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Approximate Quantum Cloaking and Almost-Trapped States

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(Received 2 June 2008; revised manuscript received 29 September 2008; published 26 November 2008)

We describe potentials which act as approximate cloaks for matter waves. These potentials are derived from ideal cloaks for the conductivity and Helmholtz equations. At most energies E, if a potential is surrounded by an approximate cloak, then it becomes almost undetectable and unaltered by matter waves originating externally to the cloak. For certain E, however, the approximate cloaks are resonant, supporting wave functions *almost trapped* inside the cloaked region and negligible outside. Applications include dc or magnetically tunable ion traps and beam switches.

DOI: 10.1103/PhysRevLett.101.220404 PACS numbers: 03.75.−b, 37.10.Ty, 41.20.Jb, 78.67.De

Introduction.—A fundamental problem is to describe the scattering of waves at energy E by a potential $V(\mathbf{r})$, as governed by the time-independent Schrödinger equation; the related inverse problem consists of determination of V from scattering data or from boundary measurements. In this Letter, for a generic energy E, we construct sequences V_n^E, $n = 1, 2, 3, \ldots$, of bounded potentials which act as approximate quantum cloaks: for any potential V_0 whose support (where it is nonzero) is surrounded by the support of V_n^E, the scattering amplitude of $V_0 + V_n^E$ goes to zero asymptotically in n, so that V_0 is undetectable by matter waves at energy E. The potentials V_n^E are centrally symmetric, layered, and supported in a spherical annulus $\{r_1 \leq r \leq r_2\}$ in $\mathbb{R}^3$, with spatial oscillations of increasing amplitude as $r \searrow r_1$ and decreasing layer thickness as $n \to \infty$ so that their local (resp., long range) effect on wave propagation becomes stronger (resp., weaker) as n increases. For generic E, the potential $V_0 + V_n^E$ has a negligible effect on matter waves originating outside of its support. Alternatively, for E close to special values, V_n^E allows the core $\{r \leq r_1\}$ to support almost trapped states and be used to form traps for ions (here denoting any charged particles), almost invisible to external matter waves. An approximate version of the dichotomy regarding ideal cloaks for the Helmholtz equation ([1] Thm. 1) holds: If E is sufficiently separated from all interior eigenvalues, then, with high probability, the approximate cloak keeps particles of energy E from entering the cloaked region, see Figs. 1 (right, red) and 2 (left); on the other hand, for E close to an eigenvalue, the cloaked region supports almost trapped states, accepting and binding such particles, leading to a new type of ion trap cf. Figs. 1 (right, blue) and 2 (right) and formula (4).

Recently, Zhang, *et al.*, [2], using ideas from transformation optics [3,4], described an ideal quantum mechanical cloak at any fixed energy E and proposed a physical implementation. The construction starts with a homogeneous, isotropic mass tensor $\hat{m}_0$ and zero potential, and subjects this pair to the singular transformation (1) below. The resulting $\hat{m}$, V yield a Schrödinger equation that is in fact the Helmholtz equation for the corresponding singular Riemannian metric and thus covered by the analysis of cloaking in [1]. The treatment there shows that potentials within the cloaked region are undetectable by exterior measurements, whether far- or near-field, and that finite energy waves must satisfy the perfectly reflective Neumann boundary condition on the inside Σ^- of the cloaking surface. The cloaking mass tensor $\hat{m}$ is highly heterogeneous and anisotropic (infinitely so at Σ^+), making such a quantum cloak challenging to construct, with ultracold atoms in an optical lattice proposed in [2] as a possible realization.

The approach in this Letter is to forgo the perfect functioning of the ideal quantum cloak, and to describe

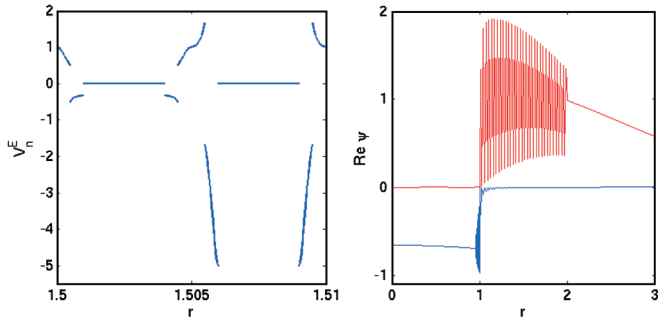

FIG. 1 (color). *Left:* The radial profile of the potential V_n^E over a typical layer $1.5 < r < 1.51$. The potential V_n^E in $\{R = 1.005 \leq r \leq 2\}$ is obtained by repeating similar profiles, with increasing amplitudes as $r \searrow R$. *Right:* $\mathrm{Re}\,\psi$ on a segment $\{(x, 0, 0): 0 \leq x \leq 3\}$ for the same cloaking potential V_n^E and two different cloaked V_0's. For the red curve, E is *not* close to an interior eigenvalue and ψ is produced by an incoming plane wave. For the blue curve, $E = E_1^{-,R}$ is the smallest Dirichlet eigenvalue on B_3 and ψ is an almost trapped state.

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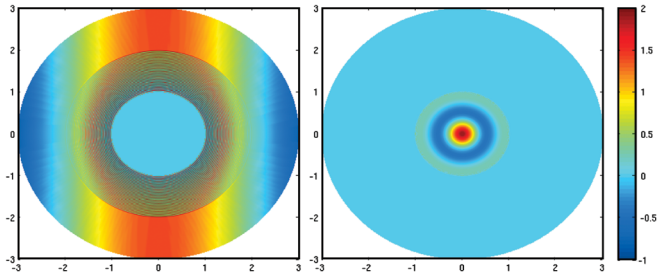

FIG. 2 (color). *Left:* Reψ in B_3 for ψ resulting from an incident plane wave. E is not near an interior Neumann eigenvalue; the matter wave passes unaltered. Moiré pattern is an artifact. *Right:* An excited almost trapped state. $E = E_j^{-,R}$ is an energy close to a Neumann eigenvalue of B_1, for which the ideal cloak supports an trapped state.

sequences of bounded potentials V_n^E, $n = 1, 2, 3, \ldots$, which act as approximate cloaks with respect to $\hat{m}_0$, thus not requiring extreme conditions to realize. The failure to cloak perfectly is in fact a controllable feature that can be taken advantage of for applications described below. The V_n^E are found by means of isotropic transformation optics, a technique we introduce for avoiding the singular and anisotropic behavior, difficult to physically realize, of material parameters that commonly occur in transformation optics-based designs [4–6]; more details and proofs can be found in [7].

Inverse scattering and conductivity cloaks.—There is an enormous literature on unique determination of a potential V from scattering of plane waves at energy E by the Schrödinger equation $(-\nabla^2 + V)\psi = E\psi$, as encoded in the scattering amplitude. Equivalently, for compactly supported V, one may consider the near-field measurements of wave functions at the boundary $\partial\Omega$ of a larger region Ω, as encoded in the Dirichlet-to-Neumann (DN) operator, $\Lambda_V^E(\psi|_{\partial\Omega}) = \partial_\nu \psi|_{\partial\Omega}$ where ν is the normal of $\partial\Omega$ [8]. Ideal cloaking gives highly singular examples of nonuniqueness, but in order to construct approximately cloaking potentials, we first need to recall the ideal electrostatic cloaking of [9]. For simplicity, we describe the cloak on $B_3 - B_1$, with $B_R = \{r \le R\}$ denoting the central ball of radius R in $\mathbb{R}^3$ so that the cloaking surface, the interface between the cloaked and uncloaked regions, is $\Sigma = \{r = 1\}$. Subjecting a homogeneous, isotropic conductivity σ_0 to a singular transformation, we constructed certain singular heterogeneous and anisotropic conductivity tensor fields on $B_3 - B_1$ which, when augmented by any conductivity bounded above and below on B_1, results in a total conductivity on B_3 giving the same electrostatic boundary measurements as σ_0. (For related results in dimension two, see [10,11].) The same construction, applied to the electric permittivity and magnetic permeability rather than the conductivity, was used to propose an electromagnetic (EM) cloak [4], and a microwave realization of a variant of that design reported [12]. Ray-based cloaking for 2D was proposed in [5], while potentials transparent for rays are in [13].

Let $F = (F^1, F^2, F^3)$: $B_3 - \{0\} \to B_3 - B_1$ be the singular transformation, for $\mathbf{r} = (x^1, x^2, x^3) \in \mathbb{R}^3$,

$$\begin{aligned} \tilde{\mathbf{r}} &= F(\mathbf{r}) = \mathbf{r}, \qquad 2 < r \le 3; \\ F(\mathbf{r}) &= \left(1 + \frac{r}{2}\right)\frac{\mathbf{r}}{r}, \qquad r \le 2, \end{aligned} \tag{1}$$

which results in the transformed version of σ_0 on $B_3 - B_1$, augmented for simplicity by $2\sigma_0$ on B_1,

$$\begin{aligned} \sigma_1 &= F_*\sigma_0, \qquad \mathbf{r} \in B_3 - B_1; \\ \sigma_1 &= 2\sigma_0, \qquad \mathbf{r} \in B_1. \end{aligned} \tag{2}$$

F_* denotes the change-of-variables action of F on tensors,

$$(F_*\sigma)^{jk}(\tilde{\mathbf{r}}) = \frac{1}{\det[\frac{\partial F^j}{\partial x^k}]} \sum_{p,q=1}^{3} \frac{\partial F^j}{\partial x^p}\frac{\partial F^k}{\partial x^q}\sigma^{pq}\bigg|_{\mathbf{r}=F^{-1}(\tilde{\mathbf{r}})}.$$

The ideal cloak σ_1 has a singularity at Σ, both in that one of the eigenvalues (corresponding to the radial direction) tends to 0 as $r \searrow 1$, and that there is a jump across Σ, within which σ_1 is nonsingular. Aside from the radius of the outer ball and the factor 2 in the second part of (2), σ_1 is the conductivity introduced in [9] and shown to be indistinguishable from σ_0, *vis-a-vis* boundary measurements at ∂B_3 of electrostatic fields.

Consider also the corresponding acoustic equation,

$$\partial_i(\sigma_1^{ij}\partial_j u) + Ea_1 u = 0, \qquad a_1 = (\det[\sigma_1^{ij}])^{-1}, \tag{3}$$

where we consider σ_1 as a mass density and a_1 as a bulk modulus. Then, using measurements at ∂B_3 of acoustic waves of frequency $\sqrt{E}$, the pair σ_1, a_1 is indistinguishable from σ_0, $a_0 = 1$ [1,14–16]. The waves u within the cloaked region have a simple description ([1] Thm. 1). Either (I) if E is not a Neumann eigenvalue of the cloaked region, then u must vanish there; or (II) if E is an eigenvalue, then u can be an associated eigenfunction there, while possibly vanishing on $B_3 - B_1$. The Dirichlet eigenvalues and eigenfunctions of (3) on B_3 can be separated into $(E_j^{+,1}, u_j^+)$ and $(E_j^{-,1}, u_j^-)$, $j = 1, 2, 3, \ldots$, with u_j^+ and u_j^- supported in $B_3 - B_1$ and B_1, resp.

Approximate cloaks and failure of cloaking.—For $1 < R \le 2$, let σ_R be given by the same formulae as in (2), but applied on $B_3 - B_R$ and B_R, resp., and similarly for a_R in (3). Observe that for each $R > 1$, σ_R and a_R are nonsingular; however, their lower (resp., upper) bounds go to 0 (resp., ∞) as $R \searrow 1$. (Similar truncations of EM cloaks have been studied in [11,17,18].)

When σ_1, a_1 are replaced by σ_R, a_R, there is a decomposition similar to the one above for (3), with eigenvalues and eigenfunctions $(E_j^{+,R}, v_j^+)$ and $(E_j^{-,R}, v_j^-)$ concentrating in $B_3 - B_1$ and B_1, resp., and $E_j^{\pm,R}$ converging to $E_j^{\pm,1}$ as $R \to 1$. The solution of the boundary value problem

$\partial_i(\sigma_R^{ij}\partial_j v) + Ea_R v = 0$ on B_3, $v|_{\partial B_3} = f$, has an eigenfunction expansion

$$v(\mathbf{r}) = \sum_{\pm}\sum_{j=1}^{\infty}\left(\int_{\partial B_3} f\frac{\partial v_j^{\pm}}{\partial \nu}dS\right)\frac{v_j^{\pm}(\mathbf{r})}{E - E_j^{\pm,R}}. \quad (4)$$

An approximate version of dichotomy (I)-(II) holds for approximate acoustic cloaks: When E is not equal to any $E_j^{\pm,1}$, one can show the DN operators for the σ_R, a_R converge to that for σ_1, a_1 as $R \to 1$; physically, this means that the boundary measurements of pressure and the normal component of the particle velocity for the approximate cloaks tend to those for the ideal cloak, which are themselves the same as for σ_0, a_0. However, if E is close to some $E_j^{-,R}$, the corresponding term in (4) may dominate the others, in which case the solution v, having a large coefficient of v_j^-, concentrates in B_1. Since v_j^- cannot vanish identically in $B_3 - B_1$, both the near-field measurements on the boundary ∂B_3 and the far-field patterns differ noticeably from those corresponding to σ_0, a_0. This interior resonance corresponds to an acoustic wave almost trapped within the cloak.

Isotropic transformation optics.—A well-known phenomenon in effective medium theory is that homogenization of isotropic material parameters may lead, in the small-scale limit, to anisotropic ones [19]. We exploit this, using ideas from [20,21], to approximate the anisotropic, almost cloaking σ_R by isotropic conductivities $\sigma_{R,\epsilon}$ so that for $\varepsilon > 0$, the pairs $\sigma_{R,\epsilon}$, a_R also function as approximate acoustic cloaks [7]. The $\sigma_{R,\epsilon}(\mathbf{r})$ are layered and spatially highly oscillating. (In the context of EM cloaking, thin concentric layers of homogeneous, isotropic media were considered in [22,23].)

Approximate Schrödinger cloaks.—The gauge transformation $\psi = \sqrt{\sigma}u$ reduces the acoustic equation (3), with nonsingular isotropic conductivity $\sigma = \sigma_{R,\epsilon}$ in place of the anisotropic σ_1, and a_R in place of a_1, to the Schrödinger equation at the same energy E, $(-\nabla^2 + V_{R,\epsilon}^E)\psi = E\psi$, where $V_{R,\epsilon}^E = \nabla^2(\sqrt{\sigma}_{R,\epsilon})/\sqrt{\sigma}_{R,\epsilon} + E(1 - a_R^{1/2}\sigma_{R,\epsilon}^{-1})$. As $\sigma_{R,\epsilon}$ is highly oscillatory, $V_{R,\epsilon}^E$ consists of a layered pattern of concentric central potential barriers and wells of increasing amplitudes and decreasing widths as $\varepsilon \searrow 0$. The radial profile of the potential over one spherical layer is in Fig. 1 (left). The boundary measurements of solutions of these Schrödinger equations at ∂B_3 coincide with those for the corresponding acoustic equations. By the convergence of the acoustic equations, we can choose $R \searrow 1$, $\epsilon \searrow 0$ so that the boundary measurements for these Schrödinger equations converge to those for the acoustic equation (3) at energy E, which in turn are the same as for the Schrödinger equation in free space. The nonresonant case is summarized by:

Approximate quantum cloaking. Let V_0 be a bounded potential on B_1, and E be neither a Dirichlet eigenvalue of the free Hamiltonian $-\nabla^2$ on B_3 nor a Neumann eigenvalue of $-\nabla^2 + V_0$ on B_1. Then, there exists a sequence of cloaking potentials V_n^E on B_3 such that the DN operators $\Lambda_{V_0+V_n^E}^E \to \Lambda_0^E$ as $n \to \infty$. I.e., at energy E, the potential $V_0 + V_n^E$ is indistinguishable by near-field measurements, asymptotically in n, from the zero potential; a similar result holds for far-field patterns. V_0 is thus approximately cloaked when surrounded by V_n^E.

As any specific measurement device has a limited precision, this means that it is possible to design a potential to cloak an object within from any single-particle measurements made using that device at energy E.

Numerics.—We use analytic expressions to compute the wave function ψ for an incident plane wave with $\psi_{\text{inc}}(x) = ae^{ik\mathbf{r}\cdot\vec{d}}$. The computations are made without reference to physical units, using $a = 1$, $E = 0.5$, $k = \sqrt{E}$. The cloak is based on $R = 1.005$, corresponding to an anisotropy ratio of σ_R at $\Sigma_R = \{r = R\}$ of 4×10^4. In the simulations, we use a cloak consisting of 20 homogenized layers inside and 30 homogenized layers outside Σ_R. Inside the cloak, we have located a centrally symmetric step potential, $W(x) = c_{\text{inn}}\chi_{[0,0.9]}(r)\Phi$. The cloaking potential V_n^E and the energy E are the same in all figures, but we vary the constant c_{inn}. In Fig. 1 (right, red) and Fig. 2 (left), we have $c_{\text{inn}} = -98.5$, and ψ is the wave produced by an incoming plane wave. In Fig. 1 (right, blue), with $c_{\text{inn}} = +1.858$, and in Fig. 2 (right), with $c_{\text{inn}} = -71.45$, there is no incoming wave, but rather an excited almost trapped state in the cloaked region.

Applications: Almost trapped states and ion traps.—A version of the dichotomy for approximate acoustic cloaks described above also holds for approximate quantum cloaks, since the u and ψ waves are equivalent by the gauge transformation. As a consequence, given an energy E, the approximate quantum cloak may be such that E either is or is not an eigenvalue for the ideal cloak. This results in B_1 becoming either (I') an almost cloaked region that with a high probability does not accept energy E particles from outside Σ; or (II') a trap that supports almost trapped states, which correspond to a particle at energy E trapped in B_1 with high probability. A design of either type could possibly be implemented by an array of ac and dc electrodes with total effective potential approximating V_n^E for large n [24]. This leads to a new type of trap for ions, differing from, e.g., the Paul [25], Penning [26], or Zajfman [27] traps, and justified on the level of quantum mechanics. Furthermore, the trap may be made tunable by including a dc electrode in the trapped region, corresponding to a Coulomb V_0; varying the charge changes whether or not E is an eigenvalue of $-\nabla^2 + V_0$ and thus which of (I') or (II') holds. Alternatively, one can switch between (I') and (II') by application of homogeneous magnetic fields; see below.

Topological ion traps.—The basic construction outlined above can be modified to make the wave function on B_1 behave as though it were confined to a compact, boundary-

less three-dimensional manifold, topologically but not metrically the three-dimensional sphere, $\mathbb{S}^3$. By suitable choice of metric, the energy level E can have arbitrary multiplicity for the interior of the resulting trap, allowing one to implement physical systems mimicking matter waves on abstract spaces. As the starting point, one uses not the original cloaking conductivity σ_1 (the single coating construction), but rather a double coating ([1] Sec. 2), which we denote here by $\sigma^{(2)}$. This is singular from both sides of Σ, and in the EM cloak setting corresponds to coating both sides of Σ with metamaterials. (See [7] Fig. 7.) By [1] Sec. 3.3, the finite energy solutions of the resulting Helmholtz equation on B_3 split into direct sums of waves on $B_3 - B_1$, as for σ_1, and waves on B_1 which are identifiable with eigenfunctions of the Laplace-Beltrami operator $-\nabla_g^2$ on $(\mathbb{S}^3, g)$ with eigenvalue $\sqrt{E}$. If one takes g to be the standard metric on $\mathbb{S}^3$, then nonground states are highly degenerate, while a generic choice of g yields nondegenerate energy levels [28]. On the other hand, by suitable choice of g, any finite number of energy levels and degeneracies can be specified [29], allowing traps supporting almost trapped states at energy E of arbitrary degeneracy.

Magnetically tunable quantum beam switch.—Consider a beam of ions of energy E, leaving an oven and traversing a tube $T = \{0 \le \rho \le \rho_0, 0 \le \theta \le 2\pi, 0 \le z \le L\}$ (in cylindrical coordinates). Treating the ions as matter waves, place in T several almost trapping traps of the type described above, centered at points z_j, $j = 1, 2, \ldots, N$ on the z-axis, forming a potential $V(\mathbf{r}) = \sum_{j=1}^{N} V_n^E(\mathbf{r} - (0, 0, z_j))$. The techniques above may be applied to the Schrödinger equation with magnetic potential $A(\mathbf{r})$ on a region Ω,

$$[-(\nabla + iA)^2 + V]\psi = E\psi \quad \text{on } \Omega,$$
$$\Lambda_{V,A}^E(f) = \partial_\nu \psi|_{\partial\Omega} + i(A \cdot \nu)f \quad \text{on } \partial\Omega. \quad (5)$$

We design the traps so that, in the absence of a magnetic field, or for small field strengths, the traps act as cloaks and thus the ions pass through T unhindered. However, if a homogeneous magnetic field is then applied to the tube, chosen so that the magnetic Schrödinger operator has E as a Neumann eigenvalue inside each trap, then there is a large probability that an ion passing the jth trap will bind to that trap. If N is large enough, then the probability that any ion traveling the length of T will become bound is ~ 1, and T thus functions as a magnetically controlled switch for the beam of ions.

Magnified magnetic fields.—For a homogeneous magnetic field with linear magnetic potential A, one can obtain a sequence of electrostatic potentials W_n for which $\lim_{n\to\infty} \Lambda_{W_n,A}^E = \Lambda_{0,\tilde{A}}^E$, with $\tilde{A}$ singular at a point. I.e., in the presence of a homogeneous magnetic field, the W_n produce far- or near-field measurements that tend, as $n \to \infty$, to those of the zero electrostatic potential in the presence of a magnetic field blowing up at a point, giving the illusion of a locally singular magnetic field [7].

A. G. and G. U. are supported by NSF, G. U. by a Walker Family Grant, M. L. by Academy of Finland, and Y. K. by EPSRC. We are grateful for discussions with A. Cherkaev and V. Smyshlyaev on homogenization and S. Siltanen on numerics, and to the referees for suggestions concerning the exposition.

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