n-l)+1 = k+(n-l)$ and $(l-1)+(n-k)+1 = l+(n-k)$. The maximality of n implies both $k+(n-l) \leq n$ and $l+(n-k) \leq n$, thence $k = l$. □

7.2.4 Veins

We now focus on some specific alternating chains of idempotents. We prove that only those ones influence the $\mathbb{SG}$-degree of our algebraic structures.

Definition 7.15. *For every s in P, a maximal alternating chain in $petal_X(s)$ is called a* vein *of this signed petal.*

Example 7.16. Consider the finite pointed ω-semigroup (T, Y) given in Example 7.11. The sequence $b > c > ca$ is a vein in $petal_Y(a)$.

Playing waiting moves inside a given vein instead of potentially being able to play through all idempotents of a signed petal will show not to be restricting. We first prove the following property.

Lemma 7.17. *Any two veins of a given signed petal share the same signs.*

Proof. Let C_1 and C_2 be two veins inside $petal_X(s)$. We prove that their $\leq$-minimal elements m_1 and m_2 have the same sign. Consider the elements $e_1 = (m_1 m_2 m_1)^\pi$ and $e_2 = (m_2 m_1 m_2)^\pi$, where π is the exponent of S_+. Then e_1 and e_2 are both idempotents satisfying $se_1 = se_2 = s$, hence $e_1, e_2 \in petal_X(s)$. Moreover, $m_1 e_1 = e_1 m_1 = e_1$ and $m_2 e_2 = e_2 m_2 = e_2$, thus $m_1 \geq e_1$ and $m_2 \geq e_2$. Since both C_1 is a maximal alternating chain, and m_1 is minimal in this chain, the elements e_1 and m_1 have the same sign. Definition 7.12 case (2) then implies that $m_1 = e_1$. By a similar argument, $m_2 = e_2$. Finally, the properties of the ω-operation yield

$$\begin{aligned} s{e_1}^\omega &= s(m_1 m_2 m_1)^\omega = s(m_1 m_2 m_2 m_1)^\omega \\ &= s m_1 m_2 (m_2 m_1 m_1 m_2)^\omega = s(m_2 m_1 m_2)^\omega = s{e_2}^\omega. \end{aligned}$$

Therefore, the idempotents $e_1 = m_1$ and $e_2 = m_2$ have the same sign, thus C_1 and C_2 have the same signs too. □

We now define a mapping from any signed petal onto one of its veins. The choice of the vein may be arbitrary, for Lemma 7.17 shows that all the veins of a given signed petal are isomorphic. This mapping will determine the strategy of an $\mathbb{SG}$-player restricting his waiting moves to the sole idempotents of such veins.

DEFINITION 7.18. *Let V be any vein $e_0 > \ldots > e_n$ inside $petal_X(s)$. We define the mapping $\sigma : petal_X(s) \longrightarrow V$ by*

$$\sigma(e) = \begin{cases} e_i & \text{if } rank(e) = i \text{ and } sign_X(e) = sign_X(e_i), \\ e_{i+1} & \text{if } rank(e) = i \text{ and } sign_X(e) \neq sign_X(e_i). \end{cases}$$

By finiteness of the set $petal_X(s)$, this mapping is effectively computable. It is onto and preserves the order $\leq$ as well as the signature, as illustrated in Figure 7.8 below.

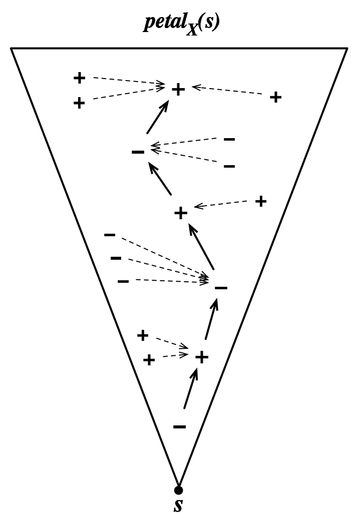


Figure 7.8: The surjection from a signed petal onto one of its veins.

We finally turn to prove that only one vein of each signed petal is significant in the description of the $\mathbb{SG}$-degree of (S, X). More precisely, we show that any $\mathbb{SG}$-player remaining indefinitely in some prefix position s can restrict his waiting moves to the idempotents of a given vein of $petal_X(s)$. To this end, we consider the imposed game $\mathbb{SG}'(X, X)$ where:

- both players are in charge of X, and are not allowed to pass their turns;
- they are both forced to play s on their first move;
- on his next moves, I is forced to play waiting moves inside $petal_X(s)$;
- on her next moves, II is forced to play waiting moves belonging exclusively to a given vein of $petal_X(s)$.

We prove that these restricting rules for Player II do actually not weaken her.

PROPOSITION 7.19. *Player II has a winning strategy in the imposed game defined above.*

PROOF. Both players are forced to play s on their first move. A winning strategy for Player II is described by induction as follows.

Player II first associates with each element e in $petal_X(s)$ a counter $\kappa(e)$. After each move of I, the integer $\kappa(e)$ will be the largest possible number of e's occurring in a factorization of I's current play. More precisely, Player II updates these counters as follows: let $(e_0, \ldots, e_{k-1})$ be the elements of $petal_X(s)$ already played by I, then for each e in $petal_X(s)$, the value of $\kappa(e)$ is set as the largest integer p such that there exists a sequence of indices

$$0 \leq i_1 \leq j_1 < i_2 \leq j_2 < \ldots < i_p \leq j_p \leq k-1$$

satisfying $e = (e_{i_1} \cdots e_{j_1}) = (e_{i_2} \cdots e_{j_2}) = \ldots = (e_{i_p} \cdots e_{j_p})$. After that, Player II computes the images on the given vein under σ of all the idempotents whose counters has increased, as described in Definition 7.18. She finally plays the $\leq$-minimum of these images. Notice that this minimum always exists since the given vein is well ordered by $\leq$.

The three following claims prove that this strategy is winning for Player II. We introduce the following notations. We set inc_∞ for the set of idempotents of $petal_X(s)$ whose counters were incremented infinitely often during the play, and we let INC_∞ be the set of $\leq$-minimal elements of inc_∞. Finally, we set

$$e_{min} = \min \{\sigma(e) \mid e \in INC_\infty\} .$$

CLAIM 7.20. *Let α be I's infinite play, and let $e \in INC_\infty$. Then $\pi_S(\alpha) = se^\omega$.*

PROOF. Since e belongs to INC_∞, its counter was incremented infinitely often during the play. Consequently, I's infinite play can be written as

$$\alpha = sv_0ev_1ev_2ev_3ev_4e \cdots ,$$

where each v_i is a finite word of $petal_X(s)^*$, for all $i \geq 0$. By idempotence of e, the infinite word α is a factorization of $\beta = sv_0ev_1eev_2eev_3eev_4ee \cdots$, and the infinite word $\gamma = sv_0(ev_1e)(ev_2e)(ev_3e) \cdots$ is a factorization of β. By Proposition 3.21, γ can be associated with a linked pair $(s, \tilde{e})$, where $\tilde{e} = eve$, for some $v \in petal_X(s)^*$. Thus $\pi_S(\gamma) = s\tilde{e}^\omega$. Moreover, by the properties of the infinite product, since γ is a factorization of β, one has $\pi_S(\gamma) = \pi_S(\beta) = s\tilde{e}^\omega$. By the same property, since α is a factorization of β, then $\pi_S(\alpha) = \pi_S(\beta) = s\tilde{e}^\omega$. Besides, notice that the element $\tilde{e}$ also appears infinitely often in a factorization of α, hence its counter was incremented infinitely often during the play, meaning that $\tilde{e} \in inc_\infty$. In addition, one has $e\tilde{e} = \tilde{e}e = \tilde{e}$, thus $e \geq \tilde{e}$. But then the minimality of e in inc_∞ implies $\tilde{e} = e$. Finally, one obtains $\pi_S(\alpha) = s\tilde{e}^\omega = se^\omega$. □

CLAIM 7.21. *Let β be II's infinite play. Then $\pi_S(\beta) = s{e_{min}}^\omega$.*

PROOF. Let $e \in INC_\infty$ such that $e_{min} = \sigma(e)$. The strategy described above guarantees that II played e_{min} infinitely often. Therefore, II's infinite play can be written as

$$\beta \quad = \quad su_0e_{min}u_1e_{min}u_2e_{min} \cdots ,$$

where each u_i is a finite word of elements of the given vein, for all $i \geq 0$. Moreover, no element $g < e_{min}$ was played by II infinitely often. Otherwise, since the set $\sigma^{-1}(g)$ is finite, there would exist f in inc_∞ such that $\sigma(f) = g$, contradicting the minimality of e_{min}. Now, since e_{min} is the $\leq$-minimal element of the given vein played infinitely often by II, every product $e_{min}u_i$ is equal to e_{min}. Proposition 3.21 then shows that the infinite word β can be associated with the linked pair (s, e_{min}). Therefore $\pi_S(\beta) = s{e_{min}}^\omega$. □

CLAIM 7.22. *One has $\pi_S(\alpha) \in X$ if and only if $\pi_S(\beta) \in X$.*

PROOF. Claim 7.21 shows that $\pi_S(\beta) = s{e_{min}}^\omega$. Now, let e be an idempotent of INC_∞ such that $\sigma(e) = e_{min}$. Claim 7.20 proves that $\pi_S(\alpha) = se^\omega$. Moreover, since σ preserves the signature, the idempotents e and e_{min} have the same sign. Therefore, $\pi_S(\alpha) = se^\omega \in X$ if and only if $\pi_S(\beta) = s{e_{min}}^\omega \in X$. □

□

7.2.5 Main veins

In this section, we prove that only some specific veins of each flower is relevant in the computation of the $\mathbb{SG}$-degree. We focus on these *main veins*.

DEFINITION 7.23. *Let $\bar{s} \in P/\mathcal{R}$. A maximal alternating chain in $flower_X(\bar{s})$ is called a* main vein *of this signed flower.*

EXAMPLE 7.24. Consider the finite pointed ω-semigroup (T, Y) given in Example 7.11. The sequence $b > c > ca$ is a main vein in $flower_Y(a)$.

Main veins satisfy the same property as veins.

LEMMA 7.25. *Any two main veins of a given signed flower share the same signs.*

PROOF. Let $C_1 \subseteq petal_X(s_1)$ and $C_2 \subseteq petal_X(s_2)$ be two main veins of $flower_X(\bar{s})$. We prove that their $\leq$-minimal elements m_1 and m_2 have the same sign. Since $s_1, s_2 \in \bar{s}$, there exist $a, b \in S_+^1$ such that $s_1a = s_2$ and $s_2b = s_1$. Now, consider the elements $e_1 = (m_1am_2bm_1)^\pi$ and $e_2 = (m_2bm_1am_2)^\pi$, where π is the exponent of S_+. Then e_1 and e_2 are idempotents satisfying $s_1e_1 = s_1$ and $s_2e_2 = s_2$, thence $e_1 \in petal_X(s_1)$ and $e_2 \in petal_X(s_2)$. Moreover, $m_1e_1 = e_1m_1 = e_1$ and $m_2e_2 = e_2m_2 = e_2$, thus $m_1 \geq e_1$ and $m_2 \geq e_2$. Furthermore, since C_1 is a maximal alternating chain, and m_1 is minimal in this chain, then e_1 and m_1 have the same sign. Definition 7.12 case (2) then implies that $m_1 = e_1$. Similarly, one has $m_2 = e_2$. Hence, the ω-associativity of the ω-product yields

$$\begin{aligned} s_1{e_1}^\omega &= s_1(m_1am_2bm_1)^\omega = s_1(m_1am_2m_2bm_1)^\omega \\ &= s_1m_1am_2(m_2bm_1am_2)^\omega = s_2{e_2}^\omega. \end{aligned}$$

Therefore, $e_1 = m_1$ and $e_2 = m_2$ have the same sign, which proves that C_1 and C_2 have the same signs too. □

As previously, we define a mapping from every signed petals of a signed flower onto a given main vein. The choice of the main vein may also be arbitrary, for Lemma 7.25 proves that mains veins of a given signed flower are all isomorphic.

We implicitly proceed in two steps: firstly, we map every signed petal onto one of its veins, as defined in Definition 7.18; secondly, we map every such vein onto a given main vein.

DEFINITION 7.26. *Let $V : e_0 > \ldots > e_n$ be a main vein of $flower_X(\bar{s})$. We define the mapping $\bar{\sigma} : flower_X(\bar{s}) \longrightarrow V$ by*

$$\bar{\sigma}(e) = \begin{cases} e_i & \text{if } rank(\sigma(e)) = i \text{ and } sign_X(e) = sign_X(e_i), \\ e_{i+1} & \text{if } rank(\sigma(e)) = i \text{ and } sign_X(e) \neq sign_X(e_i). \end{cases}$$

This mapping is onto, and preserves the natural ordering on idempotents, as well as the signature. It is illustrated in Figure 7.9. This mapping will be involved in the strategy of an $\mathbb{SG}$-player that restricts his waiting moves to the sole idempotents of some given main veins.

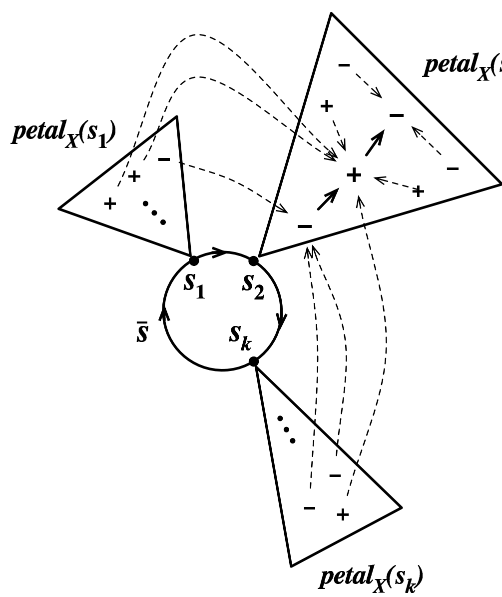


Figure 7.9: The surjection from a signed flower onto one of its main veins.

We now show that only one main vein of each signed flower matters in the computation of the $\mathbb{SG}$-degree of (S, X). In other words, any player remaining indefinitely in some $\mathcal{R}$-class of prefixes $\bar{s}$ can restrict his waiting moves to the idempotents of a given main vein inside $flower_X(\bar{s})$. We thence consider a given main vein of $flower_X(\bar{s})$ contained in $petal_X(t)$, for some $t \in \bar{s}$, and we introduce an imposed version of the game $\mathbb{SG}(X, X)$ where:

- both players are in charge of X, and cannot skip their turns;
- I is forced to only reach positions in $\bar{s}$;
- II is forced to play t on her first move, and then restrict her waiting moves to the idempotents of the given main in $petal_X(t)$.

We extend Proposition 7.19 to main veins.

PROPOSITION 7.27. *Player II has a winning strategy in this imposed game.*

PROOF. Player II fist plays t, then applies the following strategy.

She associates with each element e in $flower_X(\bar{s})$ a counter $\kappa(e)$. She updates these counters after each move of I as follows: let $(x_0, \ldots, x_{k-1})$ be the elements already played by I, then for every $t' \in \bar{s}$ and every $e \in petal_X(t')$, the value $\kappa(e)$ is the maximal number of occurrences of e appearing in position t' in a factorization of I's current play. More precisely, the value of $\kappa(e)$ is set as the largest integer p such that there exists a sequence of indices

$$0 \leq i_1 \leq j_1 < i_2 \leq j_2 < \ldots < i_p \leq j_p \leq k-1$$

satisfying

(1) $e = (x_{i_1} \cdots x_{j_1}) = (x_{i_2} \cdots x_{j_2}) = \ldots = (x_{i_p} \cdots x_{j_p})$,
(2) all the elements $x_{i_1}, x_{i_2}, \ldots, x_{i_p}$ were played in position t.

Then II computes the images on the given main vein under $\bar{\sigma}$ of all idempotents whose counters were incremented, and plays the $\leq$-minimum of those. If no element were incremented, II plays the $\leq$-largest idempotent of the given main vein. This may happen, for instance, when I passes from one prefix of the $\mathcal{R}$-class to another, and hence doesn't play an idempotent of $flower_X(\bar{s})$.

This strategy ensures that Player II increments the counter an idempotent $e \in petal_X(t')$ if and only if e appears in position t' in a factorization of I's play. The three following claims prove that this strategy is winning for Player II. We first introduce the following notations: we let inc_∞ be the set of elements in $flower_X(\bar{s})$ whose counters were incremented infinitely often during the play, and INC_∞ be the set of $\leq$-minimal elements of inc_∞. We also set

$$e_{min} = \min\left\{\bar{\sigma}(e) \mid e \in INC_\infty\right\}.$$

CLAIM 7.28. *Let α be I's infinite play, let $e \in INC_\infty$, and let $r \in \bar{s}$ be such that $e \in petal_X(r)$. Then $\pi_S(\alpha) = re^\omega$.*

PROOF. Since $e \in INC_\infty$ and $e \in petal_X(r)$, I's infinite play is of the form

$$\alpha = v_0 e v_1 e v_2 e v_3 e v_4 e \cdots,$$

where each v_i is a finite word of S_+^*, and each prefix $v_0 e \cdots e v_i$ is equal to r. By idempotence of e, the word α is a factorization of $\beta = v_0 e v_1 e e v_2 e e v_3 e e v_4 e e \cdots$, and the infinite word $\gamma = v_0(e v_1 e)(e v_2 e)(e v_3 e)\cdots$ is also a factorization of β. By Proposition 3.21, γ can be associated with a linked pair $(r, \tilde{e})$, where $\tilde{e}$ is an element of $petal_X(r)$ of the form $\tilde{e} = eve$, for some $v \in S_+^*$. Thus $\pi_S(\gamma) = r\tilde{e}^\omega$. Since γ is a factorization of β, one has $\pi_S(\gamma) = \pi_S(\beta) = r\tilde{e}^\omega$. Also, since α is a factorization of β, one obtains $\pi_S(\alpha) = \pi_S(\beta) = r\tilde{e}^\omega$. Besides, notice that $\tilde{e}$ also appears infinitely often in position r in a factorization α, hence its counter was incremented infinitely often during the play, that is $\tilde{e} \in inc_\infty$. In addition, one has $\tilde{e}e = e\tilde{e} = \tilde{e}$, thus $e \geq \tilde{e}$. But then the minimality of e in inc_∞ implies $\tilde{e} = e$. Finally, one obtains $\pi_S(\alpha) = r\tilde{e}^\omega = re^\omega$. □

CLAIM 7.29. *Let β be II's infinite play. Then $\pi_S(\beta) = t{e_{min}}^\omega$ (where t is the prefix associated with the given main vein).*

PROOF. Let $e \in INC_\infty$ such that $e_{min} = \bar{\sigma}(e)$. The strategy described above ensures that II played e_{min} infinitely often while being in position t. Therefore, II's infinite play can be written as

$$\beta = tu_0 e_{min} u_1 e_{min} u_2 e_{min} \cdots$$

where each u_i is a finite word of elements of the given main vein. Moreover, no element $g < e_{min}$ was played infinitely often by Player II. Otherwise, since the set $\bar{\sigma}^{-1}(g)$ is finite, there would exist an element $f \in inc_\infty$ such that $\bar{\sigma}(f) = g$, contradicting the minimality of e_{min}. Now, since e_{min} is the $\leq$-minimal element of the given main vein which was played infinitely often by II, every product $e_{min}u_i$ is equal to e_{min}. Proposition 3.21 then shows that β can be associated with the linked pair (t, e_{min}). Therefore, $\pi_S(\beta) = t{e_{min}}^\omega$. □

CLAIM 7.30. *One has $\pi_S(\alpha) \in X$ if and only if $\pi_S(\beta) \in X$.*

PROOF. Claim 7.29 shows that $\pi_S(\beta) = t{e_{min}}^\omega$. Now, let $e \in INC_\infty$ such that $e_{min} = \bar{\sigma}(e)$, and let r be the prefix such that $e \in petal_X(r)$. Claim 7.28 proves that $\pi_S(\alpha) = re^\omega$. Finally, since $\bar{\sigma}$ preserves the signature, the elements e and e_{min} have the same sign. Therefore, $\pi_S(\alpha) = re^\omega \in X$ if and only if $\pi_S(\beta) = t{e_{min}}^\omega \in X$. □

□

7.2.6 DAG of main veins

We now prove that the $\mathbb{SG}$-degree of (S, X) only depends on the structure of the partial ordered set $(P/\mathcal{R}, \geq_\mathcal{R})$, and on the lengths of the main veins. Consequently, we shall prune the signed DAG representation of (S, X) by focusing specifically on these two graphical features.

As a direct consequence of Proposition 7.27, we prove that an $\mathbb{SG}$-player can restrict all his waiting moves to the idempotents of some given main veins. For this purpose, we consider once again an imposed version of the game $\mathbb{SG}(X, X)$ where:

- both players are in charge of X, and cannot skip their turns;
- I plays without restriction, exactly like in a regular $\mathbb{SG}$-game.
- II is allowed to play without restriction while moving from one prefix position to another; however, every prefix position s that she reaches must be such that $petal_X(s)$ contains a main vein $V(\bar{s})$ of $flower_X(\bar{s})$, and as long as she remains in such a position s, she is forced to play waiting moves inside $V(\bar{s})$.

PROPOSITION 7.31. *Player II has a winning strategy in this imposed game.*

PROOF. Player II follows Player I as described hereafter: every time I reaches an $\mathcal{R}$-class of prefixes $\bar{s}$, Player II reaches a prefix s_i of this same $\mathcal{R}$-class $\bar{s}$ such that $petal(s_i)X$ contains a main vein V of $flower(\bar{s})X$. Then, as long as I's play remains in $\bar{s}$, II plays idempotents of V as described in Proposition 7.27. Annd so on and so forth. We prove that this strategy is winning for II. By finiteness of the partial ordering $(P/\mathcal{R}, \geq_\mathcal{R})$, Player I is forced to eventually reach an $\mathcal{R}$-class of prefixes $\bar{s}$ inside which he will remain indefinitely. Thence Player II reaches the prefix s_k associated with a given main vein of $flower_X(\bar{s})$, and plays until the end of the play as described in Proposition 7.27. She thus wins the game. □

Proposition 7.31 ensures that only one main vein of each signed flower matters in the computation of the $\mathbb{SG}$-degree. Therefore, the signed DAG representation of a finite pointed ω-semigroup can be simplified by deleting all the signed flowers, but only keeping a single main vein for each, as illustrated in Figure 7.10. Vertices denote the $\mathcal{R}$-classes of prefixes, directed edges describe the $\geq_{\mathcal{R}}$-accessibility relation, and every signed stick represents a main vein of the corresponding signed flower. In this graph representation, the $\mathcal{R}$-classes of prefixes are called *nodes*, the main vein associated with a node n is denoted by $V(n)$, and the length of $V(n)$ by $l(V(n))$.

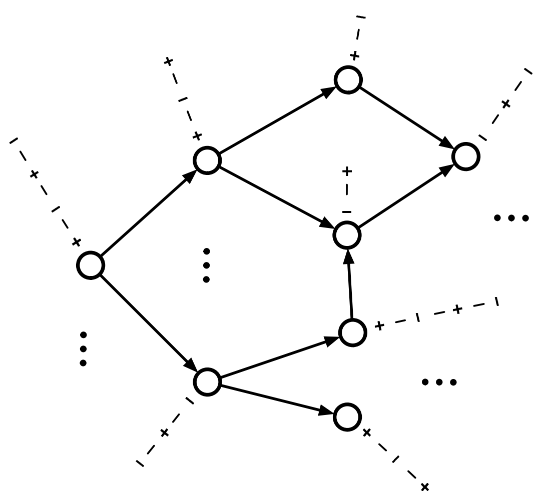


Figure 7.10: The pruned signed DAG representation of a finite pointed ω-semigroup: a labeled DAG, where each node is associated with a signed integer describing the sign and the length of its corresponding main veins.

7.3 Main algorithm

We now present the main algorithm that computes the $\mathbb{SG}$-degree of every finite pointed ω-semigroup. This algorithm works on the pruned signed DAG representation of finite pointed ω-semigroups. It associates every finite pointed ω-semigroup (S, X) with a *signed ordinal* $[\varepsilon_X]\xi_X$. We will further prove that $d_{sg}(X) = \xi_X$, and that X is self-dual if and only if $\varepsilon_X = \pm$, and X is non-self dual if and only if $\varepsilon_X \in \{+, -\}$. This algorithm is a reformulation in terms of ordinals of *Wagner's naming procedure* [41, 27, 43]. We refer to Section 1.1 for the definitions of ordinals, ordinal arithmetic, and signed ordinals.

Algorithm 7.32.

INPUT a finite pointed ω-semigroup (S, X).

OUTPUT a signed ordinal $[\varepsilon_X]\xi_X$.

(1) Compute the pruned signed DAG representation of (S, X).

(2) Define the function $n \longmapsto [\delta_n]\theta_n$ which associates to each node n the signed ordinal $[\delta_n]\theta_n$ given by

$$\delta_n = \begin{cases} + & \text{if the first element of } V(n) \text{ is positive,} \\ - & \text{if the first element of } V(n) \text{ is negative,} \end{cases} \quad \text{and} \quad \theta_n = \omega^{l(V(n))}$$

(3) Then, by backward induction, define the other function $n \longmapsto [\varepsilon_n]\xi_n$ which associates to each node n the signed ordinal $[\varepsilon_n]\xi_n$ as follows.

  (i) If n is a sink, then $[\varepsilon_n]\xi_n = [\delta_n]\theta_n$, where $[\delta_n]\theta_n$ is the signed ordinal associated with n by procedure (2).

  (ii) If n is not a sink, and $m_1, \dots, m_k$ are all the direct succesors of n already associated with their respective signed ordinals $[\varepsilon_1]\xi_1, \dots, [\varepsilon_k]\xi_k$:

  - If among $[\varepsilon_1]\xi_1, \dots, [\varepsilon_k]\xi_k$, there is only one maximal signed ordinal $[\varepsilon_{m_j}]\xi_{m_j}$, then consider the Cantor Normal Form of base ω of the ordinal ξ_{m_j}: $\xi_{m_j} = \omega^{\alpha_l} \cdot \beta_l + \ldots + \omega^{\alpha_0} \cdot \beta_0$,
    - If $\theta_n < \omega^{\alpha_0}$ or if both $\theta_n = \omega^{\alpha_0}$ and $\delta_n = \varepsilon_{m_j}$ (same signs), then set $[\varepsilon_n]\xi_n = [\varepsilon_{m_j}]\xi_{m_j}$.
    - If $\theta_n > \omega^{\alpha_0}$ or if both $\theta_n = \omega^{\alpha_0}$ and $\delta_n \neq \varepsilon_{m_j}$ (opposite signs), then set $[\varepsilon_n]\xi_n = [\delta_n](\xi_{m_j} + \theta_n)$.
  - If among $[\varepsilon_1]\xi_1, \dots, [\varepsilon_k]\xi_k$, there are two opposite maximal ordinals $[\varepsilon_{m_i}]\xi_{m_i}$ and $[\varepsilon_{m_j}]\xi_{m_j}$ (i.e. $\xi_{m_i} = \xi_{m_j}$ and $\varepsilon_{m_i} \neq \varepsilon_{m_j}$), then set $[\varepsilon_n]\xi_n = [\delta_n](\xi_{m_i} + \theta_n)$.

(4) Finally, the finite pointed ω-semigroup (S, X) is associated with the signed ordinal $[\varepsilon_X]\xi_X$ as follows: let $[\varepsilon_1]\xi_1, \dots, [\varepsilon_p]\xi_p$ be the signed ordinals associated by procedure (3) with all the respective sources $s_1, \dots, s_p$:

  - If among $[\varepsilon_1]\xi_1, \dots, [\varepsilon_p]\xi_p$, there is only one maximal signed ordinal $[\varepsilon_{max}]\xi_{max}$, then $[\varepsilon_X]\xi_X = [\varepsilon_{max}]\xi_{max}$.
  - On the other hand, if among $[\varepsilon_1]\xi_1, \dots, [\varepsilon_p]\xi_p$, there are two opposite maximal ordinals $[+]\xi_{max}$ and $[-]\xi_{max}$, then $[\varepsilon_X]\xi_X = [\pm]\xi_{max}$.

The following examples present several applications of this algorithm.

Example 7.33. We consider the finite ω-semigroup $S = (S_+, S_\omega)$ induced by the finite semigroup $S_+ = (\{0, 1, 2, 3, 4, 5\}, \max)$. It follows that $S_\omega = \{[i, j] \mid 0 \leq j \leq i \leq 5\}$. We then consider the ω-subset $X = \{[3, 0], [3, 2]\} \subseteq S_\omega$. The signed DAG representation of (S, X) is illustrated in Figure 7.11. The computation of Algorithm 7.32 is described by the signed ordinals associated with each node. The top one are assigned by procedure (2), and the below ones are computed by procedure (3). At the end, Algorithm 7.32 gives $[\varepsilon_X]\xi_X = [+]\omega^3 = [+](1, 0, 0, 0)$.

Example 7.34. Same finite ω-semigroup S as in Example 7.33, but we consider the ω-subset $Y = \{[2, 0], [2, 2], [3, 1], [4, 0], [4, 2], [4, 3], [4, 4]\} \subseteq S_\omega$. The signed DAG representation of (S, Y) is illustrated in Figure 7.12. Algorithm 7.32 gives $[\varepsilon_Y]\xi_Y = [+]\omega^2 \cdot 3 = [+](3, 0, 0)$.

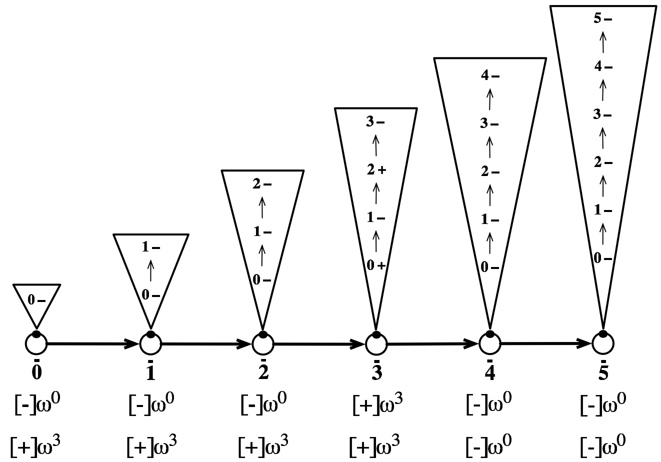


Figure 7.11: The signed DAG representation of (S, X), where each node is associated with the signed ordinals computed by procedures (2) (first line) and (3) (second line). The signed ordinal $[\varepsilon_X]\xi_X$ is the second one associated with the root, namely $[+]\omega^3$.

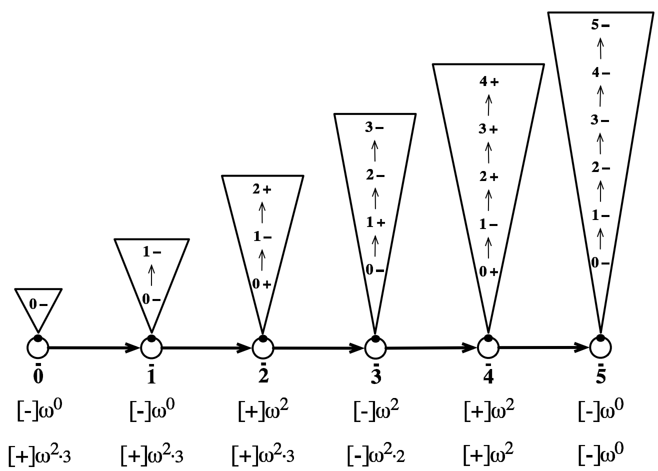


Figure 7.12: The signed DAG representation of (S, Y), where each node is associated with the signed ordinals computed by procedures (2) (first line) and (3) (second line). Algorithm 7.32 gives $[\varepsilon_Y]\xi_Y = [+]\omega^2 \cdot 3$.

Example 7.35. We consider the finite ω-semigroup $T = (T_+, T_\omega)$ induced by the finite monoid $T_+ = \{1, a, b\}$ defined by the relations $aa = ab = a$ and $ba = bb = b$. Thence $T_\omega = \{[1,1], [a,1], [a,a], [a,b], [b,1], [b,a], [b,b]\}$. We then consider the ω-subset

$$Z = \{[1,1], [a,1], [b,a], [b,b]\}.$$

The signed DAG representation of (T, Z) is illustrated in Figure 7.13. Algorithm 7.32 gives $[\varepsilon_Z]\xi_Z = [+](\omega^1 + \omega^0) = [+](\omega + 1) = [+](1, 1)$.

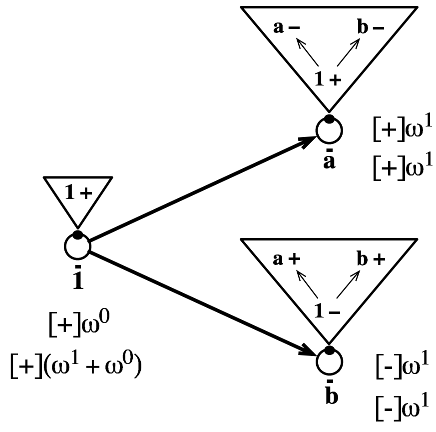


Figure 7.13: The signed DAG representation of (T, Z), where each node is associated with the signed ordinals computed by procedures (2) (first line) and (3) (second line). Algorithm 7.32 finally gives $[\varepsilon_Z]\xi_Z = [+](\omega^1 + \omega^0)$.

Example 7.36. Figure 7.14 illustrates the DAG representation of a finite pointed semigroup (S, X). Two signed ordinals are associated with each node. The top ones are given by the procedure (2). The below ones are then computed by procedure (3). The final signed ordinal associated with X is the second signed ordinal associated with the root, namely $[+](\omega^9 + \omega^4 \cdot 2)$.

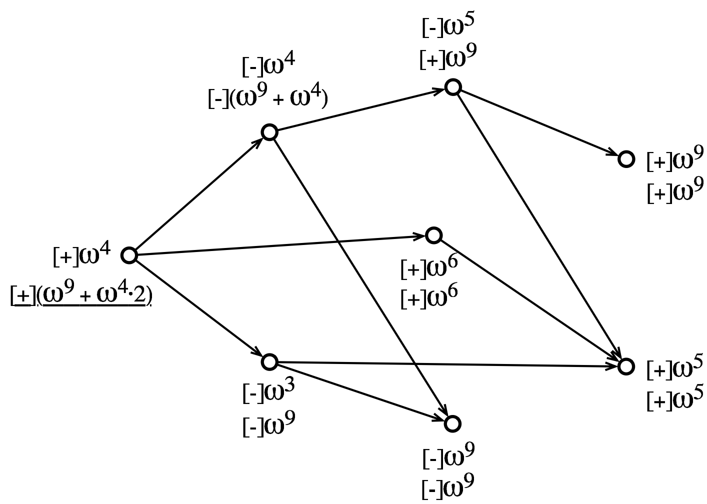


Figure 7.14: Example of a computation of Algorithm 7.32.

Example 7.37. Figure 7.15 illustrates the DAG representation of a finite pointed semigroup (T, Y). The final signed ordinal associated with Y is the second signed ordinal associated with the root, namely $[\pm](\omega^9 + \omega^4 \cdot 2)$.

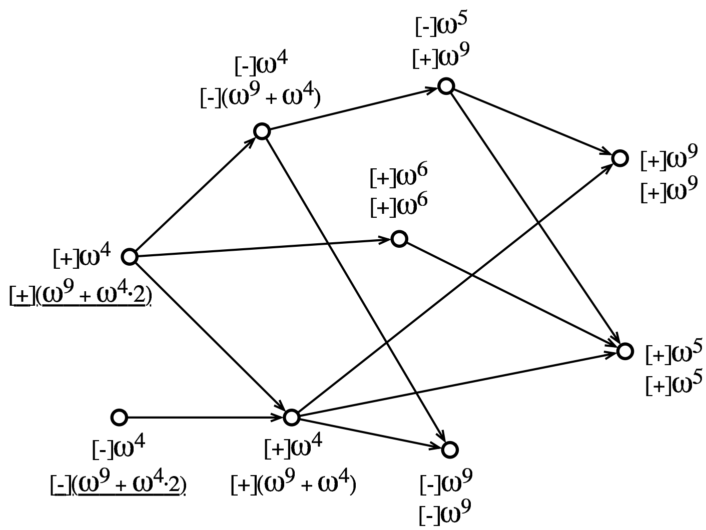


Figure 7.15: Another example of a computation of Algorithm 7.32.

Next theorem states that Algorithm 7.32 computes the precise $\mathbb{SG}$-degree of any ω-subset. The whole following section is devoted to proving this result.

Theorem 7.38. *Let (S, X) be a finite pointed ω-semigroup, and let $[\varepsilon_X]\xi_X$ be the signed ordinal asoociated with X by the main algorithm. Then $d_{sg}(X) = \xi_X$, and X is self-dual if and only if $[\varepsilon_X] = \pm$.*

Example 7.39. Let $A = \{a, b\}$, and let $K = (A^*a)^\omega$. The finite pointed ω-semigroup (S, X) given in Example 7.10 is the syntactic pointed ω-semigroup of K (see Example 3.56). Its signed DAG representation is illustrated in Figure 7.16 below. The main algorithm applied to the signed DAG representation of (S, X) gives $[\varepsilon_X]\xi_X = [-]\omega$. Therefore, X is non-self-dual and $d_{sg}(X) = \omega$. The ω-language K is thus also non-self-dual with Wagner degree equal to ω.

Example 7.40. Let $B = \{a, b, c\}$ and let $L = (a\{b, c\}^* \cup \{b\})^\omega$ be an ω-language over B. The finite pointed ω-semigroup (T, Y) given in Example 7.11 is the syntactic pointed ω-semigroup of L (see Example 3.56). Its signed DAG representation is illustrated in Figure 7.17. The main algorithm applied to the signed DAG representation of (T, Y) gives $[\varepsilon_Y]\xi_Y = [+]\omega^2$. Therefore, Y is non-self-dual and $d_{sg}(Y) = \omega^2$. The ω-language L is thence also non-self-dual with Wagner degree precisely ω^2.

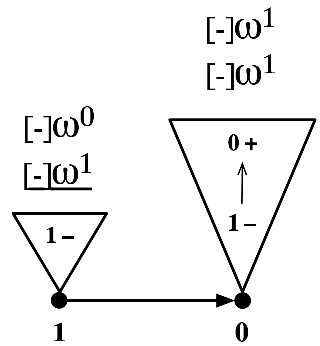


Figure 7.16: The signed DAG representation of (S, X).

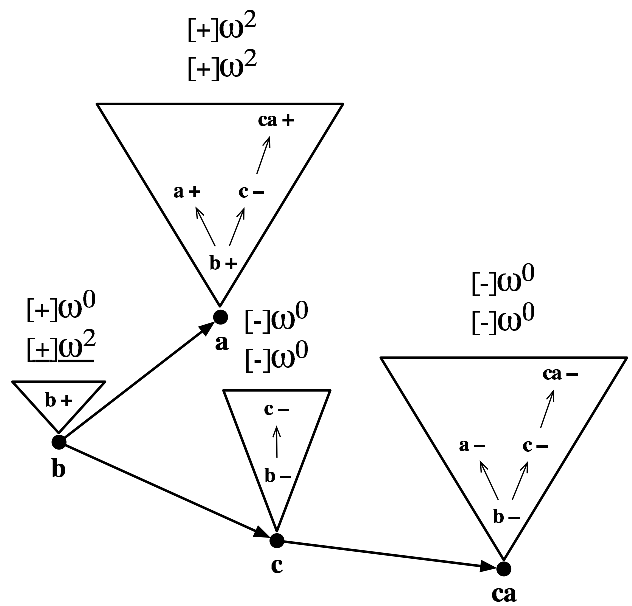


Figure 7.17: The signed DAG representation of (T, Y).

7.4 Correctness of the main algorithm

This section is entirely devoted to proving Theorem 7.38. For this purpose, we introduce three infinite two-player games involving signed ordinals and finite pointed ω-semigroups. The first one provides a game theoretical reformulation of the ordering on signed ordinals. The two other ones define two useful reductions on finite pointed ω-semigroups and signed ordinals. From this point onward, every signed ordinal is assumed to be of the form $[\varepsilon]\xi$, with $\varepsilon \in \{+, -\}$ and $0 < \xi < \omega^\omega$. Signed ordinals of the form $[\pm]\xi$ will be considered separately at the end of the section.

The following preliminary results involve the notions of *playground* and *cut* defined in Section 1.1, as well as the notations of Algorithm 7.32. Hence, if (S, X) is a finite pointed ω-semigroup, then $[\varepsilon_X]\xi_X$ denotes the signed ordinal associated with X after computation of Algorithm 7.32, and if n is a node of the signed DAG representation of (S, X), then $[\delta_n]\theta_n$ and $[\varepsilon_n]\xi_n$ are the signed ordinals associated with n by procedures (2) and (3), respectively. The first results relates the playgrounds of $[\delta_n]\theta_n$ and $[\varepsilon_n]\xi_n$, and proves that the signed ordinals $[\varepsilon_n]\xi_n$ are decreasing along the $\geq_{\mathcal{R}}$-accessibility relation between the nodes.

LEMMA 7.41. *Let (S, X) be a finite pointed ω-semigroup, and let n and n' be two nodes of the signed DAG representation of X.*

(1) *Either $pg(\xi_n) > pg(\theta_n)$, or both $pg(\xi_n) = pg(\theta_n)$ and $\varepsilon_n = \delta_n$.*
(2) *If $n \geq_{\mathcal{R}} n'$, then $[\varepsilon_n]\xi_n \geq [\varepsilon_{n'}]\xi_{n'}$.*

PROOF. We use the notations of the main Algorithm.

(1) We consider all possible values of $[\varepsilon_n]\xi_n$ computed by Algorithm 7.32. If n is a sink, then $[\varepsilon_n]\xi_n = [\delta_n]\theta_n$. Thus obviously both $pg(\xi_n) = pg(\theta_n)$ and $\varepsilon_n = \delta_n$ hold. Otherwise, if $[\varepsilon_n]\xi_n = [\varepsilon_{m_j}]\xi_{m_j}$, then $pg(\xi_n) = \alpha_0$. Therefore, either $pg(\xi_n) > pg(\theta_n)$, or both $pg(\xi_n) = pg(\theta_n)$ and $\varepsilon_n = \delta_n$. Finally, if $[\varepsilon_n]\xi_n = [\delta_n](\xi_{m_j} + \theta_n)$, then $pg(\xi_n) = pg(\theta_n)$, by definition of the ordinal sum.
(2) The signed ordinals $[\varepsilon_n]\xi_n$ are assigned recursively from the sinks to the sources of the signed DAG representation of (S, X). In both cases, if $[\varepsilon_n]\xi_n = [\varepsilon_{m_j}]\xi_{m_j}$ or if $[\varepsilon_n]\xi_n = [\delta_n](\xi_{m_j} + \theta_n)$, then $[\varepsilon_n]\xi_n$ is larger than the signed ordinals assigned to all its direct successors. Therefore, $[\varepsilon_n]\xi_n$ is larger than the signed ordinals assigned to all its successors. ☐

Next result shows that, for every node n, all the cuts of $[\varepsilon_n]\xi_n$ are reachable from n. More precisely, for every node n and every cut c of $[\varepsilon_n]\xi_n$, there exists a node n' such that both $n >_{\mathcal{R}} n'$ and $[\varepsilon_{n'}]\xi_{n'} = c$. This accessibility relation between cuts is illustrated in Figure 7.18 below. This property will be used to describe the strategy of an $\mathbb{SG}$-player moving from cut to cut.

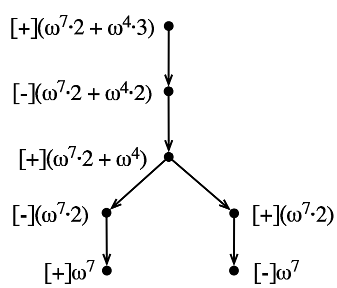


Figure 7.18: In the signed DAG representation of a finite pointed ω-semigroup, for each node n, every cut of $[\varepsilon_n]\xi_n$ is accessible from n.

LEMMA 7.42. *Let n be a node associated with the signed ordinal $[\varepsilon_n]\xi_n$, and let $[\varepsilon]\xi$ be a cut of $[\varepsilon_n]\xi_n$. Then there exists a node n' such that both $n >_{\mathcal{R}} n'$ and $[\varepsilon_{n'}]\xi_{n'} = [\varepsilon]\xi$.*

PROOF. The proof goes by induction on ξ_n. If ξ_n is of the form ω^{n_k}, then there is no possible cut of $[\varepsilon_n]\xi_n$, thence nothing to prove in this case. Otherwise, two cases may occur.

(1) Assume that $[\varepsilon_n]\xi_n = [\varepsilon_n](\omega^{n_k} \cdot p_k + \cdots + \omega^{n_0} \cdot (p_0 + 1))$, for some $k \geq 0$ and $p_0 \geq 0$. Procedure (3) of Algorithm 7.32 ensures that there exists a successor n' of n (possibly $n' = n$) such that $[\varepsilon_{n'}]\xi_{n'} = [\varepsilon_n]\xi_n$, $[\delta_{n'}]\theta_{n'} = [\varepsilon_n]\omega^{n_0}$, and $[\varepsilon_{n'}]\xi_{n'}$ was updated as follows:

$$\begin{aligned}[\varepsilon_{n'}]\xi_{n'} &= [\delta_{n'}]((\omega^{n_k} \cdot p_k + \cdots + \omega^{n_0} \cdot p_0 + \omega^{m_l} \cdot q_l + \cdots + \omega^{m_0} \cdot q_0) + \omega^{n_0}) \\ &= [\delta_{n'}](\omega^{n_k} \cdot p_k + \cdots + \omega^{n_0} \cdot (p_0 + 1)) = [\varepsilon_n]\xi_n,\end{aligned}$$

for some $n_0 > m_l > \ldots > m_0 \geq 0$, or possibly $\omega^{m_l} \cdot q_l + \cdots + \omega^{m_0} \cdot q_0 = 0$. By definition of the main algorithm, there exists a successor m of n' such that

$$[\varepsilon_m]\xi_m = [\varepsilon_m](\omega^{n_k} \cdot p_k + \cdots + \omega^{n_0} \cdot p_0 + \omega^{m_l} \cdot q_l + \cdots + \omega^{m_0} \cdot q_0),$$

where $\varepsilon_m = +$ if and only if $\varepsilon_n = -$. By the induction hypothesis, since $\xi_m < \xi_n$, the node m can access a node associated with each cut of $[\varepsilon_m]\xi_m$. Therefore, m can also access a node associated with each cut of $[\varepsilon_n]\xi_n$, and so does n.

(2) Assume that $[\varepsilon_n]\xi_n = [\varepsilon_n](\omega^{n_k} \cdot p_k + \cdots + \omega^{n_1} \cdot p_1 + \omega^{n_0})$, for some $k \geq 0$. The updating procedure (3) ensures that there exists a successor n' of n (possibly $n' = n$) such that $[\varepsilon_{n'}]\xi_{n'} = [\varepsilon_n]\xi_n$, $[\delta_{n'}]\theta_{n'} = [\varepsilon_n]\omega^{n_0}$, and $[\varepsilon_{n'}]\xi_{n'}$ was updated as follows:

$$\begin{aligned}[\varepsilon_{n'}]\xi_{n'} &= [\delta_{n'}]((\omega^{n_k} \cdot p_k + \cdots + \omega^{n_1} \cdot p_1 + \omega^{m_l} \cdot q_l + \cdots + \omega^{m_0} \cdot q_0) + \omega^{n_0}) \\ &= [\delta_{n'}](\omega^{n_k} \cdot p_k + \cdots + \omega^{n_1} \cdot p_1 + \omega^{n_0}) = [\varepsilon_n]\xi_n,\end{aligned}$$

for some $n_1 > n_0 > m_l > \ldots > m_0 \geq 0$, or also possibly $\omega^{m_l} \cdot q_l + \cdots + \omega^{m_0} \cdot q_0 = 0$.

- If $\omega^{m_l} \cdot q_l + \cdots + \omega^{m_0} \cdot q_0 \neq 0$, the main algorithm ensures that there exists a successor m of n' such that

$$[\varepsilon_m]\xi_m = [\varepsilon_m](\omega^{n_k} \cdot p_k + \cdots + \omega^{n_1} \cdot p_1 + \omega^{m_l} \cdot q_l + \cdots + \omega^{m_0} \cdot q_0).$$

  By the induction hypothesis, since $\xi_m < \xi_n$, the node m can access a node associated with each cut of $[\varepsilon_m]\xi_m$. Therefore, m can also access a node associated with each cut of $[\varepsilon_n]\xi_n$, and so does n.
- If $\omega^{m_l} \cdot q_l + \cdots + \omega^{m_0} \cdot q_0 = 0$, the main algorithm ensures that there exist two successors m and m' of n' such that

$$\begin{aligned}[\varepsilon_m]\xi_m &= [+](\omega^{n_k} \cdot p_k + \cdots + \omega^{n_1} \cdot p_1), \\ [\varepsilon_{m'}]\xi_{m'} &= [-](\omega^{n_k} \cdot p_k + \cdots + \omega^{n_1} \cdot p_1).\end{aligned}$$

  By the induction hypothesis, since $\xi_m < \xi_n$, both nodes m and m' can access a node associated with each cut of $[\varepsilon_m]\xi_m$. Finally, since $[\varepsilon_m]\xi_m$ and $[\varepsilon_{m'}]\xi_{m'}$ are the two largest cuts of n, and n can access m and m', then n can access a node associated with each cut of $[\varepsilon_n]\xi_n$. □

We now introduce three infinite two-player games. The first one provides a game theoretical characterization of the ordering on signed ordinals. The two others involve finite pointed ω-semigroups and signed ordinals.

Let $[\varepsilon_I]\xi_I$ and $[\varepsilon_{II}]\xi_{II}$ be two signed ordinals with $\varepsilon_I, \varepsilon_{II} \in \{+,-\}$. The infinite two-player game $\mathbb{O}([\varepsilon_I]\xi_I, [\varepsilon_{II}]\xi_{II})$ is defined as follows. First of all, Player I chooses a non-zero signed ordinal which is either $[\varepsilon_I]\xi_I$, or a cut of $[\varepsilon_I]\xi_I$, and Player II chooses a non-zero signed ordinal which is either $[\varepsilon_{II}]\xi_{II}$, or a cut of $[\varepsilon_{II}]\xi_{II}$. Then, the possible moves of players I and II are given as follows:

- Let $[\varepsilon]\xi$ be the last signed ordinal played by Player I. Then I can either choose a cut of $[\varepsilon]\xi$, or he can play a positive integer from his current playground: that is an integer q such that $0 \leq q \leq pg(\xi)$.
- Similarly, let $[\delta]\eta$ be the last signed ordinal played by Player II. Then II can either choose a cut of $[\delta]\eta$, or she can play a positive integer from her current playground.

In other terms, each player decreases his signed ordinal cut by cut, and plays integers of his current playground in between. Player I begins. Player II is allowed to skip her turn, provided she plays infinitely often, whereas Player I is not allowed to do so. At the end of the play, the infinite sequences respectively played by I and II consist of two finite strictly decreasing sequences of signed ordinals $[\varepsilon_{I,0}]\xi_{I,0} > \ldots > [\varepsilon_{I,m}]\xi_{I,m}$ and $[\varepsilon_{II,0}]\xi_{II,0} > \ldots > [\varepsilon_{II,n}]\xi_{II,n}$, and two infinite sequences of integers. Let i_I and i_{II} be the largest integers played infinitely often by I and II, respectively. We consider the following parity condition: Player I's play (resp. Player II's play) is said to be *accepted* if $\varepsilon_{I,m} = + \Leftrightarrow i_I$ is even (resp. $\varepsilon_{II,n} = + \Leftrightarrow i_{II}$ is even); it is called *rejected* otherwise. Then, the winning condition is given as follows: Player II wins $\mathbb{O}([\varepsilon_I]\xi_I, [\varepsilon_{II}]\xi_{II})$ if and only if I and II's plays are both accepted or both rejected. This game is illustrated in Figure 7.19.

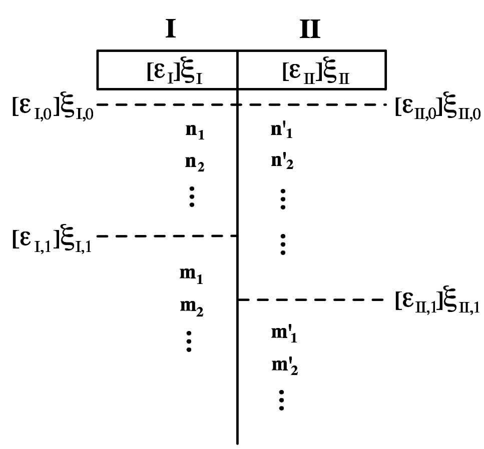


Figure 7.19: The infinite game $\mathbb{O}([\varepsilon_I]\xi_I, [\varepsilon_{II}]\xi_{II})$: players I and II first choose the respective signed ordinals $[\varepsilon_{I,0}]\xi_{I,0}$ and $[\varepsilon_{II,0}]\xi_{II,0}$, and then play either integers from their current playgrounds, or some cut of their previous signed ordinal.

This game induces the following reduction on signed ordinals:

$$[\varepsilon_I]\xi_I \leq_O [\varepsilon_{II}]\xi_{II} \text{ iff Player II has a winning strategy in } \mathbb{O}([\varepsilon_I]\xi_I, [\varepsilon_{II}]\xi_{II}).$$

As usual, we set $[\varepsilon_I]\,\xi_I <_O [\varepsilon_{II}]\,\xi_{II}$ if and only if both $[\varepsilon_I]\,\xi_I \leq_O [\varepsilon_{II}]\,\xi_{II}$ and $[\varepsilon_{II}]\,\xi_{II} \not\leq_O [\varepsilon_I]\,\xi_I$ hold, and also $[\varepsilon_I]\,\xi_I \equiv_O [\varepsilon_{II}]\,\xi_{II}$ if and only if $[\varepsilon_I]\,\xi_I \leq_O [\varepsilon_{II}]\,\xi_{II}$ and $[\varepsilon_{II}]\,\xi_{II} \leq_O [\varepsilon_I]\,\xi_I$.

Furthermore, the infinite two-player game $\mathbb{SGO}(X, [\varepsilon]\xi)$ is defined as follows. Player I plays exactly the same way as in a game $\mathbb{SG}(X, \_)$, and Player II plays as in a game $\mathbb{O}(\_, [\varepsilon]\xi)$. Player II is allowed to skip her turn, but must play infinitely often, whereas Player I is not allowed to do so. Along the play, Player I builds an infinite sequence of elements $(s_0, s_1, \ldots)$, and Player II builds a finite sequence of signed ordinals $[\varepsilon_{II,0}]\xi_{II,0} > \ldots > [\varepsilon_{II,n}]\xi_{II,n}$, and an infinite sequence of integers. The winning condition is the following: Player II wins $\mathbb{SGO}(X, [\varepsilon]\xi)$ if and only if $\pi_S(s_0, s_1, \ldots) \in X \;\Leftrightarrow\;$ her play is accepted. Once again, this game induces the following reduction relation:

$$X \leq_{OSG} [\varepsilon]\xi \text{ if and only if Player II has a winning strategy in } \mathbb{SGO}(X, [\varepsilon]\xi).$$

Finally, the infinite two-player game $\mathbb{OSG}([\varepsilon]\xi, X)$ is defined in a similar way. I plays exactly as in $\mathbb{O}([\varepsilon]\xi, \_)$, and II plays as in $\mathbb{SG}(\_, X)$. Player I begins and cannot skip his turn. Player II is allowed to skip her turn, provided she plays infinitely often. Along the play, Player I builds a finite sequence of signed ordinals $[\varepsilon_{I,0}]\xi_{I,0} > \ldots > [\varepsilon_{I,n}]\xi_{I,n}$, and an infinite sequence of integers, and Player II builds an infinite sequence $(s_0, s_1, \ldots)$ of elements of the semigroup involved. The winning condition is: Player II wins $\mathbb{OSG}([\varepsilon]\xi, X)$ if and only if Player I's play is accepted $\;\Leftrightarrow\; \pi_S(s_0, s_1, \ldots) \in X$. One more time, we define a reduction relation as follows:

$$[\varepsilon]\xi \leq_{OSG} X \text{ if and only if Player II has a winning strategy in } \mathbb{OSG}([\varepsilon]\xi, X).$$

We prove that the determinacy of these three specific games follows from the Borel Wadge determinacy.

Proposition 7.43. *For every signed ordinals $[\varepsilon]\xi$ and $[\varepsilon']\xi'$, and every Borel ω-subset X, the games $\mathbb{O}([\varepsilon]\xi, [\varepsilon']\xi')$, $\mathbb{SGO}(X, [\varepsilon]\xi)$, and $\mathbb{OSG}([\varepsilon]\xi, X)$ are determined.*

Proof. We reduce each of these games to an equivalent Wadge game with Borel winning condition. We conclude by Borel determinacy of Wadge games. More precisely, according to the rules of the $\mathbb{O}$-game, we let L be the set of infinite words of the form

$$([\varepsilon_0]\xi_0)u_0([\varepsilon_1]\xi_1)u_1 \cdots ([\varepsilon_n]\xi_n)\alpha_n,$$

where $([\varepsilon_1]\xi_1, \ldots, [\varepsilon_n]\xi_n)$ is a strictly descending sequence of signed ordinals such that each $[\varepsilon_{i+1}]\xi_{i+1}$ is a cut of $[\varepsilon_i]\xi_i$, each u_i is a finite sequence of integers bounded by $pg(\xi_i)$, and α_n is an infinite sequence of integers bounded by $pg(\xi_n)$. We then equip L with the usual topology over infinite words. Now, for every signed ordinal $[\varepsilon]\xi$, we let $L_{[\varepsilon]\xi} \subseteq L$ be the set of infinite words of the form $([\varepsilon]\xi)u_0([\varepsilon_1]\xi_1)u_1 \cdots ([\varepsilon_k]\xi_k)\alpha_k$ such that the largest integer appearing infinitely often in α_k is even if and only if $[\varepsilon_k] = +$. Then $L_{[\varepsilon]\xi}$ can be written as the conjunction of an open condition ($\mathbf{\Sigma^0_1}$) and a parity condition ($BC(\mathbf{\Sigma^0_2})$), hence it is Borel. In addition, a given player has a winning strategy in $\mathbb{O}([\varepsilon]\xi, [\delta]\eta)$ if and only if this same player has a winning strategy in $\mathbb{W}(L_{[\varepsilon]\xi}, L_{[\delta]\eta})$. Therefore, Borel Wadge determinacy implies the determinacy of $\mathbb{O}$-games. Similary, a given player has a winning strategy in the game $\mathbb{SGO}(X, [\varepsilon]\xi)$ if and only if the same player has a winning strategy in $\mathbb{W}(\pi_S^{-1}(X), L_{[\varepsilon]\xi})$, where π_S is the infinite product of the ω-semigroup involved. Once again, Borel Wadge determinacy proves that $\mathbb{SGO}$-games are determined. The last case is proved in a symmetric way. $\square$

Example 7.44. Let (S, X) be the finite pointed ω-semigroup defined in Example 7.10. We show that $[-]\omega \leq_{OSG} X$ and $X \leq_{SGO} [-]\omega$. We will further prove that these two relations imply $d_{sg}(X) = \omega$. We first describe a winning strategy for Player II in $\mathbb{OSG}([-]\omega, X)$. On his first move, Player I is forced to choose the signed ordinal $[-]\omega$, thence players I and II are forced to play elements 0 or 1. When Player I plays 1, Player II plays 0, and when Player I plays 0, Player II plays 1. Therefore, if I plays infinitely many 1's, then II plays infinitely many 0's, thus both plays are accepted. If I plays finitely many 1's, thus infinitely many 0's, then II plays finitely many 0's, thus infinitely many 1's, and hence both plays are rejected. Therefore, II wins the game, thus $[-]\omega \leq_{OSG} X$. The very same strategy is winning for Player II in the game $\mathbb{SGO}(X, [-]\omega)$, which shows that $X \leq_{SGO} [-]\omega$.

Example 7.45. Let (T, Y) be the finite pointed ω-semigroup defined in Example 7.11. We show that $[+]\omega^2 \leq_{OSG} Y$ and $Y \leq_{SGO} [+]\omega^2$ both hold. We will further prove that these two relations imply $d_{sg}(Y) = \omega^2$. First, we describe a winning strategy for Player II in $\mathbb{OSG}([+]\omega^2, Y)$. On his first move, Player I is forced to choose the signed ordinal $[+]\omega^2$. He can then play 0, 1, or 2 on his next moves. Player II can play the elements a, b, c, or ca. Hence, regardless I' first move, Player II plays a on her first move. Then, Payer II answers to the moves 0, 1, and 2 of Player I by the respective elements b, c, and ca. The signed DAG representation of (T, Y) illustrated in Figure 7.7 ensures that this strategy is winning for Player II, thus $[+]\omega^2 \leq_{OSG} Y$. Conversely, the following strategy is winning for Player II in $\mathbb{SGO}(Y, [+]\omega^2)$. II is forced to choose the signed ordinal $[+]\omega^2$ on her first move. Then, she follows I's moves in his signed DAG representation as described hereafter: every time I hits a negative idempotent of the signed DAG representation of (T, Y), II plays a 1; when I plays a positive idempotent of rank 0 or 2 in the main vein of $flower_Y(\bar{a})$, II answers with the respective elements 0 or 2. The signed DAG representation of (T, Y), illustrated in Figure 7.7, shows that this strategy is winning for Player II. Therefore $Y \leq_{SGO} [+]\omega^2$.

We now present the technical results involved in the proof of Theorem 7.38. First, we show that the $\mathbb{O}$-reduction and the classical ordering on signed ordinals coincide. Second, given a finite pointed ω-semigroup (S, X), the forthcoming lemmas 7.48, 7.49, and 7.50 prove that both relations $X \leq_{SGO} [\varepsilon_X]\xi_X$ and $[\varepsilon_X]\xi_X \leq_{OSG} X$ hold.

Lemma 7.46. *Let $[\varepsilon]\xi$ and $[\varepsilon']\xi'$ be two signed ordinals. Then $[\varepsilon]\xi \leq_O [\varepsilon']\xi'$ if and only if $[\varepsilon]\xi \leq [\varepsilon']\xi'$ (where $\leq$ is the natural ordering on signed ordinals defined in Section 1.1).*

PROOF.

($\Leftarrow$) Assume that $[\varepsilon]\xi \leq [\varepsilon']\xi'$. We prove that Player II has a winning strategy in $\mathbb{O}([\varepsilon]\xi, [\varepsilon']\xi')$. II is in charge of a larger signed ordinal than I in the game $\mathbb{O}([\varepsilon]\xi, [\varepsilon']\xi')$. Therefore, along the play, she can choose her successive signed ordinals in order that her current playground is always larger than I's. More precisely, if I lately chose $[\varepsilon_I]\xi_I$, then she can always choose a signed ordinal $[\varepsilon_{II}]\xi_{II}$ such that either $pg(\xi_{II}) > pg(\xi_I)$, or both $pg(\xi_{II}) = pg(\xi_I)$ and $\varepsilon_{II} = \varepsilon_I$. In both cases, she can suitably answer to I's integers in order to produce a play of the same acceptance. She wins the game, thus $[\varepsilon]\xi \leq_O [\varepsilon']\xi'$.

($\Rightarrow$) Assume that $[\varepsilon]\xi \not\leq [\varepsilon']\xi'$. We prove that Player I has a winning strategy in $\mathbb{O}([\varepsilon]\xi, [\varepsilon']\xi')$. First of all, every time II skips her turn, I answers by playing 0, which does not influence the acceptance of his current play. In addition, if II lately chose the signed ordinal $[\varepsilon_{II}]\xi_{II}$, then I can always choose a signed ordinal $[\varepsilon_I]\xi_I$ such that either $pg(\xi_I) > pg(\xi_{II})$, or both $pg(\xi_I) = pg(\xi_{II})$ and $\varepsilon_{II} \neq \varepsilon_I$. In both cases, he can suitably answer to II's integers in order to produce a play of the opposite acceptance. He wins the game, thus $[\varepsilon]\xi \not\leq_O [\varepsilon']\xi'$. □

REMARK 7.47. In particular, given $0 < \xi < \omega^\omega$, Player I has two winning strategies in the respective games $\mathbb{O}([+]\xi, [-]\xi)$ and $\mathbb{O}([-]\xi, [+]\xi)$. He always chooses a signed ordinal of the opposite sign as II's current one, copies every integer played by II, and plays 0 when II skips her turn. Therefore, both relations $[+]\xi \not\leq_O [-]\xi$ and $[-]\xi \not\leq_O [+]\xi$ hold.

LEMMA 7.48. *Let (S, X) be a finite pointed ω semigroup, let n be a node of X, and let $X_n = \{x \in X \mid x = se^\omega$ for some $s \in n\}$. Then*

(1) $X_n \leq_{SGO} [\delta_n]\theta_n$,

(2) $[\delta_n]\theta_n \leq_{OSG} X_n$.

PROOF. Let $V(n)$ be a main vein associated with n, and s be the prefix such that $V(n) \subseteq petal_X(s)$.

(1) We describe a winning strategy for player II in the game $\mathbb{SGO}(X_n, [\delta_n]\theta_n)$. As long as I's successive positions never reaches n, then II builds a rejecting play and wins. Otherwise, by Proposition 7.27, we may assume, without loss of generality, that I first plays the element s, and then restricts himself to playing only elements of $V(n)$. Hence, II chooses the signed ordinal $[\delta_n]\theta_n$ on her first move. Afterwards, for every idempotent e played by I, she answers by playing the rank of e in $V(n)$. The definition of $[\delta_n]\theta_n$ ensures that her current playground is large enough to do so. Moreover, again by definition of $[\delta_n]\theta_n$, I's play belongs to X_n if and only if II's play is accepted. Therefore, Player II wins the game, hence $X_n \leq_{SGO} [\delta_n]\theta_n$.

(2) We describe a winning strategy for player II in $\mathbb{OSG}([\delta_n]\theta_n, X_n)$. Since θ_n is of the form ω^k, it has no cut, hence I is forced to choose the signed ordinal $[\delta_n]\theta_n$ on his first move. Then II plays the prefix s on her first move. Afterwards, for each integer $0 \leq n \leq pg(\theta_n)$ played by I, she answers by the idempotent of $V(n)$ whose rank is precisely n. By definition of $[\delta_n]\theta_n$, I's play is accepted if and only if II's play belongs to X_n. Consequently, Player II wins the game, thus $[\delta_n]\theta_n \leq_{OSG} X_n$. □

LEMMA 7.49. *Let (S, X) be a finite pointed ω-semigroup associated with the signed ordinal $[\varepsilon_X]\xi_X$, with $\varepsilon_X \in \{+,-\}$. Then $X \leq_{SGO} [\varepsilon_X]\xi_X$.*

PROOF. We show that Player II has a winning strategy in $\mathbb{SGO}(X, [\varepsilon_X]\xi_X)$. By Proposition 7.31, we may assume that I restricts himself to only playing elements of $V(n)$ while his successive positions remain in a given node n. Hence, II first chooses the signed ordinal $[\varepsilon_X]\xi_X$, and then plays as follows. Every time I's play reaches a node n, two cases may occur.

(1) The signed ordinal $[\varepsilon_n]\xi_n$ is a cut of $[\varepsilon_X]\xi_X$. Then Player II chooses the signed ordinal $[\varepsilon_n]\xi_n$, and plays her integers as described in Lemma 7.48 (1). Lemma 7.41 (1) guarantees that her current playground is large enough to play this way.
(2) The signed ordinal $[\varepsilon_n]\xi_n$ is not a cut of $[\varepsilon_X]\xi_X$. Then $[\varepsilon_n]\xi_n$ can be written as $[\varepsilon_n](\alpha + \beta)$, where $[\varepsilon_n]\alpha$ is the largest cut of $[\varepsilon_X]\xi_X$ strictly below $[\varepsilon_n]\xi_n$. Hence, II chooses the signed ordinal $[\varepsilon_n]\alpha$. By Lemma 7.41 (2), since the signed ordinals associated to each node are decreasing along the accessibility relation, the ordinal $[\varepsilon_n]\alpha$ is indeed smaller than or equal to the previous ordinal chosen by II. In addition, this choice ensures that II's playground is larger than $pg(\xi_n)$. Thence, player II can play her integers as described in Lemma 7.48 (1).

By finiteness and acyclicity of the signed DAG representation of (S, X), I's play will eventually become confined to a certain node n'' after a finite amount of time. Then, II plays according to the corresponding signed ordinal, as described in cases (1) or (2). In both cases, Lemma 7.48 (1) ensures that she wins the game. Therefore, $X \leq_{SGO} [\varepsilon_X]\xi_X$. □

LEMMA 7.50. *Let (S, X) be a finite pointed ω-semigroup associated with the signed ordinal $[\varepsilon_X]\xi_X$, with $\varepsilon_X \in \{+,-\}$. Then $[\varepsilon_X]\xi_X \leq_{OSG} X$.*

PROOF. We describe a winning strategy for Player II in $\mathbb{OSG}([\varepsilon_X]\xi_X, X)$. Every time I chooses a signed ordinal $[\varepsilon]\xi$, II reaches one of the accessible $\leq_{\mathcal{R}}$-largest node n such that $[\varepsilon_n]\xi_n = [\varepsilon]\xi$. Lemma 7.42 ensures the existence of such a node. When I plays some integer, II answers exactly as described in Lemma 7.48 (2). By finiteness of strictly descending sequences of signed ordinals, I is forced to choose a final cut of $[\varepsilon_X]\xi_X$. Then, II reaches the suitable corresponding node, and plays as described in Lemma 7.48 (2). She thus wins the game, proving that $[\varepsilon_X]\xi_X \leq_{OSG} X$. □

The forthcoming Proposition 7.51 shows that the $\leq_O$-relation on signed ordinals coincides with the $\leq_{SG}$-relation on ω-subsets. Moreover, we prove that an ω-subset X is self-dual if and only if $\varepsilon_X = \pm$, and non-self-dual if and only if $\varepsilon_X \in \{+,-\}$. We also show that a self-dual ω-subsets and a non-self-dual one located just one level below it in the $\mathbb{SG}$-hierarchy are always associated with the same ordinals by Algorithm 7.32. Finally, the full proof of Theorem 7.38 follows from these statements.

PROPOSITION 7.51. *Let (S, X) and (T, Y) be two finite pointed ω-semigroups associated with the respective signed ordinals $[\varepsilon_X]\xi_X$ and $[\varepsilon_Y]\xi_Y$, and such that $\varepsilon_X, \varepsilon_X \in \{+,-\}$. Then $X \leq_{SG} Y$ if and only if $[\varepsilon_X]\xi_X \leq [\varepsilon_Y]\xi_Y$.*

PROOF. If $X \leq_{SG} Y$, then lemmas 7.49 and 7.50 show that $[\varepsilon_X]\xi_X \leq_{OSG} X \leq_{SG} Y \leq_{SGO} [\varepsilon_Y]\xi_Y$. By composition of strategies, one obtains $[\varepsilon_X]\xi_X \leq_O$

$[\varepsilon_Y]\xi_Y$. Therefore, Lemma 7.46 implies $[\varepsilon_X]\xi_X \leq [\varepsilon_Y]\xi_Y$. Conversely, if $[\varepsilon_X]\xi_X \leq [\varepsilon_Y]\xi_Y$, then Lemma 7.46 shows that $[\varepsilon_X]\xi_X \leq_O [\varepsilon_Y]\xi_Y$. Hence, lemmas 7.49 and 7.50 imply $X \leq_{SGO} [\varepsilon_X]\xi_X \leq_O [\varepsilon_Y]\xi_Y \leq_{OSG} Y$. By composition of strategies, it follows that $X \leq_{SG} Y$. □

Proposition 7.52. *Let (S, X) be a finite pointed ω-semigroup, and let $[\varepsilon_X]\xi_X$ be the signed ordinal associated with X by the main algorithm.*

1. *X is non-self-dual if and only if $\varepsilon_X \in \{+, -\}$,*
2. *X is self-dual if and only if $\varepsilon_X = \pm$.*

Proof. We prove that if $\varepsilon_X \in \{+, -\}$ then X is non-self-dual, and if $\varepsilon_X = \pm$, then X is self-dual. The two converse directions follow from contrapositives of these statements.

1. If $\varepsilon_X \in \{+, -\}$, Procedure (4) of Algorithm 7.32 shows that there exists a source $\bar{s}$ of the signed DAG representation of (S, X), such that $[\varepsilon_{\bar{s}}]\xi_{\bar{s}} = [\varepsilon_X]\xi_X$. Now, let s be a prefix of the $\mathcal{R}$-class $\bar{s}$, and consider the set $s^{-1}X$. The main algorithm applied on $(S, s^{-1}X)$ shows that $[\varepsilon_{s^{-1}X}]\xi_{s^{-1}X} = [\varepsilon_{\bar{s}}]\xi_{\bar{s}} = [\varepsilon_X]\xi_X$. Therefore, Proposition 7.51 shows that $s^{-1}X \equiv_{SG} X$. By Proposition 6.8, the set X is non-self-dual.
2. If $[\varepsilon_X]\xi_X = [\pm]\xi_X$, Procedure (4) shoes that there exist two sources $\bar{s}$ and $\bar{t}$ of the signed DAG representation of (S, X), such that $[\varepsilon_{\bar{s}}]\xi_{\bar{s}} = [+]\xi_X$ and $[\varepsilon_{\bar{t}}]\xi_{\bar{t}} = [-]\xi_X$. Since the signed DAG representations of (S, X) and (S, X^c) have opposite signs, there also exist two sources $\bar{s}'$ and $\bar{t}'$ of the signed DAG representation of (S, X^c) such that $[\varepsilon_{\bar{s}'}]\xi_{\bar{s}'} = [+]\xi_X$ and $[\varepsilon_{\bar{t}'}]\xi_{\bar{t}'} = [-]\xi_X$. Now, let $s \in \bar{s}$, $t \in \bar{t}$, $s' \in \bar{s}'$, and $t' \in \bar{t}'$, and consider the sets $s^{-1}X$, $t^{-1}X$, $s'^{-1}X^c$, and $t'^{-1}X^c$. One has

$$[\varepsilon_{s^{-1}X}]\xi_{s^{-1}X} = [\varepsilon_{\bar{s}}]\xi_{\bar{s}} = [+]\xi_X = [\varepsilon_{\bar{s}'}]\xi_{\bar{s}'} = [\varepsilon_{s'^{-1}X^c}]\xi_{s'^{-1}X^c},$$
$$[\varepsilon_{t^{-1}X}]\xi_{t^{-1}X} = [\varepsilon_{\bar{t}}]\xi_{\bar{t}} = [-]\xi_X = [\varepsilon_{\bar{t}'}]\xi_{\bar{t}'} = [\varepsilon_{t'^{-1}X^c}]\xi_{t'^{-1}X^c}$$

We now prove that Player II has a winning strategy in $\mathbb{SG}(X, X^c)$. Since S is finite, after finitely many moves, I is forced to reach a prefix position u belonging to some $\mathcal{R}$-class of prefixes $\bar{u}$. Hence, he becomes in charge of the set $u^{-1}X$. The maximality properties of $\bar{s}$ and $\bar{t}$ ensure that either

$$[\varepsilon_{u^{-1}X}]\xi_{u^{-1}X} = [\varepsilon_{\bar{u}}]\xi_{\bar{u}} \leq [\varepsilon_{\bar{s}}]\xi_{\bar{s}} = [\varepsilon_{s'^{-1}X^c}]\xi_{s'^{-1}X^c} \text{ or}$$
$$[\varepsilon_{u^{-1}X}]\xi_{u^{-1}X} = [\varepsilon_{\bar{u}}]\xi_{\bar{u}} \leq [\varepsilon_{\bar{t}}]\xi_{\bar{t}} = [\varepsilon_{t'^{-1}X^c}]\xi_{t'^{-1}X^c},$$

thus Proposition 7.51 shows that either $u^{-1}X \leq_{SG} s'^{-1}X^c$, or $u^{-1}X \leq_{SG} t'^{-1}X^c$. Thence, for every $u \in P(S_+)$, there exists $v \in \{s', t'\}$ such that II has a winning strategy σ_u in $\mathbb{SG}(u^{-1}X, v^{-1}X^c)$. Therefore, II first skips her turn until I reaches a prefix position u, then plays the required v, and finally applies the corresponding strategy σ_u. She wins the game $\mathbb{SG}(X, X^c)$. Therefore, $X \leq_{SG} X^c$, and X is self-dual. □

Proposition 7.53. *Let (S, X) and (T, Y) be two finite pointed ω-semigroups such that $[\varepsilon_X]\xi_X = [+]\xi$ and $[\varepsilon_Y]\xi_Y = [\pm]\xi$, for some $0 < \xi < \omega^\omega$. Then $X <_{SG} Y$, and there is no pointed ω-semigroup (U, Z) satisfying $X <_{SG} Z <_{SG} Y$.*

Proof. We first prove that $X <_{SG} Y$. Since $[\varepsilon_Y]\xi_Y = [\pm]\xi$, there exist two sources $\bar{s}$ and $\bar{t}$ of the signed DAG representation of (T, Y) such that $[\varepsilon_{\bar{s}}]\xi_{\bar{s}} =$

$[+]\xi$ and $[\varepsilon_{\bar{t}}]\xi_{\bar{t}} = [-]\xi$. Now, let $s \in \bar{s}$ and $t \in \bar{t}$, and consider the sets $s^{-1}Y$ and $t^{-1}Y$. One has

$$[\varepsilon_{s^{-1}Y}]\xi_{s^{-1}Y} = [\varepsilon_{\bar{s}}]\xi_{\bar{s}} = [+]\xi = [\varepsilon_X]\xi_X,$$
$$[\varepsilon_{t^{-1}Y}]\xi_{t^{-1}Y} = [\varepsilon_{\bar{t}}]\xi_{\bar{t}} = [-]\xi = [\varepsilon_{X^c}]\xi_{X^c},$$

thus Proposition 7.51 shows that $s^{-1}Y \equiv_{SG} X$ and $t^{-1}Y \equiv_{SG} X^c$. In particular, $X \leq_{SG} s^{-1}Y \leq_{SG} Y$, hence $X \leq_{SG} Y$. Moreover, Proposition 7.52 shows that X is non-self-dual and Y is self-dual. Therefore $X <_{SG} Y$. We now prove the second part of the proposition. Let $Z >_{SG} X$. Then $X \equiv_{SG} s^{-1}Y <_{SG} Z$, and also $X^c \equiv_{SG} t^{-1}Y <_{SG} Z$. We prove that $Y \leq_{SG} Z$, by describing a winning strategy for Player II in $\mathbb{SG}(Y, Z)$. Since T is finite, after finitely many moves, I is forced to reach a prefix position u belonging to some $\mathcal{R}$-class of prefixes $\bar{u}$. Then, he finds himself in charge of the set $u^{-1}Y$. The maximality properties of $\bar{s}$ and $\bar{t}$ ensure that either

$$[\varepsilon_{u^{-1}Y}]\xi_{u^{-1}Y} = [\varepsilon_{\bar{u}}]\xi_{\bar{u}} \leq [\varepsilon_{\bar{s}}]\xi_{\bar{s}} = [\varepsilon_{s^{-1}Y}]\xi_{s^{-1}Y} \text{ or}$$
$$[\varepsilon_{u^{-1}Y}]\xi_{u^{-1}Y} = [\varepsilon_{\bar{u}}]\xi_{\bar{u}} \leq [\varepsilon_{\bar{t}}]\xi_{\bar{t}} = [\varepsilon_{t^{-1}Y}]\xi_{t^{-1}Y},$$

and thus Proposition 7.51 shows that either $u^{-1}Y \leq_{SG} s^{-1}Y <_{SG} Z$, or $u^{-1}Y \leq_{SG} t^{-1}Y <_{SG} Z$. Hence, for every $u \in P(U_+)$, II has a winning strategy σ_u in the game $\mathbb{SG}(u^{-1}Y, Z)$. Therefore, II skips her turn until I reaches such a position u, and then applies σ_u. She wins $\mathbb{SG}(Y, Z)$, therefore $Y \leq_{SG} Z$. ☐

THEOREM 7.54. *Let (S, X) be finite pointed ω-semigroup, and let $[\varepsilon_X]\xi_X$ be the signed ordinal associated with X by Algorithm 7.32. Then $d_{sg}(X) = \xi_X$.*

PROOF. First, consider the mapping which associates every non-self-dual ω-subset X with its corresponding signed ordinal $[\varepsilon_X]\xi_X$ (with $\varepsilon_X \in \{+, -\}$). Propositions 7.52 (1) and 7.51 prove that this mapping is an embedding from the $\mathbb{FSG}$-hierarchy of non-self-dual ω-subsets into the hierarchy of signed ordinals of the form $[+]\xi$ or $[-]\xi$. The following section carries the proof that this mapping is onto. Therefore, $d_{sg}(X) = \xi_X$ holds, for every non-self-dual ω-subset. In addition, propositions 7.52 and 7.53 prove that self-dual ω-subsets and the non-self-dual ones located right below in the $\mathbb{FSG}$-hierarchy are associated with the same ordinal by the main algorithm. Therefore, $d_{sg}(X) = \xi_X$ holds, for every self-dual ω-subset. ☐

7.5 Building an ω-subset of any $\mathbb{SG}$-degree

Given any ordinal $0 < \xi < \omega^\omega$, we present two methods for building a finite pointed ω-semigroup (S, X) such that $d_{sg}(X) = \xi$, and X is non-self-dual. The first construction is direct. The second construction describes the algebraic counterpart of the ordinal operations: that is, given two pointed structures (S, X) and (T, Y), and an integer n, we successively introduce:

(1) a new finite pointed ω-semigroup $(S \oplus T, X \oplus Y)$ such that $d_{sg}(X \oplus Y) = d_{sg}(X) + d_{sg}(Y)$.
(2) a new finite pointed ω-semigroup $(S \odot n, X \odot n)$ such that $d_{sg}(X \odot n) = d_{sg}(X) \cdot n$.

(3) a new finite pointed ω-semigroup $(S \odot \omega, X \odot \omega)$ such that $d_{sg}(X \odot \omega) = d_{sg}(X) \cdot \omega$.

These algebraic operations are inspired by the set theoretical operations described by Jacques Duparc in [7, 9]. Consequentlly, starting from the empty set or the whole space, one can inductively build an ω-subset of any given $\mathbb{SG}$-degree.

7.5.1 Direct construction

We proceed in two steps. For a start, we describe ω-subsets covering all $\mathbb{SG}$-degrees strictly below ω. Thereafter, we present ω-subsets covering all the remaining $\mathbb{SG}$-degrees, namely those larger than ω and strictly below ω^ω.

$\mathbb{SG}$-degrees strictly below ω. Given any integer $n \geq 0$, we consider the finite semigroup $S_{n,+} = \{0, 1, \ldots, n\}$ equipped with the max operation. Every element is idempotent, each pair (i, j) is linked if and only if $i \geq j$, for all $i, j \in S_{n,+}$, and all the linked pairs are pairwise non-conjugate. As described in Section 3.3.2, this finite semigroup induces the finite ω-semigroup $S_n = (S_{n,+}, S_{n,\omega})$, where

$$S_{n,\omega} = \{[i, j] \mid 0 \leq j \leq i \leq n\}.$$

PROPOSITION 7.55. *Given any integer $n \geq 0$, let X be the ω-subset of S_n defined by $X = \{[i, j] \in S_{n,\omega} \mid i$ is even$\}$. Then X is non-self-dual, and $d_{sg}(X) = (n + 1)$.*

PROOF. The signed DAG representation of (S_n, X) contains $n+1$ signed flowers $flower_X(\bar{0}), \ldots, flower_X(\bar{n})$, where $\bar{i}$ denotes the $\mathcal{R}$-class of prefixes of i, for all $i \leq n$. By definition of X, the set $flower_X(\bar{i})$ contains only positive idempotents if i is even, and only negative idempotents if i is odd. The $\mathcal{R}$-classes of prefixes satisfy $\bar{i} \geq_{\mathcal{R}} \bar{j}$ whenever $i \leq j$. The signed DAG representation of the structure (S_n, X) is illustrated in Figure 7.20 below. Hence, the main algorithm gives $[\epsilon_X]\xi_X = [+](n + 1)$. Therefore, X is non-self-dual, and $d_{sg}(X) = n + 1$. □

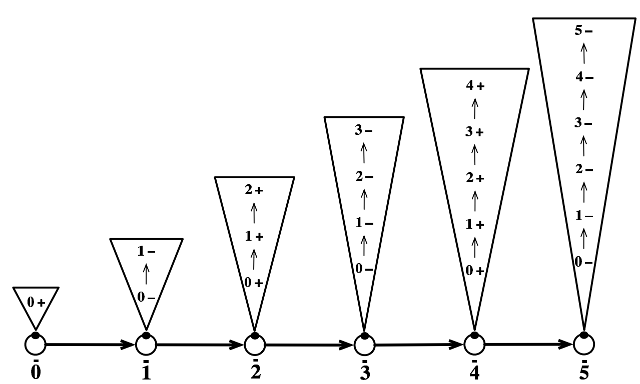


Figure 7.20: The finite pointed ω-semigroup (S_5, X).

$\mathbb{SG}$-degrees larger than ω and strictly below ω^ω. Given any signed ordinal $[\varepsilon]\xi$ such that $\omega \le \xi < \omega^\omega$, and $\varepsilon \in \{+,-\}$, we describe a corresponding finite ω-semigroup, such that playing in an $\mathbb{SG}$-game with respect to this semigroup simulates the fact of playing in an $\mathbb{O}$-game with respect to $[\varepsilon]\xi$.

More precisely, we consider the set $S_{[\varepsilon]\xi,+}$ consisting of the element $\emptyset$, as well as every triples of the form $([\varepsilon_1]\xi_1, [\varepsilon_2]\xi_2, k)$, where $[\varepsilon_1]\xi_1$ and $[\varepsilon_2]\xi_2$ are cuts of $[\varepsilon]\xi$ satisfying the relations $[\varepsilon_2]\xi_2 \le [\varepsilon_1]\xi_1$ and $k \le pg(\xi_2)$. We equip $S_{[\varepsilon]\xi,+}$ with the following operation:

$$(a,b,c)\cdot(a',b',c') = \begin{cases} \emptyset & \text{if } b \not\geq a', \\ (a,b',c') & \text{if } b \geq a' > b', \\ (a,b',\max(c,c')) & \text{if } b = a' = b', \end{cases}$$

and $\emptyset \cdot x = x \cdot \emptyset = \emptyset$, for all $x \in S_{[\varepsilon]\xi,+}$. This operation is well defined in all cases, since the relations $b \ge a'$ and $a' \not> b'$ imply either $b = a' = b'$, or $a' \not\geq b'$, which is forbidden by the definition of the elements of $S_{[\varepsilon]\xi,+}$. For a better comprehension of this product, any triple (a,b,c) of $S_{[\varepsilon]\xi,+}$ can be regarded as a decreasing interval $]a,b[$. Hence, the expression $(a,b,c)\cdot(a',b',c')$ involves the two decreasing intervals $]a,b[$ and $]a',b'[$. The definition of the operation actually states that if $b \not\geq a'$, meaning if $]a,b[$ is not strictly "above" $]a',b'[$, then the product reduces to the empty set, whereas if $]a,b[$ is strictly "above" $]a',b'[$, then the product is the new decreasing interval $]a,b'[$. The third composant is equal to c' if the interval $]a',b'[$ is nonempty, and to $\max(c,c')$ otherwise.

LEMMA 7.56. *The structure $S_{[\varepsilon]\xi,+}$ equipped with this operation is a finite semigroup.*

PROOF. The structure $S_{[\varepsilon]\xi,+}$ is finite since there are only finitely many cuts of $[\varepsilon]\xi$. We prove that the operation defined above is associative. Let $x = (a,b,c)$, $x' = (a',b',c')$, and $x'' = (a'',b'',c'')$ be elements of $S_{[\varepsilon]\xi,+}$.

- If $b \not\geq a'$ or $b' \not\geq a''$ holds, then $(x \cdot x') \cdot x'' = x \cdot (x' \cdot x'') = \emptyset$.
- If both $b \ge a'$ and $b' \ge a''$ hold, then $(x \cdot x') \cdot x'' = x \cdot (x' \cdot x'')$, since the max operation is associative. □

This finite semigroup $S_{[\varepsilon]\xi,+}$ is build in such that any play in the $\mathbb{SG}$-game with respect to $S_{[\varepsilon]\xi,+}$ simulates a play in the $\mathbb{O}$-game with respect to $[\varepsilon]\xi$. Indeed, playing the element $([\varepsilon_1]\xi_1, [\varepsilon_2]\xi_2, k)$ in the $\mathbb{SG}$-game refers to the following facts in the $\mathbb{O}$-game: the maximal signed ordinal previously played is $[\varepsilon_1]\xi_1$; we are now in charge of $[\varepsilon_2]\xi_2$; we play the integer k. Thus, in order to respect the rules of the $\mathbb{O}$-game, we require that $[\varepsilon_1]\xi_1$ and $[\varepsilon_2]\xi_2$ are cuts of $[\varepsilon]\xi$, and that both relations $[\varepsilon_2]\xi_2 \le [\varepsilon_1]\xi_1$ and $k \le pg(\xi_2)$ hold. Moreover, since playing a non-descending sequence of signed ordinals is forbidden in the $\mathbb{O}$-game, we set a trash $\emptyset \in S_{[\varepsilon]\xi,+}$ in order to collect the products of elements referring to these illegal plays. These unfortunate products are of the precise form $([\varepsilon_1]\xi_1, [\varepsilon_2]\xi_2, k) \cdot ([\varepsilon_1']\xi_1', [\varepsilon_2']\xi_2', k')$, with $[\varepsilon_2]\xi_2 \not\geq [\varepsilon_1']\xi_1'$, for they refer to the following interpretation in the $\mathbb{O}$-game: the last signed ordinal played in the left sequence is not bigger than the maximal signed ordinal played in the right sequence, so that the concatenated sequence of the signed ordinals already played is not descending, as requested. An $\mathbb{SG}$-player moving in $S_{[\varepsilon]\xi,+}$ is said to play *legally* (with respect to the rules of the $\mathbb{O}$-game) if he never reaches the position $\emptyset$. He plays *illegally* otherwise. Notice that any illegal play yields a final factorization of the form $[\emptyset, e]$, for some $e \in E(S_{[\varepsilon]\xi,+})$.

The idempotent of $S_{[\varepsilon]\xi,+}$ are precisely the element $\emptyset$, and the triples the form (a, a, c). Linked pairs are either of the form $((a, b, c), (b, b, c'))$, or of the form $(\emptyset, (a, a, c))$. All of them are pairwise non conjugate. As usual, the finite semigroup $S_{[\varepsilon]\xi,+}$ induces the finite ω-semigroup $S_{[\varepsilon]\xi} = \big(S_{[\varepsilon]\xi,+}, S_{[\varepsilon]\xi,\omega}\big)$, as described in Section 3.3.2.

PROPOSITION 7.57. *Let $[\varepsilon]\xi$ be a signed ordinal such that $\omega \leq \xi < \omega^\omega$ and $\varepsilon \in \{+, -\}$, and let $X \subseteq S_{[\varepsilon]\xi,\omega}$ be defined by*

$$X = \{[s, e] \mid s \neq \emptyset \textit{ and } e = ([\varepsilon_1]\xi_1, [\varepsilon_1]\xi_1, k) \textit{ such that } parity(k) = \varepsilon_1\}.$$

Then $d_{sg}(X) = \xi$, and X is non-self-dual.

PROOF. We both show that $X \leq_{SGO} [\varepsilon]\xi$ and $[\varepsilon]\xi \leq_{OSG} X$, meaning, by lemmas 7.49 and 7.50, that $[\varepsilon_X]\xi_X = [\varepsilon]\xi$ (where $[\varepsilon_X]\xi_X$ is the signed ordinal associated with X by Algorithm 7.32). Proposition 7.52 and Theorem 7.54 lead to the conclusion.

- We describe a winning strategy for Player II in the game $\mathbb{SGO}(X, [\varepsilon]\xi)$. As long as I plays *legally*, then II copies I in the following sense: when I plays (a, b, c), II plays the signed ordinal b and the integer c. By definitions of the semigroup $S_{[\varepsilon]\xi,+}$ and the ω-subset X, she wins if this legal situation persists until the end of the play. However, if I plays *illegally*, he will produce an infinite play which factorizes into the form $[\emptyset, e]$, for some $e \in E(S_{[\varepsilon]\xi,+})$, hence which does not belong to X. Then, since ξ is larger than ω, II can always choose (or stays with) the smallest cut $[\varepsilon']\xi'$ of $[\varepsilon]\xi$ offering her a playground strictly larger than 0. She then plays 0's until the end of the game if $\varepsilon' = -$, and 1's if $\varepsilon' = +$. She produces a rejecting play, and wins. Thence $X \leq_{SGO} [\varepsilon]\xi$.
- We describe a winning strategy for Player II in $\mathbb{OSG}([\varepsilon]\xi, X)$. II copies I as follows: when I plays the signed ordinal $[\epsilon]\xi$ and the integer k, then II plays $([\epsilon]\xi, [\epsilon]\xi, k)$. She wins the game, which means $[\varepsilon]\xi \leq_{OSG} X$. □

7.5.2 The algebraic counterpart of the ordinal operations

First of all, given two finite pointed ω-semigroups (S, X) and (T, Y), we describe another finite pointed ω-semigroup $(S \oplus T, X \oplus Y)$, such that $d_{sg}(X \oplus Y) = d_{sg}(X) + d_{sg}(Y)$. Furthermore, given an integer $n > 0$, we describe a finite pointed structure $(S \odot n, X \odot n)$, such that $d_{sg}(X \odot n) = d_{sg}(X) \cdot n$. Finally, we describe another finite pointed ω-semigroup $(S \odot \omega, X \odot \omega)$, such that $d_{sg}(X \odot \omega) = d_{sg}(X) \cdot \omega$. Consequently, starting from either the empty or the full ω-subset of $\mathbb{SG}$-degree 1, one can build by induction an ω-subset of any given $\mathbb{SG}$-degree (strictly between 0 and ω^ω).

Let $S = ((S_+, *), S_\omega)$ and $T = ((T_+, \circledast), T_\omega)$ be two finite ω-semigroups, and let $X \subseteq S_\omega$ and $Y \subseteq T_\omega$ be two non-self-dual ω-subsets. Let also $(S'_+, *')$ be a disjoint copy of the semigroup $(S_+, *)$ (i.e. $a' \in S'_+$ if and only if $a \in S_+$, and $a' *' b' = c'$ if and only if $a * b = c$). We consider the set

$$(S \oplus T)_+ = T_+^1 \cup S_+^1 \cup S_+^{'1} \cup \{0\}$$

equipped with the following operation

$$a \cdot b = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0, \\ 0 & \text{if } a \in S^1_+ \text{ and } b \in S'^1_+, \\ 0 & \text{if } a \in S'^1_+ \text{ and } b \in S^1_+, \\ a & \text{if } a \in S^1_+ \text{ and } b \in T^1_+, \\ b & \text{if } b \in S^1_+ \text{ and } a \in T^1_+, \\ a & \text{if } a \in S'^1_+ \text{ and } b \in T^1_+, \\ b & \text{if } b \in S'^1_+ \text{ and } a \in T^1_+, \\ a \circledast b & \text{if } a \text{ and } b \text{ belong to } T^1_+, \\ a * b & \text{if } a \text{ and } b \text{ belong to } S^1_+, \\ a *' b & \text{if } a \text{ and } b \text{ belong to } S'^1_+. \end{cases}$$

The element 0 is a zero; the product of any element of S^1_+ with any element of S'^1_+ is 0, and *vice versa*; the product of any two elements of either T^1_+, or S^1_+, or S'^1_+, coincides with the products of the respective monoids $(T^1_+, \circledast)$, $(S^1_+, *)$ or $(S'^1_+, *)$; elements of S^1_+ and S'^1_+ absorb the elements of T^1_+ from the left and the right, as illustrated by the following tabular:

$\cdot \nearrow$	T^1_+	S^1_+	S'^1_+	0
T^1_+	table of T^1_+	table of abs. by S^1_+	table of abs. by S'^1_+	0
S^1_+	table of abs. by S^1_+	table of S^1_+	0	0
S'^1_+	table of abs. by S'^1_+	0	table of S'^1_+	0
0	0	0	0	0

LEMMA 7.58. *The structure $((S \oplus T)_+, \cdot)$ is a semigroup.*

PROOF. The respective operations of S^1_+, S'^1_+, and T^1_+, and the absorption relations from the left and the right are associative. Adding a zero does not affect the associativity. Therefore, the operation defined on $(S \oplus T)_+$ is associative. □

The DAG representation of $(S \oplus T)_+$, consists of four sub-DAGs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$, induced by the respective elements of T^1_+, S^1_+, S'^1_+, and by 0. These DAGs satisfy the following properties:

- $\mathcal{G}_1$ is the DAG representation of T^1_+, therefore it contains the DAG representation of T_+.
- $\mathcal{G}_2$ is the DAG representation of S^1_+, possibly enriched by some new linked pairs induced by the products of the form $x \cdot y$, for $x \in S^1_+$ and $y \in T^1_+$. Hence, it contains the DAG representation of S_+.
- Similarly, $\mathcal{G}_3$ contains the DAG representation of S'_+.
- $\mathcal{G}_4$ is the single-petal flower $flower(\bar{0})$ associated with the $\mathcal{R}$-class of prefixes $\bar{0} = \{0\}$. This petal contains all idempotents of $(S \oplus T)_+$.

These DAGs are related as follows: $\mathcal{G}_1$ is not $\geq_{\mathcal{R}}$-accessible from any other $\mathcal{G}_i$; $\mathcal{G}_2$ and $\mathcal{G}_3$ are both $\geq_{\mathcal{R}}$-accessible from $\mathcal{G}_1$, but there is no $\geq_{\mathcal{R}}$-accessibility relation between them; $\mathcal{G}_4$ is $\geq_{\mathcal{R}}$-accessible from $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$. The DAG representation of $(S \oplus T)_+$ is illustrated in Figure 7.21.

As described in Section 3.3.2, the finite semigroup $(S \oplus T)_+$ can be extended to the finite ω-semigroup

$$S \oplus T = ((S \oplus T)_+, (S \oplus T)_\omega),$$

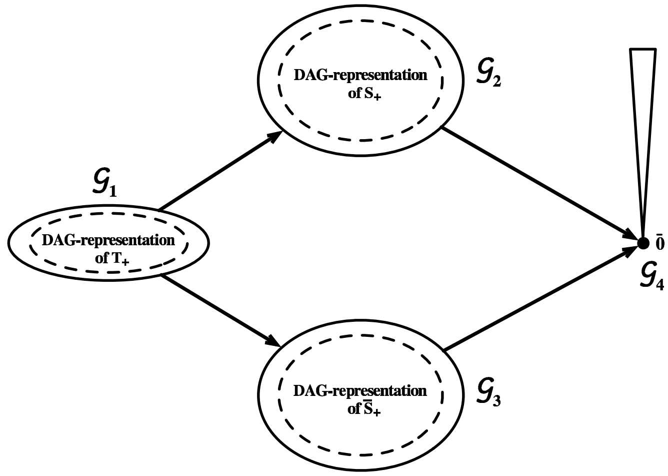


Figure 7.21: The DAG representation of $(S \oplus T)_+$. The accessibility relation between two DAGs $\mathcal{G}_i$ and $\mathcal{G}_j$ means that each node of $\mathcal{G}_j$ is $\geq_{\mathcal{R}}$-accessible from each node of $\mathcal{G}_i$.

where $(S \oplus T)_\omega = \{[s,e] \mid (s,e)$ is a linked pair of $(S \oplus T)_+\}$. Moreover, since the DAG representation of T_+ is contained in $\mathcal{G}_1$, there exists a signature of $\mathcal{G}_1$ corresponding to an ω-subset $\bar{Y} \subseteq (S \oplus T)_\omega$, such that $d_{sg}(\bar{Y}) = d_{sg}(Y)$ and $\bar{Y} \equiv_{SG} Y$. Since the DAG representation of S_+ is contained in $\mathcal{G}_2$, there also exists a signature of $\mathcal{G}_2$ corresponding to an ω-subset $\bar{X} \subseteq (S \oplus T)_\omega$, such that $d_{sg}(\bar{X}) = d_{sg}(X)$, and $\bar{X} \equiv_{SG} X$. By the same argument again, there exists a signature of $\mathcal{G}_3$ corresponding to an ω-subset $\bar{X}' \subseteq (S \oplus T)_\omega$, such that $d_{sg}(\bar{X}') = d_{sg}(X)$, but $\bar{X}' \equiv_{SG} X^c$. Using all these notations, one obtains the following result.

Proposition 7.59.

- *If $d_{sg}(X) > 1$ or $d_{sg}(Y) > 1$, by setting $X \oplus Y = \bar{X} \cup \bar{X}' \cup \bar{Y} \subseteq (S \oplus T)_\omega$, one has $d_{sg}(X \oplus Y) = d_{sg}(X) + d_{sg}(Y)$.*
- *If $d_{sg}(X) = d_{sg}(Y) = 1$, by setting $X \oplus Y = \{[0,e] \mid e \in E((S \oplus T)_+)\} \subseteq (S \oplus T)_\omega$, one has $d_{sg}(X \oplus Y) = d_{sg}(X) + d_{sg}(Y) = 2$.*

Proof. For the first case, let r_1, r_2, r_3 be the respective roots of $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$. The main algorithm applied separately to the sub-DAGs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ assigned according to $\bar{Y}$, $\bar{X}$, $\bar{X}'$ respectively gives $[\varepsilon_{r_1}]\xi_{r_1} = [+]d_{sg}(Y)$, $[\varepsilon_{r_2}]\xi_{r_2} = [+]d_{sg}(X)$, and $[\varepsilon_{r_3}]\xi_{r_3} = [-]d_{sg}(X)$. Then, the accessibility relations between these DAGs imply that $d_{sg}(X \oplus Y) = d_{sg}(X) + d_{sg}(Y)$. In the second case, the set $flower_X(\bar{0})$ contains only positive idempotents, and every other signed flower contains only negative idempotents. Therefore, the main algorithm gives $d_{sg}(X \oplus Y) = \omega^0 \cdot 2 = 2 = d_{sg}(X) + d_{sg}(Y)$. □

We now describe the algebraic counterpart of the ordinal finite multiplication. Let (S, X) be a finite pointed ω-semigroup. For any integer $n > 0$, we define the finite pointed ω-semigroup $(S \odot n, X \odot n)$ by induction on n as follows:

- $(S \odot 1, X \odot 1) = (S, X)$,
- $S \odot (n+1) = (S \odot n) \oplus S$, and $X \odot (n+1) = (X \odot n) \oplus X$.

PROPOSITION 7.60. *Let $n > 0$, then $d_{sg}(X \odot n) = d_{sg}(X) \cdot n$.*

PROOF. A direct consequence of Proposition 7.59. $\square$

We finally focus on the algebraic counterpart of the ordinal multiplication by ω. We recall that, given any ordinal ξ with Cantor normal form $\xi = \omega^{n_k} \cdot p_k + \cdots + \omega^{n_0} \cdot p_0$, the equality $\xi \cdot \omega = \omega^{n_k+1}$ holds.

Let $S = (S_+, S_\omega)$ be a finite ω-semigroup, and $X \subseteq S_\omega$, such that $d_{sg}(X) = \xi = \sum_{i=k}^{0} \omega^{n_i} \cdot p_i$. We then consider the finite monoid

$$(S \odot \omega)_+ = (S_+ \cup \{1\}, \cdot)$$

equipped with the operation of S_+ completed as follows: $a \cdot 1 = 1 \cdot a = a$, for all $a \in (S \odot \omega)_+$. The DAG representation of $(S \odot \omega)_+$, illutrated in figures 7.22 and 7.23, corresponds to the following transformation of the DAG representation of S_+:

- The flower $flower(\bar{1})$ associated with the $\mathcal{R}$-class of prefixes $\bar{1} = \{1\}$ appears. It simply consists of the single-petal $petal(1) = \{1\}$. The $\mathcal{R}$-class $\bar{1}$ can $\geq_{\mathcal{R}}$-access any other $\mathcal{R}$-class of prefixes $\bar{s}$.
- The idempotent 1, strictly $\leq$-larger than any other, appears in each petal of each flower of S_+. Therefore, the length of every chain of idempotents of S_+ is increased by 1.

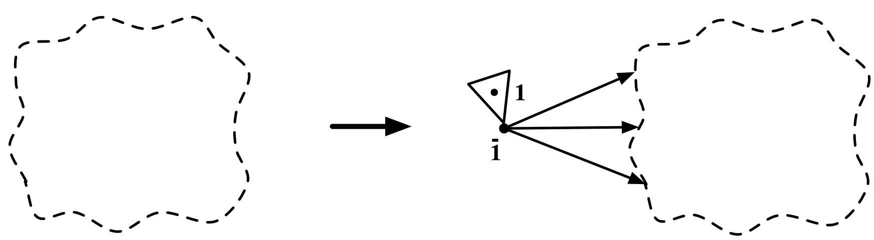


Figure 7.22: The tranformation of the DAG representation of S_+ into the one of $(S \odot \omega)_+$: the new flower $flower(\bar{1})$ associated with the $\mathcal{R}$-class of prefixes $\bar{1}$ appears.

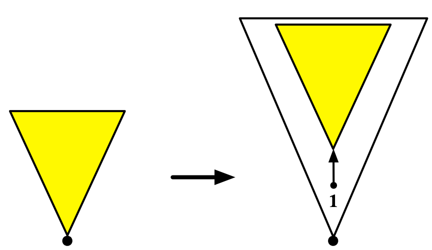


Figure 7.23: The transformation of a petal of S_+ into a petal of $(S \odot \omega)_+$: the new idempotent 1, strictly $\leq$-larger than any other, appears.

Moreover, since $d_{sg}(X) = \xi = \sum_{i=k}^{0} \omega^{n_i} \cdot p_i$, there exists at least one chain of idempotents $e_0 > \cdots > e_{n_k}$ in some petal of the DAG representation of S_+. Consequently, one can find the chain of idempotents $1 > e_0 > \cdots > e_{n_k}$ in some petal of the DAG representation of $(S \odot \omega)_+$. Finally, the monoid $(S \odot \omega)_+$ can be extended to the finite ω-semigroup

$$S \odot \omega = ((S \odot \omega)_+, (S \odot \omega)_\omega),$$

where $(S \odot \omega)_\omega) = \{[s, e] \mid (s, e) \text{ is a linked pair of } (S \odot \omega)_+\}$. Using all these notations, one obtains the following proposition.

Proposition 7.61. *Let s be a prefix of $(S \odot \omega)_+$ such that the chain of idempotents $1 > e_0 > \ldots > e_{n_k}$ belongs to petal(s). Let us also set*

$$X \odot \omega = \{[s, e_{2i}] \mid 0 \leq 2i \leq n_k\} \subseteq (S \odot \omega)_\omega,$$

then $d_{sg}(X \odot \omega) = d_{sg}(X) \cdot \omega = \omega^{n_k+1}$.

Proof. The signature according to $X \odot \omega$ yields the unique maximal alternating chain $1 > e_0 > \ldots > e_{n_k}$ of length $n_k + 1$ in the signed DAG representation of $(S \odot \omega, X \odot \omega)$. By Algorithm 7.32 and Theorem 7.54, one has $d_{sg}(X \odot \omega) = \omega^{n_k+1} = d_{sg}(X) \cdot \omega$. □

7.6 Normal forms

We now describe the *algebraic invariants* of the $\mathbb{FSG}$-hierarchy. As in [41, 34, 10], we prove that the $\mathbb{SG}$-degree of (S, X) is completely characterized by some kind of maximal alternating tree(s) contained in the signed DAG representation of (S, X) – called the *normal form* of (S, X). Then any two finite pointed ω-semigroups share the same $\mathbb{SG}$-degree if and only if they have the same normal form, up to some relation of bisimilarity. The normal form of (S, X) is a reformulation in this algebraic context of the notions of ξ-chain presented in [10], or μ_α-alternating tree described in [34], or also binary tree-like sequences of superchains described in [41].

In the sequel, the signed DAG representation of finite pointed ω-semigroups are regarded as labeled DAGs of the form $G = (V, E, p)$, where $p : V \longrightarrow \{+, -\} \times \mathbb{N}_+$ is a priority function which associates with every node n the sign and length of the main vein $V(n)$. Throughout this section, we first introduce a notion of *bisimulation* on these DAGs. We then define a *tree representation* of signed ordinals, and use this notion in order to define the *normal form* of any finite pointed ω-semigroup. We conclude by proving the invariance properties of normal forms.

Let $G = (V, E, p)$ and $G' = (V', E', p')$ be two finite DAGs, where $p : V \longrightarrow \{+, -\} \times \mathbb{N}_+$ and $p' : V' \longrightarrow \{+, -\} \times \mathbb{N}_+$ are priority functions. A *bisimulation* over G and G' is a left-and-right-total binary relation $B \subseteq V \times V'$ such that $(n, n') \in B$ if and only if

- when n and n' are sinks, then $p(n) = p'(n')$;
- when n or n' are not sinks, then $p(n) = p'(n')$, and for every edge $(n, m) \in E$, there exists an edge $(n', m') \in E'$ such that $(m, m') \in B$, and conversely, for every edge $(n', m') \in E'$, there exists an edge $(n, m) \in E$ such that $(m, m') \in B$.

When there exists a bisimultation relation over G and G', we say that G and G' are *bisimilar*, and we denote it by $G \approx G'$. As a matter of fact, the DAGs G and G' are bisimilar if and only if they *contain the same paths*, i.e. for every path in G, there exists a path in G' visiting exactly the same priorities, and conversely, for every path in G', one can also find a path in G visiting the same priorities.

The definition of bisimultation can be apprehended by means of games. To this end, we define the finite two-player game with perfect information $\mathbb{BIS}(G, G')$, where Player II tries to show that G and G' are bisimilar, whereas Player I tries to show the opposite. The rules are the following:

- On his first move, I chooses a source of either G or G'. If he chooses a source s of G, II must answer by choosing a source s' of G' such that $p(s) = p'(s')$. If he chooses a source s' of G', II must answer by choosing a source s of G such that $p(s) = p'(s')$.
- After every move of II, let $n \in V$ and $n' \in V'$ be the two nodes previously chosen respectively by I and II. Then, if it still exists, I chooses either a successor of n, or a successor of n'. If he chooses a successor m of n, then II must answer by choosing a successor m' of n' such that $p(m) = p'(m')$. If he chooses a successor m' of n', then II must answer by choosing a successor m of n such that $p(m) = p'(m')$.

If II is not able to answer correctly to I's move, she looses. If both players cannot choose a further successor node, II wins. Otherwise, the player which cannot choose a successor node whereas his opponent can do so looses the game.

PROPOSITION 7.62. *Let $G = (V, E, p)$ and $G' = (V', E', p')$ be two finite DAGs. Then $G \approx G'$ if and only if Player II has a winning strategy in $\mathbb{BIS}(G, G')$.*

PROOF. If $G \approx G'$, there exists a bisimulation relation B over G and G' which induces the following winning strategy for Player II in $\mathbb{BIS}(G, G')$: every time I chooses a node $x \in V$, II answers by an appropriate node $x' \in V'$ such that $(x, x') \in B$, and every time I chooses a node $x' \in V'$, II answers by a node $x \in V$ such that $(x, x') \in B$. Conversely, assume that Player II has a winning strategy in $\mathbb{BIS}(G, G')$. Then for every path $(x_0, \ldots, x_n)$ in G, there exists a path $(x'_0, \ldots, x'_n)$ in G' such that $p(x_i) = p'(x'_i)$, for all i; and conversely, for every path $(y'_0, \ldots, y'_n)$ in G', there exists a path $(y_0, \ldots, y_n)$ in G', such that $p(y_i) = p'(y'_i)$, for all i. The set B of such pairs (x_i, x'_i) and (y_i, y'_i) obtained by considering II's answer to every possible paths $(x_0, \ldots, x_n)$ in G and $(y_0, \ldots, y_n)$ in G' is a bisimulation over G and G'. Therefore, $G \approx G'$. □

EXAMPLE 7.63. Figure 7.24 illustrates two bisimilar labeled DAGs G and G'. The forest G' is called the *unfolding* of G: it contains exactly the same paths, but has no more vertices with more than one incoming edge. A DAG and its unfolding are always bisimilar.

We now define the *tree representation* of any signed ordinals $[\varepsilon]\xi$ by induction on the Cantor normal form of ξ. This representation is inspired by the notion of a ξ-chain introduced in Chapter 5.

(1) If $[\varepsilon]\xi$ is of the form $[+]\omega^n \cdot p$ (respectively $[-]\omega^n \cdot p$), for some integers $n \geq 0$ and $p > 0$, its tree representation consists of a "linear" sequence of p accessible nodes alternately labeled by $+n$ and $-n$ (respectively $-n$ and $+n$), as illustrated in Figure 7.25.

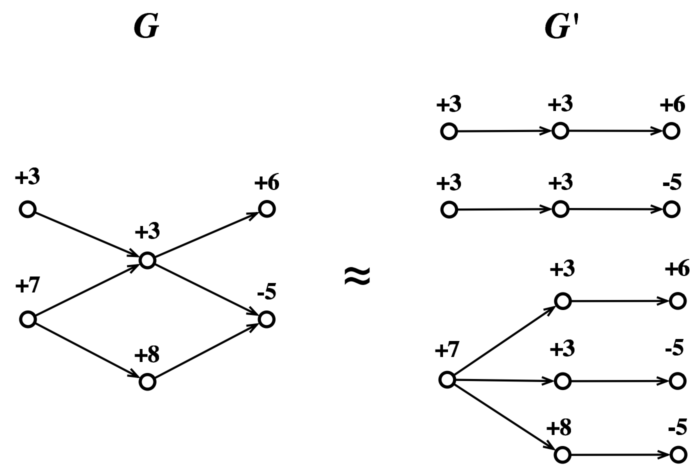


Figure 7.24: Two bisimilar labeled DAGs.

(2) If $[\varepsilon]\xi$ is of the form $[\pm]\omega^n \cdot p$, for some integers $n \geq 0$ and $p > 0$, its tree representation consists of the two disjoint tree representations of $[+]\omega^n \cdot p$ and $[-]\omega^n \cdot p$, as illustrated in Figure 7.25.

(3) If the Cantor normal form of $[\varepsilon]\xi$ is of the form $[+](\eta+\omega^n \cdot p)$ (respectively $[-](\eta + \omega^n \cdot p)$), for some $0 < \eta < \omega^\omega$, and some integers $n \geq 0$ and $p > 0$, its tree representation consists of the tree representation of $[+]\omega^n \cdot p$ (respectively $[-]\omega^n \cdot p$) related to the two disjoint tree representations of $[+]\eta$ and $[-]\eta$, as illustrated in Figure 7.26.

(4) If the Cantor normal form of $[\varepsilon]\xi$ is of the form $[\pm](\eta + \omega^n \cdot p)$, for some $0 < \eta < \omega^\omega$, and some integers $n \geq 0$ and $p > 0$, its tree representation consists of the two disjoint tree representations of $[+](\eta + \omega^n \cdot p)$ and $[-](\eta + \omega^n \cdot p)$.

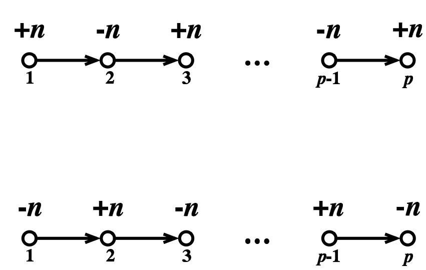


Figure 7.25: Tree representations of the signed ordinals $[+]\omega^n \cdot p$ and $[-]\omega^n \cdot p$. The union of these two graphs is the tree representation of $[\pm]\omega^n \cdot p$. Every time there is an edge from i to j, and from j to k, there is also an edge from i to k, but these transitive edges are not represented, for reasons of clarity.

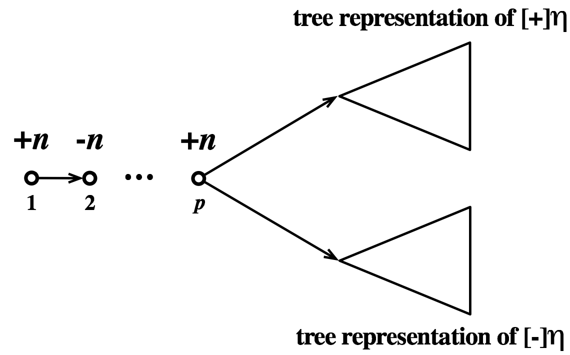


Figure 7.26: The tree representation of the signed ordinal $[+](\eta + \omega^n \cdot p)$. The tree representation of $[-](\eta + \omega^n \cdot p)$ consists of the same DAG, but with an initial sequence of nodes with opposite signs.

Example 7.64. Figure 7.27 illustrates the tree representations of the respective signed ordinals $[-](\omega^5 \cdot 4 + \omega^3 \cdot 3 + \omega^2 \cdot 5)$ and $[\pm](\omega^3 \cdot 3 + \omega^2 \cdot 5)$.

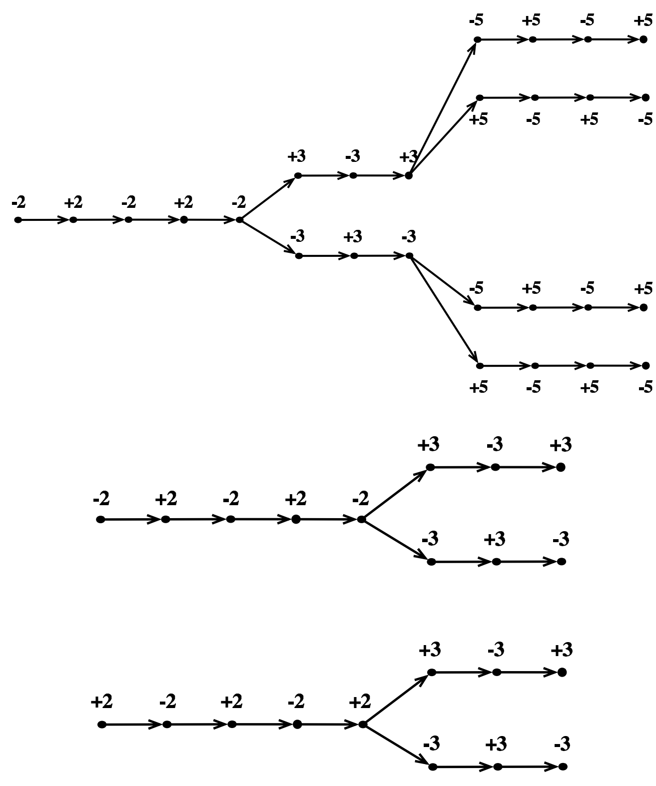


Figure 7.27: Tree representations of $[-](\omega^5{\cdot}4{+}\omega^3{\cdot}3{+}\omega^2{\cdot}5)$ and $[\pm](\omega^3{\cdot}3{+}\omega^2{\cdot}5)$.

The tree representation of $[\varepsilon]\xi$ is an encoding of the Cantor normal form of ξ, with some additional property according to the sign ε. Hence, it is uniquely determined, for each signed ordinal $[\varepsilon]\xi$. It has been defined in order to satisfy the following properties.

LEMMA 7.65. *The main algorithm applied on the tree representation of* $[\varepsilon]\xi$ *outputs precisely* $[\varepsilon]\xi$.

PROOF. The proof goes by induction on the Cantor normal form of $[\varepsilon]\xi$. We prove the result for the case $\varepsilon \in \{+,-\}$. The case $\varepsilon = \pm$ is a direct consequence. If $[\varepsilon]\xi$ is of the form $[\varepsilon]\omega^n \cdot p$, for some $n \geq 0$ and $p > 0$, the result is true. If the Cantor normal form of $[\varepsilon]\xi$ is of the form $[\varepsilon](\eta + \omega^n \cdot p)$, its tree representation consists of the tree representation of $[\varepsilon]\omega^n \cdot p$ related to the two disjoint tree representations of $[+]\eta$ and $[-]\eta$. By the induction hypothesis, the two disjoint subtree representations of $[+]\eta$ and $[-]\eta$ are associated with the respective signed ordinals $[+]\eta$ and $[-]\eta$. By definition of the Cantor normal form, ω^n is strictly below the every factor ω^i appearing in η. Therefore, the main algorithm associates the signed ordinal $[\varepsilon](\eta + \omega^n \cdot p) = [\varepsilon]\xi$ with the root of the tree representation of $[\varepsilon]\xi$. $\square$

LEMMA 7.66. *The tree representations of* $[\varepsilon]\xi$ *and* $[\varepsilon']\xi'$ *are bisimilar if and only if* $[\varepsilon]\xi = [\varepsilon']\xi'$.

PROOF. Let T and T' be the respective tree representations of $[\varepsilon]\xi$ and $[\varepsilon']\xi'$. If $[\varepsilon]\xi = [\varepsilon']\xi'$, then $T = T'$, thus obviously $T \approx T'$. Conversely, assume that $[\varepsilon]\xi \neq [\varepsilon']\xi'$. Then two cases may occur. Firstly, if $\xi = \xi'$ but $\varepsilon \neq \varepsilon'$, then T and T are the very same trees, but with opposite priorities. Therefore, T and T' do not contain the same paths, hence they are not bisimilar. Secondly, if $\xi > \xi'$, then T is a tree representation containing strictly more nodes than T', or strictly larger priorities then T'. Hence, T and T' do not contain the same paths, and they are not bisimilar. The case $\xi' > \xi$ is symmetric. $\square$

Given a finite pointed ω-semigroup (S,X), a *normal form* of (S,X) is a subgraph G of the signed DAG representation of (S,X) containing a minimal number of nodes and edges, and such that an $\mathbb{SG}$-player restricting his moves inside G is exactly as strong as if he were in charge of the whole DAG of (S,X). We prove that the normal form of (S,X) is precisely the tree representation of $[\varepsilon_X]d_{sg}(X)$ (up to bisimilarity), and hence it is unique, up to bisimilarity. Therefore, any two finite pointed ω-semigroups have the same $\mathbb{SG}$-degree if and only if they have the same normal form.

PROPOSITION 7.67. *Let* (S,X) *be a finite pointed* ω*-semigroup associated by the main algorithm with the signed ordinal* $[\varepsilon_X]\xi_X$. *Any normal form of* (S,X) *is bisimilar to the tree representation of* $[\varepsilon_X]\xi_X$.

PROOF. We use the notation of Algorithm 7.32 again. Let G be a normal form of (S,X), and G' be the tree representation of $[\varepsilon_X]\xi_X$. After computation of the main algorithm, the roots r and r' of G and G' are both associated with the signed ordinal $[\varepsilon_X]\xi_X$. Moreover, Lemma 7.42 shows that both graphs G and G' satisfy the following properties: First, for every cut $[\varepsilon]\xi$ of $[\varepsilon_X]\xi_X$, there exists a node n such that $[\varepsilon_n]\xi_n = [\varepsilon]\xi$. Second, any two nodes n and n' satisfy $n \geq_{\mathcal{R}} n'$ if and only $[\varepsilon_n]\xi_n \geq [\varepsilon_{n'}]\xi_{n'}$. In addition, by minimality of G and by definition of G', every path in G or in G' never visits a node associated with a non-cut of $[\varepsilon_X]\xi_X$; also, every path in G or in G' never visits two nodes associated with the same cut of $[\varepsilon_X]\xi_X$. All these properties ensure the existence of the following winning strategy for Player II in $\mathbb{BIS}(G, G')$: every time I moves to a successor node n, II moves to a successor node n' such that $[\varepsilon_n]\xi_n = [\varepsilon_{n'}]\xi_{n'}$. Therefore, $G \approx G'$. □

Theorem 7.68. *Let (S, X) be a finite pointed ω-semigroup, and N_X be a normal form of (S, X).*

(1) *$d_{sg}(X) = \xi$ and X is non-self-dual if and only if N_X is bisimilar to the tree representation of $[+]\xi$ or $[-]\xi$.*
(2) *$d_{sg}(X) = \xi$ and X is self-dual if and only if N_X is bisimilar to the tree representation of $[\pm]\xi$.*

Proof. If $d_{sg}(X) = \xi$ and X is non-self-dual, then $[\varepsilon_X]\xi_X$ is equal to $[+]\xi$ or $[-]\xi$. Hence, by Proposition 7.67, N_X is bisimilar to the tree representation of $[+]\xi$ or $[-]\xi$. Conversely, assume that N_X is bisimilar to the tree representation of $[\varepsilon]\xi$, with $\varepsilon \in \{+, -\}$. Proposition 7.67 shows that N_X is also bisimilar to the tree representation of $[\varepsilon_X]\xi_X$. Hence, the tree representations of $[\varepsilon_X]\xi_X$ and $[\varepsilon]\xi$ are bisimilar, and Lemma 7.66 proves that $[\varepsilon_X]\xi_X = [\varepsilon]\xi$, where $\varepsilon \in \{+, -\}$. Therefore, $d_{sg}(X) = \xi$, and X is non-self-dual. The second case is proved analogously. □

Theorem 7.69. *Let (S, X) and (T, Y) be two finite pointed ω-semigroups with normal forms N_X and N_Y, respectively. Then $X \equiv_{SG} Y$ if and only if $N_X \approx N_Y$.*

Proof. If $X \equiv_{SG} Y$, then $[\varepsilon_X]\xi_X = [\varepsilon_Y]\xi_Y$. Hence, the tree representations T_X and T_Y of $[\varepsilon_X]\xi_X$ and $[\varepsilon_Y]\xi_Y$ are equal. Proposition 7.67 then implies $N_X \approx T_X = T_Y \approx N_Y$. Conversely, by Proposition 7.67 again, one has $T_X \approx N_X \approx N_Y \approx T_Y$. Thus $T_X \approx T_Y$, and Lemma 7.66 shows that $[\varepsilon_X]\xi_X = [\varepsilon_Y]\xi_Y$. Therefore, $X \equiv_{SG} Y$. □

Corollary 7.70. *Let K and L be two ω-rational languages, let $synt(K)$ and $synt(L)$ be their syntactic images, and let N_K and N_L be the normal forms of $synt(K)$ and $synt(L)$. Then $K \equiv_W L$ if and only if $N_X \approx N_Y$.*

Proof. One has $K \equiv_W L$ if and only if $synt(K) \equiv_{SG} synt(L)$. Theorem 7.69 leads to the conclusion. □

Example 7.71. Figure 7.28 (top) illustrates the signed DAG representation of a finite pointed ω-semigroup (S, X). The two signed ordinals associated with each node are the outcomes of procedures (2) (top) and (3) (bottom) of the main algorithm. One has $[\varepsilon_X]\xi_X = [+](\omega^9 + \omega^4 \cdot 2)$. Figure 7.28 (bottom) illustrates the normal form of (S, X), which is bisimilar to the tree representation of $[+](\omega^9 + \omega^4 \cdot 2)$. One has $d_{sg}(X) = \omega^9 + \omega^4 \cdot 2$, and X is non-self-dual.

Example 7.72. Again, Figure 7.29 (top) illustrates the signed DAG representation of a finite pointed ω-semigroup (T, Y). One has $[\varepsilon_Y]\xi_Y = [\pm](\omega^9 + \omega^4 \cdot 2)$. Figure 7.29 (bottom) illustrates the normal form of (T, Y), which is bisimilar to the tree representation of $[\pm](\omega^9 + \omega^4 \cdot 2)$. In this case, one has $d_{sg}(Y) = \omega^9 + \omega^4 \cdot 2$, and X is self-dual.

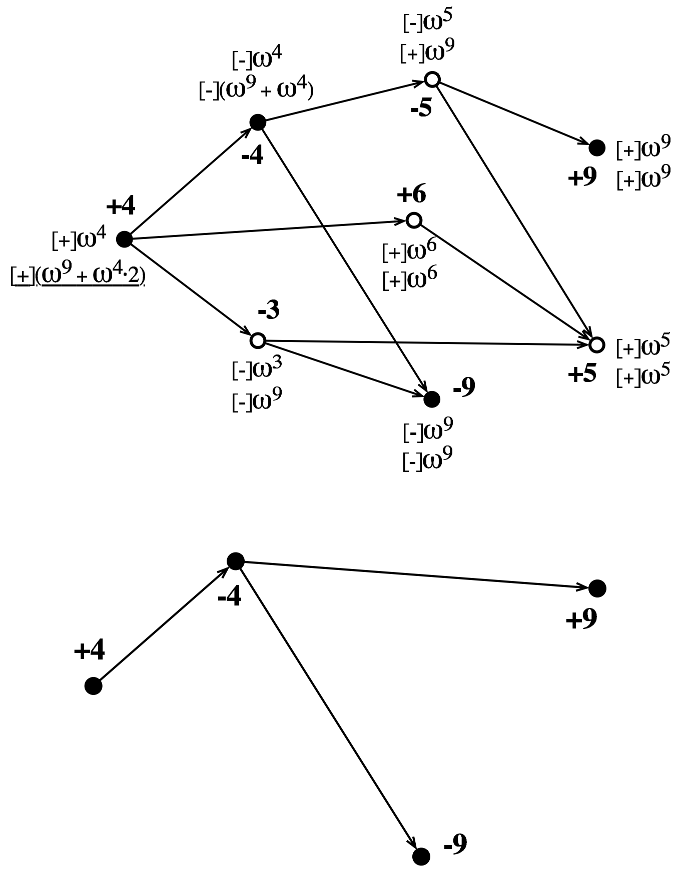
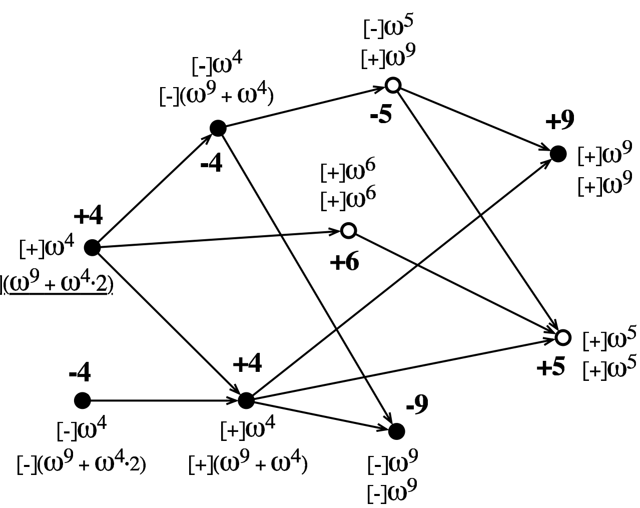


Figure 7.28: The signed DAG representation of a finite pointed ω-semigroup (S, X), and its normal form.



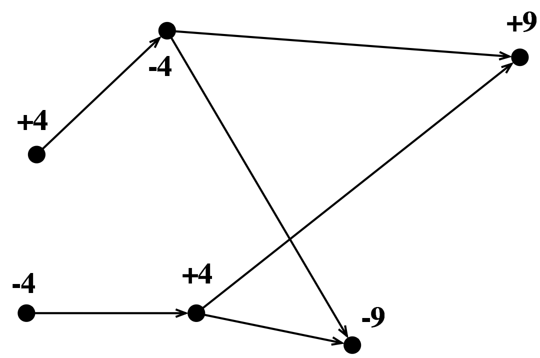


Figure 7.29: The signed DAG representation of a finite pointed ω-semigroup (T, Y), and its normal form.

EXAMPLE 7.73. Consider the finite pointed ω-semigroup

$$(S, X) = ((\{0, 1\}, \{0^\omega, 1^\omega\}), \{0^\omega\})$$

given in Example 7.10. The signed DAG representation and the normal form of (S, X) are illustrated in Figure 7.30. The normal form of (S, X) and the tree representation of $[-]\omega$ are bisimilar. Therefore, $d_{sg}(X) = \omega$, and X is non-self-dual.

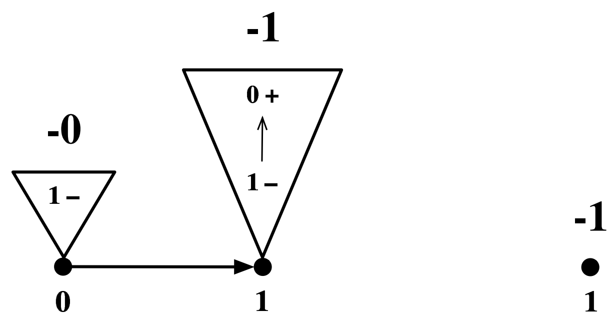


Figure 7.30: The signed DAG representation and the normal form of (S, X). The normal form is reduced to the single node 1 labeled by -1.

EXAMPLE 7.74. Consider the finite pointed ω-semigroup

$$(T, Y) = ((\{a, b, c, ca\}, \{a^\omega, (ca)^\omega, 0\}), \{a^\omega\})$$

given in Example 7.11. The signed DAG representation and the normal form of (T, Y) is illustrated in Figure 7.31. The normal form of (T, Y) is bisimilar to the tree representation of $[+]\omega^2$. Therefore, $d_{sg}(X) = \omega^2$, and X is non-self-dual.

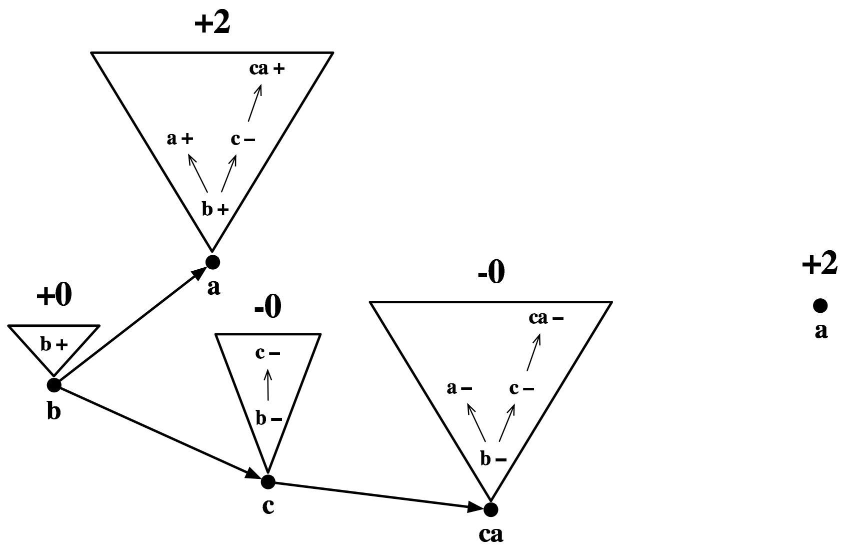


Figure 7.31: The signed DAG representation and the normal form of (T, Y). The normal form is reduced to a single node a labeled by $+2$.

Chapter 8

Computational complexity

Summary

This chapter describes an upper bound for the time complexity required in the decidability of the $\mathbb{FSG}$-hierarchy. Given a finite pointed ω-semigroup (S, X), we show that the $\mathbb{SG}$-degree of X can be performed in polynomial time. We keep on using the notations introduced in the previous chapters.

The decision algorithm. Algorithm 7.32 was proved to be the decision algorithm of the $\mathbb{FSG}$-hierarchy. Given any finite pointed ω-semigroup (S, X), it computes the signed ordinal $[\varepsilon_X]\xi_X$, such that $d_{sg}(X) = \xi_X$, and X is non-self-dual if and only if $\varepsilon_X \in \{+, -\}$. This algorithm can be reformulated as follows:

ALGORITHM 8.1. INPUT: a finite pointed ω-semigroup (S, X). OUTPUT: the signed ordinal $[\varepsilon_X]\xi_X$.

(1) construct the graph of the partial ordering $(P/\mathcal{R}, \geq_{\mathcal{R}})$;
(2) for each node n of this graph, compute the signed length of the corresponding main vein $V(n)$;
(3) run procedures (3) and (4) of Algorithm 7.32.

The input of Algorithm 8.1 consists of the finite pointed ω-semigroup (S, X), where $S = (S_+, S_\omega)$. It can be formally given by:

- the list of the n elements of S_+,
- the table of size $n \times n$ describing the operation of S_+,
- the list of pairs $(s, t) \in S_+^2$ satisfying $st^\omega \in X$.

Hence, the input has size $\mathcal{O}(n^2)$, where n is the cardinality of S_+.

For each step of this algorithm, we give a pseudocode that performs it, and then analyze its time complexity. We keep on using the usual notations, and recall that P denotes the set of prefixes of S_+, $P/\mathcal{R}$ is the set of $\mathcal{R}$-classes of prefixes, and E denotes the set of idempotents of S_+. If the semigroup S_+ has cardinality n, the time complexities for building the respective sets E and P are $\mathcal{O}(n)$ and $\mathcal{O}(n^2)$.

Step (1). Firstly, we construct the right Cayley graph of S_+. Secondly, we extract the subgraph induced by the prefix nodes. Thirdly, we compute the strongly connected components of this subgraph using Tarjan's algorithm [36].

Input: the semigroup S_+.
Output: the graph G of the partial ordering $(P/\mathcal{R}, \geq_{\mathcal{R}})$

(1) *// compute C the right Cayley graph of S_+.*
  For all $s \in S_+$
    For all $t \in S_+$
      draw an edge $C[s, st]$ from s to st
    End For
  End For
  Return C

(2) *// compute C' the subgraph of C induced by the prefix nodes*
  For all node n in C
    If $n \in P$, then keep it
    Else, delete n, and draw edges $C'[p, s]$ between
      each predecessor p and successor s of n
  End for
  Return C'

(3) *// compute G the graph of the strongly connected components of C'.*
  Compute $G = Tarjan(C')$
  Return G

Both the first and second steps of this procedure run in time $\mathcal{O}(n^2)$. Concerning the third step, Tarjan's algorithm runs in time $\mathcal{O}(|V| + |E|)$ – where V and E are respectively the sets of vertices and edges of C', that is in time $\mathcal{O}(n^2)$ in our case. Consequently, the first step of Algorithm 8.1 runs in time $\mathcal{O}(n^2)$.

Step (2). We first construct the graph of the partial order $(E, \geq)$, and then compute the signed length of a main vein associated with each $\mathcal{R}$-class of prefixes.

Input: the pointed ω-semigroup (S, X), and the graph G.
Output: for each node $n \in G$, the signed integer $[\delta_n]l_n$ associated with any main vein $V(n)$.

(1) *// construct the graph H of of the partial order $(E, \geq)$, and for each idempotent $e_i \in petal_X(s_k)$, compute its sign $\sigma_{i,k}$ according to X.*
  For all $e_i \in E$
    For all $e_j \in E \setminus \{e_i\}$
      If $e_i e_j = e_j e_i = e_j$, then
        draw an edge $H[e_i, e_j]$ from e_i to e_j
      End If
    End For
    For all $k \in K$ *//prefix index set*
      if (s_k, e_i) is a linked pair, then $\sigma_{i,k} := sign_X(e_i)$, *//see Definition 7.9*
      otherwise, $\sigma_{i,k} := *$
    End For
  End For
  Return H

(2) *// for each node n of G, compute the signed integer $[\delta_n]l_n$ associated with a given main vein $V(n)$ of n*

Topological sorting of the graph H

For all $e_i \in E$ in topological order

For all $k \in K$ *//prefix index set*

$rank_{i,k} := \max_{\{j|e_j>e_i\}} f(\sigma_{i,k}, \sigma_{j,k}, rank_{j,k})$ *// signed rank of e_i in $petal_X(s_k)$*

End For

For all $n \in G$

$[\delta_n]l_n := \max_{\{s_p \in n\}} rank_{i,p}$ *// signed length of a main vein of $flower_X(n)$*

End For

End For

Since the cardinality of E and K is bounded by n, and the signature procedure consists of two tests (see Definition 7.9), step (1) is performed in time $\mathcal{O}(n^2)$. Concerning step (2), the function f involved in the computation of the rank is defined by induction as follows:

$$f(\sigma_{i,k}, \sigma_{j,k}, rank_{j,k}) = \begin{cases} [+]0 & \text{if } e_i \text{ is a positive source of } petal_X(s_k), \\ [-]0 & \text{if } e_i \text{ is a negative source of } petal_X(s_k), \\ rank_{j,k} & \text{if } e_i \text{ is not a source and } \sigma_{i,k} = \sigma_{j,k}, \\ rank_{j,k}+1 & \text{if } e_i \text{ is not a source and } \sigma_{i,k} \neq \sigma_{j,k}, \end{cases}$$

where $[\varepsilon]n + 1 = [\varepsilon](n + 1)$, for some $n \geq 0$ and $\varepsilon \in \{+, -\}$. The topological sorting of the graph can be performed in $\mathcal{O}(n^2)$, the maximal cardinality of E and K is n, the number of predecessors of a node is also bounded by n, and the function f is performed in constant time. Hence, this procedure runs in time $\mathcal{O}(n^3)$, and the whole step (2) of Algorithm 8.1 is performed in time $\mathcal{O}(n^3)$.

Step (3). It remains to apply Algorithm 7.32.

Input: the DAG representation G of (S, X), and the signed integers $[\delta_n]l_n$, corresponding to the sign and length of the main vein associated with n, for every node n of G

Output : the signed ordinal $[\varepsilon_X]\xi_X$ revealing the $\mathbb{SG}$-degree X.

(1) Topological sorting of the graph G

For all node $n \in G$ in inverse topological order

$[\varepsilon_n]\xi_n := g(\{[\varepsilon_m]\xi_m \mid n >_{\mathcal{R}} m\})$ *// updating procedure (3) of Algorithm 7.32*

End For

(2) $[\varepsilon_X]\xi_X := h(\{[\varepsilon_s]\xi_s \mid s \text{ is a source of } G\})$ *// updating procedure (4) of Algorithm 7.32*

Since G has at most cardinality n, the topological sorting of this graph can be performed in $\mathcal{O}(n^2)$. Moreover, the functions g and h correspond to the respective procedures (3) and (4) of Algorithm 7.32. Since the number of successors of a node is bounded by n, the function g is performed in time $\mathcal{O}(n)$, for each node. Since the number of sources is bounded by n, the function h is performed in time $\mathcal{O}(n)$. Therefore, step (3) of Algorithm 8.1 runs in time $\mathcal{O}(n^2)$.

Final complexity. This analysis shows that the computational complexity of the decidability of the $\mathbb{FSG}$-hierarchy is in polynomial time in the cardinality of the finite semigroup involved.

THEOREM 8.2. *Let $S = (S_+, S_\omega)$ be a finite ω-semigroup, $X \subseteq S_\omega$, and let n be the cardinal of S_+. Then the $\mathbb{SG}$-degree of X is computable in time $\mathcal{O}(n^3)$.*

This analysis has to be compared with Thomas Wilke and Haiseung Yoo's result, which states that the Wagner degree of an ω-rational language can be computed in time $\mathcal{O}(f^2 qb + k \log k)$, if the language is recognized by a deterministic Muller automaton over an alphabet of cardinality b, with f accepting states, q states, and k strongly connected components [43]. In this case too, the decidability procedure runs in polynomial time.

Chapter 9

Additional results

Summary

In this chapter, we first present properties clarifying the DAG representation of finite semigroups. In particular, we show that a chain of idempotents of length n in a finite semigroup guarantees the existence of a linear sequence of n accessible flowers in its DAG representation.

Then, we prove that there is no family of finite ω-semigroups whose ω-subsets cover exclusively the *finite* $\mathbb{SG}$-degrees. Similarly, there is no family of finite ω-semigroups whose ω-subsets cover exclusively the $\mathbb{SG}$-degrees of the form ω^n, for all $n \geq 0$. However, there exists a family of finite ω-semigroups whose ω-subsets cover precisely all the $\mathbb{SG}$-degrees of the form $\omega^n \cdot p$, for all $n \geq 0, p > 0$.

Finally, we prove that non-self-dual ω-subsets are exactly the subsets of finite ω-semigroups built on monoids (up to $\mathbb{SG}$-equivalence). We also show that subsets of finite ω-semigroups built on left-cancelable semigroups, groups, and cyclic semigroups always have an $\mathbb{SG}$-degree of 1. As opposed to subset of finite commutative ω-semigroups, which may be of every possible $\mathbb{SG}$-degree.

9.1 The DAG representation of finite semigroups

The following results describe properties of conjugate linked pairs in the DAG representation of finite semigroups. We show that all prefixes of conjugate linked pairs always belong to the same $\mathcal{R}$-class, and moreover, idempotents of conjugate linked pairs can never form a chain. A detailed description of conjugate linked pairs in terms of Green properties can be found in Section II - 2.4 of [27].

Lemma 9.1. *Let S be a finite semigroup, and let (s, e) and (s', e') be two conjugate linked pairs of S^2. Then $s \mathcal{R} s'$ and $e \mathcal{D} e'$.*

Proof. Since (s, e) and (s', e') are conjugate, there exist $x, y \in S^1$ such that $s' = sx$, $s = s'y$, $e = xy$, and $e' = yx$. Therefore, $s \leq_{\mathcal{R}} s'$ and $s' \leq_{\mathcal{R}} s$, hence $s \mathcal{R} s'$. In addition, $e = e^2 = (xy)(xy) \leq_{\mathcal{L}} yxy \leq_{\mathcal{R}} yx = e'$ and $e' = e'^2 = (yx)(yx) \leq_{\mathcal{L}} xyx \leq_{\mathcal{R}} xy = e$, thus $e \mathcal{D} e'$. □

Lemma 9.2. *Let S be a finite semigroup, and let (s, e) and (s, f) be two conjugate linked pairs of S^2. Then $e \not> f$ and $f \not> e$.*

Proof. We show that $e \geq f$ or $f \geq e$ implies $e = f$. First of all, the relation $e \geq f$ stands by definition for $f \leq_{\mathcal{L}} e$ and $f \leq_{\mathcal{R}} e$. Besides, since (s, e) and (s, f) are conjugate, then Lemma 9.1 shows that $f \mathcal{D} e$, and Proposition 3.18 shows that $f \mathcal{J} e$. Hence, by Proposition 3.19, the relations $f \mathcal{J} e$, $f \leq_{\mathcal{R}} e$, and $f \leq_{\mathcal{L}} e$ imply both $f \mathcal{R} e$ and $f \mathcal{L} e$, that is $f \mathcal{H} e$. The class $\mathcal{H}(e)$ thus contains the two idempotents e and f, and Proposition 3.16 ensures that it is a group. Finally, since every group contains only one single idempotent, namely the identity of the group, one has $e = f$. □

We now present some properties concerning the disposition of petals and flowers in the DAG representation of finite semigroups. The first result shows that petals are closed under the $\geq$ relation, as illustrates in Figure 9.1. The second one claims that a chain of idempotents of length n ensures the existence of n distinct flowers. In particular, the length of a maximal chain of idempotents is a lower bound for the number of flowers appearing in the DAG representation of a finite semigroup. Some observations follow.

Lemma 9.3. *Let S be a finite semigroup. If (s, e) is a linked pair, then for every $e' \geq e$, the pair $(s.e')$ is also linked.*

Proof. Let (s, e) be a linked pair and let $e' \geq e$. Then $se' = (se)e' = s(ee') = se = s$. It follows that (s, e') is also a linked pair. □

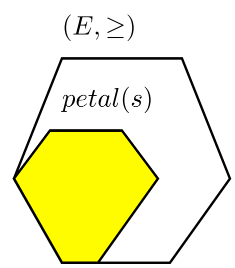


Figure 9.1: The set $petal(s)$ is a $\geq$-closed subset of the lattice of idempotents $(E, \geq)$: $e \in petal(s)$ and $e' \geq e$ implies $e' \in petal(s)$.

Proposition 9.4. *Let S be a finite semigroup, and let $e_0 > e_1 > \ldots > e_n$ be a chain of idempotents in S. Then the DAG representation of S contains at least $n+1$ flowers $flower(\bar{e}_0), flower(\bar{e}_1), \ldots, flower(\bar{e}_n)$ such that:*

- *$\bar{e}_i$ is the $\mathcal{R}$-class of prefixes of e_i, for all $i \leq n$,*
- *$\bar{e}_i >_{\mathcal{R}} \bar{e}_j$ whenever $i < j$,*
- *$flower(\bar{e}_i)$ contains the chain of idempotents $e_0 > \ldots > e_i$, for all $i \leq n$,*

as illustrated in Figure 9.2.

Proof. For each idempotent e, the pair (e, e) is obviously linked, hence every idempotent e is also a prefix. Therefore, the DAG representation of S contains the following $n+1$ flowers

$$flower(\bar{e}_0), flower(\bar{e}_1), \ldots, flower(\bar{e}_n),$$

where each $\bar{e}_i$ denotes the $\mathcal{R}$-class of e_i. Moreover, the relation $e_i > e_j$ implies $e_i >_{\mathcal{R}} e_j$, for every $i < j$. Finally, Lemma 9.3 shows that the chain $e_0 > \ldots > e_i$ is contained in $flower(\bar{e}_i)$, for all $i \leq n$. $\square$

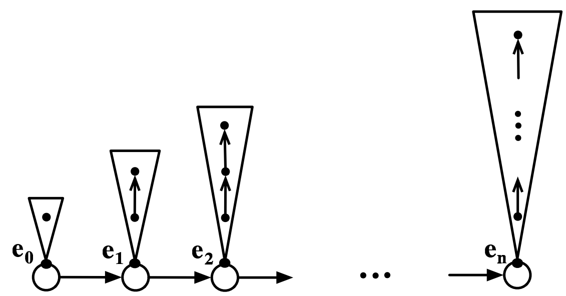


Figure 9.2: A chain of idempotents of length $n+1$ guarantees the existence of a linear sequence of $n+1$ distinct growing flowers.

The following observations conclude these complementary results.

(1) The DAG representation of a finite semigroup may contain as many flowers as elements, although all chains of idempotents have a length of only 1. Indeed, consider the semigroup $S_1 = (\{0, 1, \ldots, n\}, \cdot)$ equipped with the left absorption operation, that is $a \cdot b = a$, for every $a, b \in S_1$. Every element is idempotent, but they are all pairwise $\leq$-incomparable. Therefore, every chain of idempotents has length 1, but the DAG representation of S_1 contains the $n+1$ inaccessible flowers $flower(\bar{0}), \ldots, flower(\bar{n})$, as illustrated in Figure 9.3.

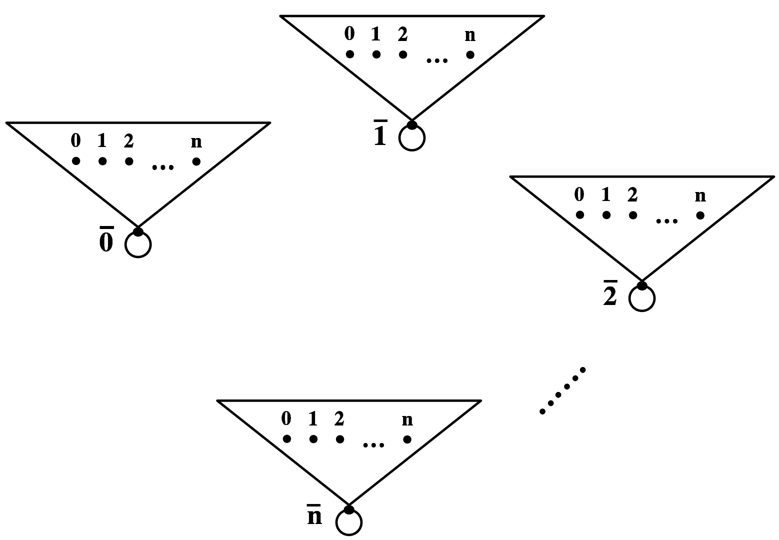


Figure 9.3: The DAG representation of S_1.

(2) The petals are not always growing along the $\geq_{\mathcal{R}}$-accessibility between flowers. For instance, consider the finite semigroup $S_2 = (\{0, 1, a, b, c, d, e\}, \cdot)$ equipped with the following operation

$\cdot$	0	1	a	b	c	d	e
0	0	0	0	0	0	0	0
1	0	1	a	b	c	d	e
a	0	a	0	d	a	0	d
b	0	b	0	0	0	0	0
c	0	c	0	e	c	0	e
d	0	d	0	0	0	0	0
e	0	e	0	0	0	0	0

The DAG representation of S_2, illustrated in Figure 9.4, shows that the petals are either growing, or decreasing, or also remaining stable along the $\geq_{\mathcal{R}}$-accessibility relation between flowers.

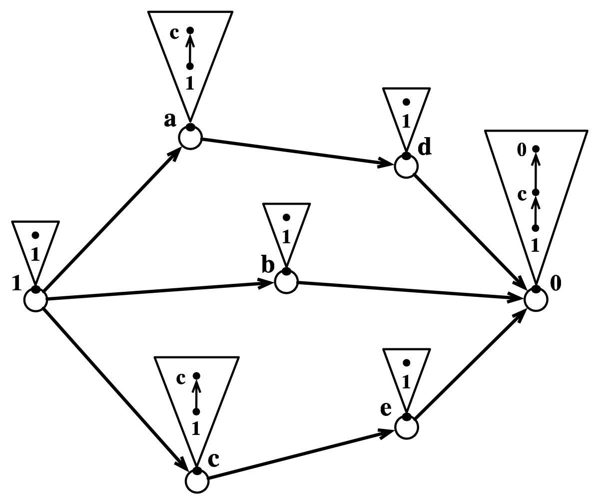


Figure 9.4: The DAG representation of S_2.

(3) A same flower may contain petals of different heights. Indeed, consider the finite semigroup $S_3 = (\{1, a, b, c\}, \cdot)$ equipped with the following operation

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	b	b	c

The DAG representation of S_3, illustrated in Figure 9.5, shows that the last flower contains two petals of different heights.

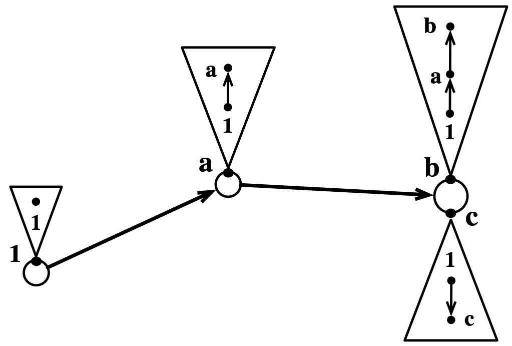


Figure 9.5: The DAG representation of S_3.

9.2 Two negative and one positive results

This section studies the existence of some specific families of ω-semigroups. If D is a set of ordinals, we say that D is $\mathbb{SG}$-*definable* if there exists a family $\mathcal{F}_D$ of finite ω-semigroups such that, up to $\mathbb{SG}$-equivalence, the ω-subsets extracted from $\mathcal{F}_D$ are exactly the ones with $\mathbb{SG}$-degrees in D. More precisely, D is $\mathbb{SG}$-definable if and only if there exists a family $\mathcal{F}_D$ of finite ω-semigroups such that:

- for any ω-subset X with $d_{sg}(X) \in D$, there exist a pointed ω-semigroup (S, Y), such that $S \in \mathcal{F}_D$ and $X \equiv_{SG} Y$;
- every ω-subset X of every ω-semigroup in $\mathcal{F}_D$ satisfies $d_{sg}(X) \in D$.

We prove that the subset of finite $\mathbb{SG}$-degrees is not $\mathbb{SG}$-definable. The $\mathbb{SG}$-degrees of the form ω^n, for $n \geq 0$, are also not $\mathbb{SG}$-definable. On the opposite, the $\mathbb{SG}$-degrees of the form $\omega^n \cdot p$, for $n \geq 0$ and $p > 0$, are $\mathbb{SG}$-definable.

PROPOSITION 9.5. $\mathbb{N}^*$ *is not* $\mathbb{SG}$*-definable.*

PROOF. First, the main algorithm shows that an ω-subset has a finite $\mathbb{SG}$-degree if and only if all its alternating chains have length 0. Now, towards a contradiction, assume that there exists a family of ω-semigroups $\mathcal{F}$ which defines all finite $\mathbb{SG}$-degrees, and let $S = (S_+, S_\omega) \in \mathcal{F}$. Then S satisfies: for every $e, f \in E(S_+)$, the relation $e \geq f$ implies $e = f$ (otherwise there could be an alternating chain of strictly positive length in S_+). We prove that the DAG representation of S_+ contains either a single flower, or several inaccessible flowers. In both cases, for every $X \subseteq S_\omega$, since the alternating chains of (S, X) have lenght 0, the main algorithm implies that $d_{sg}(X) = 1$. Hence, the family $\mathcal{F}$ only defines the $\mathbb{SG}$-degree 1, a contradiction. To this end, we first prove that $E(S_+)$ is contained in the minimal ideal of S_+. Since S_+ is finite, Proposition 3.11 shows that it has indeed a minimal ideal I. Then, let $x \in I$. The element $e = x^\pi$ is an idempotent of I, where π is the exponent of S_+. Now, let f be another idempotent, and assume that $f \notin I$. Then, both $(fef)^\pi \in I$ and $f \geq (fef)^\pi$ hold. Therefore, $f > (fef)^\pi$, contradicting the required condition on S_+. Hence, $f \in I$, which proves that $E(S_+) \subseteq I$. We finally prove that the DAG representation of S_+ contains either a single flower, or several inaccessible flowers. Let (s, e) and (t, f)

be two linked pairs of S_+^2. Since $E(S_+) \subseteq I$, then $e, f \in I$, hence $s = see \in I$ and $t = tff \in I$. Therefore, $s \mathcal{J} t \mathcal{J} e \mathcal{J} f$. Finally, either $s \geq_{\mathcal{R}} t$, but then Proposition 3.19 ensures that $s \mathcal{R} t$; or $t \geq_{\mathcal{R}} s$, and the same argument shows that $s \mathcal{R} t$; or $s \not\geq_{\mathcal{R}} t$ and $t \not\geq_{\mathcal{R}} s$, meaning that all the flowers of S_+ are pairwise inaccessible. □

PROPOSITION 9.6. *$\{\omega^n \mid n < \omega\}$ is not $\mathbb{SG}$-definable.*

PROOF. Towards a contradiction, assume that there exists a family of finite ω-semigroups $\mathcal{F}$ which defines these $\mathbb{SG}$-degrees. Let $S = (S_+, S_\omega) \in \mathcal{F}$ and $X \subseteq S_\omega$ such that $d_{sg}(X) = \omega^n$, for some $n > 0$. By the main algorithm, there exists an alternating chain of idempotents of length n in (S, X). Proposition 9.4 thus ensures that the DAG representation of S contains a linear sequence of $n + 1$ distinct flowers. Therefore, S_ω contains an ω-subset with an $\mathbb{SG}$-degree of the form $\omega^p \cdot q$, for some $p < n$ and $q > 0$. A contradiction to the required properties of the family $\mathcal{F}$. □

PROPOSITION 9.7. *$\{\omega^n \cdot p \mid n < \omega$ and $0 < p < \omega\}$ is $\mathbb{SG}$-definable.*

PROOF. Consider the family of finite ω-semigroups $\mathcal{F} = \{S_n\}_{n\in\omega}$, where $S_n = (S_{n,+}, S_{n,\omega})$ is the ω-semigroup induced by the finite semigroup

$$S_{n,+} = (\{0, 1, \ldots, n\}, \max).$$

Every element of $S_{n,+}$ is idempotent, hence also prefix. The linked pairs are of the form (i, j) with $i \geq j$, and are all pairwise non-conjugate. Moreover, both relations $i \geq j$ and $i \geq_{\mathcal{R}} j$ hold whenever $i \leq j$, and every class of prefixes $\bar{i}$ is reduced the singleton $\{i\}$. Hence, the DAG representation of $S_{n,+}$ is a linear sequence of single petal flowers growing along the $\geq_{\mathcal{R}}$-accessibility relation, as illustrated in Figure 9.6. The main algorithm ensures that every ω-subset X of $S_{n,\omega}$ satisfies $d_{sg}(X) = \omega^k \cdot p$, for some $0 \leq k \leq n$ and $p > 0$. Conversely, let X be an ω-subset such that $d_{sg}(X) = \omega^n \cdot p$, with $n \geq 0$ and $p > 0$. Then, by taking ℓ large enough, one can obviously find a finite pointed ω-semigroup (S_ℓ, Y), with $S_\ell \in \mathcal{F}$, such that $X \equiv_{SG} Y$. □

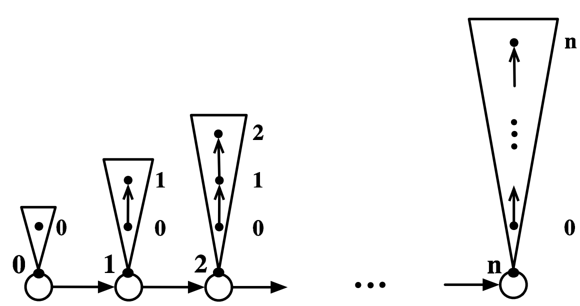


Figure 9.6: The DAG representation of the semigroup $S_{+,n}$.

9.3 Revisiting some basic algebraic concepts

This section explores some properties of finite ω-semigroups built on some specific semigroups, such as monoids, left-cancelable semigroups, groups, cyclic semigroups, and commutative semigroups.

9.3.1 Finite ω-monoids

An ω-semigroup whose first component is a monoid will be called an *ω-monoid*. We show that, up to $\mathbb{SG}$-equivalence, the non-self-dual ω-subsets are exactly the subsets of finite ω-monoids.

THEOREM 9.8. *Let $S = (S_+, S_\omega)$ be a finite ω-semigroup, and let $X \subseteq S_\omega$. The following conditions are equivalent:*

(1) *X is non-self-dual,*

(2) *there exist a finite ω-monoid $M = (M_+, M_\omega)$, and an ω-subset $Y \subseteq M_\omega$ such that $X \equiv_{SG} Y$.*

PROOF. (1) $\Rightarrow$ (2). Let X be a non-self-dual ω-subset of the finite ω-semigroup $S = (S_+, S_\omega)$. If S_+ is a monoid, there is nothing to prove. Assume that S_+ is not a monoid, and consider the monoid S_+^1. The linked pairs of S_+^1 consist of every linked pairs of S_+, as well as the pairs $(1, 1)$ and $(s, 1)$, for every $s \in S_+$. The set of idempotents of S_+^1 is given by $E(S_+^1) = E(S_+) \cup \{1\}$, and $1 > e$, for every $e \in E(S_+)$. The set of prefixes of S_+^1 is given by $P(S_+^1) = P(S_+) \cup \{1\}$, and the $\mathcal{R}$-class of prefixes $\bar{1} = \{1\}$ satisfies $\bar{1} >_{\mathcal{R}} \bar{s}$, for every $s \in P(S_+)$. Therefore, the DAG representation of S_+^1 corresponds to the DAG representation of S_+, enriched by the new flower $flower(\bar{1})$, and where the idempotent 1 has been added in every petal of S_+, as illustrated in figures 9.7 and 9.8.

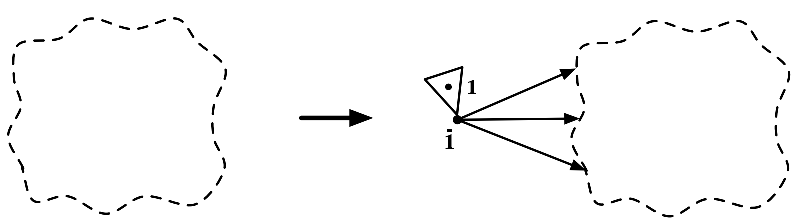


Figure 9.7: From the DAG representation of S_+ to the one of S_+^1: appearance of the "initial" flower $flower(\bar{1})$.

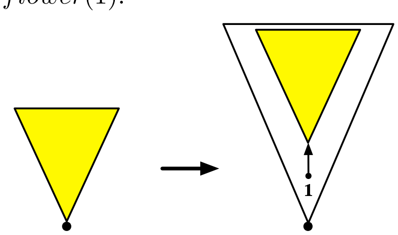


Figure 9.8: The $\leq$-largest idempotent 1 appears in every petal of S_+^1.

Now, let $[\varepsilon_X]\xi_X$ be the signed ordinal associated with X by the main algorithm. Since X is non-self-dual, one has $\varepsilon_X \in \{+,-\}$. Hence, we let $M = (S_+^1, S_\omega^1)$ be the ω-semigroup induced by S_+^1, and

$$Y = \begin{cases} X \cup \{[s,1] \mid \text{the main veins of } flower(\bar{s}) \text{ are positive}\} \cup \{[1,1]\}, & \text{if } \varepsilon_X = + \\ X \cup \{[s,1] \mid \text{the main veins of } flower(\bar{s}) \text{ are positive}\}, & \text{if } \varepsilon_X = -. \end{cases}$$

Finally, the main algorithm ensures that $[\varepsilon_X]\xi_X = [\varepsilon_Y]\xi_Y$, thus $X \equiv_{SG} Y$.

(2) ⇒ (1). Let $M = (M_+, M_\omega)$ be a finite ω-monoid, and let $Y \subseteq M_\omega$, such that $X \equiv_{SG} Y$. We describe a winning strategy for Player I in $\mathbb{SG}(Y, Y^c)$: he first plays 1 (the identity of M_+); then he copies II when she does not skip her turn, and plays 1 when II skips her turn. The two infinite words respectively produced by players I and II are identical, hence Player I wins the game. Therefore, Y is non-self-dual, and so is X.

□

9.3.2 Finite left-cancelable ω-semigroups

We first recall the definition of a left-cancelable semigroup. An ω-semigroup whose first component is a left-cancelable semigroup will naturally be called a *left-cancelable ω-semigroup*. We prove that the family of finite left-cancelable ω-semigroups contains only trivial ω-subsets.

Definition 9.9. *A semigroup S is* left-cancelable *if, for every $s, t, x \in S$, the relation $xs = xt$ implies $s = t$.*

Lemma 9.10. *Let S be a finite semigroup. Then S is left-cancelable if and only if the left multiplication by x is a bijection on S, for every $x \in S$.*

Proof. The left multiplication $\varphi_x : S \longrightarrow S$ is given by $\varphi_x(s) = xs$. Let S be a left-cancelable finite semigroup and let $x \in S$. The left-cancelability of S ensures that φ_x is injective, for every $x \in S$. In addition, since S is finite, the mapping φ_x is also onto, for every $x \in S$. Conversely, assume that φ_x is bijective, for every $x \in S$. Then the mapping φ_x is injective for every $x \in S$, meaning precisely that S is left-cancelable. □

Hence, a finite semigroup S is left-cancelable if and only if every element of S appears only once in each row of its operation table. Therefore, from a game theoretical perspective, an $\mathbb{SG}$-player in charge a left-cancelable semigroup has a unique way to reach any further position. This constraint is actually a maximal weakening, as proved below.

Proposition 9.11. *Let $S = (S_+, S_\omega)$ be a finite left cancelable ω-semigroup and let $X \subseteq S_\omega$. Then $d_{sg}(X) = 1$.*

Proof. We prove that the DAG representation of S_+ contains a unique flower and a unique idempotent in every petal. Let $s, s' \in P$. Lemma 9.10 shows that the left multiplications by s and s' are onto. Hence there exist $x, y \in S_+$ such that $sx = s'$ and $s'y = s$, and thus $s \mathrel{\mathcal{R}} s'$. In addition, let $s \in P$ and $e, e' \in petal(s)$. The relation $se = se'$ implies $e = e'$. Therefore, the main algorithm gives $d_{sg}(X) = \omega^0 = 1$. □

9.3.3 Finite ω-groups

An *ω-group* denotes an ω-semigroup whose first component is a group. As a particular case of finite left-cancelable ω-semigroups, the family of finite ω-groups also contains only trivial ω-subsets.

DEFINITION 9.12. *A semigroup S is a* group *if it contains an identity* 1*, and if for every $x \in S$, there exists $y \in S$ such that $xy = yx = 1$.*

PROPOSITION 9.13. *Let $S = (S_+, S_\omega)$ be an ω-group and let $X \subseteq S_\omega$. Then $d_{sg}(X) = 1$.*

PROOF. We prove that S_+ is left-cancelable, and conclude by Proposition 9.11. Let $s, t, x \in S_+$ such that $xs = xt$, then $s = x^{-1}xs = x^{-1}xt = t$. □

9.3.4 Finite cyclic ω-semigroups

A *cyclic ω-semigroup* denotes an ω-semigroup whose first component is a cyclic semigroup. Once again, the family of finite cyclic ω-semigroups contains only trivial ω-subsets.

DEFINITION 9.14. *Let S be a semigroup, and $R \subseteq S$. Then S is* generated by *R, or equivalently, R is a* generator *of S, denoted by $S = [R]$, if $S = \bigcup_{n\in\mathbb{N}} R^n$. The set R is an* irreducible generator *if there is no $R' \subsetneq R$ such that $S = [R']$. The semigroup S is* cyclic *if it is generated by a single element.*

LEMMA 9.15. *Let S be a finite cyclic semigroup generated by x. Then there exist two integers $i, p > 0$ such that:*

(1) *the relation $x^{i+p} = x^i$ holds,*
(2) *$S = \{x, x^2, \ldots, x^{i+p-1}\}$,*
(3) *no element of $\{x, x^2, \ldots, x^{i-1}\}$ has a right unit,*
(4) *the set $S_i = \{x^i, x^{i+1}, \ldots, x^{i+p-1}\}$ is a subgroup of S.*

PROOF.

(1) By Lemma 3.5, since S is finite, there exist two minimal integers $i, p > 0$, called the index and the period of S, such that $x^{i+p} = x^i$.
(2) The relations $S = [\{x\}]$ and $x^{i+p} = x^i$ imply that $S = \{x, x^2, \ldots, x^{i+p-1}\}$. Notice that S is commutative.
(3) Towards a contradiction, assume that there exists an element x^k which has a right unit, for some $1 \leq k < i$. Then there exists $l > 0$ such that $x^k x^l = x^{k+l} = x^k$, contradicting the minimality of the index i.
(4) By (1), for every $a, b \in S_i$, there exist $x, y \in S_i$ such that $ax = b$ and $by = a$, meaning that S_i is a group.

□

From a game theoretical perspective, an irreducible generator of S, when it exists, represents the minimal set of positions from which any other position is reachable. Cyclic semigroups have the poorest nonempty set of irreducible generators. They induce ω-semigroups with trivial ω-subsets, as proved below.

PROPOSITION 9.16. *Let $S = (S_+, S_\omega)$ be a finite cyclic ω-semigroup, and let $X \subseteq S_\omega$. Then $d_{sg}(X) = 1$.*

PROOF. Lemma 9.15 ensures that $S_+ = \{x, x^2, \ldots, x^{i+p-1}\}$, for some generator x of S_+, and some integers $i, p > 0$, and also that $S_i = \{x^i, x^{i+1}, \ldots, x^{i+p-1}\}$ is a subgroup of S. By Lemma 9.15 again, since no element of $S \setminus S_i$ has a right unit, then every element of $S \setminus S_i$ is neither a prefix of a linked pair, nor an idempotent. Therefore, the DAG representation of S consists of the unique flower induced by the group S_i. The main algorithm thus gives $d_{sg}(X) = \omega^0 = 1$. □

9.3.5 Finite commutative ω-semigroups

An ω-semigroups whose first component is a commutative semigroups is called a *commutative ω-semigroup*. We prove that the family of finite commutative ω-semigroups contains ω-subsets of every possible $\mathbb{SG}$-degrees. Furthermore, the DAG representation of finite commutative semigroups present the following properties: every flower contains a unique petal; two distinct conjugate linked pairs never appear in a same petal; the petals are always increasing along the $\geq_{\mathcal{R}}$-accessibility between flowers; there is a unique terminal flower.

LEMMA 9.17. *Let S be a finite commutative semigroup, let $s \in P(S)$, and let $s_1, s_2 \in \bar{s}$. Then $petal(s_1) = petal(s_2)$.*

PROOF. Since $s_1, s_2 \in \bar{s}$, there exist $x, y \in S^1$ such that $s_1 x = s_2$ and $s_2 y = s_1$. Now, let $e \in petal(s_1)$, then $s_2 e = s_2 yxe = s_2 yex = s_1 ex = s_1 x = s_2$, hence $e \in petal(s_2)$. Symmetrically, one has $petal(s_2) \subseteq petal(s_1)$. Therefore, $petal(s_1) = petal(s_2)$. □

LEMMA 9.18. *Let S be a finite commutative semigroup, let $s \in P(S)$, and let $e, e' \in petal(s)$. Then $[s, e] = [s, e']$ if and only if $(s, e) = (s, e')$.*

PROOF. If $(s, e) = (s, e')$, then obviously $[s, e] = [s, e']$. Conversely, if $[s, e] = [s, e']$, there exist $x, y \in S^1$ such that $e = xy$, $e' = yx$, and $sx = s$. Therefore, $e = xy = yx = e'$, that is $(s, e) = (s, e')$. □

LEMMA 9.19. *Let S be a finite commutative semigroup, and let $s_1, s_2 \in P(S)$ such that $s_1 \geq_{\mathcal{R}} s_2$. Then $petal(s_1) \subseteq petal(s_2)$.*

PROOF. Since $s_1 \geq_{\mathcal{R}} s_2$, there exists $x \in S^1$ such that $s_1 x = s_2$. Let $e \in petal(s_1)$, one has $s_2 e = s_1 xe = s_1 ex = s_1 x = s_2$, thence $e \in petal(s_2)$. Therefore, $petal(s_1) \subseteq petal(s_2)$. □

LEMMA 9.20. *Let S be a finite commutative semigroup, and let $\bar{s}_1, \bar{s}_2$ be two $\geq_{\mathcal{R}}$-minimal $\mathcal{R}$-classes of prefixes of S. Then $\bar{s}_1 = \bar{s}_2$.*

PROOF. Let $x = (s_1 s_2)^\pi = (s_2 s_1)^\pi$, where π is the exponent of S. Then x is idempotent, and hence it is also a prefix. One has $s_1 \geq_{\mathcal{R}} x$ and $s_2 \geq_{\mathcal{R}} x$, and thus $\bar{s}_1 = \bar{s}_2$, by $\geq_{\mathcal{R}}$-minimality of $\bar{s}_1$ and $\bar{s}_2$. □

Besides these properties, we prove that the family of finite commutative ω-semigroups contains ω-subsets of every possible $\mathbb{SG}$-degree.

PROPOSITION 9.21. *Let ξ be an ordinal such that $0 < \xi < \omega^\omega$. Then there exist a finite commutative ω-semigroup $S = (S_+, S_\omega)$ and an ω-subset $X \subseteq S_\omega$ such that $d_{sg}(X) = \xi$.*

Proof. For each $n \geq 0$, consider the finite *powerset ω-semigroup* $P_n = (P_{n+}, P_{n\omega})$ induced by the finite semigroup

$$P_{n+} = (\mathcal{P}(\{0, 1, \ldots n\}), \cup)\,.$$

The semigroup P_{n+} is commutative and every element is idempotent. In addition, every $\mathcal{R}$-class of prefixes is trivial: for every $s, t \in P(P_{n+})$, the relations $s \geq_{\mathcal{R}} t$ and $t \geq_{\mathcal{R}} s$ imply $s \subseteq t$ and $t \subseteq s$, thus $s = t$. In addition, every prefix s is associated with the unique petal $petal(s) = \{e \in P_{n+} \mid e \subseteq s\}$. The DAG representation of P_{n+} is illustrated in Figure 9.9. Its description ensures that, for any $0 < \xi < \omega^\omega$, there exist an integer n large enough and an ω-subset $X \subseteq P_{n\omega}$ such that $d_{sg}(X) = \xi$. □

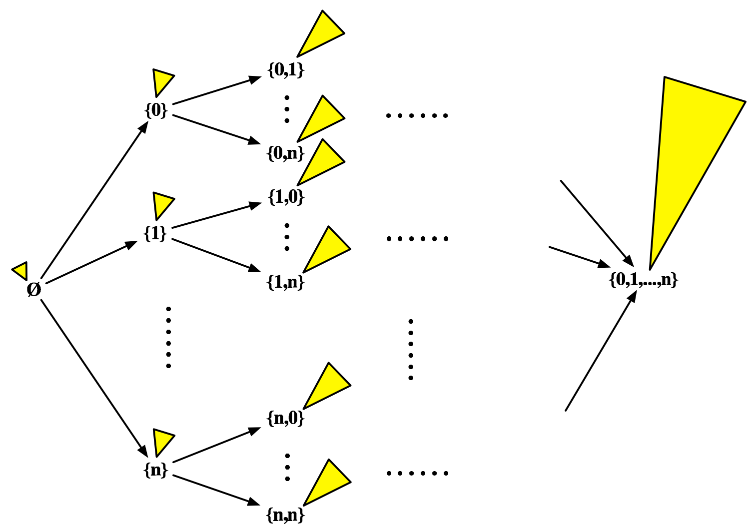


Figure 9.9: the DAG representation of the finite semigroup P_{n+}.

Finally, the different properties of the specific ω-semigroups described in this section are summarized in the following table. Each specific kind of ω-semigroups appears in front of the ω-subsets it generates.

Finite ω-monoids	non-self-dual ω-subsets
Finite left-cancelable ω-semigroups	ω-subsets of $\mathbb{SG}$-degree 1
Finite ω-groups	ω-subsets of $\mathbb{SG}$-degree 1
Finite cyclic ω-semigroups	ω-subsets of $\mathbb{SG}$-degree 1
Finite commutative ω-semigroups	ω-subsets of every $\mathbb{SG}$-degrees

Conclusion

We hope this work provides a convincing description of the algebraic counterpart of the Wagner hierarchy. In summary, based on the equivalence between ω-rational languages and finite pointed ω-semigroups, we initially proved that the Wagner degree of an ω-rational language is indeed a syntactic invariant. We then defined a Wadge-like reduction on finite pointed ω-semigroups, and proved that the resulting algebraic hierarchy is isomorphic to the Wagner hierarchy. This hierarchy has therefore a height of ω^ω, is decidable, and provides an algebraic representative of the Wagner hierarchy. In particular, an ω-rational language and its syntactic image are proven to share the same Wagner degree, and syntactic pointed ω-semigroups appeared as minimal representatives of their Wagner classes, whereas there is no convincing notion of minimal Muller automata of a given Wagner degree. Furthermore, we described a decision procedure of this hierarchy based on a graph representation of finite pointed ω-semigroups. This algorithm may thus compute the Wagner degree of any ω-rational language directly on its syntactic image. It consists of a reformulation in this algebraic context of Wagner's naming procedure [41]. Finally, we presented two methods for building a finite pointed ω-semigroups of any given degree. We also described the algebraic invariant characterizing the Wagner degree of every finite pointed ω-semigroup. These invariants are also a reformulation in this context of the notions of maximal ξ-chains presented in [10], or maximal μ_α-alternating trees described in [34], or also maximal binary tree-like sequences of superchains described in [41].

We notice by the way that the graph representation of finite pointed ω-semigroups seems more complex than the graph of Muller automata: the set of loops of a given strictly connected component in a Muller automata is a semi-lattice for inclusion, whereas the set of idempotents of a given $\mathcal{R}$-class of prefixes is not, since it contains several petals. The question of the existence of a DAG decomposition of finite ω-semigroups looking exactly as complex as the graphs of Muller automata is still open.

This work can be extended in several directions. On the one hand, we hope to widen this analysis to more sophisticated ω-languages, like the ones recognized by deterministic counters, or even deterministic pushdown automata (PDA). This would require a description of the corresponding infinite ω-semigroups, since the Wadge hierarchies of deterministic ω-languages accepted by counter automata or PDA are strictly finer than the Wagner hierarchy [8, 13]. However, an extension of this work to languages recognized by nondeterministic PDA would be very ambitious, since the Wadge hierarchy of ω-context-free languages (those recognized by nondeterministic PDA) was proven to be as complicated as the Wadge hierarchy of ω-languages accepted by nondeterministic Turing machines [14].

On the other hand, since the $\mathbb{SG}$-hierarchy restricted to free ω-semigroups coincides with the Wadge hierarchy, this work could also enlighten the Borel Wadge hierarchy itself, by characterizing Borel sets by precise algebraic properties. Many properties of the $\mathbb{SG}$-hierarchy should then be examined in the case of free ω-semigroups. For instance, we proved that a finite Borel ω-subset A is non-self-dual if and only if it is $\mathbb{SG}$-equivalent to some set B extracted from some finite ω-monoid (Theorem 9.8). This property still holds in the case of infinite ω-semigroups. In particular, when reformulated in the case of free ω-semigroups, this result states that a Borel ω-language A is non-self-dual if and only if it is $\mathbb{SG}$-equivalent to some set B extracted from some ω-monoid. Also, Proposition 9.13 shows that finite ω-groups only provide trivial ω-subsets. But this result does obviously not hold anymore in the case of infinite ω-semigroups. In the case of free ω-semigroups, one can actually prove that a Borel set A has a Wadge degree of the form ${\omega_1}^{\alpha}$, with $cof(\alpha) \neq \omega$, if and only if it is $\mathbb{SG}$-equivalent to some set B extracted from some ω-group (this result involves more sophisticated considerations about initializability, as shown in [7, 9]). Extending such results would require to provide, for any given Borel ω-language A, an $\mathbb{SG}$-equivalent set B extracted from a particular ω-semigroup which algebraically characterizes the Wadge class generated by A.

Finally, in an even more general context, one may be interested at describing the whole $\mathbb{SG}$-hierarchy, or its restriction to some specific ω-semigroups. For instance, we already mentioned that the $\mathbb{SG}$-hierarchy of pointed ω-monoids coincides with the restriction of the $\mathbb{SG}$-hierarchy to the non-self-dual ω-subsets. An extension of this kind of results to ω-groups, or other particular ω-semigroups, could be interesting.

Bibliography

[1] Alessandro Andretta. Equivalence between Wadge and Lipschitz determinacy. *Ann. Pure Appl. Logic*, 123(1-3):163–192, 2003.

[2] Julius R. Büchi. Weak second-order arithmetic and finite automata. *Z. Math. Logik Grundlagen Math.*, 6:66–92, 1960.

[3] Jérémie Cabessa and Jacques Duparc. An infinite game over ω-semigroups. In Stefan Bold Benedikt Löwe Thoralf Räsch Johan van Benthem, editor, *Foundations of the Formal Sciences V, Infinite Games*, volume 11 of *Studies in Logic*, pages 63–78, London, 2007. College Publications.

[4] Olivier Carton and Dominique Perrin. Chains and superchains in ω-semigroups. Almeida, Jorge (ed.) et al., Semigroups, automata and languages. Papers from the conference, Porto, Portugal, June 20–24, 1994. Singapore: World Scientific. 17-28 (1996)., 1996.

[5] Olivier Carton and Dominique Perrin. Chains and superchains for ω-rational sets, automata and semigroups. *Int. J. Algebra Comput.*, 7(6):673–695, 1997.

[6] Olivier Carton and Dominique Perrin. The Wagner hierarchy. *Int. J. Algebra Comput.*, 9(5):597–620, 1999.

[7] Jacques Duparc. Wadge hierarchy and Veblen hierarchy. Part I: Borel sets of finite rank. *J. Symb. Log.*, 66(1):56–86, 2001.

[8] Jacques Duparc. A hierarchy of deterministic context-free ω-languages. *Theoret. Comput. Sci.*, 290(3):1253–1300, 2003.

[9] Jacques Duparc. Wadge hierarchy and Veblen hierarchy. Part II: Borel sets of infinite rank. *J. Symbolic Logic*, ap. To appear.

[10] Jacques Duparc and Mariane Riss. The missing link for ω-rational sets, automata, and semigroups. *Int. J. Algebra Comput.*, 16(1):161–185, 2006.

[11] Samuel Eilenberg. *Automata, languages, and machines. Vol. A.* Pure and Applied Mathematics, 58. New York-London: Academic Press, a subsidiary of Harcourt Brace Jovanovich, Publishers , 1974.

[12] Samuel Eilenberg. *Automata, languages, and machines. Vol. B. With two chapters by Bret Tilson.* Pure and Applied Mathematics, 59. New York-San Francisco-London: Academic Press, a subsidiary of Harcourt Brace Jovanovich, Publishers , 1976.

[13] Olivier Finkel. An effective extension of the Wagner hierarchy to blind counter automata. In *Computer science logic (Paris, 2001)*, volume 2142 of *Lecture Notes in Comput. Sci.*, pages 369–383. Springer, Berlin, 2001.

[14] Olivier Finkel. Borel ranks and Wadge degrees of context free omega languages. In *New Computational Paradigms, First Conference on Computability in Europe, CiE*, volume 2142 of *Lecture Notes in Comput. Sci.*, pages 129–138, Berlin, 2005. Springer.

[15] David Gale and Frank M. Stewart. Infinite games with perfect information. In *Contributions to the theory of games, vol. 2*, Annals of Mathematics Studies, no. 28, pages 245–266. Princeton University Press, Princeton, N. J., 1953.

[16] James A. Green. On the structure of semigroups. *Annals of Mathematics*, 54(1):163–172, 1951.

[17] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[18] Samuel C. Kleene. Representation of events in nerve nets and finite automata. In *Automata Studies*, Annals of Mathematics Studies, no. 34, pages 3–42. Princeton University Press, Princeton, N. J., 1956.

[19] Kenneth Kunen. *Set theory. An introduction to independence proofs. 2nd print.* Studies in Logic and the Foundations of Mathematics, 102. Amsterdam-New York-Oxford: North-Holland. XVI, 313 p. , 1983.

[20] Richard E. Ladner. Application of model theoretic games to discrete linear orders and finite automata. *Information and Control*, 33(4):281–303, 1977.

[21] Donald A. Martin. Borel determinacy. *Ann. of Math. (2)*, 102(2):363–371, 1975.

[22] Yiannis N. Moschovakis. *Descriptive set theory.* Studies in Logic and the Foundations of Mathematics, Vol. 100. Amsterdam, New York, Oxford: North-Holland Publishing Company. XII, 637 p. , 1980.

[23] Yiannis N. Moschovakis. *Notes on set theory.* Undergraduate Texts in Mathematics. New York, NY: Springer-Verlag. 286 p., 1994.

[24] David E. Muller. Infinite sequences and finite machines. In *FOCS*, pages 3–16. IEEE, 1963.

[25] Dominique Perrin and Jean-Eric Pin. First-order logic and star-free sets. *J. Comput. System Sci.*, 32(3):393–406, 1986.

[26] Dominique Perrin and Jean-Éric Pin. Semigroups and automata on infinite words. In *Semigroups, formal languages and groups (York, 1993)*, pages 49–72. Kluwer Acad. Publ., Dordrecht, 1995.

[27] Dominique Perrin and Jean-Éric Pin. *Infinite words*, volume 141 of *Pure and Applied Mathematics*. Elsevier, 2004.

[28] Jean-Éric Pin. *Variétés de langages formels*. Masson, Paris, 1984.

[29] Jean-Éric Pin. *Varieties of formal languages*. North Oxford, London et Plenum, New-York, 1986. (Traduction de Variétés de langages formels).

[30] Jean-Éric Pin. Logic, semigroups and automata on words. *Annals of Mathematics and Artificial Intelligence*, 16:343–384, 1996.

[31] Jean-Eric Pin. Positive varieties and infinite words. In C.L. Lucchesi and A.V. Moura, editors, *Latin'98*, volume 1380 of *Lecture Notes in Comput. Sci.*, pages 76–87. Springer Verlag, Berlin, Heidelberg, New York, 1998.

[32] Joseph G. Rosenstein. *Linear orderings.* Pure and Applied Mathematics, 98. New York etc.: Academic Press, a Subsidiary of Harcourt Brace Jovanovich, Publishers. XVII, 487 p. , 1982.

[33] Jacques Sakarovitch. Monoides pointés. *Semigroup Forum*, 18(3):235–264, 1979.

[34] Victor Selivanov. Fine hierarchy of regular ω-languages. *Theor. Comput. Sci.*, 191(1-2):37–59, 1998.

[35] Michael Sipser. Introduction to the theory of computation. *SIGACT News*, 27(1):27–29, 1996.

[36] Robert Tarjan. Depth-first search and linear graph algorithms. *SIAM Journal on Computing*, 1(2):146–160, 1972.

[37] Wolfgang Thomas. Star-free regular sets of ω-sequences. *Inform. and Control*, 42(2):148–156, 1979.

[38] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proc. Lond. Math. Soc., II. Ser.*, 42:230–265, 1936.

[39] William W. Wadge. Degrees of complexity of subsets of the Baire space. *Notice A.M.S.*, pages A714–A715, 1972.

[40] William W. Wadge. *Reducibility and determinateness on the Baire space*. PhD thesis, University of California, Berkeley, 1983.

[41] Klaus Wagner. On ω-regular sets. *Inform. and Control*, 43(2):123–177, 1979.

[42] Thomas Wilke. An Eilenberg theorem for ∞-languages. In *Automata, languages and programming (Madrid, 1991)*, volume 510 of *Lecture Notes in Comput. Sci.*, pages 588–599. Springer, Berlin, 1991.

[43] Thomas Wilke and Haiseung Yoo. Computing the Wadge degree, the Lifshitz degree, and the Rabin index of a regular language of infinite words in polynomial time. In Peter D. Mosses, Mogens Nielsen, and Michael I. Schwartzbach, editors, *TAPSOFT '95: Theory and Practive of Software Development*, volume 915 of *Lecture Notes in Computer Science*, pages 288–302, Aarhus, Denmark, 1995. Springer.