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FIBRE TILINGS

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For the sixtieth birthday of Peter M. Gruber

Abstract. Generalizing an earlier notion of secondary polytopes, Billera and Sturmfels introduced the important concept of fibre polytopes, and showed how they were related to certain kinds of subdivision induced by the projection of one polytope onto another. There are two obvious ways in which this concept can be extended: first, to possibly unbounded polyhedra, and second, by making the definition a categorical one. In the course of these investigations, it became clear that the whole subject fitted even more naturally into the context of finite tilings which admit strong duals. In turn, this new approach provides more unified and perspicuous explanations of many previously known but apparently quite disparate results.

§1. *Introduction.* In two papers [**4, 5**], Billera and Sturmfels generalized the notion of secondary polytopes introduced in [**6**] to that of fibre polytopes. They demonstrated there how fibre polytopes are related to coherent subdivisions induced by the projection of one polytope onto another, and further showed what results from the iteration of the fibre polytope construction.

There are two respects in which the original description is somewhat less than completely satisfactory. First, the definition of fibre polytope depended on a metrical setting. In consequence, there was no natural extension of the definition to unbounded polyhedra, although many of the concepts involved immediately generalize. One aim of the present paper is to provide a more categorical setting for fibre polytopes; Bernd Sturmfels himself (private communication) has said that he also thinks of fibre polytopes in such a way.

Connexions have also been noticed between fibre polytopes and techniques such as Gale diagrams; see [**3, 4**]. In the latter context, particularly in the author's representation theory of **[12]**, there is present a symmetry in the formulation which is lacking in the general theory. This paper will restore the symmetry, in showing that the proper setting is that of a certain class of finite tilings, namely those admitting strong duals (we distinguish "strong" from the slightly weaker "orthogonal"). These are analogues of infinite tilings discussed in **[16]**; indeed, there are many parallels between that paper and this, and we frequently use the same terms for corresponding concepts, even when the exact definitions are occasionally a little different. It turns out that there are fibre tilings which reflect many of the properties of fibre polytopes; unfortunately, their definition is not even as categorical as that of fibre polyhedra.

Nevertheless, in this more general setting, many earlier results can be seen in a new light. As just one striking example, we have a very short explanation of a description (due, independently, to Schneider and Sturmfels—see Example 8.2) of certain decompositions of a sum of polytopes into sums of their faces. As another, we generalize the notion of mixed polytopes described by Schneider in **[23]** to mixed tilings (we are grateful for an early sight of this latter paper). We also present new results about zonotopes, and reformulate old ones about hyperplane arrangements.

§2. *Polyhedra and complexes.* Our setting will be finite-dimensional vector spaces over a fixed ordered field $\mathbb{F}$; any topological references will refer to the order topology induced by $\mathbb{F}$. The reader will lose no generality in thinking of this field as the real numbers $\mathbb{R}$. For the general background to polyhedra and polytopes, good references are **[8, 26]** (we often quote results and follow definitions from these without specific mention).

The *dual* space $\mathbb{X}^*$ of such a vector space $\mathbb{X}$ consists of the linear functionals on $\mathbb{X}$; the result of applying $u \in \mathbb{X}^*$ to $x \in \mathbb{X}$ is written $\langle x, u\rangle = \langle u, x\rangle$. (The notation emphasizes the symmetry between $\mathbb{X}$ and $\mathbb{X}^*$, namely $\mathbb{X}^{**} (= (\mathbb{X}^*)^*) = \mathbb{X}$.

A *(closed) half-space* of $\mathbb{X}$ is a set of the form

$$H^-(u, \beta) := \{x \in \mathbb{X} \mid \langle x, u\rangle \leqslant \beta\}$$

for some $u \in \mathbb{X}^* \setminus \{o\}$ and $\beta \in \mathbb{F}$. A *polyhedron* is a non-empty intersection of finitely many half-spaces of $\mathbb{X}$. We denote by $\mathcal{Q}(\mathbb{X})$ the family of polyhedra in $\mathbb{X}$.

As usual, the *face* of $P \in \mathcal{Q}(\mathbb{X})$ in direction $u \in \mathbb{X}^*$ is

$$F(P, u) := \{x \in P \mid \langle x, u\rangle = \eta(P, u)\},$$

with

$$\eta(P, u) := \max\{\langle x, u\rangle \mid x \in P\}$$

the *support functional* of P in direction u, whenever this is finite. (As a general rule, we allow such functions to take values in $\overline{\mathbb{F}} := \mathbb{F} \cup \{\pm\infty\}$, so that we can always write "max" in this context; note that $\eta(P, u)$ is attained if it is finite.) We denote by $\mathcal{F}(P)$ the family of non-empty faces of a polyhedron P (including

$P = F(P, o)$ itself), so that $\mathcal{F}(P) \subseteq \mathcal{Q}(\mathbb{X})$; we often also write $F \leqslant P$ instead of $F \in \mathcal{F}(P)$.

By a *(polyhedral) complex* in $\mathbb{X}$, we mean a non-empty finite family $\mathcal{B} \subseteq \mathcal{Q}(\mathbb{X})$ of polyhedra, such that, if $G \in \mathcal{B}$, then $\mathcal{F}(G) \subseteq \mathcal{B}$, and, if $F, G \in \mathcal{B}$, then $F \cap G \in \mathcal{F}(G) \cup \{\emptyset\}$. (Observe that our definition differs slightly from the usual one: in our usage, we find it convenient to insist that all members of a complex be non-empty, and so we exclude $\emptyset$.) The *body* of $\mathcal{B}$ is $|\mathcal{B}| := \bigcup \mathcal{B}$ (often called—confusingly, in this context—the *polyhedron* of $\mathcal{B}$), which is the underlying point-set of the complex $\mathcal{B}$. An important example of a complex is $\mathcal{F}(P)$, with $P \in \mathcal{Q}(\mathbb{X})$.

If $\mathcal{B}$ and $\mathcal{C}$ are complexes such that $|\mathcal{B}| \cap |\mathcal{C}| \neq \emptyset$, then we write

$$\mathcal{B} \wedge \mathcal{C} := \{F \cap G \mid F \in \mathcal{B} \text{ and } G \in \mathcal{C}\} \setminus \{\emptyset\}$$

for the *meet* of $\mathcal{B}$ and $\mathcal{C}$. (As before, we always wish to exclude the empty set.) We can regard a non-empty affine subspace A as a polyhedral complex, identifying it with $\{A\}$; in this sense, when $\mathcal{B}$ is a complex such that $\mathcal{B} \wedge A \neq \emptyset$ (so that $|\mathcal{B}| \cap A \neq \emptyset$), then $\mathcal{B} \wedge A$ is a complex. If $\mathcal{C} = \mathcal{B} \wedge \mathcal{D}$ for some other complex $\mathcal{D}$, then we write $\mathcal{B} \sqsubseteq \mathcal{C}$.

If $\mathcal{B}$ and $\mathcal{C}$ satisfy $|\mathcal{B}| = |\mathcal{C}|$, then $\mathcal{B} \sqsubseteq \mathcal{C}$ means that $\mathcal{C}$ *refines* $\mathcal{B}$, or is a *refinement* of $\mathcal{B}$, in the usual sense; in other words,

$$\mathcal{B} \subseteq \{|\mathcal{E}| \mid \mathcal{E} \subseteq \mathcal{C}\},$$

so that each member of $\mathcal{B}$ is a union of members of $\mathcal{C}$. Similarly, if $|\mathcal{B}| = |\mathcal{C}|$, then $\mathcal{B} \wedge \mathcal{C}$ is the *common refinement* of $\mathcal{B}$ and $\mathcal{C}$. However, we often loosely refer to (common) refinements when the underlying bodies $|\mathcal{B}|$ and $|\mathcal{C}|$ do not coincide, even though, strictly speaking, this is inaccurate.

Polarity plays an important rôle in our discussions. We call a polyhedron $C \in \mathcal{Q}(\mathbb{X})$ a *(polyhedral) cone* if $\lambda x \in C$ for each $x \in C$ and $\lambda \geqslant 0$; that is, C coincides with its *positive hull* pos C, namely the set of all non-negative linear combinations of points of C. We denote by $\mathcal{C}(\mathbb{X})$ the family of all cones in $\mathbb{X}$. The *polar* of $C \in \mathcal{C}(\mathbb{X})$ is

$$C^* := \{u \in \mathbb{X}^* \mid \langle x, u \rangle \leqslant 0 \text{ for all } x \in C\} \in \mathcal{C}(\mathbb{X}^*).$$

A well-known fact (which we shall not prove here) is that $C^{**}(= (C^*)^*) = C$. Something we often appeal to is the following.

PROPOSITION 2.1. *There is an inclusion-reversing correspondence $F \longleftrightarrow \widehat{F}$ between $\mathcal{F}(C)$ and $\mathcal{F}(C^*)$.*

Proof. The result is familiar, but we give a brief proof. We first make the fairly obvious remark that, if $C, D \in \mathcal{C}$, then

$$(C + D)^* = C^* \cap D^*. \tag{2.1}$$

(We shall use this in its dual form $(C \cap D)^* = C^* + D^*$ as well.) Next, for $S \subseteq \mathbb{V}$, we write

$$S^{\perp} := \{u \in \mathbb{V}^* \mid \langle x, u \rangle = 0 \text{ for all } x \in S\},$$

so that

$$S^{\perp\perp}(= (S^{\perp})^{\perp}) = \operatorname{lin} S.$$

Now let $F \leqslant C$. The corresponding face of C^* is

$$\widehat{F} := \{u \in C^* \mid \langle x, u \rangle = 0 \text{ for all } x \in F\} = C^* \cap F^{\perp}.$$

It follows that

$$\widehat{F}^{\perp} = (C^* \cap F^{\perp})^{\perp} = (C^*)^{\perp} + F^{\perp\perp} = \operatorname{lineal} C + \operatorname{lin} F = \operatorname{lin} F,$$

where lineal C is the *lineality space* of C, which (in this case) consists of the face of apices of C. We deduce that

$$\widehat{\widehat{F}} = C^{**} \cap \widehat{F}^{\perp} = C \cap \operatorname{lin} F = F,$$

as required. □

REMARK 2.2. Note also that

$$\operatorname{lin} \widehat{F} = \widehat{F}^{\perp\perp} = (\operatorname{lin} F)^{\perp} = F^{\perp},$$

so that the foregoing description is symmetric between F and $\widehat{F}$.

If $F \leqslant P$, then the set

$$N(F, P) := \{u \in \mathbb{X}^* \mid F \subseteq F(P, u)\} \in \mathcal{C}(\mathbb{X}^*)$$

is called the *normal cone* to P at F. In fact, if we write

$$A(F, P) := \operatorname{pos}(P - F)$$

for the *angle cone* of P at F (this definition differs a little from our usual one, but in the context of this paper it is more convenient), then

$$N(F, P) = A(F, P)^*. \tag{2.2}$$

The family $\mathcal{N}(P)$ of normal cones to P forms its *normal fan*; it is a complex.

Closely related to (2.2) is a basic result about polar cones. This was claimed in [**14**, p. 112], but the proof amounted to little more than an assertion. Because it so central, we prove the result properly here.

PROPOSITION 2.3. *Let* $C \in \mathcal{C}(\mathbb{X})$ *and* $C^* \in \mathcal{C}(\mathbb{X}^*)$ *be polar cones, and let* $F \leqslant G \leqslant C$. *Then*

$$A(F, G)^* = A(\widehat{G}, \widehat{F}).$$

Proof. First, we have

$$A(F, G) = \operatorname{pos}(G - F) = \operatorname{pos} G - \operatorname{pos} F = G - F = G + F - F = G + \operatorname{lin} F.$$

since $F \leqslant G$. Hence, using (2.1) and replacing G by C, we deduce that

$$A(F, C)^* = (C + \operatorname{lin} F)^* = C^* \cap F^{\perp} = \widehat{F},$$

as in the proof of Proposition 2.1. Again, because $F \leqslant G$, we have

$$A(F, G) = G - F = (C - F) \cap \operatorname{lin} G,$$

so that, because $\widehat{G} \leqslant \widehat{F}$, there follows

$$A(F, G)^* = (C - F)^* + G^{\perp} = \widehat{F} + \operatorname{lin} \widehat{G} = \widehat{F} + \widehat{G} - \widehat{G} = \widehat{F} - \widehat{G} = A(\widehat{G}, \widehat{F}),$$

as claimed. $\square$

We call $P, Q \in \mathcal{Q}(\mathbb{X})$ *isomorphic* if their face complexes $\mathcal{F}(P)$ and $\mathcal{F}(Q)$ are isomorphic as sets partially ordered by inclusion. In addition, we say that P and Q are *strongly isomorphic*, written $P \approx Q$, if this isomorphism is induced by parallelism of corresponding faces: $F(P, u) \longleftrightarrow F(Q, u)$ for each $u \in \mathbb{X}^*$. Something important to bear in mind, which follows from the definition, is

PROPOSITION 2.4. *Let $P, Q \in \mathcal{Q}(\mathbb{X})$. Then $P \approx Q$ if and only if $\mathcal{N}(P) = \mathcal{N}(Q)$.*

§3. *Linear mappings.* In preparation for later results, it is useful to look at the relationships induced between pairs of polyhedra and complexes by linear mappings. Most earlier literature discusses these relationships in terms of internal sections and projections. However, we wish to present the material in as categorical way as possible, and so our context will be that of linear mappings between (possibly) different vector spaces.

In this section, we can often work in greater generality than we need later. Throughout, let $\mathbb{X}, \mathbb{Y}$ be finite-dimensional vector spaces over $\mathbb{F}$, and let $\Theta : \mathbb{X} \to \mathbb{Y}$ be a linear mapping. Recall that the *dual* mapping $\Theta^* : \mathbb{Y}^* \to \mathbb{X}^*$ is such that $\langle x, v\Theta^* \rangle = \langle x\Theta, v \rangle$ for all $x \in \mathbb{X}$ and $v \in \mathbb{Y}^*$.

If $K \in \mathcal{Q}(\mathbb{Y})$, then, as usual,

$$K\Theta^{-1} := \{x \in \mathbb{X} \mid x\Theta \in K\}$$

is the inverse image of K under Θ. Similarly, if $K \in \mathcal{Q}(\mathbb{X})$, then

$$K\Theta := \{x\Theta \mid x \in K\}$$

is the image of K under Θ. We begin with a general form of a well-known result which relates polarity, sections and projections of cones. However, for completeness, and because it plays such a central rôle, we give a proof.

LEMMA 3.1. *Let $K \in \mathcal{C}(\mathbb{Y})$. Then*

$$K\Theta^{-1} = (K^*\Theta^*)^*. \tag{3.1}$$

Proof. Indeed, we have

$$\begin{aligned} x \in (K^*\Theta^*)^* &\Longleftrightarrow \langle x, u\rangle \leqslant 0 && \text{for all } u \in K^*\Theta^* \\ &\Longleftrightarrow \langle x, v\Theta^*\rangle \leqslant 0 && \text{for all } v \in K^* \\ &\Longleftrightarrow \langle x\Theta, v\rangle \leqslant 0 && \text{for all } v \in K^* \\ &\Longleftrightarrow x\Theta \in K^{**} = K \\ &\Longleftrightarrow x \in K\Theta^{-1}, \end{aligned}$$

as claimed. □

Note that (3.1) can also be expressed in the form

$$(K\Theta^{-1})^* = K^*\Theta^*.$$

In our first application of this lemma, let $K \in \mathcal{Q}(\mathbb{Y})$, and suppose that $P = K\Theta^{-1}$ is the inverse image of K under Θ. If $F \in \mathcal{F}(P)$, then there is a unique $J \in \mathcal{F}(K)$ such that $(\operatorname{relint} F)\Theta = \operatorname{relint}(F\Theta) \subseteq \operatorname{relint} J$; we call J the *carrier* of $F\Theta$, written $J = \operatorname{carr}(F\Theta, K)$. Further, write

$$\mathcal{N}(K;\Theta) := \{N(J,K) \in \mathcal{N}(K) \mid J = \operatorname{carr}(F\Theta, K) \text{ for some } F \in \mathcal{F}(P)\}. \quad (3.2)$$

Then we have

THEOREM 3.2. *With the notation above,*

$$\mathcal{N}(K\Theta^{-1}) = \mathcal{N}(K;\Theta)\Theta^*.$$

By the latter expression, we just mean the set of images under Θ^* of normal cones in $\mathcal{N}(K;\Theta)$.

If $\mathcal{K}$ is a complex in $\mathbb{Y}$ such that $|\mathcal{K}| \cap \operatorname{im}\Theta \neq \emptyset$, then we similarly write

$$\mathcal{K}\Theta^{-1} := \{G\Theta^{-1} \mid G \in \mathcal{K}\} \setminus \{\emptyset\}.$$

It is clear that $\mathcal{K}\Theta^{-1}$ is a complex in $\mathbb{X}$; further,

$$\mathcal{K}\Theta^{-1} = (\mathcal{K} \wedge \operatorname{im}\Theta)\Theta^{-1}.$$

For the image of a polyhedron under a linear mapping, we then have

THEOREM 3.3. *With the notation above, let $K \in \mathcal{Q}(\mathbb{X})$. Then*

$$\mathcal{N}(K\Theta) = \mathcal{N}(K)(\Theta^*)^{-1}.$$

For proofs of these theorems, we just apply Lemma 3.1 to the angle and normal cones of faces of K. See also, for example, [**26**].

Finally, let $\mathcal{K}$ be a complex in $\mathbb{X}$. There is no obvious sense in which $\mathcal{K}\Theta$ is a complex, although later on (for certain complexes of a special type) we shall give this a meaning. However, we do have the *common refinement complex*

$\bigwedge(\mathcal{K}\Theta)$, which is defined as follows. If $y \in |\mathcal{K}|\Theta$, let

$$G(y) := \bigcap\{F\Theta \mid F \in \mathcal{K} \text{ and } y \in F\Theta\};$$

then

$$\bigwedge(\mathcal{K}\Theta) := \{G(y) \mid y \in |\mathcal{K}|\Theta\}.$$

It is clear that $\bigwedge(\mathcal{K}\Theta)$ is indeed a complex in $\mathbb{Y}$, with $|\bigwedge(\mathcal{K}\Theta)| = |\mathcal{K}|\Theta$.

§4. *Fibre polyhedra.* We next introduce fibre polyhedra. In future, our setting will always be the following. We have a short exact sequence of vector spaces and linear mappings

$$\mathbb{O} \longrightarrow \mathbb{X} \xrightarrow{\Phi} \mathbb{V} \xrightarrow{\Psi} \mathbb{Y} \longrightarrow \mathbb{O};$$

recall that this means that Φ is injective, Ψ is surjective, and $\operatorname{im}\Phi = \ker\Psi$. We assume that neither Φ nor Ψ is trivial or an isomorphism. The sequence of dual spaces and mappings

$$\mathbb{O} \longleftarrow \mathbb{X}^* \xleftarrow{\Phi^*} \mathbb{V}^* \xleftarrow{\Psi^*} \mathbb{Y}^* \longleftarrow \mathbb{O}$$

is also short exact.

Our notation for fibre polyhedra will differ considerably from that of **[4]**, not least because we place the emphasis of the definition on the sectional aspect. For the moment, therefore, we ignore the purpose for which fibre polytopes were first introduced.

We need to recall some notation and terminology. The *vector* (or *Minkowski*) *sum* of polyhedra $P, Q \in \mathcal{Q}(\mathbb{X})$ is

$$P + Q := \{x + y \mid x \in P, y \in Q\};$$

P and Q are *summands* of $P + Q$. We should observe the following

LEMMA 4.1. *If $P, Q \in \mathcal{Q}(\mathbb{X})$, then*

$$\mathcal{N}(P + Q) = \mathcal{N}(P) \wedge \mathcal{N}(Q).$$

For a singleton set $\{t\}$, we write $P + t := P + \{t\}$, which is the *translate* of P by $t \in \mathbb{X}$. The *scalar multiple* of $P \in \mathcal{Q}(\mathbb{X})$ by $\lambda \in \mathbb{F}$ is

$$\lambda P := \{\lambda x \mid x \in P\}.$$

Here, we usually have $\lambda \geqslant 0$. We call $P, Q \in \mathcal{Q}(\mathbb{X})$ *homothetic* if $Q = \lambda P + t$ for some $\lambda > 0$ and translation vector $t \in \mathbb{X}$. Finally, we write $P \preceq Q$ if P is (homothetic to) a summand of Q. Note that $P \preceq Q \preceq P$ implies that $P \approx Q$.

Lemma 4.1 directly leads to

PROPOSITION 4.2. *If $P, Q \in \mathcal{Q}(\mathbb{X})$, then $P \preceq Q$ if an only if $\mathcal{N}(P) \sqsubseteq \mathcal{N}(Q)$.*

Let $\Phi: \mathbb{X} \to \mathbb{V}$ be an injective linear mapping, as above, and let $K \in \mathcal{Q}(\mathbb{V})$. If $b \in K$, then we call $(K - b)\Phi^{-1}$ a *section* of K under Φ. Alternatively, we refer to $(K - b)\Phi^{-1}$ as the *fibre over* $p := b\Psi$, with $\Psi: \mathbb{V} \to \mathbb{Y}$ the surjective linear mapping as above, and denote it by $K(p)$, even though $K(p)$ is now only determined up to translation. A *fibre polyhedron* of K with respect to Φ is then a polyhedron $P \in \mathcal{Q}(\mathbb{X})$ such that $K(p) \preceq P$ for each $p \in K\Psi$, and, if $P' \in \mathcal{Q}(\mathbb{X})$ also satisfies this property, then $P \preceq P'$. As the definition makes clear, a fibre polyhedron is better thought of as a strong isomorphism class rather than as a single polyhedron. We write $\mathrm{Fib}(K; \Phi)$ for the fibre polyhedron, bearing this qualification in mind. Observe also that the definition makes $\mathrm{Fib}(K; \Phi)$ *universal* for the property that each section of K by Φ is homothetic to one of its summands.

We can construct a fibre polyhedron $\mathrm{Fib}(K; \Phi)$ as follows. The different fibres $K(p)$ fall into finitely many strong isomorphism classes; indeed, if $G \in \bigwedge(\mathcal{F}(K)\Psi)$ and $p, p' \in \operatorname{relint} G$, then $K(p) \approx K(p')$. We then define $\mathrm{Fib}(K; \Phi)$ to be the sum of one representative from each of these strong isomorphism classes. Observe that, if $G, G' \in \bigwedge(\mathcal{F}(K)\Psi)$ are such that $G \leqslant G'$, and if $p \in \operatorname{relint} G, p' \in \operatorname{relint} G'$, then $K(p) \preceq K(p')$; thus we may confine the sum to representatives derived from each top-dimensional cell G.

By Proposition 2.4, strongly isomorphic polyhedra have the same normal fans. The fact that $\mathrm{Fib}(K; \Phi)$ really defines a strong isomorphism class is underlined by the main result of the section.

THEOREM 4.3. *Let $\Phi: \mathbb{X} \to \mathbb{V}$ be injective, and let $K \in \mathcal{Q}(\mathbb{V})$. Then*

$$\mathcal{N}(\mathrm{Fib}(K; \Phi)) = \bigwedge(\mathcal{N}(K)\Phi^*).$$

Proof. This follows directly from Theorem 3.2. □

However, there are interesting consequences of Theorem 4.3.

COROLLARY 4.4. *Strongly isomorphic polyhedra have the same fibre polyhedra.*

At first sight, this may be a little surprising. For example, it is easy to see that, while a regular 3-dimensional cube has hexagonal central sections (perpendicular to a long diagonal), if we sufficiently elongate one of its edges (to form a rectangular box), then parallel sections can be pentagons at worst. Thus sections of one polyhedron need not be homothetic to summands of any parallel sections of some strongly isomorphic polyhedron.

Indeed, Corollary 4.4 can be strengthened to

THEOREM 4.5. *With the assumptions of Theorem 4.3, let $K, K' \in \mathcal{Q}(\mathbb{V})$ be such that $K \preceq K'$. Then*

$$\mathrm{Fib}(K;\,\Phi) \preceq \mathrm{Fib}(K';\,\Phi).$$

Proof. Since $\mathcal{N}(K) \sqsubseteq \mathcal{N}(K')$ by Proposition 4.2, we see at once that

$$\mathcal{N}(\mathrm{Fib}(K;\,\Phi)) = \bigwedge(\mathcal{N}(K)\Phi^*) \sqsubseteq \bigwedge(\mathcal{N}(K')\Phi^*) = \mathcal{N}(\mathrm{Fib}(K';\,\Phi)),$$

and now Proposition 4.2 again yields the result. □

We can write Theorem 4.5 in an alternative form.

COROLLARY 4.6. *With the assumptions of Theorem 4.3, let $K_1, K_2 \in \mathcal{Q}(\mathbb{V})$. Then*

$$\mathrm{Fib}(K_1;\,\Phi) + \mathrm{Fib}(K_2;\,\Phi) \preceq \mathrm{Fib}(K_1 + K_2;\,\Phi).$$

Proof. This follows directly from Theorem 4.5 and the fact that, if $P_1, P_2, Q \in \mathcal{Q}(\mathbb{X})$ are such that $P_j \preceq Q$ for $j = 1, 2$, then $P_1 + P_2 \preceq Q$. □

REMARK 4.7. Equality will not generally hold in Corollary 4.6, as easy examples show.

Billera & Sturmfels [5] introduced the concept of *iterated fibre polytopes*. There is a natural generalization to fibre polyhedra.

THEOREM 4.8. *Let $\Theta\colon \mathbb{X}_1 \to \mathbb{X}_2$ and $\Phi_2\colon \mathbb{X}_2 \to \mathbb{V}$ be injective linear mappings, let $\Phi_1 := \Theta\Phi_2$, and let $K \in \mathcal{Q}(\mathbb{V})$. Then*

$$\mathrm{Fib}(K;\,\Phi_1) \preceq \mathrm{Fib}(\mathrm{Fib}(K;\,\Phi_2);\,\Theta).$$

Proof. We just look at the normal fans. We have

$$\begin{aligned}
\mathcal{N}(\mathrm{Fib}(\mathrm{Fib}(K;\,\Phi_2);\,\Theta)) &= \bigwedge(\mathcal{N}(\mathrm{Fib}(K;\,\Phi_2))\Theta^*)\\
&= \bigwedge\Big(\Big(\bigwedge(\mathcal{N}(K)\Phi^*{}_2)\Big)\Theta^*\Big)\\
&\sqsupseteq \bigwedge(\mathcal{N}(K)\Phi_2^*\Theta^*)\\
&= \mathcal{N}(\mathrm{Fib}(K;\,\Phi_1)),
\end{aligned}$$

as claimed, since $\Phi_2^*\Theta^* = (\Theta\Phi_2)^* = \Phi_1^*$. □

Let us illustrate this last result by means of an example.

EXAMPLE 4.9. The truncated octahedron is a fibre polytope of a 7-simplex T. To see this, note that suitably chosen normals to the facets of T can be projected to the 8 vertices of a 3-cube C, and the common refinement introduces the 6 normals to the facets of C as further facet normals of the fibre polytope. The fibre polygon of the truncated octahedron in a general direction will therefore be a 14-gon. However, a fibre polygon of T itself can be at worst an octagon.

REMARK 4.10. Were the definition of fibre polyhedron truly categorical, then we would have *functorial* behaviour under linear mappings, meaning that equality would prevail in Theorem 4.8.

§5. *Coherent subdivisions.* Fibre polytopes were devised in order to explain certain features of subdivisions of polytopes. We give the appropriate background, suitably generalized.

Let $Q \in \mathcal{Q}(\mathbb{Y})$. A polyhedral complex $\mathcal{C}$ is a *subdivision* of Q if $|\mathcal{C}| = Q$, with $|\mathcal{C}|$ as before the underlying point-set of $\mathcal{C}$. We are interested solely in the case where $Q = K\Psi$, with $K \in \mathcal{Q}(\mathbb{V})$ and $\Psi\colon \mathbb{V} \to \mathbb{Y}$ a surjective linear mapping. We then say that the subdivision $\mathcal{C}$ of Q is Ψ*-induced* if $\mathcal{C} = \mathcal{E}\Psi$ for some subfamily $\mathcal{E} \subseteq \mathcal{F}(K)$, in the sense that, if $F, G \in \mathcal{E}$ are such that $F\Psi \leqslant G\Psi$ in $\mathcal{C}$, then

$$F = (F\Psi)\Psi^{-1} \cap G.$$

In writing this, we demand that Ψ induce a one-to-one correspondence between $\mathcal{E}$ and $\mathcal{C}$; however, we do not require that Ψ be one-to-one on each face of $\mathcal{E}$.

The important concept here is that of Ψ-coherence. To explain this, we have to introduce the connexion with fibre polyhedra. So, we again have our short exact sequence

$$\mathbb{O} \longrightarrow \mathbb{X} \xrightarrow{\Phi} \mathbb{V} \xrightarrow{\Psi} \mathbb{Y} \longrightarrow \mathbb{O}$$

of spaces and mappings. Let $F \leqslant \operatorname{Fib}(K; \Phi)$. Then $F = F(\operatorname{Fib}(K; \Phi), q)$ for any point (normal vector) $q \in \operatorname{relint} N(F, \operatorname{Fib}(K; \Phi))$, where $N(F, \operatorname{Fib}(K; \Phi)) \in \bigwedge(\mathcal{N}(K)\Phi^*)$, and this gives rise to a face $F(p, q) := F(K(p), q)$ of each fibre $K(p)$ of K (or of each section $(K - b)\Phi^{-1}$ with $p = b\Psi$); note that this face is independent of the particular choice of q. In turn, writing

$$J(p, q) := \operatorname{carr}(F(p, q)\Phi + b, K)$$

for the corresponding carrier in K, we have

THEOREM 5.1. *The family* $\mathcal{E}(F) =:= \{J(p, q) \in \mathcal{F}(K) \mid p \in K\Psi\}$ *is a subfamily of* $\mathcal{F}(K)$ *such that* $\mathcal{C}(F) := \mathcal{E}(F)\Psi$ *is a* Ψ*-induced subdivision of* $K\Psi$.

We say that such a subdivision $\mathcal{C}(F) = \mathcal{C}(q)$ (the latter form of the dependence is often useful) is Ψ-*coherent*. An obvious consequence of the definition is

THEOREM 5.2. *There is a one-to-one correspondence $F \longleftrightarrow \mathcal{C}(F)$ between $\mathcal{F}(\mathrm{Fib}(K; \Phi))$ and Ψ-coherent subdivisions of $K\Psi$, such that $F \leqslant G$ if and only if $\mathcal{C}(F) \sqsupseteq \mathcal{C}(G)$.*

We shall discuss that part of the theorem concerning refinements in later sections. It must be emphasized that it is far from being the case that all induced subdivisions are coherent. For instance, every triangulation of a polytope is induced from a projection of a suitable simplex, but not every one is coherent. The investigation of triangulations in this light is a lively topic of research.

§6. *Tilings*. We now change tack, and introduce the main objects of study of this paper; our terminology will largely follow that of **[16]**, although the context there was somewhat different. A complex $\mathcal{K}$ in $\mathbb{V}$ whose underlying point-set $K := |\mathcal{K}|$ is convex is called a (*finite*) *tiling*. In other words, $\mathcal{K}$ is a subdivision of K, as we have previously defined the term; clearly, K is a polyhedron. The faces of $\mathcal{K}$ of maximal dimension are called its *tiles*.

We call the tiling $\mathcal{K}$ *proper* if $\mathrm{aff}\,|\mathcal{K}| = \mathbb{V}$, and each face of $\mathcal{K}$ is line-free (this need not hold of $|\mathcal{K}|$ itself). If $\mathcal{K}$ is *improper*, then it lies in a proper affine subspace of $\mathbb{V}$, or one of its tiles contains a line (and hence all of them do).

We call another tiling $\mathcal{K}^*$ in $\mathbb{V}^*$ *dual* to $\mathcal{K}$ if there is a one-to-one inclusion-reversing correspondence $F \longleftrightarrow \widehat{F}$ between $\mathcal{K}$ and $\mathcal{K}^*$. In addition, $\mathcal{K}^*$ is a *strong* dual of $\mathcal{K}$ if, for any faces $F, G \in \mathcal{K}$ such that $F \leqslant G$, the corresponding faces $\widehat{F}, \widehat{G} \in \mathcal{K}^*$ satisfy the condition of Proposition 2.3, namely

$$A(\widehat{G}, \widehat{F}) = A(F, G)^*.$$

The usual definition of the angle cone $A(F, G)$ has its face of apices spanned by $\mathrm{aff}\, F$ (rather than being translated to contain o); an alternative definition, which takes this into account (and bear in mind (2.2)), is

$$N(\widehat{G}, \widehat{F}) = N(F, G)^*,$$

which, if more general, is somewhat less natural. Observe that this definition of strong duality generalizes the more restrictive one of **[16]**.

It follows from the definition that, if we write

$$G_{\|} := \mathrm{lin}(G - G)$$

for the linear subspace parallel to G, then $G_{\|} \leqslant \mathbb{V}$ and $\widehat{G}_{\|} \leqslant \mathbb{V}^*$ are orthogonal complementary subspaces, so that

$$\widehat{G}_{\|} = (G_{\|})^{\perp} = \{v \in \mathbb{V}^* | \langle x, v \rangle = 0 \text{ for all } x \in G_{\|}\}.$$

This latter condition defines a slightly weaker condition, which we might refer to as *orthogonal* duality (although many authors use this to mean a condition which ultimately implies strong duality—see, for example, [1]). However, strong duality imposes an extra orientation condition: if T is a tile of a proper tiling $\mathcal{K}$ with facet S, and if $a^* = \widehat{T}$ and $E^* = \widehat{S}$ are the corresponding vertex and edge of $\mathcal{K}^*$, then E^* points from a^* in the direction of an outer normal to T at S.

To clarify this notion, we give a simple example, which nevertheless motivates much of what follows.

EXAMPLE 6.1. If $K \in \mathcal{Q}(\mathbb{V})$, then $\mathcal{F}(K)$ is a tiling—it has just one tile K: if K is full-dimensional and line-free, then $\mathcal{F}(K)$ is proper. It has a strong dual, namely its normal fan $\mathcal{N}(K)$. Note especially that strong duality of tilings differs from the usual duality of polytopes induced by polarity.

It is appropriate to extend some vocabulary to the new situation. We call two tilings $\mathcal{K}_1$ and $\mathcal{K}_2$ *strongly isomorphic*, written $\mathcal{K}_1 \approx \mathcal{K}_2$, if corresponding faces of each are themselves strongly isomorphic. In particular, *homothetic* tilings, of the form $\mathcal{K}' = \lambda\mathcal{K} + t$ with $\lambda > 0$ and $t \in \mathbb{V}$, are strongly isomorphic. We then have the obvious

LEMMA 6.2. *Strong duality preserves strong isomorphism classes of tilings, so that $\mathcal{K}_1 \approx \mathcal{K}_2$ implies that $\mathcal{K}_1^* \approx \mathcal{K}_2^*$.*

We now move on to sections and projections of tilings; these will be appropriate analogues of the corresponding concepts for polyhedra which we discussed in Section 3. The first notion is just what one would expect. Let $\mathcal{K}$ be a tiling in $\mathbb{V}$, and let $\Phi\colon \mathbb{X} \to \mathbb{V}$ be an injective linear mapping such that $\operatorname{im}\{\Phi \cap |\mathcal{K}| \neq \emptyset$. Then we define the *section tiling* $\mathcal{K}\Phi^{-1}$ by

$$\mathcal{K}\Phi^{-1} := \{J\Phi^{-1} \mid J \in \mathcal{K}\} \setminus \{\emptyset\}. \tag{6.1}$$

However, we need strong duality to make sense of the second, and this introduces a concept of coherently induced tiling.

THEOREM 6.3. *Let $\mathcal{K}$ be a tiling in $\mathbb{V}$ which has a strong dual. If the injective linear mapping $\Phi\colon \mathbb{X} \to \mathbb{V}$ is such that the section tiling $\mathcal{K}\Phi^{-1}$ exists, then it has a strong dual.*

Proof. Write $\mathcal{B} := \mathcal{K}\Phi^{-1}$, so that the faces of $\mathcal{B}$ are precisely the polyhedra $F = J\Phi^{-1}$, with $J \in \mathcal{K}$ such that $\operatorname{im}\Phi \cap \operatorname{relint} J \neq \emptyset$ (it should be clear that we can confine our attention to such J). Define $\widehat{F} := \widehat{J}\Phi^*$, with $\widehat{J}$ as usual the face of $\mathcal{K}^*$ corresponding to J, and write $\mathcal{B}^* := \{\widehat{F} \mid F \in \mathcal{B}\}$. We show that $\mathcal{B}^*$ is the required strong dual of $\mathcal{B}$.

So, let $F \leqslant G$ be faces of $\mathcal{B}$, and, as above, define $J \leqslant K$ to be the faces of $\mathcal{K}$ such that

$$(\operatorname{relint} F)\Phi \subseteq \operatorname{relint} J, \qquad (\operatorname{relint} G)\Phi \subseteq \operatorname{relint} K.$$

With $\widehat{F}$, and so on, denoting the corresponding dual faces, we have

$$A(F, G) = A(J, K)\Phi^{-1}, \qquad A(\widehat{G}, \widehat{F}) = A(\widehat{K}, \widehat{J})\Phi^*.$$

Since $A(\widehat{K}, \widehat{J}) = A(J, K)^*$ by assumption, the required result $A(\widehat{G}, \widehat{F}) = A(F, G)^*$ follows at once from Lemma 3.1. □

Observe that it is not really important in Theorem 6.3 that Φ be injective; if it is not, though, it is a little inappropriate to talk about sections.

Theorem 6.3 actually defines the concept of the *projection tiling* $\mathcal{K}^*\Phi^*$. Strictly speaking, and in analogy to the situation for polyhedra, perhaps we ought to refer to $\mathcal{K}^*\Phi^*$ as a Φ^*-*coherent* (*induced*) tiling—this just arises from strong dualization of the operation of taking a section.

We should observe that the alternative, and more direct, concept of projection tiling, namely the common refinement tiling $\bigwedge(\mathcal{K}\Psi)$, with $\Psi\colon \mathbb{V} \to \mathbb{Y}$ (say) a surjective linear mapping, can be defined for any finite tiling $\mathcal{K}$ in $\mathbb{V}$. We shall see later that this does play a rôle when $\mathcal{K}$ has a strong dual, but for the moment it is not very relevant.

There is a connexion between sections and projections which will prove useful below. For the moment, we merely phrase things in terms of linear mappings, and largely leave the geometric implications until later.

THEOREM 6.4. *In the diagram of vector spaces and linear mappings*

$$\begin{array}{ccccc}
\mathbb{O} & \longrightarrow & \mathbb{U} & \xrightarrow{\Phi} & \mathbb{V} \\
 & & \Psi\big\downarrow & & \Theta\big\downarrow \\
\mathbb{O} & \longrightarrow & \mathbb{X} & \xrightarrow{\Omega} & \mathbb{Y} \\
 & & \big\downarrow & & \big\downarrow \\
 & & \mathbb{O} & & \mathbb{O}
\end{array}$$

(*a*) $\mathbb{Y}, \Omega$ *and* Θ *can be reconstructed from* $\mathbb{U}, \mathbb{V}, \mathbb{X}, \Phi$ *and* Ψ;
(*b*) $\mathbb{U}, \Phi$ *and* Ψ *can be reconstructed from* $\mathbb{X}, \mathbb{Y}, \mathbb{V}, \Omega$ *and* Θ.

Proof. Of course, by this we mean that (for example) we can fill in the part of the diagram involving $\mathbb{Y}, \Omega$ and Θ, given the remaining data.

The result is well known in many contexts, but for completeness we give a proof. For (a), we define

$$\mathbb{Y} := \mathbb{V}/(\ker \Psi)\Phi,$$

and let $\Theta\colon \mathbb{V} \to \mathbb{Y}$ be the natural projection, so that $\ker \Theta = (\ker \Psi)\Phi$. We then define $\Omega : \mathbb{X} \to \mathbb{Y}$ by

$$\Omega := \Psi^{-1}\Phi\Theta.$$

Now

$$u \in \ker \Psi \Longleftrightarrow u\Phi \in (\ker \Psi)\Phi = \ker \Theta,$$

because Φ is one-to-one, and since Ψ is onto, this shows simultaneously that Ω is well defined and one-to-one.

For (b), we can dualize the diagram, appeal to (a), and then dualize again.

However, a direct proof is as easy. We define

$$\mathbb{U} := (\operatorname{im}\Omega)\Theta^{-1},$$

and $\Phi\colon \mathbb{U} \to \mathbb{V}$ to be the natural embedding. We then set

$$\Psi := \Phi\Theta\Omega^{-1}.$$

Again, the definitions immediately imply that Ψ is well defined and onto. □

What Theorem 6.4 says, loosely, is that the operations of taking sections and projections commute. This suggests an interesting question, although it is not immediately relevant to the present topic. A polytope with n facets can be expressed as a section of an $(n-1)$-simplex, while one with n vertices is a projection of an $(n-1)$-simplex. We therefore pose

QUESTION 6.5. Given a general polytope P, what is the smallest dimension $m := m(P)$ such that P can be obtained as a section of a projection (or a projection of a section) of an m-simplex?

One might suspect that $m = n - 1$ if P is simple with n facets (or simplicial with n vertices). Observe that $m(P^*) = m(P)$, if P^* is a polar dual of P; it is not immediately clear whether $m(P)$ depends only on the combinatorial type of P.

§7. *Liftable tilings.* We now move on to the concept of liftability of tilings. We shall define this in a way which is most immediately amenable to our present purposes. A *polyhedral function* $f\colon \mathbb{V} \to \overline{\mathbb{F}} := \mathbb{F} \cup \{\pm\infty\}$ is a convex function whose *epigraph*

$$\operatorname{epi} f := \{(x, \xi) \in \mathbb{V} \times \mathbb{F} \mid \xi \geqslant f(x)\}$$

is a polyhedron. Its (*effective*) *domain* is

$$\operatorname{dom} f := \{x \in \mathbb{V} \mid f(x) < \infty\}.$$

If f is *proper*, meaning that $\operatorname{dom} f \neq \emptyset$, and $f(x) \neq -\infty$ for any $x \in \mathbb{V}$, then the faces of $\operatorname{epi} f$ lying in the graph of f naturally form a polyhedral complex $\mathcal{G}(f)$, called the *graph complex* of f. We say that the tiling $\mathcal{K}$ is *liftable* if there exists a polyhedral function f, such that $\mathcal{K}$ is the image of the graph complex $\mathcal{G}(f)$ under the natural projection $\Pi\colon (x, \xi) \mapsto x$ of $\mathbb{V} \times \mathbb{F}$ on $\mathbb{V}$. (Alternatively, $\mathcal{K}$ is called a *regular* subdivision of $|\mathcal{K}|$.)

There is a central connexion between strong duality and liftability. In preparation for this, we recall one further concept, and introduce another. First, the *conjugate* f^* of a polyhedral function f is defined for $y \in \mathbb{V}^*$ by

$$f^*(y) = \max\{\langle x, y\rangle - f(x) \mid x \in \mathbb{V}\}.$$

(We can write "max" rather than the usual "sup" here, because f is polyhedral, and we allow $\pm\infty$ as values of functions.) Observe that $f^{**}(= (f^*)^*) = f$. Then

$f^*\colon \mathbb{V}^* \to \overline{\mathbb{F}}$ is also polyhedral. This follows from the geometric picture of conjugacy. Define

$$C := \operatorname{cl}\operatorname{pos}\{(x, \xi, -1) \in \mathbb{V} \times \mathbb{F}^2 \mid (x, \xi) \in \operatorname{epi} f\},$$
$$C^* := \operatorname{cl}\operatorname{pos}\{(y, -1, \eta) \in \mathbb{V}^* \times \mathbb{F}^2 \mid (y, \eta) \in \operatorname{epi} f^*\}.$$

(The closure operations are very mild, and amount to adjoining a copy (rec epi f) $\times\{0\}$ of the recession cone of the epigraph of f, and an analogous copy of rec epi f^*.) Then, as the notation suggests, C and C^* are indeed polar cones. (For further details about conjugacy, consult **[20]**, although this geometric picture—which actually applies to arbitrary closed convex functions—is absent there.)

The second idea is closely tied in with the geometric description of conjugacy. Let $\mathcal{K}$ be a tiling in $\mathbb{V}$. The *cone over* $\mathcal{K}$, denoted by cone $\mathcal{K}$, sits in $\mathbb{V} \times \mathbb{F}$, and is such that

$$\operatorname{cone}\mathcal{K} := \bigcup\{\mathcal{F}(\operatorname{cl}\operatorname{pos}(F \times \{-1\})) \mid F \in \mathbb{V}\}. \tag{7.1}$$

Our main result here is

THEOREM 7.1. *A tiling $\mathcal{K}$ in $\mathbb{V}$ has a strong dual if and only if it is liftable.*

Proof. A similar result can be found in **[16]**; however, a different approach better elucidates the geometry.

First, suppose that $\mathcal{K}$ is liftable, say to the graph complex $\mathcal{G}(f)$ of the polyhedral function f. Then the projection $\mathcal{K}^*$ of the graph complex $\mathcal{G}(f^*)$ of the conjugate function f^* is the required strong dual. To see this, let $C \subseteq \mathbb{V} \times \mathbb{F}^2$ be the cone derived as above from epi f, and let C^* be its polar. Then the polarity correspondence between $\mathcal{F}(C)$ and $\mathcal{F}(C^*)$ induces the required duality between $\mathcal{K}$ and $\mathcal{K}^*$. Indeed, the projection from the epigraph onto the tiling, and the injection of the epigraph into the cone, correspond to (an obvious) part of the diagram

$$\begin{array}{ccccc}
\mathbb{O} & \longrightarrow & \mathbb{V}\times\mathbb{F} & \longrightarrow & \mathbb{V}\times\mathbb{F}^2 \\
 & & \downarrow & & \downarrow \\
\mathbb{O} & \longrightarrow & \mathbb{V} & \longrightarrow & \mathbb{V}\times\mathbb{F} \\
 & & \downarrow & & \downarrow \\
 & & \mathbb{O} & & \mathbb{O}
\end{array}$$

Fitting the appropriate objects into this diagram gives

$$\begin{array}{ccc}
\mathcal{F}(\operatorname{epi} f) & \longrightarrow & \mathcal{F}(C) \\
\downarrow & & \downarrow \\
\mathcal{K} & \longrightarrow & \operatorname{cone}\mathcal{K}
\end{array}$$

(Strictly speaking, we should translate the two cones appropriately.) Dualizing the latter picture gives

$$\begin{array}{ccc}
\operatorname{cone}\mathcal{K}^* & \longleftarrow & \mathcal{F}(C^*) \\
\uparrow & & \uparrow \\
\mathcal{K}^* & \longleftarrow & \mathcal{F}(\operatorname{epi} f^*)
\end{array}$$

and appealing to Theorem 6.3 yields the claim. Note that cone $\mathcal{K}^* = \mathcal{N}(\operatorname{epi} f)$.

For the converse, let $\mathcal{K}^*$ be a strong dual of $\mathcal{K}$. For each tile $F \in \mathcal{K}$, we pick a point $a^* = a^*(F) \in$ relint $\widehat{F}$, with $\widehat{F} \in \mathcal{K}^*$ the face corresponding to F. Next, choose $f^*(a^*) \in \mathbb{F}$ (to be subject to subsequent conditions), and define $f\colon F \to \mathbb{F}$ by

$$f(x) := \langle x, a^* \rangle - f^*(a^*).$$

We are thus implicitly defining the conjugate f^* at the same time. The constants $f^*(a^*)$ must satisfy certain compatibility conditions, namely, if $F \leqslant G$ and $a^* \in$ relint $\widehat{F}$ and $b^* \in$ relint $\widehat{G}$ are the chosen points, then

$$\langle x, a^* \rangle - f^*(a^*) = \langle x, b^* \rangle - f^*(b^*)$$

for any $x \in F$. In fact, these compatibility conditions fix the $f^*(a^*)$ up to a constant. Indeed, the normal cone to epi f at its face

$$\widetilde{F} := \{(x, f(x)) \mid x \in F\}$$

is just $N(\widetilde{F}, \operatorname{epi} f) = \operatorname{cl} \operatorname{pos}(\widehat{F} \times \{-1\})$—the compatibility conditions ensure that these faces fit together correctly—and the existence of the normal cone implies the local, and hence global, convexity of f. (Of course, this fits in with what we had above.) The full definition of $f\colon \mathbb{V} \to \overline{\mathbb{F}}$ is thus

$$f(x) = \begin{cases} \langle x, a^*(F) \rangle - f^*(a^*(F)), & \text{if } x \in F \text{ for some } F \in \mathcal{K}, \\ \infty, & \text{otherwise,} \end{cases}$$

which gives the required polyhedral lifting function. □

Observe that $\mathcal{K}^*$ determines f up to a choice of the initial *height*, given by arbitrarily fixing some particular $f^*(a^*(F))$.

It is clear that strongly isomorphic epigraphs yield strongly isomorphic tilings. However, the converse is generally false; we only obtain strongly isomorphic epigraphs from strongly isomorphic tilings if we lift by means of the *same* strong dual.

We demonstrated in **[16]** various additional properties of the infinite analogues of liftable tilings. These properties carry over, and are basic to our investigations.

Let $\mathcal{K}$ be a liftable tiling in $\mathbb{V}$. If the injective linear mapping $\Phi\colon \mathbb{X} \to \mathbb{V}$ is such that $|\mathcal{K}| \cap \operatorname{im} \Phi \neq \emptyset$, then Φ induces the section tiling $\mathcal{B} = \mathcal{K}\Phi^{-1}$, which is such that $\mathcal{B}\Phi := \mathcal{K}$ wedge im Φ. If f is the polyhedral function which lifts $\mathcal{K}$ into the graph complex $\mathcal{G}(f)$, then the corresponding lifting function g of $\mathcal{B}$ is given by

$$g(x) := f(x\Phi).$$

Alternatively, if we define the injective linear mapping $\overline{\Phi}\colon \mathbb{X} \times \mathbb{F} \to \mathbb{V} \times \mathbb{F}$ by

$$(x, \xi)\overline{\Phi} := (x\Phi, \xi),$$

then epi $G = (\operatorname{epi} f)\overline{\Phi}^{-1}$. We thus have an alternative view of Theorem 6.3, namely,

LEMMA 7.2. *The section tiling of a liftable tiling $\mathcal{K}$ in $\mathbb{V}$ determined by an injective linear mapping $\Phi\colon \mathbb{X} \to \mathbb{V}$ is liftable.*

We need to take more care over liftability of projections. We want there to exist a tiling $\mathcal{C}$ in $\mathbb{Y}$ which is induced from the liftable tiling $\mathcal{K}$ in $\mathbb{V}$ by a surjective linear mapping $\Psi\colon \mathbb{V} \to \mathbb{Y}$. In analogy to sections, we expect that this will be induced by a corresponding projection of the epigraph epi f of a lifting function f. Now we know that there are many different liftings of $\mathcal{K}$; as we have seen, though, up to height these are completely determined by the choice of a strong dual $\mathcal{K}^*$. What we have to avoid is the following situation. Exactly as for Φ, the projection Ψ induces a corresponding surjective linear mapping $\overline{\Psi}\colon \mathbb{V} \times \mathbb{F} \to \mathbb{Y} \times \mathbb{F}$ by

$$(z, \zeta)\overline{\Psi} := (z\Psi, \zeta).$$

While f is proper, it need not be the case that (epi f) $\overline{\Psi}$ is itself the epigraph of a proper polyhedral function h. When it is, we write $\mathcal{C} = \mathcal{K}\Psi$ for the tiling in $\mathbb{Y}$ given by the graph complex $\mathcal{G}(h)$, which we call a *projection* tiling.

What will ensure this condition? Since (epi f) $\overline{\Psi}$ is always polyhedral, it is clear that, for h to be proper, we must have an affine function $\langle, w\rangle + \gamma$ on $\mathbb{Y}$, for some $w \in \mathbb{Y}^*$ and $\gamma \in \mathbb{F}$, such that $h \geqslant \langle\cdot, w\rangle + \gamma$. That is, for all $(y, \eta) \in$ epi h, we should have

$$\eta \geqslant \langle y, w\rangle + \gamma.$$

Pulling this back to $\mathbb{V}$, it says that, for all $(z, \zeta) \in$ epi f,

$$\zeta \geqslant \langle z\Psi, w\rangle + \gamma = \langle z, w\Psi^*\rangle + \gamma.$$

In other words, there must exist a $v := w\Psi^* \in$ im Ψ^* such that $f \geqslant \langle\cdot, v\rangle + \gamma$.

Now v satisfies this last condition exactly when $v \in$ dom $f^* = |\mathcal{K}^*|$, from the definition of the strong dual $\mathcal{K}^*$ derived from the conjugate function f^*. We conclude that we have shown

LEMMA 7.3. *A projection tiling $\mathcal{K}\Psi$ exists exactly when* im $\Psi^* \cap |\mathcal{K}^*| \neq \emptyset$, *in which case its lifting function h is given by*

$$h(y) = \min\{f(z) | z \in y\Psi^{-1}\}.$$

Indeed, we can verify this by conjugacy of the lifting functions. Let $g(x) = f(x\Phi)$ be the lifting function of $\mathcal{K}\Phi^{-1}$, as above. We can then show directly that the lifting function of $\mathcal{K}^*\Phi^*$ is

$$g^*(u) = \min\{f^*(v) | v \in u(\Phi^*)^{-1}\},$$

namely, that of $\mathcal{K}^*\Phi^*$.

The reader will have noticed that the condition which allows a section tiling for Φ is just the dual to that which allows a projection tiling for Ψ. Of course, Theorem 6.3 shows that this is no accident.

It is appropriate to end the section with a remark about the special case of Ψ-coherent subdivisions arising from a polyhedron K; we thus employ the notation of Section 5. Our normal vector $q \in \mathbb{X}^*$ (corresponding to $F \leqslant \mathrm{Fib}(K; \Phi)$) gives a hyperplane $H := q^\perp = \{x \in \mathbb{X} \mid \langle x, q \rangle = 0\}$ in $\mathbb{X}$, so that $H\Phi < \operatorname{im} \Phi = \ker \Psi$ has codimension 1 in $\ker \Psi$. We may then think of Ψ as a composition $\Psi = \Psi'\Pi$ of a linear mapping Ψ' with $\ker \Psi' = H\Phi$, and a projection Π with 1-dimensional kernel $L \leqslant \mathbb{W} := \mathbb{V}\Psi'$, say. As an outer normal, q determines a half-line L^+ of L, and $K\Psi' + L^+$ can then be thought of as the epigraph of a polyhedral function h on $\mathbb{Y}$. This function h is then exactly that which lifts the Ψ-coherent subdivision $\mathcal{C}(F)$ of $K\Psi$. It can be observed that $\Pi \colon \mathcal{G}(h) \to \mathcal{C}(F)$ is one-to-one on each face of $\mathcal{G}(h)$; it is Ψ' itself which has collapsed any faces of $\mathcal{K}(q)$ onto lower-dimensional faces of $\mathcal{G}(h)$.

§8. *Operations on tilings.* The liftability property of tilings which possess strong duals enables us to generalize to them various operations on polyhedra. Throughout, let $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2, \ldots$ denote such tilings in $\mathbb{V}$, and suppose that they lift to the graph complexes corresponding to the polyhedral functions $f, f_1, f_2, \ldots$.

We first recall various well-known operations on polyhedral functions. The meaning of the *left scalar multiple* λf (for $\lambda > 0$) and *sum* $f_1 + f_2$ is obvious. The conjugate operations are the *right scalar multiple* $f\lambda$ (again for $\lambda > 0$) given by

$$(f\lambda)(x) := \lambda f(\lambda^{-1}x),$$

and the *infimal convolute* $f_1 \square f_2$ given by

$$(f_1 \square f_2)(x) := \min\{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x\}.$$

What is happening here is more easily explained in terms of the epigraphs, since we have

$$\operatorname{epi}(f\lambda) = \lambda \operatorname{epi} f, \qquad \operatorname{epi}(f_1 \square f_2) = \operatorname{epi} f_1 + \operatorname{epi} f_2.$$

Recall that *affine* functions are those of the form $x \mapsto \langle x, a^* \rangle + \beta$, for some $a^* \in \mathbb{V}^*$ and $\beta \in \mathbb{F}$.

In the following table, we define operations on tilings which correspond to those on polyhedral functions.

Function	Tiling
λf	$\mathcal{K}$
$f_1 + f_2$	$\mathcal{K}_1 \wedge \mathcal{K}_2$
$f\lambda$	$\lambda\mathcal{K}$
$f_1 \square f_2$	$\mathcal{K}_1 + \mathcal{K}_2$

We need to make several remarks.

- The substitution $f \mapsto \lambda f + \langle\,, a^*\rangle + \beta$ does not change $\mathcal{K}$.
- The correspondence $f_1 + f_2 \longleftrightarrow \mathcal{K}_1 \wedge \mathcal{K}_2$ does not depend on the liftings of $\mathcal{K}_1$ and $\mathcal{K}_2$.
- Under conjugacy, $(\lambda f)^* = f^*\lambda$ and $(f_1 + f_2)^* = f_1^* \,\Box\, f_2^*$.
- In particular, $(\lambda f + \langle\,, a^*\rangle + \beta)^*(y) = (f^*\lambda)(y - a^*) - \beta$, giving $\lambda(\mathcal{K}^* + a^*)$.
- The *sum* $\mathcal{K}_1 + \mathcal{K}_2$ of $\mathcal{K}_1$ and $\mathcal{K}_2$ is defined by the table, and depends on the choice of liftings of $\mathcal{K}_1$ and $\mathcal{K}_2$.

The last fact is awkward. Notice, though, that the fourth point illustrates the fact that strong duals are not unique.

From the relationship between liftable tilings and polyhedral functions, together with the conjugacy relations above, we deduce the following duality relationship between meets and sums.

THEOREM 8.1. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be liftable tilings in $\mathbb{V}$. Then*

$$(\mathcal{K}_1 \wedge \mathcal{K}_2)^* = \mathcal{K}_1^* + \mathcal{K}_2^*,$$

if either side of the equation is defined.

Let us illustrate this latter idea, generalizing a result of Schneider [[**22**], Theorem 4.1] and Sturmfels (see [**10**]) for polytopes, of which the particular case $r = 2$ was proved earlier by Betke [**2**].

EXAMPLE 8.2. Let $P_1, \ldots, P_r$ be polyhedra in $\mathbb{V}$. If $v_1, \ldots, v_r \in \mathbb{V}^*$ are arbitrary vectors such that the complex

$$(\mathcal{N}(P_1) - v_1) \wedge \cdots \wedge (\mathcal{N}(P_r) - v_r) \neq \emptyset,$$

then it has a strong dual, whose underlying point-set is $P_1 + \cdots + P_r$. Indeed, the dual is just a particular case of $\mathcal{F}(P_1) + \cdots + \mathcal{F}(P_r)$, and so its tiles are sums of faces F_j of the P_j. Exactly which F_j occur can be read off from the relation

$$(\operatorname{relint} N(F_1, P_1) - v_1) \cap \cdots \cap (\operatorname{relint} N(F_r, P_r) - v_r) \neq \emptyset.$$

If the v_j are chosen in sufficiently general position, then these sums $F_1 + \cdots + F_r$ will be direct. (In [**10, 22**], there was an extra condition $v_1 + \cdots + v_r = o$; this arose because the decomposition was obtained as a coherent one from the projection of $P_1 \times \cdots \times P_r$ onto $P_1 + \cdots + P_r$.)

We can proceed to the limit $\lambda \searrow 0$ for the dilatate $\lambda\mathcal{K}$ (even if $\mathcal{K}$ is not liftable). What we obtain is the *recession tiling* rec $\mathcal{K}$ of $\mathcal{K}$, which is defined by

$$\operatorname{rec} \mathcal{K} := \{\operatorname{rec} F \mid F \in \mathcal{K}\}. \tag{8.1}$$

If we lift $\mathcal{K}$ by f, say, then the lifting function of rec $\mathcal{K}$ is just the *recession function* rec f of f, whose epigraph is

$$\operatorname{epi}(\operatorname{rec} f) := \operatorname{rec}(\operatorname{epi} f). \tag{8.2}$$

The strong dual of rec $\mathcal{K}$ is

$$(\operatorname{rec} \mathcal{K})^* = \mathcal{F}(|\mathcal{K}^*|),$$

as can be seen in several ways, one of which is that $(\operatorname{rec} f)^*$ is the (convex) indicator function of dom $f^* = |\mathcal{K}^*|$ (see [20]). As a further consequence. $|\operatorname{rec} \mathcal{K}|^* = |\operatorname{rec} \mathcal{K}^*|$

Observe that we may combine these various operations. For example, with fixed liftings of $\mathcal{K}_1, \ldots, \mathcal{K}_r$ and $\lambda_1, \ldots, \lambda_r \geqslant 0$, we can define

$$\lambda_1 \mathcal{K}_1 + \cdots + \lambda_r \mathcal{K}_r,$$

with $0\mathcal{K}_j$ interpreted as rec $\mathcal{K}_j$ if $\lambda_j = 0$. Note that, for $\lambda_1, \ldots, \lambda_r > 0$, all these tilings have the same strong dual $\mathcal{K}_1^* \wedge \cdots \wedge \mathcal{K}_r^*$. Furthermore, $\mathcal{K} + \operatorname{rec} \mathcal{K} = \mathcal{K}$ for each tiling $\mathcal{K}$.

Generalizing the corresponding notion for polyhedra, for two tilings $\mathcal{K}$ and $\mathcal{L}$ we write $\mathcal{K} \preceq \mathcal{L}$ if $\mathcal{K}$ is homothetic to a summand of $\mathcal{L}$. There follows directly from the definition a result closely connected to Theorem 8.1.

PROPOSITION 8.3. *If the (liftable) tilings $\mathcal{K}, \mathcal{L}$ satisfy $\mathcal{K} \preceq \mathcal{L}$, then strong duals $\mathcal{K}^*, \mathcal{L}^*$ can be chosen so that $\mathcal{K}^* \sqsubseteq \mathcal{L}^*$.*

Proof. Indeed, we can lift $\mathcal{K}, \mathcal{L}$ by the polyhedral functions f, g (say), such that epi $f \preceq$ epi g Then

$$\operatorname{cone} \mathcal{K}^* = \mathcal{N}(\operatorname{epi} f) \sqsubseteq \mathcal{N}(\operatorname{epi} g) = \operatorname{cone} \mathcal{L}^*,$$

which yields $\mathcal{K}^* \sqsubseteq \mathcal{L}^*$ immediately. □

It is clear that the image of a liftable tiling under an affinity (non-singular affine mapping) is also liftable. In fact, more is true. A *projective mapping* on $\mathbb{V}$ is one of the form

$$x \mapsto x\Phi := (\langle x, c\rangle + \delta)^{-1}(x\Phi + b),$$

where $\Psi: \mathbb{V} \to \mathbb{V}$ is a linear mapping, $b \in \mathbb{V}, c \in \mathbb{V}^*$ and $\delta \in \mathbb{F}$. This is naturally identifiable with the class of linear mappings $\widetilde{\Phi}: \mathbb{V} \times \mathbb{F} \to \mathbb{V} \times \mathbb{F}$. given by

$$(x, \eta)\widetilde{\Phi} := \lambda(x\Psi + \eta b, \langle x, c\rangle + \eta\delta),$$

where $\lambda \neq 0$. Then Φ is called a *projectivity* if $\widetilde{\Phi}$ is non-singular.

If $\mathcal{K}$ is a tiling in $\mathbb{V}$, we say that the projectivity Φ is *permissible* for $\mathcal{K}$ if $|\mathcal{K}| \cap H(c, -\delta) = \emptyset$ (the hyperplane $H(c, -\delta) := \{x \in \mathbb{V} \mid \langle x, c\rangle = -\delta\}$ is "sent to infinity" by Φ). However, a slight generalization of this condition is appropriate. We say that Φ is *weakly permissible* for $\mathcal{K}$ if

$$(\text{relint } |\mathcal{K}|) \cap H(c, -\delta) = \emptyset;$$

we then define

$$\mathcal{K}\Phi := \bigcup \{\mathcal{F}(\text{cl}(F\Phi)) | F \in \mathcal{K}\} \setminus \{\emptyset\}. \tag{8.3}$$

This definition allows some faces of $\mathcal{K}$ to be sent to infinity, while other faces of $\mathcal{K}\Phi$ come from infinity.

PROPOSITION 8.4. *If $\mathcal{K}$ is a liftable tiling in $\mathbb{V}$, and $\Phi: \mathbb{V} \to \mathbb{V}$ is a projectivity which is weakly permissible for $\mathcal{K}$, then $\mathcal{K}\Phi$ is a liftable tiling.*

Proof. To see this, just observe that, in the notation above (with $\lambda > 0$),

$$\mathcal{K}\Phi \times \{-1\} = (\text{cone } \mathcal{K})\widetilde{\Phi} \cap (\mathbb{V} \times \{-1\}).$$

As we noted above, (cone $\mathcal{K}$)$\widetilde{\Phi}$ is a liftable tiling, since cone $\mathcal{K}$ is. $\square$

We may also combine tilings in different spaces. Let $\mathcal{K}$ be a tiling in $\mathbb{V}$ and $\mathcal{L}$ a tiling in $\mathbb{W}$. Then the *product* tiling in $\mathbb{V} \times \mathbb{W}$ is

$$\mathcal{K} \times \mathcal{L} := \{F \times G \,|\, F \in \mathcal{K} \text{ and } G \in \mathcal{L}\}.$$

The following is clear.

THEOREM 8.5. *If $\mathcal{K}$ in $\mathbb{V}$ and $\mathcal{L}$ in $\mathbb{W}$ are liftable tilings, then $\mathcal{K} \times \mathcal{L}$ is liftable, with strong dual*

$$(\mathcal{K} \times \mathcal{L})^* = \mathcal{K}^* \times \mathcal{L}^*$$

in $\mathbb{V}^* \times \mathbb{W}^*$.

§9. *Fibre tilings.* We are now ready to describe the main concept which the paper has been aiming at. We recall that our overall framework is the short exact sequence of spaces and mappings

$$\mathbb{O} \longrightarrow \mathbb{X} \xrightarrow{\Phi} \mathbb{V} \xrightarrow{\Psi} \mathbb{Y} \longrightarrow \mathbb{O},$$

together with its dual sequence

$$\mathbb{O} \longleftarrow \mathbb{X}^* \xleftarrow{\Phi^*} \mathbb{V}^* \xleftarrow{\Psi^*} \mathbb{Y}^* \longleftarrow \mathbb{O}.$$

To fit in with the set-up of Section 4, we phrase our central definitions in the following way. Let $\mathcal{K}$ be a liftable tiling in $\mathbb{V}$, and fix a polyhedral lifting function f of $\mathcal{K}$, or alternatively a strong dual $\mathcal{K}^*$ of $\mathcal{K}$. It is convenient now to insist that $o \in |\mathcal{K}|$ (which is equivalent to $o \in \text{dom} f$), and (by adding to f a suitable affine function) that f be bounded below by some constant. The same is then true of the conjugate f^* of f, so that $o \in |\mathcal{K}^*|$ for the corresponding strong dual.

Define the lifting $\overline{\Phi}$ of Φ as in Section 7. It is clear that the fibre polyhedron $\mathrm{Fib}(\mathrm{epi} f; \overline{\Phi})$ is the epigraph of a polyhedral function g on $\mathbb{X}$; we define the *fibre tiling* $\mathrm{Fib}(\mathcal{K}; \Phi)$ to be that whose lifted graph complex is $\mathcal{G}(g)$. Of course, just as with fibre polyhedra, it is clear that the definition at best only yields a strong isomorphism class of tilings. In a moment, we shall see that this is exactly the case.

Before we do this, let us look at some properties of $\mathrm{Fib}(\mathcal{K}; \Phi)$, as we have just defined it. For each $b \in K := |\mathcal{K}|$, write $p := b\Psi \in \mathbb{Y}$. Define the section tiling $\mathcal{B}(p) := (\mathcal{K} - b)\Phi^{-1}$ in $\mathbb{X}$, exactly as in the proof of Theorem 6.3, with $\mathcal{K} - b$ replacing $\mathcal{K}$. In other words, using the notation of that proof, we define the corresponding lifting function $g(b, \cdot)$ by $g(b, x) := f(x\Phi + b)$, or

$$\mathrm{epi}\, g(b, \cdot) := (\mathrm{epi} f - (b, 0))\overline{\Phi}^{-1}.$$

Note that our definition is devised so that it actually gives a translation class of tilings $\mathcal{B}(p)$, since we identify those $b \in p\Psi^{-1}$. Because of the way we have yoked them together, these epigraphs $\mathrm{epi}\, g(b, \cdot)$ fall into finitely many strong isomorphism classes. Indeed, it is appropriate at this point to make the following remark.

LEMMA 9.1. *Let* $F, G \in \bigwedge(\mathcal{K}\Psi)$ *satisfy* $F \leq G$. *If* $p \in \mathrm{relint}\, F$ *and* $q \in \mathrm{relint}\, G$, *then* $\mathcal{B}(p) \preceq \mathcal{B}(q)$.

Proof. This is a direct consequence of the fact that, if $q = c\Psi$ (say), then the lifting functions $g(b, \cdot)$ and $g(c, \cdot)$ are such that $\mathrm{epi}\, g(b, \cdot) \preceq \mathrm{epi}\, g(c, \cdot)$. □

Compare also the alternative picture in Section 6 of the common refinement of the projection tilings arising from different choices of parallel sections of $\mathcal{K}^*$ by Ψ^*. Note, though, that $\mathcal{K}^*$ corresponds to the conjugate f^*, which depends on the choice of the lifting function f.

We can thus take $\mathcal{B}(p)$ to be such that $\mathcal{G}(g(b,))$ is the corresponding graph complex. The fibre tiling is now a sum of a representative from each strong isomorphism class of tilings $\mathcal{B}(p)$.

Let us list various properties of fibre tilings which follow from the definition.

THEOREM 9.2. *The fibre tiling* $\mathrm{Fib}(\mathcal{K}; \Phi)$ *is a liftable tiling in* $\mathbb{X}$. *Further, every section* $(\mathcal{K} - b)\Phi^{-1}$, *with* $b \in |\mathcal{K}|$, *is homothetic to a summand of* $\mathrm{Fib}(\mathcal{K}; \Phi)$.

What we have already done hints at the exact analogue of Theorem 4.3 for tilings.

THEOREM 9.3. *One choice of strong dual of the fibre tiling* $\mathrm{Fib}(\mathcal{K}; \Phi)$ *is*

$$\mathrm{Fib}(\mathcal{K}; \Phi)^* = \bigwedge(\mathcal{K}^*\Phi^*).$$

Proof. Indeed, the dual of the sum of sections is the corresponding common refinement of projections, which is just what the theorem claims.

Note that each projection $\widehat{G}\Phi^*$ of a face $\widehat{G} \in \mathcal{K}^*$ will appear; it arises from any section $(\mathcal{K} - b)\Phi^{-1}$ for which $(\operatorname{relint} G - b) \cap \operatorname{im}\Phi \neq \emptyset$. Whether this G appears is independent of the particular choice of b; it really depends on the position of $p \in \bigwedge(\mathcal{K}\Psi)$ rather than that of $b \in p\Psi^{-1}$. □

Theorem 9.3 makes a point which, perhaps, far from clearly follows from the definition. There are many different lifting functions f for the tiling $\mathcal{K}$. Even though different liftings lead to strongly isomorphic strong dual tilings $\mathcal{K}^*$, it is not the case that such tilings $\mathcal{K}^*$ project to strongly isomorphic common refinement tilings $\bigwedge(\mathcal{K}^*\Phi^*)$. This already demonstrates that, while the parallel with fibre polyhedra is very nice, something is seriously lacking. Further, Theorem 9.2 says that every section $(\mathcal{K} - b)\Phi^{-1}$, with $b \in |\mathcal{K}|$, is homothetic to a summand of $\operatorname{Fib}(\mathcal{K}; \Phi)$. However, there is no reason why we should not take different liftings of these sections for our representative sample of points b, and so form a quite different sum tiling. This will satisfy the properties of Theorem 9.2, but our given $\operatorname{Fib}(\mathcal{K}; \Phi)$ will not necessarily be homothetic to a summand of it. In other words, the definition of fibre tilings loses the universality property of that of fibre polyhedra.

Note, however, the *dual* picture:

$$\left(\bigwedge(\mathcal{K}\Psi)\right)^* = \operatorname{Fib}(\mathcal{K}^*; \Psi^*).$$

This says that any two fibre tilings $\operatorname{Fib}(\mathcal{K}^*; \Psi^*)$ are strongly isomorphic, no matter what initial choice is made of $\mathcal{K}^*$. But this is clear: the epigraphs of the corresponding lifting functions are strongly isomorphic, since these are all determined by $\mathcal{K}$.

Perhaps we can underline the reason why things behave better for fibre polyhedra, using the language we have developed above. The only tilings which are strongly isomorphic to a normal fan $\mathcal{N}(K)$ are its translates. Hence the common refinements of its projections are also translates, and so strongly isomorphic. Thus Theorem 9.3 yields a single strong isomorphism class of fibre "tilings", namely (the face complexes of) the fibre polyhedra.

Just as we had for fibre polyhedra, there is a close connexion between faces of the fibre tiling $\operatorname{Fib}(\mathcal{K}; \Phi)$ and Ψ-coherent tilings $\mathcal{K}\Psi$.

THEOREM 9.4. *There is a one-to-one correspondence $F \longleftrightarrow \mathcal{C}(F)$ between faces $F \in \operatorname{Fib}(\mathcal{K}; \Phi)$ and Ψ-coherent tilings $\mathcal{C}(F)$ of the form $\mathcal{K}\Psi$, such that $F \leq G$ if and only if $\mathcal{C}(F) \sqsupseteq \mathcal{C}(G)$.*

Proof. All we have to do is chase through the chain connecting F to $\mathcal{C}(F)$. First, we have the correspondence $F \longleftrightarrow \widehat{F}$ between $\operatorname{Fib}(\mathcal{K}; \Phi)$ and $\bigwedge(\mathcal{K}^*\Phi^*)$. In turn, a choice $q \in \operatorname{relint} \widehat{F}$ gives a section $\mathcal{K}^*(q) := (\mathcal{K}^* - c)(\Psi^*)^{-1}$ of $\mathcal{K}^*$ in $\mathbb{Y}^*$, with $c \in |\mathcal{K}^*|$ such that $c\Phi^* = q$. Finally, the strong dual $\mathcal{C}(F) = \mathcal{C}(q)$ of $\mathcal{K}^*(q)$ is the required Ψ-coherent projection. Lemma 9.1, in the dual form using $\widehat{F} \geq \widehat{G}$, yields the assertion about refinements. □

§10. *Iterated fibre tilings.* Theorem 4.8 describes how polyhedra behave under iteration of the fibre process. In Section 5, we associated the fibre polyhedron Fib(K; Φ) with Ψ-coherent subdivisions of $K\Psi$, where we continue to use the notation introduced in Section 4. It is therefore natural to ask about the relationship between corresponding iterated coherent subdivisions.

In fact, in view of Section 9, we look at everything in the context of liftable tilings, and regard those arising from iterated fibre polyhedra as a special case. We make the same assumptions about our strong dual tilings $\mathcal{K}$ in $\mathbb{V}$ and $\mathcal{K}^*$ in $\mathbb{V}^*$ as we did in Section 9.

We set up the situation in a diagram of spaces and mappings; the notation extends that of Theorem 4.8 in the obvious way.

$$\begin{tikzcd} & \mathbb{O} \arrow[d] & & \mathbb{O} \arrow[d] & & & \\ \mathbb{O} \arrow[r] & \mathbb{X}_1 \arrow[r, "\Phi_1"] \arrow[d, "\Theta"'] & & \mathbb{V} \arrow[r, "\Psi_1"] \arrow[d, "I"'] & \mathbb{Y}_1 \arrow[r] \arrow[d, "\Upsilon"'] & \mathbb{O} \\ \mathbb{O} \arrow[r] & \mathbb{X}_2 \arrow[r, "\Phi_2"] & & \mathbb{V} \arrow[r, "\Psi_2"] \arrow[d] & \mathbb{Y}_2 \arrow[r] \arrow[d] & \mathbb{O} \\ & & & \mathbb{O} & \mathbb{O} & \end{tikzcd} \tag{10.1}$$

In (10.1), each sequence is exact at intermediate spaces, and $I\colon \mathbb{V} \to \mathbb{V}$ is the identity map. The crucial thing to observe is that there is such a map Υ as shown, and that it is indeed surjective. This follows directly from the fact that $\ker \Psi_1 = \operatorname{im} \Phi_1 \leq \operatorname{im} \Phi_2 = \ker \Psi_2$.

For $j = 1, 2$, if we pick $q_j \in |\bigwedge(\mathcal{K}^*\Phi_j^*)|$, then we obtain a corresponding Ψ_j-coherent tiling $\mathcal{C}_j(q_j)$ in $\mathbb{Y}_j$. In view of $\Phi_1 = \Theta\Phi_2$, and hence $\Phi_1^* = \Phi_2^*\Theta^*$, the natural way to relate two such subdivisions is by

$$q_1 = q_2\Theta^*.$$

Dualizing the construction of Section 7, if we now pick $c \in |\mathcal{K}^*|$ such that $q_2 = c\Phi_2^*$, then $q_1 = c\Phi_1^*$. In other words, the same c will serve to define the dual tilings $\mathcal{C}_1(q_1)^*$ and $\mathcal{C}_2(q_2)^*$. The required connexion is now clear.

THEOREM 10.1. *With the notation of (10.1), the injective linear mapping* $\Theta\colon \mathbb{X}_1 \to \mathbb{X}_2$ *yields Υ-coherent projections from Ψ_1-coherent tilings in $\mathbb{Y}_1$ derived from $\mathcal{K}$ to Ψ_2-coherent tilings in $\mathbb{Y}_2$.*

Proof. In fact, the dual map Υ^* is such that

$$\mathcal{C}_2(q_2)^* = \mathcal{C}_1(q_1)^*(\Upsilon^*)^{-1},$$

in the notation employed above. □

Notice, though, that when it comes to looking at iterated fibre tilings, the same considerations apply here as in Section 4. Namely, we have

$$\mathrm{Fib}(\mathcal{K}; \Theta\Phi_2) \preceq \mathrm{Fib}(\mathrm{Fib}(\mathcal{K}; \Phi_2); \Theta).$$

In a similar (and exactly dual) way, we have

$$\bigwedge(\mathcal{K}\Psi_1\Upsilon) \sqsubseteq \bigwedge\Big(\big(\bigwedge(\mathcal{K}\Psi_1)\big)\Upsilon\Big).$$

In the special case when $\mathcal{K} = \mathcal{F}(K)$ with $K \in \mathcal{Q}(\mathbb{V})$, Theorem 10.1 relates coherent subdivisions corresponding to iterated fibre polyhedra. Observe also that the whole picture is preserved under duality: in (10.1), just replace spaces and mappings by their duals, and reverse the directions of all the arrows, noting that injections and surjections are also interchanged. Hence, at the cost of a certain loss of generality, we have restored a symmetry which was lacking in the framework of fibre polyhedra and coherent subdivisions, but which is present in the context of representations, as we shall see in Section 12.

§11. *Mixed tilings.* In the next four sections, we discuss various examples of fibre tilings. Generalizing earlier work of Weil [**25**] and Goodey and Weil [**7**], which was motivated by questions of translative integral geometry, Schneider [**23**] has defined the concept of mixed polytopes. He observed there (see [[**23**], Section 2]) that mixed polytopes admitted an alternative description as fibre polytopes. This suggests a further generalization to tilings, although our notion loses the quantitative aspects of the original.

Let $\mathcal{B}_1, \ldots, \mathcal{B}_r$ be liftable tilings in $\mathbb{X}$. The *mixed tiling* $\operatorname{Mix}(\mathcal{B}_1, \ldots, \mathcal{B}_r)$ is defined in the following way; as usual, the definition is actually of a strong isomorphism class of tilings. Let $\mathbb{V} := \mathbb{X}^r$, let $\Phi\colon \mathbb{X} \to \mathbb{V}$ be the *diagonal* mapping

$$x\Phi := (x, \ldots, x),$$

and set $\mathcal{B} := \mathcal{B}_1 \times \cdots \times \mathcal{B}_r$. Then we define

$$\operatorname{Mix}(\mathcal{B}_1, \ldots, \mathcal{B}_r) := \operatorname{Fib}(\mathcal{B}; \Phi). \tag{11.1}$$

Observe what is happening here. The general fibre is

$$(\mathcal{B} - b)\Phi^{-1} = (\mathcal{B}_1 - b_1)\wedge \cdots \wedge(\mathcal{B}_r - b_r),$$

where $b = (b_1, \ldots, b_r)$. These fibres lie in finitely many strong isomorphism classes, and $\operatorname{Mix}(\mathcal{B}_1, \ldots, \mathcal{B}_r)$ is a sum of representatives of these classes. Moreover, $\operatorname{Mix}(\mathcal{B}_1, \ldots, \mathcal{B}_r)$ is universal for the property that each common refinement $(\mathcal{B}_1 - b_1)\wedge \cdots \wedge(\mathcal{B}_r - b_r)$ is homothetic to one of its summands.

The dual operation is also of interest (compare [[**23**], Section 3]). The dual Φ^* of Φ is the *sum* mapping:

$$(y_1, \ldots, y_r)\Phi^* = y_1 + \cdots + y_r$$

for $(y_1, \ldots, y_r) \in \mathbb{V}^* = (\mathbb{X}^*)^r$. Each choice of $b \in \mathbb{V}$ such that $(\mathcal{B} - b)\Phi^{-1} \neq \emptyset$ gives a corresponding dual sum, which we can write (compare the discussion of Section 8) as $(\mathcal{B}_1 - b_1)^* + \cdots + (\mathcal{B}_r - b_r)^*$; this is a subdivision of $|\mathcal{B}_1^*| + \cdots + |\mathcal{B}_r^*|$, whose faces are sums $G_1 + \cdots + G_r$, with $G_i \in \mathcal{B}_i^*$ for $i = 1, \ldots, r$. Then

$$(\mathrm{Mix}(\mathcal{B}_1, \ldots, \mathcal{B}_r))^* = \bigwedge((\mathcal{B}_1^* \times \cdots \times \mathcal{B}_r^*)\Phi^*)$$

is the common refinement (dualizing the fibre tiling construction) of these sums.

In **[23]**, Schneider also considers the effect on the mixed polytope of dilatations applied to the components. However, since the resulting descriptive formula involves the volumes (and other metrical invariants) of the component polytopes, it is clear that there will be no obvious generalization to mixed tilings.

§12. *Representations.* We next treat the connexion between fibre tilings and representations; for background details, consult [**8, 12, 13, 26**], and, for the specific relationships discussed here, see [**3**].

For our purposes, the technique of representations is best described in (more or less) geometric terms. Let $P \in \mathcal{Q}(\mathbb{X})$ be a polyhedron; we suppose that P is full-dimensional and pointed. Then, for some ordered set $U = (u_1, \ldots, u_n)$ of normal vectors in $\mathbb{X}^*$ (which must, in fact, span $\mathbb{X}^*$ linearly), and some $b = (\beta_1, \ldots, \beta_n) \in \mathbb{F}^n$, we can write P in the form

$$P(U, b) := \{x \in \mathbb{X} \mid \langle x, u_j \rangle \leqslant \beta_j \text{ for } j = 1, \ldots, n\}.$$

When U is fixed, we write $\mathcal{Q}(\mathbb{X}; U)$ for the family of all such polyhedra $P(U, b)$.

We find it convenient to change the conventions of our earlier papers. Let $E = (e_1, \ldots, e_n)$ be a basis of the n-dimensional space $\mathbb{V}$, and choose the basis $E^* = (e_1^*, \ldots, e_n^*)$ of the dual space $\mathbb{V}^*$ to satisfy

$$\langle e_j, e_k^* \rangle = -\delta_{jk}$$

for $j, k = 1, \ldots, n$, with δ_{jk} the familiar Kronecker delta. That is, E^* is the *negative* of the usual dual basis of $\mathbb{V}^*$. The reason for this choice is that $K := \mathrm{pos}\, E$ and $K^* := \mathrm{pos}\, E^*$ are polar cones as defined in Section 2. This then implies that $\mathcal{F}(K^*) = \mathcal{N}(K)$, so that the face complexes $\mathcal{F}(K)$ and $\mathcal{F}(K^*)$ are orthogonal dual tilings, each with a single full-dimensional cell.

If $\Phi^*: \mathbb{V}^* \to \mathbb{X}^*$ is the linear mapping induced by the mappings $e_k^* \mapsto u_k$ for $k = 1, \ldots, n$, then its dual $\Phi: \mathbb{X} \to \mathbb{V}$ is given by

$$x\Phi = -\sum_{i=1}^{n} \langle x, u_i \rangle e_i. \tag{12.1}$$

If we identify b with $\sum_{i=1}^{n} \beta_i e_i$, and use "$\geqq$" (and so on) to imply the corresponding inequalities holding between the coordinates of vectors in $\mathbb{V}$ with respect to E, then $x \in P$ is equivalent to $x\Phi + b \geqq o$, or $x\Phi \in K - b$. In other words,

$$P = (K - b)\Phi^{-1},$$

in accord with the convention of this paper.

With $\Psi\colon \mathbb{V} \to \mathbb{Y}$ as in previous sections, and $\overline{u}_j := e_j\Psi$ for $j = 1, \ldots, n$, the ordered set $\overline{U} := (\overline{u}_1, \ldots, \overline{u}_n)$ is called a *linear transform* of U. The point

$$p := b\Psi = \sum_{i-1}^{n} \beta_i \overline{u}_i \in \mathbb{Y}$$

is called the *representative* of P. Of course, what the definition does is identify P (under Φ) with the section $p\Psi^{-1}$ of K. Further, this identification actually identifies translation classes of polyhedra, which was the original motivating force in **[12]** for the introduction of representations.

If we write $\mathcal{K} := \mathcal{F}(K)$, then the combinatorics (and indeed more) of the polyhedra $P(U, b) \in \mathcal{Q}(\mathbb{X}, U)$ for different choices of b are determined by the different positions of p in the common refinement complex $\bigwedge(\mathcal{K}\Psi)$. As we have seen in Section 9, p will also yield a Φ^*-coherent complex derived from $\mathcal{K}^* = \mathcal{F}(K^*)$; of course, this will just be $\mathcal{N}(P)$. Similarly, choosing $q \in \bigwedge(\mathcal{K}^*\Phi^*)$ gives polyhedra $P(\overline{U}, c) \in \mathcal{Q}(\mathbb{Y}^*, \overline{U})$ with $q = c\Psi$.

We do not generally think of p as a possible normal vector to the corresponding fibre polyhedron Fib $(K^*; \Psi^*)$. But of course we can, and this shows that the (near) one-to-one correspondence between cells of $\bigwedge \mathcal{K}\Psi$ and strong isomorphism classes of polyhedra in $\mathcal{Q}(\mathbb{X}; U)$ is reflected by the facial structure of another polyhedron in $\mathbb{Y}^*$—actually a sum of polyhedra in $\mathcal{Q}(\mathbb{Y}^*; \overline{U})$. (The correspondence is not quite one-to-one; we recall that distinct points p can correspond to the same polyhedron P, when facet-normals u_j become redundant.)

§13. *Diagrams.* One motivation for the investigations of this paper was a connexion between fibre polytopes and diagram theory. In **[3]**, this appeared to relate fibre polyhedra to representations, as we saw in Section 12. However, the original picture of secondary polytopes in **[6]** looked at subdivisions (and more especially triangulations) of a polytope Q induced from expressing it as a linear projection of a simplex; Gale diagrams also appeared in **[4]**. We shall see that there is a very close relationship between such subdivisions and a Gale diagram of Q. In an analogous way, there are similar relationships involving centrally symmetric polytopes and their central diagrams, and zonotopes and their zonal diagrams, the latter of which we shall treat in Section 14.

Our setting in the first context is the following. Let $Q \in \mathcal{P}(\mathbb{Y})$ be a d-polytope with n vertices; we assume that dim $\mathbb{Y} = d$. Then we can express Q in the form $Q = T\Psi$, with $T \in \mathcal{P}(\mathbb{V})$ an $(n-1)$-simplex and $\Psi\colon \mathbb{V} \to \mathbb{Y}$ a linear mapping; again, we suppose that dim $\mathbb{V} = n - 1$. If Q has vertex-set $V = (v_1, \ldots, v_n)$, then we suppose the vertex-set $E = (e_1, \ldots, e_n)$ of T labelled so that $v_j = e_j\Psi$ for each j.

The normal fan $\mathcal{N}(T)$ of T is formed by n basic cones in $\mathbb{V}^*$, each spanned by $n - 1$ out of n vectors $e_1^*, \ldots, e_n^*$, with e_j^* an outer normal to the facet of T opposite e_j. If, with our usual conventions, we write $\overline{v}_j := e_j^*\Phi^* \in \mathbb{X}^*$, then we call $\overline{V} := (\overline{v}_1, \ldots, \overline{v}_n)$ a *Gale diagram* of Q; we thus assume that dim $\mathbb{X} = n - d - 1$. Note that our formulation is rather more general than that arising from affine transforms. (Compare also the somewhat different geometric description of Gale diagrams given in [**18**].)

Thinking of $\mathcal{N}(T)$ as a strong dual of $\mathcal{F}(T)$, it follows that a point $q \in \mathbb{X}^*$ will give rise to a Ψ-coherent subdivision $\mathcal{D}(q)$ of Q, and conversely. Moreover, we are able to read off from the position of q relative to $\overline{V}$ the combinatorics of $\mathcal{D}(q)$. Namely, if we denote by $\widetilde{W} \subseteq \overline{V}$ the *complement* of $W \subseteq V$, in the sense that their index sets are complementary, then W is the vertex-set of a face conv W of $\mathcal{D}(q)$ if and only if $q \in$ relint pos $\widetilde{W}$. From the particular choice $q = o$, which gives $\mathcal{D}(o) = \mathcal{F}(Q)$, we obtain the usual Gale diagram relationship; this is more often expressed in terms of *convex* hulls of subsets of $\overline{V}$, but in the wider context we see that this formulation is inappropriate.

The idea of inserting a new point into a Gale diagram to give a triangulation is an old one; it was implicitly used in, for example, [**19**], and is described in [**13**, 3B9]. However, what was had in mind there was expressing Q as the vertex-figure of a $(d+1)$-polytope with $n+1$ vertices, with the triangulation induced by radial projection of the antistar of the new vertex from that vertex. The Gale diagram of this new polytope is actually obtained by inserting $-q$ into the original Gale diagram $\overline{V}$.

We should also observe here that the fibre polytope Fib$(T; \Phi)$ is just the secondary polytope, as defined in [**6**], so that the common refinement complex $\bigwedge(\mathcal{N}(T)\Phi^*)$, which is that given by the cones spanned by subsets of $\overline{V}$, indeed determines the coherent subdivisions (and, in particular, the triangulations) of Q induced by Ψ. (Carl Lee has pointed out to us that [**13**] thus, in effect, anticipates [**6**].)

Before we leave this topic, we just note that a regular subdivision of a polytope lifts to the lower surface of the projection of a simplex, and so is a coherent projection of the simplex.

We shall say little about central diagrams. The core polytope K is now a cross-polytope, whose normal cones are spanned by the faces of a cube. This implies that the diagram relationship for subdivisions of centrally symmetric polytopes $K\Psi$ is correspondingly complicated. We refer the reader to [**18**] for further details, and in particular their geometric origins.

§14. *Zonotopes.* Zonotopes form a rather special class of polytopes. An exactly analogous situation to that of Gale diagrams arises from zonotopes; however, the connexion between zonotopes and arrangements of hyperplanes makes the whole theory that much richer. Recall that a zonotope in $\mathbb{Y}$ is a sum $Z = S_1 + \cdots + S_n$ of n (line) segments, which for present purposes we take to be of the form $S_j := \operatorname{conv}\{-b_j, b_j\}$ for $j = 1, \ldots, n$. We denote by $\mathcal{Z}(\mathbb{Y}) \subseteq \mathcal{P}(\mathbb{Y})$ the family of zonotopes in $\mathbb{Y}$. With $(e_1, \ldots, e_n)$ a basis of $\mathbb{V}$, let $E_j := \operatorname{conv}\{-e_j, e_j\}$ for $j = 1, \ldots, n$, and write $C := E_1 + \cdots + E_n$, so that (combinatorially) C is an n-cube. The linear mapping $\Psi : \mathbb{V} \to \mathbb{Y}$ such that $b_j = e_j\Psi$ then expresses $Z = C\Psi$ as a linear image of the cube.

If $(e_1^*, \ldots, e_n^*)$ is the dual basis of $\mathbb{V}^*$, then the 2^n maximal cones of the normal fan $\mathcal{N}(C)$ of C are of the form $\operatorname{pos}\{\varepsilon_1 e_1^*, \ldots, \varepsilon_n e_n^*\}$, where $\varepsilon_j = \pm 1$ for $j = 1, \ldots, n$. With our usual conventions, if $\overline{b}_j := e_j^*\Phi^* \in \mathbb{X}^*$ (so that $\overline{B} :=$

$(\bar{b}_1, \ldots, \bar{b}_n)$ is a linear transform of $B := (b_1, \ldots, b_n)$), then $(\pm\bar{b}_1, \ldots, \pm\bar{b}_n)$ is called a *zonal diagram* of Z.

The position of a point $q \in \mathbb{X}^*$ relative to the zonal diagram then gives a subdivision of Z by translates of its subzonotopes (formed by summing subsets of its defining segments); the special choice $q = o$ will give rise to the usual zonal diagram relationships. For further details of these, see [**11**]. Observe that all these subdivisions will be liftable; their strong duals are given by partitions of $\mathbb{Y}$ by affine arrangements of hyperplanes (see [**1**], where the liftability of these partitions is proved directly).

Indeed, it is this latter aspect which we can tie in with our general picture of operations on tilings; when we do this, we find that we can discard much of the apparatus just described. The face-complex of a segment S_j in $\mathbb{Y}$ is an improper tiling, whose strong dual is a *hyperplane tiling*, namely, the tiling of $\mathbb{Y}^*$ induced by a single hyperplane H_j whose normal is parallel to the segment. If we take $S_j = \operatorname{conv}\{b_j, -b_j\}$ as above, and lift it by f_j such that

$$f_j(y) = \langle y, u_j \rangle \quad \text{for } y \in S_j,$$

then

$$H_j = \{w \in \mathbb{Y}^* \mid \langle w - u_j, b_j \rangle = 0\}.$$

Thus liftable zonal decompositions of zonotopes, which are sums of segments in the more general sense of sums of their face-complexes, are strong duals to common refinements of hyperplane tilings, or, in other words, to the tilings induced by affine arrangements of hyperplanes.

Observe also that (in analogy to the corresponding result for regular subdivisions of simplices) a regular zonal subdivision of a zonotope can be regarded as a coherent projection of a cube.

Fibre polytopes of zonotopes are also special. In [**4**] (see also [**26**]), it was shown that the fibre polytope of a cube is a zonotope. In fact, as a generalization of this result (which Lou Billera tells us he and Sturmfels were actually aware of), we have

THEOREM 14.1. *The fibre polytope of a zonotope is a zonotope.*

Proof. Actually, we should really phrase this as: among the fibre polytopes of a zonotope is a zonotope. To see this, we change notation from the discussion above, but still work in our general context. Let $Z \in \mathcal{Z}(\mathbb{V})$ be a zonotope. Then $\mathcal{N}(Z)$ is the common refinement of a family of hyperplane tilings determined by linear hyperplanes in $\mathbb{V}^*$. Thus the faces of $\mathcal{N}(Z)$ fall into linear subspaces of $\mathbb{V}^*$. It follows that

$$\mathcal{N}(\mathrm{Fib}(Z; \Phi)) = \bigwedge(\mathcal{N}(Z)\Phi^*)$$

is similarly the common refinement of a family of linear hyperplane tilings in $\mathbb{X}^*$, so that the fibre polytope $\mathrm{Fib}(Z; \Phi)$ is itself a zonotope. □

The special case of fibre polytopes of the cube is of further interest, since it can be tied in with another notion. In each linear subspace of a vector space $\mathbb{W}$

over an ordered field $\mathbb{F}$ is a well-defined concept of volume in $\mathbb{F}$ of polytopes (see, for example, [**9, 15**]); this is unique, up to a (positive) scaling factor. Indeed, since volume is closely tied in with determinants (with respect to fixed bases), it even makes sense to talk about *oriented* volume.

However, the fact that volume is only specified up to a scaling factor makes clear that there is no natural definition of surface area of a polytope. Nevertheless, we can define the *area vector array* $A(P)$ of a polytope $P \in \mathbb{W}$. Let G be a facet of P, which we suppose to be full-dimensional. The *oriented area vector* $a(G)$ of G is the outer normal vector to P at G, scaled so that

$$\langle x, a(G) \rangle = \mathrm{vol}(G \times \mathrm{conv}\{-x, o\})$$

for $x \in \ \mathrm{aff}\ G$. (The bracketed term is a prism with one base G, and the other a translate of G whose affine hull contains o.) Then $A(P)$ is just the array of area vectors of the facets of P, ordered in some fashion (for example, the natural order, if P is regarded as a polyhedron with given facet normals, as in previous sections).

Let $P \in \mathcal{P}(\mathbb{W})$ be full-dimensional, and, for each facet G of P, define the line segment $S(G)$ by $S(G) := \mathrm{conv}\{-a(G), a(G)\}$. We now define the *projection zonotope* of P to be

$$\mathrm{Proj}\ P := \tfrac{1}{2} \sum_G S(G) \in \mathcal{Z}(\mathbb{W}^*). \tag{14.1}$$

In geometric terms, this says that, for $x \in \mathbb{W}$,

$$\mathrm{vol}(P + \mathrm{conv}\{o, x\}) - \mathrm{vol} P = \eta(\mathrm{Proj}\ P, x)$$

gives the support functional of Proj P. Observe that the orientation of the area vectors now plays a minor rôle.

Now let C, Z, B and $\overline{B}$ be as before, define $\overline{S}_j := \mathrm{conv}\{-\overline{b}_j, \overline{b}_j\}$ for $j = 1, \ldots, n$, and set

$$\overline{Z} := \overline{S}_j + \cdots + \overline{S}_n; \tag{14.2}$$

then we call $\overline{Z}$ a zonotope *associated* with Z. There is a nice connexion between fibre zonotopes and associated zonotopes.

THEOREM 14.2. *With the standard notation, if $C \in \mathcal{Z}(\mathbb{V})$ is a cube and $\overline{Z} \in \mathcal{Z}(\mathbb{X}^*)$ is a zonotope associated with the zonotope $Z = C\Psi \in \mathcal{Z}(\mathbb{Y})$, then*

$$\mathrm{Fib}(C; \Phi) = \mathrm{Proj}\ \overline{Z}.$$

Proof. This is a straightforward consequence of the description of the fibre polytope of a general zonotope, given in the proof of Theorem 14.1. $\square$

In fact, much less is required by our combinatorial definition—any set of outer normal vectors to the facets of $\overline{Z}$ will serve to determine the segments whose sum is the fibre zonotope. In the metrical setting of regular cubes in euclidean spaces and their (internal) orthogonal projections, Theorem 14.2 actually holds for the more restricted definition of the projection polytope in terms of an integral, up to a scaling factor. However, because this more refined result does not fit into the present context, we shall postpone further discussion of it until [**17**].

§15. *Liftability criteria.* Hitherto, we have dealt with liftable tilings, which we have shown in Theorem 7.1 to be just those with strong duals. However, we have not considered criteria which would enable us to identify such tilings. We give one condition in this section, and briefly mention others. A further sufficient condition will be discussed in Section 16.

Our criterion uses representation theory. Let $\mathcal{B}$ be a finite tiling in $\mathbb{X}$. We shall lose no generality in assuming that $\mathcal{B}$ is proper. Suppose that $\mathcal{B}$ has vertices $v_1, \ldots, v_m$, and that its unbounded edges have (distinct) directions $v_{m+1}, \ldots, v_n$. Defining $\mathbf{N} := \{1, \ldots, n\}$, each face of $\mathcal{B}$ can be expressed as a generalized convex hull conv $V(\mathbf{J})$, where $V(\mathbf{J}) := \{v_j \mid j \in \mathbf{J}\}$, for some $\mathbf{J} \subseteq \mathbf{N}$ (by this, we mean the sum of the convex hull of the vertices and the positive hull of the directions in the set). Let $\mathcal{J}$ denote the family of such subsets $\mathbf{J}$.

Define

$$\widetilde{v}_j := \begin{cases} (v_j, -1), & \text{if } j = 1, \ldots, m, \\ (v_j, 0), & \text{if } j = m+1, \ldots, n, \end{cases}$$

let $V := (\widetilde{v}_1, \ldots, \widetilde{v}_n) \subseteq \mathbb{X} \times \mathbb{F}$, and, for $\mathbf{J} \subseteq \mathbf{N}$, let $V(\mathbf{J}) := \{\widetilde{v}_j \mid j \in \mathbf{J}\}$. If $\mathcal{B}$ is liftable, say by the polyhedral function f, then

$$\text{cone } \mathcal{B} = \{\text{pos } V(\mathbf{J}) \mid \mathbf{J} \in \mathcal{J}\}$$

is the normal fan to epi f^*, with f^* the conjugate of f.

Now let $\overline{V} = (\overline{v}_1, \ldots, \overline{v}_n)$ be a linear transform of V (actually, an affine transform of $(v_1, \ldots, v_n)$ if $m = n$). If $p^* \in$ {pos $\overline{\mathrm{V}}$ represents epi f^*, as in Section 12, then

$$p^* \in \text{ relint pos } \overline{V}(\mathbf{N} \setminus \mathbf{J}) \Longleftrightarrow \mathbf{J} \in \mathcal{J}. \tag{15.1}$$

We now have our criterion.

THEOREM 15.1. *With the notation above, the tiling $\mathcal{B}$ is liftable if and only if*

$$\bigcap\{\text{relint pos} \overline{V}(\mathbf{N} \setminus \mathbf{J}) | \mathbf{J} \in \mathcal{J}\} \neq \emptyset.$$

Proof. We saw above that the condition is necessary. For its sufficiency, let p^* be a point in the given set. Then p^* represents a polyhedron in $\mathbb{X}^* \times \mathbb{F}$ with V as the set of outer facet normals and cone $\mathcal{B}$ as its normal fan. This polyhedron is thus of the form epi f^*, as above, and hence $\mathcal{B}$ itself is liftable. □

Notice that the condition of Theorem 15.1 can be checked without knowing anything about possible strong duals or lifting functions.

Alternative (known) criteria, on the other hand, usually do directly imply the existence of a strong dual. For instance, one can ask that $\mathcal{B}$ be a pegged tiling in the sense of **[16]** (see also **[1]**, although that argument is not quite complete), or that it have a positive stress on its faces of codimension 1 (see **[21]** and the references therein), which implies the pegging condition.

§16. *Locally simple tilings.* We shall not discuss general criteria for liftability of a tiling; there is a survey of these in **[21]**. However, there is a striking case of a liftable tiling where there is essentially no dependence on the particular choice of a strong dual tiling $\mathcal{K}^*$ to $\mathcal{K}$. Building on earlier (unpublished) work of Whiteley, Rybnikov **[21]** has proved the following result. To explain this, we need a little more terminology.

Let $\mathcal{K}$ be a proper tiling in $\mathbb{V}$. Following the spirit of **[8]** (rather than **[21]**), we say that $\mathcal{K}$ is *k-simple* if each interior face of $\mathcal{K}$ of *codimension k* is contained in $k+1$ tiles (the minimum possible). The *star* of a face $F \in \mathcal{K}$ is defined to be

$$\operatorname{star}(F, \mathcal{K}) := \{\operatorname{pos}(G - F) \mid G \in \mathcal{K} \text{ and } F \leqslant G\}. \tag{16.1}$$

Thus the star describes the local structure of the tiling $\mathcal{K}$ around its face F. It is convenient here to allow $F = \emptyset$, and define $\operatorname{star}(\emptyset, \mathcal{K}) := \mathcal{K}$. We then have

THEOREM 16.1. *Let* $\dim \mathbb{V} \geqslant 3$, *and let* $\mathcal{K}$ *be a proper tiling in* $\mathbb{V}$. *If, in addition,* $\mathcal{K}$ *satisfies*

(*a*) $\mathcal{K}$ *is* 2-*simple,*
(*b*) $\operatorname{star}(F, \mathcal{K})$ *is liftable for each interior face* $F \in \mathcal{K}$ *of codimension* 3, *then* $\mathcal{K}$ *is liftable. In particular, if* $\mathcal{K}$ *is* 3-*simple, then it is liftable.*

It is easy to see that any two strong duals of $\mathcal{K}$ must be homothetic, because their bounded 2-faces are triangles.

Of course, in the case of simple tilings, where each vertex (interior only) is surrounded by $n+1$ tiles, we are now back in one of the main areas of study, since the dual tiling is a (liftable) triangulation of a simplicial polytope. The question of which triangulations of (simplicial) polytope are liftable has, as we have already observed, occupied much attention in recent years.

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