HIGHER-ORDER EXPANSIONS FOR COMPOUND DISTRIBUTIONS AND RUIN PROBABILITIES WITH SUBEXPONENTIAL CLAIMS

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Abstract

Let X_i $(i = 1, 2, \ldots)$ be a sequence of subexponential positive independent and identically distributed random variables. In this paper we offer two alternative approaches to obtain higher-order expansions of the tail of $\sum_{i=1}^n X_i$ and subsequently for ruin probabilities in renewal risk models with claim sizes X_i. In particular, these emphasize the importance of the term $\mathbb{P}(\sum_{i=1}^n X_i > s, \max(X_1, \ldots, X_n) \le s/2)$ for the accuracy of the resulting asymptotic expansion of $\mathbb{P}(\sum_{i=1}^n X_i > s)$. Furthermore, we present a more rigorous approach to the often suggested technique of using approximations with shifted arguments. The cases of a Pareto-type, Weibull and Lognormal distribution for X_i are discussed in more detail and numerical investigations of the increase in accuracy by including higher-order terms in the approximation of ruin probabilities for finite realistic ranges of s are given.

1 Introduction

We consider compound sums $S_N = X_1 + \cdots + X_N$ with independent identically distributed (i.i.d.) random variables X_i with subexponential distribution, i.e.

$$\lim_{s \to \infty} \mathbb{P}(X_1 + X_2 > s)/\mathbb{P}(X_1 > s) = 2,$$

with tail probabilities $\mathbb{P}(X_i > s) = 1 - F(s) = \overline{F}(s)$. The claim number N is independent of the claim sizes $X_i, i = 1, 2, \ldots$ and is assumed to satisfy $\mathbb{E}\left[(1+\epsilon)^N\right] < \infty$ for an $\epsilon > 0$, i.e. N is light-tailed.
It is well known that the numerical computation of the total claims distribution $G(s) = \mathbb{P}(S_N \le s)$ is usually time-consuming because of the fat tail of the claim size distribution. This applies to both commonly suggested methods of computation: the integral equation for densities (see e.g. [26, Th.4.4.4 and (4.4.18)]) as well as Panjer's method ([26, Th.4.4.2]) with discretized claim size distribution. For the tail probabilities $\overline{G}(s) = 1 - G(s)$, another way of approximation are asymptotic expansions that are theoretically valid for large

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values of s: for a first-order asymptotic formula see e.g. Teugels and Veraverbeke [27], Greenwood [19], von Bahr [30] and Embrechts and Veraverbeke [15]. Later on second-order asymptotic formulas were considered (see e.g. Omey and Willekens [24] and Grübel [20]). Further work on approximations of compound distributions in the heavy-tailed case can be found in Mikosch & Nagaev [22], Willekens [31] and Baltrūnas [5]. Higher-order asymptotic expansions were finally given in Geluk et al. [18], Borovkov and Borovkov [14], Barbe and McCormick [7] and Barbe et al. [11].

In this paper we want to review the available higher-order asymptotic expansions of $\overline{G}(s)$ and complement some of these results. Further we give a numerical study that highlights the advantages and disadvantages of higher-order asymptotic expansions as an approximation for $\overline{G}(s)$. These approximations for compound distributions also lead to corresponding approximations for ruin probabilities in the collective renewal risk model with i.i.d. claim sizes X_i and i.i.d. interclaim times, as the Pollaczek-Khintchine formula expresses the ruin probability as a geometric compound (see Section 6).

The paper is organized as follows: in Section 2 we give an overview of available results on higher-order approximations with a particular emphasis on the used methodology, including a discussion of the main steps of the proof of Barbe & McCormick [7] and Barbe et al. [11]. In Section 3 we present two alternative approaches to derive asymptotic expansions and extend the expansion given in [7] by adding an additional term. In Section 4 we show how asymptotic approximations in terms of derivatives can be transformed to asymptotic expansions in terms of the original function with shifted arguments. In Section 5 an example is given of how asymptotic expansions for the light-tailed case and higher-order asymptotic expansions can be combined. In Section 6 we then exploit the connections to ruin probabilities. Finally, in Section 7 extensive numerical illustrations on the performance of these approximations for a finite realistic range of s are given. Some of the more technical proofs are postponed to the Appendix.

2 Asymptotic approximations of compound sums

Unless stated otherwise, in the sequel we will always consider an i.i.d. sequence $(X_i)_{i\geq 1}$ with subexponential distribution function F (and X as a generic random variable with the same distribution function) and an independent integer-valued random variable with $\mathbb{E}\left[(1+\epsilon)^N\right] < \infty$ for some $\epsilon > 0$.

The classical first-order asymptotic approximation for $P(X_1 + \cdots + X_N > s) = \overline{G}(s) = a_1(s) + o(\overline{F}(s))$, is given by

$$a_1(s) := E[N]\,\overline{F}(s)$$

(see e.g. [26, Th.2.5.4]). A second-order asymptotic

$$\overline{G}(s) = a_2(s) + o(f(s)) \tag{1}$$

was provided in Omey & Willekens [24] and [31] under the assumption that $\mathbb{E}[X] < \infty$ and F has a continuous density f that is long-tailed, dominantly varying and with an upper Matuszewska index $\alpha(f) < -1$, i.e. for all $x > 0$, $\limsup_{s\to\infty} f(x+s)/f(s) = 1$, $\limsup_{s\to\infty} f(xs)/f(s) < \infty$ and

$$\alpha(f) := \lim_{x\to\infty} \frac{\log\left(\limsup_{s\to\infty} \frac{f(xs)}{f(s)}\right)}{\log(x)} < -1.$$

In this case

$$a_2(s) := a_1(s) + 2\,\mathbb{E}\left[\binom{N}{2}\right]\mathbb{E}\left[X\right] f(s).$$

If $\overline{F}(s)$ is regularly varying with index $-\alpha$ for $0 < \alpha \le 1$ ($\overline{F}(s) \in \mathcal{R}_{-\alpha}$, i.e. for all $x > 0$, $\lim_{s\to\infty} \overline{F}(xs)/\overline{F}(s) = x^{-\alpha}$) with a regularly varying density $f(s)$, a second-order approximation for $\overline{G}(s)$ is given in Omey & Willekens [23] by

$$\hat{a}_2(s) = \begin{cases} a_1(s) - \frac{(2-\alpha)\Gamma(2-\alpha)}{(\alpha-1)\Gamma(3-2\alpha)}\mathbb{E}\left[\binom{N}{2}\right] f(s)\int_0^s \overline{F}(y)\ \mathrm{d}y, & 0 < \alpha < 1, \\ a_1(s) + 2\,\mathbb{E}\left[\binom{N}{2}\right] f(s)\int_0^s \overline{F}(y)\ \mathrm{d}y, & \alpha = 1. \end{cases}$$

See [16], [17] and [9] for more details about $\hat{a}_2(s)$.
If $f(s)$ is not continuous, but f is of bounded total variation, it follows from Grübel [20] that if there exists a monotonically decreasing function $\tau(s)$ with

$$\sup_{s>0} \frac{\tau(s)}{\tau(2s)} < \infty, \quad \tau(s) = O(s^{-4}), \quad V_s^{s+1} f = o(\tau(s)),$$

where V_s^{s+1} denotes the total variation of f in the interval $(s, s+1]$, then $\overline{G}(s) = a_2(s) + o(\tau(s))$. The particular choice $\tau(s) = \sup_{y\ge s} f(y)$ then also gives (1).

In Willekens [31] and Willekens & Teugels [32] it is shown that if $\overline{F}(s) \in \mathcal{R}_{-\alpha}$, $\alpha > 2$ and $f'(s)$ is also regularly varying, or alternatively $f''(s)$ exists, is asymptotically decreasing and there exists a function $\chi(s) \in \mathcal{R}_\gamma$, $0 < \gamma \le 1$ such that for all $x \in \mathbb{R}$

$$\lim_{s\to\infty} \frac{f''(s + x\chi(s))}{f''(s)} = e^{-x},$$

then a third-order asymptotic $\overline{G}(s) = a_3(s) + o(f'(s))$ with

$$a_3(s) := a_2(s) - \frac{\mathbb{E}\left[N(X_1 + \cdots + X_{N-1})^2\right]}{2} f'(s)$$

holds. Baltrūnas & Omey [6] consider distribution functions $F \in S^*$ (e.g. [21]), i.e.

$$\lim_{s\to\infty} \int_0^s \frac{\overline{F}(s-x)}{\overline{F}(s)} \overline{F}(x)\ \mathrm{d}x = 2\mathbb{E}\left[X_1\right] < \infty$$

and show the quite general result that $a_2(s)$ is a second-order asymptotic if additionally for all $x > 0$

$$0 < \liminf_{s\to\infty} \frac{f(xs)\overline{F}(x)}{f(x)\overline{F}(sx)} \le \limsup_{s\to\infty} \frac{f(xs)\overline{F}(s)}{f(s)\overline{F}(xs)} < \infty, \quad \lim_{s\to\infty} \frac{f(s+x)\overline{F}(s)}{f(s)\overline{F}(s+x)} = 1$$

$$\text{and} \quad \frac{\overline{F}(s/2)^2}{f(s)} = o(1).$$

Note that heavy-tailed Weibull and lognormal distributions are examples of distributions fulfilling these requirements.

For cases where a density of X does not necessarily exist, the following results (for a fixed number of summands) have been established: Geluk et al. [18] derive a higher-order asymptotic expansion for the sum of two regularly varying random variables X_1 and X_2 with distribution function $F_1(s)$ and $F_2(s)$, respectively, for which the second-order behavior is given and of the form

$$\lim_{s\to\infty} \frac{\overline{F}_i(xs)/\overline{F}_i(s) - x^{-\alpha}}{b_i(s)} = x^{-\alpha}\frac{x^\rho - 1}{\rho}, \quad i = 1, 2,$$

where b_i is a function with $b_i(s) \to 0$ $(s \to \infty)$. If $0 < \alpha < 1$ it is then shown that

$$\mathbb{P}(X_1+X_2 > s) = \sum_{i=1}^{2} \overline{F}_i(s) - \frac{\Gamma(1-\alpha)^2}{\Gamma(1-2\alpha)}\overline{F}_1(s)\overline{F}_2(s) + o(1)\left(\sum_{i=1}^{2} b_i(s)\overline{F}_i(s) + \overline{F}_1(s)\overline{F}_2(s)\right).$$

If $\alpha \geq 1$ and $\mathbb{E}[X_i^\alpha] < \infty$ then

$$\mathbb{P}(X_1 + X_2 > s) = \sum_{i=1}^{2}\sum_{j=0}^{\lfloor\alpha\rfloor} \frac{\Gamma(\alpha+j)\mathbb{E}\left[X_{3-i}^j\right]}{\Gamma(\alpha)j!}\frac{\overline{F}_i(s)}{s^j} + o(1)\sum_{i=1}^{2}\left(b_i(s)\overline{F}_i(s) + \frac{\overline{F}_i(s)}{s^{\lfloor\alpha\rfloor}}\right).$$

If $\alpha \in \mathbb{N}$ and $\mathbb{E}[X_i^\alpha] = \infty$ then

$$\begin{aligned}\mathbb{P}(X_1 + X_2 > s) = \sum_{i=1}^{2}&\left(\sum_{j=0}^{\alpha-1} \frac{\Gamma(\alpha+j)\mathbb{E}\left[X_{3-i}^j\right]}{\Gamma(\alpha)j!}\frac{\overline{F}_i(s)}{s^j} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\frac{1}{s^\alpha}\int_0^s \overline{F}_{3-i}(y)\,\mathrm{d}y\right)\\ &+ o(1)\sum_{i=1}^{2}\left(b_i(s)\overline{F}_i(s) + \int_0^s \overline{F}_{3-i}(y)\,\mathrm{d}y\right).\end{aligned}$$

On the other hand, if $\alpha \geq 1$ is not an integer and $\mathbb{E}[X_i^\alpha] = \infty$, then for a constant h_α

$$\begin{aligned}\mathbb{P}(X_1 + X_2 > s) = &\sum_{i=1}^{2}\sum_{j=0}^{\alpha-1} \frac{\Gamma(\alpha+j)\mathbb{E}\left[X_{3-i}^j\right]}{\Gamma(\alpha)j!}\frac{\overline{F}_i(s)}{s^j} + h_\alpha\overline{F}_1(s)\overline{F}_2(s)\\ &+ o(1)\sum_{i=1}^{2}\left(b_i(s)\overline{F}_i(s) + \overline{F}_1(s)\overline{F}_2(s)\right).\end{aligned}$$

Next, consider real-valued i.i.d. random variables $X_1, X_2, \ldots$ with $F(s) = s^{-\alpha}\ell(s)$, where $\ell(s)$ is a slowly varying function, and $\mathbb{E}[X_1] = 0$, $\mathbb{E}\left[X_1^2\right] = 1$ and $\mathbb{E}[|X_1|^a] < \infty$ for some $2 \leq a \leq \alpha$. Under the assumption that for a $k \leq a$ there exist functions $\ell_0(s), \ldots, \ell_k(s)$ and $\delta_k(x, s)$ such that

$$\ell(s(1+x)) = \ell(s)\left(\sum_{j=0}^{k-1} x^j\ell_j(s) + y^k\ell_k(s)(1+\delta_k(x,s))\right),$$

and $\lim_{(x,s)\to(0,\infty)} \delta_k(x,s) = 0$, Borovkov & Borovkov [14] showed that for a $c > 0$ uniformly in $n \le cs^2/\log(s)$,

$$\overline{G}(s) = n\overline{F}(s)\left(1 + \sum_{j=2}^{k} \frac{(-1)^j}{s^j}\, U_j(s)\, \mathbb{E}\left[S_{n-1}^j\right] + o(n^{k/2}s^{-k})\right),$$

where

$$U_j(s) = \sum_{m=0}^{j} \binom{-\alpha}{m} \ell_{j-m}(s).$$

Following another approach, Barbe & McCormick [7, 10] (see also [8, 9]) recently used the concept of smoothly varying functions to derive higher-order expansions:

Definition 2.1. *A function $h(s)$ is smoothly varying with index $-\alpha$ and order $m \in \mathbb{N}$ ($h \in SR_{-\alpha,m}$) if $h(s)$ is eventually m-times continuously differentiable and $h^{(m)}(s)$ is regularly varying with index $-\alpha - m$.*
For a non-integer $u > 0$ with $u = m + r$, $m \in \mathbb{N}$ and $r \in [0,1)$, $h(s)$ is smoothly varying with index $-\alpha$ and order u ($h \in SR_{-\alpha,u}$), if $h \in SR_{-\alpha,m}$ and

$$\lim_{\delta\to 0} \limsup_{s\to\infty} \sup_{0<|x|<\delta} \left| \frac{h^{(m)}(s(1-x)) - h^{(m)}(s)}{|x|^r h^{(m)}(t)} \right| = 0.$$

If $\overline{F}(s) \in \mathrm{SR}_{-\alpha,u}$ then it is shown in [7] that for $k < \min(\alpha+1, u+1)$, $\overline{G}(s) = a_k(s) + o\left(\overline{F}^{(k-1)}(s)\right)$ with

$$a_k(s) := a_1(s) + \sum_{j=0}^{k-2} \frac{(-1)^j \mathbb{E}\left[N(X_1 + \cdots + X_{N-1})^{j+1}\right]}{(j+1)!} f^{(j)}(s). \tag{2}$$

Further, if the hazard rate $h(s) = f(s)/\overline{F}(s) \in \mathrm{SR}_{-\alpha,m}$, $0 < \alpha \le 1$, $\lim_{s\to\infty} sh(s) = \infty$ and $\liminf_{s\to\infty} sh(s)/\log(s) > 0$, then it is shown in [11] that for $k \le m+1$, $\overline{G}(s) = a_k(s) + o\left(\overline{F}^{(k-1)}(s)\right)$.

Remark 2.1. *The above conditions on the hazard rate are for instance fulfilled with $m = \infty$ for the Weibull and lognormal distribution.*

Remark 2.2. *For a distribution function with $h(s) \in SR_{-\alpha,\infty}$ we have for every $k > 0$ that $\overline{G}(s) = a_k(s) + O(|F^{(k)}(s)|)$. Note however, that in many cases*

$$\lim_{j\to\infty} \left| \frac{(-1)^k \mathbb{E}\left[N(X_1 + \cdots + X_{N-1})^j\right] \overline{F}^{(j)}(s)}{j!} \right| \ge \lim_{j\to\infty} \left| \frac{(-1)^k \mathbb{E}\left[N\right] \mathbb{E}\left[X_1^j\right] \overline{F}^{(j)}(s)}{j!} \right| = \infty,$$

so that $\lim_{k\to\infty} |a_k(s)| = \infty$ for a fixed s (indeed, this is for instance the case for lognormal or Weibull F).

Before we outline the ideas of the proofs given in [7, 11] one should mention that all proofs for higher order expansion (except the proof in [20] which uses Banach algebra techniques) use some decomposition of $\mathbb{P}(S_n > s)$ and then asymptotically evaluate involved convolution integrals.
For a distribution function $F(s)$ and an $0 < \eta < 1$, define the operators

$$M_c F(s) = F(s/c) \quad \text{and} \quad T_{F,\eta} g(s) = \int_0^{\eta s} g(s-x) dF(x)$$

for a constant c and a function $g(s)$, respectively. The main idea in the proofs of [7, 11] is to use the decomposition

$$\begin{aligned}\mathbb{P}(S_n > s) =& \mathbb{P}(S_{n-1} \leq (1-\eta)s, S_n > s) + \mathbb{P}(X_n \leq \eta s, S_n > s) + \mathbb{P}(S_{n-1} > (1-\eta)s, X_n > \eta s) \\ =& T_{F^{*n-1},1-\eta}\overline{F}(s) + T_{F,\eta}\overline{F^{*n-1}}(s) + \left(M_{1/\eta}\overline{F}\; M_{1/(1-\eta)}\overline{F^{*n-1}}\right)(s),\end{aligned}$$

and further by recursion

$$\mathbb{P}(S_n > s) = \sum_{j=1}^{n} T_{F,\eta}^{j-1} T_{F^{*(n-j)},1-\eta}\,\overline{F}(s) + \sum_{j=1}^{n} T_{F,\eta}^{j-1}\left(M_{1/(1-\eta)}\overline{F^{*(n-j)}}\; M_{1/\eta}\overline{F}\right)(s). \tag{3}$$

Now one makes use of finite Taylor expansions to get for $m > 0$, $0 \leq i \leq k$, $\tau \in \{\eta, 1-\eta\}$ and an $\tilde{R}_{k,i}(s) = o\left(\overline{F}^{(k)}(s)\right)$

$$T_{F^{*m},\tau}\overline{F}^{(i)}(s) = \sum_{j=0}^{k-i} \frac{(-1)^j \overline{F}^{(i+j)}(s)}{j!} \int_0^{\tau s} x^j \;\mathrm{d}F^{*m}(x) + \int_0^{\tau s} \tilde{R}_{k,i}(s-x)\;\mathrm{d}F^{*m}(x). \tag{4}$$

One can show that for $j \leq k$

$$\overline{F}^{(j)}(s) \int_{\tau s}^{\infty} x^j \;\mathrm{d}F^{*m}(x) = o\left(\overline{F}^{(k)}(s)\right), \tag{5}$$

which for regularly varying $\overline{F}$ is a consequence of the monotone density and Karamata's Theorem (e.g. [12]). Further one can show that for $\tilde{R}(s) = o\left(\overline{F}^{(k)}(s)\right)$

$$\int_0^{\tau s} \tilde{R}(s-x)\;\mathrm{d}F^{*m}(x) = o\left(\overline{F}^{(k)}(s)\right), \tag{6}$$

which is easily seen to be true for regularly varying $\overline{F}(s)$. The proofs for $F(s) \notin \mathcal{R}$ are more involved (for details see [10]). In [7] Laplace characters are defined by

$$L_{F,m} = \sum_{j=0}^{m} \frac{(-1)^j}{j!}\mu_{F,j} D^j, \quad \text{i.e.} \quad L_{F,m} g(s) = \sum_{j=0}^{m} \frac{(-1)^j}{j!}\mu_{F,j} g^{(j)}(s),$$

where $\mu_{F,k} = \int_0^\infty x^k dF(x)$. This concept is useful as for two distribution functions F_1 and F_2 one has

$$L_{F_1,m} L_{F_2,m} g(s) = L_{F_1 * F_2} g(s),$$

where in the multiplication, terms of D^j with $j > m$ are omitted. Applying (4), (5) and (6) we get

$$T_{F,\eta}^{j-1} T_{F^{*(n-j)},1-\eta} \overline{F}(s) = L_F^{j-1} L_{F^{*(n-j)}} \overline{F}(s) + o\left(\overline{F}^{(k)}(s)\right) = L_{F^{*(n-1)}} \overline{F}(s) + o\left(\overline{F}^{(k)}(s)\right).$$

Finally, one has to show that the conditioning on $N = n$ is feasible which can be quite intricate.

3 Higher-order results for regularly varying distributions

Let $\lceil \alpha \rceil$ denote the smallest integer k with $\alpha \leq k$. For $\overline{F}(s) \in \mathrm{SR}_{-\alpha,\lceil\alpha\rceil}$, in Barbe & McCormick [7] an asymptotic expansion of $\overline{G}(s)$ up to order m for $m < \alpha$ was provided. In this section we present two alternative approaches for obtaining expansions and use them to derive the next asymptotic term (note that for $0 < \alpha \leq 1$ the latter is done in Omey & Willekens [23], and that Geluk et al. [18] provide an O result). Since the step from fixed n to a random N is straight-forward in this case (cf. Barbe et al. [11]), we will only focus on the proof for fixed n.

At first we show that for an asymptotic expansion up to order $\overline{F}(s)^2$ the term $\mathbb{P}(S_n > s, M_n \leq s/2)$ can be neglected (although it will turn out in the numerical illustrations in Section 7 that for smaller values of s this term is crucial for the accuracy of the asymptotic approximation).

Lemma 3.1. *Assume that $\overline{F}(s) \in \mathcal{R}_{-\alpha}$ with $\alpha > 0$, then for $n \geq 2$*

$$\mathbb{P}\left(S_n > s, M_n \leq s/2\right) = o(\overline{F}(s)^2).$$

Proof.

$$\begin{aligned}
\mathbb{P}\left(S_n > s, M_n \leq s/2\right) &\leq n\, \mathbb{P}\left(S_n > s, M_n \leq s/2, M_n = X_n\right) \\
&\leq n\, \mathbb{P}\left(S_{n-1} > s/2, M_{n-1} \leq s/2, X_n > s/n\right) \\
&= n\, \mathbb{P}\left(S_{n-1} > s/2, M_{n-1} \leq s/2\right) \mathbb{P}\left(X_n > s/n\right) \\
&= n\left(\mathbb{P}\left(S_{n-1} > s/2\right)) - \mathbb{P}\left(M_{n-1} > s/2\right)\right) \mathbb{P}\left(X_n > s/n\right) = o(\overline{F}(s)^2).
\end{aligned}$$

□

Under some additional conditions on $F(s)$, an exact asymptotic for $\mathbb{P}(S_n > s, M_n \leq s/2)$ can be given:

Lemma 3.2. *Assume that $\overline{F}(s) \in SR_{-\alpha,2}$ with $\alpha > 2$, then*

$$\mathbb{P}(S_n > s, M_n \leq s/2) \sim \binom{n}{2} \mathbb{E}\left[S_{n-2}^2\right] f(s/2)^2$$

For a proof we refer to the Appendix.

If $\overline{F}(s)$ is not regularly varying, one can show that $\mathbb{P}\left(S_n > s, M_n \leq s/2\right) = O(\overline{F}(s/2)^{2-\epsilon})$ for an $\epsilon > 0$ under quite general conditions:

Lemma 3.3. *Assume that the following assumptions hold.*

(A1) $F(s)$ *has some finite power mean* $\mu(\gamma) = E[X^\gamma] < \infty, 0 < \gamma < 1$;

(A2) for the hazard function $R(s) = -\log(\overline{F}(s))$ *we can find an eventually concave function* $h_0(s)$ *with* $h_0(s) \sim -\log(\overline{F}(s))$;

(A3) for all $\delta > 0$ *and* $c > 0$ *we have* $\lim_{s\to\infty} R(s)\overline{F}(cs/R(s))^\delta = 0$.

Then for all $0 < \varepsilon < 1$ *and* $K > 1$ *we can find a constant* $M = M(\varepsilon, K) > 0$ *such that for all* $s > M$ *and* $n \geq 2$

$$\mathbb{P}\left(S_n > Ks, M_n \leq s\right) \leq (1+\varepsilon)^n \overline{F}(s)^{K(1-\varepsilon)}, \tag{7}$$

where $M_n = \max(X_1, ..., X_n)$.

Proof. Cf. Appendix. □

3.1 A simple approach based on another decomposition

Denote with $M_n := \max_{1\leq i\leq n} X_i$ and $S_{(n-1)} := S_n - M_n$. Instead of the decomposition (3), consider the decomposition

$$\mathbb{P}(S_n > s) = \mathbb{P}(S_{(n-1)} > s/2, M_n > s/2) \\ + \mathbb{P}(S_n > s, M_n \leq s/2) + \mathbb{P}(S_n > s, S_{(n-1)} \leq s/2). \tag{8}$$

We now look at the third summand in (8):

Lemma 3.4. *Assume that* $\overline{F}(s) \in SR_{-\alpha,\lceil\alpha\rceil}$. *Denote with* $k := \lceil\alpha\rceil - 1$. *If* $\alpha \neq k+1$ *then*

$$\mathbb{P}(S_n > s, S_{(n-1)} \leq s/2) - a_{k+1}(s) \sim -\binom{n}{2}\overline{F}(s)^2\Big((1-2\alpha)\ B(1-\alpha, 1-\alpha) + 2^{2\alpha}\Big),$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ *is the Beta function (for* $a > -1$ *and* $b > -1$ *it can be written as* $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$, *for other values it is defined by analytic continuation of this integral, cf. [2]) and* a_{k+1} *is defined as in (2). If* $\alpha = k+1$ *then*

$$\mathbb{P}(S_n > s, S_{(n-1)} \leq s/2) - a_{k+1}(s) \sim \frac{n(-1)^{k+1}\overline{F}^{(k+1)}(s)}{(k+1)!}\int_0^{s/2} x^{k+1}\ dF^{*(n-1)}(x).$$

Proof. Cf. Appendix. □

Theorem 3.5. *Let* $X_1, \ldots, X_n$ *be i.i.d. random variables with common distribution function* $\overline{F}(s) \in SR_{-\alpha,\lceil\alpha\rceil}$. *If* $k < \alpha < k+1$ *then*

$$\mathbb{P}(S_n > s) = a_{k+1}(s) - \binom{n}{2}\overline{F}(s)^2\,(1-2\alpha)\ B(1-\alpha, 1-\alpha) + o(\overline{F}(s)^2).$$

If $\alpha = k+1$ *then*

$$\mathbb{P}(S_n > s) = a_{k+1}(s) + \frac{n(-1)^{k+1}\overline{F}^{(k+1)}(s)}{(k+1)!}\int_0^{s/2} x^{k+1}\ dF^{*(n-1)}(x) \\ + o\left(\overline{F}^{(k+1)}(s)\int_0^{s/2} x^{k+1}\ dF^{*(n-1)}(x)\right).$$

Proof. Denote with $X_{(1)} \le \cdots \le X_{(n)}$ the order statistics of $X_1, \ldots, X_n$, then

$$\begin{aligned}\mathbb{P}(S_{(n-1)} > s/2, M_n > s/2) &= \mathbb{P}(X_{(n)} > s/2, X_{(n-1)} > s/2) \\ &\quad + \mathbb{P}(X_{(n)} > s/2, X_{(n-1)} \le s/2, S_{(n-1)} > s/2) \\ &= \mathbb{P}(X_{(n)} > s/2, X_{(n-1)} > s/2) \\ &\quad + n\,\mathbb{P}(X_n > s/2)\mathbb{P}(M_{n-1} \le s/2, S_{n-1} > s/2).\end{aligned}$$

It follows that as $s \to \infty$

$$\mathbb{P}(S_{(n-1)} > s/2, M_n > s/2) \sim 2^{2\alpha}\binom{n}{2}\overline{F}(s)^2.$$

Lemma 3.4 now completes the proof. $\square$

Remark 3.1. *Note that if $\alpha = k + 1/2$, then $B(1-\alpha, 1-\alpha) = 0$ in which case we do not get the next term in the asymptotic expansion. For a stable distribution F, the next asymptotic term in this particular situation was given in Omey & Willekens [23].*

Remark 3.2. *If for an integer $k > 0$, $\overline{F}(s) \in SR_{-(k+1),k+1}$ and $\mathbb{E}[X_1] < \infty$, then*

$$\frac{n(-1)^{k+1}\overline{F}^{(k+1)}(s)}{(k+1)!}\int_0^{s/2} x^{k+1}\ dF^{*(n-1)}(x) \sim \frac{n(-1)^{k+1}\overline{F}^{(k+1)}(s)}{(k+1)!}\mathbb{E}\left[(S_{n-1})^{k+1}\right].$$

If $\mathbb{E}[X_1] = \infty$ then

$$\frac{n(-1)^{k+1}\overline{F}^{(k+1)}(s)}{(k+1)!}\int_0^{s/2} x^{k+1}\ dF^{*(n-1)}(x) \sim \frac{n(n-1)(-1)^{k+1}\overline{F}^{(k+1)}(s)}{(k+1)!}\int_0^{s/2} x^{k+1}\ dF(x).$$

Theorem 3.5 was based on the fact that $\overline{F}(s)$ has $k+1$ continuous derivatives. On the other hand, Geluk et al. [18] gave higher-order asymptotic expansions for the sum of two random variables if $F(s)$ fulfills a second-order regular variation condition. A similar result is provided in the following:

Corollary 3.6. *Let $X_1, \ldots, X_n$ be i.i.d. random variables with common distribution function $F(s)$. Assume that there exists a function $H(s) \in SR_{-\alpha,\lceil\alpha\rceil}$ such that $|\overline{F}(s) - H(s)|$ is regularly varying with index $-(\alpha+\rho)$, $\rho > 0$. If $k < \alpha < k+1$ then*

$$\begin{aligned}\mathbb{P}(S_n > s) = n\overline{F}(s) + \sum_{i=1}^{k}\frac{n(-1)^i H^{(i)}(s)}{j!}\mathbb{E}\left[S_{n-1}^j\right] - \binom{n}{2}\overline{F}(s)^2\,(1-2\alpha)\ B(1-\alpha,1-\alpha) \\ + o\Big(|\overline{F}(s) - H(s)| + \overline{F}(s)^2\Big).\end{aligned}$$

If $\alpha = k+1$ then

$$\begin{aligned}\mathbb{P}(S_n > s) = n\overline{F}(s) + \sum_{i=1}^{k}\frac{n(-1)^i H^{(i)}(s)}{j!}\mathbb{E}\left[S_{n-1}^j\right] + \frac{n(-1)^{k+1}H^{(k+1)}(s)}{(k+1)!}\int_0^{s/2} x^{k+1}\ dF^{*(n-1)}(x) \\ + o(n|\overline{F}(s) - H(s)|) + o\left(\frac{n(-1)^{k+1}H^{(k+1)}(s)}{(k+1)!}\int_0^{s/2} x^{k+1}\ dF^{*(n-1)}(x)\right).\end{aligned}$$

Proof. Similarly to [18] we approximate

$$\int_0^{s/2} \overline{F}(s-x)\ \mathrm{d}F^{*(n-1)}(x) \approx \int_0^{s/2} H(s-x)\ \mathrm{d}F^{*(n-1)}(x).$$

The asymptotic expansion of

$$\int_0^{s/2} H(s-x)\ \mathrm{d}F^{*(n-1)}(x)$$

can be evaluated as in the proof of Theorem 3.5. Since $|\overline{F}(s) - H(s)|$ is regularly varying and $H(s)$ is eventually continuous, $\overline{F}(s) - H(s)$ is eventually positive or negative. Hence we get by the Uniform Convergence Theorem

$$\begin{aligned}
&\int_0^{s/2} \overline{F}(s-x) - H(s-x)\ \mathrm{d}F^{*(n-1)}(x) \\
&\sim \left(\overline{F}(s) - H(s)\right) \int_0^{s/2} (1 - x/s)^{-(\alpha+\rho)}\ \ \mathrm{d}F^{*(n-1)}(x) \\
&\sim (\overline{F}(s/2) - H(s/2)) \left(1 - 2^{\alpha+\rho}\overline{F}^{*(n-1)}(s/2) + \frac{\alpha+\rho}{s} \int_0^{s/2} \left(1 - \frac{x}{s}\right)^{-(\alpha+\rho+1)} \overline{F}^{*(n-1)}(x)\ \mathrm{d}x \right) \\
&= (\overline{F}(s/2) - H(s/2)) \left(1 - 2^{\alpha+\rho}\overline{F}^{*(n-1)}(s/2) + (\alpha+\rho) \int_0^{1/2} (1-x)^{-(\alpha+\rho+1)}\, \overline{F}^{*(n-1)}(sx)\ \mathrm{d}x \right) \\
&= (\overline{F}(s/2) - H(s/2))\,(1 + o(1))\,.
\end{aligned}$$

□

3.2 A recursive scheme

Assume that $\eta = 1/2$. In [7] the main idea was to use the decomposition (3), and then focus on

$$T_{F^{*(n-1)},1/2}F(s).$$

Instead, by focusing on

$$T_{F,1/2}F^{*(n-1)}(s),$$

we can follow a slightly different approach:

Lemma 3.7. *For all $s > 0$ and $n \geq 2$ there exist functions $\bar{R}_1$ and $\bar{R}_2$ with*

$$\begin{aligned}
&\bar{R}_1(s) \leq \mathbb{P}(X_1 > s/2)^2, \\
&\bar{R}_2(s) \leq \mathbb{P}(S_n > s, M_n \leq s/2) \quad \textit{and}
\end{aligned}$$

$$\mathbb{P}\{S_n > s\} = \frac{n}{n-1} \int_0^{s/2} P(S_{n-1} > s - x)\ dF(x) + n(\bar{R}_1(s) + \bar{R}_2(s)).$$

Proof.

$$
\begin{aligned}
\frac{1}{n}\mathbb{P}(S_n > s) &= \mathbb{P}(S_n > s, X_n = M_n) \\
&= \mathbb{P}(S_n > s, X_n = M_n, X_1 \le s/2) + \bar{R}_1(s) \\
&= \int_0^{s/2} \mathbb{P}(S_{n-1} > s - x, X_{n-1} = M_{n-1} \ge x)\ \mathrm{d}F(x) + \bar{R}_1(s) \\
&= \int_0^{s/2} \mathbb{P}(S_{n-1} > s - x, X_{n-1} = M_{n-1})\ \mathrm{d}F(x) + \bar{R}_1(s) + \bar{R}_2(s) \\
&= \frac{1}{n-1}\int_0^{s/2} P(S_{n-1} > s - x)\ \mathrm{d}F(x) + \bar{R}_1(s) + \bar{R}_2(s).
\end{aligned}
$$

□

Before we go on we investigate the asymptotic behaviour of $\mathbb{P}(S_n > s, M_n \le s/2)$:

By Lemma 3.1 and 3.3 we get that for every $\epsilon > 0$

$$
\overline{F}^{*n}(s) = \frac{n}{n-1}\int_0^{s/2} \overline{F}^{*(n-1)}(s-x)\ \mathrm{d}F(x) + O(\overline{F}(s/2)^{2-\epsilon}).
$$

Hence we can get an approximation of $\overline{F}^{*n}(s)$ from an approximation of $\overline{F}^{*(n-1)}(s)$ where the error we add in every step is asymptotically small.

Lemma 3.8. *Let $F(s)$ be a distribution with density $f(s)$. Assume that there exists functions $g_i(s)$, $i = 0, \ldots, k$ for which the following assumptions hold: Assume there exist $\varepsilon > 0$, a function $K(s)$ and an $M > 0$ such that for $s > M$*

$$
(1+\epsilon)^n K(s) \ge \mathbb{P}(S_n > s, M_n \le s/2) \quad \text{and} \quad K(s) \ge \overline{F}(s/2)^2
$$

with

$$
\int_0^{s/2} K(s-x) f(x)\ dx \le (1+\epsilon) K(s).
$$

If for $s > M$ and constants $a_{i,j}$ the inequalities

$$
\left| \int_0^{s/2} \overline{F}(s-x) f(x)\ dx - \overline{F}(s) - \sum_{j=0}^{k} a_{-1,j} g_j(s) \right| \le K(s), \tag{9}
$$

$$
\left| \int_0^{s/2} g_i(s-x) f(x)\ dx - g_i(s) - \sum_{j=i+1}^{k} a_{i,j} g_j(s) \right| \le K(s), \quad i = 0, \ldots k \tag{10}
$$

hold, then for $s > 2^{n-2}M$ and $n = 2, 3, \ldots$

$$
\left| \overline{F}^{*n}(s) - n\overline{F}(s) - \sum_{i=0}^{k} A_i^{(n)} g_i(s) \right| \le V(1+\epsilon)^{n-1} n^{k+3} K(s). \tag{11}
$$

where

$$V = 1 + \sum_{i=0}^{k} \sum_{j=1}^{i+1} \sum_{-1=j_0<\cdots<j_j=i} \left| \prod_{l=0}^{j-1} a_{j_l,j_{l+1}} \right|,$$

and for $i \geq 0$

$$A_i^{(n)} = \sum_{j=1}^{i+1} \frac{1}{j!}(n)_{j+1} \sum_{-1=j_0<\cdots<j_j=i} \prod_{l=0}^{j-1} a_{j_l,j_{l+1}},$$

where $(n)_j = n(n-1)\cdots(n-(j-1))$.

Proof. We prove the assertion (11) of the theorem by induction on n. For $n = 2$ we use

$$\overline{F}^{*2}(s) = 2\int_0^{s/2} \overline{F}(s-x)f(x)dx + \overline{F}(s/2)^2$$

and take from (9) with $|\hat{R}(s)| \leq K(s)$

$$\begin{aligned} 2\int_0^{s/2} \overline{F}(s-x)f(x)dx &= 2\overline{F}(s) + 2\sum_{i=0}^{k} a_{-1,i}g_i(s) + 2\hat{R} \\ &= 2\overline{F}(s) + \sum_{i=0}^{k} A_i^{(n)} g_i(s) + 2\hat{R}. \end{aligned}$$

Finally, the error $2\hat{R}(s) + \overline{F}(s/2)^2$ is bounded by $3K(s) \leq Vn^{k+3}K(s)$.
Assume now that (11) holds for some $n \geq 2$. Then with $|\hat{R}_i(s)| \leq K(s)$, $(i \geq 1)$ and

$$|\hat{R}_0| \leq \overline{F}(s/2)^2 + (1+\varepsilon)^n K(s)$$

we obtain

$$\begin{aligned} \overline{F}^{*(n+1)}(s) =& \frac{n+1}{n} \int_0^{s/2} \overline{F}^{*n}(s-x)f(x)dx + (n+1)\hat{R}_0(s) \\ =& \frac{n+1}{n} \int_0^{s/2} \left[n\overline{F}(s-x) + \sum_{i=0}^{k} A_i^{(n)} g_i(s-x) \right] f(x)dx \\ &+ (n+1)\hat{R}_0(s) + V(n+1)n^{k+2}(1+\varepsilon)^n \hat{R}_1(s) \\ =& (n+1)\overline{F}(s) + \frac{n+1}{n} \sum_{i=0}^{k} \left(na_{-1,i} + A_i^{(n)} + \sum_{j=0}^{i-1} A_j^{(n)} a_{j,i} \right) g_i(s) \\ &+ 2(n+1)(1+\varepsilon)^n \hat{R}_3(s) + (n+1)Vn^{k+2}(1+\varepsilon)^n \hat{R}_1(s) + \frac{n+1}{n} \left(n + \sum_{i=0}^{k} A_i^{(2)} \right) \hat{R}_2(s). \end{aligned}$$

Since

$$\left| \frac{n+1}{n} \sum_{i=0}^{k} A_i^{(n)} \right| \leq (n+1)n^{k+1}(V-1)$$

we get that the error term is less then ($n \geq 2$, $k \geq 0$)

$$V(1+\epsilon)^n(n+1)(n^{k+2}+n^{k+1}+2)K(s) \leq V(1+\epsilon)^n(n+1)^{k+3}K(s).$$

At last note that

$$\begin{aligned}
&\frac{n+1}{n}\left(na_{-1,i}+A_i^{(n)}+\sum_{j=0}^{i-1}A_j^{(n)}a_{j,i}\right)\\
=&\frac{n+1}{n}\left(A_i^{(n)}+na_{-1,1}+\sum_{j=0}^{i-1}\sum_{l=1}^{j+1}\frac{1}{l!}(n)_{l+1}\sum_{-1=l_0<\cdots<l_l=j<l_{l+1}=i}\prod_{m=0}^{l}a_{l_m,l_{m+1}}\right)\\
=&\frac{n+1}{n}\left(A_i^{(n)}+\sum_{j=1}^{i+1}\frac{1}{(j-1)!}(n)_j\sum_{-1=j_0<\cdots<j_j=i}\prod_{l=0}^{j-1}a_{l_j,l_{j+1}}\right)\\
=&\sum_{j=1}^{i+1}\sum_{-1=j_0<\cdots<j_j=i}\frac{n+1}{n}\left(\frac{1}{(j-1)!}(n)_j\prod_{l=0}^{j-1}a_{l_j,l_{j+1}}+\frac{1}{j!}(n)_{j+1}\prod_{l=0}^{j-1}a_{j_l,j_{l+1}}\right)\\
=&\sum_{j=1}^{i+1}\sum_{-1=j_0<\cdots<j_j=i}\frac{n+1}{n}\frac{n(n)_j}{j!}\prod_{l=0}^{j-1}a_{j_l,j_{l+1}}\\
=&\sum_{j=1}^{i+1}\sum_{-1=j_0<\cdots<j_j=i}\frac{1}{j!}(n+1)_{j+1}\prod_{l=0}^{j-1}a_{j_l,j_{l+1}}=A_i^{(n+1)}.
\end{aligned}$$

□

For the choice $g_i(s)=\overline{F}^{(i)}(s)$ it becomes clear that one has to evaluate $T_{F,1/2}\overline{F}^{(i)}(s)$ and $T_{F,1/2}K(s)$, so that we end up in the same situation as in [7, 11].

4 Asymptotics with a shifted argument

We have seen that for higher-order asymptotic expansions, in principle one needs the derivatives of $\overline{F}(s)$ or of a related function. In this section we want to give an asymptotic expansion which only consists of $\overline{F}(s)$ evaluated at different arguments, i.e

$$\mathbb{P}(S_N>s)\sim x_1\,\mathbb{E}[N]\;\overline{F}(s-k_1)+\cdots+x_m\,\mathbb{E}[N]\;\overline{F}(s-k_m)=:a_m^S(s).$$

for constants $x_1,\ldots,x_m$ and $k_1,\ldots,k_m$. This has the advantage that we do not need to evaluate the derivatives of $\overline{F}(s)$. Further, under appropriate conditions the approximation $a_m^S(s)$ can be of the same asymptotic order as $a_{2m}(s)$ as defined in (2). To that end, assume that

$$\mathbb{P}(S_N>s)=\mathbb{E}[N]\,\overline{F}(s)+\sum_{i=1}^{m-1}\frac{(-1)^ic_i\overline{F}^{(i)}(s)}{i!}+o\left(\overline{F}^{(m-1)}(s)\right),$$

where according to (2), $c_i=\mathbb{E}\left[N(X_1+\cdots+X_{N-1})^i\right]$. For any k, a Taylor expansion of $\overline{F}(s-k)$ at $s_0=s$ yields

$$\mathbb{P}(S_N>s)=\;\mathbb{E}[N]\;\overline{F}(s-k)+\sum_{i=1}^{m-1}\frac{(-1)^i(c_i-\;\mathbb{E}[N]\;k^i)\overline{F}^{(i)}(s)}{i!}+o\left(\overline{F}^{(m-1)}(s)\right).$$

So when we choose m different values of k_i $(i = 1, \dots, m)$, multiply the corresponding equation by x_i and add them all, one obtains $a_m^S(s)$ as an approximation of $\mathbb{P}(S_N > s)$ with an asymptotic decay of order $o\left(\overline{F}^{(m-1)}(s)\right)$, if x_i $(i = 1, \dots, m)$ are chosen as the solution of the equations

$$\sum_{j=1}^{m} x_j = 1 \quad \text{and} \quad \sum_{j=1}^{m} (c_i - \mathbb{E}[N]\, k_j^i)\, x_j = 0 \quad (i = 1, \dots, m),$$

or equivalently

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ k_1 & k_2 & \cdots & k_m \\ \vdots & \vdots & \ddots & \vdots \\ k_1^{m-1} & k_2^{m-1} & \cdots & k_m^{m-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 1 \\ c_1/\mathbb{E}[N] \\ \vdots \\ c_{m-1}/\mathbb{E}[N] \end{pmatrix}.$$

Since the above matrix is of Vandermonde type, it is regular as long as $k_i \neq k_j$ for $i \neq j$. Now we still have the freedom to choose the values of k_j. If

$$\mathbb{P}(S_N > s) = \mathbb{E}[N]\, \overline{F}(s) + \sum_{i=1}^{2m-1} \frac{(-1)^i c_i \overline{F}^{(i)}(s)}{i!} + o\left(\overline{F}^{(2m-1)}(s)\right),$$

we can try to choose k_j $(j = 1, \dots, m)$ such that

$$\sum_{j=1}^{m} (c_{m+i} - \mathbb{E}[N]\, k_j^{m+i})\, x_j = 0 \quad \text{for} \quad i = 1, \dots, m.$$

Example 4.1. *For $m = 1$ this results in $k_1 = c_1/\mathbb{E}[N] = \mathbb{E}[X]\, \mathbb{E}\left[N^2 - N\right]/\mathbb{E}[N]$ and subsequently*

$$\mathbb{P}(S_N > s) = \mathbb{E}[N]\; \overline{F}\left(s - \mathbb{E}[X]\, \left(\frac{E[N^2]}{E[N]} - 1\right)\right) + o(f(s)).$$

If furthermore N is Poisson(λ)-distributed, then this results in the simple formula

$$\mathbb{P}(S_N > s) = \lambda\, \overline{F}\big(s - \lambda\, \mathbb{E}[X]\big) + o(f(s)),$$

which seems to have been used in actuarial practice; so the above reasoning provides a formal justification for its use.

Remark 4.1. *If $\overline{F}(s)$ is not sufficiently differentiable, then there might at least exist a function $H(s)$ with*

$$\overline{F}(s) = H(s) + o\left(H^{(2m-1)}(s)\right) \quad \text{and} \quad \mathbb{P}(S_N > s) = \sum_{i=0}^{2m-1} \frac{(-1)^i c_i H^{(i)}(s)}{i!} + o\left(H^{(2m-1)}(s)\right),$$

in which case the above procedure still establishes an asymptotic expansion $a_m^S(s)$ of $\mathbb{P}(S_N > s)$ only involving the tails $\overline{F}(s - k_i)$. If the c_i do not depend on H (but only on F), then we only need the existence of H (cf. Corollary 3.6).

5 Approximations of geometric sums

In the numerical illustrations in Section 7, we will see that even higher-order expansions can lead to poor estimates for $\overline{G}(s)$ when s is not sufficiently large. In Thorin & Wikstad [29, 28] and Asmussen & Binswanger [3], approximations of $\overline{G}(s)$ were given that work reasonably for small s but do not have the right asymptotic behavior, so that it is difficult to judge when to switch from this approximation to the above asymptotic expansion for increasing s. In this section we give an example how one can use the decomposition (8) to combine approximations for small s and asymptotic expansions. We will concentrate on geometric compounds, i.e.

$$\overline{G}(s) = (1-\rho)\sum_{n=0}^{\infty} \rho^n \overline{F}^{*n}(s).$$

for a $\rho > 0$. For an approximation for small s we will use an asymptotic for $\overline{G}(s)$ for $\rho \to 1$ and fixed s which was recently developed in Blanchet and Glynn [13]. In a ruin-theoretic setting (where $\overline{G}(s)$ represents the ruin probability, cf. Section 6) this would correspond to low safety loading. The crucial idea in [13] is to approximate $\overline{G}(s)$ by

$$\mathbb{P}\left(\sum_{n=0}^{N} X_i I_{\{X_i \le s/(1-\rho)\}} > s\right), \tag{12}$$

and to use a uniform renewal theorem (developed in [13]) to get

$$\mathbb{P}\left(\sum_{n=0}^{N} X_i I_{\{X_i \le s/(1-\rho)\}} > s\right) \sim \frac{1-\rho}{\rho\hat{\theta}_s^{1-\rho}\frac{d\phi_s^{1-\rho}(\theta)}{d\theta}\big|_{\theta=\hat{\theta}_s^{1-\rho}}} e^{-s\hat{\theta}_s^{1-\rho}} =: a^{BG}(s) \quad \text{as } \rho \to 1,$$

where for $k > 0$

$$\phi_s^k(\theta) := \int_0^{s/k} e^{\theta x} dF(x) + \overline{F}(s/k),$$

and $\hat{\theta}_s^k$ is the solution of $\phi_s^k(\theta) = \rho^{-1}$.

Remark 5.1. *For fixed s and ρ the approximation*

$$\mathbb{P}\left(\sum_{n=0}^{N} X_i I_{\{X_i \le s/k\}} > s\right) \approx \frac{1-\rho}{\rho\hat{\theta}_s^{1-\rho}\frac{d\phi_s^{1-\rho}(\theta)}{d\theta}\big|_{\theta=\hat{\theta}_s^{1-\rho}}} e^{-s\hat{\theta}_s^{1-\rho}}$$

is exactly the Cramér-Lundberg approximation (see e.g. [25]) for a compound geometric distribution with claim sizes $Y_i = X_i I_{\{X_i \le s/k\}}$. This approximation is known to be reasonable in the light tailed case.

A natural question is the quality of the approximation when s is large.

Lemma 5.1. *Assume that $\overline{F}(s) \in SR_{-\alpha,1}$, $\alpha > 0$ and $-\log(\overline{F}(s))$ is eventually concave then for any $k > 0$*

$$\frac{1-\rho}{\rho\hat{\theta}_s^{k}\frac{d\phi_s^{k}(\theta)}{d\theta}\big|_{\theta=\hat{\theta}_s^{k}}} e^{-s\hat{\theta}_s^{k}} \sim \frac{\alpha^k \overline{F}(s/k)^k}{(\rho^{-1}-1)^k(-\log(\overline{F}(s/k)))^{k+1}}.$$

Proof. Cf. Appendix. □

Hence for large s the above approximation developed in [13] may not be satisfactory. Motivated by Lemma 5.1 and (8) we want to approximate, now for a $k \geq 1$,

$$\mathbb{P}(S_N > s, M_N \leq s/k) \approx \frac{1-\rho}{\rho \hat{\theta}_s^k \frac{d\phi_s^k(\theta)}{d\theta}\big|_{\theta=\hat{\theta}_s^k}} \, e^{-s\hat{\theta}_s^k}.$$

Note that

$$\left\{ \sum_{i=0}^{n} X_i I_{\{X_i \leq s/k\}} > s \right\} \setminus \{S_n > s, M_n \leq s/k\} \subset \left\{ S_{(n-1)} > s, M_n > s/k \right\}.$$

If we choose $k = 1$ then clearly

$$\mathbb{P}(S_N > s) = \mathbb{P}(M_N > s) + \mathbb{P}\left(S_N > s, M_N \leq s\right).$$

and by the above we arrive at the asymptotic

$$a_1^G(s) := \mathbb{P}(M_N > s) + \frac{1-\rho}{\rho \hat{\theta}_s^1 \frac{d\phi_s^1(\theta)}{d\theta}\big|_{\theta=\hat{\theta}_s^1}} \, e^{-s\hat{\theta}_s^1} = \frac{\rho \overline{F}(s)}{1-\rho F(s)} + \frac{1-\rho}{\rho \hat{\theta}_s^1 \frac{d\phi_s^1(\theta)}{d\theta}\big|_{\theta=\hat{\theta}_s^1}} \, e^{-s\hat{\theta}_s^1}.$$

Further we get under the conditions of Lemma 5.1 that

$$\overline{G}(s) = a_1^G(s) + O(\overline{F}(s)/\log^2 \overline{F}(s)).$$

If we choose $k = 2$ we have, similarly to (8),

$$\mathbb{P}(S_n > s) = \mathbb{P}(S_n > s, M_n > s/2) + \mathbb{P}(S_n > s, M_n \leq s/2).$$

As can be seen from Lemma 3.1 and 3.3, for every $\epsilon > 0$

$$|\mathbb{P}(S_n > s) - \mathbb{P}(S_n > s, M_n > s/2)| = O\left(\overline{F}(s)^{2-\epsilon}\right).$$

It follows that a higher order asymptotic up to order $O\left(\overline{F}(s)^{2-\epsilon}\right)$ for $\mathbb{P}(S_n > s)$ is a higher order asymptotic of $\mathbb{P}(S_n > s, M_n > s/2)$, too. We have chosen to use $a_1^S(s)$ to approximate $\mathbb{P}(S_n > s, M_n > s/2)$, hence we get

$$a_2^G(s) := a_1^S(s) + \frac{(1-\rho)}{\rho \hat{\theta}_s^2 (\phi_s^2)'(\hat{\theta}_s^2)} e^{-s\hat{\theta}_s^2}$$

as an approximation to $\mathbb{P}(S_N > s)$. Note that this approximation is asymptotically of the same order as $a_1^S(s)$ itself.

Remark 5.2. *One should note that for the evaluation of $a_k^G(s)$ one has to find θ_s^k and evaluate the derivative of the moment generating function of F, so the evaluation of $a_k^G(s)$ is not as straight-forward as the evaluation of the asymptotic expansions.*

6 Expansions for ruin probabilities

The above approximations for geometric compound distributions immediately yield corresponding approximations for ruin probabilities $\psi(s)$ in the collective renewal risk model with i.i.d. claim sizes X_i, i.i.d. interclaim times T_i, initial capital s and constant premium intensity $c > \mathbb{E}[X]/\mathbb{E}[T]$, as the Pollaczek-Khintchine formula gives for every $s \geq 0$

$$\psi(s) = (1-\rho)\sum_{n=1}^{\infty} \rho^n \overline{H_0^{*n}}(s),$$

where $H_0(s) = H^+(s)/H^+(\infty)$ and H^+ is the defective distribution of the ladder height of the random walk $\sum_{i=1}^{n}(X_i - cT_i)$ with $\rho = H^+(\infty) < 1$ (see e.g. Rolski et al. [26, Th.6.5.1]). If the interclaim times T_i are exponential with parameter λ (i.e. the compound Poisson risk model), then the ladder height distribution H^+ can be expressed through the integrated tail distribution function

$$F_I(s) = \frac{1}{\mathbb{E}[X]}\int_0^s \overline{F}(x)dx$$

resulting in

$$\psi(s) = \left(1 - \frac{\lambda\mathbb{E}[X]}{c}\right)\sum_{n=1}^{\infty}\left(\frac{\lambda\mathbb{E}[X]}{c}\right)^n \overline{F_I^{*n}}(s).$$

Hence the results of the previous sections can be applied whenever F_I admits expansions for convolutions. This approach was exploited by Baltrūnas [4, 5] to obtain second-order asymptotic approximations for the renewal risk model.
Using the above results it is now straight-forward to write down higher-order expansions. For instance, consider a compound Poisson model, where the claim size distribution $F(s)$ has a density $f(s)$ such that its negative derivative $-f'(s)$ is regularly varying with index $-\alpha-2, \alpha > 3$, (so that the first three moments $\mu_k := \mathbb{E}\left[X^k\right]$ $(k = 1, 2, 3)$ exist), then the expansion for the infinite horizon ruin probability $\psi(s)$ of third-order reads

$$\psi(s) = \frac{\rho}{1-\rho}\overline{F}_I(s) + \frac{\rho^2}{(1-\rho)^2}\frac{\mu_2}{2\mu^2}\overline{F}(s) + \left(\frac{\rho^2}{(1-\rho)^2}\frac{\mu_3}{3\mu^2} + \frac{\rho^3}{(1-\rho)^3}\frac{\mu_2^2}{4\mu^3}\right) f(s) + o(f(s)). \tag{13}$$

7 Numerical computation of compound distributions

7.1 Upper and lower bounds

For the Panjer method we first discretize the distribution $F(s)$ as follows: we start with a step size $\Delta > 0$ and define the point probabilities $f_i(k\Delta)$ for F_i by

$$f_1(k\Delta) = \int_{k\Delta}^{(k+1)\Delta} f(x)dx, k = 0, 1, 2, ...$$

$$f_2(k\Delta) = \int_{(k-1)\Delta}^{k\Delta} f(x)dx, k = 1, 2, ...$$

These are upper and lower bounds for $F(s)$ in the sense that for all $s \geq 0$

$$\overline{F}_1(s) \leq \overline{F}(s) \leq \overline{F}_2(s).$$

The corresponding approximations for $G(s)$ are upper and lower bounds:

$$G_1(s) \leq G(s) \leq G_2(s),$$

where for $i = 1, 2$

$$G_i(s) = \sum_{n=0}^{\infty} \mathbb{P}\{N = n\} F_i^{*n}(s), s \geq 0.$$

For the computation of the point probabilities $g_i(k\Delta)$ of $G_i(s)$ we use Panjer's recursion:

$$\begin{aligned} g_1(0) &= \sum_{n=0}^{\infty} \mathbb{P}\{N = n\} f_1(0)^n \\ g_1((k+1)\Delta) &= \sum_{j=1}^{k+1} \left(a + b\frac{j}{k+1}\right) \frac{f_1(j\Delta) g_1((k+1-j)\Delta)}{(1 - af_1(0))}, k = 0, 1, 2, ... \end{aligned}$$

and

$$\begin{aligned} g_2(0) &= \mathbb{P}\{N = 0\} \\ g_2((k+1)\Delta) &= \sum_{j=1}^{k+1} \left(a + b\frac{j}{k+1}\right) f_2(k\Delta) g_2((k+1-j)\Delta), k = 0, 1, 2, ... \end{aligned}$$

We need two different recursions because of $f_1(0) > 0$. It turns out that the method of upper and lower bounds is – apart from its programming effort – quite efficient.

Remark 7.1. *For fixed stepsize Δ we get by the subexponentiality of F that*

$$\lim_{s\to\infty} \frac{\overline{F}_1(s)}{\overline{F}(s)} = \lim_{s\to\infty} \frac{\overline{F}_2(s)}{\overline{F}(s)} = 1.$$

and hence

$$\lim_{s\to\infty} \frac{\overline{G}_1(s)}{\overline{G}_2(s)} = 1.$$

7.2 Higher-Order Approximations

In this section we want to give numerical illustrations of the performance of the discussed estimators. We will reproduce the result of Abate et al. [1] that the first-order asymptotic formula is of little use in the range of finite s that is of interest in real world applications. It turns out that higher-order results can improve the approximation considerably (see Example 7.1), but it is also demonstrated that in other cases the improvement may be marginal (see Example 7.3) or that higher-order asymptotic expansions even do not improve the estimate at all (cf. Example 7.2). To assess the performance of the approximations we compare it to the upper bounds $a_u(s)$ and lower bounds $a_l(s)$ based on Panjer recursion described in Section 7.1. The curves in the following plots show the relative

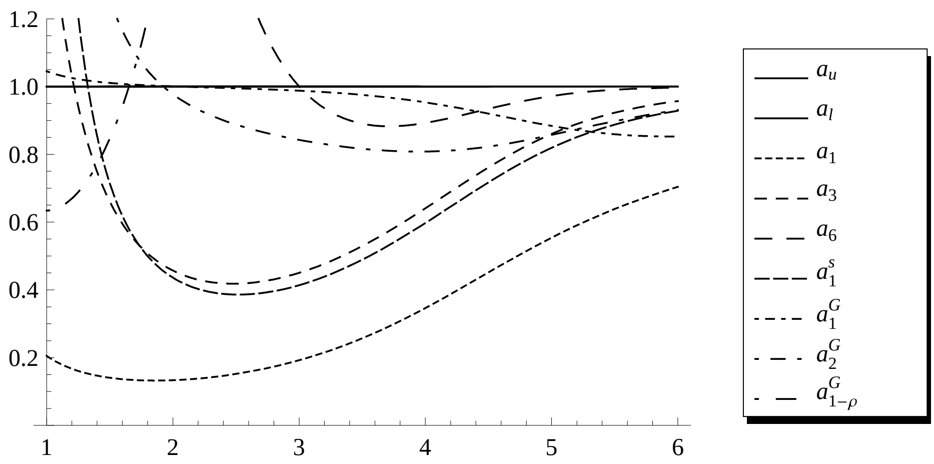

Figure 1: Relative probabilities for geometric compound distributions with $\rho = 0.7$ and lognormal claimsize with $\mu = 0$ and $\sigma = 1$

probabilities $a_i(s)/a(s)$, where $a(s) := (a_u(s) + a_l(s))/2$ is the average of the upper and the lower bound, and the x-axis is in logarithmic units $(-\log_{10}(a(s)))$. Hence the x-values in the range of 1 and 6 correspond to tail (ruin, respectively) probabilities between 10% and 0.0001%, which should include all values of s that are of practical interest.

Example 7.1. *Consider first for $F(s)$ a lognormal distribution with $\mu = 0$ and $\sigma = 1$ and N has a geometric distribution with $\rho = 0.7$. Figure 1 gives the asymptotic approximations $a_i(s)$ discussed in the previous sections for $\mathbb{P}(X_1 + \cdots + X_N > s)$, plotted as $a_i(s)/a(s)$. Values on the x-axis between 2 to 4 correspond to tail probabilities between 1% and 0.01%. One observes that the first-order approximation $a_1(s)$ dramatically underestimates the actual tail probability. The third-order approximation $a_3(s)$ and the shifted first-order approximation $a_1^s(s)$ improve the approximation considerably but are still too crude for practical purposes. The relative error for the sixth-order approximation $a_6(s)$ may already be useful in this case. The best approximation is given by the geometric estimator a_1^G which has a relative error below 5% for tail probabilities larger than 0.01%, but gets worse for smaller tail probabilities. The estimator $a_2^G(s)$ provides approximation with less than 20% relative error. As expected by its construction, the estimator of [13] is not helpful at all for large s.*

Example 7.2. *Let now $F(s)$ be a Weibull distribution function with parameter $\beta = 1/4$, (i.e. $\overline{F}(s) = e^{-s^{1/4}}$) and N a Poisson distribution with $\lambda = 2$. In Figure 2 we see that the first-order asymptotic approximation $a_1(s)$ is quite reasonable for all considered values. The second-order asymptotic $a_2(s)$ and the first-order shifted approximation $a_1^s(s)$ provide excellent approximations for the real tail probabilities in the range below 0.1% and are still reasonable in the range below 1%. On the other hand the sixth-order approximation $a_6(s)$ is only useful for tail probabilities below 0.01%, so that in this case the use of orders higher than 2 in the expansion is actually counterproductive (one reason is that for every fixed s, $\lim_{n\to\infty} a_n(s) = \infty$).*

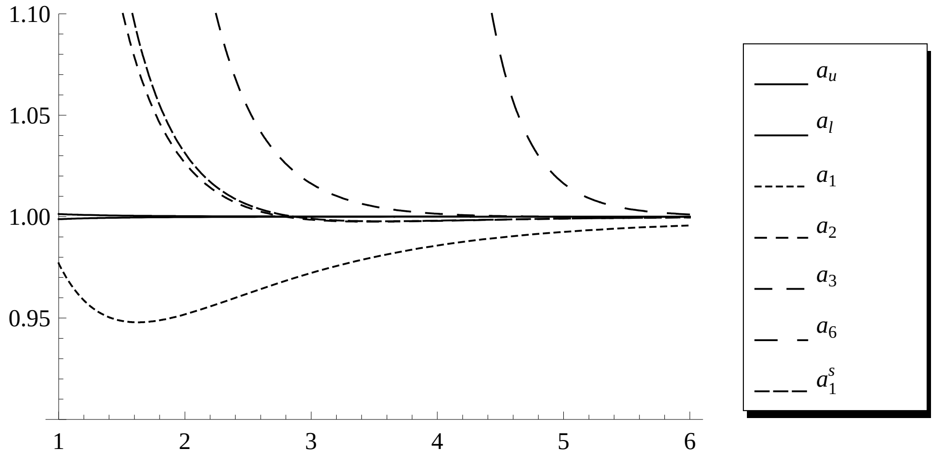

Figure 2: Relative probabilities for Poisson compound distributions with $\lambda = 2$ with Weibull claim size $\beta = 0.25$

The final two examples consider ruin probabilities in the compound Poisson model as described in Section 6.

Example 7.3. *In Figure 3 the claims are Pareto with parameter* $\alpha = 3.2$*, the Poisson parameter is* $\lambda = 2$*, and the premium rate is* $c = 3.5$*. Here the behaviour of the asymptotic is similar to the one in Figure 1. One should note that for ruin probabilities greater than* 1% *none of the asymptotic approximations gives satisfactory estimates.*

Example 7.4. *In Figure 4 the claims are Pareto with parameter* $\alpha = 6.5$*, the Poisson parameter is* $\lambda = 1$*, and the premium rate is* $c = 1.5$ *(which corresponds to* $\rho = 0.79$ *in the Pollaczek-Khintchine formula). The main aim of this example is to show how valuable the asymptotic results can be. The lines corresponding to the asymptotic approximations are (except for ruin probabilities larger than* 1%*) nearly identical with the* x*-axis and hence of no practical value. On the other hand one can see that the approximation* $a_1^G(s)$ *can not be distinguished from the lines of the upper and lower bound and is extremely accurate. The approximation* $a_2^G(s)$ *is similarly accurate except for small ruin probabilities (where it is quite inaccurate). Figure 5 depicts the logarithm of the approximations with respect to* s*. One can see that the behaviour of* $\log_{10}(\psi(s))$ *changes at* $s \approx 60$ *and the asymptotic expansions only work after that point. The picture also includes a Monte Carlo estimate of* $\mathbb{P}(S_N > s, M_n > s/2)$*, which illustrates that for moderate values of* s *the asymptotic approximations are roughly able to capture the part* $\mathbb{P}(S_N > s, M_N > s/2)$ *of the decomposition, but not the whole quantity* $\psi(s)$*.*

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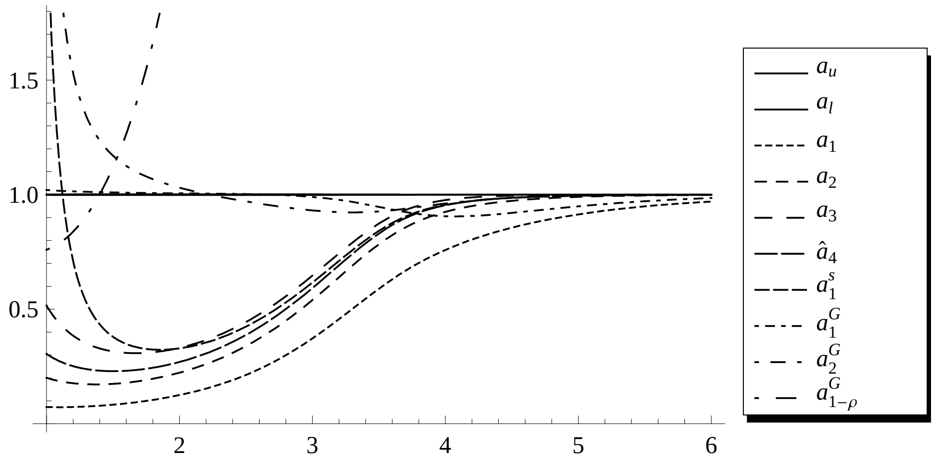

Figure 3: Relative probabilities for ruin probabilities with $\lambda = 2$, $c = 3.5$ and Pareto claim sizes with $\alpha = 3.2$.

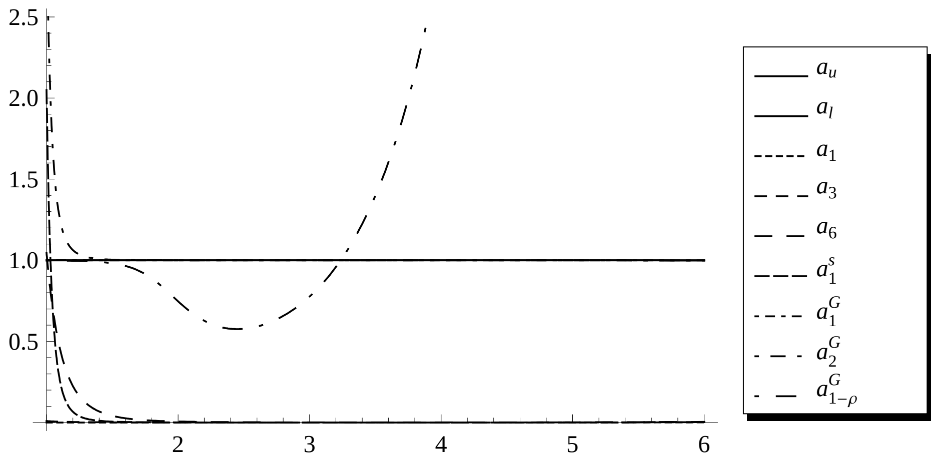

Figure 4: Relative probabilities for ruin probabilities with $\lambda = 1$, $c = 1.5$ and Pareto claimsizes with $\alpha = 6.5$.

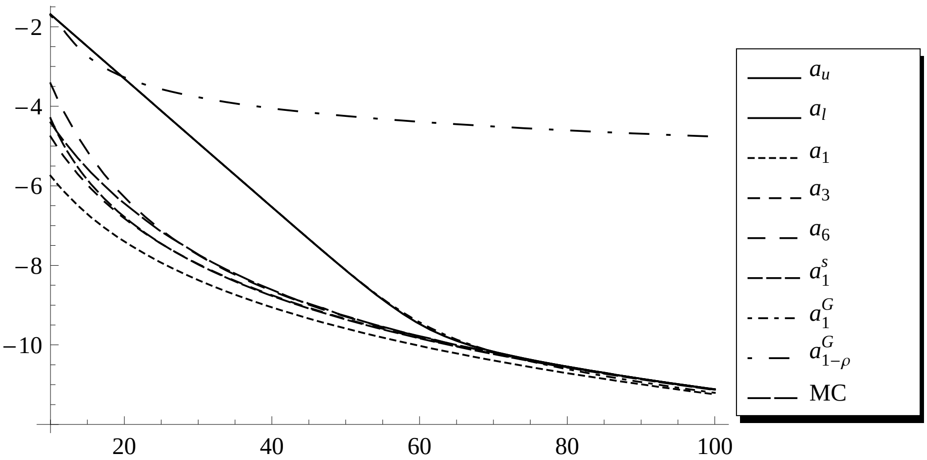

Figure 5: Absolute log probabilities for ruin probabilities with $\lambda = 1$, $c = 1.5$ and Pareto claim sizes with $\alpha = 6.5$, with a Monte Carlo estimate of $\mathbb{P}(S_N > s, M_N > s/2)$ added.

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A Proof of Lemma 3.2

We now consider the asymptotic behavior of $\mathbb{P}(S_n > s, M_n \leq s/2)$ and start with an auxiliary result. Note that for the proof of Lemma 3.2 we will condition on the case that the two largest elements of $X_1, \ldots, X_n$ are X_{n-1} and X_n.

Lemma A.1. *Let $X_1, \ldots, X_n$ be i.i.d. random variables with common distribution function $F(s)$. Assume that $F(s) \in SR_{-\alpha,2}$ with $\alpha > 2$ then*

$$\lim_{t\to\infty}\lim_{s\to\infty} \frac{\mathbb{P}(S_n > s, S_{n-2} \leq t, M_n \leq s/2)}{f(s/2)^2} = \frac{1}{2}\mathbb{E}\left[S_{n-2}^2\right].$$

Proof. At first note that for $x \geq 0$ and $\xi_s \in (s/2 - x, s/2)$ by Taylor expansion

$$\lim_{s\to\infty} \frac{\mathbb{P}(s/2 \geq X_1 > s/2 - x)}{f(s/2)} = \lim_{s\to\infty} \frac{xf(s/2) - \frac{x^2}{2}f'(\xi_s)}{f(s/2)} = x.$$

Define the measure

$$\nu_s\left(Y_1 \leq x_1, Y_2 \leq x_2, Y_3 \leq x_3\right) = \frac{\mathbb{P}(s/2 \geq X_n > s/2 - x_1, s/2 \geq X_{n-1} > s/2 - x_2, S_{n-2} \leq x_3)}{f(s/2)^2}.$$

ν_s converges vaguely to the measure

$$\nu\left(Y_1 \leq x_1, Y_2 \leq x_2, Y_3 \leq x_3\right) = x_1 x_2 F^{*(n-2)}(x_3).$$

We have that

$$\lim_{s\to\infty} \frac{\mathbb{P}(S_n > s, S_{n-2} \le t, M_n \le s/2)}{f(s/2)^2} = \lim_{s\to\infty} \nu_s \left(Y_1 + Y_2 - Y_3 < 0, Y_3 \le t\right) = \nu \left(Y_1 + Y_2 - Y_3 < 0, Y_3 \le t\right).$$

Finally note that

$$\nu \left(Y_1 + Y_2 - Y_3 < 0, Y_3 \le t\right) = \int_0^t \int_0^{x_3} \int_0^{x_3 - x_2} \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}F^{*(n-2)}(x_3) = \int_0^t \frac{x^2}{2} \, \mathrm{d}F^{*(n-2)}(x).$$

□

Proof of Lemma 3.2: We have to show that

$$\lim_{t\to\infty} \lim_{s\to\infty} \frac{\mathbb{P}(S_n > s, M_n \le s/2, X_n > M_{n-1}, X_{n-1} > M_{n-2}, S_{n-2} > t)}{f(s/2)^2} = 0$$

At first notice that for an arbitrary $m > 0$

$$\mathbb{P}(X_n > s/m, X_{n-1} > s/m, S_{n-2} > s/m) \sim (n-2)m^{3\alpha}\overline{F}(s)^3 = o(f(s/2)^2).$$

Hence we will assume that $S_{n-2} \le s/m$ and $X_i > ((m-2)/2m)s$ for $i = n-1, n$. As in the proof of Lemma A.1 we can show that there exists a constant c_1 such that for all $k \le s/(m+1)$

$$\mathbb{P}(s/2 \ge X_1 > s/2 - k) \le c_1 k f(s).$$

We have that

$$\begin{aligned}
&\mathbb{P}(S_n > s, M_n \le s/2, S_{n-2} < s/m) \\
&\le \sum_{k=\lfloor t \rfloor}^{\lfloor \frac{s}{m} \rfloor} \mathbb{P}(s/2 \ge X_1 > s/2 - (k+1))^2 \mathbb{P}(k+1 \ge S_{n-2} > k) \\
&\le c_1^2 f(s)^2 \sum_{k=\lfloor t \rfloor}^{\lfloor \frac{s}{m} \rfloor} (k+1)^2 \mathbb{P}(k+1 \ge S_{n-2} > k) \\
&\le c_1^2 f(s)^2 \int_{\lfloor t \rfloor}^{\infty} (x+2)^2 dF^{*(n-2)}.
\end{aligned}$$

Since $\alpha > 2$ the integral in the last equation tends to 0 as $t \to \infty$. □

B Proof of Lemma 3.3

If F is long-tailed with $\sup_{s\geq 0}\frac{\overline{F}(s)}{\overline{F}(s+1)} \leq K_1$ then for $n > s$ we get

$$\begin{aligned}\int_{(0,s]} \frac{1}{\overline{F}(x)}\,\mathrm{d}F(x) &\leq \sum_{i=0}^{n-1} \frac{F\left((i+1)\frac{s}{n}\right) - F\left(i\frac{s}{n}\right)}{\overline{F}\left((i+1)\frac{s}{n}\right)} = -\sum_{i=0}^{n-1} \frac{\overline{F}\left(i\frac{s}{n}\right)}{\overline{F}\left(i\frac{s}{n}+\frac{s}{n}\right)}\left(\frac{\overline{F}\left((i+1)\frac{s}{n}\right)}{\overline{F}\left(i\frac{s}{n}\right)} - 1\right)\\ &\leq -K_1 \sum_{i=0}^{n-1} \log\left(\frac{\overline{F}\left((i+1)\frac{s}{n}\right)}{\overline{F}\left(i\frac{s}{n}\right)}\right) = -K_1\left(\log\left(\overline{F}(s)\right) - \log\left(\overline{F}(0)\right)\right).\end{aligned}$$

From assumption (A3) of Lemma 3.3 it follows that

$$\lim_{s\to\infty} \frac{-\log(\overline{F}(s))}{s} = 0, \tag{14}$$

since from

$$\limsup_{s\to\infty} \frac{-\log(\overline{F}(s))}{s} > 0$$

the existence of a sequence s_n with $-\log(\overline{F}(s))/s > \epsilon$ and $\lim_{n\to\infty} s_n = \infty$ would follow, and hence

$$\lim_{n\to\infty} R(s_n)\overline{F}(cs_n/R(s_n))^{\delta} \geq R(s_n)\overline{F}(c/\epsilon)^{\delta} = \infty.$$

Proof of Lemma 3.3: The proof is a variation of the proof of Theorem 1 in [22, p.71]. For $t > 0$ let $M(t) = \int_{-\infty}^{s} \exp(tx)dF(x)$. We set

$$t = -s^{-1}(1-\varepsilon)\log(\overline{F}(s)).$$

Then, according to (14), $t \to 0$ when $s \to \infty$, and hence for sufficiently large s the function $h_0(x)$ from assumption (A3) is concave on $(1/t, \infty)$, and for $x > 1/t$

$$h_0(x)(1-\varepsilon/2) \leq -\log(\overline{F}(x)) \leq \frac{h_0(x)}{(1-\varepsilon/2)}.$$

The function

$$g_0(x) := tx - (1-\varepsilon/2)h_0(x)$$

is convex on the interval $[1/t, s]$ and hence assumes its maximum in either $1/t$ or s. With $g(x) = tx + \log(\overline{F}(x))$ we have

$$M(t) = \int_{-\infty}^{s} \frac{g(x)}{\overline{F}(x)} dF(x), \tag{15}$$

$$g(x) \leq g_0(x). \tag{16}$$

Furthermore,

$$\begin{aligned} g_0(1/t) &= 1 - (1-\varepsilon/2)h_0(1/t) \leq 1 + (1-\varepsilon/2)^2 \log(\overline{F}(1/t)),\\ g_0(s) &= -(1-\varepsilon)\log(\overline{F}(s)) - (1-\varepsilon/2)h_0(s)\\ &\leq -(1-\varepsilon)\log(\overline{F}(s)) + (1-\varepsilon/2)^2\log(\overline{F}(s)) = \varepsilon^2/4\log(\overline{F}(s)).\end{aligned}$$

We now split the integral $M(t)$ as follows:

$$\begin{aligned} M(t) &= \int_{-\infty}^{1/t} \exp(tx)dF(x) + \int_{1/t}^{s} \exp(tx)dF(x) \\ &= M_1(t) + M_2(t). \end{aligned}$$

One obtains

$$\begin{aligned} M_1(t) &\leq 1 + t^{\gamma} \max_{0\leq v\leq 1} \frac{e^v - 1}{v^{\gamma}} \mu(\gamma), \\ M_2(t) &\leq \int_{1/t}^{s} \frac{\exp(g_0(x))}{\overline{F}(x)} dF(x) \leq e^{\max(g_0(1/t), g_0(s))} \int_{1/t}^{s} \frac{1}{\overline{F}(x)} dF(x). \end{aligned}$$

Assumption (A3) implies that $M_2(t) \to 0$. Hence for sufficiently large s we have

$$M(t) \leq 1 + \varepsilon.$$

Finally, the Markov inequality implies the assertion:

$$\begin{aligned} \mathbb{P}(S_n > Ks, M_n \leq s) &\leq (1+\varepsilon)^n \exp(-Kts) \\ &= (1+\varepsilon)^n \overline{F}(s)^{K(1-\varepsilon)}. \end{aligned}$$

This inequality holds for sufficiently large s, say $s > M(\varepsilon, K)$, simultaneously for all $n \geq 2$.
□

Since the Markov inequality is rather crude, one cannot expect a sharp upper bound. The bound, however, holds for a large class of subexponential distributions, as shown in the following:

Lemma B.1. *In each of the following cases, conditions (A1)-(A3) hold:*

1. *$F(s)$ is a heavy-tailed Weibull distribution with density*

$$f(s) = bs^{b-1}\exp(-s^b), \quad s > 0,\ 0 < b < 1.$$

2. *$F(s)$ is lognormal with density*

$$f(s) = s^{-1}(2\pi\sigma^2)^{-1/2}\exp(-(\log(s)-\mu)^2/(2\sigma^2)), \quad s > 0, \mu \in \mathbb{R}, \sigma^2 > 0.$$

3. *$F(s)$ with tail $\overline{F}(s) = L(s)\overline{G}(s)$, where $\overline{G}(s)$ is the tail of a distribution function $G(s)$ that satisfies (A1)-(A3) and*

$$\lim_{s\to\infty} \frac{\log(L(s))}{\log(\overline{G}(s))} = 0. \tag{17}$$

Proof. It is clear that (A1) is true in all four cases.

1. The Weibull distribution has hazard function $R(s) = s^b$ which is concave for $b < 1$. Furthermore, $h(s) = -s/\log(\overline{F}(s)) = s^{1-b}$, and $\overline{F}(ch(s))^\delta = \exp(-\delta(ch(s))^b)$. On the other hand, $R(s) = s^b$ which is killed by the factor $\exp(-\delta(ch(s))^b)$.

2. Write $\phi(s)$ for the standard normal density and $\Phi(s)$ for its distribution function. Then for $s > 0$
$$\overline{F}(s) = 1 - \Phi\left(\frac{\log(s) - \mu}{\sigma}\right) \sim \frac{\sigma}{\log(s) - \mu}\phi\left(\frac{\log(s) - \mu}{\sigma}\right).$$
Furthermore,
$$R(s) = -\log(\overline{F}(s)) \sim \frac{1}{2\sigma^2}\log(s)^2 =: h_0(s),$$
with eventually concave function $h_0(s)$. Thus $h(s) = -s/\log(\overline{F}(s))$ grows faster than a positive power of s. Therefore, for any $c, \delta > 0$ the term $R(s)$ is killed by the factor $\overline{F}(ch(s))^\delta$ since for $\varepsilon > 0$ the term $\exp(-\log(s^\varepsilon)^2)$ decreases faster than any negative power of s.

3. (A1)-(A2) are clear. Note that from (17) it follows that for every $\epsilon > 0$ there exists a K_ϵ such that
$$L(s) \le K_\epsilon \overline{G}(s)^{-\epsilon}$$
To prove (A3) notice that for an $\epsilon > 0$ and large s
$$R(s)\overline{F}(cs/R(s))^\delta) \le -(1+\epsilon)K_{\delta/2}\log(\overline{G}(s))\overline{G}(c(1-\epsilon)/(-\log(\overline{G}(s)))^{\delta/2}$$
which, by assumption (A3) for $G(s)$, converges to 0.

□

C Proof of Lemma 3.4

Proof. As in [7] we start with a Taylor expansion of $\overline{F}(s)$.

$$\begin{aligned}\mathbb{P}(S_n > s, S_{(n-1)} \le s/2) =& n\int_0^{s/2} \overline{F}(s-x)\,\mathrm{d}F^{*(n-1)}(x)\\ =& n\sum_{j=0}^k \frac{(-1)^j\overline{F}^{(j)}(s)}{j!}\int_0^{s/2} x^j\,\mathrm{d}F^{*(n-1)}(x)\\ &+ n\int_0^{s/2}\frac{(-1)^{k+1}x^{k+1}\overline{F}^{(k+1)}(\xi_x^s)}{(k+1)!}\,\mathrm{d}F^{*(n-1)}(x),\end{aligned}$$

for a $\xi_x^s \in (s-x, s)$. Note that for $\zeta_x^s \in (0, x)$:

$$\begin{aligned}&\int_0^{s/2}\frac{(-1)^{k+1}x^{k+1}\overline{F}^{(k+1)}(\xi_x^s)}{(k+1)!}\,\mathrm{d}F^{*(n-1)}(x)\\ &\quad= \left(\frac{s}{2}\right)^{k+1}\overline{F}^{(k+1)}(s)\int_0^1 (-1)^{k+1}\frac{x^{k+1}\overline{F}^{(k+1)}(s(1-\zeta_x^s/2))}{(k+1)!\overline{F}^{(k+1)}(s)}\,\mathrm{d}F^{*(n-1)}(sx/2)\end{aligned}$$

At first we have to evaluate

$$\lim_{s\to\infty} \frac{x^{k+1}\overline{F}^{(k+1)}(s(1-\zeta_x^s/2))}{(k+1)!\overline{F}^{(k+1)}(s)}.$$

If $\zeta_x^\infty := \lim_{s\to\infty} \zeta_x^s$ exists, then by Karamata's theorem we get that

$$\lim_{s\to\infty} \frac{x^{k+1}\overline{F}^{(k+1)}(s(1-\zeta_x^s/2))}{(k+1)!\overline{F}^{(k+1)}(s)} = \frac{x^{k+1}(1-\zeta_x^\infty/2)^{-\alpha-k-1}}{(k+1)!}.$$

By the definition of ζ_x^s in the remainder term of a Taylor expansion we get that for every s,

$$\frac{x^{k+1}\overline{F}^{(k+1)}(s(1-\zeta_x^s/2))}{(k+1)!\overline{F}^{(k+1)}(s)} = \frac{\overline{F}(s(1-x/2))}{(-1)^{k+1}(s/2)^{k+1}\overline{F}^{(k+1)}(s)} - \sum_{j=0}^{k} \frac{x^j\overline{F}^{(j)}(s)}{(-1)^{k+1-j}(s/2)^{k+1-j}j!\overline{F}^{(k+1)}(s)},$$

and hence

$$\begin{aligned}\lim_{s\to\infty} \frac{x^{k+1}\overline{F}^{(k+1)}(s(1-\zeta_x^s/2))}{(k+1)!\overline{F}^{(k+1)}(s)} &= \lim_{s\to\infty} \frac{\overline{F}(s(1-x/2))}{(-1)^{k+1}(s/2)^{k+1}\overline{F}^{(k+1)}(s)} \\ &\quad - \lim_{s\to\infty} \sum_{j=0}^{k} \frac{x^j\overline{F}^{(j)}(s)}{(-1)^{k+1-j}(s/2)^{k+1-j}j!\overline{F}^{(k+1)}(s)} \\ &= \frac{2^{k+1}(1-x/2)^{-\alpha}\Gamma(\alpha)}{\Gamma(\alpha+k+1)} - \sum_{j=0}^{k} \frac{x^j 2^{k+1-j}\Gamma(\alpha+j)}{j!\Gamma(\alpha+k+1)} \\ &= \frac{2^{k+1}\Gamma(\alpha)}{\Gamma(\alpha+k+1)} \sum_{j=k+1}^{\infty} \left(\frac{x}{2}\right)^j \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)}.\end{aligned}$$

Note that the convergence is uniform for $x \in (0,1)$ and that ζ_x^∞ can be evaluated through

$$(1-x/2)^{-\alpha} = \sum_{j=0}^{k} \frac{\Gamma(\alpha+j)}{j!\Gamma(\alpha)} \left(\frac{x}{2}\right)^j + \frac{\Gamma(\alpha+k+1)(1-\zeta_x^\infty/2)^{-\alpha-k-1}}{(k+1)!\Gamma(\alpha)} \left(\frac{x}{2}\right)^{k+1}$$

Hence we have to evaluate the integral

$$I(s) := \frac{\overline{F}^{(k+1)}(s)\Gamma(\alpha)}{\Gamma(\alpha+k+1)} \int_0^1 (-1)^{k+1} \left(\frac{sx}{2}\right)^{k+1} \sum_{j=k+1}^{\infty} \left(\frac{x}{2}\right)^{j-(k+1)} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)} \, \mathrm{d}F^{*(n-1)}(sx/2).$$

Assume that $\alpha \neq k+1$. By partial integration and the Uniform Convergence Theorem we get

$$
\begin{aligned}
I(s) =& \frac{s}{2}\frac{\overline{F}^{(k+1)}(s)\Gamma(\alpha)}{\Gamma(\alpha+k+1)}\int_0^1 (-1)^{k+1}\left(\frac{sx}{2}\right)^k \overline{F}^{*(n-1)}(sx/2)\sum_{j=k+1}^{\infty} j\left(\frac{x}{2}\right)^{j-(k+1)}\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)}\ \mathrm{d}x \\
&-\left(\frac{s}{2}\right)^{k+1}\frac{(-1)^{k+1}\overline{F}^{(k+1)}(s)\Gamma(\alpha)\overline{F}^{*(n-1)}(s/2)}{\Gamma(\alpha+k+1)}\sum_{j=k+1}^{\infty}\left(\frac{1}{2}\right)^{j-(k+1)}\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)} \\
\sim& \left(\frac{s}{2}\right)^{k+1}\frac{(-1)^{k+1}\overline{F}^{(k+1)}(s)\Gamma(\alpha)\overline{F}^{*(n-1)}(s/2)}{\Gamma(\alpha+k+1)}\int_0^1 \sum_{j=k+1}^{\infty}\left(\frac{x}{2}\right)^{j-\alpha-1}\frac{j\Gamma(\alpha+j)}{2^{k-\alpha}\Gamma(\alpha)\Gamma(j+1)}\ \mathrm{d}x \\
&-(n-1)\overline{F}(s)\overline{F}(s/2)\sum_{j=k+1}^{\infty}\left(\frac{1}{2}\right)^{j}\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)} \\
\sim&(n-1)\overline{F}(s)\overline{F}(s/2)\sum_{j=k+1}^{\infty}\frac{j}{j-\alpha}\frac{\Gamma(\alpha+j)}{2^j\Gamma(\alpha)\Gamma(j+1)} \\
&-(n-1)\overline{F}(s)\overline{F}(s/2)\sum_{j=k+1}^{\infty}\frac{\Gamma(\alpha+j)}{2^j\Gamma(\alpha)\Gamma(j+1)} \\
&=(n-1)\overline{F}(s)\overline{F}(s/2)\sum_{j=k+1}^{\infty}\frac{\alpha}{j-\alpha}\frac{\Gamma(\alpha+j)}{2^j\Gamma(\alpha)\Gamma(j+1)}.
\end{aligned}
$$

For $j \le k$ we have

$$
\int_{s/2}^{\infty} x^j\ \mathrm{d}F^{*(n-1)}(x) = \int_{s/2}^{\infty} jx^{j-1}\overline{F}^{*(n-1)}(x)\ \mathrm{d}x + \left(\frac{s}{2}\right)^j \overline{F}^{*(n-1)}(s/2)
$$

and

$$
\lim_{s\to\infty}\frac{\int_{s/2}^{\infty} jx^{j-1}\overline{F}^{*(n-1)}(x)\ \mathrm{d}x}{(s/2)^j\overline{F}^{*(n-1)}(s/2)} = \frac{j}{\alpha-j}.
$$

Hence

$$
\begin{aligned}
-\sum_{j=0}^{k}\frac{(-1)^j\overline{F}^{(j)}(s)}{j!}\int_{s/2}^{\infty} x^j\ \mathrm{d}F^{*(n-1)}(x) \sim& \sum_{j=0}^{k}\frac{(-1)^j\overline{F}^{(j)}(s)(s/2)^j\overline{F}^{*(n-1)}(s/2)}{j!}\frac{j}{j-\alpha} \\
&-\sum_{j=0}^{k}\frac{(-1)^j\overline{F}^{(j)}(s)(s/2)^j\overline{F}^{*(n-1)}(s/2)}{j!} \\
&\sim (n-1)\overline{F}(s/2)\overline{F}(s)\sum_{j=0}^{k}\frac{(-1)^j\overline{F}^{(j)}(s)(s/2)^j}{\overline{F}(s)j!}\frac{\alpha}{j-\alpha} \\
&\sim (n-1)\overline{F}(s)\overline{F}(s/2)\sum_{j=0}^{k}\frac{\alpha}{j-\alpha}\frac{\Gamma(\alpha+j)}{2^j\Gamma(\alpha)\Gamma(j+1)}.
\end{aligned}
$$

Collecting all terms we get

$$\begin{aligned}\mathbb{P}(S_n > s, S_{(n-1)} \leq s/2) - a_k(s) \sim &(n-1)\overline{F}(s/2)\overline{F}(s) \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+j)}{2^j \Gamma(\alpha)\Gamma(j+1)} \frac{\alpha}{j-\alpha} \\ =& -(n-1)\overline{F}(s/2)\overline{F}(s) \left(\frac{2^{-\alpha}\pi \cot(\pi\alpha)\Gamma(2\alpha)}{\Gamma(\alpha)^2} + 2^{\alpha-1} \right) \\ =& -(n-1)\overline{F}(s/2)\overline{F}(s) \left(2^{-\alpha-1}(1-2\alpha) \ \mathrm{B}(1-\alpha, 1-\alpha) + 2^{\alpha-1} \right).\end{aligned}$$

For $\alpha = k+1$ with $j \neq k+1$ we can proceed as above to get that these terms are $O(\overline{F}(s)^2)$. By Karamata's Theorem this is dominated by

$$\begin{aligned}\frac{\overline{F}^{(k+1)}(s)\Gamma(\alpha)}{\Gamma(\alpha+k+1)} \int_0^1 (-1)^{k+1} \left(\frac{sx}{2}\right)^{k+1} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)} \ \mathrm{d}F^{*(n-1)}(sx/2) \\ = \frac{(-1)^{k+1}\overline{F}^{(k+1)}(s)}{(k+1)!} \int_0^{s/2} x^{k+1} \ \mathrm{d}F^{*(n-1)}(x).\end{aligned}$$

□

D Proof of Lemma 5.1

At first we need an auxiliary result.

Lemma D.1. *Assume that $\overline{F}(s) \in SR_{-\alpha,1}$, $\alpha > 0$ and $-\log(\overline{F}(s))$ is eventually concave, then the solution $\theta(s)$ of*

$$\int_0^{s/k} e^{\theta(s)x} f(x)dx + \overline{F}(s/k) - \rho^{-1} = 0 \tag{18}$$

fulfills

$$\theta(s) = (1+\epsilon(s)) \frac{-k\log(\overline{F}(s/k))}{s}$$

for $s \to \infty$ with

$$\epsilon(s) = \frac{\log\left(\frac{1-\rho^{-1}}{\alpha}\log(\overline{F}(s/k))\right)}{-\log(\overline{F}(s/k))} + o\left(\frac{1}{-\log(\overline{F}(s/k))}\right).$$

Proof. Choose $\delta_2 > \delta_1 > 0$ then

$$\begin{aligned}\int_0^{s/k} e^{\theta(s)x} f(x)dx =& \int_0^{\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx \\ &+ \int_{\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}}^{\frac{s}{2k}} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx \\ &+ \int_{\frac{s}{2k}}^{\frac{s}{k}} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx.\end{aligned}$$

We have that

$$F\left(\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}\right) \le \int_0^{\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx$$
$$\le e^{\delta_1(1+\epsilon(s))} F\left(\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}\right).$$

Using Potter bounds we get for constants $K > 0$ and $\delta_2 > 0$

$$\int_{\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}}^{\frac{s}{2k}} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx = \int_{\frac{\delta_1}{-\log(\overline{F}(s/k))}}^{\frac{1}{2}} \overline{F}(s/k)^{-x(1+\epsilon(s))} \frac{s}{k} f\left(\frac{sx}{k}\right) dx$$
$$\le K\overline{F}(s/k)^{\frac{-(1+\epsilon(s))}{2}} \frac{s}{k} f\left(\frac{s}{k}\right)\left(\frac{-\log((\overline{F}(s/k))}{\delta_1}\right)^{\alpha+1+\delta_2} \xrightarrow{s\to\infty} 0.$$

By letting $\delta_1 \to 0$ we get that we have to show

$$\lim_{s\to\infty} \int_{\frac{s}{2k}}^{\frac{s}{k}} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx = \rho^{-1} - 1.$$

Let $b(s) = (-(1+\epsilon(s))\log(\overline{F}(s/k)))$ then we get by the Uniform Convergence Theorem

$$\int_{\frac{s}{2k}}^{\frac{s}{k}} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx = \frac{s}{kb(s)} \int_{\frac{b(s)}{2}}^{b(s)} e^x f\left(\frac{sx}{kb(s)}\right) \mathrm{d}x$$
$$\sim \frac{s}{k} f\left(\frac{s}{k}\right)(b(s))^{\alpha} \int_{\frac{b(s)}{2}}^{b(s)} e^x x^{-(\alpha+1)} \mathrm{d}x$$
$$= \frac{s}{k} f\left(\frac{s}{k}\right)(-b(s))^{\alpha}\left(\Gamma(-\alpha, -b(s)/2) - \Gamma(-\alpha, -b(s))\right)$$
$$\sim \frac{\frac{s}{k} f\left(\frac{s}{k}\right) e^{b(s)}}{b(s)} = \frac{\frac{s}{k} f\left(\frac{s}{k}\right) \overline{F}(s/k)^{-(1+\epsilon(s))}}{-\log(\overline{F}(s/k))(1+\epsilon(s))} \sim \alpha \frac{e^{-\epsilon(s)\log(\overline{F}(s/k)}}{-\log(\overline{F}(s/k)}$$

□

Proof of Lemma 5.1. Define $\epsilon(s)$ as in Lemma D.1. From Lemma D.1 it follows that we have to show for any $k > 0$

$$\hat{\theta}_s^k \frac{d\phi_s^k(\theta)}{d\theta}\Big|_{\theta=\hat{\theta}_s^k} = \hat{\theta}_s^k \int_0^{s/k} x\overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)\,\mathrm{d}x \sim \frac{1-\rho}{\rho}\log(\overline{F}(s/k)).$$

For $\delta_1 > 0$ and $\delta_2 < 1$ we have

$$\hat{\theta}_s^k \int_0^{s/k} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx = \hat{\theta}_s^k \int_0^{\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}} x\overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx$$
$$+ \hat{\theta}_s^k \int_{\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}}^{\frac{\delta_2 s}{k}} x\overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx$$
$$+ \hat{\theta}_s^k \int_{\frac{\delta_2 s}{k}}^{\frac{s}{k}} x\overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx.$$

We have

$$\hat{\theta}_s^k \int_0^{\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}} x\overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx \le \delta_1(1+\epsilon(s))e^{\delta_1(1+\epsilon(s))}.$$

Using Potter bounds we get

$$\begin{aligned}
&\hat{\theta}_s^k \int_{\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}}^{\frac{\delta_2 s}{k}} x\overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx \\
&\qquad \le \delta_2(-\log(\overline{F}(s/k))(1+\epsilon(s)))\overline{F}(s/k)^{\delta_2(1+\epsilon(s)}\overline{F}\left(\frac{\delta_1 s}{-k\log(\overline{F}(s/k))}\right) \xrightarrow{s\to\infty} 0.
\end{aligned}$$

One further obtains

$$\begin{aligned}
&\delta_2(1+\epsilon(s))(-\log(\overline{F}(s/k)) \int_{\frac{\delta_2 s}{k}}^{\frac{s}{k}} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx \\
&\le \hat{\theta}_s^k \int_{\frac{\delta_2 s}{k}}^{\frac{s}{k}} x\overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx \\
&\le (1+\epsilon(s))(-\log(\overline{F}(s/k)) \int_{\frac{\delta_2 s}{k}}^{\frac{s}{k}} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx.
\end{aligned}$$

As in the proof of Lemma D.1 we can now show that

$$\lim_{s\to\infty} \int_{\frac{\delta_2 s}{k}}^{\frac{s}{k}} \overline{F}(s/k)^{-x(1+\epsilon(s))k/s} f(x)dx = \rho^{-1} - 1.$$

The assertion follows with $\delta_1 \to 0$ and $\delta_2 \to 1$. □