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VALUATIONS AND TENSOR WEIGHTS ON POLYTOPES

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For the 65th birthday of Rolf Schneider.

Abstract. Let $\mathbb{V}$ be a finite-dimensional vector space over a square-root closed ordered field $\mathbb{F}$ (this restriction permits an inner product with corresponding norm to be imposed on $\mathbb{V}$). Many properties of the family $\mathcal{P} := \mathcal{P}(\mathbb{V})$ of convex polytopes in $\mathbb{V}$ can be expressed in terms of *valuations* (or *finitely additive measures*). Valuations such as volume, surface area and the Euler characteristic are translation invariant, but others, such as the moment vector and inertia tensor, display a polynomial behaviour under translation. The common framework for such valuations is the *polytope* (or *Minkowski*) *ring* $\Pi := \Pi(\mathbb{V})$, and its quotients under various powers of the ideal T of Π which is naturally associated with translations. A central result in the theory is that, in all but one trivial respect, the ring Π/T is actually a graded algebra over $\mathbb{F}$. Unfortunately, while the quotients Π/T^{k+1} are still graded rings for $k > 1$, they now only possess a rational algebra structure; to obtain an algebra over $\mathbb{F}$, some (weak) continuity assumptions have to be made, although these can be achieved algebraically, by factoring out a further ideal A, the *algebra ideal*.

An apparently unrelated topic concerns the *piecewise polynomials* on $\mathbb{V}$ (or, more strictly, on the dual space $\mathbb{V}^*$), which are functions f on $\mathbb{V}$ whose restrictions $f|_C$ to each (closed) cone C of some complete fan (complex) $\mathcal{C}$ of convex cones with apex the origin o are polynomials. The piecewise polynomials form a graded algebra under pointwise addition and multiplication.

A bridge between these two notions is provided by the *weight algebra* $\mathcal{W} := \mathcal{W}(\mathbb{V})$. Initially, this attaches symmetric tensors to faces of polytopes $P \in \mathcal{P}$; these satisfy certain *Green-Minkowski connexions*, which generalize (in a sense) the Minkowski relations for facet areas of a polytope. The algebra properties of $\mathcal{W}$ correspond to fundamental geometric operations on polytopes. More importantly, weights provide a concrete representation of the underlying

abstract space for valuations which have a polynomial behaviour, and then naturally link up with piecewise polynomials. In principle, as in Π, there is still the need to pass to suitable quotient spaces, but in the weight algebra $\mathcal{W}$ such quotients almost sit as subspaces.

Motivated by duality and other considerations, and in particular a product on valuations introduced by Alesker, it turns out that there is a quite different *fibre* product which can be imposed on $\mathcal{W}$; this has its origins in a modified construction of fibre polytopes.

§1. *Introduction.* The universal group for valuations on the family $\mathcal{P} := \mathcal{P}(\mathbb{V})$ of convex polytopes in a finite-dimensional vector space $\mathbb{V}$ over an arbitrary ordered field $\mathbb{F}$ is the polytope or Minkowski ring $\Pi := \Pi(\mathbb{V})$. Although the multiplication in Π (which is induced by Minkowski addition) is apparently unimportant for valuations themselves, it is, as we shall see, an extremely useful feature.

The universal group for the translation invariant valuations on $\mathcal{P}$ is the polytope algebra, which we described in [**19**] (see also [**29**]). This algebra is Π/T, where $T = T(\mathbb{V}) \leqslant \Pi$ is the ideal which naturally corresponds to translations. A key geometric construction (adapted from one originally due to Thorup in [**14**] – see Theorem 3.6 below) shows that Π/T is (in all but one trivial respect) a graded algebra over $\mathbb{F}$.

In [**23**], we described the weight algebra $\Omega = \Omega(\mathbb{V})$, which provides an alternative, and perhaps more concrete, approach to the same circle of ideas (in this context, it is convenient to assume that $\mathbb{F}$ is square-root closed, so that the structure of an inner product space with norm can be imposed on $\mathbb{V}$). Together with [**20**, **22**], there was established a (near) isomorphism between Π/T and Ω; in particular, the resulting machinery yields an easier route to the proof in [**20**] of the necessity of the conditions of the g-theorem, which describes the possible f-vectors (sequences of numbers of faces) of simple polytopes. (It is worth remarking here that all these algebras are graded, and that isomorphisms only fail in grade 0, which corresponds to the Euler characteristic – naturally, therefore, the grade zero terms arising from the original geometry are just $\mathbb{Z}$, whereas, in the more algebraic context, the rational numbers $\mathbb{Q}$ or the base field $\mathbb{F}$ are appropriate.)

The universal groups for the wider classes of valuations which exhibit polynomial behaviour under translation (for example, moment vectors and inertia tensors) are the quotients Π/T^{k+1} for some $k \geqslant 1$; here, the situation is rather different. These higher polytope algebras still have algebra structures, but now only over $\mathbb{Q}$. However, the corresponding valuations which are most interesting are also weakly continuous (we define this term later). It turns out that there is a further algebra ideal A such that the quotients $\Pi(T^{k+1} + A)$ have a full $\mathbb{F}$-algebra structure, and this is exactly what is needed to impose weak continuity. Moreover, just as the polytope algebra admits a concrete expression in terms of scalar-valued weights, it turns out that these higher polytope algebras (modulo weak continuity) can be phrased in terms of tensor-valued weights.

The basic tensor-valued weights are the tensorials of polytopes, which (up to scaling factors) generalize the mass moments, whose initial terms are volume, moment vector and inertia tensor. The various tensorials of a given polytope and its faces are not independent. The ways in which they are related, by means of the Green-Minkowski connexions, which generalize the Minkowski relations for scalar weights, provide the defining conditions for the tensor-weight algebra $\mathcal{W} = \mathcal{W}(\mathbb{V})$. A central result of this paper is that $\mathcal{W}$ is indeed an algebra; the appropriate definition of multiplication closely follows that of **[23]**, and involves showing that tensor weights behave as they should under linear mappings (for this latter result, a weakened Green-Minkowski relation for weights serves the purpose). By looking at the subalgebra $\mathcal{W}(P)$ associated with a simple polytope P, we prove that $\mathcal{W}$ is generated by the space $\mathcal{W}_1$ of its weights of degree 1. This will incidentally show that $\mathcal{W}(P)$ is isomorphic to the face ring of the dual simplicial polytope P^*.

In **[2]** (a preliminary account of which appeared in **[1]**), Alesker introduced a product of translation covariant valuations. Motivated by this, and by a homomorphism on weights induced by the fibre polytope construction found in **[26]**, we define here a new fibre product of weights; though analogous to Alesker's product, it is not quite the same.

A yet different algebra is that of the piecewise polynomial functions on $\mathbb{V}$, which has been used in **[4, 7, 8]** to investigate the structure of Π and its quotients Π/T^{k+1}. We show that this algebra is also isomorphic to $\mathcal{W}$; this will be straightforward, since we have a direct correspondence between natural generators of each. However, $\mathcal{W}$ has the advantage over the polynomial algebra, in that the translation quotients are just obtained by truncating the tensor weights in a natural way, setting to zero tensors of grade greater than k. The equivalences classes then have unique representatives, which is obviously more convenient for purposes of calculation.

An earlier and somewhat different approach to part of the material of this paper first appeared in **[24]**.

§2. *The polytope ring*. For the most part, we use the standard notation of the theory of convex polytopes, although we have simplified some of it; for general background material, see **[6, 12, 37]**.

We begin by describing the basic properties of what has sometimes been called the *Minkowski ring* (see **[15, 9, 29]**). We prefer the term *polytope ring* for two reasons: first, it is the precursor of the polytope algebra of **[19]** (see also **[29]** – we shall not repeatedly quote these basic references) and, second, the alternative term puts overmuch emphasis on the multiplication in the ring.

We work here in a d-dimensional vector space $\mathbb{V}$ over an ordered field $\mathbb{F}$. We later assume that $\mathbb{F}$ is (positive) square-root closed. This is not a very serious restriction, since we can always embed our initial ordered field in its square-root closure. More to the point, it is convenient to be able to impose an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{V}$, with a corresponding norm $\| \cdot \|$; only for emphasis will we subsequently distinguish between $\mathbb{V}$ and its dual space $\mathbb{V}^*$. We write $\mathcal{P} := \mathcal{P}(\mathbb{V})$ for the family of non-empty convex polytopes in $\mathbb{V}$.

The *polytope ring* $\Pi := \Pi(\mathbb{V})$ is defined as follows.

- As an abelian group, Π has a generator $[P]$ for each $P \in \mathcal{P}$; further, it is natural to set $[\emptyset] := 0$.

- The generators satisfy the relations $[P \cup Q] + [P \cap Q] = [P] + [Q]$ whenever P, $Q \in \mathcal{P}$ are such that $P \cup Q$ is also convex (this corresponds to the *valuation property*).
- The multiplication on Π is induced by Minkowski addition on $\mathcal{P}$, namely, $[P] \cdot [Q] := [P + Q]$ for $P, Q \in \mathcal{P}$; the multiplication is extended by linearity.

In this context, we recall that a *valuation* is a mapping φ from $\mathcal{P}$ into some abelian semigroup, such that

$$\varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q),$$

whenever $P, Q \in \mathcal{P}$ are such that $P \cup Q \in \mathcal{P}$ also. Thus a valuation on $\mathcal{P}$ induces an additive (semi-)group homomorphism on Π, and conversely. Further, the *Minkowski* or *vector sum* of P and Q is

$$P + Q := \{x + y | x \in P, y \in Q\}.$$

When $Q = \{y\}$ is a point-set, we briefly write $P + y := P + \{y\}$ for the *translate* of P by y; we also employ the abbreviation $[y] := [\{y\}]$.

There is an implied assertion here that the addition and multiplication on Π defined in this manner are compatible, that is, that the distributive laws hold. Indeed, this is proved in, for example, [**19**, Lemma 6], if we ignore those parts which refer to translation invariance. However, an easier approach uses the following.

THEOREM 2.1 *Linear mappings on* $\mathbb{V}$ *induce ring homomorphisms on* Π.

Proof. For completeness, we repeat the proofs of, for example, [**19**, **21**, **27**]. Let Θ: $\mathbb{V} \to \mathbb{W}$ be a linear mapping. For the moment, we ignore the distributive laws.

For addition, let $P, Q \in \mathcal{P}(\mathbb{V})$ be such that $P \cup Q \in \mathcal{P}(\mathbb{V})$ also. It is clear that

$$(P \cup Q)\Theta = P\Theta \cup Q\Theta.$$

It is also clear that

$$(P \cap Q)\Theta \subseteq P\Theta \cap Q\Theta.$$

For the opposite inclusion, let $z \in P\Theta \cap Q\Theta$. Thus, there are $x \in P$ and $y \in Q$ such that $z = x\Theta = y\Theta$. Since $P \cup Q$ is convex, there exists $0 \leqslant \lambda \leqslant 1$ such that $(1 - \lambda)x + \lambda y \in P \cap Q$. Then

$$z = (1 - \lambda)x\Theta + \lambda y\Theta = ((1 - \lambda)x + \lambda y)\Theta \in (P \cap Q)\Theta,$$

as was to be shown.

For multiplication, it is obvious that

$$(P + Q)\Theta = P\Theta + Q\Theta.$$

We now pass to the corresponding polytope classes, and the theorem then follows. □

We then deduce

THEOREM 2.2 *Under the given addition and multiplication,* Π *is a commutative ring, with identity* $1 := [o]$.

Proof. We apply Theorem 2.1 to the sum mapping $\Sigma\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, given by

$$(x, y)\Sigma := x + y. \tag{2.1}$$

Since it is clear that

$$(P_1 \cup P_2) \times Q = (P_1 \times Q) \cup (P_2 \times Q), \quad (P_1 \cap P_2) \times Q = (P_1 \times Q) \cap (P_2 \times Q),$$

whether or not $P_1 \cup P_2$ is convex, the distributive laws immediately follow. $\square$

A useful observation is the following. With a set $S \subseteq \mathbb{V}$ can be identified its *characteristic function* $\delta(S, \cdot)\colon \mathbb{V} \to \{0, 1\}$, defined by

$$\delta(S, x) := \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \notin S. \end{cases}$$

The group generated by the functions $\delta(P, \cdot)$ with $P \in \mathcal{P}$ is denoted $\mathcal{X} = \mathcal{X}(\mathbb{V})$. Then we have a result which, in a slightly different form, is originally due to Groemer [**11**].

LEMMA 2.3 *As an abelian group*, $\Pi \cong \mathcal{X}$.

In fact, Groemer goes on to define a multiplication on $\mathcal{X}$, which is just that induced by Minkowski addition.

We later need the *scalar multiple* of P by λ, namely

$$\lambda P := \{\lambda x \mid x \in P\};$$

this induces the *dilatation* on Π, defined by

$$\lambda \circ [P] := [\lambda P]$$

on the generators of Π.

Much of our treatment actually extends to the ring $\Gamma = \Gamma(\mathbb{V})$ derived in a similar way from the family $\mathcal{Q} = \mathcal{Q}(\mathbb{V})$ of polyhedra (polyhedral sets) in $\mathbb{V}$. However, only a small additional subfamily of $\mathcal{Q}$ is needed here. Recall that $K \in \mathcal{Q}$ is a *cone* with *apex* a if $(1 - \lambda)a + \lambda K \subseteq K$ for each $\lambda \geqslant 0$. Of most importance are those cones with apex the origin o; the family of such cones in $\mathbb{V}$ is denoted $\mathcal{C} = \mathcal{C}(\mathbb{V})$, and the corresponding subring of Γ is denoted $\Xi = \Xi(\mathbb{V})$. (Note that we have changed the notation to Ξ from Σ of earlier papers, since we have just employed Σ with a quite different meaning.)

The *support functional* $\eta(P, \cdot)$ of $P \in \mathcal{P}$ is defined as usual for $u \in \mathbb{V}^*$ (or in $\mathbb{V}$, if we do not wish to distinguish the dual space) by

$$\eta(P, u) := \sup\{\langle x, u\rangle \mid x \in P\}.$$

There is the natural convention $\eta(\emptyset, \cdot) \equiv -\infty$; in fact, in all but this case, we can replace "sup" by "max" in the definition (and even here with the obvious meaning). The corresponding *face* of P *in direction* u is

$$F(P, u) := \{x \in P \mid \langle x, u\rangle = \eta(P, u)\}.$$

We do not generally count the empty set $\emptyset$ as a face of a polyhedron P, but we do think of P as a face of itself (formally, we take $u = o$ in the definition). If F is a face of $P \in \mathcal{P}$, then we write $F \leqslant P$ or $P \geqslant F$, with $F < P$ meaning additionally that $F \neq P$. If F is a facet of P (that is, a face of codimension 1), then we write $F \lhd P$ or $P \rhd F$. Alternatively, $\mathcal{F}(P)$ denotes the family of (non-empty) faces of P; we write $\mathcal{F}_k(P)$ for the subfamily of its k-faces (those of dimension k).

Finally, the *normal cone* to P at its face F is

$$N(F, P) := \{u \in \mathbb{V}^* \mid F \leqslant F(P, u)\}.$$

The family $\mathcal{N}(P) := \{N(F, P) \mid F \leqslant P\}$ of normal cones to faces of P forms a complex, called the *normal fan* of P.

For future reference, we also mention here that, for $\lambda \geqslant 0$,

$$\eta(\lambda P, \cdot) = \lambda \eta(P, \cdot).$$

Further, if $t \in \mathbb{V}$, then clearly $\eta(t, \cdot) = \langle t, \cdot \rangle$.

Since

$$\eta(P \cup Q, \cdot) = \max\{\eta(P, \cdot), \eta(Q, \cdot)\}$$

for all P, $Q \in \mathcal{P}$, and

$$\eta(P \cap Q, \cdot) = \min\{\eta(P, \cdot), \eta(Q, \cdot)\}$$

whenever $P \cup Q$ is also convex, while

$$\eta(P + Q, \cdot) = \eta(P, \cdot) + \eta(Q, \cdot),$$

we see that the mapping $P \mapsto \gamma(P, \cdot) := \exp(\eta(P, \cdot))$ (the latter regarded as a formal power – we could even write $X^{\eta(P, \cdot)}$, with X an indeterminate) induces a ring homomorphism on Π. Notice that we have the natural convention $\gamma(\emptyset, \cdot) \equiv 0$. (With Minkowski addition as the semigroup operation, we are actually forming the semigroup ring of $\mathcal{P}$.)

REMARK 2.4 In the wider context of polyhedra, there are two advantages over [**8**, **29**] in defining

$$\gamma(P, \cdot) := \exp(-\eta(P, \cdot))$$

for $P \in \mathcal{Q}$. First, the only special convention needed is that $\gamma(\emptyset, \cdot) \equiv 0$, because, if u is not a normal vector to a support hyperplane of P, so that $\eta(P, u) = \infty$, then $\gamma(P, u) = 0$. Second, there results a pleasing symmetry, in the following sense. The convex conjugate of the support functional is the convex indicator functional $\eta^*(P, \cdot)$, for which

$$\eta^*(P, x) := \begin{cases} 0, & \text{if } x \in P, \\ \infty, & \text{if } x \notin P. \end{cases}$$

In turn, the characteristic functional of P is $\delta(P, \cdot) = \exp(-\eta^*(P, \cdot))$.

A central result in the abstract theory is the following; we refer to [**9**, **15**, **29**] for proofs.

THEOREM 2.5 *The homomorphism γ is an injection; that is, if $x \in \Pi$ is such that $\gamma(x, \cdot) \equiv 0$, then $x = 0$.*

We do not need many properties of γ, although it lurks in the background of much of what we do later. However, there is one application which we shall use in Section 4. The *Euler map* $\varepsilon: \Pi \to \Pi$ is defined on its generators by

$$[P]\varepsilon := \sum_{F \leqslant P} (-1)^{\dim F}[F]. \tag{2.2}$$

A little work shows that ε is an involutory automorphism of Π. We do not reproduce the details here; however, let us remark that the easiest route to showing that ε is an endomorphism is the observation that the class [relint P] of $P \in \mathcal{P}$ is well defined by

$$[\operatorname{relint} P] = (-1)^{\dim P}[P]\varepsilon.$$

In Section 4, we shall appeal to a result which is easily proved by applying γ to the product $[P] \cdot [-P]\varepsilon$.

THEOREM 2.6 *The class* $[P]$ *of* $P \in \mathcal{P}$ *is invertible in* Π; *its inverse is*

$$[P]^{-1} = (-1) \circ [P]\varepsilon = [-P]\varepsilon.$$

We conclude the section by mentioning another endomorphism of Π.

THEOREM 2.7 *The mapping* $P \mapsto F(P, u)$ *induces an endomorphism of* Π.

Proof. We make the natural definition: the endomorphism φ on the generators of Π is given by $[P]\varphi := [F(P, u)]$. The properties of the support functional listed above easily lead to the result. □

§3. *Ideals and weak continuity*. A crucial relationship in Π is that between the *cylinder ideal* Z and the *translation ideal* T, whose definitions are

$$Z := \langle [P] - 1 \mid P \in \mathcal{P} \rangle,$$
$$T := \langle [t] - 1 \mid t \in \mathbb{V} \rangle,$$

where $\langle S \rangle$ here denotes the ideal in Π generated by its subset S; recall that we make the identification $1 := [o]$. We show the following important result, which was already mentioned in [**21**, §3.5] (with an unfortunate mistake in the direction of the inclusion), although its significance was not recognized there.

THEOREM 3.1 $Z^{d+m} \leqslant T^m$ *for each* $k \geqslant 0$.

In fact, the result is probably better stated as $Z^{d+m} \leqslant Z^d T^m$ (or even $Z^{d+m} = Z^d T^m$).

We begin the proof with an almost obvious remark.

LEMMA 3.2 *Let* $u_0, \ldots, u_d$ *be non-zero vectors in* $\mathbb{V}$, *and for* $j = 0, \ldots, d$, *let* $H_j^+ := \{x \in \mathbb{V} \mid \langle x, \varepsilon_j u_j \rangle \geqslant 0\}$ *with* $\varepsilon_j = \pm 1$. *Then there are choices of the* ε_j *such that* $\bigcap_{j=0}^{d} \operatorname{int} H_j^+ = \emptyset$.

Proof. We may assume that the u_j are in linearly general position (that is, that no d lie on any hyperplane), otherwise the result reduces to one in a lower dimension, and we can appeal to induction – the lemma is trivial if $d = 1$. There are now non-zero numbers $\lambda_0, \ldots, \lambda_d$, such that $\sum_{j=0} \lambda_j u_j = o$; we then set

$\varepsilon_j := \operatorname{sign} \lambda_j$. If we had a vector $a \in \bigcap_{j=0}^{d} \operatorname{int} H_j^+$, then taking inner products with a yields the contradiction

$$0 = \langle a, o \rangle = \sum_{j=0}^{d} \lambda_j \langle u_j, a \rangle = \sum_{j=0}^{d} |\lambda_j| \langle \varepsilon_j u_j, a \rangle > 0.$$

The lemma follows at once. □

We next have an easy remark.

LEMMA 3.3 *Let F be a face of a polytope P, and let v be a vertex of F. Then $N(v, P) \subseteq N(v, F)$.*

Proof. This is clear; any hyperplane which supports P at v must also support F. □

We are now in a position to prove Theorem 3.1. We first observe that there is nothing to prove if $m = 0$. Further, the general result follows from the case $m = 1$, since that will assert that we can replace any product of $d + 1$ terms $[P] - 1$, with $P \in \mathcal{P}$, by a sum of products of d such terms, with a translation term $[t] - 1$. Moreover, observe that Π is generated by simplex classes (a neat proof of the corresponding dissection result can be found in [**36**]).

LEMMA 3.4 *Let $P_0, \ldots, P_d$ be simplices, with $\dim P_j \geqslant 1$ for each $j = 0, \ldots, d$. Then there are facets F_j of P_j for $j = 0, \ldots, d$, such that, for each direction u, there is at least one j for which $\eta(F_j, u) = \eta(P_j, u)$.*

Proof. To see this, pick any edge E_j of P_j for $j = 0, \ldots, d$, and let u_j be a non-zero vector parallel to E_j for $j = 0, \ldots, d$. Changing signs of the u_j as necessary, we can assume that the condition of Lemma 3.2 is satisfied (with u_j in place of $\varepsilon_j u_j$). Let v_j be the vertex of E_j for which $H_j^+ = N(v_j, E_j)$ is the normal cone. Then $\operatorname{int} N(v_j, E_j) = \operatorname{int} H_j^+$, so that Lemmas 3.2 and 3.3 imply that

$$\bigcap_{j=0}^{d} \operatorname{int} N(v_j, P_j) \subseteq \bigcap_{j=0}^{d} \operatorname{int} N(v_j, E_j) = \bigcap_{j=0}^{d} \operatorname{int} H_j^+ = \emptyset.$$

Now let F_j be the facet of P_j opposite to v_j for $j = 0, \ldots, d$. The support hyperplanes to P_j which also support F_j are just those whose outer normals do not lie in $\operatorname{int} N(v_j, P_j)$. That is, for each direction u, there is at least one $j = 0, \ldots, d$ for which P_j and F_j have the same support hyperplane with outer normal u, which is the lemma. □

The final step is

LEMMA 3.5 *Theorem 3.1 is true if $m = 1$.*

Proof. Since Π is generated by the classes of simplices, we need only show that, if $P_0, \ldots, P_d$ are non-empty simplices, then $x := ([P_0] - 1) \cdots ([P_d] - 1) \in T$. We use induction on the dimensions of the P_j. If one of them has dimension 0, then the result already holds, so that we can suppose that all have positive dimension. Choose facets F_j of P_j for $j = 0, \ldots, d$ which satisfy the condition of Lemma 3.4. We now use Theorem 2.5, which states that the mapping $P \mapsto \gamma(P, \cdot) := \exp(\eta(P, \cdot))$ induces a separating ring homomorphism on Π. For each direction u, there is a j such that $\eta(P_j, u) = \eta(F_j, u)$, and so it follows that

$$\gamma(([P_0] - [F_0]) \cdots ([P_d] - [F_d]), \cdot) \equiv 0;$$

in other words,

$$y := ([P_0] - [F_0]) \cdots ([P_d] - [F_d]) = 0.$$

If we now write $[P_j] - [F_j] = ([P_j] - 1) - ([F_j] - 1)$ for each j, expand the product $y = ([P_0] - [F_0]) \cdots ([P_d] - [F_d])$ accordingly, and subtract $y = 0$ from x, we see that we have expressed the original product as a sum of such product terms, in each of which the dimension of at least one P_j has been lowered by 1. This is the required inductive step, which completes the proof. □

Many of the valuations φ in which we are interested have additional properties, usually involving translation. For instance, an important sub-class consists of those which are *translation invariant*, meaning that $\varphi(P + t) = \varphi(P)$ for each polytope P and translation vector t. Among such valuations are volume, surface area and the Euler characteristic. Others, such as the moment vector, are *translation covariant*, so that $\varphi(P + t) = \varphi(P) + \psi(P)t$ for some valuation ψ; here, φ takes values in $\mathbb{V}$ itself, and ψ then turns out to be translation invariant. A third example is the inertia tensor, which exhibits a quadratic behaviour under translation.

In general, following **[31]**, we define recursively a valuation φ to be *polynomial of degree m* if $P \mapsto \varphi(P + t) - \varphi(P)$ is polynomial of degree $m - 1$ for each translation vector $t \in \mathbb{V}$; the recursion begins with the case $m = 0$ of translation invariance. We loosely refer to this property as *translation covariance* also, particularly when the degree of polynomiality is immaterial. The abstract abelian group for such valuations turns out, as we shall see, to be the quotient Π/T^{m+1}.

However, before we go further, it is convenient to impose another condition on valuations. Let $U = (u_1, \ldots, u_n)$ be a (for the moment) fixed set of normal vectors in $\mathbb{V}$ (or, more strictly, $\mathbb{V}^*$) which spans $\mathbb{V}$ positively, and write $\mathcal{P}(U)$ for the family of (non-empty) polytopes in $\mathbb{V}$ of the form

$$P(U, b) := \{x \in \mathbb{V} \mid \langle x, u_j \rangle \leqslant \beta_j \text{ for } j = 1, \ldots, n\},$$

where $b = (\beta_1, \ldots, \beta_n) \in \mathbb{F}^n$ is called the *support vector* of $P(U, b)$; the individual β_j are the *support parameters*. A valuation φ on $\mathcal{P}$ is called *weakly continuous* if the mapping $b \mapsto \varphi(P(u, b))$ is continuous on $\mathcal{P}(U)$ for each such set U of normals.

Our aim, later on, is to describe a suitable abstract model for weakly continuous translation covariant valuations. At this stage, though, we wish to rephrase the weak continuity condition algebraically. To this end, define the *algebra ideal* $A \leqslant Z^2$ by

$$A := \langle ([\lambda P] - 1)([Q] - 1) - ([P] - 1)([\lambda Q] - 1) \mid P, Q \in \mathcal{P} \text{ and } \lambda > 0 \rangle. \quad (3.1)$$

Note that

$$([P+Q]-1)-([P]-1)-([Q]-1)=([P]-1)([Q]-1)\in Z^2$$

for P, $Q \in \mathcal{P}$, from which it is not hard to deduce that

$$([\lambda P]-1)([Q]-1)-([P]-1)([\lambda Q]-1)\in Z^3$$

when $\lambda \in \mathbb{Q}$. A crucial result of **[19]**, based on an earlier one in **[14]** (with a nice geometric proof by Thorup – see also **[33]** for a different approach), can be rewritten in the following form.

THEOREM 3.6 *With the notation introduced previously*,

$$A \leqslant Z^3 + T.$$

Again, it may be preferable to express this result in the form $A \leqslant Z^3 + ZT$.

As a consequence, we have the main result of **[19]**.

COROLLARY 3.7 *Except at grade* 0, *the quotient* Π/T *is a graded algebra over the base field* $\mathbb{F}$.

The exception is that, at grade 0, we have $\mathbb{Z}$ (the Euler characteristic) rather than $\mathbb{F}$.

We now introduce the core object of study in this paper. With a temporary notation (which we shall later discard), define $\bar{\Pi}_r := \bar{\Pi}_r(\mathbb{V})$ by

$$\bar{\Pi}_r := \begin{cases} \mathbb{F}, & \text{if } r=0, \\ Z/Z^2, & \text{if } r=1, \\ Z^r/(Z^{r+1}+Z^{r-2}A), & \text{if } r\geqslant 2, \end{cases} \tag{3.2}$$

with the obvious convention $Z^0 := \Pi$. Then we have

THEOREM 3.8 *The ring*

$$\bar{\Pi} := \bigoplus_{r\geqslant 0} \bar{\Pi}_r$$

is a graded algebra over $\mathbb{F}$.

Proof. The multiplication in $\bar{\Pi}$ is clear; note that, indeed,

$$\bar{\Pi}_r \cdot \bar{\Pi}_s = \bar{\Pi}_{r+s}$$

for $r, s \geqslant 0$, so long as we further take a product of any two elements modulo A where appropriate. So, we need to show that each $\bar{\Pi}_r$ is a vector space over $\mathbb{F}$, and that the multiplication is compatible with the vector space structure.

For $\bar{\Pi}_1$, we can follow the argument of **[19]**, except that we do not factor out by translation. So, if $x \in \bar{\Pi}_1$ and $\lambda \in \mathbb{F}$, we define

$$\lambda x := \begin{cases} \lambda \circ x, & \text{if } \lambda \geqslant 0, \\ -(-\lambda)\circ x, & \text{if } \lambda < 0. \end{cases}$$

Some easy manipulation (which we shall not go into) shows that this does induce a vector space structure on $\bar{\Pi}_1$; the details can be found in [**19**, Lemma 27] (again, ignoring those parts which refer to translation invariance).

Also following [**19**], it can be seen that $\bar{\Pi}_1$ is isomorphic to the linear space of differences of support functionals on $\mathcal{P}$.

For $\bar{\Pi}_2$, we observe that

$$(\lambda \circ x) \cdot y \equiv x \cdot (\lambda \circ y) \ (\mathrm{mod}\, A)$$

for $x, y \in \bar{\Pi}_1$ and $\lambda \geqslant 0$, and this permits us to define

$$\lambda(xy) := (\lambda x)y$$

unambiguously for $x, y \in \bar{\Pi}_1$ and $\lambda \in \mathbb{F}$, with λx as defined previously. Since $\bar{\Pi}_2$ is generated by such products xy, this extends to a vector space structure on $\bar{\Pi}_2$. The extension to $\bar{\Pi}_r$ for larger r is now routine. □

Before we end this section, we should make some further remarks about the ideal A. By itself, A is not particularly useful; it only really comes into force when we take a quotient by some power of the cylinder ideal Z (or, equivalently in view of Theorem 3.1, some power of the translation ideal T). Suppose that we are working in Π/Z^{m+1} for some m. The remark after the definition of A in (3.1) easily leads to a rational algebra structure (except, as usual, in grade 0), and we can define (again following [**19**])

$$\log P := \sum_{r=1}^{m} \frac{(-1)^{r-1}}{r} ([P] - 1)^r.$$

Then $\log P \in \bar{\Pi}_1$ and, crucially for A,

$$\log P \equiv [P] - 1 \ (\mathrm{mod}\, Z^2).$$

Thus $\log P \cdot \log Q \equiv ([P] - 1)([Q] - 1) \ (\mathrm{mod}\, Z^3)$, with obvious implications for the rôle played by A in the definition of $\bar{\Pi}_r$ for $r \geqslant 2$.

§4. *Summand subalgebras.* A strong combinatorial thread runs through much of what we do in this paper, although it has not become apparent hitherto. For many reasons, simple polytopes are natural objects of study; recall that a d-polytope P is *simple* if each of its vertices belongs to exactly d of its facets. We shall see below that the space of summands of a simple polytope is a natural object of study in the present context.

We have already introduced the basic notions of Minkowski linear combinations. They play a central rôle in this paper, and so we now discuss them in more detail. We largely follow [**12**, Chapter 14], to which we refer for further details, but we slightly vary some terminology, and also introduce notation which is more convenient for our purposes.

If $P, Q \in \mathcal{P}$ are such that $P = Q + Q'$ for some $Q' \in \mathcal{P}$, then we call Q a *summand* of P. Note that a translate of a summand is also a summand. We write $Q \preceq P$ if Q is a summand of λP for some $\lambda > 0$; observe that the relation $\preceq$ is clearly transitive. Further, we write $P \approx Q$ for the equivalence relation given by $P \preceq Q \preceq P$, and call P and Q *strongly isomorphic*.

Various properties of this relation follow directly from the definition, if we bear in mind that $(\lambda + \mu)P = \lambda P + \mu P$ for $\lambda, \mu \geqslant 0$. The key observation is that, if $Q_j \preceq P_j$ for $j = 1, 2$, then $Q_1 + Q_2 \preceq P_1 + P_2$. In particular, if $P_1 = P_2 = P$, then

$$Q_1, Q_2 \preceq P \;\Rightarrow\; Q_1 + Q_2 \preceq P,$$
$$Q \preceq P \text{ and } \lambda \geqslant 0 \;\Rightarrow\; \lambda Q \preceq P.$$

It follows that

$$\bar{\mathcal{K}}(P) := \{Q \in \mathcal{P} \mid \mathcal{Q} \preceq \mathcal{P}\}$$

has the natural structure of a convex cone (indeed, it is not hard to see that it is polyhedral); we call $\bar{\mathcal{K}}(P)$ the (*closed*) *type-cone* of P. Note that, if $Q \preceq P$, then $\bar{\mathcal{K}}(Q) \subseteq \bar{\mathcal{K}}(P)$; in fact, $\bar{\mathcal{K}}(Q) \leqslant \bar{\mathcal{K}}(P)$. In particular, if $Q \approx P$, then $\bar{\mathcal{K}}(Q) = \bar{\mathcal{K}}(P)$.

From the definition in Section 2, it follows that, for $P, Q \in \mathcal{P}$ and $\lambda \geqslant 0$,

$$F(P + Q, u) = F(P, u) + F(Q, u),$$
$$F(\lambda P, u) = (P, u),$$

for each vector $u \in \mathbb{V}$. Hence, if $Q \preceq P$, then $F(Q, u) \preceq F(P, u)$ for each normal vector u.

In terms of normal fans, a necessary and sufficient condition for $Q \preceq P$ is that $\mathcal{N}(P)$ be a refinement of $\mathcal{N}(Q)$; indeed, for a Minkowski sum, $\mathcal{N}(P + Q)$ is the common refinement of $\mathcal{N}(P)$ and $\mathcal{N}(Q)$. Thus, if $P \approx Q$, then $\mathcal{N}(P) = \mathcal{N}(Q)$, and there is an isomorphism between their faces given by $F(P, u) \leftrightarrow F(Q, u)$ for each vector u (this accounts for the term "strongly isomorphic"). We denote by $\mathcal{K}(P)$ the *strong isomorphism class* of $P \in \mathcal{P}$; that is,

$$\mathcal{K}(P) := \{Q \in \mathcal{P} \mid Q \approx P\}.$$

Looking at strong isomorphism from the viewpoint of normal cones accounts for the alternative term *normally equivalent* sometimes used instead of strongly isomorphic (see, for instance, [**9**]); other terms employed have been *analogous* (particularly in translations from Russian – but this is so obviously a generally useful word, and so should not be used in a specialized context) and *locally similar*. In a natural sense, of course, $\bar{\mathcal{K}}(P) = \operatorname{cl} \mathcal{K}(P)$ is the closure of $\mathcal{K}(P)$ in the weak sense of limits of support vectors in any fixed class $\mathcal{P}(U)$.

It is clear that, if $P \in \mathcal{P}(U)$ and $Q \preceq P$, then $Q \in \mathcal{P}(U)$ also. However, it will not generally be the case that, if $P, Q \in \mathcal{P}(U)$, then $P + Q \in \mathcal{P}(U)$.

Let F and G be two polyhedra. If $F \lhd G$, then we denote by $u(F, G)$ the unit outer normal vector to G at F, always taken intrinsically in the linear subspace

$$G_{\parallel} := \operatorname{lin}(G - G) \leqslant \mathbb{V} \tag{4.1}$$

parallel to G (it is at this point that it is important to identify $\mathbb{V}^*$ with $\mathbb{V}$, and to assume that $\mathbb{F}$ is square-root closed). When F is not a facet of G, we can adopt the convention that $u(F, G) := o$.

We may also observe here that, if Q is any polytope, and P is obtained from Q by any sufficiently small parallel displacements of its facet hyperplanes, then $\mathcal{N}(P)$ refines $\mathcal{N}(Q)$, so that $Q \preceq P$. In particular, there is always a simple polytope P with the same number of facets as Q such that $Q \preceq P$. A useful remark for the future is the following.

LEMMA 4. 1 *If $Q_1, \ldots, Q_m \in \mathcal{P}$ are any polytopes, then there exists a simple d-polytope P such that $Q_j \in \bar{\mathcal{K}}(P)$ for each $j = 1, \ldots, m$.*

Proof. Define $Q := Q_1 + \cdots + Q_m$. By adding any d-polytope to Q if necessary, we may assume that $\dim Q = d$. We may then suppose that $Q \in \mathcal{P}(U)$ for a suitable set U of normal vectors. If P is obtained from Q by any sufficiently small change in its support vector, then P is a simple polytope such that $\mathcal{N}(P)$ refines $\mathcal{N}(Q)$, and hence $Q \preceq P$. Since $Q_j \preceq Q$ for each j, it follows that $Q_j \preceq P$ also, as required. □

When we work within Π, on any given occasion we are only working in practice with the classes of finitely many polytopes. We therefore conclude from Lemma 4.1 that we may always confine our attention to appropriate subrings or corresponding quotient subalgebras. Let $\mathcal{K}$ be a strong isomorphism class of polytopes. Then we define

$$\Pi(\mathcal{K}) := \langle [P] \cdot [Q]^{-1} \mid P, Q \in \mathcal{K} \rangle \leqslant \Pi,$$
$$\bar{\Pi}(\mathcal{K}) := \langle \log Q \mid Q \in \mathcal{K} \rangle \leqslant \bar{\Pi}.$$

It is easy to see that $\Pi(\mathcal{K})$ is a subring of Π and, by definition, $\bar{\Pi}(\mathcal{K})$ is a subring of $\bar{\Pi}$.

For the remainder of this section, let P be a fixed simple d-polytope, with $\mathcal{K} := \mathcal{K}(P)$ its strong isomorphism class. We write $U = (u_1, \ldots, u_n)$ for its set of facet normals, and $b = (\beta_1, \ldots, \beta_n) \in \mathbb{F}^n$ for its support vector, as defined in Section 3; thus $P = P(U, b)$. Let $e_j \in \mathbb{F}^n$ be the jth standard coordinate basis vector. Note that, if we replace P by a sufficiently large dilatate $\lambda P = P(U, \lambda b)$ (with $\lambda > 0$), then $\lambda b + e_j$ is the support vector of a polytope $(P, \lambda b + e_j) \in \mathcal{K}$. (Any small enough perturbation of the support vector of a simple polytope preserves its strong isomorphism class.)

We now define

$$E_j := [P(U, \lambda b + e_j)][P(U, \lambda b)]^{-1} \in \Pi(\mathcal{K}). \tag{4.2}$$

LEMMA 4.2 *The class E_j is independent of b and (suitably large) λ.*

Proof. Suppose that $P(U, c)$, $P(U, \mu c + e_j) \in \mathcal{K}$ also, for some $c \in \mathbb{F}^n$ and $\mu > 0$. Then

$$\begin{aligned} E_j &= [P(U, \lambda b + e_j)][P(U, \lambda b)]^{-1} \\ &= ([P(U, \lambda b + e_j)][P(U, \mu c)]) \cdot ([P(U, \lambda b)][P(U, \mu c)])^{-1} \\ &= [P(U, \lambda b + e_j) + P(U, \mu c)] \cdot [P(U, \lambda b) + P(U, \mu c)]^{-1} \\ &= [P(U, \lambda b + \mu c + e_j)] \cdot [P(U, \lambda b + \mu c)]^{-1}, \end{aligned}$$

which is now symmetric between (b, λ) and (c, μ). □

In what follows, we assume that any vectors $b, c \in \mathbb{F}^n$ are chosen sufficiently large so that all relevant polytopes are in $\mathcal{K}$.

LEMMA 4.3 *The class E_j is invertible in $\Pi(\mathcal{K})$.*

Proof. If c is such that $P(U, c)$, $P(U, c - e_j) \in \mathcal{K}$, then

$$[P(U, c - e_j)][P(U, c)]^{-1} = ([P(U, (c - e_j) + e_j)][P(U, c - e_j)])^{-1} = E_j^{-1},$$

with $c - e_j$ instead of b (or λb) in the definition. This is as required. □

For $P \in \mathcal{K}$, we have $[P]^k = [kP]$ for integer $k \geqslant 0$, with $[P]^0 = [o] = 1$. Using $[P]^{-1} = [-P]\varepsilon$, we can clearly define $[P]^q$ for any $q \in \mathbb{Q}$. We naturally extend the definition to all $\lambda \in \mathbb{F}$ by writing

$$[P]^\lambda := \begin{cases} [\lambda P], & \text{if } \lambda \geqslant 0, \\ [\lambda P]\varepsilon, & \text{if } \lambda < 0. \end{cases}$$

A routine verification of cases shows that

LEMMA 4.4 *If $P \in \mathcal{P}$, then*

$$[P]^{(\lambda + \mu)} = [P]^\lambda \cdot [P]^\mu$$

for all $\lambda, \mu \in \mathbb{F}$.

The definition above clearly extends to E_j^λ for each j and all $\lambda \in \mathbb{F}$. We then have

THEOREM 4.5 *Let $P = P(U, b)$ be a simple polytope, with the conventions for U and b adopted earlier. Then*

$$[P] = \prod_{j=1}^{n} E_j^{\beta_j}.$$

So far as $\bar{\Pi}$ is concerned, with the same simple polytope class $\mathcal{K}$, we clearly have $\bar{\Pi}_1(\mathcal{K}) \cong \mathbb{F}^n$, and we can regard $\bar{\Pi}(P)$ as a quotient algebra of the polynomial algebra $\mathbb{F}[e_1, \ldots, e_n]$, where we can make the identification $e_j := \log P(U, \lambda b + e_j) - \log P(U, \lambda b)$ for each j (again, with suitably large $\lambda > 0$). In fact, we then have

$$\log P = \sum_{i=1}^{n} \beta_i e_i.$$

A central structural result is

LEMMA 4.6 *Let P be a simple d-polytope, and let $F_{j(1)}, \ldots, F_{j(r)}$ be facets of P such that $F_{j(1)} \cap \cdots \cap F_{j(r)} = \emptyset$. Then*

(*a*) $([P_{j(1)}] - [P]) \cdots ([P_{j(r)}] - [P]) = 0$ *in* Π, *where* $P = P(U, b)$ *and* $P_j = P(U, b + e_j)$, *with b replacing λb for suitably large λ if necessary,*
(*b*) $e_{j(1)} \cdots e_{j(r)} = 0$ *in* $\bar{\Pi}$.

Proof. Let $u \in \mathbb{V}$ be arbitrary. Then $u \in N(v, P)$, for some vertex v of P. Now $\eta(P_k, u) \neq \eta(P, u)$ only when $v \in F_k$. Hence, under the given conditions, for each vector $u \in \mathbb{V}$ there is at least one $j = 1, \ldots, r$ such that $\eta(P_j, u) = \eta(P, u)$. The two results claimed follow at once. (Recall that $\log P$ is identified with $\eta(P, \cdot)$.) □

In the polynomial algebra $\mathbb{F}[X_1, \ldots, X_n]$, define the ideal M by

$$M := \langle X_{j(1)} \cdots X_{j(r)} \mid F_{j(1)} \cap \cdots \cap F_{j(r)} = \emptyset \rangle,$$

where we continue to employ the same notation. The graded algebra $\mathbb{F}[X_1, \ldots, X_n]/M$ is called the *face-ring* of the simple polytope P (or, more strictly, of the dual simplicial polytope P^*). It follows immediately from Theorem 4.6 that

COROLLARY 4.7 *If P is a simple d-polytope in $\mathbb{V}$, then the subalgebra $\bar{\Pi}(P)$ of $\bar{\Pi}$ is isomorphic to a quotient of the face-ring of P.*

As a matter of fact, $\bar{\Pi}(P)$ is actually isomorphic to the face-ring. We could prove this here, but we shall be able to do it more readily later in Section 11.

§5. *Translation covariant valuations.* Suppose that the valuation φ on $\mathcal{P}$ is polynomial of degree $m \geqslant 1$, so that $\varphi'(\cdot, t)$, defined for $t \in \mathbb{V}$ by

$$\varphi'(P, t) = \varphi(P + t) - \varphi(P),$$

is polynomial of degree $m - 1$. Recalling that Π is the universal group for valuations on $\mathcal{P}$, we see that, when we lift φ and φ' to homomorphisms on Π and use linearity, we have

$$\varphi'([P], t) = \varphi([P]([t] - 1)),$$

with $\varphi'([\cdot], t)$ polynomial of degree $m - 1$. By induction, it follows that, for (fixed) $t_1, \ldots, t_k \in \mathbb{V}$, the valuation

$$\varphi^{(k)}([\cdot], t_1, \ldots, t_k) := \varphi([\cdot]([t_1] - 1) \cdots ([t_k] - 1))$$

is polynomial of degree $m - k$, and thus, in particular, vanishes whenever $k > m$. We conclude that we have shown

THEOREM 5.1 *If φ is a valuation on $\mathcal{P}$ which is polynomial of degree m, then, as an induced homomorphism on Π,*

$$\ker \varphi \geqslant T^{m+1}.$$

Theorem 3.1 implies that $Z^{d+k} = Z^d T^k$ for each $k \geqslant 0$; it follows that we can henceforth work in Π/Z^{d+m+1}. Thus we can write

$$[P] = \exp(\log P) = \sum_{s=0}^{d+m} \frac{1}{s!} (\log P)^s,$$

with the sth term

$$[P]_s := \frac{1}{s!} (\log P)^s \in Z^s / Z^{s+1}$$

(or in $\bar{\Pi}_s$, if we additionally factor out the algebra ideal A, which we shall do shortly). Note that $[P]_0 = 1$ and $[P]_1 = \log P$.

We deduce that a translation covariant valuation φ admits a decomposition

$$\varphi = \sum_{s=0}^{d+m} \varphi_s$$

for some m, where each φ_s is (non-negative) homogeneous of degree s, in that

$$\varphi_s(\lambda P) = \lambda^s \varphi_s(P) \tag{5.1}$$

for each $P \in \mathcal{P}$ and non-negative $\lambda \in \mathbb{Q}$ (with the appropriate convention $0^0 = 1$). If φ is, in addition, weakly continuous, then (5.1) will hold for all $\lambda \geqslant 0$.

Finally, if we consider a Minkowski linear combination $P = \sum_{i=1}^{k} \lambda_i P_i$, then

$$\varphi_s(P) = \varphi_s((\lambda_1 \log P_1 + \cdots + \lambda_k \log P_k)^s)$$

is a homogeneous polynomial of total degree s in $\lambda_1, \ldots, \lambda_k$ for non-negative $\lambda_j \in \mathbb{Q}$, and generally for $\lambda_j \geqslant 0$ if φ is weakly continuous.

We summarize the foregoing discussion in

THEOREM 5.2 *Let φ be a weakly continuous translation covariant valuation on $\mathcal{P}$. If $P_1, \ldots, P_k \in \mathcal{P}$, then $\varphi(\lambda_1 P_1 + \cdots + \lambda_k P_k)$ admits a polynomial expansion in the variables $\lambda_1, \ldots, \lambda_k \geqslant 0$. If φ is polynomial of degree m, then the polynomial expansion has total degree (at most) $d + m$.*

REMARK 5.3 The following is worth noting. Suppose that φ is a weakly continuous valuation on $\mathcal{P}$ which is homogeneous of degree r. We then see that

$$\varphi = \varphi|_{\bar{\Pi}_r}$$

(that is, $\varphi(x) = 0$ if $x \in \bar{\Pi}_s$ for any $s \neq r$). Hence we can define

$$\varphi(P_1, \ldots, P_r) := \varphi(\log P_1 \cdots \log P_r),$$

the expression on the right, of course, corresponding to the lift of φ to $\bar{\Pi}$. Then it is exactly factoring out additionally by the algebra ideal A which makes sense of

$$\varphi(\lambda P_1, P_2, \ldots, P_r) = \lambda \varphi(P_1, P_2, \ldots, P_r)$$

for all $\lambda \geqslant 0$ if $r \geqslant 2$.

§6. *Tensorials.* The results of Section 5 show that weakly continuous translation covariant valuations on $\mathcal{P}$ admit polynomial expansions in their translation vector arguments. Since polynomial functions on $\mathbb{V}$ can be factored through the tensor algebra over $\mathbb{V}$ (with appropriate coefficients, which here are themselves valuations), in this section we introduce tensors, and an important family of tensor-valued valuations.

Indeed, in preparation both for the separation theorem for $\bar{\Pi}$ and for our later definition of tensor weights, we now discuss the functions on polytopes which are the general terms in the sequence volume, moment vector, inertia tensor, and so on. To do this, we must first introduce symmetric tensors.

The space of *symmetric tensors* on $\mathbb{V}$ is just the graded polynomial algebra

$$\mathbb{T} = \mathbb{T}(\mathbb{V}) := \mathbb{F}[e_1, \ldots, e_d],$$

with $\{e_1, \ldots, e_d\}$ any (linear) basis of $\mathbb{V}$. The subspace of symmetric s-tensors (that is, polynomials in $\mathbb{T}$ of degree s) is denoted $\mathbb{T}_s$. Thus $\mathbb{T}_0 \cong \mathbb{F}$, and $\mathbb{T}_1 \cong \mathbb{V}$ itself. As a vector space over $\mathbb{F}$, we have

$$\dim \mathbb{T}_s = \binom{d+s-1}{s}.$$

It is occasionally helpful to identify an s-tensor with a homogeneous polynomial function on the dual space $\mathbb{V}^*$ of total degree s. Note, however, that we do not usually distinguish between the dual spaces $\mathbb{V}$ and $\mathbb{V}^*$, which we identify by means of the inner product. However, at one place in Section 8 we shall find it helpful to make this distinction. In the same spirit, there is one point at which it is perhaps preferable to distinguish between tensors on $\mathbb{V}$ and those on $\mathbb{V}^*$. In particular, we wish to talk about the family $\mathbb{T}_{s,1}$ of tensors of type $(s, 1)$, where the "1" indicates a component from the dual $\mathbb{V}^*$. However, this concept occurs only once, and then incidentally.

Because we wish to emphasize the fact that $\mathbb{T} := \bigoplus_{s \geqslant 0} \mathbb{T}_s$ is an algebra, we always write the (tensor) product of two tensors a and b as ab, rather than as $a \otimes_{\text{sym}} b$ (with $\otimes_{\text{sym}}$ denoting the symmetric tensor over $\mathbb{F}$), thus suppressing the tensor symbol. (In contrast, an unadorned $\otimes$ is the usual non-symmetric tensor product over $\mathbb{Z}$.) In particular, if $x \in \mathbb{V} = \mathbb{T}_1$, then x^s is a *basic* s-tensor.

So far, we have not needed the extra assumption that our ground field $\mathbb{F}$ is square-root closed. Even here there are ways to avoid this, although they are somewhat clumsy; however, the convenience of being able to work in an inner product space $\mathbb{V}$ with a corresponding norm far outweighs the disadvantage of loss of generality.

Let $\mathcal{P}$ be a k-polytope, that is, a polytope of dimension k. The *s-tensorial* $M_s(P)$ of P is defined by

$$M_s(P) := \frac{1}{s!} \int_P x^s dx \in \mathbb{T}_s,$$

where the integral is with respect to the ordinary k-dimensional volume measure (scaled by the unit cube derived from an orthonormal basis) in k-flats (affine subspaces of dimension k). The reason for scaling by $1/s!$ will become apparent later, when we consider the abstract analogues of tensorials; without the scaling, an alternative term for the concept is a *mass moment*. As a function of P, the s-tensorial is (positive) homogeneous of degree $k + s$, in that

$$M_s(\lambda P) = \lambda^{k+s} M_s(P)$$

for all $\lambda \geqslant 0$. We thus see that $M_0 = \text{vol}_k$ is ordinary k-volume, while $M_1(P)$ is the moment vector of P and $M_2(P)$ is half the usual inertia tensor of P. When $k = 0$, the appropriate definition is that the 0-volume of a singleton point-set is 1, so that

$$M_s(v) = \frac{1}{s!} v^s$$

for $v \in \mathbb{V}$. Finally, we find it convenient to define $M_s(P) := 0$ when $s < 0$.

A particular fact about tensorials will be central to our investigations. Before stating the general result, let us motivate it by means of a few examples.

As before, if P is a k-polytope and $F \lhd P$, then $u(F, P)$ denotes the outer unit normal vector to P at F. The *Minkowski relation* for areas of facets says that

$$\sum_{F \lhd P} \mathrm{vol}_{k-1}(F) u(F, P) = o,$$

the zero vector. Perhaps less familiar is

$$\sum_{F \lhd P} \langle M_1(F), u(F, P) \rangle = k \, \mathrm{vol}_k(P);$$

this shows that the k-volume of P is determined by the moment vectors of its facets.

It turns out that there is a general family of such relations (although the last is not a typical example). What we call the *Green-Minkowski connexion* between tensorials, or *GMC* for short, is the following.

THEOREM 6.1 *Let $k \geqslant 1$, let P be a k-polytope, and let $t \in P_{\|}$. Then*

$$\sum_{F \lhd P} M_s(F) \langle u(F, P), t \rangle = M_{s-1}(P) t.$$

Proof. This is actually just Green's Theorem [**10**], applied to the function x^s. We have

$$\langle t, \nabla \rangle x^s = \langle t, \nabla_P \rangle x^s = s x^{s-1} t,$$

where ∇_P is the divergence restricted to $P_{\|}$, so that

$$\begin{aligned} \sum_{F \lhd P} M_s(F) \langle u(F, P), t \rangle &= \sum_{F \lhd P} \left\{ \frac{1}{s!} \int_F x^s \, dx \right\} \langle u(F, P), t \rangle \\ &= \frac{1}{s!} \int_P s x^{s-1} t \, dx \\ &= M_{s-1}(P) t, \end{aligned}$$

as claimed. □

When we recall that $\mathbb{T}$ embeds in its field of fractions (because the product of two non-zero tensors is clearly non-zero), we see that we can actually solve the Green-Minkowski connexion for $M_{s-1}(P)$, which we therefore conclude is completely determined by the $M_s(F)$ for the facets F of P. We may observe that, to solve for $M_{s-1}(P)$ in this way, we need only consider a single non-zero $t \in P_{\|}$.

A variant of the relation of Theorem 6.1 involves tensors in $\mathbb{T}_{s,1}$ (we recall that the suffix "1" refers to a tensor component in $\mathbb{V}^*$). We obtain this by removing the dependence on the vector $t \in P_{\|}$. If $\dim P = k$, let $\{e_1, \ldots, e_k\}$ be a basis of $P_{\|}$, and define

$$q(P) := \sum_{i=1}^{k} e_i e_i^* \in \mathbb{T}_{1,1},$$

with $\{e_1^*, \ldots, e_k^*\}$ the corresponding dual basis.

COROLLARY 6.2 *Let $k \geqslant 1$, and let P be a k-polytope. Then, for each s,*

$$\sum_{F \lhd P} M_s(F) u(F, P) = M_{s-1}(P) q(P).$$

A general k-polytope P can be dissected into k-simplices, and so, to find $M_s(P)$, it is enough to be able to calculate the s-tensorial of a k-simplex. Although we shall not need the formula (and in any case it is more easily proved using Theorem 18.2 or, rather, its expression in terms of weights), we give a proof here to illustrate how to apply the Green-Minkowski connexion.

THEOREM 6.3 *Let $P = \operatorname{conv}\{w_0, \dots, w_k\}$ be a k-simplex in $\mathbb{V}$. Then*

$$M_s(P) = \frac{k!}{(k+s)!} \operatorname{vol}_k(P) \sum_{s(0)+\cdots+s(k)=s} w_0^{s(0)} \cdots w_k^{s(k)}.$$

Proof. We use Theorem 6.1, with $s+1$ instead of s and $t := \|w_1 - w_0\|^{-1}(w_1 - w_0)$. The case $k = 0$ is trivial, since $M_s(w_0) = (1/s!) w_0^s$. We therefore suppose that $k > 0$, and make the inductive assumption that the result holds for $(k-1)$-simplices and each $s \geqslant 0$.

For $j = 0, \dots, k$, let $F_j := \operatorname{conv}\{w_0, \dots, \widehat{w}_j, \dots, w_k\}$ be a facet of P (as usual, the notation means that w_j is omitted), and let $u_j := u(F_j, P)$ be the corresponding unit facet normal. Now

$$\varphi := \langle t, u_1 \rangle \operatorname{vol}_{k-1}(F_1) = -\langle t, u_0 \rangle \operatorname{vol}_{k-1}(F_0)$$

is the common $(k-1)$-volume of the orthogonal projection of F_0 and F_1 on the hyperplane orthogonal to t; it thus follows that

$$\|w_1 - w_0\| \varphi = k \operatorname{vol}_k(P).$$

Since $\langle t, u_j \rangle = 0$ for $j \geqslant 2$, we use the inductive assumption to deduce that

$$\begin{aligned}
tM_s(P) &= \sum_{j=0}^{k} \langle t, u_j \rangle M_{s+1}(F_j) \\
&= \langle t, u_0 \rangle \frac{(k-1)!}{(k+s)!} \operatorname{vol}_{k-1}(F_0) \sum_{s(1)+\cdots+s(k)=s+1} w_1^{s(1)} \cdots w_k^{s(k)} \\
&\quad + \langle t, u_1 \rangle \frac{(k-1)!}{(k+s)!} \operatorname{vol}_{k-1}(F_1) \sum_{s(0)+s(2)+\cdots+s(k)=s+1} w_0^{s(0)} w_2^{s(2)} \cdots w_k^{s(k)} \\
&= \frac{(k-1)!}{(k+s)!} \varphi \sum_{r \geqslant 0} (w_1^{r+1} - w_0^{r+1}) \sum_{s(2)+\cdots+s(k)=s-r} w_2^{s(2)} \cdots w_k^{s(k)} \\
&= \frac{(k-1)!}{(k+s)!} \varphi (w_1 - w_0) \sum_{s(0)+\cdots+s(k)=s} w_0^{s(0)} \cdots w_k^{s(k)} \\
&= t \frac{k!}{(k+s)!} \operatorname{vol}_k(P) \sum_{s(0)+\cdots+s(k)=s} w_0^{s(0)} \cdots w_k^{s(k)},
\end{aligned}$$

and on cancelling t we obtain the required formula for $M_s(P)$. □

§7. *Cone groups and separation.* In this section, we describe the central separation theorem for the algebra $\bar{\Pi}$. We shall not, in fact, prove the theorem at this stage; the proof will depend upon a characterization of certain subalgebras of the weight algebra $\mathcal{W}$ which will be the object of study of the next few sections, together with Theorem 4.7.

First, we must introduce the cone groups. Let L be a linear subspace of $\mathbb{V}$, and denote by $\mathcal{C}(L)$ the family of polyhedral cones in L with apex o. Then the *cone group* $\widehat{\Xi}_L$ is the abelian group with a generator $\langle K\rangle$ for each $K \in \mathcal{C}(L)$, which is called the *class* of K. These generators satisfy the relations $\langle K \cup K'\rangle + \langle K \cap K'\rangle = \langle K\rangle + \langle K'\rangle$ whenever $K \cup K' \in \mathcal{C}(L)$ also, and $\langle K\rangle = 0$ if $\dim K < \dim L$; thus $\widehat{\Xi}_L$ is the abstract group for *simple* valuations on $\mathcal{C}(L)$. If we define the additive subgroup $\Xi_<(L)$ of $\Xi(L)$ by

$$\Xi_<(L) := \langle [K] \mid K \in \mathcal{C}(L) \text{ and } \dim K < \dim L\rangle,$$

then, as an abelian group, we have

$$\widehat{\Xi}_L := \Xi(L)/\Xi_<(L). \tag{7.1}$$

Thus, for $K \in \mathcal{C}(L)$, we have $\langle K\rangle := [K] + \Xi_<(L)$. Of course, $\widehat{\Xi}_L$ has a rather trivial group structure, and we can think of one of its elements as a finite family of cones with integer multiplicities (which may be negative), where lower-dimensional cones in L are ignored. The *full* cone group $\widehat{\Xi}$ is then defined to be

$$\widehat{\Xi} = \widehat{\Xi}(\mathbb{V}) := \bigoplus_{L \leqslant \mathbb{V}} \widehat{\Xi}_L.$$

It is also useful to set

$$\widehat{\Xi}_s := \bigoplus_{\dim L = s} \widehat{\Xi}_L.$$

If F is a face of a polytope $P \in \mathcal{P}$, let $L := \operatorname{lin} N(F,P)$, with $N(F,P)$ the normal cone to P at F, and write $\widehat{n}(F,P) := \langle N(F,P)\rangle$ for the class of $N(F,P)$ in $\widehat{\Xi}_L$; that is, we take the class intrinsically. Note that, if $\dim F = k$, then $\dim N(F,P) = d - k$, so that $\widehat{n}(F,P) \in \widehat{\Xi}_{d-k}$; in particular, if P is a d-polytope, then $N(P,P) = \{o\}$.

Finally, we define the function $V_{k,s}$ on $\mathcal{P}$ by

$$V_{k,s}(P) := \sum_{F \in \mathcal{F}_k(P)} M_s(F) \otimes \widehat{n}(F,P) \in \mathbb{T}_s \otimes \widehat{\Xi}_{d-k}. \tag{7.2}$$

It is a variant of a standard result (compare [**19**, **22**, **23**]) that $V_{k,s}$ is a weakly continuous translation covariant valuation, and thus induces a (group) homomorphism on $\bar{\Pi}$ (and hence on Π also), which we denote by the same symbol.

The basic separation result, for which see Sections 11 and 13, is

THEOREM 7.1 *If $x \in \bar{\Pi}$ is such that $V_{k,s}(x) = 0$ for all $k = 0, \ldots, d$ and $s = 0, \ldots, m$, then $x \in T^{m+1}$.*

§8. *Tensor weights.* As an important part of the background for the proof of Theorem 7.1, we must provide an algebra as the target for the homomorphisms

$V_{k,s}$. Of course, the images of the $V_{k,s}$ must fit together to give an algebra under the operations induced by those on $\bar{\Pi}$; however, it is not immediately obvious what those operations will actually be.

In this section, we generalize the notion of (scalar) weight introduced in [**23**], and in the next we show that certain families of these weights form an algebra. Much of the treatment will parallel that of [**23**]; however, we follow a slightly simpler approach suggested by Paterson [**30**]. As in [**23**], for the most part we work with weights on fixed polytopes; the extension to the more abstract context is provided by standard results, which we mention when appropriate.

Let P be a polytope in $\mathbb{V}$. A (*tensor*) *weight* a on P is a mapping a: $\mathcal{F}(P) \to \mathbb{T}$ which satisfies the *Green-Minkowski relations* (*GMR*), in that there exists an a': $\mathcal{F}(P) \to \mathbb{T}$ such that

$$\sum_{F \lhd G} a(F)\langle u(F,G), t\rangle = a'(G)t, \tag{8.1}$$

for each face G of P; the conventions for the unit normal vectors $u(F,G)$ are those introduced in Section 6. In Section 10, we shall show that a' is itself a weight; however, for the moment we do not assume this. The vector space of all weights on P is denoted $\bar{\mathcal{W}}(P)$ (the reason for this notation will be made clear later). Comparing degrees shows that the GMR associates s-tensors $a(F)$ on k-faces F with $(s-1)$-tensors $a'(G)$ on $(k+1)$-faces G. The subspace of $\bar{\mathcal{W}}(P)$ consisting of those weights in $\bar{\mathcal{W}}(P)$ whose values are tensors of degree s on k-faces is denoted $\bar{\mathcal{W}}_{k,s}(P)$.

Defining $q(P) \in \mathbb{T}_{1,1}$ as in Section 6 enables us to express (8.1) in the alternative form

$$\sum_{F \lhd G} a(F)u(F,G) = a'(G)q(G), \tag{8.2}$$

for each face G of P.

We may set everything in a more abstract context by identifying a weight a on P with the element

$$\sum_{F \leqslant P} a(F) \otimes \widehat{n}(F,P) \in \mathbb{T} \otimes \widehat{\Xi}. \tag{8.3}$$

As far as the GMR are concerned, the thing to bear in mind is that $F \lhd G \leqslant P$ if and only if $N(F,P) \rhd N(G,P)$; moreover, we then have

$$u(F,G) = -u(N(G,P), N(F,P)).$$

It follows that, if we are given an element of $\mathbb{T} \otimes \widehat{\Xi}$, say

$$\sum_{K \in \mathcal{C}} b(K) \otimes \langle K\rangle,$$

with b: $\mathcal{C} \to \mathbb{T}$, then this corresponds to a weight exactly when, for each $K' \in \mathcal{C}$, we have

$$\sum_{K \rhd K'} b(K)u(K',K) = -q(K'_{\perp})b'(K'),$$

for some function $b'\colon \mathcal{C} \to \mathbb{T}$. Here,

$$P_\perp := (P_\|)^\perp$$

for $P \in \mathcal{Q}$.

The next result shows why we can make such an identification.

PROPOSITION 8.1 *Let $P, Q \in \mathcal{P}$ be such that $P \preceq Q$. Then there is a natural embedding $\bar{\mathcal{W}}(P) \hookrightarrow \bar{\mathcal{W}}(Q)$.*

Proof. We define $\Theta\colon \mathcal{F}(Q) \to \mathcal{F}(P)$ as follows. For $G \leqslant Q$, pick $u \in \operatorname{relint} N(G, Q)$, and set $G\Theta := F(P, u) \leqslant P$; this satisfies $\operatorname{relint} N(G, Q) \subseteq \operatorname{relint} N(G\Theta, P)$. Define

$$b(G) := \begin{cases} a(G\Theta), & \text{if } \dim G = \dim G\Theta, \\ 0, & \text{if } \dim G > \dim G\Theta. \end{cases}$$

Note that this is well defined, because $G\Theta$ is independent of the particular choice of u; the second case arises if $N(G, Q)$ has smaller dimension than $N(G\Theta, P)$. We make exactly the same definitions for the mapping a' associated with a by GMR, giving b' defined on $\mathcal{F}(Q)$.

We must now verify GMR for Q; that is, we have to show that, for each $J \leqslant Q$,

$$\sum_{G \lhd J} b(G)u(G, J) = b'(J)q(J).$$

There are various possibilities, which must be dealt with separately; we label the cases, so that we can refer to them again in the proof of Theorem 12.2. However, since $F(P, u) \preceq F(Q, u)$ for each normal vector u, we readily see that it is enough to consider the case $J = Q$ itself, for which $Q\Theta = P$; moreover, we lose no generality in assuming that $\dim Q = d$.

Case (a): $\dim P = d$. Note here that $q(P) = q(Q)$. Among the $G \lhd Q$, certain ones will have $\dim(G\Theta) = d - 1$, and then $u(G\Theta, P) = u(G, Q)$. More importantly, each $F \lhd P$ will arise as $F = G\Theta$ (for some $G \lhd Q$) in this way. There then follows

$$\begin{aligned} \sum_{G \lhd Q} b(G)u(G, Q) &= \sum_{G \lhd Q} a(G\Theta)u(G\Theta, P) \\ &= \sum_{F \lhd P} a(F)u(F, P) \\ &= a'(P)q(P) = b'(Q)q(Q), \end{aligned}$$

as required.

Case (b): $\dim P = d - 1$. Now Q has just two facets G_1, G_2 for which $\dim(G_j\Theta) = d - 1$; indeed, we have here $G_j\Theta = P$ for $j = 1, 2$. Moreover, $u(G_1, Q)$ and $u(G_2, Q)$ are the two unit normals to the hyperplane aff P; hence

$u(G_1, Q) = -u(G_2, Q)$, and thus

$$\begin{aligned}\sum_{G \lhd J} b(G)u(G, J) &= a(G_1\Theta)u(G_1\Theta, J\Theta) + a(G_2\Theta)u(G_2\Theta, J\Theta)\\ &= a(P)(u(G_1, Q) + u(G_2, Q))\\ &= 0 = b'(Q)q(Q),\end{aligned}$$

as we need.

Case (c): $\dim(P) \leqslant d - 2$. Here, $b(G) = 0$ for all $G \lhd Q$, because $G\Theta = P$ for such G, and hence $\dim(G\Theta) \leqslant \dim G - 1$. Thus all terms in the GMR vanish.

This completes the proof. □

There is an immediate consequence, although this is really an elementary observation.

COROLLARY 8.2 *If $P \approx Q$, with the strong isomorphism given by $F \leftrightarrow G$, then there is a natural isomorphism $a \mapsto b$ between $\bar{\mathcal{W}}(P)$ and $\bar{\mathcal{W}}(Q)$ given by $b(G) := a(F)$ for each $a \in \bar{\mathcal{W}}(P)$.*

It is obvious from their definition that weights are designed to mimic tensorials; *a priori*, the various tensorials of the faces of P induce weights on P. In fact, the core of the paper consists in showing that, in a sense, weights are just linear combinations of tensorials. However, a little care will be needed. A natural guess is that a weight on P will be an image (under the maps $V_{k,s}$) of an element of $\bar{\Pi}(P)$. Unfortunately, this is generally untrue unless P is simple; we shall give an easy counter-example in Section 10.

The parallel with tensorials does motivate one important definition. Central to our arguments will be the way that weights behave under linear mappings. In the translates of a linear subspace L, volume vol_L is uniquely specified by reference to a unit cube with respect to any orthonormal basis of L. If Φ is a linear mapping on $\mathbb{V}$ (the target space need not be specified), then, for each k-dimensional linear subspace L of $\mathbb{V}$, there is a constant $\kappa(L, \Phi) \geqslant 0$, called the *volume ratio*, such that

$$\mathrm{vol}_k(Q\Phi) = \kappa(L, \Phi)\,\mathrm{vol}_k(Q),$$

for each polytope Q lying in a flat parallel to L (this constant is the volume of $Q\Phi$, when Q is a unit cube in L). Further, Φ induces a linear mapping, also denoted Φ, on the algebra $\mathbb{T}$ of symmetric tensors on $\mathbb{V}$; if $a \in \mathbb{T}$, then its image under Φ is thus written $a\Phi$. Finally, if the weight $a(G)$ is attached to a face G of a polytope Q, then the weight attached to $G\Phi$ will be

$$(a\Phi)(G\Phi) := \kappa(G, \Phi)a(G)\Phi. \tag{8.4}$$

In other words, we apply the linear mapping to the tensor value of the weight, and then further scale by the appropriate volume ratio.

As we should expect, we have

LEMMA 8.3 *The Green-Minkowski relations are preserved under non-singular linear mappings.*

Proof. Here, we need to distinguish between the effects of a linear mapping on a vector space and on its dual. We first observe that, in verifying that GMR is preserved for a face G of a polytope P, we clearly need only assume that our linear mapping Φ is non-singular on G itself. Hence it is enough to suppose that $G = P$, which is full-dimensional. Further, since orthogonal mappings (with respect to our inner product) obviously preserve GMR, it is sufficient to consider the case when $\Phi\colon \mathbb{V} \to \mathbb{V}$ is an invertible linear mapping.

The right side of GMR transforms under Φ into

$$\kappa(P,\Phi)(a(P)t)\Phi = \kappa(P,\Phi)(a(P)\Phi)(t\Phi) = ((a\Phi)(P\Phi))(t\Phi).$$

For the left side, consider the contribution from a facet F of P. We pick an orthonormal basis $\{e_1,\ldots,e_d\}$ of $\mathbb{V}$, in such a way that $\{e_2,\ldots,e_d\}$ is a basis of the hyperplane $L := F_\|$ parallel to F. Thus

$$\kappa(P,\Phi) = |\det(e_1\Phi,\ldots,e_d\Phi)|,$$

with the determinant taken relative to the chosen basis $\{e_1,\ldots,e_d\}$. The dual basis $\{e_1^*,\ldots,e_d^*\}$ is transformed by the adjoint $\Phi^* = (\Phi^{-1})^\top$ of Φ; we therefore have

$$1 = \langle e_1^*, e_1\rangle = \langle u(F,P), e_1\rangle = \langle u(F,P)\Phi^*, e_1\Phi\rangle.$$

But since $u(F,P)\Phi \perp (F\Phi)_\| = L\Phi$, we have $u(F,P)\Phi = \lambda u(F\Phi,P\Phi)$ for some $\lambda(>0)$. On the other hand, direct calculation of the volume ratios gives

$$\kappa(P,\Phi) = \kappa(F,\Phi)\langle u(F\Phi,P\Phi), e_1\Phi\rangle,$$

since facet normals are always unit vectors, and hence

$$\kappa(P,\Phi)(u(F,P)\Phi^*) = \kappa(F,\Phi)u(F\Phi,P\Phi).$$

Collecting these terms together, we conclude that

$$\begin{aligned}\kappa(P,\Phi)(a(F)\Phi)\langle u(F,P),t\rangle &= \kappa(P,\Phi)(a(F)\Phi)\langle u(F,P)\Phi^*, t\Phi\rangle\\ &= (\kappa(F,\Phi)(a(F)\Phi))\langle u(F\Phi,P\Phi), t\Phi\rangle\\ &= (a\Phi)(F\Phi)\langle u(F\Phi,P\Phi), t\Phi\rangle,\end{aligned}$$

which provides the required contribution to the left side of GMR. □

Lemma 8.3 shows that the image of a weight under a non-singular linear mapping is also a weight. We also need the same result to hold for singular linear mappings. Since we can factor such a mapping into a composition of an (internal) orthogonal projection, a non-singular linear mapping and an isometric injection (which is trivial in this context), the next step is

LEMMA 8.4 *Let $\mathbb{W} \leqslant \mathbb{V}$ be a linear hyperplane, and let a be a weight on a polytope $P \in \mathcal{P}(\mathbb{V})$. Then orthogonal projection $\Phi\colon \mathbb{V} \to \mathbb{V}$ induces a weight $a\Phi \in \bar{\mathcal{W}}(P\Phi)$.*

Proof. We must describe $a\Phi$ on each face of $P\Phi$. It is clear from Theorem 8.5 that we may suppose Φ to be singular on P (and that P be full-dimensional, although this is less important). Moreover, the inverse image under Φ of a face of $P\Phi$ is either of the form F on which Φ is non-singular, or of the form G on which Φ is singular. In the former case, Theorem 8.3 gives us $(a\Phi)(F\Phi)$. In the latter, $G\Phi$ admits two dissections into a union of faces $F\Phi$, with each $F \lhd G$. Let the unit vector t span $\ker \Phi$. We claim that

$$(a\Phi)(G\Phi) := \sum_{\langle t, u(F,G)\rangle > 0} \kappa(F,\Phi)(a\Phi)(F\Phi) = \sum_{\langle t, u(F,G)\rangle < 0} \kappa(F,\Phi)(a\Phi)(F\Phi)$$

defines $a\Phi$ on $G\Phi$. Note that $\kappa(F,\Phi) = |\langle t, u(F,G)\rangle|$, so that facets F with $\langle t, u(F,G)\rangle = 0$ do not contribute to either sum.

First, it follows directly from applying Φ to the GMR (8.1) that the two sums coincide, since $t\Phi = o$. It remains to check that this $a\Phi$ satisfies GMR on $G\Phi$. Now Φ is one-to-one on each $F \lhd G$, so that Theorem 8.5 implies that the GMR holds for each $F\Phi$. If $J \lhd F$, then there are two possibilities. If $J\Phi \cap \operatorname{relint}(G\Phi) \neq \emptyset$, then $J \lhd F'$ for some other $F' \lhd G$, and the contributions from $(a\Phi)(J\Phi)$ to GMR for $G\Phi$ cancel. There remain the contributions from those J with $J\Phi \subset \operatorname{relbd}(G\Phi)$, and these sum to give GMR for $G\Phi$ itself. □

We remark that the tensorials themselves satisfy Lemmas 8.3 and 8.4, as indeed they must.

Putting together Lemmas 8.3 and 8.4, we deduce

THEOREM 8.5 *If $P \in \mathcal{P}(\mathbb{V})$, then a linear mapping $\Phi: \mathbb{V} \to \mathbb{W}$ induces a homomorphism from $\bar{\mathcal{W}}(P)$ to $\bar{\mathcal{W}}(P\Phi)$.*

We denote this induced homomorphism by the same symbol Φ.

REMARK 8.6 Observe that, if the linear mapping Φ is non-singular on a polytope P, then

$$M_s(P\Phi) = (M_s(P))\Phi,$$

as we have defined the effect of such mappings on weights.

§9. *Weight algebras.* We now discuss under what circumstances we can multiply two weights together, in a way that generalizes what we can do for tensorials. A weight $a \in \bar{\mathcal{W}}(P)$ is said to satisfy the *Green-Minkowski connexion* (*GMC*) if, in the GMR (8.1), $a' = a$; that is,

$$\sum_{F \lhd G} a(F)\langle u(F,G), t\rangle = a(G)t, \tag{9.1}$$

for each $G \leqslant P$. We write $\mathcal{W}(P)$ for the subspace of those $a \in \bar{\mathcal{W}}(P)$ which satisfy GMC. As with $\bar{\mathcal{W}}(P)$, if the $a(F)$ are s-tensors on k-faces F, then the $a(G)$ are $(s-1)$-tensors on $(k+1)$-faces G; thus the (*total*) *degree* $r := s + k$ of a is well defined. We denote by $\mathcal{W}_r(P)$ the subspace of GMC-weights on P of degree r.

Further, we call *a quasi-scalar* if $a' = 0$; the family of quasi-scalar weights on P is denoted $\bar{\Omega}(P)$. We call $a \in \bar{\mathcal{W}}(P)$ a *scalar* weight if $a: \mathcal{F}(P) \to \mathbb{F}$, and denote the family of them by $\Omega(P)$ (this accords with the notation of [**23**]); clearly, the GMR (8.1) implies that $\Omega(P) \leqslant \bar{\Omega}(P)$.

LEMMA 9.1 *If $P \in \mathcal{P}(\mathbb{V})$ and $Q \in \mathcal{P}(\mathbb{W})$ are polytopes, then weights $a \in \bar{\mathcal{W}}(P)$ and $b \in \bar{\mathcal{W}}(Q)$ induce a weight $a \times b \in \bar{\mathcal{W}}(P \times Q)$ on the direct (cartesian) product $P \times Q$ if either a and b satisfy the GMC, or are (quasi-)scalar.*

Proof. The definition of $a \times b$ is again that which we are forced to adopt, if we wish weights to mimic tensorials. Namely, we set

$$(a \times b)(F \times G) := a(F)b(G),$$

for all faces $F \in \mathcal{F}(P)$ and $G \in \mathcal{F}(Q)$. The astute reader will notice that our scaling of tensorials (so that we use these rather than mass moments) is designed to facilitate this definition; otherwise, the degrees of the tensors would need to be specified, and binomial coefficients would need to be introduced. Naturally, we call $a \times b$ the *direct product* of a and b. Observe that $a(F)b(G) \in \mathbb{T}(\mathbb{V} \times \mathbb{W})$ in an obvious way (note that $\mathbb{T}(\mathbb{V}) \hookrightarrow \mathbb{T}(\mathbb{V} \times \mathbb{W})$ naturally under $x \mapsto (x, o)$, and similarly for $\mathbb{T}(\mathbb{W})$).

It clearly suffices to check GMR (8.1) on $P \times Q$ itself. A general vector $t \in (P \times Q)_{\|}$ is uniquely expressible as $t = t_P + t_Q = (t_P, t_Q)$, with $t_P \in P_{\|}$ and $t_Q \in Q_{\|}$. The facets of $P \times Q$ are of two kinds, namely $F \times Q$ with $F \lhd P$, or $P \times G$, with $G \lhd Q$. The corresponding unit normal vectors are

$$\begin{aligned} u(F \times Q, P \times Q) &= (u(F,P), o), \\ u(P \times Q, P \times Q) &= (o, u(G,Q)). \end{aligned}$$

Of course, we then have

$$\begin{aligned} \langle u(F \times Q, P \times Q), t\rangle &= \langle u(F,P), t_P\rangle, \\ \langle u(P \times Q, P \times Q), t\rangle &= \langle u(G,Q), t_Q\rangle. \end{aligned}$$

For GMR, in general we have

$$\begin{aligned} &\sum_{F \lhd P} (a \times b)(F \times Q)\langle u(F \times Q, P \times Q), t\rangle \\ &\qquad + \sum_{G \lhd Q} (a \times b)(P \times G)\langle u(P \times G, P \times Q), t\rangle \\ &= \sum_{F \lhd P} a(F)b(Q)\langle u(F,P), t_P\rangle + \sum_{G \lhd Q} a(P)b(G)\langle u(G,Q), t_Q\rangle \\ &= a'(P)b(Q)t_P + a(P)b'(Q)t_Q. \end{aligned}$$

We now address the two cases separately. If a and b satisfy GMC, so that $a' = a$ and $b' = b$, then the last expression is $a(P)b(Q)t$, and hence $a \times b$ also satisfies GMC. If a and b are (quasi-)scalar, so that $a' = b' = 0$, then the last expression is 0, and hence $a \times b$ is also (quasi-)scalar. □

We can now establish the algebra properties which we announced earlier.

THEOREM 9.2 *For each $P, Q \in \mathcal{P}$, there is a multiplication $\mathcal{W}(P) \otimes \mathcal{W}(Q) \to \mathcal{W}(P + Q)$. In particular, $\mathcal{W}(P)$ is an algebra over $\mathbb{F}$ for each $P \in \mathcal{P}$. Moreover, the direct limit $\mathcal{W} = \mathcal{W}(\mathbb{V})$ of the $\mathcal{W}(P)$ under $\preceq$ exists.*

Proof. As in (2.1), let $\Sigma\colon \mathbb{V}\times\mathbb{V}\to\mathbb{V}$ be the sum mapping, given by

$$(x,y)\Sigma := x+y.$$

For $a\in\mathcal{W}(P)$ and $b\in\mathcal{W}(Q)$, we define

$$ab := (a\times b)\Sigma,$$

with $\Sigma\colon \mathcal{W}(P\times Q)\to\mathcal{W}(P+Q)$ the corresponding induced mapping of Lemma 8.4. Since $a\times b\in\mathcal{W}(P\times Q)$ by Lemma 9.1, we see that indeed $ab\in\mathcal{W}(P+Q)$.

This is the multiplication asked for; note that the distributive laws follow automatically.

By Proposition 8.2, $\mathcal{W}(P+P)=\mathcal{W}(2P)=\mathcal{W}(P)$ for each $P\in\mathcal{P}$, and so we obtain the required product on $\mathcal{W}(P)$. Clearly, this satisfies

$$ab = ba,$$
$$(a+a')b = ab+a'b,$$
$$(\lambda a)b = \lambda(ab),$$

for each $a,a',b\in\mathcal{W}(P)$ and $\lambda\in\mathbb{F}$, and hence $\mathcal{W}(P)$ is an algebra over $\mathbb{F}$.

Finally, let $i_Q\in\mathcal{W}(Q)$ be given by

$$i_Q(G) := \begin{cases} 1, & \text{if } G \text{ is a vertex of } Q,\\ 0, & \text{otherwise.}\end{cases} \tag{9.2}$$

It is easy to check that multiplication by i_Q embeds $\mathcal{W}(P)$ in $\mathcal{W}(P+Q)$. In fact, in the picture of (8.3), the embedding is just given by the corresponding refinement of normal cones (which induces an addition on $\widehat{\Xi}$). □

Let us expand this last comment a little. Consider first a product weight ab on $P+Q$ itself. A typical contribution to $(ab)(P+Q)$ comes from a pair of faces $F\leqslant P$ and $G\leqslant Q$ of complementary dimension (in $P+Q$); this will be

$$\sigma(F,G)a(F)b(G),$$

where $\sigma(F,G):=\kappa(F\times G,\Sigma)$ gives the volume of the sum of unit cubes (of the appropriate dimension) in $F_{\|}$ and $G_{\|}$. Later on, we shall need an explicit description of which such pairs of faces contribute to the product. This is given in terms of the normal fans of P and Q. We choose a general translation vector $v\in\mathbb{V}^*$, and pick those faces $F\leqslant P$ and $G\leqslant Q$ for which $\operatorname{relint} N(F,P)\cap \operatorname{relint}(N(G,Q)-v)$ is a single point. (For a proof, see [**3**] or, for a more general result, [**13**, **34**]; the latter is set in an even more general context in [**25**].)

The analogous calculations, carried out intrinsically, give $(ab)(F+G)$ when $F\leqslant P$ and $G\leqslant Q$, and in (8.3) we have

$$N(F+G,P+Q) = N(F,P)\cap N(G,Q).$$

When $b=i_Q$, the contribution of i_Q to $\mathbb{T}\otimes\widehat{\Xi}$ is carried on full-dimensional cones alone; indeed,

$$i_Q\mapsto 1\otimes\langle\mathbb{V}\rangle,$$

since the union of these cones is just $\mathbb{V}$ itself. Thus multiplying $a \in \mathcal{W}(P)$ by i_Q just corresponds to subdividing the normal cones $N(F, P)$ of P (if we now think of normal cones to $P + Q$), but addition in $\widehat{\Xi}$ shows that we actually obtain a again.

Exactly the same arguments show that $\bar{\Omega}(P)$ and $\Omega(P)$ are algebras over $\mathbb{F}$ (the latter was established in [**23**]), and so we shall not repeat the details.

REMARK 9.3 The tensorials M_s themselves, while motivating the definitions we have used, do not behave in such an elementary way as weights. In fact, an easy calculation shows that

$$M_s(P \times Q) = \sum_{k=0}^{s} M_k(P) M_{s-k}(Q);$$

we use here

$$\frac{1}{s!}\binom{s}{k} = \frac{1}{k!}\,\frac{1}{(s-k)!},$$

which shows why we incorporated the scaling factor $1/s!$ in the definition of M_s. If we consider the formal power series

$$M(P; \tau) := \sum_{s \geqslant 0} M_s(P)\tau^s,$$

then we deduce that

$$M(P \times Q; \tau) = M(P; \tau) M(Q; \tau).$$

§10. *Simple polytopes*. An important step in establishing the isomorphism theorem of Section 11 is to show that, if P is a simple polytope, then the subalgebra $\mathcal{W}(P)$ is generated by its first weight space. In the course of this, we also prove a kind of separation theorem for $\mathcal{W}(P)$. The following argument appeals to [**16**] (in the dual form), which gives an easier approach than the analogous one to that in [**19**].

Let $P \in \mathcal{P}(\mathbb{V})$ be a simple d-polytope. Recall that this means that each vertex of P lies in exactly d facets (the minimum number). As before, if $U = (u_1, \ldots, u_n)$ is the (ordered) set of unit facet normals to P, we write P in the form

$$P = P(U, b) := \{x \in \mathbb{V} \mid \langle x, u_j \rangle \leqslant \beta_j \text{ for } j = 1, \ldots, n\},$$

where $b := (\beta_1, \ldots, \beta_n) \in \mathbb{F}^n$. If, as in Section 4, we replace b by a suitably large positive multiple, we may clearly assume that each $P(U, b + e_j) \approx P$, where $e_j \in \mathbb{F}^n$ is the jth standard basis vector for $j = 1, \ldots, n$.

Just as with the subalgebra $\bar{\Pi}(P)$, for each $j = 1, \ldots, n$, we have an element in $\mathcal{W}_1(P)$ which we can identify with e_j. This is the 1-weight on P corresponding to unit displacement of the jth facet hyperplane

$$H_j := H(u_j, \beta_j) := \{x \in \mathbb{V} \mid \langle x, u_j \rangle = \beta_j\}$$

which meets P in the facet F_j (say); we use the same symbol e_j for it. It is clear that the weight e_j is *localized* to F_j, in that it vanishes on every face of P which does not meet F_j. It follows that, for $1 \leqslant j(1) \leqslant \cdots \leqslant j(r) \leqslant n$, the product

$e_{j(1)} \cdots e_{j(r)}$ is localized to the intersection $F_{j(1)} \cap \cdots \cap F_{j(r)}$, and so vanishes unless the intersection is non-empty.

Notice, as well, that

$$t \longleftrightarrow (\langle t, u_1 \rangle, \ldots, \langle t, u_n \rangle) = \sum_{i=1}^{n} \langle t, u_i \rangle e_i,$$

for $t \in \mathbb{V}$, embeds $\mathbb{T}$ into the corresponding *translation subalgebra* $\mathcal{T}(P)$ of $\mathcal{W}(P)$.

We call a (non-zero) vector $v \in \mathbb{V}^*$ *general* (with respect to P – it is useful to distinguish the dual space here) if no hyperplane

$$H(v, \eta) := \{x \in \mathbb{V} \mid \langle x, v \rangle = \eta\}$$

contains more than one vertex of P. It is clear that such general directions v exist; indeed, v only needs to avoid the finitely many hyperplanes in $\mathbb{V}^*$ of the form

$$\{w \in \mathbb{V}^* \mid \langle x, w \rangle = \langle y, w \rangle\},$$

where $x, y \in \operatorname{vert} P$.

We now *sweep* the hyperplane $H(v, \eta)$, or, rather, the associated half-space

$$H^-(v, \eta) := \{x \in \mathbb{V} \mid \langle x, v \rangle \leqslant \eta\},$$

through P; that is, we increase η from $-\infty$ to ∞. We say that $x \in \operatorname{vert} P$ is of *type* r if $H^-(v, \eta)$, with $\eta := \langle x, v \rangle$, contains exactly r *downward* edges of P through x. Thus x is the top vertex (with respect to v) of an r-face F of P; we call F a *distinguished* face of P for the direction v. The complementary set of $d - r$ *upward* edges through x which lie in the other half-space $H^+(v, \eta)$ bounded by $H(v, \eta)$ similarly determines a $(d - r)$-face F' of P, which is distinguished for $-v$. Now F' is the intersection of r (distinct) facets of P, say

$$F' = F_{j(1)} \cap \cdots \cap F_{j(r)},$$

with $1 \leqslant j(1) < \cdots < j(r) \leqslant n$. Since the product $e_F := e_{j(1)} \cdots e_{j(r)}$ is localized to F', and since $e_{j(s)}$ takes a positive (scalar) value on the downward edge which does not lie in $F_{j(s)}$ for each $s = 1, \ldots, r$, we easily see that, if G is an r-face of P lying in $H^-(v, \eta)$, then $e_F(G) \in \mathbb{F}$ satisfies

$$e_F(G) \begin{cases} > 0, & \text{if } G = F, \\ = 0, & \text{otherwise.} \end{cases}$$

Now let $a \in \mathcal{W}(P)$. We up-date a as we sweep through P. When the sweeping half-space acquires the vertex x, and corresponding distinguished r-face F, we suppose that $a(G) = o$ whenever $G \leqslant P$ is such that $G \subseteq \operatorname{int} H^-(v, \eta)$ (that is, $G \subseteq H^-(v, \eta)$, but $x \notin G$). It is fairly clear that $a(F)$ determines $a(G)$ for each face $G \leqslant P$ with $x \in G \subseteq H^-(v, \eta)$; we shall return to this point in the wider context of Section 12. We then redefine

$$a' := a - (a(F)/e_F(F))e_F$$

to be the new a. Then $a'(G) = o$ whenever $G \subseteq H^-(v, \eta)$ (including all those G with $x \in G$). We conclude at once that we have proved

THEOREM 10.1 *If P is a simple polytope, then the weights e_F, with F running over the distinguished faces of P when P is swept in some general direction v, form a basis of $\mathcal{W}(P)$ as a module over the translation subalgebra $\mathcal{T}(P)$ (or over $\mathbb{T}$).*

We call $\mathcal{B}_v := \{e_F \mid F \leqslant P \text{ distinguished for } v\}$ the *sweep basis* of $\mathcal{W}(P)$ for the direction v.

Since the translations themselves are linear combinations of the basic 1-weights e_j, an immediate consequence is

COROLLARY 10.2 *The subalgebra $\mathcal{W}(P) \leqslant \mathcal{W}$ is generated by $e_1, \ldots, e_n$, and hence is generated by $\mathcal{W}_1(P)$.*

For completeness, we recall some further facts, and note how they are reflected in $\mathcal{W}(P)$. Associated with a simple d-polytope P are certain combinatorial invariants $h_r(P)$. We define these as follows. For $j = 0, \ldots, d$, let $f_j(P)$ denote the number of j-faces of P, and set

$$f(P, \tau) := \sum_{j=0}^{d} f_j(P)\tau^j,$$

with τ an indeterminate. Define the new polynomial $h(P, \tau) = \sum_{r=0}^{d} h_r(P)\tau^r$ by

$$h(P, \tau) := f(P, \tau - 1).$$

Then we have

PROPOSITION 10.3 *For each $r = 0, \ldots, d, h_r(P)$ is the number of vertices of P of type r with respect to any sweep, or the number of distinguished r-faces of P.*

Proof. This result is (by now) very well known. At a vertex x of type r, with distinguished face F, the sweeping half-space $H^-(v, \eta)$ acquires each of the j-faces of F which contain x. Thus the change in $h(P, \tau)$ at x is

$$\Delta h(P, \tau) = \Delta f(P, \tau - 1) = \sum_{j=0}^{r} \binom{r}{j} (\tau - 1)^j = \tau^r,$$

and the desired conclusion follows at once. $\square$

Reversing the direction of the sweep, and so changing v to $-v$, leads at once to the *Dehn-Sommerville equations*

$$h_r(P) = h_{d-r}(P) \qquad \text{for } r = 0, \ldots, d.$$

The *Hilbert function* of $\mathcal{W}(P)$ counts the dimension of $\mathcal{W}_r(P)$ for each $r \geqslant 0$. It is usually expressed as a formal power series; from the results above we deduce

PROPOSITION 10.4 *If P is a simple d-polytope, then the Hilbert function of $\mathcal{W}(P)$ is*

$$\frac{h(P, \tau)}{(1 - \tau)^d}.$$

Exactly the same arguments apply to $\Omega(P)$ and $\bar{\Omega}(P)$ (and, indeed, with a forward look to Section 12, to $\bar{\mathcal{W}}(P)$ as well). Each distinguished face F in a sweep basis gives a corresponding basis element e_F of $\Omega(P)$, $\bar{\Omega}(P)$ and $\bar{\mathcal{W}}(P)$, which takes a scalar value on F. We conclude

THEOREM 10.5 *Let $P \in \mathcal{P}(\mathbb{V})$ be a simple d-polytope. Then*

(*a*) *$\mathcal{B}_v$ is a basis for $\Omega(P)$ over $\mathbb{F}$;*
(*b*) *$\mathcal{B}_v$ is a basis for $\bar{\Omega}(P)$ as a module over $\mathbb{T}$;*
(*c*) *$\mathcal{B}_v$ is a basis for $\bar{\mathcal{W}}(P)$ as a module over $\mathbb{T}$.*

In the last case, we must interpret each e_F as an element of $\mathcal{W}(P)$.

The dimensions of subspaces of $\Omega(P)$ and $\bar{\Omega}(P)$ are easily calculated. Since

$$\dim \mathbb{T}_s = \binom{s+d-1}{s},$$

writing $\bar{\Omega}_{k,s}$ for the space of s-tensor quasi-scalar weights on k-faces of P, and using the known value $\dim \bar{\Omega}_{0,k}(P) = \dim \Omega_k(P) = h_k(P)$ from **[20, 23]**, we thus have

THEOREM 10.6 *For $k = 0, \ldots, d$ and each $s \geqslant 0$,*

$$\dim \bar{\Omega}_{k,s}(P) = \binom{s+d-1}{s} h_k(P).$$

Let us now do similar calculations for the dimensions of the spaces $\bar{\mathcal{W}}_{k,s}(P)$. At this point, we make a forward reference to Theorem 12.1. This says, in effect, that $\bar{\mathcal{W}}_{k+1,s-1}(P)$ is the image of $\bar{\mathcal{W}}_{k,s}(P)$ under the GMC. The kernel of the GMC consists of $\bar{\Omega}_{k,s}$. Hence,

$$\dim \bar{\mathcal{W}}_{k,s}(P) = \dim \bar{\mathcal{W}}_{s-1,k+1}(P) + \dim \bar{\Omega}_{k,s}(P).$$

By an induction argument, there follows at once

THEOREM 10.7 *For $k = 0, \ldots, d$ and each $s \geqslant 0$,*

$$\dim \bar{\mathcal{W}}_{k,s}(P) = \sum_{j=k}^{d} \binom{d+s+k-j-1}{s+k-j} h_j(P).$$

§11. *The isomorphism theorem.* We are now in a position to prove the central results in the theory of tensor-valued weights. The main one depends on the following

LEMMA 11.1 *Let $P \in \mathcal{P}(\mathbb{V})$ be a simple polytope. Then*

$$\bar{\Pi}(P) \cong \mathcal{W}(P).$$

Proof. Define the *canonical r-weight* of $Q \in \mathcal{K}(P)$ by

$$q_r(F) := M_{r-\dim F}(G),$$

where $G \leqslant Q$ corresponds to $F \leqslant P$ under the strong isomorphism $P \approx Q$. (Recall the convention that $M_s \equiv 0$ if $s < 0$.) It is clear that the mapping

$Q \mapsto q_r$ is a weakly continuous valuation which is polynomial of degree r. As we saw in Theorem 6.1, each q_r satisfies GMC, so that $q_r \in \mathcal{W}(P)$. Furthermore,

$$q_r = \frac{1}{r!}(q_1)^r,$$

because this relation certainly holds on the vertices of P.

Now suppose that $P = P(U, b)$ as in Section 4, and recall from there the identification

$$e_j := \log P_j - \log P,$$

with $P_j := P(U, b + e_j)$ for $j = 1, \ldots, n$. Setting $Q := P_j$, we can further identify e_j with $q_1 - p_1$, and, as we saw in Corollary 10.2, $\mathcal{W}(P)$ is generated by $e_1, \ldots, e_n$. Finally, Proposition 10.4 tells us, in effect, that $\mathcal{W}(P)$ is isomorphic to the face ring of the simplicial polytope P^* dual to P, while Corollary 4.7 says that $\bar{\Pi}$ is a quotient of this face ring. The claim of the lemma follows at once. □

Lemma 4.1 shows that, given any $Q_1, \ldots, Q_m \in \mathcal{P}$, there is some simple polytope P such that $Q_j \preceq P$ for each j. The main isomorphism theorem is an immediate consequence.

THEOREM 11.2 *Let $\mathbb{V}$ be a finite dimensional vector space over the square-root closed ordered field $\mathbb{F}$. Then the graded algebras $\bar{\Pi}(\mathbb{V})$ and $\mathcal{W}(\mathbb{V})$ are isomorphic.*

In view of our alternative picture of tensor weights as elements of $\mathbb{T} \otimes \widehat{\Xi}$, we can rephrase Theorem 11.2 in the following way.

THEOREM 11.3 *Let $\mathbb{V}$ be as in Theorem 11.2. For each $r \geqslant 0$, define the r-class of a polytope $P \in \mathcal{P}(\mathbb{V})$ by*

$$p_r := \sum_{s=0}^{r} \sum_{\dim F = r-s} M_s(F) \otimes \widehat{n}(F, P) = \sum_{k+s=r} V_{k,s}(P),$$

with $V_{k,s}$ as in (7.2). Then, as weights, these r-classes generate $\mathcal{W}_r(\mathbb{V})$. In particular, the mappings $P \mapsto p_r$ induce isomorphisms from $\bar{\Pi}_r(\mathbb{V})$ to $\mathcal{W}_r(\mathbb{V})$.

We end the section with a far-ranging generalization of the main theorem of [**17**], which follows at once from Theorem 11.3. Write $\mathcal{V}(\mathbb{V}, \mathbb{W})$ for the family of weakly continuous translation covariant valuations $\varphi\colon \mathcal{P}(\mathbb{V}) \to \mathbb{W}$ into the (finite dimensional) vector space $\mathbb{W}$ over $\mathbb{F}$.

THEOREM 11.4 *Let $\mathbb{V}$ be as in Theorem 11.2, let $\mathbb{W}$ be another vector space over $\mathbb{F}$, and let $\varphi \in \mathcal{V}(\mathbb{V}, \mathbb{W})$. Then there are homomorphisms $\psi\colon \widehat{\Xi}(\mathbb{V}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{T}(\mathbb{V}), \mathbb{W})$ such that*

$$\varphi(P) = \sum_{F \leqslant P} \sum_{s \geqslant 0} M_s(F)(\widehat{n}(F, P)\psi_s),$$

where all but finitely many ψ_s vanish.

It must be borne in mind here that valuations pay no attention to the multiplicative structure of Π, or of $\mathcal{W}$. Thus there will not necessarily be any relationships among the different ψ_s.

§12. *Extensions.* So far as (weakly continuous translation covariant) valuations are concerned, it is only the additive structure of $\bar{\mathcal{W}}$ which is significant. In other words, as Theorem 11.4 shows, the vector spaces $\bar{\mathcal{W}}_{s,k}$ of s-tensor weights on k-faces of polytopes can act quite independently as source spaces for valuations. Nevertheless, Theorem 11.2 shows that it is actually the subspaces of $\mathcal{W}$, whose elements are related by the GMC, which are universal.

For the behaviour of a valuation on any finite family of polytopes, we can confine our attention to a subalgebra $\bar{\mathcal{W}}(P)$, for some $P \in \mathcal{P}$. However, only when P is simple can we guarantee that $\bar{\mathcal{W}}(P)$ is generated over $\mathbb{T}$ by $\mathcal{W}_1(P)$. Indeed, if P is a simplicial d polytope which is not a simplex, and so has (say) $n \geqslant d+2$ facets, then P has $(d-1)$-weights which do not lie in $\mathcal{W}_1(P)^{d-1}$.

The *extension problem* is the following. Let $P \in \mathcal{P}$, and let $a\colon \mathcal{F}_r(P) \to \mathbb{T}$ satisfy the GMR, so that there exists $b\colon \mathcal{F}_{r+1}(P) \to \mathbb{T}$ such that

$$\sum_F a(F)u(F,G) = b(G)q(G) \tag{12.1}$$

for each such $G \in \mathcal{F}_{r+1}(P)$. There arise two questions. First, does b itself satisfy the GMR on $(r+2)$-faces of P? Second, if so, is a the restriction of a (GMC-)weight in $\mathcal{W}(P)$ to $\mathcal{F}_r(P)$?

We can answer these questions in a special case; however, in the wider context of the whole space $\bar{\mathcal{W}}$ of tensor weights, this answers both questions positively.

THEOREM 12.1 *Let P be a simple d-polytope, let $0 \leqslant r < d$, and suppose that $a\colon \mathcal{F}_r(P) \to \mathbb{T}$ satisfies the GMR (12.1) on $\mathcal{F}_{r+1}(P)$. Then a is the restriction of a GMC-weight in $\mathcal{W}(P)$ to $\mathcal{F}_r(P)$.*

Proof. There is nothing to prove when $r = d-1$, but we need to bear this case in mind. When $r \leqslant d-2$, we first have to show that, for each $J \in \mathcal{F}_{r+2}(P)$, there exists $c(J) \in \mathbb{T}$ such that $\sum_G b(G)u(G,J) = c(J)q(J)$. It is thus clear that we need only consider the case $r = d-2$, so that $J = P$.

A good part of the argument refers to what we did earlier; the basic idea is to restrict a to ridges ($(d-2)$-faces) and b to facets of P which contain a single vertex. So, we sweep P by a hyperplane in a general direction, ending at the final vertex x, say. Vertices of P of type $s < d-2$ do not concern us. At a vertex of type $d-2$ we strip off the appropriate multiple of the corresponding element of the sweep basis in $\mathcal{W}_{d-2}(P)$; at a vertex of type $d-1$ we do the same, noting that the GMC deals with the corresponding $(d-2)$-faces as well.

We end up with a and b localized to the final vertex x in the sweep. Let the facets of P through x be $F_1, \ldots, F_d$, with corresponding unit outer normal vectors $u_j := u(F_j, P)$ for $j = 1, \ldots, d$. Let $G_{jk} := F_j \cap F_k$, and write for short

$$a_{jk} := a(G_{jk}), \quad b_j := b(F_j),$$

for each j and k. The unit outer normal to F_j at G_{jk} is

$$v_{jk} := u(G_{jk}, F_j) = (1 - \langle u_k, u_j \rangle^2)^{-1/2}(u_k - \langle u_k, u_j \rangle u_j).$$

Let $(u_1^*, \ldots, u_d^*)$ be the dual basis to $(u_1, \ldots, u_d)$, so that $\langle u_j, u_k \rangle = \delta_{jk}$ for each j and k. Then $u_j^* \in \mathbb{V}$ is a vector along the edge of P into x which does not lie in F_j. From the GMR for a, we thus have

$$\begin{aligned} b_j u_k^* &= \sum_{i \neq j} a_{ji} \langle v_{ji}, u_k^* \rangle \\ &= (1 - \langle u_k, u_j \rangle^2)^{-1/2} a_{jk} \\ &= b_k u_j^*, \end{aligned}$$

by symmetry, for each $j, k = 1, \ldots, d$. It follows that we can write

$$b_j = c u_j^*,$$

for some $c \in \mathbb{T}$ (recall that $\mathbb{T}$ can be embedded in its field of fractions, so that division in $\mathbb{T}$ is unique). But now we immediately have

$$\sum_{j=1}^{d} b_j u_j = c \sum_{j=1}^{d} u_j u_j^* = cq$$

with $q := q(P)$, so that b satisfies GMR (with c the corresponding value on P).

This part of the proof shows that (reverting to a more suitable notation), if a is defined on r-faces of P satisfying GMR, then a is defined on s-faces for $s = r, \ldots, d$. But now we can strip off a, beginning with P. If, for some $s = r, \ldots, d$, we have subtracted weights so that a vanishes on faces of dimension greater than s, then, because the new a still satifies GMR on $\mathcal{F}_s(P)$, it follows that a is a quasi-scalar weight on $\mathcal{F}_s(P)$, that is, a $\mathbb{T}$-linear combination of sweep-basis elements in $\mathcal{W}_s(P)$. Once we have worked back to r-faces we are done, since a was not specified on faces of lower dimension. This completes the proof. □

We can now appeal to the discussion of Section 8, and deduce from Theorem 12.1 the corresponding extension result for general polytopes.

THEOREM 12.2 *Let $P, Q \in \mathcal{P}$ with Q a simple d-polytope such that $P \preceq Q$. Let $0 \leqslant r < d$, and suppose that $a: \mathcal{F}_r(P) \to \mathbb{T}$ satisfies the GMR (12.1) on $\mathcal{F}_{r+1}(P)$. Then a is the restriction of a GMC-weight in $\mathcal{W}(Q)$ to $\mathcal{F}_r(P)$.*

Proof. Choose any simple d-polytope Q such that $P \preceq Q$. By Proposition 8.1, there exist induced weights a', b' on Q. We thus know from Theorem 12.1 that b' itself satisfies the GMR, with corresponding weight c'. As in the previous theorem, the only important case is $r = d - 2$, so that c' just takes its value on Q itself (we do not assume that $\dim P = d$, and so this does cover all cases). In case (a) of Proposition 8.1, for which $\dim P = d$, we see immediately that $c(P) := c'(Q)$ is the correct definition; exactly as in that case (with b', c'

instead of d', b'), because $q(P) = q(Q)$ we have

$$\sum_{F \lhd P} b(F)u(F,P) = \sum_{G \lhd Q} b'(G)u(G,Q) = c'(Q)q(Q) = c(P)q(P),$$

as required. Similarly, when $\dim P = d - 1$ the proof follows case (b) of the proposition, while if $\dim P \leqslant d - 2$ it follows case (c); in both cases, $c(P) = 0$. □

REMARK 12.3 The method of proof of Theorem 12.1 shows that, on a simple polytope, a GMR-weight extends to faces of lower dimension as well. This is not the case for polytopes in general. If P is a simplicial d-polytope (all its proper faces are simplices), then the restriction of $\mathcal{W}_{d-1}(P)$ to the facets of P is 1-dimensional – it just consists of multiples of the $(d-1)$-volumes (areas) of the facets. However, if P is not a simplex, then $\bar{\mathcal{W}}_{d-1}(P)$ is $(f - d)$-dimensional, where $f > d + 1$ is the number of facets of P.

§13. *Translation quotients.* In Section 6, we wrote $\mathbb{T}_s = \mathbb{T}_s(\mathbb{V})$ for the space of symmetric s-tensors over $\mathbb{V}$. We have seen also that $\mathbb{T}$ embeds naturally in $\mathcal{W} = \mathcal{W}(\mathbb{V})$. It is then clear that the quotient $\mathcal{W}/\mathbb{T}_{r+1} \cong \bar{\Pi}/T^{r+1}$. For a given polytope P, which we usually take to be simple, it follows that we have sub-algebras $\mathcal{W}^r(P) := \mathcal{W}(P)/(\mathcal{W}(P) \cap \mathcal{T}_{r+1}(P))$, with $\mathcal{T}(P)$ the image of $\mathbb{T}$ in $\mathcal{W}(P)$, which correspond to the valuations which are polynomial of degree at most r. It is clear that $\mathcal{W}^r(P) \leqslant \bar{\mathcal{W}}(P)$ in a natural way. The multiplication is just that induced from $\mathcal{W}(P)$ itself; this works in spite of the fact that the GMC (9.1) does not always hold (because tensor weights of degree r on s-faces do not come from those of degree $r+1$ on $(s-1)$-faces) – this is taken care of by factoring out $\mathbb{T}_{r+1}$.

The claim of Theorem 7.1 follows immediately. Theorem 11.3 gives the basic isomorphism theorem, and factoring out by the space tensors $\mathbb{T}_{m+1}$ of degree $m+1$ just corresponds to restriction to those $V_{k,s}$ for which $s \leqslant m$.

§14. *Weights as valuations.* It is a special case of Theorem 7.1 that a weakly continuous translation covariant valuation $\varphi: \mathcal{P}(\mathbb{V}) \to \mathbb{F}$ is expressible in the form

$$P\varphi = \sum_{s=0}^{r} \sum_{F \leqslant P} \langle M_s(F), \hat{n}(F,P)\psi_s \rangle \tag{14.1}$$

for some r and some homomorphisms $\psi_s: \hat{\Xi} \to \mathbb{T}_s^*$, the grade s term of the dual tensor algebra $\mathbb{T}^* := \mathbb{T}(\mathbb{V}^*)$, and $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathbb{T}$ and $\mathbb{T}^*$.

Suppose, however, that we are interested in the restriction of such valuations to a given class $\mathcal{K}(P)$ (or its closure $\bar{\mathcal{K}}(P)$), with P a simple polytope. With $v \in \mathbb{V}^*$ a general direction, we have a sweep-basis $\mathcal{B}_v$ of $\mathcal{W}(P)$ as a module over $\mathbb{T}$. We write the elements of $\mathcal{B}_v$ as $b_1, \cdots, b_m$, in the order in which the corresponding m vertices of P are met by the sweep hyperplane. We similarly have a sweep basis $\mathcal{B}_{-v}$ where we reverse the sweep direction. We label its elements $b_m^*, \ldots, b_1^*$, again in the sweep order, so that b_j and b_j^* arise from the same vertex of P (met in opposite directions). We then have

LEMMA 14.1 *For* $1 \leqslant j < k \leqslant m$,

$$(b_j^* b_k)(P) = 0,$$

while for $1 \leqslant j \leqslant m$,

$$(b_j^* b_j)(P) > 0$$

is a scalar.

Proof. For the first assertion, if b_k is localized to the face $G_k \leqslant P$, so that

$$b_k = \prod_{F_i \geqslant G} e_i,$$

and if b_j^* is similarly localized to G_j^*, then $b_j^* b_k = 0$ because $G_j^* \cap G_k = \emptyset$. For the second assertion, G_j^* and G_j are faces of complementary dimensions meeting in the jth vertex, and so their product gives a positive scalar on P. □

REMARK 14.2 Lemma 14.1 almost says that $\mathcal{B}_v$ and $\mathcal{B}_{-v}$ are dual bases of $\mathcal{W}(P)$ as a $\mathbb{T}$-module; more exactly, it is easy (in principle) to construct such a basis dual to $\mathcal{B}_v$ as linear combinations of elements of $\mathcal{B}_{-v}$.

A particularly interesting case of duality which has already been used extensively (see, for example, [**20**, **23**]) is the fact that (in the sense we have been considering) $\Omega(P)$ is its own dual; more specifically, $\Omega_r(P)^* = \Omega_{d-r}(P)$. To be more precise, for $a \in \Omega_r(P)$ and $b \in \Omega_s(P)$, we can define

$$\langle a, b \rangle := (ab)(P),$$

the value that the product in $\Omega(P)$ takes on P itself. This will vanish unless $r + s = d$. More generally, for $a, b \in \Omega(\mathbb{V})$, say $a \in \Omega(P)$ and $b \in \Omega(Q)$ with $P, Q \in \mathcal{P}(\mathbb{V})$, we can define

$$\langle a, b \rangle := (ab)(P + Q) = (ab)(\mathbb{V}), \tag{14.2}$$

in an obvious sense when we proceed to the direct limit.

In fact, this extends even further. As before, we write $\bar{\Omega}(P)$ for the space of quasi-scalar weights on the simple d-polytope P. We have seen in Theorem 10.5 that an element of $\bar{\Omega}(P)$ is just a $\mathbb{T}$-linear combination of elements of $\Omega(P)$. We can therefore define $\bar{\Omega}^*(P)$ similarly to be the space of $\mathbb{T}^*$-linear combinations of elements of $\Omega(P)$, for a moment retaining the distinction between $\mathbb{T}$ and $\mathbb{T}^*$. There is an obvious pairing $\bar{\Omega}(P) \otimes \bar{\Omega}^*(P) \to \mathbb{F}$; just take the value of the weight product on P itself in the natural way, and then apply the pairing $\mathbb{T} \otimes \mathbb{T}^* \to \mathbb{F}$ which is induced by the duality between $\mathbb{V}$ and $\mathbb{V}^*$. Because the weights are quasi-scalar, there is no problem with the product weight $a \times b$ in Lemma 9.1, even though $a(F)b(G) \in \mathbb{T} \otimes \mathbb{T}^*$; indeed, it is quite natural to apply the pairing at this stage, so that $a(F)b(G) \mapsto \langle a(F), b(G) \rangle$. With this definition, we can draw an immediate conclusion.

THEOREM 14.3 *The vector spaces* $\bar{\Omega}(P)$ *and* $\bar{\Omega}^*(P)$ *are duals.*

REMARK 14.4 We have only sketched this last notion, because, while it is very suggestive, it does not lead to the generalization which we need in the next section.

In a slightly different direction, there is a further generalization.

THEOREM 14.5 *Let P be a simple polytope. Then, regarded as a $\mathbb{T}$-module, the space $\mathcal{W}(P)$ is self-dual.*

Proof. Again, this is a direct consequence of Lemma 14.1 (and the subsequent remark), about which no more needs to be said. $\square$

§15. *Dual maps.* It is a staple of vector space theory that, if $\Lambda\colon \mathbb{V} \to \mathbb{W}$ is a linear mapping between vector spaces, then there is a dual mapping $\Lambda^*\colon \mathbb{W}^* \to \mathbb{V}^*$, such that

$$\langle x\Lambda, y\rangle = \langle x, y\Lambda^*\rangle$$

for all $x \in \mathbb{V}$ and $y \in \mathbb{W}^*$. We have seen in Theorem 8.5 that a linear mapping induces a corresponding homomorphism on the spaces of GMR-weights. Since it is only the vector space structure (rather than the algebra structure) which is germane in this context, it is natural to ask about the induced linear mapping on the dual spaces. Unfortunately, our description of the dual space in (14.1) is not very helpful from this point of view, because it is not clear what the induced dual mappings on the spaces $\mathrm{Hom}_{\mathbb{F}}(\widehat{\Xi}, \mathbb{T}^*)$ are.

There is one special case where we can explicitly describe the dual mapping. This is for the spaces $\Omega = \Omega(\mathbb{V})$, where we have constructed the pairing $\Omega \times \Omega \to \mathbb{F}$ given by $\langle a, b\rangle := (ab)(P + Q)$, if $a \in \Omega(P)$ and $b \in \Omega(Q)$. Of course, two weights can always be taken to lie in $\Omega(P)$ for a common $P \in \mathcal{P}$, and, indeed, we have seen in Section 14 that we can proceed to the direct limit $\Omega(\mathbb{V})$; thus we can write $\langle a, b\rangle := (ab)(\mathbb{V})$ with an obvious meaning.

Now suppose that we have a linear mapping $\Phi\colon \mathbb{V} \to \mathbb{W}$. We have used the same symbol Φ to indicate the induced (algebra) mapping from $\Omega(\mathbb{V})$ to $\Omega(\mathbb{W})$. We now ask: what is the induced dual linear mapping $\bar{\Phi}\colon \Omega(\mathbb{W}) \to \Omega(\mathbb{V})$, such that

$$\langle a\Phi, b\rangle = \langle a, b\bar{\Phi}\rangle \tag{15.1}$$

for all $a \in \Omega(\mathbb{V})$ and $b \in \Omega(\mathbb{W})$? We use the distinctive notation $\bar{\Phi}$ rather than a more conventional Φ^*, in part because we are working in inner product spaces, for which Φ^* would be the mapping between the weight algebras induced by the dual mapping $\Phi^*\colon \mathbb{W} \to \mathbb{V}$. However, the notation will be consistent with that of the next section.

Before we give the definition, we lay the foundations for it. We begin with a result which lies at the heart of what we do. First, let $L, M \leqslant \mathbb{V}$ be such that L and M are subspaces of complementary dimension; thus L and $M^\perp$ (the orthogonal complement of M) have the same dimension. We write $\sigma(L, M) := \kappa(L \times M, \Sigma)$, the volume ratio of the projection of the product onto the sum; thus, if $A \subseteq L$ and $B \subseteq M$ are unit boxes, then $\sigma(L, M) = \mathrm{vol}(A + B)$. Further, recall this definition from number theory: if

A is the matrix with rows $a_1, \ldots, a_r \in \mathbb{V}$, thought of as coordinate vectors with respect to some orthonormal basis, then

$$\operatorname{Det} A := (\det AA^\top)^{1/2}.$$

Moreover, to save space, we write $(e_1, e_2, \ldots)$ for the matrix with *rows* $e_1, e_2, \ldots$. We then have

LEMMA 15.1 *Let $L, M \leqslant \mathbb{V}$ be subspaces of complementary dimension, and let $\{e_1, \ldots, e_d\}$ and $\{f_1, \ldots, f_d\}$ be orthonormal bases of $\mathbb{V}$ such that (for some r) $L = \operatorname{lin}\{e_1, \ldots, e_r\}$ and $M = \operatorname{lin}\{f_{r+1}, \ldots, f_d\}$. Then*

$$\sigma(L, M) = |\det(\langle e_i, f_j\rangle) \mid 1 \leqslant i, j \leqslant r)|.$$

Proof. By definition, $\sigma(L, M)$ is the absolute value of

$$\begin{aligned}\det(e_1, \ldots, e_r, f_{r+1}, \ldots, f_d) &= \pm\det((e_1, \ldots, e_r, f_{r+1}, \ldots, f_d) \\ &\qquad \times [f_1^\top \cdots f_r^\top f_{r+1}^\top \cdots f_d^\top]) \\ &= \pm\det\begin{bmatrix} (\langle e_i, f_j\rangle) \mid 1 \leqslant i, j \leqslant r) & * \\ 0 & I_{d-r}\end{bmatrix},\end{aligned}$$

which gives the result claimed. □

Motivated by what must be the case when $\mathbb{W} = \mathbb{V}$ and Φ is invertible, we regard $\Omega(\mathbb{W})$ as a subspace of $\mathbb{F} \otimes \widehat{\Xi}(\mathbb{W})$, and so define $b\bar{\Phi}$, for $b \in \Omega(\mathbb{W})$, by

$$\Bigl(\sum_{K \in \mathcal{C}(\mathbb{W})} b(K) \otimes \langle K\rangle\Bigr)\bar{\Phi} := \sum_{K \in \mathcal{C}(\mathbb{W})} \kappa(K, \Phi^*) b(K) \otimes \langle K\rangle\Phi^* \in \mathbb{F} \otimes \widehat{\Xi}(\mathbb{V}), \quad (15.2)$$

where

$$\langle K\rangle\Phi^* := \begin{cases} \langle K\Phi^*\rangle, & \text{if } \Phi^* \text{ is non-singular on } K, \\ 0, & \text{otherwise.}\end{cases} \qquad (15.3)$$

noting that this is actually taken care of by the volume ratio $\kappa(K, \Phi^*)$. We then have

THEOREM 15.2 *The mapping $\bar{\Phi}$ defined by (15.2) satisfies the duality property (15.1).*

Proof. To simplify the calculations involved, we split the proof into three parts, according to the nature of Φ: first, $\Phi: \mathbb{V} \to \mathbb{V}$ is invertible; second, Φ is orthogonal projection onto a hyperplane $\mathbb{W} \leqslant \mathbb{V}$; third, Φ is isometric injection into a hyperplane $\mathbb{V} \leqslant \mathbb{W}$. Every linear mapping is a composition of these special kinds.

Recall first how we calculate $\langle a, b\rangle$. Regard $a \in \Omega_r(P)$ for some $P \in \mathcal{P}$, and the $(d-r)$-weight b as an element of $\mathbb{F} \otimes \widehat{\Xi}$, as above; to aid geometric visualization, however, it is convenient to think of $b \in \Omega_{d-r}(Q)$ for some $Q \in \mathcal{P}$ as well. Choose a general vector $v \in \mathbb{V}$. Then $F \in \mathcal{F}_r(P)$ contributes to the product $\langle a, b\rangle$ precisely when

$$\operatorname{relint} N(F, P) \cap (\operatorname{relint} K - v) \neq \emptyset$$

for some r-cone K (which we visualize as $N(G, Q)$ for some $Q \in \mathcal{F}_{d-r}(Q)$), and then the contribution is $\sigma(N(F, P), K)a(F)b(K)$. Here, $\sigma(N(F, P), K)$ is an abbreviation for $\sigma(N(F, P)_{\|}, K_{\|})$.

In each case, our subsequent calculations replace P by $P\Phi$ and a by $a\Phi$.

For the first case, we let our typical contribution come from $F\Phi$ (with $F \in \mathcal{F}_r(P)$ as before) and $G \in \mathcal{F}_{d-r}(Q)$ such that $K = N(G, Q)$. Following Lemma 15.1, if $\{e_1, \ldots, e_r\}$ and $\{f_1, \ldots, f_r\}$ are orthonormal bases of $F_{\|}$ and $K_{\|}$, respectively, then $\{e_1\Phi, \ldots, e_r\Phi\}$ is a basis of $(F\Phi)_{\|}$, which is such that

$$\kappa(F, \Phi) = \mathrm{Det}(e_1\Phi, \ldots, e_r\Phi).$$

It follows at once that

$$|\det(\langle e_i\Phi, f_j\rangle) \mid 1 \leqslant i, j \leqslant r) \mid a(F)b(K)$$

is the contribution of $F\Phi$ and K (or G) to $\langle a\Phi, b\rangle$. Now, in this case, the dissection of $P\Phi + Q$ given by a normal vector v corresponds to that of $P + Q\Phi^{-1}$ given by $v\Phi^*$. Since

$$\langle e_i\Phi, f_j\rangle = \langle e_i, f_j\Phi^*\rangle$$

(bearing in mind here that we should think of $f_j \in \mathbb{V}^*$ for each j), the same argument shows that the contribution from F and $K\Phi^*$ to $\langle a, b\bar{\Phi}\rangle$ is

$$|\det(\langle e_i, f_j\Phi^*\rangle) \mid 1 \leqslant i, j \leqslant r) \mid a(F)b(K),$$

which is exactly what we want.

In the second case, we can work in $\mathbb{V}$, in which case Φ^* is just the injection of $\mathbb{W}$ into $\mathbb{V}$ (or, more strictly, of $\mathbb{W}^*$ into $\mathbb{V}^*$). The general vector $v \in \mathbb{W}$ used to calculate $\langle a\Phi, b\rangle$ must now be lifted slightly into general position v' in $\mathbb{V}$, say by adding a small vector in $\mathbb{W}^{\perp}$; we think of this as lying on the "positive" side of $\mathbb{W}$. So far as a face $F' \in \mathcal{F}_r(P\Phi)$ is concerned, there are two possibilities. If $F' = F\Phi$ for some r-face $F \leqslant P$, then v' will pick out F for

$$\operatorname{relint} N(F, P) \cap (\operatorname{relint}(K\Phi^* - v') \neq \emptyset.$$

Essentially the same argument as in the previous paragraph will show that the contribution to $\langle a, b\bar{\Phi}\rangle$ is the same. Otherwise, F' will arise from a $(r+1)$-face of P, and each r-face F in the "upper surface" of this face will contribute. Again, though, Lemma 15.1 will ensure that the total contribution to $\langle a, b\bar{\Phi}\rangle$ will be the same.

Finally, the last case is trivial; it is really the same as the first, except that now Φ^* is orthogonal projection from $\mathbb{W}$ to $\mathbb{V}$ (or from $\mathbb{W}^*$ to $\mathbb{V}^*$), and a general vector in $\mathbb{V}$ (giving a suitable dissection) lifts to a general vector in $\mathbb{W}$ giving essentially the same dissection. □

We can now generalize Theorem 15.2 in a somewhat artificial way. However, this description, combined with results from **[26]**, will motivate what we do in the next section. We saw in Theorem 14.3 that the spaces $\bar{\Omega}(P)$ and $\bar{\Omega}^*(P)$ are duals. In fact, the abstract description of the product of weights in terms of the interaction between normal fans enables us to talk about the pairing $\bar{\Omega}(\mathbb{V})$ and $\bar{\Omega}(\mathbb{V}^*)$ (with some looseness of language). For $a \in \bar{\Omega}(\mathbb{V})$ and $b \in \bar{\Omega}(\mathbb{V}^*)$, we define $\langle a, b\rangle$ to be the value given by Section 14 on $\mathbb{V}$ itself (that is, on $P + Q$ if $a \in \bar{\Omega}(P)$ and $b \in \bar{\Omega}^*(Q)$).

It is then natural to ask the following question. Let $\Phi\colon \mathbb{V} \to \mathbb{W}$ be a linear mapping. Then Φ induces a linear mapping $\Phi\colon \bar{\Omega}(\mathbb{V}) \to \bar{\Omega}(\mathbb{W})$. What, if any, linear mapping $\bar{\Phi}\colon \bar{\Omega}(\mathbb{W}^*) \to \bar{\Omega}(\mathbb{V}^*)$ is induced by Φ, in the sense that, if $a \in \bar{\Omega}(\mathbb{V})$ and $b \in \bar{\Omega}(\mathbb{W}^*)$, then $\langle a\Phi, b\rangle = \langle a, b\bar{\Phi}\rangle$?

To define $\bar{\Phi}$, we again return to the picture of the weight b as

$$b = \sum_{K \in \mathcal{C}(\mathbb{W}^*)} b(K) \otimes \langle K\rangle,$$

where the (finite) sum runs over cones, and the $b(K) \in \mathbb{T}^* = \mathbb{T}(\mathbb{W}^*)$, interpreted as weights, satisfy the GMR. Then the appropriate definition of $\bar{\Phi}$ is

$$b\bar{\Phi} := \sum_{K \in \mathcal{C}} \kappa(K, \Phi^*) b(K)\Phi^* \otimes \langle K\phi^*\rangle. \tag{15.4}$$

With the pairing of Section 14, there follows at once

$$\langle a\Phi, b\rangle = \langle a, b\bar{\Phi}\rangle, \tag{15.5}$$

for all $a \in \bar{\Omega}(\mathbb{V})$ and $b \in \bar{\Omega}(\mathbb{W})$.

Remark 15.3 A comment is in order here. It is only the artificial definition of $\bar{\Omega}^*$ which ensures that $b\bar{\Phi}$ is actually a weight.

§16. *The fibre product.* In [**1**] (see also [**2**]), Alesker defined a multiplication on smooth translation covariant valuations on the space of convex bodies in a euclidean space. In [**26**], we described a mapping on tensor weights induced by the fibre polytope construction. While the two ideas are closely connected by the concept which we shall shortly introduce, it is far from clear exactly how the appropriate homomorphism induced by a general linear map should behave; the fibre polytope construction employed was tied in with the structure of an inner product space (no harm results if we call this structure *euclidean*).

We begin with some background. In keeping with the more general setting of [**25**], we have (proper) orthogonal complementary subspaces $\mathbb{X}$ and $\mathbb{Y}$ of an inner product space $\mathbb{V}$, and let $\Phi\colon \mathbb{X} \to \mathbb{V}$ be the (isometric) injection and $\Psi\colon \mathbb{V} \to \mathbb{Y}$ the orthogonal projection. Thus we have a short exact sequence

$$\mathbb{O} \longrightarrow \mathbb{X} \xrightarrow{\Phi} \mathbb{V} \xrightarrow{\Psi} \mathbb{Y} \longrightarrow \mathbb{O}$$

of spaces and mappings. In [**26**], we redefined the original notion in [**5**] of the *fibre polytope* $\operatorname{fib}(P;\Phi)$ to be

$$\operatorname{fib}(P;\Phi) := \int_{\mathbb{Y}} ((P - y) \cap \mathbb{X})\, dy,$$

where we replace an empty integrand by o. We showed there that the mapping $P \mapsto \operatorname{fib}(P;\Phi)$ induces a natural homomorphism $\bar{\Phi}\colon \mathcal{W}(P) \to \mathcal{W}(\operatorname{fib}(P;\Phi))$. Indeed, let $\Phi^*\colon \mathbb{V} \to \mathbb{X}$ be the (dual) orthogonal projection (as usual, we identify a space with its dual when we have an inner product). The mapping Φ^* induces a corresponding homomorphism on the tensor algebra $\mathbb{T} := \mathbb{T}(\mathbb{V})$, which we denote by the same symbol.

The result from [**26**] is the following.

THEOREM 16.1 *If* $\Phi: \mathbb{X} \hookrightarrow \mathbb{V}$ *is an isometric injection, define* $\bar{\Phi}: \mathbb{T}(\mathbb{V}) \otimes \widehat{\Xi}(\mathbb{V}) \to \mathbb{T}(\mathbb{X}) \otimes \widehat{\Xi}(\mathbb{X})$ *on its generators by*

$$(a \otimes \langle K \rangle)\bar{\Phi} := \kappa(K, \Phi^*)(a\Phi^*) \otimes \langle K \rangle \Phi^*.$$

Then $\bar{\Phi}$ *induces a vector space homomorphism from* $\mathcal{W}(\mathbb{V})$ *to* $\mathcal{W}(\mathbb{X})$. *If* $a \in \mathcal{W}(P)$ *for some* $P \in \mathcal{P}(\mathbb{V})$, *then* $a\bar{\Phi} \in \mathcal{W}(\mathrm{fib}(P; \Phi))$.

Proof. For completeness, we briefly sketch the proof from [**26**]. Let $P \in \mathcal{P}(\mathbb{V})$, and let $a \in \mathcal{W}(P)$. Using Proposition 8.1 and Theorem 9.2, we may assume that P is simple, and that a is a canonical weight on P. The weight induced by a on $\mathrm{fib}(P, \Phi)$ is the natural one, obtained by integrating over the fibres. Notice, by the way, that GMC holds for each such fibre, and so the corresponding weight contribution to $\mathrm{fib}(P; \Phi)$ must also satisfy GMC; that is, it will indeed be a weight.

A face $F \leqslant P$ will make a contribution only if $F\Psi$ is full-dimensional in $\mathbb{Y}$, so that $\dim(F\Psi) = m := \dim \mathbb{Y}$. Such a face we call *general*; the rest are *singular*. We wish to apply Fubini's theorem, but we need a scaling factor, called the *volume ratio*, defined as follows. Pick an orthonormal basis $\{b_1, \ldots, b_k\}$, say, of $F_{\|}$, so that $\{b_{m+1}, \ldots, b_k\}$ is a basis of $F_{\|} \cap \mathbb{X}$. Let C be the unit m-cube based on $\{b_1, \ldots, b_m\}$; then the scaling factor is the volume of $C\Psi$. A little thought shows that, if $K := N(F, P)$ is the normal cone to P at F, then the scaling factor is, in fact, $\kappa(K, \Phi^*)$, namely, that of the projection Φ^* on K. (To see this, note that, if we extend to an orthonormal basis $\{b_1, \ldots, b_d\}$ of $\mathbb{V}$, then $\{b_{k+1}, \ldots, b_d\}$ is an orthonormal basis of $\mathrm{lin}\, K$.)

Direct calculation shows that

$$\int_{F\Psi} M_s(F \cap (\mathbb{X} + y))\, dy = \kappa(K, \Phi^*) M_s(F).$$

Now

$$M_s((F - y) \cap \mathbb{X}) = (M_s(F \cap (\mathbb{X} + y)))\Phi^*,$$

and so, taking the orthogonal projection Φ^* outside the integral, we see that F contributes $\kappa(K, \Phi^*) M_s(F)\Phi^*$ to the corresponding weight on $\mathrm{fib}(P; \Phi)$.

It is clear that the same calculation carries over to a singular face F; here we have $\kappa(K, \Phi^*) = 0$.

Now the normal fan of $\mathrm{fib}(P; \Phi)$, which consists of the complex of normal cones to its faces, is just the common refinement of the fan induced by the projection of the normal fan of P under Φ^* (see, for example, [**25**]). Finally, if $F \leqslant P$, then the contribution of its normal cone is thus $N(F, P)\Phi^*$. The desired conclusion follows. □

We call $\bar{\Phi}$ the *fibre map*. Of course, $\bar{\Phi}$ is certainly not an algebra homomorphism, if only because it does not preserve degrees. It also bears emphasizing that it is the euclidean structure that makes the fibre map construction work in a unique way.

Let us note that $\bar{\Phi}$ preserves dimensions of normal cones rather than (as in the homomorphisms of weight algebras introduced previously) those of faces. If, as before, $\dim \mathbb{V} = d$ and $\dim \mathbb{Y} = m$ (this turns out to be a better choice than specifying $\dim \mathbb{X}$), then we have $\bar{\Phi}: \mathcal{W}_{s,k}(P) \to \mathcal{W}_{s+m,k-m}(\mathrm{fib}(P; \Phi))$. Another thing to observe that *all* normal cones contribute to the image under $\bar{\Phi}$; contrast this with the situation for the homomorphism of (8.4), where choices have to be made of cells in induced subdivisions.

An important property of $\bar{\Phi}$ is the following. We refer to [**26**] for the fairly elementary proof.

THEOREM 16.2 *The fibre map* $\bar{\Phi}: \mathcal{W}(\mathbb{V}) \to \mathcal{W}(\mathbb{X})$ *is surjective.*

As we said, it is not obvious how to define the homomorphism induced by a general linear mapping. The root of the problem is very basic. The original definition of the fibre polytope is a bit of a hybrid: the injection $\Phi: \mathbb{X} \to \mathbb{V}$ leads to a mapping $\mathcal{P}(\mathbb{V}) \to \mathcal{P}(\mathbb{X})$, and while the mapping $\bar{\Phi}$ does arise in a natural way (given the setting of inner product spaces), it is unnatural in that it acts on tensors in one space and cones in its dual. (We point out that Φ^* should operate on $\mathbb{T}^*$ rather than on $\mathbb{T}$.)

All that said, there are circumstances where we have a natural decomposition of a space into orthogonal subspaces; one of these leads to what we call the *fibre product* $\boxtimes$ on $\mathcal{W}$ which, while motivated by Alesker's product of valuations in [**1**, **2**], does not actually correspond to it exactly. Let $\Delta: \mathbb{V} \to \mathbb{V} \times \mathbb{V}$ be the *diagonal mapping*, given by

$$z\Delta := (x, x).$$

Its dual is therefore the sum mapping Σ, given by

$$(x, y)\Sigma := x + y.$$

These are not quite an isometric injection and orthogonal projection, but they are up to scaling; these are given by $(1/\sqrt{2})\Delta$ and $(1/\sqrt{2})\Sigma$, respectively. Observe that $\mathbb{V}\Delta$ has a natural complement $\{(x, -x) \mid x \in \mathbb{V}\}$. If, as before, we define the volume ratio of cones J and K in $\mathbb{V}$ to be

$$\sigma(J, K) := \kappa(J \times K, \Sigma),$$

then we have a product $\boxtimes: \mathcal{W} \times \mathcal{W} \to \mathcal{W}$, induced by

$$a \boxtimes b \leftrightarrow (a(J) \otimes \langle J \rangle) \boxtimes (b(K) \otimes \langle K \rangle) := \sigma(J, K) a(J) b(K) \otimes \langle J + K \rangle. \tag{16.1}$$

The definition of $\boxtimes$ makes the following theorem transparent; the only comment needed for the proof is the obvious $\sigma(K_1, K_2)\sigma(K_1 + K_2, K_3) = \sigma(K_1, K_2 + K_3)\sigma(K_2, K_3)$.

THEOREM 16.3 *The fibre product* $\boxtimes$ *on* $\mathcal{W}$ *turns it into a commutative algebra over* $\mathbb{F}$. *It has a unity, namely,* $1 \otimes \langle \{o\} \rangle$.

The unity corresponds to a full-dimensional unit volume.

In contrast to the situation for the standard product on $\mathcal{W}$, the fibre product does not generally preserve the subsets $\mathcal{W}(P)$ (to avoid confusion between the two products, we resile from calling $\mathcal{W}(P)$ a subalgebra here). However, if $\mathcal{C}$ is a finite family of polyhedral cones in $\mathbb{V}$, let us write $\mathcal{W}(\mathcal{C})$ for the family of weights of the form

$$\sum_{K \in \mathcal{C}} a(K) \otimes \langle K \rangle.$$

Then the following is clear.

THEOREM 16.4 *Let $\mathcal{C}$ be a finite family of cones in $\mathbb{V}$. Then the set of weights $\mathcal{W}(\mathcal{C})$ is closed under $\boxtimes$ if and only if $J + K \in \mathcal{C}$ whenever $J, K \in \mathcal{C}$ are such that $\sigma(J, K) > 0$.*

Observe that any such finite family generates another finite family which is closed under $+$. The general setting here is thus a family $\mathcal{C}(U)$ generated by the half-lines $\operatorname{pos}\{u_j\}$, with $U = (u_1, \ldots, u_n) \subseteq \mathbb{V}$.

To answer a question which we have implicitly raised, we need one more idea. Call a d-polytope $P \in \mathcal{P}(\mathbb{V})$ *monotypic* if every polytope in $\mathbb{V}$ with the same set $U := U(P)$ of outer facet normals is (strongly) isomorphic to P. Monotypic polytopes, together with related classes, were investigated in [**28**]. Then we have

THEOREM 16.5 *Let $P \in \mathcal{P}(\mathbb{V})$. Then $\mathcal{W}(P)$ is closed under the fibre product $\boxtimes$ if and only if P is monotypic.*

Proof. What is required here is that every (non-degenerate) sum $J + K$ of normal cones to P be dissectable into normal cones of P. In particular, this must happen for every cone of the form $\operatorname{pos}\{u_j\} + \operatorname{pos}\{u_k\}$, with $u_j, u_k \in U(P)$, and the monotypicity of P then easily follows (with reference to [**28**]). $\square$

§17. *Mixed polytopes.* Building on earlier work, in [**35**] Schneider described what he called mixed polytopes of $P_1, \ldots, P_k$ in a euclidean space $\mathbb{E}$; their support functionals are the coefficients (suitably normalized) of $\lambda_1, \ldots, \lambda_k \geqslant 0$ in

$$\int_{\mathbb{E}^{k-1}} \eta(\lambda_1 P_1 \cap (\lambda_2 P_2 + x_2) \cap \cdots \cap (\lambda_k P_k + x_k), \cdot)\, dx_2 \cdots dx_k,$$

with our previous convention that sets $\eta(\emptyset, \cdot) = 0$. In [**25**], we modified his original definition (which is not quite symmetric in $P_1, \ldots, P_k$), to bring it into line with the general constructions which we introduced there.

Schneider himself related his mixed polytopes to particular cases of fibre polytopes. Our formulation is the following. We have k polytopes $P_1, \ldots, P_k$ in d-dimensional euclidean space $\mathbb{E}$. Then, by (our) definition, the *mixed polytopes* of $P_1, \ldots, P_k$ are the coefficients of the powers of $\lambda_1, \ldots, \lambda_k \geqslant 0$ in the expansion of $\operatorname{fib}(\lambda_1 P_1 \times \cdots \times \lambda_k P_k; \Delta)$, where $\Delta\colon \mathbb{E} \to \mathbb{V} := \mathbb{E}^k$, given by

$$x\Delta := (x, \ldots, x),$$

is the *diagonal mapping*; this is an isometric injection only up to a scaling factor, but will serve the purpose. The orthogonal complement of $\mathbb{X} := \operatorname{im} \Delta$ in $\mathbb{V}$ is

$$\mathbb{Y} = \{y = (x_1, \ldots, x_k) \in \mathbb{V} \mid x_1 + \cdots + x_k = o\}.$$

We obtain a fibre over the typical point $y \in \mathbb{Y}$ just when

$$(\lambda_1 P_1 - x_1) \cap \cdots \cap (\lambda_k P_k - x_k) \neq \emptyset;$$

note that this restores the symmetry among $P_1, \ldots, P_k$. The dual of Δ is the *sum mapping* $\Sigma: \mathbb{V} \to \mathbb{E}$, given by

$$(x_1, \ldots, x_k)\Sigma := x_1 + \cdots + x_k;$$

again, up to a scaling factor, this is an orthogonal projection.

In [**26**], we pointed out that our techniques would enable us to calculate the mixed polytopes explicitly. With the introduction of the fibre product in Section 16, we can now go further. Again as we remarked in [**26**], we wish to calculate the 1-weight, which arises as the projection of a $((k-1)d+1)$-weight in $\mathbb{V}$ under the induced fibre map $\bar{\Sigma}$. If we want the coefficient of, say, $\lambda_1^{r_1} \cdots \lambda_k^{r_k}$, with $r_1 + \cdots + r_k = (k-1)d + 1$, this will come from the $((k-1)d+1)$-volumes of faces $F_1 \times \cdots \times F_k$ of $P_1 \times \cdots \times P_k$, with $\dim F_j = r_j$ for $j = 1, \ldots, k$. In effect, these are the canonical r_j-weights p_{j,r_j} of P_j for each j, from which we deduce

THEOREM 17.1 *The* 1-*weights of the mixed polytopes of* $P_1, \ldots, P_k$, *are*

$$p_{1,r_1} \boxtimes \cdots \boxtimes p_{k,r_k},$$

where $r_1, \ldots, r_k \geqslant 0$ *with* $r_1 + \cdots + r_k = (k-1)d + 1$.

§18. *Piecewise polynomials.* A function f on the dual space $\mathbb{V}^*$ is *piecewise polynomial* if there exists a fan (complex) $\mathcal{N}$ of polyhedral cones in $\mathbb{V}^*$ with apex o whose union is the whole space (that is, $\mathcal{N}$ is complete), such that the restriction $f|_C$ of f to each (closed) cone $C \in \mathcal{N}$ is a polynomial function. It is clear that piecewise polynomial functions are continuous, and that they form an algebra under the usual addition and multiplication of functions. This algebra has been studied extensively; we cite [**4**, **7**, **8**] for more background.

For simplicity in the following discussion, we shall confine our attention to a fixed fan $\mathcal{N} = \mathcal{N}(P)$, which is the normal fan to some simple d-polytope in $\mathcal{P}(\mathbb{V})$ (this accounts for our setting in the dual space), and the corresponding subalgebra $\mathcal{R} = \mathcal{R}(P)$. This is not a severe restriction, since any given complete fan can be refined to one of such a form $\mathcal{N}(P)$; of course, refinement (which is implicit in both addition and multiplication) preserves piecewise polynomiality, and at any time we can only be looking at finitely many functions.

In [**7**, **8**], Brion has related piecewise polynomials to the polytope algebras. As we remarked in Section 3, the quotients Π/T^{k+1} only have a rational algebra structure when $k \geqslant 1$ (as usual, we gloss over the fact that the zero-grade space is only isomorphic to $\mathbb{Z}$); [**8**, §4.4] shows this explicitly. However, what we do have is

THEOREM 18.1 *If* P *is a simple* d-*polytope, then the algebras* $\mathcal{R}(P)$ *and* $\mathcal{W}(P)$ *are naturally isomorphic.*

Proof. We only sketch the proof, since the details are very similar to those in the papers cited above. We begin by observing that tensors on $\mathbb{V}$ correspond directly to (global) polynomial functions on $\mathbb{V}^*$, and so the component of a tensor weight

$$\sum_{v \in \operatorname{vert} P} a(v) \otimes \widehat{n}(v, P)$$

on the vertex-set of a polytope P gives a corresponding piecewise polynomial f on the fan $\mathcal{N}(P)$ in $\mathbb{V}^*$; the GMC for the edges of P enforces continuity. Indeed, if $v, w \in \operatorname{vert} P$ are adjacent, so that the normal cones $N(v, P)$ and $N(w, P)$ share a facet $N(E, P)$ with $E = \operatorname{conv}\{v, w\}$, then $a(v) - a(w)$ is divisible by $v - w$, so that the associated polynomials coincide on $N(E, P)$. Conversely, we saw in Section 12 that we only need the GMR for edges to show that we have a weight, which implies that a piecewise polynomial in $\mathcal{R}(P)$ yields an associated weight.

Suppose, as before, that $U = (u_1, \ldots, u_n) \subseteq \mathbb{V}^*$ is the set of (unit) outer normals to the facets of P. For each $j = 1, \ldots, n$, define the piecewise linear function f_j by

$$f_j = \eta(P(U, b + e_j), \cdot) - \eta(P(U, b), \cdot),$$

where (again as before) $P = P(U, b)$, with the components of the support vector b chosen so that $P(U, b + e_j) \approx P$. Following [**8**], we can describe f_j explicitly. First, $f_j(u_j) = 1$. Second, if $C \in \mathcal{N}(P)$ contains u_j, let D be its facet with $u_j \notin D$; then f_j vanishes on D. Finally, $f|_C \equiv 0$ if $C \in \mathcal{N}$ with $u_j \notin C$. We identified the e_j with elements of $\mathcal{W}_1(P)$ which generate $\mathcal{W}(P)$ as a module over $\mathbb{T}$; it is now not hard to see that $e_j \leftrightarrow f_j$ induces the required isomorphism between $\mathcal{W}(P)$ and $\mathcal{R}(P)$. $\square$

The following discussion actually expands this sketch a little. In [**8**, §2.2], Brion produces a linear mapping $\pi_P\colon \mathcal{R}(P) \to \mathcal{R}(\mathbb{V}^*)$ of grade $-d$. We demonstrate that π_P just gives the correspondence between the components of a weight a on the vertices of a simple d-polytope P and the component on P itself, through iteration of the GMC.

We define π_P as follows. There are d edges of P through each $v \in \operatorname{vert} P$. Let $w_1(v), \ldots, w_d(v)$ be vectors directed along these edges towards v, and chosen so that $\det(w_1(v), \ldots, w_d(v)) = \pm 1$. If $f \in \mathcal{R}(P)$, then

$$f\pi_P := \sum_{v \in \operatorname{vert} P} \frac{f}{\langle w_1(v), \cdot\rangle \cdots \langle w_d(v), \cdot\rangle}. \tag{18.1}$$

Initially, it is far from clear that $f\pi_P$ is even well defined, let alone a polynomial. However, we prove

THEOREM 18.2 *Let P be a simple d-polytope. Then the mapping π_P on $\mathcal{R}(P)$ of (18.1) induces a corresponding mapping $\pi_P\colon \mathcal{W}(P)|_{\operatorname{vert} P} \to \mathcal{W}(P)|_P$ between components of weights in $\mathcal{W}(P)$.*

Proof. Translated into terms of weights, we have

$$a\pi_P := \sum_{v \in \operatorname{vert} P} a(v)/w_1(v) \cdots w_d(v),$$

again assuming that this is well defined.

We need to follow the GMC from vert P through to P, and show that it has the same effect as π_P. First, recall that, if $w_1, \ldots, w_r \in \mathbb{V}$, then (as before) we define $\operatorname{Det}(w_1, \ldots, w_r)$ by

$$\operatorname{Det}(w_1, \ldots, w_r)^2 := |\langle w_i, w_j\rangle|,$$

so that

$$\mathrm{Det}(w_1, \ldots, w_d) = |\det(w_1, \ldots, w_d)|.$$

For any given vertex v, we can clearly normalize the $w_j := w_j(v)$ so that $\mathrm{Det}(w_1, \ldots, w_r) = 1$ for $r = 1, \ldots, d$ (we shall order the w_j suitably later). Thus we may assume inductively that, for each (proper) face $F < P$, the analogous mapping $\pi_F \colon \mathcal{W}(P)|_{\mathrm{vert}\, F} \to \mathcal{W}(P)|_F$ has the required properties. (It should be borne in mind that $\mathcal{W}(P)|_{\mathcal{F}(F)} = \mathcal{W}(F)$.) Moreover, since π_P clearly preserves the module structure over $\mathbb{T}$, it is enough to calculate its effect on elements of a sweep basis of $\mathcal{W}(P)$.

We first treat the basic d-weight a. This is supported by the last vertex v encountered in the sweep, and the faces of P which contain it. Let u_j be the unit normals to the jth facet F_j of P which contain v, so that w_j points along the edge of P through v which does not lie in F_j. If $\mathrm{Det}(w_1, \ldots, w_{d-1}) = 1$, then

$$\mathrm{Det}(w_1, \ldots, w_d) = 1 \Longrightarrow \langle w_d, u_d \rangle = 1.$$

In the GMC (9.1), we choose $t := w_d$. Our inductive assumption then leads to

$$a(P) w_d = a(F_d) \langle w_d, u_d \rangle = a(v)/w_1 \cdots w_{d-1},$$

since $\langle w_d, u_j \rangle = 0$ for $j = 1, \ldots, d-1$, as we wanted to show.

We next look at a basic $(d-1)$-weight. This is localized to an edge $E = \mathrm{conv}\{v, v'\}$, say. If F, F' are the two facets of P whose unit normals are u, u', and which contain v, v' (respectively) but not E, then we take $t := v - v'$ in the GMC (9.1). With w_d again pointing along E to v, we have

$$\langle w_d, u \rangle = 1 \Longrightarrow w_d = \langle v - v', u \rangle^{-1} (v - v'),$$

and similarly

$$w_d' = \langle v' - v, u' \rangle^{-1} (v' - v) = \langle v - v', u' \rangle^{-1} (v - v'),$$

with w_d' playing the analogous rôle for v'. The GMC then yields

$$a(F)/w_d + a(F')/w_d' = (a(F)\langle v - v', u \rangle + a(F')\langle v - v', u' \rangle)/(v - v') = a(P) = o,$$

as we wanted.

Finally, if a is a basic r-weight with $r \leqslant d-2$, then the calculations from GMC yield $a(F) = o$ whenever $\dim F \geqslant d-1$, once again agreeing with those arising from π_P. $\square$

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