ISSN 0143-4543

University of Aberdeen Business School

Median Independent Inequality Orderings

Ramses H. Abul Naga* and Tarik Yalcin †

October 19, 2010

Discussion Paper 2010-03
October 2010

Editor: Dr W David McCausland
www.abdn.ac.uk/business/

* University of Aberdeen.
† Université de Lausanne.

Median Independent Inequality Orderings

Ramses H. Abul Naga [*] and Tarik Yalcin [†, ‡]

October 19, 2010

Abstract

To date, inequality orderings for ordered response data are only suitable for comparing distributions that share a common median state. In this paper we propose a methodology for comparing distributions irrespective of their medians. We set out to do so by introducing a general pre-ordering and equivalence relation defined over distributions with different median responses, leading us naturally to derive a partial ordering over equivalence classes. We then discuss the implications of our results for the axiomatic derivation of inequality indices for ordered response data.

Keywords: ordered response data, equivalence relations, inequality orderings, inequality measures.

JEL codes: I3, I1.

[*]Business School and Health Economics Research Unit (University of Aberdeen).

[†]Institute of Health Economics and Management (Université de Lausanne).

[‡]We wish to thank Gérard Antille, Frank Cowell and Ajit Mishra for discussions. We are are particularly grateful to Malcolm Pemberton for detailed comments on an earlier draft of the paper.

1. Introduction

Ordered response data are purely ordinal data which do not possess an appropriate measurement scale. Typical examples of such data include self-reported health status, happiness, educational attainment, socio-economic status and the degree of satisfaction with life. Such data are collected from responses to diverse questions such as: "Taken all together, how would you say things are these days – would you say that you are very happy, pretty happy, or not too happy?" (The United States General Social Survey), or "On the whole are you very satisfied, fairly satisfied, not very satisfied or not at all satisfied with the life you lead?" (Euro-barometer Survey Series.) Ordered response data have become increasingly the subject of investigation by economists in relation to topics such as how income inequality affects happiness (Alesina et al. 2004), the evolution of happiness inequality (Stevenson and Wolfers, 2008) and the cross-sectional variation in inequality of self-reported health status (Allison and Foster, 2004.)

The classical approach pertaining to the measurement of income inequality (e.g. Atkinson, 1970, Shorrocks, 1980 and Cowell, 2000) is however not suited to the context of ordered response data since the computation of Lorenz curves and income inequality measures requires calculation of sums, means and other moments which are not defined for ordinal data [1]. Instead researchers have used order statistics and related functions such as the cumulative distribution to order distributions and measure the underlying level of inequality [2]. Allison and Foster (2004) for instance replace the Lorenz ordering by a Median Preserving Spread relation, suited for the comparison of distributions of ordered response data which share a common median response. Self-reported health data (the context discussed by Allison and Foster) typically exhibit identical medians; but there are clearly other contexts where cumulative distributions do not share this convenient property.

In Table 1, we report findings from Jones et al. (2010), where respondents in the World Health Survey were asked to report their perception of treatment in the health system of their country of residence. The median response in Austria and Denmark was the highest state (very good), whereas the median response in France and Spain was the penultimate state (good). The first two rows of the table

[1] This point is extensively discussed in the health economics literature. See for instance Allison and Foster (2004) and Zheng (2006).

[2] See for instance Allison and Foster (2004), Zheng (2006), Apouey (2007) and Abul Naga and Yalcin (2008).

are comparable according to the median preserving spread relation, and likewise, the third and fourth row are comparable. However, no comparison can be made between Austria and Spain, Austria and France, Denmark and Spain or Denmark and France [3]. Income inequality comparisons are not restricted to distributions with identical means and, similarly, in the context of ordered response data *we need to know how to order two distributions X and Y independently of their medians.* This is an important gap in the literature, which we set out to address in this paper.

A central question in this paper is how to generalize earlier work in a way that the new ordering remains identical to the median preserving spread criterion when the distributions we are comparing do happen to exhibit an identical median state. Also, we inquire as to what additional properties must inequality indices for ordered response data satisfy above those that are available for indices that compare distributions which share a common median response. Finally, this leads us to examine the indices already available to date, and to inquire as to which of these belong to the resulting new class of median independent inequality indices.

Our starting point in the paper is to observe that regardless of the actual median response, inequality must be zero when everyone is located at the median. This allows us to define an equivalence relation on a subset of the set of cumulative distributions. We call this restricted set *the set of egalitarian distributions.* By considering neighboring distributions to particular elements of this set, we obtain a general equivalence relation between distributions with differing medians. In turn this allows us to obtain the *equivalence class* of any given cumulative distribution function X (the set of distributions that are equivalent to X.) We then define a partial ordering on equivalence classes. This will ensure that cumulative distributions which are equivalent will exhibit the same level of inequality.

Sections 2 and 3 of the paper contain preliminary definitions and results that prepare the ground for the core sections. In Section 4 we generalize the relation between the elements of the set of egalitarian distributions to obtain a general equivalence relation across distributions which need not possess a common median. In Section 5 we derive our proposed partial ordering over equivalence classes and discuss how this allows us to test in a simple fashion that a given distribution X and all members of its equivalence class, exhibit less inequality than another distribution Y and its entire equivalence class. In Section 6 we discuss the implications of our results for the axiomatic derivation of inequality indices for ordered

[3] The methodology we set out below in fact enables us to compare all pairs of distributions from Table 1, with the exception of Spain and Austria.

response data. We also examine in this section which of the indices available to date belong to the new class of median independent inequality measures. We conclude the paper in Section 7. A detailed appendix gathers the proofs of the main results of the paper.

2. Order properties of the set of cumulative distributions

Our starting point is to consider a situation whereby the economic status of a person is measured according to an ordinal scale $c \doteq (c_1, ..., c_n)$. We denote $C \doteq \{c : 0 < c_1 < c_2 < ... < c_n < \infty\}$ the set of ordered increasing scales. Because the scale is entirely arbitrary, calculations of summary statistics (mean and other moments) will not form the basis of our measurement of inequality. Instead, following earlier work, the cumulative proportions underlying each outcome will be the key inputs to our inequality indices.

Let $\Lambda \doteq \{F = (f_1, f_2, ..., f_n) : 0 \leq f_1 \leq .. \leq f_n = 1\}$ denote the set of cumulative distribution functions ($cdfs$) defined over n ordered states, and let $X = [x_1, ..., x_{n-1}, 1]$ be a cdf. The notion of Median Preserving Spread, in short $MedPS$, (Allison and Foster 2004) is central in ordering distributions which share a common median response:

Definition 2.1 *Let $X \doteq [x_1, ..., x_{n-1}, 1]'$ and $Y \doteq [y_1, ..., y_{n-1}, 1]'$ be any two elements of Λ. Y is a Median Preserving Spread of X, denoted $X \lhd Y$, if and only if the two distributions satisfy the following three conditions:*

($AF1$) $med\,(X) = med\,(Y) = m$

($AF2$) for all $i < m$, $x_i \leq y_i$

($AF3$) for all $i \geq m$, $x_i \geq y_i$

We endow the set of cumulative distributions with a partial order $(\Lambda, \prec_{AF})$ thus enabling us to order two distributions X and Y when the latter is obtained from the former via a sequence of median preserving spreads:

Definition 2.2 *Let X and Y be any two elements of Λ. $X \prec_{AF} Y$, if and only if there exists a finite sequence of distributions $Q_0, ..., Q_k$ such that $Q_0 = X$, $Q_k = Y$, and $Q_{l-1} \lhd Q_l$ for all $l = 1, ..., k$.*

An inequality index for ordered response data is a function

$$\Delta\,(X, c) : \Lambda \times C \to \mathbb{R}_+ \tag{2.1}$$

which preserves the above order in the sense that $X \prec_{AF} Y$ entails $\Delta\,(X, c) \leq \Delta\,(Y, c)$ for any given scale $c \in C$. Examples of such inequality indices are given

in Table 1. Further key properties that such indices are required to satisfy will be discussed in Section 6.

For ease of presentation, we specialize our discussion to the context $n = 3$ [4], and we simply refer to Λ as the set of cumulative distributions. We shall adopt the definition that a state m is the median of $X = [x_1, x_2, 1]'$ if $x_{m-1} \leq 0.5$ and $x_m \geq 0.5$.

Let $X = [x_1 \ x_2 \ x_3]'$ be some element of Λ and define the following three sets

$$\Lambda_1 \doteq \{(x_1, x_2) : 0.5 \leq x_1 \leq x_2 \leq x_3 = 1\} \tag{2.2}$$
$$\Lambda_2 \doteq \{(x_1, x_2) : 0 \leq x_1 \leq 0.5 \leq x_2 \leq x_3 = 1\} \tag{2.3}$$
$$\Lambda_3 \doteq \{(x_1, x_2) : 0 \leq x_1 \leq x_2 \leq 0.5 \leq x_3 = 1\} \tag{2.4}$$

Λ_i denotes the set of cumulative distributions with median state i, and $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. Since $x_3 = 1$, we can visualize Λ and its three subsets in two-dimensional space (see Figure 1.)

Example 1 Consider the following three distributions

$$F = \begin{pmatrix} 0.7 \\ 0.8 \\ 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0.8 \\ 0.8 \\ 1 \end{pmatrix} \qquad H = \begin{pmatrix} 0.2 \\ 0.3 \\ 1 \end{pmatrix}$$

Then $F, X \in \Lambda_1$ while $H \in \Lambda_3$. That is, $med(F) = med(X) = 1$ while $med(H) = 3$. Observe furthermore that $X \prec_{AF} F$ while H is not comparable to F or to X according to the relation $\prec_{AF}$ since F and X do not share a common median response state with H. ◯

Returning to Figure 1, consider the following four distributions:

$$\tilde{\Pi} = [0.5 \quad 0.5 \quad 1]' \tag{2.5}$$
$$\hat{\Pi}_1 = [1 \quad 1 \quad 1]' \tag{2.6}$$
$$\hat{\Pi}_2 = [0 \quad 1 \quad 1]' \tag{2.7}$$
$$\hat{\Pi}_3 = [0 \quad 0 \quad 1]' \tag{2.8}$$

For a given Λ_j, the ordering $\prec_{AF}$ entails that $\hat{\Pi}_j$ is the most egalitarian distribution, and $\tilde{\Pi}$ exhibits most inequality. Alternatively we may state that each Λ_j

[4] All results in this paper generalize in the context $n > 3$. See the Appendix section for further detail.

has a minimal element $\hat{\Pi}_j$ and a maximal element $\tilde{\Pi}$ in a way that each $F \in \Lambda_j$ is bounded by these two elements: $\hat{\Pi}_j \prec_{AF} F$ and $F \prec_{AF} \tilde{\Pi}$. Thus, arrows in the various subsets of Λ in Figure 1 indicate the direction of a median preserving spread. We define

$$\mathbb{P} = \left\{\hat{\Pi}_1, \hat{\Pi}_2, \hat{\Pi}_3\right\} \tag{2.9}$$

as the set of egalitarian distributions. We close this section by observing that the relation $\prec_{AF}$ does not allow us to compare any two elements of $\mathbb{P}$ or as a matter of fact, any two distributions which belong to the interior of distinct subsets of Λ, [5].

3. A general pre-ordering

Observe that $\hat{\Pi}_1$, $\hat{\Pi}_2$ and $\hat{\Pi}_3$ share a common property with regards to measuring inequality: all individuals report the same value for our variable of interest. Though these distributions are not identical, we may wish to consider them *equivalent* in the sense that each $\hat{\Pi}_j$ achieves the lowest possible level of inequality within Λ_j

Equivalence relations are used to formalize a similarity between different elements of a set. While we do not specify the equivalence relation we shall be working with at this stage, we note that any methodology for measuring inequality is founded on a quasi-ordering, or pre-ordering [6] [7]. Our purpose in this paper is to order distributions X and Y with possibly different medians in a way that the ordering remains equivalent to $(\Lambda, \prec_{AF})$ when X and Y have the same median.

To do so, we shall jointly define an equivalence relation and a pre-order over a set, via the formulation of two axioms. For a simple starting example, consider the set of integers $\mathbb{Z}$, and let s, s_1 and t be any elements of $\mathbb{Z}$. Then our first axiom may take the form $t \preceq s \Leftrightarrow t \equiv s_1$ and $|s_1| \leq |s|$, while the second axiom would state that $t \equiv s \Leftrightarrow |t| \leq |s|$ and $|s| \leq |t|$ (that is, $t \equiv s \Leftrightarrow |t| = |s|$.)

Returning to our specific problem, consider a general equivalence relation $\equiv_R$ and a pre-order $\preceq$ over the set of cumulative distributions, which are jointly defined by the following two axioms:

[5] By *interior*, we are excluding comparisons between distributions which may be on the common boundary of two subsets of Λ.

[6] We use the term pre-order to denote a reflexive and transitive relation.

[7] The Lorenz pre-order for instance provides the rationale for the Pigou-Dalton transfer principle in the literature pertaining to the measurement of income inequality.

•$AXIOM\ 1 : X \equiv_R Y_1$ and $Y_1 \prec_{AF} Y_2 \Leftrightarrow X \preceq Y_2$ for any $X, Y_1, Y_2 \in \Lambda$ (inequality aversion)

•$AXIOM\ 2 : X \preceq Y$ and $Y \preceq X \Leftrightarrow X \equiv_R Y$ for any $X, Y \in \Lambda$ (equivalence)

We shall develop in the sections below specific relations which satisfies the above axioms over the set of cumulative distributions.

$AXIOM\ 1$ makes it clear that we want to generalize an existing methodology for comparing distributions with different medians, and this is to be achieved by adding more structure to the set of cumulative distributions, via formalizing an equivalence relation $\equiv_R$.

$AXIOM\ 2$ has clearly its analogue in the income inequality literature. If equality of Lorenz curves is used to define a relation $\equiv_L$ between two income distributions, then $\equiv_L$ may readily be verified to be a reflexive, symmetric and transitive relation.

Our first result is formulated with the help of the concept of an equivalence class for a given distribution X:

$$\kappa_R(X) = \{Y \in \Lambda : X \equiv_R Y\} \tag{3.1}$$

That is, $\kappa_R(X)$ gathers all the distributions that are equivalent to X in the light of the definition of equivalence embodied in $\equiv_R$ [8]. The set of equivalence classes (the quotient set) induced by $\equiv_R$ will be denoted Λ/R:

$$\Lambda/R = \{S \subseteq \Lambda : S = \kappa_R(X) \text{ for some } X \in \Lambda\} \tag{3.2}$$

Proposition 3.1 *Let X, Y_1 and Y_2 be any members of Λ and let $\equiv_R$ denote any equivalence relation over the set Λ with resulting equivalence classes $\kappa_R(.)$. Then the following two statements are equivalent:*

(i) $(\Lambda, \preceq)$ is a pre-ordering which satisfies $AXIOM\ 1$ and $AXIOM\ 2$.

(ii) $(\Lambda/R, \prec_R)$ defined by the relation $\kappa_R(X) \prec_R \kappa_R(Y_2) \Leftrightarrow X \equiv_R Y_1$ and $Y_1 \prec_{AF} Y_2$, is a partial ordering over the set of equivalence classes.

This result is an instance of a more general theorem which enables a pre-ordered set to induce a partial order over equivalence classes (Harzheim, 2005; Theorem 4.9, p.17). The Proposition provides the basis for understanding the meaning of a specific partial ordering that we define on equivalence classes (see

[8] For example, we will introduce in the next section an equivalence relation $\equiv_E$ which will enable us to formalize the intuitive property that $\hat{\Pi}_1$, $\hat{\Pi}_2$ and $\hat{\Pi}_3$ are all equivalent, and belong to a same equivalence class.

below): we may define a specific pre-ordering which satisfies $AXIOMS$ 1 and 2, or equivalently, we may define a partial ordering over equivalence classes as defined in the above Proposition. We shall develop the latter approach in the next sections.

4. The equivalence relation

In this section we shall begin by formalizing the similarity between the elements $\hat{\Pi}_1$, $\hat{\Pi}_2$ and $\hat{\Pi}_3$ of the set of equivalent distributions $\mathbb{P}$, via an equivalence relation. We then extend this equivalence relation to the entire set of cumulative distributions Λ. The resulting relation, $(\Lambda, \equiv_E)$, is used to define the set of distributions which are equivalent to a particular distribution X, namely the equivalence class $\kappa_E(X)$.

Consider again the set of egalitarian distributions $\mathbb{P}$. For the purpose of measuring inequality, all three distributions in this set share a fundamental property: every member of distribution $\hat{\Pi}_j$ reports being at the respective median m_j (but medians differ across each $\hat{\Pi}_j$.) Also, returning to Figure 1, we may observe that each of these distributions may be shown to be at a maximum distance from the most polarized distribution $\tilde{\Pi}$ [9]. Let $|Z|$ denote the absolute value of a vector Z. Formally, we have

$$\left|\hat{\Pi}_j - \tilde{\Pi}\right| = [\ 0.5 \quad 0.5 \quad 0\ ]' \quad for\ all\ \hat{\Pi}_j \in \mathbb{P} \tag{4.1}$$

Accordingly, we can take these distributions as being related in that inequality is always zero when everyone is at the median. More specifically the relation

$$\hat{\Pi}_j \sim \hat{\Pi}_l \Longleftrightarrow \left|\hat{\Pi}_j - \tilde{\Pi}\right| = \left|\hat{\Pi}_l - \tilde{\Pi}\right| \tag{4.2}$$

defined over $\mathbb{P}$ is reflexive, symmetric and transitive, and accordingly is an equivalence relation. How do we generalize this concept of equivalence to other distributions?

Unless otherwise stated, we shall adopt the convention that F, G_1, G_2 and H are distributions chosen such that $F \in \Lambda_1$, $G_1, G_2 \in \Lambda_2$ and $H \in \Lambda_3$. Now consider any ε and ω such that $0 < \omega < \varepsilon < 0.5$. According to the $\prec_{AF}$ relation,

[9] Though we do not specify any explicit metric for the set Λ, note though for example that a Eucledian metric, as well as the usual metric (see footnote 10) would support this statement.

we have in Λ_1 : $\hat{\Pi}_1 = (1 \quad 1 \quad 1)' \prec_{AF} (1-\varepsilon \quad 1-\omega \quad 1)' \doteq F$. Likewise, in Λ_3 : $\hat{\Pi}_3 = (0 \quad 0 \quad 1)' \prec_{AF} (\omega \quad \varepsilon \quad 1)' \doteq H$.

Observe that

$$\left|F - \tilde{\Pi}\right| = \left| \begin{array}{c} 1-\varepsilon-0.5 \\ 1-\omega-0.5 \\ 1-1 \end{array} \right| = \left(\begin{array}{c} 0.5-\varepsilon \\ 0.5-\omega \\ 0 \end{array} \right) \tag{4.3}$$

$$\left|H - \tilde{\Pi}\right| = \left| \begin{array}{c} \omega-0.5 \\ \varepsilon-0.5 \\ 1-1 \end{array} \right| = \left(\begin{array}{c} 0.5-\omega \\ 0.5-\varepsilon \\ 0 \end{array} \right) \tag{4.4}$$

so that $\left|H - \tilde{\Pi}\right|$ differs from $\left|F - \tilde{\Pi}\right|$ in that the vector $\left|H - \tilde{\Pi}\right|$ is a permutation of the elements of $\left|F - \tilde{\Pi}\right|$.

Returning to Figure 1, observe that within the square shaped set Λ_2, starting from $\hat{\Pi}_2$ we may generate two distinct median preserving spreads G_1 and G_2 of $(0 \quad 1 \quad 1)'$. Specifically, for $0 < \omega < \varepsilon < 0.5$, we have both $\hat{\Pi}_2 \prec_{AF} (\varepsilon, 1-\omega, 1)' \doteq G_1$ and $\hat{\Pi}_2 \prec_{AF} (\omega, 1-\varepsilon, 1)' \doteq G_2$. We may observe again:

$$\left|G_1 - \tilde{\Pi}\right| = \left| \begin{array}{c} \varepsilon-0.5 \\ 1-\omega-0.5 \\ 1-1 \end{array} \right| = \left(\begin{array}{c} 0.5-\varepsilon \\ 0.5-\omega \\ 0 \end{array} \right) = \left|F - \tilde{\Pi}\right| \tag{4.5}$$

$$\left|G_2 - \tilde{\Pi}\right| = \left|H - \tilde{\Pi}\right| \tag{4.6}$$

so that $\left|G_1 - \tilde{\Pi}\right|$ and $\left|G_2 - \tilde{\Pi}\right|$ are identical up to permutation. Each of F, G_1 and G_2, and H are respectively neighboring distributions of $\hat{\Pi}_1$, $\hat{\Pi}_2$, and $\hat{\Pi}_3$ [10]. But the above equalities also entail that all four vectors, namely, $\left|G_1 - \tilde{\Pi}\right|$, $\left|G_2 - \tilde{\Pi}\right|$, $\left|F - \tilde{\Pi}\right|$ and $\left|H - \tilde{\Pi}\right|$ are identical up to permutation.

[10] If we endow Λ with a specific metric, we can formalize the concept that the distribution $F = [\ 1-\varepsilon \quad 1-\omega \quad 1\ ]'$ is a neighboring distribution to $\hat{\Pi}_1$. Let

$$d(X,Y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$$

denote the usual metric, then (Λ, d) becomes a metric space.

Define $B(X,r) \doteq \{Y \in \Lambda : d(X,Y) < r\}$ as the open ball in Λ, centred at X, and with radius r. Then since $\omega < \varepsilon$, we have that $F \in B\left(\hat{\Pi}_1, 2\varepsilon\right)$, $G_1, G_2 \in B\left(\hat{\Pi}_2, 2\varepsilon\right)$ and $H \in B\left(\hat{\Pi}_3, 2\varepsilon\right)$; that is each of F, G_1, G_2 and H is contained within an open ball of radius 2ε and centred at the most egalitarian distribution of the relevant subset of Λ.

Define $\mathbb{M}$ as the set of permutation matrices which permutate any but the last component of an n-dimensional vector. We generalize the relation between $\hat{\Pi}_1, \hat{\Pi}_2$ and $\hat{\Pi}_3$ over $\mathbb{P}$ as follows:

Definition 4.1 *Let* $X \in \Lambda_j$ *and* $Y \in \Lambda_l$ *Then* $X \equiv_E Y \Leftrightarrow \left|X - \tilde{\Pi}\right| = M\left|Y - \tilde{\Pi}\right|$ *where* $M \in \mathbb{M}$

The relation $(\Lambda, \equiv_E)$ embodies a certain symmetry in our judgements on the level of inequality:

Example 2: Consider the following distributions:

$$F = \begin{pmatrix} 0.7 \\ 0.8 \\ 1 \end{pmatrix} \in \Lambda_1, \quad H = \begin{pmatrix} 0.2 \\ 0.3 \\ 1 \end{pmatrix} \in \Lambda_3$$

Then $F \equiv_E H$ since $\left|F - \tilde{\Pi}\right| = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left|H - \tilde{\Pi}\right|$. $\bigcirc$

The relation $(\Lambda, \equiv_E)$ generalizes the earlier relation $(\mathbb{P}, \sim)$ in the sense that it preserves the equivalence between the members of $\mathbb{P}$. Furthermore,

Proposition 4.2 *The relation* $(\Lambda, \equiv_E)$ *is an equivalence relation.*

We use the above result firstly to define and obtain the equivalence class of any $X \in \Lambda$. Then, in the section below, we shall specialize the result of Proposition 3.1 in the context of $\equiv_E$ to obtain the required partial ordering on the set of equivalence classes, thus enabling us to compare distributions with different medians.

We define then the equivalence class of any $X \in \Lambda$ as follows:

$$\kappa_E(X) = \left\{Y \in \Lambda,\ M \in \mathbb{M} : \left|X - \tilde{\Pi}\right| = M\left|Y - \tilde{\Pi}\right|\right\} \tag{4.7}$$

We observe that the equivalence class of any member of the set of egalitarian distributions is precisely the entire set $\mathbb{P}$:

$$\kappa_E\left(\hat{\Pi}_j\right) = \left\{\hat{\Pi}_1, \hat{\Pi}_2, \hat{\Pi}_3\right\} = \mathbb{P} \qquad j = 1, 2, 3 \tag{4.8}$$

Returning to Example 2, it is instructive to examine in the light of the above definitions the equivalence class of the distribution F :

Example 3 For the distribution $F = \begin{pmatrix} 0.7 \\ 0.8 \\ 1 \end{pmatrix}$ of Example 2, we have

$$\kappa_E(F) = \{F, G_1, G_2, H\}$$

where

$$H = \begin{pmatrix} 0.2 \\ 0.3 \\ 1 \end{pmatrix}, G_1 = \begin{pmatrix} 0.3 \\ 0.8 \\ 1 \end{pmatrix}, G_2 = \begin{pmatrix} 0.2 \\ 0.7 \\ 1 \end{pmatrix}$$

◯

5. Ordering distributions from different equivalence classes

If X and Y have different median responses, yet they are in the same equivalence class, we shall need to ensure that inequality indices will give $\Delta(X, c) = \Delta(Y, c)$ for all $c \in C$. The question we shall attend to here is how to order a given representative of $\kappa_E(X)$ and one of $\kappa_E(Y)$ when the equivalence classes $\kappa_E(X)$ and $\kappa_E(Y)$ are distinct.

To pave the way for deriving our partial ordering over equivalence classes, we return to $AXIOMS$ 1 and 2 of Section 3, where we substitute the definition 4.1 of $\equiv_E$ for the general equivalence relation $\equiv_R$. This first entails:

Definition 5.1 *Let X and Y be elements of Λ with respective equivalence classes $\kappa_E(X)$ and $\kappa_E(Y)$. Then $\kappa_E(X) \prec_E \kappa_E(Y)$ iff there exists $Y^* \in \kappa_E(Y)$ such that $X \prec_{AF} Y^*$.*

This is one of several equivalent definitions of $\kappa_E(X) \prec_E \kappa_E(Y)$:

Proposition 5.2 *Let X and Y be two cumulative distributions with equivalence classes $\kappa_E(X)$ and $\kappa_E(Y)$ respectively. Then the following three statements are equivalent:*

(i) $\kappa_E(X) \prec_E \kappa_E(Y)$.

(ii) There exist $\hat{U} \in \kappa_E(X)$ *and* $\hat{V} \in \kappa_E(Y)$ *such that* $\hat{U} \prec_{AF} \hat{V}$.

(iii) For all $U \in \kappa_E(X)$ *there exists* $V \in \kappa_E(Y)$ *such that* $U \prec_{AF} V$.

In words, when $\kappa_E(X)$ is more egalitarian than $\kappa_E(Y)$ it is the case that there is a distribution $\hat{U}$ which is equivalent to X and a distribution $\hat{V}$ which is equivalent to Y, such that $\hat{U}$ and $\hat{V}$ have identical medians, and furthermore $\hat{U} \prec_{AF} \hat{V}$. In turn, (iii) above ensures that every distribution U of $\kappa_E(X)$ will

be comparable to some distribution V in the equivalence class of Y such that $U \prec_{AF} V$. We illustrate the above result by means of the following example:

Example 4 Recall the distribution $F = \begin{pmatrix} 0.7 \\ 0.8 \\ 1 \end{pmatrix}$ of Example 3 with equivalence class $\kappa_E(F) = \{F, G_1, G_2, H\}$. Now consider a distribution $X = \begin{pmatrix} 0.8 \\ 0.9 \\ 1 \end{pmatrix}$, with equivalence class

$$\kappa_E(X) = \left\{ \overset{\cdot}{X}, B_1, B_2, D \right\}$$

such that

$$D = \begin{pmatrix} 0.1 \\ 0.2 \\ 1 \end{pmatrix}, B_1 = \begin{pmatrix} 0.2 \\ 0.9 \\ 1 \end{pmatrix}, B_2 = \begin{pmatrix} 0.1 \\ 0.8 \\ 1 \end{pmatrix}$$

We may observe that $X \prec_{AF} F$, $B_1 \prec_{AF} G_1$, $B_2 \prec_{AF} G_2$ and $D \prec_{AF} H$. In other terms, $\kappa_E(X) \prec_E \kappa_E(F)$ $\bigcirc$

We next substitute $\equiv_E$ for the general equivalence relation $\equiv_R$ in Proposition 3.1 to state the following result:

Proposition 5.3 *The relation $\prec_E$ defined over the set of equivalence classes Λ/E is a partial ordering.*

Ideally, in order to make the above methodology easily applicable one needs a simple empirical criterion for testing that either of the three conditions of Proposition 5.2 is satisfied. We begin by observing from Lemma $A1$ of the Appendix that when $X \prec_{AF} Y$ we have that $\left|X - \tilde{\Pi}\right| \geqslant \left|Y - \tilde{\Pi}\right|$, and furthermore, if $med(X) = med(Y)$ and $\left|X - \tilde{\Pi}\right| \geqslant \left|Y - \tilde{\Pi}\right|$, this entails $X \prec_{AF} Y$.

In the general case where X and Y do not necessarily have identical median states, we obtain the following simple criterion for ordering equivalence classes:

Proposition 5.4 *Let X and Y be any two distributions in Λ. Then the equivalence class $\kappa_E(X)$ is more egalitarian than the equivalence class $\kappa_E(Y)$ if and only if there exists $M \in \mathbb{M}$ such that $\left|X - \tilde{\Pi}\right| \geq M \left|Y - \tilde{\Pi}\right|$.*

Note that in the criterion of Lemma A1-*ii* where X and Y have identical medians, the permutation matrix M of the above Proposition is simply the identity matrix.

The result of Proposition 5.3 also entails that there exist least, and most, egalitarian equivalence classes:

Corollary 5.5 *For any distribution X with equivalence class $\kappa_E(X)$, and for any $\hat{\Pi} \in \mathbb{P}$, we have $\kappa\left(\hat{\Pi}\right) \prec_E \kappa_E(X)$ and $\kappa_E(X) \prec_E \kappa_E\left(\tilde{\Pi}\right)$*

The above result may be used to normalize inequality indices to take on a minimal value of 0, and if required a maximal value of 1.

6. Median independent inequality measures

We now return to the general case where Λ is the set of cumulative distributions defined over $n \geq 3$ states, and we open this section with a discussion of a problem that is inherent to the analysis of ordered response data. The problem is the following: consider ranking two distributions X and Y using an inequality index $\Delta(.,c)$, in relation to some scale c^1 and then again using another scale c^2, where $c^1, c^2 \in C$. Then, unless suitable restrictions are placed on the inequality index $\Delta(.,c)$, the ordering of the two distributions could fail to remain invariant to changes in the measurement scale. Accordingly, we restrict our discussion below to indices which are order invariant to the choice of scale:

- $ORDER-SCALINV$: $\Delta(X,c^1) \leq \Delta\left(Y,c^1\right) \Leftrightarrow \Delta\left(X,c^2\right) \leq \Delta(Y,c^2)$ for all $X, Y \in \Lambda$ and all $c^1, c^2 \in C$ (order invariance to scale)

Referring to Blackorby et al. (1978; Theorem 3.2 a) and Abul Naga and Yalcin (2008), $\Delta(X,c)$ is order invariant to scale if and only if Δ is strictly separable in X :

$$\Delta(X,c) = \Gamma[\phi(X),c] \tag{6.1}$$

where $\phi : \Lambda \to \mathbb{R}$, $\Gamma : \mathbb{R} \times C \to \mathbb{R}_+$ and Γ is increasing in ϕ. From here on therefore we assume that the inequality index is order invariant to scale, and we turn our attention to two key properties the inequality index must satisfy in relation to the ordering $(\Lambda/E, \prec_E)$. Firstly, as $\Delta(\ ,\ )$ is a numerical representation of the partial ordering over equivalence classes, we must first ensure that if the equivalence class of X is more egalitarian than the equivalence class of Y, there is less inequality under X than Y :

- $EQUAL : \kappa_E(X) \prec_E \kappa_E(Y) \Rightarrow \Delta(X,c) \leq \Delta(Y,c)$ for all $c \in C$ (inequality aversion)

Next, we must ensure that if X and Y are equivalent distributions, the level of inequality is identical under these two distributions:

- $EQUIV:\ X \equiv_E Y \Rightarrow \Delta(X,c) = \Delta(Y,c) \quad \text{for all } c \in C$ (equivalence)

To examine $EQUIV$ consider the set of functions $\phi : \Lambda \to \mathbb{R}_+$ which are constant on the equivalence class of X:

$$\Phi_E = \{\phi : \Lambda \to \mathbb{R}_+ : X \equiv_E Y \Rightarrow \phi(X) = \phi(Y)\} \tag{6.2a}$$

Φ_E is the set of functions which are symmetric in $|x_1 - 0.5|, ..., |x_n - 0.5|$.

Example 5 Let $\alpha, \beta, k > 0$ be constants and let $c \in C$. Consider the following functions that map Λ to $\mathbb{R}_+$:

$$\phi_1(X) = \sum_{i=1}^{n} |x_i - 0.5|^{\alpha} + k \tag{6.3}$$

$$\phi_2(X) = k \prod_{i=1}^{n-1} |x_i - 0.5|^{\alpha} \tag{6.4}$$

$$\phi_3(X) = \min_{i=1,...,n} \{|x_i - 0.5|\} \tag{6.5}$$

$$\phi_4(X) = \max_{i=1,...,n} \{|x_i - 0.5|\} \tag{6.6}$$

$$\phi_5(X) = \sum_{i=m}^{n} (x_i - x_{i-1})c_i - \sum_{i=2}^{m-1} (x_i - x_{i-1})c_i - x_1 c_1 \tag{6.7}$$

$$\phi_6(X) = \sum_{i=1}^{m-1} x_i^{\alpha} - \sum_{i=m}^{n} x_i^{\beta} + k \tag{6.8}$$

Then $\phi_1, \phi_2, \phi_3, \phi_4$ are elements of the set Φ_E while ϕ_5, ϕ_6 do not belong to the set Φ_E as they are not symmetric in the arguments $|x_1 - 0.5|, ..., |x_n - 0.5|$. ◯

The lemma below, adapted from MacLane and Birkhoff (1999, Theorem 19; p.35) allows us to conclude that $\phi(.)$ will satisfy the axiom $EQUIV$ if and only if $\phi(X) = \gamma[\kappa_E(X)]$, where $\gamma(.)$ maps the equivalence class of X to $\mathbb{R}_+$:

Lemma 6.1 *Given the equivalence relation* $\equiv_E$ *on the set* Λ*, let* $\phi : \Lambda \to \mathbb{R}_+$ *be any member of the set of functions* Φ_E*. Then there is exactly one function* $\gamma : \Lambda/E \to \mathbb{R}_+$ *for which* $\phi(X) = \gamma[\kappa_E(X)]$

Put more simply, the function $\phi(.)$ may be seen as a composition of a first mapping which projects X to its equivalence class [11], and a second mapping $\gamma(.)$ from Λ/E to $\mathbb{R}_+$.

Recall from Proposition 5.4 that $\kappa_E(X) \prec \kappa_E(Y)$ if and only if $\left|X - \tilde{\Pi}\right| \geq M\left|Y - \tilde{\Pi}\right|$ for some $M \in \mathbb{M}$. Clearly then, if the function $\phi : \Lambda \to \mathbb{R}_+$ were to satisfy both *EQUIV* and *EQUAL*, $\phi(X)$ would be required to be symmetric *and* decreasing in $|x_1 - 0.5|, ..., |x_n - 0.5|$. We summarize this discussion with the following Proposition:

Proposition 6.2 $\Delta(X, c)$ *satisfies the axioms* $ORDER-SCALINV, EQUIV$ *and* $EQUAL$ *if and only if there exist functions* $\Gamma : ran(\phi) \times C \to \mathbb{R}$ *and* $\phi : \Lambda \to \mathbb{R}_+$ *such that*

$$\Delta(X, c) = \Gamma[\phi(|x_1 - 0.5|, ..., |x_n - 0.5|), c] \tag{6.9}$$

where $\phi(\cdot)$ *is symmetric and decreasing in* $|x_1 - 0.5|, ..., |x_n - 0.5|$ *and* $\Gamma(\cdot)$ *is increasing in* $\phi(\cdot)$ [12].

This result enables us quite simply to construct new families of median independent inequality indices. It is conceivable, for example, to construct a new measure based on the function $\phi_2(X)$ of Example 5,

$$\Delta(\Pi, c) = 1 - (0.5)^{-(n-1)\alpha} \prod_{i=1}^{n-1} |x_i - 0.5|^{\alpha} \qquad \alpha > 0 \tag{6.10}$$

where the function is normalized to take on values between 0 and 1. Alternatively, any member of the set Φ_E can be used for this purpose, and we have collected several of its elements in Example 5.

The result also allows us to identify which of the existing inequality indices for ordered response data satisfy the new ordering relation $(\Lambda/E, \prec_E)$ of this paper. Firstly, we note that all of the indices presented in Table 2 were constructed for the purpose of preserving the ordering relation $(\Lambda, \prec_{AF})$, that requires a common median response when comparing a given pair of distributions. Recall from

[11] This first mapping is also known as the *natural projection* of X to $\kappa_E(X)$.

[12] For a given function $\phi(.)$, $ran(\phi)$ denotes the range of ϕ.

Example 5, that the functions ϕ_5 and ϕ_6 do not belong to the set Φ_E as they are not symmetric in their arguments $|x_i - 0.5|$. The Allison and Foster (2004) measure Δ_{AF} is based on the function $\phi_5(.)$, and therefore does not preserve the ordering relation introduced in this paper. Likewise, the alphabeta family of Abul Naga and Yalcin (2008) is based on the function $\phi_6(.)$ of Example 5, and accordingly does not satisfy the ordering relation $\prec_E$. Unlike the above two indices, the Apouey (2007) measure and the Absolute value measure of Abul Naga and Yalcin (2008), based on the function ϕ_1 of Example 5, are symmetric functions of $|x_1 - 0.5|, ..., |x_n - 0.5|$. Using the results of Lemma 6.1 and Proposition 6.2, we may conclude that these indices preserve the ordering relation $\prec_E$ that we have introduced in this paper.

Also, the result of Proposition 6.2 reveals that the class of inequality indices that are order-preserving for this new relation is quite large and awaits further exploration.

7. Concluding comments

Because the Lorenz ordering and related income inequality measures are not suitable for the analysis of ordered response data, researchers have adopted a median preserving spread ordering, and constructed related inequality measures for the analysis of such data. The median preserving spread concept however is a relation restricted to the comparison of distributions that exhibit common median responses. We have thus studied in Section 3 a general pre-ordering and equivalence relation that together extend the median preserving relation of Allison and Foster (2004) to the comparison of cumulative distributions on the basis of their equivalence classes, and irrespective of their medians.

Clearly, the set $\mathbb{P}$ of egalitarian distributions ought to constitute one such equivalence class. The starting point of Section 4 was to formalize the similarity between members of this set. This was provided to us by selecting a reference distribution–the most polarized distribution $\tilde{\Pi}$ was chosen– and by observing that all members of $\mathbb{P}$ were at an equal distance from this reference distribution. By then extending comparisons to neighboring distributions of $\mathbb{P}$, we have generalized the equivalence relation $(\mathbb{P}, \sim)$ to obtain our relation $(\Lambda, \equiv_E)$.

In Section 5 we have obtained several equivalent definitions of the resulting partial ordering $(\Lambda/E, \prec_E)$ over equivalence classes, thus allowing us to compare distributions independently from their medians. In the specific case where two distributions exhibit an identical median response, the methodology set out in this

paper ensures that the ordering $\prec_E$ coincides with the median preserving spread relation $\prec_{AF}$ of Allison and Foster (2004). Finally, in Section 6 we have obtained the required functional form inequality indices must satisfy to be order-preserving for the relation $\prec_E$. The result enables the derivation of new families of median independent inequality indices for ordered response data.

8. Appendix

The appendix gathers proofs of the main results stated in the paper. In the general case where $n \geq 3$, we define the most polarized distribution,

$$\tilde{\Pi} = [0.5, ...0.5, 1]' \tag{8.1}$$

the n egalitarian distributions,

$$\begin{aligned} \hat{\Pi}_1 &= [1, ..., 1]' && (8.2)\\ \hat{\Pi}_2 &= [0, 1, ..., 1]' && (8.3)\\ &\vdots \\ \hat{\Pi}_n &= [0, ..., 0, 1]' && (8.4) \end{aligned}$$

and $\mathbb{P} = \{\hat{\Pi}_1, ..., \hat{\Pi}_n\}$ is the set of egalitarian distributions.

Let $i, m, l \in \mathbb{N}$. We define the index sets

$$\begin{aligned} \mathbb{N}_{ml} &= \{i \in \mathbb{N} : m \leq i < l\} && (8.5)\\ \mathbb{N}_{lm} &= \{i \in \mathbb{N} : l \leq i < m\} && (8.6) \end{aligned}$$

If A and B are sets, we define $A \oplus B$ as the set that gathers all the elements of A and B. If for instance, $A = \{a, b\}$ and $B = \{b, c)$, then $A \oplus B = \{a, b, b, c\}$.

We begin by stating and proving four preliminary lemmas. In what follows let $P(X) = [p_1(X), ..., p_n(X)]'$ be a vector of n functions, where $p_n(X) = 1$ and for $i = 1, ..., n-1$ $p_i(X) : \Lambda \to [0, 1]$. The notation below is standard in the literature on the theory of majorization (cf. Marshall and Olkin, 1979): we let

$$0 \leq p_{(1)}(X) \leq p_{(2)}(X) \leq ... \leq p_{(n)}(X) = 1 \tag{8.7}$$

be the values of the n functions ordered in ascending fashion. Also define

$$P_{\uparrow} = [p_{(1)}(X), p_{(2)}(X), ..., p_{(n)}(X)]' \tag{8.8}$$

and observe that given the definition of $P(X)$, the resulting vector $P_{\uparrow}$ is an element of Λ.

Lemma A1 *Let X and Y be any two elements of Λ. Then:*
(i) $X \prec_{AF} Y \Longrightarrow \left|X - \tilde{\Pi}\right| \geq \left|Y - \tilde{\Pi}\right|$
(ii) $med(X) = med(Y)$ *and* $\left|X - \tilde{\Pi}\right| \geq \left|Y - \tilde{\Pi}\right|$ *entail* $X \prec_{AF} Y$.

Proof *(i)* If $X \prec_{AF} Y$ and m denotes the median state of both distributions, this entails

$$x_i \leq y_i \qquad i = 1, ..., m-1 \tag{a1}$$
$$x_i \geq y_i \qquad i = m, ..., n \tag{a2}$$

Thus (a1) entails for all $i = 1, ..., m-1$ that $|x_i - 0.5| \geq |y_i - 0.5|$, while (a2) also entails for all $i = m, ..., n$ that $|x_i - 0.5| \geq |y_i - 0.5|$. That is, $\left|X - \tilde{\Pi}\right| \geq \left|Y - \tilde{\Pi}\right|$. The converse is generally incorrect. For example, if $n = 3$ and $X = \begin{pmatrix} 0.8 \\ 0.8 \\ 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 0.3 \\ 0.8 \\ 1 \end{pmatrix}$, then X is not comparable to Y yet $\left|X - \tilde{\Pi}\right| \geq \left|Y - \tilde{\Pi}\right|$.

(ii) If $med(X) = med(Y) = m$, then $\left|X - \tilde{\Pi}\right| \geq \left|Y - \tilde{\Pi}\right|$ entails (a1) and (a2). Thus $X \prec_{AF} Y$. $\square$

Lemma A2 *For all $m = 1, ..., n$ and $l = 1, ..., n$ and for all $X \in \Lambda_m$, there exists at least one distribution $Y \in \Lambda_l$ such that $X \equiv_E Y$.*

Proof Observe first that $\tilde{\Pi}$ lies at the intersection of all subsets of Λ. The reflexive property of the equivalence relation $\equiv_E$ then ensures that the result holds immediately for $\tilde{\Pi}$. For a given $X \neq \tilde{\Pi}$, where $X \in \Lambda_m$, we shall therefore consider three cases depending on whether l is smaller, greater, or equal to m.

Consider first the case $l < m$. Let $p_i(X) = 1 - x_i$ if $l \leq i < m$, and $p_i(X) = x_i$ otherwise. Then, since $|1 - x_i - 0.5| = |x_i - 0.5|$, we have that $|P(X) - \tilde{\Pi}| = |X - \tilde{\Pi}|$, Also, we have by construction $P(X)_{\uparrow} \in \Lambda$. Since $P(X)_{\uparrow}$ is a permutation of the vector $P(X)$, we have $|P(X)_{\uparrow} - \tilde{\Pi}| = M|X - \tilde{\Pi}|$ for some $M \in \mathbb{M}$. Hence it follows that $P(X)_{\uparrow} \equiv_E X$. Note furthermore that for $i \geq l$, $p_i(X) \geq 0.5$. Hence $P(X)_{\uparrow}$ has $l - 1$ elements taking values smaller than 0.5. and $med(P(X)_{\uparrow}) = l$. Hence for $Y = P(X)_{\uparrow}$, we have that $Y \equiv_E X$, where $Y \in \Lambda_l$.

Next, consider the case $l > m$. Let $p_i(X) = 1 - x_i$ if $m \leq i < l$, and $p_i(X) = x_i$ otherwise. Repeating the steps related to the case $l < m$ above, we find $P(X)_\uparrow \in \Lambda_l$, and $P(X)_\uparrow = M|X - \tilde{\Pi}|$ for some $M \in \mathbb{M}$. Hence, for $Y = P(X)_\uparrow$, we have that $Y \equiv_E X$, where $Y \in \Lambda_l$.

Finally, when $l = m$, the reflexive property of the equivalence relation entails $X \equiv_E X$. $\square$

Lemma A3 *Let U and V be any two elements of Λ_m, with $U \prec_{AF} V$.*

For the case $l < m$ and for $X = U, V$ define the vector $P(X)$ with elements

$$p_i(X) = 1 - x_i \qquad i \in \mathbb{N}_{lm} \tag{8.9}$$
$$p_i(X) = x_i \qquad i \in \{1, ..., n\} \backslash \mathbb{N}_{lm} \tag{8.10}$$

For the case $l > m$ and for $X = U, V$ define the vector $P(X)$ with elements

$$p_i(X) = 1 - x_i \qquad i \in \mathbb{N}_{ml} \tag{8.11}$$
$$p_i(X) = x_i \qquad i \in \{1, ..., n\} \backslash \mathbb{N}_{ml} \tag{8.12}$$

Then, defining $U^* = P(U)$ and $V^* = P(V)$ we have:
(i) $U^*_\uparrow, V^*_\uparrow \in \Lambda_l$
(ii) $U^*_\uparrow \prec_{AF} V^*_\uparrow$,
(iii) $U^*_\uparrow \in \kappa_E(U)$ and $V^*_\uparrow \in \kappa_E(V)$

Proof Consider first the case $l < m$. For (i), the proof is as in the proof of Lemma *A*2 above. Next consider *(ii)*. Since U and V are elements of Λ_m, with $U \prec_{AF} V$, we have $u_i \leq v_i$ for $i = 1, ..., m-1$, while $u_i \geq v_i$ for $i = m, ..., n$. Accordingly this entails

$$1 - u_i \geq 1 - v_i \qquad l \leq i < m \tag{8.13}$$
$$u_i \geq v_i \qquad i = m, ..., n \tag{8.14}$$

Construct the sets $S_1 = \{1 - u_i : l \leq i < m\}$, $T_1 = \{1 - v_i : l \leq i < m\}$, $S_2 = \{u_i : i = m, .., n\}$ and $T_2 = \{v_i : i = m, .., n\}$. Finally define $S = S_1 \oplus S_2$ and $T = T_1 \oplus T_2$. Observe that each element of T can be assigned a distinct upper bound in the set S, so that no two elements of T are attributed the same upper bound.

Let $V^*_\uparrow = P(V)$. Then, $V^*_\uparrow = [v^*_{(1)}, ..., v^*_{(n)}]'$. Define $u^*_i = u_i$ for $i < l$. Next define $u^*_l \in S$ as the least upper bound (*L.U.B*) of $v^*_{(l)}$ from the set S. Likewise, define u^*_{l+1} as the *L.U.B* of $v^*_{(l+1)}$ in the set $S \backslash \{u^*_l\}$. Since $v^*_{(l+1)} \geq v^*_{(l)}$, it follows

that $u^*_{l+1} \geq u^*_l$. Likewise, we may define u^*_{l+2} as the *L.U.B* of $v^*_{(l+2)}$ in the set $S\backslash\{u^*_l, u^*_{l+1}\}$, ..., and u^*_{n-1} as the *L.U.B* of $v^*_{(n-1)}$ in the set $S\backslash\{u^*_l, ...u^*_{n-2}\}$. Defining $U^*_\uparrow = [u_1, ..., u_{l-1}, u^*_l, ..., u^*_{n-1}, 1]'$, this construction gives us that $U^*_\uparrow \prec_{AF} V^*_\uparrow$ where $U^*_\uparrow, V^*_\uparrow \in \Lambda_l$.

Now assume to the contrary that for some j such that $l \leq j \leq n$, $v^*_{(j)}$ does not have a least upper bound in the set $S\backslash\{u^*_l, ...u^*_{j-1}\}$. Without loss of generality, assume $j = n$. Then it must be that one of the inequalities [8.13–8.14] above was violated, or equivalently that $U \not\prec_{AF} V$. This establishes (ii).

For (iii) use Lemma A2 to conclude that for $M_1, M_2 \in \mathbb{M}$ we have $\left|U^* - \tilde{\Pi}\right| = M_1 \left|U - \tilde{\Pi}\right|$ and $\left|V^* - \tilde{\Pi}\right| = M_2 \left|V - \tilde{\Pi}\right|$, so that $U^* \equiv_E U$ and $V^* \equiv_E V$.

The proof is similar in the context where $l \geq m$. $\square$

Lemma A4 *Let M_i, M_j be any two elements of $\mathbb{M}$ and let I denote the identity matrix. Then we have:*

(i) $M_i \in \mathbb{M} \Longleftrightarrow M_i^{-1} \in \mathbb{M}$ for all $M_i \in \mathbb{M}$
(ii) $M_i M_j \in \mathbb{M}$ and $M_j M_i \in \mathbb{M}$ for all M_i and $M_j \in \mathbb{M}$
(iii) $I \in \mathbb{M}$

Proof This follows from the result that the set of permutation matrices forms a group under multiplication (Maclane and Birkhoff, 1999, ch. 2) and that $\mathbb{M}$ is a subgroup of this set.

Proof of Proposition 4.2 Let $F, G, H \in \Lambda$. Then:
(i) *REFLEX* $\left|F - \tilde{\Pi}\right| = I \left|F - \tilde{\Pi}\right|$. Thus $F \equiv_E F$.

(ii) *SYM* : let $\left|F - \tilde{\Pi}\right| = M \left|G - \tilde{\Pi}\right|$ where $M \in \mathbb{M}$. Then since from Lemma A4 $M^{-1} \in \mathbb{M}$ we have $M^{-1} \left|F - \tilde{\Pi}\right| = M^{-1} M \left|G - \tilde{\Pi}\right| = \left|G - \tilde{\Pi}\right|$. Thus $F \equiv_E G \Longleftrightarrow G \equiv_E F$

(iii) *TRANSI*: Let $\left|F - \tilde{\Pi}\right| = M_1 \left|G - \tilde{\Pi}\right|$ and $\left|G - \tilde{\Pi}\right| = M_2 \left|H - \tilde{\Pi}\right|$, where M_1 and M_2 are elements of $\mathbb{M}$. Then $\left|F - \tilde{\Pi}\right| = M_1 M_2 \left|H - \tilde{\Pi}\right|$. Lemma A4 entails that $M_1 M_2 \in \mathbb{M}$ so that $F \equiv_E G$ and $G \equiv_E H \Longrightarrow F \equiv_E H$. $\square$

Proof of Proposition 5.2 We shall prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$ Let $\kappa_E(X) \prec_E \kappa_E(Y)$. Then, from Definition 5.1, there exists $Y^* \in \kappa_E(Y)$ such that $X \prec_{AF} Y^*$. Setting $\hat{U} = X$ and $\hat{V} = Y^*$, we have the required result.

$(ii) \Rightarrow (iii)$ Let $\hat{U} \in \kappa_E(X)$ and $\hat{V} \in \kappa_E(Y)$ such that $\hat{U} \prec_{AF} \hat{V}$. By assumption, $\hat{U}$ and $\hat{V}$ belong to the same subset Λ_j of Λ. From Lemma $A2$, $\hat{U}$ has an equivalent distribution U, and $\hat{V}$ has an equivalent distribution V, in each Λ_l. We can construct U and V as indicated in Lemma $A3$, by setting $U = U^*_{\uparrow}$ and $V = V^*_{\uparrow}$ such that $U \in \kappa_E(X),\ V \in \kappa_E(Y)$ and $U \prec_{AF} V$.

$(iii) \Rightarrow (i)$ We have $\kappa_E(X) = \kappa_E(U) \prec_E \kappa_E(V) = \kappa_E(Y)$. Thus $\kappa_E(X) \prec_E \kappa_E(Y)$ as required. $\square$

Proof of Proposition 5.3 This follows from Proposition 3.1 since $\equiv_E$ is an equivalence relation.

Proof of Proposition 5.4 $(\Rightarrow)$ $\kappa_E(X) \prec_E \kappa_E(Y)$ entails, from Proposition 5.2, that there exists $X^* \in \kappa_E(X)$ with $X^* \prec_{AF} Y$. From Lemma $A1$, it follows that $\left|X^* - \tilde{\Pi}\right| \geq \left|Y - \tilde{\Pi}\right|$, i.e. $\left|X - \tilde{\Pi}\right| = M\left|X^* - \tilde{\Pi}\right| \geq M\left|Y - \tilde{\Pi}\right|$ for some $M \in \mathbb{M}$.

$(\Leftarrow)$ Assume that $\left|X - \tilde{\Pi}\right| \geq M\left|Y - \tilde{\Pi}\right|$. From Lemma A2 there exists $Y^* \in \kappa_E(Y)$ such that $med(Y^*) = med(X)$, and $\left|Y - \tilde{\Pi}\right| = M^*\left|Y^* - \tilde{\Pi}\right|$. In turn, we have $\left|X - \tilde{\Pi}\right| \geq MM^*\left|Y^* - \tilde{\Pi}\right|$. Definition 5.1 therefore entails $\kappa_E(X) \prec_E \kappa_E(Y^*) = \kappa_E(Y)$ as required. $\square$

Proof of Proposition 6.2 $(\Rightarrow)$ Let $\Delta(X, c)$ satisfy the scale invariance axiom $ORDER-SCALINV$. Then Theorem 3.2a of Blackorby et al. (1978) entails that the inequality index is of the form $\Delta(X, c) = \Gamma[\phi(X), c]$ where Γ is increasing in $\phi(.)$. Together $ORDER - SCALINV$ and $EQUIV$ entail that $\Delta(X, c) = \Gamma[\phi(X), c]$ where $\phi(X)$ is a symmetric function of $|x_1 - 0.5|, ..., |x_n - 0.5|$. Taken together the three axioms entail that $\phi(X)$ is symmetric and decreasing in each of $|x_1 - 0.5|, ..., |x_n - 0.5|$.

$(\Leftarrow)$ Conversely, let $\Delta(\Pi, c)$ be of the form (6.1) where $\phi(X)$ is a symmetric and decreasing function of $|x_1 - 0.5|, ..., |x_n - 0.5|$. Then it can be verified straightforwardly that $\Delta(\Pi, c)$ satisfies the three axioms $ORDER-SCALINV$, $EQUIV$ and $EQUAL$. $\square$

9. References

Abul Naga R. and T. Yalcin (2008): "Inequality Measurement for Ordered Response Health Data" *Journal of Health Economics* 27, 1614-1625.

Allison R. A. and Foster J. (2004): "Measuring Health Inequalities Using Qualitative Data", *Journal of Health Economics* 23, 505-524.

Alesina A., R. Di Tella and R. MacCulloch (2004): " Inequality and Happiness: are Europeans and Americans Different?" *Journal of Public Economics* 88, 2009-2042.

Apouey B. (2007): "Measuring Health Polarization with Self-Assessed Health Data", *Health Economics* 16, 875-894.

Atkinson A. (1970): "On the Measurement of Inequality", *Journal of Economic Theory* 2, 244–263.

Blackorby C., D. Primont and R. Russell (1978), *Duality, Separability and Functional Structure: Theory and Economic Applications*, New York, North-Holland.

Cowell F. (2000): "Measurement of Inequality", in Atkinson A. and F. Bourguignon eds: *A Handbook of Income Distribution*, Elsevier.

Harzheim E. (2005): *Ordered Sets*, Elsevier.

Jones A., N. Rice, S. Robone and P. Rosa Dias (2010): " Inequality and Polarization in Health Systems Responsiveness: a Cross-Country Analysis", Discussion Paper, Centre of Health Economics, University of York.

MacLane S. and G. Birkhoff (1999): *Algebra*, American Mathematical Society Chelsea Publishing.

Marshall A. and I. Olkin (1979): *Inequalities: Theory of Majorization and its Applications*, San Diego, Academic Press.

Stevenson B. and J. Wolfers (2008): "Happiness Inequality in the United States", IZA discussion paper 3624.

Shorrocks A. (1980): "The Class of Additively Decomposable Inequality Measures", *Econometrica* 48, 613-625.

Zheng B. (2006): "Measuring Health Inequality and Health Opportunity", discussion paper 06-06, Department of Economics, University of Denver.

Table 1: Respect and communication in health treatment: cumulative distributions for selected European countries

Country	Very bad	Bad	Moderate	Good	Very good	Median state
Austria	0.00	0.01	0.06	0.38	1	Very good
Denmark	0.03	0.05	0.08	0.39	1	Very good
Spain	0.01	0.02	0.08	0.69	1	Good
France	0.03	0.05	0.14	0.52	1	Good

Source: Table 1 of Jones et al. (2010); Respondents in the World Health Survey were asked to rate their experience of the health system in relation to respectful treatment and communication.

Table 2: Some inequality measures for ordered response data

Measure	Definition	Median independent
Allison and Foster (2004)	$\Delta_{AF} = \sum_{i=m}^{n}(x_i - x_{i-1})c_i - \sum_{i=2}^{m-1}(x_i - x_{i-1})c_i - x_1 c_1$	No
Apouey (2007)	$\Delta_{AP} = 1 - \frac{2^{\alpha}}{n-1}\sum_{i=1}^{n-1}\lvert x_i - 0.5\rvert^{\alpha}$ where $0 \le \alpha \le 1$	Yes
Abul Naga and Yalcin (2008)	$\Delta_{AV} = 1 - \frac{2\sum_{i=1}^{n}\lvert x_i - 0.5\rvert - 1}{n-1}$	Yes
Abul Naga and Yalcin (2008)	$\Delta_{\alpha,\beta} = \frac{\sum_{i<m} x_i^{\alpha} - \sum_{i\ge m} x_i^{\beta} + (n+1-m)}{k_{\alpha,\beta} + (n+1-m)}$ where $\alpha, \beta \ge 1$	No

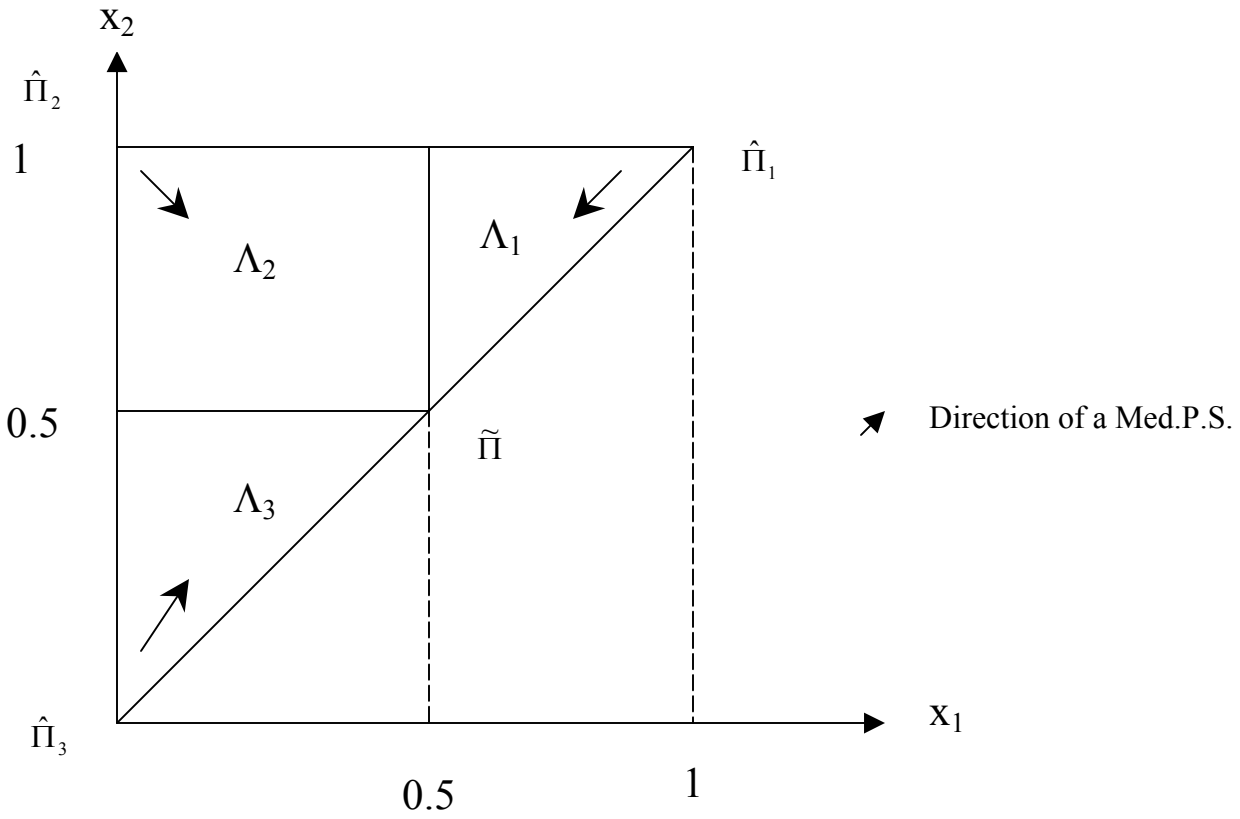

Figure 1 : Order properties of the set of cumulative distributions