ERROR TERM IMPROVEMENTS FOR VAN DER CORPUT TRANSFORMS

BY

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DISSERTATION

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Abstract

We improve the error term in the van der Corput transform for exponential sums $\sum^*_{a \le n \le b} g(n) e^{2\pi i f(n)}$. For many smooth functions g and f, we can show that the largest factor of the error term is given by a simple explicit function, which can be used to show that previous results, such as those of Karatsuba and Korolev, are sharp. Of particular note, the methods of this paper avoid the use of the truncated Poisson formula, and thus can be applied to much longer intervals $[a, b]$ with far better results. As an example of the strength of these results, we provide a detailed analysis of the error term in the case $g(x) = 1$ and $f(x) = (x/3)^{3/2}$.

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Introduction

We are interested here in estimating the error term Δ associated with the van der Corput transform,

$$\sum_{a\le n\le b}^{*} g(n)e(f(n)) = \sum_{f'(a)\le r\le f'(b)}^{*} \frac{g(x_r)e(f(x_r) - rx_r + \frac{1}{8})}{\sqrt{f''(x_r)}} + \Delta, \tag{1.1}$$

where f and g are several times continuously differentiable functions with $f''(x) > 0$ for $x \in [a, b]$ and where x_r is defined by $f'(x_r) = r$, $x_r \in [a, b]$.[1] A starred sum indicates that if a limit of summation is an integer, the corresponding summand is multiplied by 1/2. The function $e(x)$ denotes $e^{2\pi ix}$.

The van der Corput transform is best-known in number theory for being the crucial element of Process B in the theory of exponent pairs and is sometimes simply referred to as "Process B." As a method of estimating exponential sums, the van der Corput transform is often presented alongside other methods of Weyl, van der Corput, and Vinogradov. Direct application of the van der Corput transform can take a complicated sum to one more amenable to estimation techniques or it can reduce the number of terms and therefore make computational estimations easier. The van der Corput transform itself is involutive—applying it to the right-hand side of (1.1) will simply return the sum on the left-hand side of (1.1)—so one gains nothing by applying it twice in a row; but one could alternate applications of the van der Corput transform with other techniques (such as the Process A of the theory of exponent pairs) to achieve better results. This alternation method is still a fruitful ground for modern research. Recently, Cellarosi [3] attained interesting new results in the classical case of $g(x) = 1$ and $f(x) = \alpha x^2$, where the alternating technique employed is simply reducing α modulo 1; Nakai [30, 31, 32] has investigated the possibility of using an analogous method when $f(x)$ is cubic or quartic; and Hiary [13, 14] has used a similar iterative procedure to improve numerical computations of the truncated theta function and the Riemann zeta function. The van der Corput transform has also seen recent use in physical applications, including quantum optics and wave processes (see [19] and the papers cited there for more details).

[1] Since $f''(x)$ is assumed to be positive, $f'(x)$ is surjective and hence x_r is unique.

Van der Corput [37] originally showed that, given

$$|f''(x)| \asymp \lambda_2, \qquad |f^{(3)}(x)| \ll \lambda_3, \qquad g(x) = 1, \qquad \text{for } x \in [a,b],$$

the error term can be bounded like

$$\Delta = O(\lambda_2^{-1/2}) + O(\log(f'(b) - f'(a) + 2)) + O((b-a)\lambda_2^{1/5}\lambda_3^{1/5}).$$

(Here we use the Landau and Vinodradov asymptotic notations which will be defined explicitly in Section 2.) Phillips [33] improved this error under the additional assumptions

$$|f^{(4)}(x)| \ll \lambda_4 \qquad \text{for } x \in [a,b] \qquad \text{and} \qquad \lambda_3^2 = \lambda_2\lambda_4;$$

in this case, we can replace $O((b-a)\lambda_2^{1/5}\lambda_3^{1/5})$ with $O((b-a)\lambda_3^{1/3})$.

The form of the error term found in most modern texts on analytic number theory [11, 17, 20, 27][2] has its roots in the work of Kolesnik [23] and Heath-Brown [12], although the results of the latter authors required the function f to be complex-analytic, an assumption which has since been circumvented. This moderately-difficult-to-prove form of the error term suffices for many basic applications of the van der Corput transform. We present this modern bound on the error in the notation used by Huxley.

Theorem 1.1 (Lemma 5.5.3 in [16])**.** *Suppose that $f(x)$ is real and four times continuously differentiable on $[a,b]$. Suppose that there are positive parameters M and T, with $M \geq b-a$, such that, for $x \in [a,b]$, we have*

$$f''(x) \asymp T/M^2, \qquad f^{(3)}(x) \ll T/M^3, \qquad \textit{and} \qquad f^{(4)}(x) \ll T/M^4.$$

Let $g(x)$ be a real function of bounded variation V on the closed interval $[a,b]$. Then

$$\sum_{a \leq n \leq b} g(n)e(f(n)) = \sum_{f'(a) \leq r \leq f'(b)} \frac{g(x_r)e(f(x_r) - rx_r + \frac{1}{8})}{\sqrt{f''(x_r)}} + O\left((V + |g(a)|)\left(\frac{M}{\sqrt{T}} + \log(f'(b) - f'(a) + 2)\right)\right),$$

where x_r is the unique solution in $[a,b]$ to $f'(x_r) = r$. The implicit constant in the big-O term depends on the implicit constants in the relations between T, M, and the derivatives of $f(x)$.

The error term $M/\sqrt{T}$ here corresponds to the $\lambda_2^{-1/2}$ term in the estimates of van der Corput and

[2] Curiously, [36] skips this form of the error term entirely, although it contains some finer estimates of interest.

Phillips.

Unfortunately, for many interesting cases, the above error is insufficient. As Huxley [16, p. 475] notes, when applying the van der Corput transform to a multi-dimensional exponential sum, "...the error terms and the truncation error in the Poisson summation formula may add up to more than the estimate for the reflected sum." Thus, finer error terms, useful for a broad spectrum of problems including computation and physical applications, have been given by various people, including Kolesnik [22], Liu [25], Redouaby and Sargos [35], Karatsuba and Korolev [19], and Blanc [2]. Liu [25] extends an (unfortunately not well-known) earlier work of Min [26], removing the latter's condition that $f(x)$ be an algebraic function. Redouaby and Sargos show that the conditions on $f^{(4)}(x)$ and $g''(x)$ can be removed without greatly increasing the bound on the error term. The work of Karatsuba and Korolev is the only one to give the implicit constants in the big-O terms explicitly.

It is difficult to state most of these other forms of the error term in full detail; we will, however provide the following inexplicit form of Karatsuba and Korolev's result as an example of the comparative strength of these errors compared with Theorem 1.1. (We supplement this result with a simplified error term that comes from Karatsuba and Voronin [21].)

Theorem 1.2. *Suppose that $f(x)$ and $g(x)$ are real-valued functions with $f \in C^4[a,b]$ and $g \in C^2[a,b]$. Suppose there are positive constants M, T, and U, with $M \asymp b-a$, such that, for $x \in [a,b]$,*

$$f''(x) \asymp T/M^2, \qquad |f^{(3)}(x)| \ll T/M^3, \qquad |f^{(4)}(x)| \ll T/M^4 \tag{1.2}$$

$$|g(x)| \ll U, \qquad |g'(x)| \ll U/M, \qquad |g''(x)| \ll U/M^2. \tag{1.3}$$

Then,

$$\sum_{a \le n \le b}^{*} g(n)e(f(n)) = \sum_{f'(a) \le r \le f'(b)}^{*} \frac{g(x_r)e(f(x_r) - rx_r + 1/8)}{\sqrt{f''(x_r)}} + O\left(U\left(T(a) + T(b)\right)\right)$$
$$+ O\left(U\left(\log\left(f'(b) - f'(a) + 2\right) + \frac{M}{T} + \frac{T}{M^2} + 1\right)\right),$$

where x_r is the unique solution to $f'(x_r) = r$ in the interval $[a,b]$ and

$$T(\mu) = \begin{cases} 0, & \text{if } \|f'(\mu)\| = 0. \\ \min\left\{\dfrac{M}{\sqrt{T}}, \dfrac{1}{\|f'(\mu)\|}\right\}, & \text{if } \|f'(\mu)\| \ne 0. \end{cases} \tag{1.4}$$

The size of the implicit constant in the big-O term depends on the implicit constants in the relations of M,

T, U, and the derivatives of $f(x)$ and $g(x)$.

If, in addition, $M \ll T \ll M^2$ and $M \geq b-a$, we may remove the terms $O(T/M^2+1)$.

The $M/\sqrt{T}$ term again makes an appearance in this theorem. It cannot, in fact, be completely removed. If b is not an integer, but $f'(b)$ is an integer, then changing b to $b-\epsilon$ for some very small $\epsilon > 0$ will not change the value of the sum on the left-hand side of (1.1), but removes a term of size $M/\sqrt{T}$ from the right-hand side.

Blanc [2] gives the only instance (that the author is aware of) of an explicit main term for the error. We give it below.

Theorem 1.3. *Suppose that $f(x)$ and $g(x)$ are real-valued functions with $f \in C^5[a,b]$ and $g \in C^2[a,b]$. Suppose there are positive constants M, T, U, and c, with $M \geq b-a$, $c \geq 1$, $c^{-1}M^2 \leq T \leq cM^3$, such that, for $x \in [a,b]$,*

$$\begin{aligned} f''(x) &\geq c^{-1}TM^{-2}, & \\ |f^{(r)}(x)| &\leq cTM^{-r}, & \text{for } r = 2,3,4,5, \\ |g^{(r)}(x)| &\leq cUM^{-r}, & \text{for } r = 0,1,2. \end{aligned}$$

Then,

$$\sum_{a \leq n \leq b}^{*} g(n)e(f(n)) = \sum_{r=\lfloor f'(a)\rfloor+1}^{\lfloor f'(b)\rfloor} \frac{g(x_r)e(f(x_r)-rx_r+\frac{1}{8})}{\sqrt{f''(x_r)}} + \mathcal{R}(b) - \mathcal{R}(a) + O(U),$$

where

$$R(\mu) = g(\mu)e(f(\mu)+1/4)\int_0^\infty \frac{\sinh\left(2\pi s(f'(\mu))z\right)}{\sinh(\pi z)} e\left(-\frac{f''(\mu)}{2}z^2\right)\, dz,$$

$s(\cdot)$ is the sawtooth function, and the implicit constant depends only on c.

The function $R(\mu)$ can be difficult to estimate in a simple way, although it does satisfy the type of bounds in the Karatsuba and Korolev paper.

Van der Corput's results in [38] also deserve a mention here, as they are of an entirely different flavor from those listed above and because they are generally not very well-known.[3] Instead of having a coefficient of $g(x_r)/\sqrt{f''(x_r)}$ in the transformed sum, he has a more general set of coefficients. In addition, his error

[3]This may be largely van der Corput's fault. Robert Schmidt, in reviewing van der Corput's paper for Zentralblatt, remarked that because van der Corput provided no context for his results, neither in terms of past results or future goals, that the paper was unlikely to spark much interest, and indeed it has so far only been cited once elsewhere, in [4]. Due to the desire to avoid repeating van der Corput's mistake, the complicated main theorem of this paper will not be presented in the introduction.

term—which would require too many definitions to state succinctly here—bears no resemblance to any of the other error estimates cited or formulated in this paper.

While van der Corput's results in [38] are quite complicated to use, they are aesthetically pleasing. As we remarked above, the van der Corput transform is involutive, but in all the other results given above as well as the main result of this paper, one can obtain very different error terms when one applies the transform to the right-hand side of (1.1) instead of the left, assuming that the conditions necessary to apply the results would even still hold. In [38], van der Corput shows that the conditions are still satisfied and the error term unchanged regardless of which side one applies his transform to; his *theorem* is involutive.

The van der Corput transform and its error has been studied in much more general settings than we go into here: of particular interest, Jutila [18] considered sums of the form $\sum b(n)g(n)e(f(n))$ for certain multiplicative functions $b(n)$ (see also [16, Ch. 20]), and Krätzel [24] considered the van der Corput transform of a convergent infinite series.

One may ask what the best possible error term could be. Given the frequent restriction in theorems on the van der Corput transform that $f''(x) \asymp \lambda_2$ (or, equivalently $f''(x) \asymp T/M^2$), it is not surprising that the case where g is constant and f is quadratic (that is, $f''(x) = \lambda_2$) is the most-commonly studied special case of the van der Corput transform and the one with the best error terms [3, 4, 9, 29, 39]. Fedotov and Klopp [9][4] have given the error term in this case as an explicit integral, and in [10] they investigate this result further. But perhaps most amazing are the results of Coutsias and Kazarinoff [4]: they showed that for positive integers n, we have

$$\left| \sum_{k=0}^{N}{}^{*} e\left(\omega \cdot \frac{k^2}{2}\right) - \frac{e(\operatorname{sgn}(\omega)/8)}{\sqrt{|\omega|}} \sum_{k=0}^{n}{}^{*} e\left(-\frac{1}{\omega} \cdot \frac{k^2}{2}\right) \right| \leq C \left| N - \frac{n}{\omega} \right|$$

for $0 < |\omega| < 1$, $N = [\![n/\omega]\!]$ is the nearest integer to n/ω, and $1 < C < 3.14$ is a particular constant. Not only is the error bounded, but it shrinks to zero as n/ω nears an integer.

The Coutsias-Kazarinoff result suggests that the van der Corput transform should be very accurate; in particular, the van der Corput transform for nice enough functions f and g shouldn't have compounding error terms, such as the $\log(f'(b) - f'(a) + 2)$ seen in most of the other results mentioned above.

The main theorems of this paper (and Theorem 4.4, especially) confirm this hypothesis, allowing the van der Corput transform to be applied on very long intervals with a much higher degree of accuracy than in previous results.

As a quick example of this, consider the following well-known transform which appears in Iwaniec and

[4]This paper contains a small error in line (0.4) that helped to spark the author's investigation into the van der Corput transform.

Kowalski's book [17, p. 211] (a version of this transform also appears in [25]). Given $X > 0$, $N > 0$, and $\alpha > 1$, $\nu > 1$, consider

$$\sum_{N \le n \le \nu N}^{*} \left(\frac{\alpha}{n}\right)^{\frac{1}{2}} e\left(\frac{X}{\alpha}\left(\frac{n}{N}\right)^{\alpha}\right) = \sum_{M \le m \le \mu M}^{*} \left(\frac{\beta}{m}\right)^{\frac{1}{2}} e\left(\frac{1}{8} - \frac{X}{\beta}\left(\frac{m}{M}\right)^{\beta}\right) + \Delta, \tag{1.5}$$

where $1/\alpha + 1/\beta = 1$, $\mu^{\beta} = \nu^{\alpha}$, and $MN = X$. Using a form of Theorem 1.1, Iwaniec and Kowalski[5] give

$$\Delta = O(N^{-1/2} \log(M+2) + M^{-1/2} \log(N+2)),$$

with an implicit constant dependent on α and ν.

Using the results of this paper, we may improve this to the following.

Corollary 1.4. *Provided $N \gg 1$ and $M \gg 1$, in equation* (1.5)*, we may take*

$$\Delta = O(N^{-1/2} + M^{-1/2}) \tag{1.6}$$

with an implicit constant dependent only *on α and the lower bounds on N and M.*

Corollary 1.4 allows us to take ν as large as we like (and hence our intervals as large as we like) without increasing the bound on Δ. This, as well as the loss of the logarithmic factor, are common to applications of the results in this paper.

More powerful results are possible. In many cases, the main theorems of this paper also allow one to extract the next term in the asymptotic for the van der Corput transform. Under the same general hypotheses of Theorems 1.1 and 1.2, we can improve the error terms to the following.

Corollary 1.5. *Suppose that $f(x)$ and $g(x)$ are real-valued functions with $f \in C^4[a,b]$ and $g \in C^2[a,b]$. Suppose there exist constants M, T, and U satisfying $M = b - a \gg 1$, $T \gg 1$, and the bounds on lines* (1.2) *and* (1.3)*. Moreover, assume that $f''(a)$ and $f''(b)$ are both < 1. Then, we have*

$$\sum_{a \le n \le b}^{*} g(n) e(f(n)) = \sum_{f'(a) \le r \le f'(b)}^{*} \frac{g(x_r) e(f(x_r) - r x_r + 1/8)}{\sqrt{f''(x_r)}} - \mathcal{T}(b) + \mathcal{T}(a) + O\left(\frac{U}{\sqrt{T}}\left(1 + \frac{M}{T}\right)\right),$$

[5] A typo in the book has the logarithms in opposite places.

where $\mathcal{T}(\mu)$ equals

$$\begin{cases} \dfrac{g(\mu)f^{(3)}(\mu)e(f(\mu))}{6\pi i f''(\mu)^2} - \dfrac{g'(\mu)e(f(\mu))}{2\pi i f''(\mu)} + O\left(U\left(1+\dfrac{1}{M}\right)\right), & \text{if } \|f'(\mu)\| = 0, \\ O\left(\dfrac{UM}{\sqrt{T}}\right), & \text{if } 0 < \|f'(\mu)\| \leq \sqrt{f''(\mu)}, \\ g(\mu)e(f(\mu) + [\![f'(\mu)]\!]\,\mu)\left(-\dfrac{1}{2\pi i\langle f'(\mu)\rangle} + \psi\left(\mu, \langle f'(\mu)\rangle\right)\right) \\ \quad + O\left(\dfrac{U}{M\|f'(x)\|^2}\left(1+\dfrac{T}{M}\right) + \dfrac{UT}{M^2\|f'(x)\|^3}\right), & \text{if } \|f'(\mu)\| \geq \sqrt{f''(\mu)}. \end{cases}$$

Here, $[\![x]\!]$ represents the nearest integer[6] *to x; $\langle x\rangle$, the difference between x and the nearest integer to x, namely $x - [\![x]\!]$; and $\|x\|$, the distance between x and the nearest integer to x, so that $\|x\| = |\langle x\rangle|$. The function $\psi(x,\epsilon)$ equals*

$$-\frac{1}{2\pi i}\lim_{R\to\infty}\sum_{0<|r|<R}\frac{e(rx)}{r+\epsilon} \qquad \text{for } |\epsilon| \leq \frac{1}{2},$$

which converges and is uniformly bounded for all real x.

The size of the implicit constant in the big-O term depends on the implicit constants in the relations of M, T, U, and the derivatives of $f(x)$ and $g(x)$.

This in particular suggests that the size of $T(\mu)$ in Karatsuba and Korolev's result is optimal when $\|f'(\mu)\| \geq \sqrt{f''(\mu)}$. When $\|f'(\mu)\| = 0$—that is, when $f'(\mu)$ is an integer—the term $\mathcal{T}(\mu)$ can be more simply bounded by $O(UM/T)$.

Corollary 1.5 can, in certain cases, be used to improve the Kusmin-Landau inequality, a common result in the study of exponential sums. A short history of this theorem is given on page 20 of [11].

Theorem 1.6 (The Kusmin-Landau inequality—Theorem 2.1 in [11])**.** *Suppose $f \in C^1[a,b]$ and $f'(x)$ is monotonic on an interval $[a,b]$. Moreover suppose $\|f'(x)\| \geq \theta > 0$ on $[a,b]$. Then, we have*

$$\left|\sum_{a\leq n\leq b} e\left(f(n)\right)\right| \leq \cot\left(\frac{\pi\theta}{2}\right).$$

Corollary 1.7. *Suppose f, T, and M satisfy the conditions of Corollary 1.5.*

If $\theta = \min_{z\in[a,b]}\|f'(z)\|$ is positive, $\|f'(a)\| > \sqrt{f''(a)}$, and $\|f'(b)\| > \sqrt{f''(b)}$, then

$$\sideset{}{^*}\sum_{a\leq n\leq b} e(f(n)) = \frac{e(f(b))}{2\pi i\langle f'(b)\rangle} - \frac{e(f(a))}{2\pi i\langle f'(a)\rangle} + O\left(\frac{1}{M\theta^2}\left(1+\frac{T}{M}\right) + \frac{T}{M^2\theta^3} + \frac{1}{\sqrt{T}}\left(1+\frac{M}{T}\right)\right),$$

where $\langle x\rangle$ represents the difference between x and the nearest integer to x.

[6] If $x = n + 1/2$ for an integer n, then it doesn't matter if we let $[\![x]\!]$ equal n or $n+1$ provided that we do so consistently.

Corollary 1.7 follows immediately as a special case of Corollary 1.5. Moreover, it suggests that the constant in the classical Kusmin-Landau inequality may not be optimal. The function $\cot(\pi\theta/2)$ grows like $2/\pi\theta$ as θ approaches zero; however, Corollary 1.7 suggests that the growth should be at worst like $1/\pi\theta$ as θ approaches zero.

While Corollaries 1.5 and 1.7 strengthen many previous results, they are still constrained to apply to short intervals. The results of this paper do not give similarly simple conditions and error terms when the size of the interval is allowed to grow arbitrarily large. But, given a specific sum, we can show very great improvements as one endpoint of the interval tends towards infinity. Consider the specific transform

$$\sum_{1\le n\le N}^{*} e\left((n/3)^{3/2}\right) = \sum_{(1/12)^{1/2}\le r\le (N/12)^{1/2}}^{*} (24r)^{1/2}\cdot e(-4r^3+1/8)+\Delta, \tag{1.7}$$

where N is an integer. (We have kept the $(1/12)^{1/2}$ due to it's natural appearance in the application of the van der Corput transform. It may be replaced by $1/2$ with no change in value on the right-hand side, however.)

If we apply the form of the error term from Theorem 1.1, then we obtain the following result, which was included as an example in [27].

Corollary 1.8. *In line* (1.7)*, we have*

$$\Delta = O(N^{1/4}).$$

If we apply Theorem 1.2 from Karatsuba and Korolev instead, we obtain much finer results.

Corollary 1.9. *In line* (1.7)*, we have*

$$\Delta = \begin{cases} O\left((\log N)^2\right) + O\left(\min\left\{N^{1/4}, \dfrac{1}{\|(N/12)^{1/2}\|}\right\}\right), & \text{if } \|(N/12)^{1/2}\| \neq 0, \\ O\left((\log N)^2\right), & \text{if } \|(N/12)^{1/2}\| = 0. \end{cases}$$

In this case, the $(\log N)^2$ term comes from needing to break the sum on the left-hand side of (1.7) into roughly diadic intervals, each of which contributes an error term of size around $\log N$. One must also choose the endpoints of these intervals to be values which are 12 times a square, in order to keep the error term $T(\mu)$ zero.

Using the results of this paper, we can show that the result of Karatsuba and Korolev is almost sharp, in that we will extract an explicit term of size $N^{1/4}$ when $N^{1/4}$ is smaller than $\|(N/12)^{1/2}\|^{-1}$ and extract an explicit term of size $\|(N/12)^{1/2}\|^{-1}$ when $\|(N/12)^{1/2}\|^{-1}$ is smaller than $N^{1/4}$.

Corollary 1.10. *In line* (1.7)*, we have*

$$\Delta = c + \operatorname{sgn}(N' - N)\, e\left((N'/3)^{3/2}\right) \int_{\phi_{r'}(N-N')}^{+\infty} e\left(\frac{1}{2} f''(N') y^2\right)\, dy$$
$$- e\left((N/3)^{3/2}\right) \psi\left(N, \left\langle (N/12)^{1/2} \right\rangle\right) + O(N^{-1/2}).$$

where

$$r' = \left[\left[(N/12)^{1/2}\right]\right], \qquad N' = 12(r')^2, \qquad \phi_{r'}(x) = \left(\frac{f(N'+x) - f(N') - r'x}{\frac{1}{2} f''(N')}\right)^{1/2},$$

where $f(x) = (x/3)^{3/2}$ *and* c *is a particular constant*[7]*. The remaining functions are all as in Corollary 1.5.*

One way to understand these new error terms is via the geometry of the associated sums. Given functions f and g, one typically considers the curve $S(t) : [0, \infty) \to \mathbb{R}^2$ given by

$$S(t) = \sum_{1 \le n \le t} g(n) e\left(f(n)\right) + \{t\} g(\lfloor t \rfloor + 1) e\left(f(\lfloor t \rfloor + 1)\right),$$

where $\lfloor x \rfloor$ represents the floor of a real number x and $\{x\}$ represents the fractional part of x. Geometric aspects of these curves have been well-studied [1, 6, 7, 28].

Especially when f'' is small, these curves generate a series of spiral-like figures. The van der Corput transform of the sum associated to $S(t)$ then generates a new curve that can be seen to connect the center point of successive spirals by straight lines like in Figure 1.1. Thus the van der Corput transform can be seen as smoothing (if f'' is small) or roughening (if f'' is big) the curve. An easily accessible explanation for why this occurs is given in [5].

For our example case (1.7), the corresponding curve is displayed in Figure 1.2.

[7] Mathematica calculations with $N = 48,000,000$ give $c \approx -0.280 - 0.186i$.

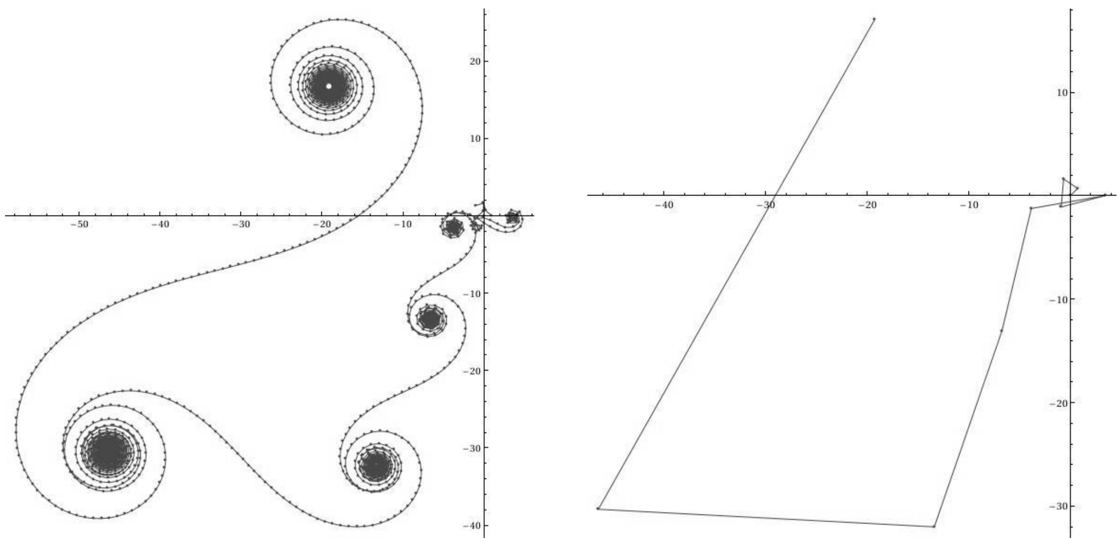

Figure 1.1: The curves associated to the exponential sum $\sum_{n\leq N} e(n\log n)$ and it slightly extended van der Corput transform $\sum_{n\leq \log(N)+1} \sqrt{e^{n-1}} \cdot e(-e^{n-1}+1/8)$ with $N=4000$.

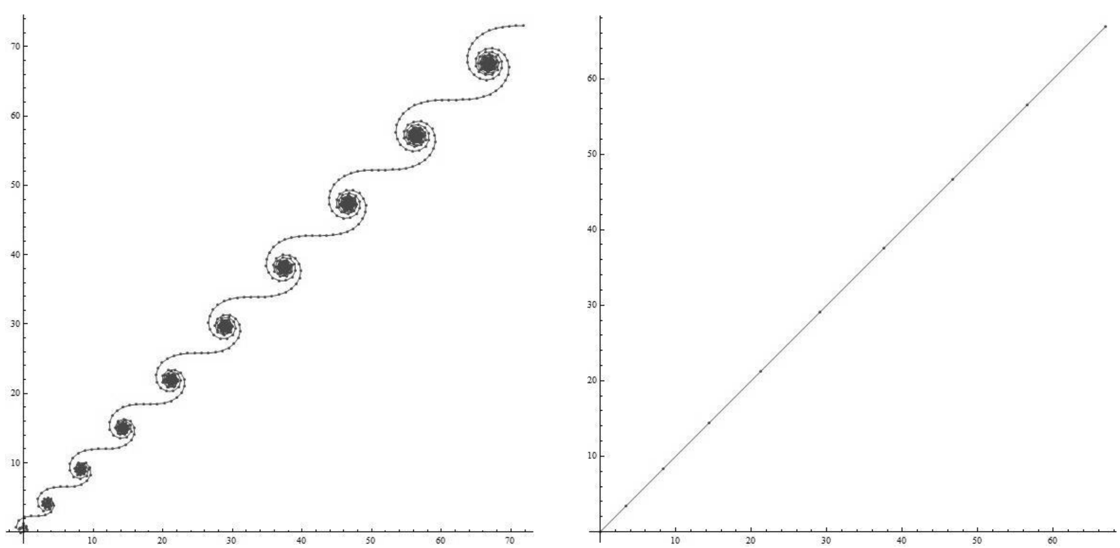

Figure 1.2: The curves associated to the exponential sums on the left-hand and right-hand side of (1.7) with $N=1,2,\ldots,1200$.

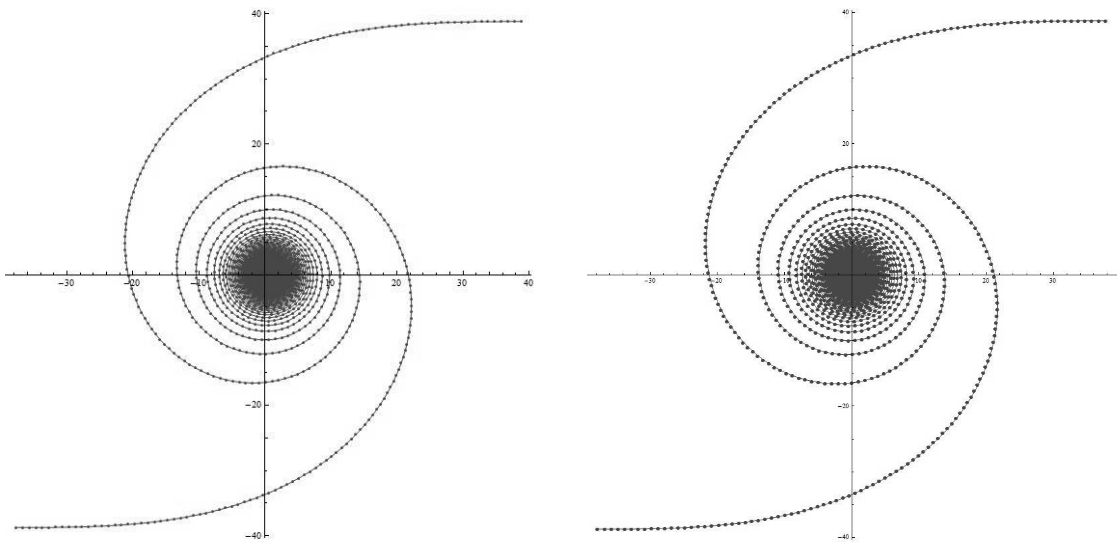

Figure 1.3: The 500th spiral of the curve associated to the left-hand side of (1.7) and its approximation by Corollary 1.10. These pictures are off-set closer to the origin.

Chapter 2

Notation

We will frequently use the Landau and Vinogradov asymptotic notations. The big-O notation $f(x) = O(g(x))$ (equivalently, $f(x) \ll g(x)$) means that there exists some constant c such that $|f(x)| \leq c|g(x)|$ on the domain in question. By $O(f(x)) = O(g(x))$, we mean that a function which is asymptotically bounded by $f(x)$ will also be asymptotically bounded by $g(x)$. The little-o notation $f(x) = o(g(x))$ as $x \to a$—typically with a equal to ∞—means that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

By $f(x) \asymp g(x)$, we shall mean that $g(x) \ll f(x) \ll g(x)$.

We will need several functions that relate a real number x to the integers nearby it. Let $\lfloor x \rfloor$ denote the usual floor of a real number x, the largest integer less than or equal to x, and let $\lceil x \rceil$ denote the usual ceiling of a real number x, the smallest integer greater than or equal to x. Let $\{x\} := x - \lfloor x \rfloor$ denote the fractional part of a real number x. Let $s(x) := \{x\} - 1/2$ denote the sawtooth function, and let

$$\psi(x) := \begin{cases} s(x), & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

denote the "smoothed" sawtooth function, with Fourier series

$$\psi(x) = -\frac{1}{\pi} \sum_{r=1}^{\infty} \frac{\sin(2\pi r x)}{r}.$$

We also wish to have a modified sawtooth function, given by

$$\psi(x, \epsilon) := -\frac{1}{2\pi i} \lim_{R \to \infty} \sum_{0<|r|<R} \frac{e(rx)}{r + \epsilon} \qquad \text{for } |\epsilon| \leq \frac{1}{2}.$$

The convergence of $\psi(x, \epsilon)$ will be guaranteed by Proposition 5.10.

Let $\langle x \rangle := s(x + 1/2)$ denote the difference between x and the nearest integer to x, let $[\![x]\!] := x - \langle x \rangle$

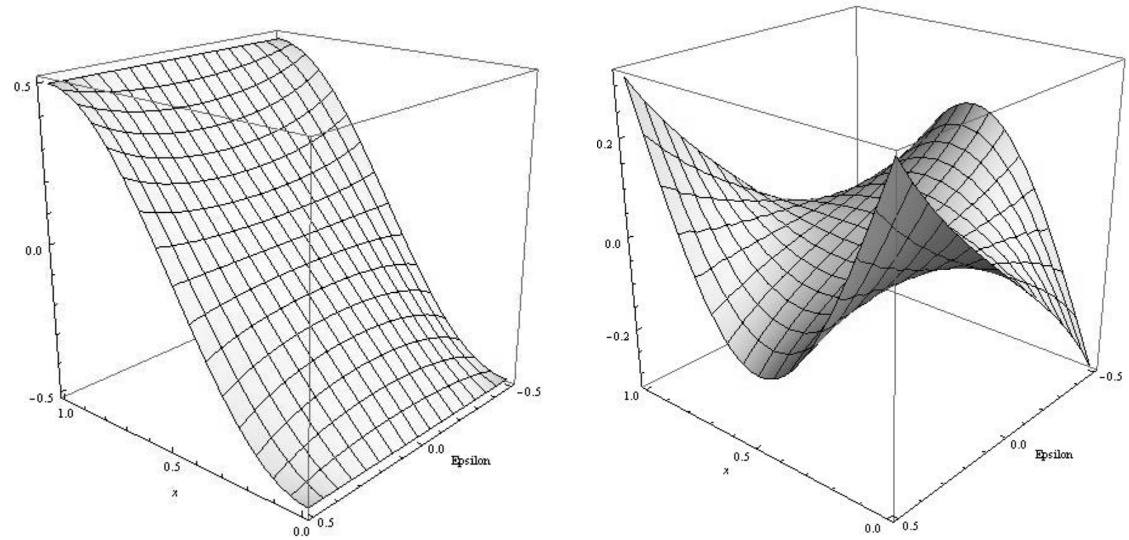

Figure 2.1: The real (left) and imaginary (right) parts of the function $\psi(x, \epsilon)$

denote the nearest integer to x, and let $\|x\| := \min\{1 - \{x\}, \{x\}\}$ denote the distance from x to $[\![x]\!]$. Let $\|x\|^*$ be given by

$$\|x\|^* = \begin{cases} \|x\|, & \text{if } \|x\| \neq 0, \\ 1, & \text{if } \|x\| = 0. \end{cases}$$

A few more definitions will simplify the (nonetheless still complicated) statement of the theorem. For visibility's sake, we will avoid writing (x) all the time when the choice of argument is always the same. Given functions f and g, let

$$\begin{aligned} H &= gf^{(3)} + 3g'f'' & G &= 12gg''(f'')^2 \\ W_\pm &= \frac{(2g'')^2 g'}{4\pi^2(H \pm \sqrt{H^2 - G})^2} - \frac{(2g'')^3(f''g)}{4\pi^2(H \pm \sqrt{H^2 - G})^3} & W_0 &= -\frac{H^2 f^{(3)}}{108\pi^2 g(f'')^5} \\ r_\pm &= f' - \frac{H \pm \sqrt{H^2 - G}}{2g''} & r_0 &= f' - \frac{3g(f'')^2}{H}, \end{aligned} \tag{2.1}$$

and given an integer r, let

$$h_r = \frac{(f' - r)g' - gf''}{4\pi^2(f' - r)^3}. \tag{2.2}$$

Chapter 3

Heuristics for the van der Corput transform

The van der Corput transform is a simple two-step process: the Poisson summation formula is first applied to obtain a sum of integrals, and then each integral is estimated using stationary phase methods. In particular, the Poisson summation formula (see Lemma 5.9) gives

$$\sum_{a\le n\le b}^{*} g(n)e\left(f(n)\right) = \lim_{R\to\infty}\sum_{r=-R}^{R}\int_a^b g(x)e\left(f(x)-rx\right)\ dx. \tag{3.1}$$

An oscillatory integral, like

$$\int_a^b g(x)e\left(f(x)\right)\ dx, \tag{3.2}$$

is said to have a stationary phase point if there exists an $x' \in [a,b]$ such that $f'(x') = 0$; if such a point exists, then we expect (3.2) to be roughly

$$\frac{g(x')e(f(x')+1/8)}{\sqrt{f''(x')}}\cdot c(x'),$$

plus some small error, where $c(x')$ equals $1/2$ if x' equals a or b and equals 1 otherwise. In our case, the stationary phase point of the integrand

$$g(x)e\left(f(x)-rx\right)$$

occurs at $x = x_r$, which is defined by $f'(x_r) - r = 0$. Provided $f'(a)\le r\le f'(b)$, this stationary phase point will be inside the interval of integration, and so we expect

$$\sum_{a\le n\le b}^{*} g(n)e\left(f(n)\right) \approx \sum_{f'(a)\le r\le f'(b)}^{*}\frac{g(x_r)e(f(x_r)-rx_r+1/8)}{\sqrt{f''(x_r)}}.$$

The error term for the van der Corput transform thus comes from the error in estimating all the integrals of (3.1).

The first problem in doing such an estimation comes from the integrals without a stationary phase point.

Typically the method one uses to estimate such an integral is an application of integration by parts.

$$\int_a^b e\left(f(x) - rx\right)\ dx = \left.\frac{e(f(x) - rx)}{2\pi i(f'(x) - r)}\right]_a^b + \int_a^b \frac{f''(x)}{2\pi i(f'(x) - r)^2} e\left(f(x) - rx\right)\ dx$$

(We will assume that $g(x) = 1$ for a while to simplify the heuristics.) The terms $e(f(x) - rx)/2\pi i(f'(x) - r)$ at a and b are referred to as the first-order endpoint contributions and exhibit a great deal of cancellation (see Proposition 5.10); for example, if $f'(a)$ is an integer, then

$$\lim_{R\to\infty} \sum_{\substack{|r|\leq R \\ r\neq f'(a)}} -\frac{e(f(a) - ra)}{2\pi i(f'(a) - r)} = \psi(a) e\left(f(a)\right),$$

and similarly if $f'(b)$ is an integer.

However, we are still left with the integral

$$\int_a^b \frac{f''(x)}{2\pi i(f'(x) - r)^2} e\left(f(x) - rx\right)\ dx$$

to evaluate. We could estimate the integral by taking absolute values of the integrand, but we would then obtain an estimate of the size $O(|f'(a) - r|)^{-1}) + O(|f'(b) - r|^{-1})$, which is of the same order as the first-order endpoint contributions, but without the cancellation that would allow them to be nicely summed.

One possible solution would be to apply integration by parts a second time: this gives

$$\int_a^b \frac{f''(x)}{2\pi i(f'(x) - r)^2} e\left(f(x) - rx\right)\ dx = \left.\frac{f''(x) e(f(x) - rx)}{(2\pi i)^2(f'(x) - r)^3}\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{f''(x)}{(2\pi i)^2(f'(x) - r)^3}\right) e\left(f(x) - rx\right)\ dx, \quad (3.3)$$

with a second-order endpoint contribution and a new integral. The second-order endpoint terms are absolutely convergent, and if $f'(a)$ and $f'(b)$ are integers, they sum to $O(f''(a)) + O(f''(b))$, which is a good estimate if f'' is small at the endpoints. On the other hand, if we try to bound the new integral in (3.3) by taking absolute values of the integrand, we get terms of the same order of magnitude as the second-order endpoint contributions (which, unlike the first-order endpoints, are summable). But in addition, we have the total variation of $f''(x)/(f'(x) - r)^3$ on $[a, b]$, which may be roughly bounded by the sum of the moduli of local maxima and minima of $f''(x)/(f'(x) - r)^3$ on $[a, b]$.

Unless $f(x)$ is quadratic—as in Coutsias and Kazarinoff's case[1]—it is difficult to find good estimates on

[1]Coutsias and Kazarinoff actually analyze the resulting integrals and endpoint contributions in the quadratic case when inegration by parts is repeated many times.

these integrals. More applications of integration by parts or the presence of a non-constant g only make things worse. Therefore, these terms are often avoided entirely by applying the truncated Poisson formula instead of the full Poisson summation formula.

Proposition 3.1 (Truncated Poisson formula—Proposition 8.7 in [17]). *Let $f \in C^2[a,b]$ be a real function with $f''(x) > 0$ on the interval $[a,b]$. We then have*

$$\sum_{a<n<b} e\left(f(n)\right) = \sum_{\alpha-\epsilon<r<\beta+\epsilon} \int_a^b e\left(f(x) - rx\right)\ dx + O\left(\epsilon^{-1} + \log(\beta-\alpha+2)\right), \tag{3.4}$$

where α, β, and ϵ are any numbers with $\alpha \le f'(a) \le f'(b) \le \beta$ and $0 < \epsilon \le 1$, the implied constant being absolute.

The truncated Poisson formula is the source of the $O(\log(f'(b) - f'(a) + 2))$ error term in many of the results mentioned above. An explicit version of the truncated Poisson formula for non-trivial g is given in Lemma 7 of [19].

After applying the truncated Poisson formula with $\beta = f'(b)$ and $\alpha = f'(a)$, one is left with approximately $f'(b) - f'(a)$ integrals with which one hopes to apply stationary phase estimates. These estimates work best on a small interval around the stationary phase point, where the second derivative of f is fairly constant, and the higher derivatives of f are small. So we would like to break the integrals in (3.4) into several pieces, such as

$$\int_a^b = \int_a^{a'} + \int_{a'}^{b'} + \int_{b'}^b \tag{3.5}$$

where the point of stationary phase is near the middle of $[a', b']$. However, to make this effective, we would again require good estimates on integrals with no stationary phase point (the integrals from a to a' and b' to b). In addition, one is faced with possible first-order (and higher) endpoint contributions at a' and b'. (While estimates of stationary phase integrals benefit from integrating on a small interval, they suffer again if the interval is too small. The $O(T(a) + T(b))$ terms on line (1.4) that appear in stronger results on the van der Corput transform arise from bounding the stationary phase integrals where x_r is closest to a or b.)

Because of this, many results on the van der Corput transform seek to treat the entire integral from a to b as a single stationary phase integral, hence the common conditions, as in Theorem 1.1, that $f''(x)$ has constant order on the entire interval $[a,b]$ and that higher derivatives of f be likewise small on the entire interval $[a,b]$.

The techniques of this paper seek to overcome some of these difficulties.

First, we use the recent and powerful stationary phase estimates first proved by Huxley [15, 16] and refined by Redouaby and Sargos [34]. These estimates show that stationary phase integrals, such as the middle term in (3.5), also generate first-order endpoint contributions. In fact, these contributions directly cancel the first-order endpoint contributions at a' and b' generated by the first and third term in (3.5). Thus we no longer need to treat the full integral from a to b with stationary phase estimates and so can replace the global restrictions on the derivatives of f and g (such as those in Theorem 1.2) with local restrictions.

In particular, we replace the standard assumptions

$$f''(x) \asymp TM^{-2}, \qquad |f^{(3)}(x)| \ll TM^{-3}, \qquad \text{and} \qquad |f^{(4)}(x)| \ll TM^{-4}$$

for $x \in [a,b]$, with an assumption that looks like

$$f''(z) \asymp T(x)M(x)^{-2}, \qquad |f^{(3)}(z)| \ll T(x)M(x)^{-3}, \qquad \text{and} \qquad |f^{(4)}(z)| \ll T(x)M(x)^{-4}$$

for $z \in [x - M(x), x + M(x)]$ and $x \in [a,b]$, with some function $M(x)$ discussed in more detail in Section 4.1.

Second, to avoid use of the truncated Poisson summation formula, we develop a method to get reasonable bounds on the integrals arising from applying integration by parts twice, as in (3.3). Instead of looking at one integer r at a time and counting the contribution of $f''(x)/(f'(x)-r)^3$ for each x's which give local maxima and minima, we instead look at each x and imagine a possible fixed real-valued r that causes the point x to be a critical point of this function. This then defines a function $r(x)$, and we estimate the sum over all values of x where the function $r(x)$ takes integer values using a variant of Euler-Maclaurin summation (see Proposition 5.8 and Section 7.5).

Third, since the previous two techniques frequently benefit from f'' being large, we develop a method to deal with the large second-order endpoint contributions in this case. The reason for these terms being large is that

$$\frac{f''(a)}{(f'(a)-r)^3}$$

is large when r is close to $f'(a)$ (and likewise at b). However, if $f''(a)$ is large and $f^{(3)}(a)$ is not too large in comparison, then a small shift in a to, say, $a + \epsilon$ should greatly increase the size of the denominator while keeping the numerator roughly the same size. Therefore, by altering the endpoints of the integrals in the Poisson summation formula by a small amount when r is close to $f'(a)$ or $f'(b)$, we can reduce the size of the second-order endpoint contributions.

With these techniques, the main theorems of this paper give the van der Corput transform in the following form.

$$\sum_{a\le n\le b}^{*} g(n)e\left(f(n)\right) = \sum_{f'(a)\le r\le f'(b)}^{*} \frac{g(x_r)e(f(x_r)-rx_r+\frac{1}{8})}{\sqrt{f''(x_r)}} - \mathcal{D}(b)+\mathcal{D}(a)+O(\Delta_1+\Delta_2+\Delta_3+\Delta_4)$$

The $\mathcal{D}$ terms will usually equal the first-order endpoint contributions and will be explicit in many cases. The term Δ_1 will estimate the error that arises when there exists an x_r close to a or b and is analogous to the $T(a)+T(b)$ terms present in the results of Kolesnik, Liu, and Karatsuba and Korolev. The term Δ_2 will estimate the second-order endpoint contributions at a and b. The term Δ_3 (and a bit of Δ_4) will estimate the remaining error in applying stationary-phase methods and the second-order endpoint contributions at a' and b'. The remainder of Δ_4 will estimate the size of the integrals that arise from applying integration by parts twice.

The techniques outlined in this section allow for a number of interesting extensions to the van der Corput transform.

Since we replace the global restrictions on the derivatives of f and g with local restrictions, we can apply the van der Corput transform to many new sums, including those where f and g have moderate oscillations. Likewise, we can now directly apply the van der Corput transform to sums where f'' is large.

The function $f(f'^{-1}(x)) - xf'^{-1}(x)$ that appears on the right-hand side of the van der Corput transform (1.1) is sometimes referred to as the van der Corput reciprocal of the function f. (Redouaby and Sargos [35] and the papers cited within have studied the van der Corput reciprocal in much greater detail, including useful asymptotics for how the reciprocal changes as f is perturbed.) If the second derivative of f is large, then the second derivative of the van der Corput reciprocal of f will be small, so the results cited above, if they cannot be applied directly to a sum, can be—and frequently are—applied “backwards” to the van der Corput transform of the sum.

For example, if we attempted to apply the form of the error from Theorem 1.2 directly to the sum

$$\sum_{0\le n\le N} e(\alpha\cdot\beta^n)$$

for some $\beta>1$, then the error term $O(\log(f'(b)-f'(a)+2))$ would be at least $O(N)$, the size of the trivial bound on the sum (and that doesn’t even take into account the size of the implicit constant!). Theorem 4.1 allows us to apply the transform to this sum directly and obtain a $O(1)$ error.

In general, if $f''\ge 1$ is reasonably large compared to g and both are free of wild oscillations, then Δ should be no larger than $O(\max_{a\le x\le b} g(x))$.

As a final note, the results of this paper do not give an improvement in all possible cases. The strength of Corollary 1.10 comes from looking at the transformation in both directions. In one way, f'' is very small, and in the other, it is very large, and different techniques are employed in each case. But if, for example, f'' varies between a very large value and a value very close to zero, then the results of this paper may give very poor error terms. Another case where the results of this paper do not give any improvement over the other theorems cited is

$$\sum_{a\leq n\leq b}^{*} \sin(\alpha n)e(\beta n^2);$$

here the oscillations in the $\sin(\alpha n)$ are simply too frequent.

We will make the frequent notational convention that a and b denote the endpoints of an interval and in this context, the variable μ will refer to an arbitrary endpoint of such an interval. We shall also generally assume that for the function $f(x)$, the second derivative $f''(x)$ is always positive in every context in which it occurs, unless otherwise noted.

Chapter 4

Details of the main theorems

4.1 The auxiliary function $M(x)$ and condition (M)

We will use a function $M(x)$ to measure the length of an interval around x where the functions $f''(x)$ and $g(x)$ are well-approximated by their Taylor polynomials up to the second degree. The larger $M(x)$ is, the more linear the functions $f''(x)$ and $g(x)$ will appear at the point x.

To be more concrete, by condition (M), we shall refer to the existence of bounded, positive-valued, continuously differentiable functions $M(x)$ and $U(x)$ on $[a, b]$, along with several associated positive constants C_2, C_{2-}, C_3, C_4, D_0, D_1, D_2, δ, which together satisfy several conditions:

(I) we have $\delta < 1$, and for

$$\eta := \frac{3\delta}{2C_{2-}^2 C_2 C_3},$$

we also have $\eta \geq 1$;

(II) If $c(a) = 1$, where

$$c(x) := \begin{cases} 0, & \text{if } \mathbb{Z} \cap (f'(x) - f''(x), f'(x) + f''(x)) \setminus \{f'(x)\} = \emptyset, \text{ and} \\ 1, & \text{otherwise,} \end{cases}$$

then $M(a)$ must be at most $b - a$, and if $c(b) = 1$, then $M(b) \leq b - a$ as well;

(III) If $J := [a - c(a) \cdot M(a), b + c(b) \cdot M(b)]$, then $f \in C^4[J]$ and $g \in C^3[J]$; and,

(IV) If I_x denotes the intersection $[x - M(x), x + M(x)] \cap J$, then all the following inequalities hold for every $x \in a, b]$ and $z \in I_x$:

$$\frac{1}{C_{2-}} f''(x) \leq f''(z) \leq C_2 f''(x), \qquad |g(z)| \leq D_0 U(x),$$

$$|f^{(3)}(z)| \leq C_3 \frac{f''(x)}{M(x)}, \qquad |g'(z)| \leq D_1 \frac{U(x)}{M(x)},$$

$$|f^{(4)}(z)| \le C_4 \frac{f''(x)}{M(x)^2}, \qquad \text{and} \qquad |g''(z)| \le D_2 \frac{U(x)}{M(x)^2}.$$

For any given $f(x)$, $g(x)$, a, and b, there are infinitely many possible choices of the functions $M(x)$, $U(x)$, and the associated constants that satisfy the above conditions. Therefore, for the remainder of the paper, we shall assume that if $f(x)$, $g(x)$, a, and b remain unchanged, the particular functions $M(x)$ and $U(x)$ and associated constants we reference will be likewise unchanged. We shall also assume that if $f(x)$ and $g(x)$ are unchanged, but a and b allowed to vary, then, first, the associated constants will be unchanged, and, second, given two *sufficiently large* intervals $[a_1, b_1]$ and $[a_2, b_2]$ with corresponding auxiliary functions $M_1(x)$, $U_1(x)$ and $M_2(x)$, $U_2(x)$, we have $M_1(x) = M_2(x)$ and $U_1(x) = U_2(x)$ on $[a_1, b_1] \cap [a_2, b_2]$. The sufficiently large condition is needed since for small intervals, the controlling factor in the size of $M(a)$ and $M(b)$ might not be the properties of f or g but instead the length of the interval $[a, b]$ due to part (II).

Since condition (M) is rather intricate, we pause a moment to illuminate it further.

(I) The requirement of $\delta < 1$ in condition (M) part (I) guarantees that the Taylor approximation to f'' has good properties (see Proposition 5.5, Lemma 6.5, and the remark following the lemma). Note that the condition on η implies that C_3 must be less than 2.

The restriction on η is somewhat malleable. If $\eta < 1$, then we can weaken the above restrictions by replacing $M(x)$, C_3, C_4, D_1, D_2, and η, by $\eta M(x)$, ηC_3, $\eta^2 C_4$, ηD_1, $\eta^2 D_2$, and 1, respectively. The term ηC_3 must again be less than 2 (in fact, it will be less than 3/2).

(II) In Section 3, we described a process of sometimes shifting the endpoint μ of an integral by a certain amount. The function $c(\mu)$ is an indicator function of whether we will shift endpoints or not, and $M(\mu)$ is the distince the endpoint will be shifted. The restriction $M(\mu) \le b - a$ guarantees that there is enough room in our interval to do the shift we want. Often, this will not be a difficult restriction unless $b - a$ is very small.

(III) The restriction of condition (M) part (III) to have f and g be several times continuously differentiable on the larger interval J is often no restriction at all. Many applications of the van der Corput transform have $f(x)$ and $g(x)$ be polynomials, exponentials, logarithms, or other C^∞ functions that exist on large domains.

The bounds on the derivatives of f'' and g in condition (M) part (IV) allow us to use estimates on stationary phase integrals that will be discussed in Section 5. By expecting the properties of f'' and g to extend to the larger interval J, we can also extend the integrals under consideration to this larger interval.

(While typical stationary phase results require only that g is twice continuously differentiable, we require three times in order for the functions $W'_\pm$ and $r'_\pm$ to exist.)

(IV) To understand the complex system of inequalities in condition (M) part (IV), it is helpful to rewrite them in terms of Taylor approximations. The first three inequalities then become

$$\begin{aligned}
&f''(z) \asymp f''(x),\\
&f''(z) = f''(x)\left(1 + O\left(\frac{C_3}{M(x)}(z-x)\right)\right),\\
&f''(z) = f''(x)\left(1 + O\left(\frac{C_4}{M(x)^2}(z-x)^2\right)\right) + f^{(3)}(x)(z-x) \qquad (4.1)
\end{aligned}$$

for $z \in I_x$, with implicit constant 1 in the second and third lines. The equality on line (4.1) gives definite form to our earlier statement that the larger $M(x)$ can be, the more linear f'' appears at the point x.

Alternately, one can interpret $M(x)$ as somehow representing the rate of decay as one takes successive derivatives. At the point $z = x$, the various inequalities imply—among other things—that $|f^{(3)}(x)| \ll f''(x)/M(x)$ and $|f^{(4)}(x)| \ll f''(x)/M(x)^2$.

Here are some examples of the function $M(x)$ in different circumstances. In each case we assume that the associated constants always take the same values. We will also assume that a and b are both positive and that if $U(x)$ is taken to equal $g(x)$, then $g(x)$ is bounded away from 0.

- If $f(x)$ and $g(x)$ are polynomials, then there exists an $\epsilon > 0$ such that condition (M) is satisfied with $M(x) = \epsilon \cdot x$ and $U(x) = g(x)$.

- If $f(x)$ and $g(x)$ are power functions (a constant times x^α for some $\alpha \in \mathbb{R}$), then there exists an $\epsilon > 0$ such that condition (M) is satisfied with $M(x) = \epsilon \cdot x$ and $U(x) = g(x)$.

- If $f(x)$ and $g(x)$ are exponential functions ($\alpha \cdot \beta^x$, with $\beta > 1$), then there exists an $\epsilon > 0$ such that condition (M) is satisfied with $M(x) = \epsilon$ and $U(x) = g(x)$. (This holds regardless of whether a and b are positive or not.)

- If $f(x) = t(\log x)/2\pi$ and $g(x) = x^{-\sigma}$, which corresponds to the Riemann Zeta function at $z = \sigma + it$, then there exists an $\epsilon > 0$ independent of t such that condition (M) is satisfied with $M(x) = \epsilon x$ and $U(x) = g(x)$.

- If $f(x) = \alpha x^2 + \beta x^{-1}\sin(\gamma x)$ and $g(x) = 1$, then there exists an $\epsilon > 0$ such that condition (M) is satisfied with $M(x) = \epsilon\sqrt{x}$ and $U(x) = 1$.

- If $f(x)$ is a power function and $g(x) = \sin(\alpha x)$ for some constant α, then there exists an $\epsilon > 0$ such that condition (M) is satisfied with $M(x) = \epsilon$ and $U(x) = 1$.

4.2 The assumptions

Assume that condition (M) holds for some function $M(x)$ and $U(x)$. Let J be as in condition (M) part (III). We assume that $f''(x)$ is a positive-valued function, bounded away from 0 on the interval J, and that $g(x)$ is a real-valued function on J as well. (The case $f''(x) < 0$ may be considered by taking the conjugate of the sum.) This will guarantee that $f'(x)$ is a continuous, monotonic function, so that $x_r := f'^{-1}(r)$ is well defined.

Recall the definitions

$$H(x) = g(x)f^{(3)}(x) + 3g'(x)f''(x) \quad \text{and} \quad G(x) = 12g(x)g''(x)(f''(x))^2.$$

Let $J_\pm$ be the union of all intervals $[a', b'] \subset J$ such that $G(x) \neq 0$ and $H(x)^2 - G(x) \geq 0$ for $x \in [a', b']$, let J_0 be the union of all intervals $[a', b'] \subset J$ such that $g''(x) = 0$, $g(x) \neq 0$, and $H(x) \neq 0$ for $x \in [a', b']$, and let J_{null} be the set of points $x \in J$ such that $g(x) = 0$ but $g'(x) \neq 0$ and $g''(x) \neq 0$. It is possible that $J_\pm$ will contain isolated points due to $H(x)^2 - G(x)$ having a zero on an interval where it is otherwise negative, and it is also possible that J_0 will contain isoluted points due to $g''(x)$ having a zero on an interval where it is otherwise non-zero; we denote the set of isolated points in $J_\pm$ and J_0 by $J_\pm^*$ and J_0^* respectively. Let $\partial J_\pm$ and ∂J_0 denote the endpoints of the non-zero-length intervals contained in their respective sets. In particular, $\partial J_\pm \cap J_\pm^* = \varnothing$ and $\partial J_0 \cap J_0^* = \varnothing$.

For our final assumption, suppose that if (a', b') is an interval contained in $J_\pm$, then either $H(x)^2 - G(x)$ equals 0 on the whole interval and $H(x)$ does not equal 0 at any point on the interval nor does it tend to 0 at the endpoints, or $H(x)^2 - G(x)$ does not equal 0 at any point on the interval nor does it tend to 0 at the endpoints; and if (a', b') is an interval contained in J_0, then $g(x)$ does not tend to 0 at the endpoints of this interval.

4.3 Statement of the main theorem and its variants

Theorem 4.1. *Assume all the conditions of Section 4.2 hold. Then*

$$\sum_{a \leq n \leq b}^{*} g(n)e(f(n)) = \sum_{f'(a) \leq r \leq f'(b)}^{*} \frac{g(x_r)e(f(x_r) - rx_r + \frac{1}{8})}{\sqrt{f''(x_r)}} - \mathcal{D}(b) + \mathcal{D}(a) + \Delta,$$

where

$$\mathcal{D}(x) = \mathcal{D}_1(x) + \mathcal{D}_2(x),$$

$$\mathcal{D}_1(x) = \begin{cases} g(x)e(f(x) + [\![f'(x)]\!]\, x)\times \\ \quad \times\left(-\dfrac{1}{2\pi i\langle f'(x)\rangle} + \psi(x, \langle f'(x)\rangle)\right), & \textit{if } f''(x) < \|f'(x)\|, \\ g(x)e(f(x) + [\![f'(x)]\!]\, x)\psi(x, \langle f'(x)\rangle), & \textit{if } \|f'(x)\| \le f''(x) < 1 - \|f'(x)\|, \\ O(\theta_1 U(x)), & \textit{if } 1 - \|f'(x)\| \le f''(x) < 1, \\ O\left(U(x)\left(\theta_2 + \dfrac{\theta_3}{M(x)} + \dfrac{\theta_4}{\sqrt{f''(x)}M(x)}\right)\right), & \textit{if } f''(x) \ge 1, \end{cases}$$

$$\mathcal{D}_2(x) = \begin{cases} \dfrac{g(x)f^{(3)}(x)e(f(x))}{6\pi i f''(x)^2} - \dfrac{g'(x)e(f(x))}{2\pi i f''(x)}, & \textit{if } \|f'(x)\| = 0, \\ 0, & \textit{if } \|f'(x)\| \ne 0, \end{cases}$$

and

$$\Delta = \sum_{i=1}^{3} O(\Delta_i(a) + \Delta_i(b)) + O(\Delta_4) + O(\Delta_5),$$

$$\Delta_1(x) = \begin{cases} \min\left\{\theta_5 \dfrac{U(x)}{\sqrt{f''(x)}}, \theta_6 \dfrac{U(x)}{\|f'(x)\|}\right\}, & \textit{if } \|f'(x)\| \ne 0, \quad m_x \ge 1, \\ \theta_7 \dfrac{U(x)}{f''(x)^2(b-a)^3}, & \textit{if } \|f'(x)\| = 0, \\ 0, & \textit{if } \|f'(x)\| \ne 0, \quad m_x = 0, \end{cases}$$

$$\Delta_2(x) = \frac{U(x)}{f''(x)^2 M(x)^3}(\theta_8 + \theta_9\sqrt{f''(x)}M(x)) + \frac{U(x)m_x}{f''(x)^2 M(x)^3}\left(\theta_{10} + \theta_{11} f''(x)M(x)^2\right)$$
$$+ \begin{cases} \theta_{12}\dfrac{U(x)}{M(x)}\min\left\{2, \dfrac{1}{f''(x)}\right\} \\ \quad + \theta_{13}U(x)\min\left\{4f''(x), \dfrac{1}{f''(x)}\right\}, & \textit{if } \|f'(x)\| = 0 \textit{ or } m_x \ge 1, \\ \dfrac{\theta_{12}}{2}\dfrac{U(x)}{M(x)\|f'(x)\|^2} + \dfrac{\theta_{13}}{2}\dfrac{U(x)f''(x)}{\|f'(x)\|^3}, & \textit{otherwise}, \end{cases}$$

$$\Delta_3(a) = \int_{\overline{a}}^{b} \frac{U(x)}{f''(x)(x-a)^3}\left(\theta_{14} + \frac{\theta_{15}}{f''(x)M(x)} + \frac{\theta_{16}}{f''(x)(x-a)}\right)\, dx$$
$$+ \theta_{17}\left(\frac{U(\overline{a})}{f''(\overline{a})^2(\overline{a}-a)^3} + \frac{U(b)}{f''(b)^2(b-a)^3}\right),$$

$$\Delta_3(b) = \int_{a}^{\overline{b}} \frac{U(x)}{f''(x)(b-x)^3}\left(\theta_{14} + \frac{\theta_{15}}{f''(x)M(x)} + \frac{\theta_{16}}{f''(x)(b-x)}\right)\, dx$$
$$+ \theta_{17}\left(\frac{U(\overline{b})}{f''(\overline{b})^2(b-\overline{b})^3} + \frac{U(a)}{f''(a)^2(b-a)^3}\right),$$

$$\Delta_4 = \int_a^b \frac{U(x)}{f''(x)M(x)^3}\left(\theta_{18} + \theta_{19}\sqrt{f''(x)}M(x)\right)\left(1 + \frac{|U'(x)|}{U(x)\cdot f''(x)} + \frac{\theta_{20} + 4|M'(x)|}{f''(x)M(x)}\right)\, dx,$$

$$\Delta_5 = \mathcal{K}(J_0, W_0, r_0) + \mathcal{K}(J_\pm, W_+, r_+) + \mathcal{K}(J_\pm, W_-, r_-) + \sum_{x \in J_{null}} \frac{g''(x)^2}{36\pi^2 g'(x) f''(x)^2},$$

and

$$\mathcal{K}(I, W, r) = \int_I (|W(x)||r'(x)| + |W'(x)|)\ dx + \sum_{x \in I^*} |W(x)| \\ + 2 \sum_{\substack{x \in I \text{ and } r' \text{ changes sign at } x, \\ \text{or } x \in \partial I}} |\psi(r(x)) \cdot W(x)|\,.$$

The number m_x equals the cardinality of the set

$$\mathbb{Z} \cap (f'(x) - f''(x), f'(x) + f''(x)) \setminus \{f'(x)\}.$$

In particular,

$$m_x = \begin{cases} 0 & f''(x) \le \|f'(x)\| \\ O(1 + f''(x)) & f''(x) > \|f'(x)\| \end{cases}.$$

Also, $\overline{a}$ equals the smallest value in the interval $[a + \min\{M(a), C_2^{-1}\}, b]$ such that $f'(\overline{a})$ is an integer; if no such value exists, then we may take $\Delta_3(a) = 0$. Similarly, $\overline{b}$ equals the largest value in the interval $[a, b - \min\{M(b), C_2^{-1}\}]$ such that $f'(\overline{b})$ is an integer, and if no such value exists, then we may take $\Delta_3(b) = 0$.

The functions $M(x)$ and $U(x)$ are as in condition (M), the sets $J_\pm$, $J_\pm^*$, J_0, and J_0^* are all as in Section 4.2, and the functions ψ, s, W_0, r_0, $W_\pm$, $r_\pm$, $\langle\cdot\rangle$, $[\![\cdot]\!]$, and $\|\cdot\|$ are all as in Section 2.

The convergence of the integrals in the $\mathcal{K}$ terms will be guaranteed by Proposition 5.11

The implicit constants in the big-O terms are all 1, with the various θ terms dependent upon the constants C_2, C_{2^-}, C_4, D_0, D_1, D_2, δ in condition (M), as follows:

$$\theta_1 = 21,$$

$$\theta_2 = \frac{8C_{2^-}^{1/2}(D_0 + 2D_1)}{\sqrt{\pi}} + \frac{21D_0}{2\pi} + \frac{3(D_0 + 2D_0C_2 + 3D_1 + C_3)}{4\pi^2} + 21,$$

$$\theta_3 = \frac{2D_0}{\pi(2 - C_3)}, \qquad \theta_4 = \frac{2(D_0 + 2D_1)}{\pi^2}, \qquad \theta_5 = \frac{8C_{2^-}^{1/2}(D_0 + 2D_1)}{\sqrt{\pi}},$$

$$\theta_6 = \frac{21D_0}{2\pi}, \qquad \theta_7 = \frac{D_0}{2\pi^2(1-\delta)^3},$$

$$\theta_8 = \frac{D_0}{\pi^2(1-\delta)^3} + \frac{1}{4\pi^2}\left(D_1\left(\frac{2}{2 - C_3}\right)^2 + D_0\left(\frac{2}{2 - C_3}\right)^3\right),$$

$$\theta_9 = \frac{\sqrt{66}\cdot D_0\left(\frac{4}{3}C_{2-}^4C_2^2C_3^2 + (1+\delta)C_{2-}^3\left(\frac{1}{6}C_2C_3^2+\frac{1}{4}C_2^2C_4\right)\right)}{\pi(1-\delta)^5}$$
$$+ \frac{C_{2-}(C_2D_2+C_3D_1)}{2\pi}\left(1+\frac{2C_{2-}}{\pi}\right) + \frac{2(C_3C_{2-}^3D_0+3C_2C_{2-}^4C_3D_1+C_{2-}C_2^2D_2+C_{2-}^2D_2)}{\pi^2},$$
$$\theta_{10} = \frac{1}{2\pi^2}\left(D_1\left(\frac{2}{2-C_3}\right)^2 + D_0\left(\frac{2}{2-C_3}\right)^3 + \frac{C_{2-}^2D_0}{(1-\delta)^3}\right),$$
$$\theta_{11} = \frac{D_0}{\pi(2-C_3)}, \qquad \theta_{12} = \frac{3D_1}{\pi^2}, \qquad \theta_{13} = \frac{5D_0}{2\pi^2}, \qquad \theta_{14} = \frac{D_0}{2\pi(1-\delta)^3},$$
$$\theta_{15} = \frac{D_1+2C_3D_0}{2\pi^2(1-\delta)^3}, \qquad \theta_{16} = \frac{3D_0}{2\pi^2(1-\delta)^3}, \qquad \theta_{17} = \frac{D_0}{4\pi^2(1-\delta)^3},$$
$$\theta_{18} = \frac{D_0}{\pi^2(1-\delta)^3} + \frac{1}{2\pi^2}\left(D_1\left(\frac{2}{2-C_3}\right)^2 + D_0\left(\frac{2}{2-C_3}\right)^3\right),$$
$$\theta_{19} = \frac{\sqrt{66}\cdot D_0\left(\frac{4}{3}C_{2-}^4C_2^2C_3^2 + (1+\delta)C_{2-}^3\left(\frac{1}{6}C_2C_3^2+\frac{1}{4}C_2^2C_4\right)\right)}{\pi(1-\delta)^5}$$
$$+ \frac{C_{2-}(C_2D_2+C_3D_1)}{2\pi}\left(1+\frac{2C_{2-}}{\pi}\right) + \frac{2(C_3C_{2-}^3D_0+3C_2C_{2-}^4C_3D_1+C_{2-}C_2^2D_2+C_{2-}^2D_2)}{\pi^2},$$
$$\theta_{20} = 2C_3.$$

Before continuing to further refinements of the main theorem, let us pause to assure the reader that the vast collection of error terms above are more psychologically daunting than they are computationally difficult. In particular, provided $f''(x) \gg 1$ and $M(x) \gg 1$ on $[a,b]$, the error terms $\mathcal{D}(x)$ and $\Delta_i(x)$ for $i = 1,2,3$ are all bounded by $O(\max_{[a,b]} U(x))$, where the implicit constant is dependent on the implicit constants in the lower bounds of f'' and M. We shall demonstrate this in more detail in the proof of Corollary 1.4.

There are multiple ways of getting nice bounds on the terms in $\Delta_3(x)$. If f'' and M are generally quite large, then we can often assume no more that $\overline{a}-a, b-\overline{b} \gg 1$ to obtain good bounds (see the proof of Corollary 1.4). Alternately, if f'' is small, then we can show that most of the time, $\overline{a}-a$ and $b-\overline{b}$ are very large (see the proof of Corollary 1.5).

Also, we expect that if $g(x)$ and $f''(x)$ share the same rate of decay in their derivatives, then the various functions H, G, and the W and r functions should all be relatively well-behaved. If g and f'' are both power functions, taking derivatives of each entails multiplying by a constant and dividing by x. Thus, we have that $g(x)f^{(3)}(x)$ is a constant multiple of $g'(x)f''(x)$, hence why H is a power function as well.

Under fairly strong and yet quite common conditions, we can remove much of the complication from the assumptions of Section 4.2 and from Δ_4.

Theorem 4.2. *Assume that f and g be real-valued functions satisfying $f''(x) > 0$ on $[a,b]$ and that condition (M) holds for a function $M(x)$ such that $m_\mu = 0$, where m_μ is defined as in Theorem 4.1, and such that*

$M(\mu) \geq b - a$, for μ equal to both a and b.

Under these conditions, the result of Theorem 4.1 holds, with Δ now just equal to

$$\Delta = O\left(\Delta_1(a) + \Delta_1(b) + \Delta_2'(a) + \Delta_2'(b) + \Delta_3(a) + \Delta_3(b) + \Delta_4\right)$$

and with no Δ_5 term. Here we have

$$\Delta_2'(x) = \frac{U(x)}{f''(x)^2 M(x)^3}(\theta_8' + \theta_9'\sqrt{f''(x)}M(x)) + \theta_{12}'\frac{U(x)}{M(x)\|f'(x)\|^{*2}} + \theta_{13}'\frac{U(x)f''(x)}{\|f'(x)\|^{*3}},$$

with

$$\begin{aligned}
\theta_8' &= \frac{D_0}{\pi^2(1-\delta)^3}, \\
\theta_9' &= \frac{\sqrt{66}\cdot D_0\left(\frac{4}{3}C_{2-}^4 C_2^2 C_3^2 + (1+\delta)C_{2-}^3\left(\frac{1}{6}C_2 C_3^2 + \frac{1}{4}C_2^2 C_4\right)\right)}{\pi(1-\delta)^5} \\
\theta_{12}' &= \frac{2D_1 + D_2}{\pi^2}, \\
\theta_{13}' &= \frac{3(D_0 + 2D_0 C_2 + 3D_1 + C_3)}{4\pi^2}.
\end{aligned}$$

Moreover, we may replace θ_{18} in the definition of Δ_4 with

$$\theta_{18}' = \frac{D_0}{\pi^2(1-\delta)^3}.$$

We may relax the condition that $\eta > 1$ in condition (M), but, if we do, then we must replace all instances of $M(x)$, C_3, C_4, D_1, and D_2 with $\eta M(x)$, ηC_3, $\eta^2 C_4$, ηD_1, and $\eta^2 D_2$, respectively, in all places except *θ_{12}' and θ_{13}'.*

Theorem 4.3. *Assume that the conditions of Theorem 4.1 hold, and that, in addition, there exists an integer r' such that $0 < |r' - f(b)| < 1/2$ and $0 < |x_{r'} - b| \leq M(x_{r'})$. If $x_{r'} > b$, then we assume that condition (M) holds on the interval $[a, r']$ as well. (We could also assume that the conditions of Theorem 4.2 hold and permit that $m_b = 1$ with one term that corresponds precisely to r'.)*

Then the conclusion of Theorem 4.1 (respectively, Theorem 4.2) holds, but with $\mathcal{D}(b)$ replaced by

$$\mathcal{D}'(b) = \mathcal{D}_1'(b) + \mathcal{D}_3'(b) + \mathcal{D}_4'(b),$$

where

$$\mathcal{D}_1'(b) = \begin{cases} g(b)e(f(b) + [\![f'(b)]\!]\, b)\psi(b, \langle f'(b)\rangle), & \text{if } f''(b) < 1 - \|f'(b)\|, \\ O(\theta_1 U(b)) & \text{if } 1 - \|f'(b)\| \le f''(b) < 1, \\ O\left(U(b)\left(\theta_2 + \dfrac{\theta_3}{M(b)} + \dfrac{\theta_4}{\sqrt{f''(b)}M(b)}\right)\right), & \text{if } f''(b) \ge 1, \end{cases}$$

$$\mathcal{D}_3'(b) = \operatorname{sgn}(b - x_{r'})g(x_{r'})e(f(x_{r'}) - r'x_{r'}) \int_{\phi_{r'}(b - x_{r'})}^{+\infty} e\left(\frac{1}{2}f''(x_{r'})y^2\right)\, dy$$
$$+ \frac{g(x_{r'})f^{(3)}(x_{r'})e(f(b) - r'b)}{6\pi i f''(x_{r'})^2} - \frac{g'(x_{r'})e(f(b) - r'b)}{2\pi i f''(x_{r'})} + O\left(\theta_{21}\frac{U(x_{r'})}{f''(x_{r'})M(x_{r'})^2} \cdot |b - x_{r'}|\right),$$

$$\mathcal{D}_4'(b) = O\left(\theta_{22}\frac{U(x_{r'})}{f''(x_{r'})^2}\left(\frac{1}{M(x_{r'})^3} + \frac{1}{(x_{r'} - a)^3}\right)\right) + O\left(\frac{\theta_{23}U(x_{r'})}{f''(x_{r'})^{3/2}M(x_{r'})^2}\right),$$

where

$$\phi_{r'}(x) = \left(\frac{f(x_{r'} + x) - f(x_{r'}) - r'x}{\frac{1}{2}f''(x_{r'})}\right)^{1/2}.$$

Moreover, each occurence of $\|f'(x)\|$ *in* $\Delta_1(x)$ *and* $\Delta_2(x)$ *can be replaced with* $1 - \|f'(x)\|$*. If* $m_x = 1$ *and* $|x_{r'} - b| < f''(b)$*, then we can instead let* $\Delta_1(x) = 0$*. In Theorem 4.2, we replace* $\|f'(x)\|^*$ *with*

$$\begin{cases} 1, & \text{if } \|f'(x)\| = 0, \\ 1 - \|f'(x)\|, & \text{if } \|f'(x)\| \ne 0. \end{cases}$$

If $\overline{b} = x_{r'}$*, then we may replace* $\overline{b}$ *by the largest number in the interval* $[a, x_{r'})$ *such that* $f'(\overline{b})$ *is an integer.*

The new implicit constants are given as follows:

$$\theta_{21} = \frac{(9C_3^2 + 2C_2C_4)C_{2-}^3 D_0}{24\pi} + \frac{C_{2-}D_0}{12\pi(1-\delta)^3}\left(2C_{2-}^4 C_2^2 C_3^2 + \left(1 + \frac{\delta}{2}\right)C_{2-}^3\left(\frac{C_2C_3^2}{6} + \frac{C_2^2C_4}{4}\right)\right)$$

$$\theta_{22} = \frac{D_0}{2\pi^2(1-\delta)^3},$$

$$\theta_{23} = \frac{\sqrt{66}\cdot D_0\left(\frac{4}{3}C_{2-}^4 C_2^2 C_3^2 + (1+\delta)C_{2-}^3\left(\frac{1}{6}C_2C_3^2 + \frac{1}{4}C_2^2C_4\right)\right)}{\cdot\pi(1-\delta)^5}$$
$$+ \frac{C_{2-}(C_2D_2 + C_3D_1)}{2\pi}\left(1 + \frac{2C_{2-}}{\pi}\right) + \frac{2\left(C_3C_{2-}^3 D_0 + 3C_2C_{2-}^4 C_3 D_1 + C_{2-}C_2^2 D_2 + C_{2-}^2 D_2\right)}{\pi^2}$$

Note that since $x_{r'} \ne b$, we have that $\mathcal{D}_2(b) = 0$ automatically.

We conclude with two additional theorems which extend the results of Theorem 4.1 in specific directions. The following theorem shows that if the various error terms from Theorem 4.1 are nicely bounded as b tends to infinity, then many of them can be replaced by a constant plus a $o(1)$ error.

Theorem 4.4. *Assume all the conditions of Section 4.2 hold with a fixed and b tending to infinity, and assume that the interval $[a,b]$ is large enough so that the function $M(x)$ does not change as b increases. In particular, this will require that $b-a \geq M(a)$.*

Let K_b be the set of all $x \in [a,b]$ such that $x+M(x) > b$, where $M(x)$ is as in condition (M). Let $K_b' := K_b \cap [a, \bar{b}]$, where $\bar{b}$ equals the largest value in the interval $[a, b-\min\{M(b), C_2^{-1}\}]$ such that $f'(\bar{b})$ is an integer, and if no such value exists, then $K_b' = \emptyset$. As in Section 4.2, let ∂S for an arbitrary set S denote the endpoints of all non-zero-length intervals contained in S and let S^* denote the isolated points of S. Then let

$$\begin{aligned}\Delta_9 = &\int_{K_b'} \frac{U(x)}{f''(x)(b-x)^3}\left(\theta_{14} + \frac{\theta_{15}}{f''(x)M(x)} + \frac{\theta_{16}}{M(x)(b-x)}\right)\, dx \\ &+ \sum_{x \in \partial K_b' \cup K_b'^*} \frac{D_0}{2\pi^2(1-\delta)^3} \cdot \frac{U(x)}{f''(x)^2(b-x)^3} \\ &+ 2\int_{K_b} \frac{U(x)}{f''(x)M(x)^3}\left(2\theta_{17} + \theta_{19}\sqrt{f''(x)}M(x)\right)\left(1 + \frac{|U'(x)|}{U(x)\cdot f''(x)} + \frac{\theta_{20} + 4|M'(x)|}{f''(x)M(x)}\right)\, dx \\ &+ 2\sum_{x \in \partial K_b \cup K_b^*} \frac{U(x)}{f''(x)^2 M(x)^3}\left(2\theta_{17} + \theta_{19}\sqrt{f''(x)}M(x)\right),\end{aligned}$$

and assume that Δ_9 tends to 0 as b tends to ∞.

Suppose we have

$$\Delta_6(b) = \theta_{17}\frac{U(b)}{f''(b)^2(b-a)^3} + \frac{U(b)}{f''(b)^2 M(b)^3}\left(2\theta_{17} + \theta_{19}\sqrt{f''(b)}M(b)\right)$$

and that $\Delta_6(b)$ tends to 0 for some increasing sequence $\{b^{(i)}\}$ tending to ∞. Suppose in addition, we have a (possibly different) increasing sequence $\{b^{(i)}\}$ tending to infinity on which $\Delta_2(b^{(i)})$ tends to 0 and for which $b^{(i+1)} - b^{(i)} \leq \max\{M(b^{(i)}), M(b^{(i+1)})\}$.

In addition, assume that the sums and integrals in $\Delta_3(a)$, Δ_4, and Δ_5 terms all converge as b tends to infinity in Theorem 4.1.

Then, we have that

$$\sum_{a \leq n \leq b}^{*} g(n)e(f(n)) = \sum_{f'(a) \leq r \leq f'(b)}^{*} \frac{g(x_r)e(f(x_r) - rx_r + \frac{1}{8})}{\sqrt{f''(x_r)}} - \mathcal{D}(b) + c + \Delta$$

as a remains fixed and b tends to infinity, where c is some constant and

$$\Delta = O\left(\Delta_1(b) + \Delta_2(b) + \Delta_5([b,\infty)) + \Delta_6(b) + \Delta_7([b,\infty)) + \Delta_8([b,\infty)) + \Delta_9\right),$$

where $\Delta_5([b,\infty))$ is defined as Δ_5 was in Theorem 4.1 but as though the interval in question were $[b,\infty)$ not $[a,b]$. Also, we let

$$\Delta_7([b,\infty)) = \int_b^\infty \frac{U(x)}{f''(x)(x-a)^3}\left(\theta_{14} + \frac{\theta_{15}}{f''(x)M(x)} + \frac{\theta_{16}}{f''(x)(x-a)}\right) dx,$$
$$\Delta_8([b,\infty)) = 2\int_b^\infty \frac{U(x)}{f''(x)M(x)^3}\left(2\theta_{17} + \theta_{19}\sqrt{f''(x)}M(x)\right)\left(1 + \frac{|U'(x)|}{U(x)\cdot f''(x)} + \frac{\theta_{20} + 4|M'(x)|}{f''(x)M(x)}\right) dx,$$

and all other functions are as in Theorem 4.1.

An analogous result holds if b is fixed and a tends to $-\infty$. This can also be combined with the result of Theorem 4.3.

Much of the accuracy gleaned from Theorems 4.1 and 4.2 comes when $f''(x)$ is small. If $f''(x)$ is large, instead, then $\mathcal{D}(x)$ shifts from being an explicit term to a big-O term. However, with additional work it can be made somewhat explicit.

Theorem 4.5. *Assume the conditions of Section 4.2 hold. Suppose that $M(a)$, $f''(a) \geq 1$, and let C and L be real numbers satisfying*

$$f''(a)^{-1/2} \ll C < M(a) \qquad \text{and} \qquad \sqrt{f''(a)} \ll L < \min\{f''(a), C\cdot f''(a), f''(a) - \|a\|\}.$$

Also let $\epsilon = \langle a\rangle$ and $\epsilon' = \langle f'(a)\rangle$.

Then we may replace

$$\mathcal{D}_1(a) = O\left(U(a)\left(1 + \frac{1}{M(a)} + \frac{1}{\sqrt{f''(a)}M(a)}\right)\right)$$

in the statement of Theorem 4.1 or 4.2 with

$$\mathcal{D}_1(a) = \mathcal{D}_5(a) + O\left(\frac{U(a)f''(a)C^4L}{M(a)} + \frac{U(a)L}{f''(a)C} + \frac{U(a)f''(a)}{L^2} + \frac{U(a)}{f''(a)C^2} + \frac{U(a)}{M(a)}\right), \tag{4.2}$$

where $\mathcal{D}_5(a)$ can be estimated in multiple different ways.

- *If $\epsilon = 0$, then*

$$\mathcal{D}_5(a) = O\left(\frac{U(a)|\epsilon'|L}{f''(a)} + \frac{U(a)|\epsilon'|(1+|\epsilon'|C)\log(1+f''(a))}{\sqrt{f''(a)}} + U(a)f''(a)|\epsilon'|^3C^4\right).$$

- *If $\epsilon' = 0$, then*

$$\mathcal{D}_5(a) = \psi(a)g(a)e(f(a) - f'(a)a).$$

- *If* $\epsilon' = 0$ *and* $|\epsilon| > C$, *then*
$$\mathcal{D}_5(a) = O\left(\frac{U(a)}{(|\epsilon| - C)\sqrt{f''(a)}}\right).$$

Suppose that $\epsilon' = 0$ *and* $M(a) \le f''(a)^7$. *We can then optimize the above results, so that* $\mathcal{D}_1(a)$ *equals*

$$\begin{cases} \psi(a)g(a)e(f(a) - f'(a)a) + O(U(a)f''(a)^{1/3}|\epsilon|^{2/3}) & \\ \qquad + O\left(\frac{U(a)}{M(a)^{2/15}f''(a)^{1/15}} + \frac{U(a)}{M(a)}\right), & \text{if } |\epsilon| \le f''(a)^{-1/2}, \\ \emptyset\left(\frac{U(a)}{f''(a)^{1/3}|\epsilon|^{2/3}}\right) + O\left(\frac{U(a)}{M(a)^{2/15}f''(a)^{1/15}} + \frac{U(a)}{M(a)}\right), & \text{if } |\epsilon| \ge f''(a)^{-1/2}. \end{cases}$$

In all cases, the implicit constants depend on the constants associated to condition (M) *and on the implicit constants in the bounds on* C *and* L. *This theorem holds true if* a *is everywhere replaced by* b.

Remark 4.1. The results of Theorems 4.4 and 4.5 can be combined to provide a more explicit $\mathcal{D}(b)$ term as b tends to infinity.

Chapter 5

Necessary lemmas and useful propositions

The first and second derivative tests below are well-known and often accompany any discussion of the van der Corput transform.

Lemma 5.1 (First derivative test—Lemma 5.1.2 in [16]). *Let $f(x)$ be real-valued and differentiable on the open interval (α, β) with $f'(x)$ monotone and $f'(x) \geq \kappa > 0$ on (α, β). Let $g(x)$ be real, and let V be the total variation of $g(x)$ on the closed interval $[\alpha, \beta]$ plus the maximum modulus of $g(x)$ on $[\alpha, \beta]$. Then*

$$\left| \int_{\alpha}^{\beta} g(x) e(f(x)) \, dx \right| \leq \frac{V}{\pi \kappa}$$

Lemma 5.2 (Second derivative test—Lemma 5.1.3 in [16]). *Let $f(x)$ be real-valued and twice differentiable on the open interval (α, β) with $f''(x) \geq \lambda > 0$ on (α, β). Let $g(x)$ be real, and let V be the total variation of $g(x)$ on the closed interval $[\alpha, \beta]$ plus the maximum modulus of $g(x)$ on $[\alpha, \beta]$. Then*

$$\left| \int_{\alpha}^{\beta} g(x) e(f(x)) \, dx \right| \leq \frac{4V}{\sqrt{\pi \lambda}}.$$

Proposition 5.3 (Condition (M) version of first and second derivative tests). *Suppose condition (M) holds. If for some $x \in J$, we have $[\alpha, \beta] \subset I_x := J \cap [x - M(x), x + M(x)]$, then*

$$\left| \int_{\alpha}^{\beta} g(y) e(f(y)) \, dy \right| \leq \theta_{24} \frac{U(x)}{\min_{x \in [\alpha, \beta]} |f'(x)|} \quad \text{and} \quad \left| \int_{\alpha}^{\beta} g(y) e(f(y)) \, dy \right| \leq \theta_{25} \frac{U(x)}{\sqrt{f''(x)}},$$

where

$$\theta_{24} = \frac{D_0 + 2D_1}{\pi} \qquad \text{and} \qquad \theta_{25} = \frac{4C_{2^-}^{1/2}(D_0 + 2D_1)}{\sqrt{\pi}}.$$

Proof. Using condition (M), we can bound V in Lemmas 5.1 and 5.2 by

$$\max\{g(\alpha), g(\beta)\} + \int_{\alpha}^{\beta} |g'(y)| \, dy \leq D_0 U(x) + 2M(x) \cdot D_1 \frac{U(x)}{M(x)} \leq (D_0 + 2D_1) U(x).$$

Also by condition (M), we have that the minimum of $f''(z)$ on I_x is at worst $\geq C_{2^-}^{-1} f''(x)$. □

In some cases, the first derivative test needs to be made explicit, for which we have the following result.

Proposition 5.4. *Let $f \in C^3([\alpha,\beta])$, and $g \in C^2([\alpha,\beta])$, and define $h_r(x)$ as in (2.2).*

Suppose that $f'(x) \neq r$ on an interval $[\alpha,\beta]$, and let

$$K_r(\alpha,\beta) := \sum_{\substack{x\in[\alpha,\beta] \\ h_r'(x)=0}} |h_r(x)|\,.$$

Then we have

$$\int_\alpha^\beta g(x)e(f(x)-rx)\ dx = \left[\frac{g(x)}{2\pi i(f'(x)-r)}e(f(x)-rx)\right]_\alpha^\beta + O(K_r(\alpha,\beta)) + O(2|h_r(\alpha)|) + O(2|h_r(\beta)|)$$

Proof. We begin by applying integration by parts twice to our original integral.

$$\begin{aligned}\int_\alpha^\beta g(x)e(f(x)-rx)\ dx &= \left[\frac{g(x)}{2\pi i(f'(x)-r)}e(f(x)-rx)\right]_\alpha^\beta \\ &\quad - \int_\alpha^\beta \frac{1}{2\pi i}\frac{(f'(x)-r)g'(x)-g(x)f''(x)}{(f'(x)-r)^2}e(f(x)-rx)\ dx \qquad (5.1)\\ &= \left[\frac{g(x)}{2\pi i(f'(x)-r)}e(f(x)-rx)\right]_\alpha^\beta \\ &\quad - \left[-h_r(x)e(f(x)-rx)\right]_\alpha^\beta \qquad (5.2)\\ &\quad + O\left(\int_\alpha^\beta \left|\frac{d}{dx}h_r(x)\right|\ dx\right). \qquad (5.3)\end{aligned}$$

The two terms on line (5.2) are bounded trivially by $O(|h_r(a)|) + O(|h_r(b)|)$.

The remaining integral on line (5.3) is bounded by the total variation of $h_r(x)$, which is at most the modulus of $h_r(x)$ at every critical point added to $|h_r(\alpha)|$ and $|h_r(\beta)|$. $K_r(\alpha,\beta)$ is, by definition, the sum of the moduli $h_r(x)$ at every critical point (since the derivative—by the assumption that $f'(x) \neq r$ on $[\alpha,\beta]$—always exists), so the proof is complete. □

By making use of the following proposition, we can sometimes avoid the K_r term in Proposition 5.4. An alternate form of the resulting error terms can be found in Lemma 5.5.5 of [16].

Proposition 5.5. *Suppose condition (M) holds. If $\epsilon = \pm 1$, then for all x such that $x, x+\epsilon\cdot M(x) \in J$ and all r such that either $f'(x) - r = 0$ or $\operatorname{sgn}(f'(x)-r) = \operatorname{sgn}(\epsilon)$, we have*

$$|f'(x+\epsilon\cdot M(x)) - r| \geq |f'(x)-r| + \left(1-\frac{C_3}{2}\right)M(x)f''(x).$$

Proof. By Taylor's remainder theorem, we can write

$$f'(x+\epsilon\cdot M(x)) - r = (f'(x)-r) + \epsilon\cdot M(x)f''(x) + O\left(\frac{1}{2}M(x)^2 \max_{y\in[x,x+\epsilon\cdot M(x)]} f^{(3)}(y)\right) \tag{5.4}$$

with implicit constant 1.

By condition (M), we have

$$\max_{y\in[x,x+\epsilon\cdot M(x)]} |f^{(3)}(y)| \le \frac{C_3 f''(x)}{M(x)}.$$

Therefore the big-O term in (5.4) is at most $C_3M(x)f''(x)/2$. Since $C_3 < 2$ and since $f'(x) - r$ is either 0 or shares the same sign as ϵ, this gives the result. □

Proposition 5.6. *Suppose f and g satisfy condition (M) for some function $M(x)$ and its associated constants. Let $\epsilon \in [-1,1]\setminus\{0\}$. Assume that x_r is not in the interval from x to $x+\epsilon\cdot M(x)$, and that* $\operatorname{sgn}(f'(x)-r) = \operatorname{sgn}(\epsilon)$. *Then we have that*

$$\int_{x+\epsilon\cdot M(x)}^{x} g(y)e(f(y)-ry)\ dy = \frac{g(y)}{2\pi i(f'(y)-r)}e(f(y)-ry)\Bigg]_{y=x+\epsilon\cdot M(x)}^{x} + O\left(\theta_{26}\cdot\frac{U(x)f''(x)}{|f'(x)-r|^3}\right) + O\left(\theta_{27}\cdot\frac{U(x)}{M(x)(f'(x)-r)^2}\right).$$

We also have the following bound if $\epsilon = \pm 1$:

$$\frac{g(y)}{2\pi i(f'(y)-r)}e(f(y)-ry)\Bigg]_{y=x+\epsilon\cdot M(x)} = O\left(\theta_{28}\cdot\frac{U(x)}{M(x)f''(x)}\right).$$

Here,

$$\theta_{26} = \frac{D_0 + 2D_0C_2 + 3D_1 + C_3}{4\pi^2}, \qquad \theta_{27} = \frac{2D_1 + D_2}{4\pi^2}, \qquad \text{and} \qquad \theta_{28} = \frac{D_0}{\pi(2-C_3)}$$

Proof. As in Proposition 5.4, we apply integration by parts twice to our starting integral:

$$\int_{x+\epsilon\cdot M(x)}^{x} g(y)e(f(y)-ry)\ dy = \left[\frac{g(y)}{2\pi i(f'(y)-r)}e(f(y)-ry)\right]_{y=x+\epsilon\cdot M(x)}^{x} + O\left(|h_r(x)|\right) + O\left(|h_r\left(x+\epsilon\cdot M(x)\right)|\right) + O\left(\int_{x+\epsilon\cdot M(x)}^{x}\left|\frac{d}{dy}\frac{(f'(y)-r)g'(y)-g(y)f''(y)}{(2\pi i)^2(f'(y)-r)^3}\right|\ dy\right). \tag{5.5}$$

The relation

$$\frac{g(y)}{2\pi i(f'(y)-r)}e(f(y)-ry)\Big]_{y=x+\epsilon\cdot M(x)} = O\left(\frac{D_0}{\pi(2-C_3)}\cdot\frac{U(x)}{M(x)f''(x)}\right)$$

holds by Proposition 5.5.

Using condition (M) and Proposition 5.5, we have the following additional bounds.

$$\begin{aligned}
|h_r(x)| &\le \frac{D_1U(x)}{4\pi^2M(x)(f'(x)-r)^2}+\frac{D_0U(x)f''(x)}{4\pi^2|f'(x)-r|^3}\\
|h_r(x+\epsilon\cdot M(x))| &\le \frac{g'(x+\epsilon\cdot M(x))}{4\pi^2(f'(x+\epsilon\cdot M(x))-r)^2}+\frac{g(x+\epsilon\cdot M(x))f''(x+\epsilon\cdot M(x))}{4\pi^2|f'(x+\epsilon\cdot M(x))-r|^3}\\
&\le \frac{D_1U(x)}{4\pi^2M(x)(f'(x)-r)^2}+\frac{D_0C_2U(x)f''(x)}{4\pi^2|f'(x)-r|^3}.
\end{aligned}$$

Finally, to estimate the integral in (5.5) first notice that

$$\begin{aligned}
\left|\frac{d}{dy}\frac{(f'(y)-r)g'(y)-g(y)f''(y)}{(2\pi i)^2(f'(y)-r)^3}\right| &\le \frac{g''(y)}{4\pi^2(f'(y)-r)^2}+\frac{3g'(y)f''(y)}{4\pi^2|f'(y)-r|^3}\\
&\quad+\frac{g(y)f^{(3)}(y)}{4\pi^2|f'(y)-r|^3}+\frac{3g(y)f''(y)^2}{4\pi^2(f'(y)-r)^4}.
\end{aligned}$$

On the interval between x and $x+\epsilon\cdot M(x)$, the maximum of the first term is

$$\le \frac{D_2}{4\pi^2}\cdot\frac{U(x)}{M(x)^2(f'(x)-r)^2},$$

and since we are integrating over an interval of length $M(x)$, the contribution of this term is at most

$$\frac{D_2}{4\pi^2}\cdot\frac{U(x)}{M(x)(f'(x)-r)^2}.$$

Similarly, one can show that

$$\left|\int_{x+\epsilon\cdot M(x)}^{x}\frac{3g'(y)f''(y)}{4\pi^2|f'(y)-r|^3}+\frac{g(y)f^{(3)}(y)}{4\pi^2|f'(y)-r|^3}\,dy\right|\le\frac{3D_1+C_3}{4\pi^2}\cdot\frac{U(x)f''(x)}{|f'(x)-r|^3}.$$

We estimate the remaining integral by using $f''(y)\le C_2f''(x)$ on I_x:

$$\begin{aligned}
\left|\int_{x+\epsilon\cdot M(x)}^{x}\frac{3g(y)f''(y)^2}{4\pi^2(f'(y)-r)^4}\,dy\right| &\le \frac{3D_0C_2}{4\pi^2}\cdot U(x)f''(x)\left|\int_{x+\epsilon\cdot M(x)}^{x}\frac{f''(y)}{(f'(y)-r)^4}\,dy\right|\\
&\le \frac{D_0C_2}{4\pi^2}\cdot U(x)f''(x)\left|\left[\frac{1}{(f'(y)-r)^3}\right]_{x+\epsilon\cdot M(x)}^{x}\right|
\end{aligned}$$

$$\leq \frac{D_0 C_2}{4\pi^2} \cdot \frac{U(x) f''(x)}{|f'(x) - r|^3}.$$

□

Remark 5.1. If $g(x)$ is a constant, then we may remove the term

$$O\left(\frac{U(x)}{M(x)(f'(x) - r)^2}\right)$$

from the statement of Proposition 5.6. This would remove the term $U(x)/M(x)\|f'(x)\|^2$ from $\Delta_2(x)$ in Theorem 4.1. However, very often one finds a term of this same size occuring in $\Delta_3(x)$ regardless.

The primary technique we shall use to evaluate sums will be a version the Euler-Maclaurin summation formula; however, the Euler-Maclaurin formula itself sums over all integers in an interval $[a, b]$, while we will often want to sum over all values x in some interval $[\alpha, \beta]$ for which $F(x)$ is an integer (for some function F).

Lemma 5.7 (Euler-Maclaurin summation, first derivative version—page 10 in [20])**.** *Suppose f is a differentiable function on $[\alpha, \beta]$. Then*

$$\sum_{\alpha \leq n \leq \beta}^{*} f(n) = \int_{\alpha}^{\beta} f(x)\ dx - \psi(\beta) f(\beta) + \psi(\alpha) f(\alpha) + \int_{\alpha}^{\beta} \psi(x) f'(x)\ dx.$$

Proposition 5.8. *Suppose G is a real-valued, differentiable function on $[a, b]$. Suppose $F : [a, b] \to [\alpha, \beta]$ is an onto differentiable function, then*

$$\sum_{\alpha \leq n \leq \beta} G\left(F^{-1}(n)\right) = \int_{a}^{b} G(y) \cdot |F'(y)|\ dy + O\left(\int_{a}^{b} |G'(y)|\ dy\right)$$
$$+ O\left(2 \sum{}' |\psi(F(x)) \cdot G(x)|\right) + O\left(\frac{|G(a)| + |G(b)|}{2}\right)$$

where $G(F^{-1}(n))$ is a sum over all $G(x)$ for $x \in F^{-1}(n)$ and $\sum'$ runs over all x that are local extrema of F on $[a, b]$. For the purposes of this proposition, endpoints are considered to be local extrema.

Proof. Let a', b' be two successive extrema for F (that is, there are no extrema in the interval $[a', b']$). Then $F|_{[a',b']}$ is monotonic and hence $F|_{[a',b']}^{-1}$ is 1-1. Let $[\alpha', \beta']$ be the image of $[a', b']$ under F.

Then, by applying Lemma 5.7 and then the change of variables $y = F|_{[a',b']}^{-1}(x)$, we have

$$\sum_{\alpha' \leq n \leq \beta'}^{*} G\left(F|_{[a',b']}^{-1}(n)\right) = \int_{\alpha'}^{\beta'} G\left(F|_{[a',b']}^{-1}(x)\right)\ dx + \int_{\alpha'}^{\beta'} \psi(x) \frac{G'(F|_{[a',b']}^{-1}(x))}{F'(F|_{[a',b']}^{-1}(x))}\ dx$$

$$
\begin{aligned}
&- \psi(\beta')G\left(F|_{[a',b']}^{-1}(\beta')\right) + \psi(\alpha')G\left(F|_{[a',b']}^{-1}(\alpha')\right) \\
= &\int_{a'}^{b'} G(y) \cdot |F'(y)| \ dy + O\left(\frac{1}{2}\int_{a'}^{b'} |G'(y)| \ dy\right) \\
&+ O\left(|\psi(F(a'))G(a')|\right) + O\left(|\psi(F(b'))G(b')|\right)
\end{aligned}
$$

If $F(b') = \beta'$ and if b' is not an endpoint, then by summing over the interval $[a', b']$ and the interval $[b', c']$, where c' is the next extrema after b', we see that the term $G(b')$ is countedwith the correct multiplicity. Therefore, summing over all such intervals gives the desired left-hand side, up to possibly two terms of the size $G(a)/2$ and $G(b)/2$. □

Lemma 5.9 (Lemma 5.4.2 in [16][1]—Poisson summation)**.** *Let $f \in C^2([a, b])$ for real numbers $a < b$, then*

$$
\sum_{a \le n \le b}^{*} f(n) = \lim_{R\to\infty} \sum_{r=-R}^{R} \int_a^b f(x)e(rx) \ dx.
$$

Proof. We start with the Euler-Maclaurin summation formula as

$$
\sum_{\alpha \le n \le \beta}^{*} f(n) = \int_\alpha^\beta f(x) \ dx - \psi(\beta)f(\beta) + \psi(\alpha)f(\alpha) + \int_\alpha^\beta \psi(x)f'(x) \ dx.
$$

But by integration by parts, we also have

$$
\begin{aligned}
\int_\alpha^\beta \psi(x)f'(x) \ dx &= -\frac{1}{\pi} \lim_{R\to\infty} \int_\alpha^\beta \sum_{r=1}^{R} \frac{\sin(2\pi rx)}{r} f'(x) \ dx \\
&= \psi(\beta)f(\alpha) - \psi(\alpha)f(\alpha) + \lim_{R\to\infty} \sum_{r=1}^{R} \int_\alpha^\beta 2\cos(2\pi rx)f(x) \ dx \\
&= \psi(\beta)f(\alpha) - \psi(\alpha)f(\alpha) + \lim_{R\to\infty} \sum_{r=1}^{R} \int_\alpha^\beta (e(rx) + e(-rx))f(x) \ dx,
\end{aligned}
$$

which completes the proof. □

We end this section with two more specific propositions. The first will give us good bounds on the partial sums and tails of $\psi(x, \epsilon)$, our modified sawtooth function, as well as guarantee its convergence. The second will show that the integrals in Theorem 4.1 involving the W and r functions will converge.

[1] Huxley assumes a and b are integers, so we provide the proof when a, b are real.

Proposition 5.10. *Suppose $\beta > \alpha \geq 1$, $\epsilon \in [-1/2, 1/2]$, and $x \in \mathbb{R}$. Then*

$$\left| \sum_{\alpha \leq |m+\epsilon| \leq \beta} \frac{e(mx)}{m+\epsilon} \right| \leq \begin{cases} 21 & \text{if } \alpha\|x\| \leq 1 \\ \dfrac{35}{2\alpha\|x\|} & \text{otherwise.} \end{cases}$$

In particular, the sum is always less than 21. Also, the sum is convergent as β tends to ∞ and the same bounds hold in this case.

Proof. We begin with a quick calculation we will need later. Suppose z is a real number at least 1.

$$\begin{aligned} \sum_{n \geq z} \frac{1}{n(n-1/2)} &\leq \frac{1}{z(z-1/2)} + \int_z^\infty \frac{1}{y(y-1/2)}\, dy \\ &\leq \frac{1}{z-1/2} + \int_z^\infty \frac{1}{(y-1/2)^2}\, dy \\ &= \frac{4}{2z-1}. \end{aligned}$$

Without loss of generality, we may replace that x with $\langle x \rangle$ and so may assume that $x \in [-1/2, 1/2)$. We start with the assumption that β is finite.

Next, we remove all appearances of the ϵ from the sum. We use $C_{\alpha,x,\epsilon}$ and $C'_{\alpha,x,\epsilon}$ to denote constants which only depend on the variables α, x, and ϵ.

$$\begin{aligned} \sum_{\alpha \leq |m+\epsilon| \leq \beta} \frac{e(mx)}{m+\epsilon} &= \sum_{\alpha \leq |m| \leq \beta} \frac{e(mx)}{m+\epsilon} + C_{\alpha,x,\epsilon} + O\left(\frac{2}{\beta - 1/2}\right) \\ &= \sum_{\alpha \leq |m| \leq \beta} \frac{e(mx)}{m} - \sum_{\alpha \leq |m| \leq \beta} e(mx)\left(\frac{\epsilon}{m(m+\epsilon)}\right) + C_{\alpha,x,\epsilon} + O\left(\frac{4}{2\beta-1}\right) \\ &= \sum_{\alpha \leq |m| \leq \beta} \frac{e(mx)}{m} - \sum_{\alpha \leq |m|} e(mx)\left(\frac{\epsilon}{m(m+\epsilon)}\right) + O\left(\frac{4}{2\beta-1}\right) + C_{\alpha,x,\epsilon} + O\left(\frac{4}{2\beta-1}\right) \\ &= \sum_{\alpha \leq |m| \leq \beta} \frac{e(mx)}{m} + C'_{\alpha,x,\epsilon} + C_{\alpha,x,\epsilon} + O\left(\frac{8}{2\beta-1}\right). \end{aligned}$$

Here, $C_{\alpha,x,\epsilon}$ contains the terms where $|m| < \alpha \leq |m+\epsilon|$ or $|m+\epsilon| < \alpha \leq |m|$. In particular, it is no larger than $4/(2\alpha - 1)$. By our earlier calculations, $C'_{\alpha,x,\epsilon}$ is bounded by $4/(2\alpha-1)$ as well.

Now we pause a moment to show that we may let $\beta = \infty$ with no problems of convergence. The sum

$$\sum_{1 \leq |m|} \frac{e(mx)}{m}$$

is, up to a constant multiple, the Fourier series for the sawtooth function $\psi(x)$ and converges for all x. This implies that

$$\lim_{\beta\to\infty} \sum_{\alpha\le|m|\le\beta} \frac{e(mx)}{m}$$

converges for all α and x. Therefore, since

$$\lim_{\beta\to\infty} \left(\sum_{\alpha\le|m+\epsilon|\le\beta} \frac{e(mx)}{m+\epsilon} - \sum_{\alpha\le|m|\le\beta} \frac{e(mx)}{m} \right) = C'_{\alpha,x,\epsilon} + C_{\alpha,x,\epsilon}$$

we have that

$$\sum_{\alpha\le|m|} \frac{e(mx)}{m+\epsilon}$$

converges for all α, x, and ϵ.

If $x = 0$, then

$$\sum_{\alpha\le|m|\le\beta} \frac{e(mx)}{m} = 0,$$

and hence

$$\left| \sum_{\alpha\le|m+\epsilon|\le\beta} \frac{e(mx)}{m+\epsilon} \right| = \frac{8}{2\alpha-1} + \frac{8}{2\beta-1} \le \frac{16}{2\alpha-1}.$$

This proves the proposition in the case $x = 0$, so we may assume for the remainder of the proof that $x \neq 0$.

Now we remove the absolute value in the condition on the sum and apply summation by parts.

$$\begin{aligned}
\sum_{\alpha\le|m|\le\beta} \frac{e(mx)}{m} &= 2i \sum_{\alpha\le m\le\beta} \frac{\sin(2\pi mx)}{m} \\
&= 2i \sum_{\alpha\le m\le\beta} \left(\frac{1}{m} - \frac{1}{m+1}\right) \left(\sum_{\alpha\le k\le m} \sin(2\pi kx) \right) + 2i \frac{1}{\lfloor\beta\rfloor+1} \sum_{\alpha\le m\le\beta} \sin(2\pi mx) \\
&= 2i \sum_{\alpha\le m\le\beta} \frac{1}{m^2+m} \cdot \frac{\sin(\pi(m+\lceil\alpha\rceil)x)\sin(\pi(m-\lceil\alpha\rceil+1)x)}{\sin(\pi x)} \\
&\qquad + \frac{2i}{\lfloor\beta\rfloor+1} \cdot \frac{\sin(\pi(\lfloor\beta\rfloor+\lceil\alpha\rceil)x)\sin(\pi(\lfloor\beta\rfloor-\lceil\alpha\rceil+1)x)}{\sin(\pi x)}.
\end{aligned}$$

Since we assumed $x \in [-1/2, 1/2]$, we have that $|\sin(\pi x)| \ge |2x|$. For the numerators of the functions, we use that $|\sin(y)| \le \min\{1, |y|\}$. For the sum, we have the following estimate:

$$\begin{aligned}
&\left| 2i \sum_{\alpha\le m\le\beta} \frac{1}{m^2+m} \cdot \frac{\sin(\pi(m+\lceil\alpha\rceil)x)\sin(\pi(m-\lceil\alpha\rceil+1)x)}{\sin(\pi x)} \right| \\
&\qquad \le 2 \sum_{\alpha\le m\le\beta} \frac{1}{m^2+m} \min\left\{ \frac{1}{2\|x\|}, (m^2+m)\|x\| \right\}
\end{aligned}$$

$$\leq 2 \sum_{\alpha \leq m \leq 1/\|x\|} \|x\| + \sum_{m \geq \max\{\alpha, 1/\|x\|\}} \frac{1}{m^2 \|x\|}$$

If $\alpha\|x\| \leq 1$, then we get the following bound:

$$\begin{aligned}
&\left| 2i \sum_{\alpha \leq m \leq \beta} \frac{1}{m^2+m} \cdot \frac{\sin(\pi(m+\lceil\alpha\rceil)x)\sin(\pi(m-\lceil\alpha\rceil+1)x)}{\sin(\pi x)} \right| \\
&\quad \leq 2\|x\| \cdot \|x\|^{-1} + \|x\|^{-1} \sum_{m \geq \|x\|^{-1}} \frac{1}{m^2} \leq 4.
\end{aligned}$$

If $\alpha\|x\| > 1$, then instead we obtain

$$\left| 2i \sum_{\alpha \leq m \leq \beta} \frac{1}{m^2+m} \cdot \frac{\sin(\pi(m+\lceil\alpha\rceil)x)\sin(\pi(m-\lceil\alpha\rceil+1)x)}{\sin(\pi x)} \right| \leq \|x\|^{-1} \sum_{m \geq \alpha} \frac{1}{m^2} \leq \frac{2}{\alpha\|x\|}.$$

Likewise, for the additional term, we have

$$\begin{aligned}
&\left| \frac{2i}{\lfloor\beta\rfloor+1} \cdot \frac{\sin(\pi(\lfloor\beta\rfloor+\lceil\alpha\rceil)x)\sin(\pi(\lfloor\beta\rfloor-\lceil\alpha\rceil+1)x)}{\sin(\pi x)} \right| \\
&\quad \leq \frac{1}{\beta} \min\left\{ \frac{1}{2\|x\|}, 2\beta^2\|x\| \right\} \\
&\quad = \min\left\{ \frac{1}{2\beta\|x\|}, 2\beta\|x\| \right\} \\
&\quad = \min\left\{ \frac{1}{2\beta\|x\|}, 1 \right\}.
\end{aligned}$$

So, in total, if $\alpha\|x\| \leq 1$, then we have

$$\left| \sum_{\alpha \leq |m+\epsilon| \leq \beta} \frac{e(mx)}{m+\epsilon} \right| \leq \frac{16}{2\alpha-1} + 4 + 1 \leq 21.$$

And otherwise, we have

$$\begin{aligned}
\left| \sum_{\alpha \leq |m+\epsilon| \leq \beta} \frac{e(mx)}{m+\epsilon} \right| &\leq \frac{16}{2\alpha-1} + \frac{2}{\alpha\|x\|} + \frac{1}{2\beta\|x\|} \\
&\leq \frac{16}{\alpha\|x\|} + \frac{2}{\alpha\|x\|} + \frac{1}{2\alpha\|x\|} \\
&= \frac{35}{2\alpha\|x\|}.
\end{aligned}$$

This completes the proof. □

Proposition 5.11. *Suppose the conditions of Section 4.2 hold, then the function* $|W_0'|+|W_0\cdot r_0|$ *is integrable on* J_0 *and the function* $|W_+'|+|W_-'|+|W_+\cdot r_+'|+|W_-\cdot r_-'|$ *is integrable on* $J_\pm$.

Proof. Let the functions H, G, W_0, $W_\pm$, r_0, and $r_\pm$ be as in (2.1).

The functions H and G are continuous and bounded on J. The functions W_0 and r_0 are continuous inside J_0, and the functions $W_\pm$ and $r_\pm$ are continuous inside $J_\pm$. The only possible barrier to integrability is the presence of a zero in the denominator of one of these functions at an endpoint of an interval in the corresponding set.

The only terms in the denominator of W_0' are g and f'', which are both bounded away from 0 on J_0. The only terms in the denominator of W_0 are g and f'' again, but r_0' has a factor of H^2 in the denominator; however, in $|W_0\cdot r_0|$, the H^2 in the denominator of r_0' is cancelled by the factor of H^2 in the numerator of W_0, so this too has no zeroes in the denominator at the endpoints of J_0. The case of an interval (a',b') where H^2-G is identically zero but H does not equal nor tend to zero is dealt with in the same way.

Therefore, we will assume we are in the case where $H^2(x)-G(x)$ does not equal nor tend to 0 on the interval in question for the remainder of this proof.

We have

$$r_\pm' = f'' - \left(\frac{H\pm\sqrt{H^2-G}}{2g''}\right)'$$

and

$$\begin{aligned} W_\pm' = &-\left(\frac{H\pm\sqrt{H^2-G}}{2g''}\right)^{-3}(f^{(3)}g+f''g')+\left(\frac{H\pm\sqrt{H^2-G}}{2g''}\right)^{-2}g'' \\ &+3\left(\frac{H\pm\sqrt{H^2-G}}{2g''}\right)^{-4}\left(\frac{H\pm\sqrt{H^2-G}}{2g''}\right)'(f''g) \\ &-2\left(\frac{H\pm\sqrt{H^2-G}}{2g''}\right)^{-3}\left(\frac{H\pm\sqrt{H^2-G}}{2g''}\right)'g'. \end{aligned}$$

In addition,

$$\left(\frac{H\pm\sqrt{H^2-G}}{2g''}\right)' = \frac{H\pm\sqrt{H^2-G}}{2g''^2}g^{(3)} - \frac{1}{2g''\sqrt{H^2-G}}\left(H'\left(\sqrt{H^2-G}\pm H\right)\mp\frac{G'}{2}\right) \tag{5.6}$$

So, the only terms in the denominator of $r_\pm'$ are g'' and $\sqrt{H^2-G}$, and the only terms in the denominator of $W_\pm'$ are $H\pm\sqrt{H^2-G}$ and $\sqrt{H^2-G}$ as all the g'' terms cancel.

The technique required will change slightly depending on whether we consider the + terms or the − terms. As G tends to 0 (which is itself implied by g'' tending to 0), $H\pm\sqrt{H^2-G}$ will tend to either $2H$ or

0. By the assumptions of Section 4.2, if G tends to 0 at the endpoint of an interval in $J_{\pm}$, then H cannot tend to 0 at the same point.

If the sign is chosen so $H \pm \sqrt{H^2 - G}$ tends to $2H$, then the terms $H \pm \sqrt{H^2 - G}$ and $\sqrt{H^2 - G}$ in the denominator of $W'_{\pm}$ do not vanish, and the two copies of g'' in the denominator of $r'_{\pm}$ are cancelled by the two copies of g'' in the numerator of $W_{\pm}$. Therefore, in this case, there are no zeroes in the denominator.

For the remaining case, when $H \pm \sqrt{H^2 - G} \to 0$, we have that

$$\begin{aligned} \frac{H \pm \sqrt{H^2 - G}}{2g''} &= \frac{|H|}{2g''} \left(\frac{H}{|H|} \pm \sqrt{1 - \frac{G}{H^2}} \right) \\ &= \frac{|H|}{2g''} \left(\mp \frac{G}{2H^2} + O\left(\frac{G^2}{H^4} \right) \right) \\ &= \frac{3g(f'')^2}{H} (1 + O(g'')), \end{aligned} \tag{5.7}$$

where the implcit constant depends on the size of H near this point. In particular, since g and f'' do not approach 0 at the endpoints of $J_{\pm}$, equation (5.7) implies that a copy of $H \pm \sqrt{H^2 - G}$ tending to 0 in the denominator of a function can be cancelled by a copy of g'' in the numerator to prevent the presence of a zero in the denominator. In particular, $W_{\pm}$ will not have a zero in the denominator.

By (5.7), the only term in $|W'_{\pm}| + |W_{\pm} \cdot r'_{\pm}|$ that could produce a zero in the denominator is the derivative

$$\left(\frac{H \pm \sqrt{H^2 - G}}{2g''} \right)'. \tag{5.8}$$

Therefore, it suffices to show that (5.8) has no zeroes in the denominator at an endpoint of $J_{\pm}$.

By applying (5.7) to line (5.6) and noting that in this case $\pm(H^2 - G)^{-1/2} = -H^{-1}(1 + O(G))$, we have

$$\begin{aligned} \left(\frac{H \pm \sqrt{H^2 - G}}{2g''} \right)' &= \frac{3g(f'')^2 g^{(3)}}{g''} \left(\frac{1}{H} \pm \frac{1}{\sqrt{H^2 - G}} \right) \mp \frac{3g(f'')^2 H'}{2\sqrt{H^2 - G}} \pm \frac{12g'(f'')^2 + 24gf''f^{(3)}}{4\sqrt{H^2 - G}} + O(1) \\ &= \frac{3g(f'')^2 g^{(3)}}{g''} \cdot O\left(\frac{G}{H} \right) \mp \frac{3g(f'')^2 H'}{2\sqrt{H^2 - G}} \pm \frac{12g'(f'')^2 + 24gf''f^{(3)}}{4\sqrt{H^2 - G}} + O(1) \\ &= \mp \frac{3g(f'')^2 H'}{2\sqrt{H^2 - G}} \pm \frac{12g'(f'')^2 + 24gf''f^{(3)}}{4\sqrt{H^2 - G}} + O(1). \end{aligned}$$

Since $\sqrt{H^2 - G}$ does not tend to 0 on $J_{\pm}$, the derivative (5.8) has no zeroes in the denominator. □

Chapter 6

The Redouaby-Sargos estimates

It is crucial to the van der Corput transform to have very good evaluations of integrals that contain stationary phase points. The most powerful results known to the author are those of Redouaby and Sargos in [34]. They do not provide explicit bounds on their big-O constants, which we would like to provide.

For the duration of this section, unless otherwise stated, we shall consider a C^4 function $f : [a, b] \to \mathbb{R}$ and a C^2 function $g : [a, b] \to \mathbb{R}$. We shall assume that for certain constants, $C_2, C_{2^-}, C_3, C_4, D_0, D_1, D_2, M$ and δ which satisfy $\delta < 1$, $b - a \le M$, and for all $x \in [a, b]$, we have

$$\frac{1}{C_{2^-}} \cdot \frac{T}{M^2} \le f''(z) \le C_2 \frac{T}{M^2}, \qquad |g(z)| \le D_0 U,$$
$$|f^{(3)}(z)| \le C_3 \frac{T}{M^3}, \qquad |g'(z)| \le D_1 \frac{U}{N},$$
$$|f^{(4)}(z)| \le C_4 \frac{T}{M^4}, \qquad \text{and} \qquad |g''(z)| \le D_2 \frac{U}{N^2}.$$

Also, we will define

$$\eta := \frac{3\delta}{2C_{2^-}^2 C_2 C_3}.$$

In addition, we shall assume that there exists $c \in [a, b]$ with $f'(c) = 0$, and that $f''(x) > 0$ on $[a, b]$. We make two last definitions:

$$\alpha = \frac{1}{2} f''(c) \quad \text{and} \quad \phi(t) = \operatorname{sgn}(t) \left(\frac{f(c+t) - f(c)}{\alpha} \right)^{1/2}, \quad t \in [a - c, b - c]. \tag{6.1}$$

6.1 initial lemmas

Lemma 6.1 (Lemma 4 in [34])**.** *Let $\phi : [a - c, b - c] \cap [-\eta M, \eta M] \to \mathbb{R}$ be as above. This function is a C^3 function, satisfying the following properties:*

$$\phi(t) = t \left(1 + O\left(\frac{\theta_{29}|t|}{2M} \right) \right) = t \left(1 + O\left(\frac{\delta}{2} \right) \right),$$

$$\phi'(t) = 1 + O\left(\frac{\theta_{29}|t|}{M}\right) = 1 + O(\delta),$$
$$\phi''(t) = O\left(\frac{\theta_{29}}{M}\right) \qquad \textit{and} \qquad \phi''(0) = \frac{f^{(3)}(c)}{3f''(c)},$$
$$\phi^{(3)}(t) = O\left(\frac{\theta_{30}}{M^2}\right),$$

where all big-O constants are 1 uniformly.

Here,

$$\theta_{29} = \frac{2C_{2-}^2 C_2 C_3}{3} \qquad \textit{and} \qquad \theta_{30} = C_{2-}^3\left(\frac{C_2 C_3^2}{6} + \frac{C_2^2 C_4}{4}\right).$$

Proof. We can write out the derivatives of $\phi(t)$ explicitly as

$$\phi(t) = \frac{\operatorname{sgn}(t)}{\alpha^{1/2}} \cdot (f(c+t) - f(c))^{1/2},$$
$$\phi'(t) = \frac{\operatorname{sgn}(t)}{\alpha^{1/2}} \cdot \frac{f'(c+t)}{2(f(c+t) - f(c))^{1/2}},$$
$$\phi''(t) = \frac{\operatorname{sgn}(t)}{\alpha^{1/2}} \cdot \frac{-\frac{1}{4}(f'(c+t))^2 + \frac{1}{2}(f(c+t) - f(c))f''(c+t)}{(f(c+t) - f(c))^{3/2}},$$
$$\phi^{(3)}(t) = \frac{\operatorname{sgn}(t)}{\alpha^{1/2}} \cdot \frac{\frac{3}{8}f'(c+t)^3 - \frac{3}{4}(f(c+t) - f(c))f'(c+t)f''(c+t) + \frac{1}{2}(f(c+t) - f(c))^2 f^{(3)}(c+t)}{(f(c+t) - f(c))^{5/2}}.$$

And, by a simple Taylor series expansion of $f(c+t)$, we have

$$\phi(0) = 0 \qquad \text{and} \qquad \phi'(0) = 1.$$

Now the Taylor expansion of $f(c+t) - f(c)$ has zero constant and linear terms, and since

$$\frac{T}{M^2 C_{2-}} \leq \frac{d}{dt}\left(f(c+t) - f(c)\right) \leq \frac{TC_2}{M^2}$$

we have

$$\frac{T}{M^2 C_{2-}} t \leq f'(c+t) \leq \frac{TC_2}{M^2} \cdot t \qquad \text{and} \qquad \frac{T}{2M^2 C_{2-}} \cdot t^2 \leq f(c+t) - f(c) \leq \frac{TC_2}{2M^2} \cdot t^2.$$

Now consider the numerator of $\phi''(t)$. We take it's derivative to obtain

$$\frac{d}{dt}\left(-\frac{1}{4}\left(f'(c+t)\right)^2 + \frac{1}{2}\left(f(c+t) - f(c)\right)f''(c+t)\right)$$
$$= -\frac{1}{2}f'(c+t)f''(c+t) + \frac{1}{2}f'(c+t)f''(c+t) + \frac{1}{2}(f(c+t) - f(c))f^{(3)}(c+t)$$

$$= \frac{1}{2}\left(f(c+t) - f(c)\right) f^{(3)}(c+t).$$

From this, we can see that $\phi''(0) = f^{(3)}(c)/3f''(c)$. And by using the upper and lower bounds on $f(c+t)$ and its derivatives, we have

$$|\phi''(t)| \leq \frac{1}{\left(\frac{T}{2M^2C_{2-}}\right)^{1/2}} \cdot \frac{\frac{1}{6}\frac{T^2C_2C_3}{M^5}|t|^3}{\left(\frac{T}{2M^2C_{2-}}t^2\right)^{3/2}} = \frac{2C_{2-}^2C_2C_3}{3M}.$$

This gives the first three results by the Taylor remainder theorem.

Bounding $\phi^{(3)}(t)$ nicely requires a fair bit more work. In particular we need to show that the numerator that appeared in our earlier derivation is bounded by a constant times t^5. To do this, we first look at the derivative of the numerator to obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{3}{8}f'(c+t)^3 - \frac{3}{4}\left(f(c+t)-f(c)\right)f'(c+t)f''(c+t) + \frac{1}{2}\left(f(c+t)-f(c)\right)^2 f^{(3)}(c+t)\right) \\ &\quad = \frac{3}{8}f''(c+t)\left[f'(c+t)^2 - 2\left(f(c+t)-f(c)\right)f''(c+t)\right] + \frac{1}{4}\left(f(c+t)-f(c)\right)f'(c+t)f^{(3)}c+t) \\ &\qquad + \frac{1}{2}(f(c+t)-f(c))^2 f^{(4)}(c+t). \end{aligned}$$

The last summand can be seen to be bounded by

$$\frac{1}{2}\left(\frac{TC_2}{2M^2}t^2\right)^2 \cdot \frac{TC_4}{M^4} = \frac{T^2C_2^2C_4}{2^3M^8} \cdot t^4.$$

We take the derivative of the remaining two terms again to obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{3}{8}f''(c+t)\left[f'(c+t)^2 - 2(f(c+t)-f(c))f''(c+t)\right] + \frac{1}{4}(f(c+t)-f(c))f'(c+t)f^{(3)}(c+t)\right) \\ &\quad = \frac{5}{8}f^{(3)}(c+t)\left[f'(c+t)^2 - 2(f(c+t)-f(c))f''(c+t)\right] + \frac{1}{4}(f(c+t)-f(c))f'(c+t)f^{(4)}(c+t). \end{aligned}$$

By using the bound on the term in brackets that we found before, we see that this is bounded in size by

$$\frac{5}{8} \cdot \frac{TC_3}{M^3} \cdot \frac{2T^2C_2C_3}{3M^5} \cdot |t|^3 + \frac{1}{4} \cdot \frac{TC_2}{2M^2} \cdot t^2 \cdot \frac{TC_2}{M^2} \cdot |t| \cdot \frac{TC_4}{M^4} = \frac{T^3}{M^8}\left(\frac{5C_2C_3^2}{12} + \frac{C_2^2C_4}{8}\right)|t|^3.$$

If we now combine these estimates into a bound on $\phi^{(3)}(t)$ we obtain

$$|\phi^{(3)}(t)| \leq \frac{1}{\left(\frac{T}{2M^2C_{2-}}\right)^{1/2}} \cdot \frac{\frac{T^3}{M^8}\left(\frac{C_2C_3^2}{48}+\frac{C_2^2C_4}{32}\right)|t|^5}{\left(\frac{T}{2M^2C_{2-}}t^2\right)^{5/2}} = \frac{C_{2-}^3}{M^2}\left(\frac{C_2C_3^2}{6}+\frac{C_2^2C_4}{4}\right),$$

which completes the proof. □

Lemma 6.2 (Lemma 5 in [34]). *Let ψ be the inverse function to ϕ. Then ψ has the expansion*

$$\psi(y) = y - \frac{1}{2}\phi''(0)y^2 + G(y)$$

on the interval $[Y_1, Y_2]$ which is the image of ϕ. Here, we have that $G \in C^3[Y_1, Y_2]$ and

$$|G^j(y)| \leq \theta_{31}\frac{|y|^{3-j}}{(3-j)! \cdot M^2}, \quad j = 0, 1, 2, 3, \quad y \in [Y_1, Y_2],$$

where

$$\theta_{31} = \frac{\frac{4}{3}C_{2-}^4C_2^2C_3^2 + (1+\delta)C_{2-}^3\left(\frac{1}{6}C_2C_3^2 + \frac{1}{4}C_2^2C_4\right)}{(1-\delta)^5}$$

Proof. We have that the first three derivatives of $\psi(y)$ satisfy

$$\begin{aligned}
\psi'(y) &= [\phi'(\psi(y))]^{-1}, \\
\psi''(y) &= [\phi'(\psi(y))]^{-3} \cdot (-\phi''(\psi(y))), \\
\psi^{(3)}(y) &= [\phi'(\psi(y))]^{-5}\left(3[\phi''(\psi(y))]^2 - \phi'(\psi(y)) \cdot \phi^{(3)}(\psi(y))\right).
\end{aligned}$$

From these equation sand $\phi(0) = 0$, $\phi'(0) = 1$, it is clear that the Taylor series for $\psi(y)$ begins with $0 + y - \frac{1}{2}\phi''(0)y^2$. Therefore, it suffices to show that

$$|\psi^{(3)}(y)| \leq \frac{\theta_{31}}{M^2}.$$

We know that $\psi(y)$ must be in the domain of ϕ. Therefore, by Lemma 6.1 the desired bound holds. □

6.2 The main result

Lemma 6.3 (Lemma 6 in [34]). *Assume the conditions at the start of this section, and in addition, assume $c \in [a,b]$ satisfies $0 < c - a \leq \eta M$ and $f'(c) = 0$. Then*

$$\int_a^c e(f(x))\,dx = e^{i\pi/4}\frac{e(f(c))}{2\sqrt{f''(c)}} - \frac{f^{(3)}(c)e(f(c))}{6\pi i f''(c)^2} - \frac{e(f(a))}{2\pi i f'(a)} + \frac{e(f(a))}{4\pi i\alpha\phi(a-c)} - e(f(c))\int_{-\phi(a-c)}^{+\infty} e(\alpha y^2)\,dy + O\left(\frac{\theta_{32}}{f''(c)^{3/2}M^2}\right),$$

where α and $\phi(t)$ are as in (6.1), with $a - c \leq t \leq 0$, and

$$\theta_{32} = \frac{\sqrt{33}}{\sqrt{2}\cdot\pi}\cdot\theta_{31} = \frac{\sqrt{33}}{\sqrt{2}\cdot\pi}\cdot\frac{\frac{4}{3}C_{2-}^4 C_2^2 C_3^2 + (1+\delta)C_{2-}^3\left(\frac{1}{6}C_2C_3^2 + \frac{1}{4}C_2^2C_4\right)}{(1-\delta)^5}.$$

If we have $0 < b - c \leq \eta M$, then

$$\int_c^b e(f(x))\,dx = e^{i\pi/4}\frac{e(f(c))}{2\sqrt{f''(c)}} + \frac{f^{(3)}(c)e(f(c))}{6\pi i f''(c)^2} + \frac{e(f(b))}{2\pi i f'(b)} - \frac{e(f(b))}{4\pi i\alpha\phi(c-b)} - e(f(c))\int_{\phi(c-b)}^{+\infty} e(\alpha y^2)\,dy + O\left(\frac{\theta_{32}}{f''(c)^{3/2}M^2}\right),$$

Proof. We begin by considering the integral on the interval $[a, c-\epsilon]$ and at the end of the proof we will allow ϵ to tend to 0. By integration by parts, we have for sufficiently small ϵ

$$\int_a^{c-\epsilon} e(f(x))\,dx = \left[\frac{e(f(x))}{2\pi i f'(x)}\right]_a^{c-\epsilon} + \frac{1}{2\pi i}\int_a^{c-\epsilon}\frac{f''(x)}{f'(x)^2}e(f(x))\,dx.$$

In the last integral we make a change of variables $y = \phi(x-c)$. With ψ as in Lemma 6.2, we have

$$\int_a^{c-\epsilon}\frac{f''(x)}{f'(x)^2}e(f(x))\,dx = \frac{e(f(c))}{2\alpha}\int_{\phi(a-c)}^{\phi(-\epsilon)}\left(\frac{\psi'(y)}{y^2} - \frac{\psi''(y)}{y}\right)e(\alpha y^2)\,dy.$$

By Lemma 5, we have

$$\frac{\psi'(y)}{y^2} - \frac{\psi''(y)}{y} = \frac{1}{y^2} + \frac{G'(y)}{y^2} - \frac{G''(y)}{y} = \frac{1}{y^2} + \mathcal{H}(y),$$

where $\mathcal{H} \in C^1[\phi(a-c), 0)$, and on their domain, $\mathcal{H}$ and $\mathcal{H}'$ are bounded by

$$|\mathcal{H}(y)| \leq \frac{3\theta_{31}}{2M^2} \qquad \text{and} \qquad |\mathcal{H}'(y)| \leq \frac{4\theta_{31}}{|y|M^2}. \tag{6.2}$$

If we let

$$J_1(\epsilon) := \int_{\phi(a-c)}^{\phi(-\epsilon)} \frac{e(\alpha y^2)}{y^2}\, dy \qquad \text{and} \qquad J_2(\epsilon) := \int_{\phi(a-c)}^{\phi(-\epsilon)} \mathcal{H}(y) e(\alpha y^2)\, dy,$$

then we have shown so far

$$\int_a^{c-\epsilon} e(f(x))\, dx = \left[\frac{e(f(x))}{2\pi i f'(x)}\right]_a^{c-\epsilon} + \frac{e(f(c))}{4\pi i \alpha} \left(J_1(\epsilon) + J_2(\epsilon)\right). \tag{6.3}$$

We focus next on bounding $J_2(\epsilon)$. Let $\delta = \sqrt{11/3} \cdot \alpha^{-1/2}$ and split the integral of $J_2(\epsilon)$ in the following way:

$$\int_{\phi(a-c)}^{\phi(-\epsilon)} = \int_{-\delta}^{\phi(-\epsilon)} + \int_{-2\delta}^{-\delta} + \cdots + \int_{-2^K\delta}^{-2^{K-1}\delta} + \int_{\phi(a-c)}^{-2^K\delta}.$$

Using (6.2) to bound the integrand directly, we obtain

$$\left| \int_{-\delta}^{\phi(-\epsilon)} \mathcal{H}(y) e(\alpha y^2)\, dy \right| \le \sqrt{33} \cdot \frac{\theta_{31}}{2\alpha^{1/2} M^2}.$$

For the remaining integrals, we use the first derivatives test (Lemma 5.1) to obtain

$$\begin{aligned}
\left| \int_{-2^k\delta}^{-2^{k-1}\delta} \mathcal{H}(y) e(\alpha y^2)\, dy \right| &\le \frac{|\mathcal{H}(-2^k\delta)| + 2^{k-1}\delta \cdot \max_{y\in[-2^k\delta, -2^{k-1}\delta]} |\mathcal{H}'(y)|}{\pi |2\alpha \cdot (-2^{k-1}\delta)|} \\
&\le \frac{\frac{3}{2}\theta_{31} M^{-2} + 2^{k-1}\delta \cdot 4\theta_{31} |-2^{k-1}\delta|^{-1} M^{-2}}{\sqrt{11/3} \cdot 2^k \pi \alpha^{1/2}} \\
&= \sqrt{33} \cdot \frac{\theta_{31}}{2^{k+1}\pi\alpha^{1/2} M^2}.
\end{aligned}$$

Summing these together we have

$$\begin{aligned}
\left| \frac{e(f(c))}{4\pi i \alpha} J_2(\epsilon) \right| &\le \frac{1}{4\pi\alpha} \left(\frac{\sqrt{33} \cdot \theta_{31}}{2\alpha^{1/2} M^2} + \sum_{k=1}^{K} \frac{\sqrt{33} \cdot \theta_{31}}{2^{k+1}\pi\alpha^{1/2} M^2} \right) \\
&\le \frac{\theta_{31}}{4\pi\alpha^{3/2} M^2} \left(\frac{\sqrt{33}}{2} + \sqrt{33} \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \right) \\
&= \frac{\sqrt{33} \cdot \theta_{31}}{4\pi\alpha^{3/2} M^2} = \frac{\theta_{32}}{f''(c)^{3/2} M^2}.
\end{aligned} \tag{6.4}$$

For $J_1(\epsilon)$, we undo the integration by parts we performed at the start of this proof, now that the non-quadratic part of f has been removed. This gives

$$\frac{e(f(c))}{4\pi i \alpha} J_1(\epsilon) = e(f(c)) \int_{\phi(a-c)}^{\phi(-\epsilon)} e(\alpha y^2)\, dy - \frac{e(f(c))}{4\pi i \alpha} \left[\frac{e(\alpha y^2)}{y}\right]_{\phi(a-c)}^{\phi(-\epsilon)}. \tag{6.5}$$

Combining (6.3), (6.4), and (6.5), we have

$$\int_a^{c-\epsilon} e(f(x))\,dx = -\frac{e(f(a))}{2\pi i f'(a)} + \frac{e(f(a))}{4\pi i\alpha\phi(a-c)} + e(f(c))\int_{\phi(a-c)}^{\phi(-\epsilon)} e(\alpha y^2)\,dy + A_\epsilon + O\left(\frac{\theta_{32}}{f''(c)^{3/2}M^2}\right),$$

where the implicit constant is 1, and where

$$A_\epsilon = \frac{e(f(c-\epsilon))}{2\pi i}\left(\frac{1}{f'(c-\epsilon)} - \frac{1}{2\alpha\phi(-\epsilon)}\right) \longrightarrow -\frac{f^{(3)}(c)e(f(c))}{6\pi i f''(c)^2} \qquad \text{as } \epsilon \to 0.$$

The proof is completed by noting that

$$\int_{\phi(a-c)}^{0} e(\alpha y^2)\,dy = \int_0^{+\infty} e(\alpha y^2)\,dy - \int_{-\phi(a-c)}^{+\infty} e(\alpha y^2)\,dy.$$

□

Lemma 6.4 (Lemma 9 in [34])**.** *Assume the hypotheses at the start of this section. Then, we have*

$$\int_a^c g(x)e(f(x))\,dx = g(c)\int_a^c e(f(x))\,dx + \frac{g(c)-g(a)}{2\pi i f'(a)}e(f(a)) + \frac{g'(c)}{2\pi i f''(c)}e(f(c)) + O\left(\frac{UM^2}{NT^{3/2}}\left(\theta_{33} + \theta_{34}\frac{M}{N}\right)\right)$$

$$\int_c^b g(x)e(f(x))\,dx = g(c)\int_c^b e(f(x))\,dx + \frac{g(b)-g(c)}{2\pi i f'(b)}e(f(b)) - \frac{g'(c)}{2\pi i f''(c)}e(f(c)) + O\left(\frac{UM^2}{NT^{3/2}}\left(\theta_{33} + \theta_{34}\frac{M}{N}\right)\right),$$

where

$$\theta_{33} = \frac{C_{2-}C_3D_1}{4\pi}\left(1 + \frac{2C_{2-}}{\pi}\right) + \frac{C_3C_{2-}^3D_0 + 3C_2C_{2-}^4C_3D_1}{\pi^2},$$

$$\theta_{34} = \frac{C_{2-}C_2D_2}{4\pi}\left(1 + \frac{2C_{2-}}{\pi}\right) + \frac{C_{2-}C_2^2D_2 + C_{2-}^2D_2}{\pi^2}.$$

This result still holds if we weaken the assumptions at the start of this section so that f is only C^3 instead of C^4, and the condition on $f^{(4)}(x)$ may likewise be ignored.

Proof. First, note that

$$\int_a^c g(x)e(f(x))\,dx = g(c)\int_a^c e(f(x)) + J_3,$$

where

$$J_3 = \int_a^c (g(x) - g(c))e(f(x))\ dx = \left[\frac{(g(x) - g(c))e(f(x))}{2\pi i f'(x)}\right]_a^{c-0} + \frac{1}{2\pi i}\int_a^c \rho(x)e(f(x))\ dx$$

by integration by parts, with

$$\rho = \frac{(g - g(c))f'' - g'f'}{f'^2}.$$

In order to bound the new integral $\int \rho(x)e(f(x))\ dx$, we want to have a bound on ρ and it's derivative.

Consider the numerator of $\rho(x)$, and take its derivative. This gives

$$\left|(g(x) - g(c))f^{(3)}(x) - g''(x)f'(x)\right| \le \frac{UD_1}{N}\cdot|x-c|\cdot\frac{TC_3}{M^3} + \frac{UD_2}{N^2}\cdot\frac{TC_2}{M^2}\cdot|x-c|,$$

and hence, by the Mean Value Theorem,

$$|(g(x) - g(c))f''(x) - g'(x)f'(x)| \le \frac{UT}{2NM^3}\left(D_1C_3 + D_2C_2\frac{M}{N}\right)\cdot|x-c|^2.$$

Therefore,

$$\begin{aligned}|\rho(x)| &= |(g(x) - g(c))f''(x) - g'(x)f'(x)|\cdot|f'(x)|^{-2}\\ &\le \frac{UT}{2NM^3}\left(D_1C_3 + D_2C_2\frac{M}{N}\right)\cdot|x-c|^2\cdot\left(\frac{T}{M^2C_{2^-}}|x-c|\right)^{-2}\\ &= \frac{UMC_{2^-}}{2NT}\left(D_1C_3 + D_2C_2\frac{M}{N}\right) \qquad (6.6)\end{aligned}$$

For $\rho'(x)$, we have

$$\rho'(x) = \frac{(g(x) - g(c))f^{(3)}(x)}{f'(x)^2} - \frac{g''(x)}{f'(x)} + 2\frac{g'(x)f'(x)f''(x) - (g(x) - g(c))f''(x)^2}{f'(x)^3}.$$

The first summand we may bound by

$$\left|\frac{(g(x) - g(c))f^{(3)}(x)}{f'(x)^2}\right| \le \frac{UD_0}{N}|x-c|\cdot\frac{TC_3}{M^3}\cdot\left(\frac{T}{M^2C_{2^-}}|x-c|\right)^{-2} = \frac{UMD_0C_3C_{2^-}^2}{TN}|x-c|^{-1}.$$

The second summand we may bound by

$$\left|-\frac{g''(x)}{f'(x)}\right| \le \frac{UD_2}{N^2}\cdot\left(\frac{T}{M^2C_{2^-}}|x-c|\right)^{-1} = \frac{UM^2D_2C_{2^-}}{TN^2}|x-c|^{-1}.$$

For the third summand, we first bound the derivative of the numerator to obtain

$$\begin{aligned}
&\left|\frac{d}{dx}\left(g'(x)f'(x)f''(x)-(g(x)-g(c))f''(x)^2\right)\right| \\
&\quad=\left|g''(x)f'(x)f''(x)+g'(x)f'(x)f^{(3)}(x)-2(g(x)-g(c))f''(x)f^{(3)}(x)\right| \\
&\quad\le \frac{UD_2}{N^2}\cdot\left(\frac{TC_2}{M^2}\right)^2|x-c|+\frac{UD_1}{N}\cdot\frac{TC_2}{M^2}|x-c|\cdot\frac{TC_3}{M^3}+2\frac{UD_1}{N}|x-c|\cdot\frac{TC_2}{M^2}\cdot\frac{TC_3}{M^3} \\
&\quad=\frac{UT^2}{M^5N}\left(3D_1C_2C_3+D_2C_2^2\frac{M}{N}\right)|x-c|.
\end{aligned}$$

Thus the third summand is bounded by

$$\frac{UT^2}{M^5N}\left(3D_1C_2C_3+D_2C_2^2\frac{M}{N}\right)|x-c|^2\cdot\left(\frac{T}{M^2C_{2-}}|x-c|\right)^{-3}=\frac{UMC_{2-}^3}{TN}\left(3D_1C_2C_3+D_2C_2^2\frac{M}{N}\right)|x-c|^{-1}.$$

Thus, in total, we have

$$|\rho'(x)|\le\frac{UM}{TN}\left(D_0C_3C_{2-}^2+3D_1C_2C_{2-}^3C_3+\left(D_2C_2^2+D_2C_{2-}\right)\frac{M}{N}\right)|x-c|^{-1}. \tag{6.7}$$

Now, to bound the term

$$\frac{1}{2\pi i}\int_a^c \rho(x)e(f(x))\,dx$$

and complete the proof, we first break the integral up into the following pieces, letting $\delta = M/T^{1/2}$,

$$\int_a^c=\int_{c-\delta}^c+\int_{c-2\delta}^{c-\delta}+\int_{c-4\delta}^{c-2\delta}+\cdots+\int_a^{c-2^K\delta}.$$

The integral from $c-\delta$ to c is bounded trivially, giving

$$\left|\frac{1}{2\pi i}\int_{c-\delta}^c \rho(x)e(f(x))\,dx\right|\le\frac{UM^2C_{2-}}{4\pi NT^{3/2}}\left(D_1C_3+D_2C_2\frac{M}{N}\right).$$

Each of the remaining integrals is bounded using the first derivative test using (6.6) and (6.7), giving

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{c-2^k\delta}^{c-2^{k-1}\delta}\rho(x)e(f(x))\,dx\right| \\
&\quad\le\frac{|\rho(c-2^k\delta)|+2^{k-1}\delta\cdot\max_{x\in[c-2^k\delta,c-2^{k-1}\delta]}|\rho'(x)|}{2\pi^2\dfrac{T}{M^2C_{2-}}\cdot 2^{k-1}\delta} \\
&\quad\le\frac{UM^2C_{2-}^2}{2^{k+1}\pi^2NT^{3/2}}\left(C_3D_1+C_2D_2\frac{M}{N}\right)
\end{aligned}$$

$$+\frac{UM^2C_{2-}}{2^k\pi^2NT^{3/2}}\left(C_3C_{2-}^2D_0+3C_2C_{2-}^3C_3D_1+\left(C_2^2D_2+C_{2-}D_2\right)\frac{M}{N}\right).$$

Summing these all together gives

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_a^c\rho(x)e(f(x))\;dx\right| &\leq \frac{UM^2}{NT^{3/2}}\left(\frac{C_{2-}C_3D_1}{4\pi}\left(1+\frac{2C_{2-}}{\pi}\right)+\frac{C_3C_{2-}^3D_0+3C_2C_{2-}^4C_3D_1}{\pi^2}\right)\\ &\quad+\frac{UM^3}{N^2T^{3/2}}\left(\frac{C_{2-}C_2D_2}{4\pi}\left(1+\frac{2C_{2-}}{\pi}\right)+\frac{C_{2-}C_2^2D_2+C_{2-}^2D_2}{\pi^2}\right)\\ &= \frac{UM^2}{NT^{3/2}}\left(\theta_{33}+\theta_{34}\frac{M}{N}\right),\end{aligned}$$

which completes the proof. □

6.3 The "long" stationary phase integral

Lemma 6.5 (Lemma 2 in [34]). *Assume all conditions from the start of this section.*

If $0 < c-a \leq \eta M$, *then*

$$\begin{aligned}\int_a^c e(f(x))\;dx &= \frac{e(f(c)+1/8)}{2\sqrt{f''(c)}}-\frac{f^{(3)}(c)e(f(c))}{6\pi i f''(c)^2}-\frac{e(f(a))}{2\pi i f'(a)}\\ &\quad+O\left(\frac{1}{2\pi^2f''(c)^2(c-a)^3(1-\delta)^3}\right)+O\left(\frac{\theta_{32}}{f''(c)^{3/2}M^2}\right).\end{aligned}$$

If $0 < b-c \leq \eta M$, *then*

$$\begin{aligned}\int_c^b e(f(x))\;dx &= \frac{e(f(c)+1/8)}{2\sqrt{f''(c)}}+\frac{f^{(3)}(c)e(f(c))}{6\pi i f''(c)^2}+\frac{e(f(a))}{2\pi i f'(a)}\\ &\quad+O\left(\frac{1}{2\pi^2f''(c)^2(b-c)^3(1-\delta)^3}\right)+O\left(\frac{\theta_{32}}{f''(c)^{3/2}M^2}\right).\end{aligned}$$

The implicit constants are 1 *uniformly in both cases.*

Remark 6.1. The above result is a one-sided form of Huxley's result [15]. Redouaby and Sargos include the additional assumption that $T, M \geq 1$; however, this condition is never used in the lemmas we cite, and so may be safely ignored. This was confirmed in private correspondence with Sargos.

Proof. We use Lemma 6.3. If $a \neq c$, then, by integration by parts, we have

$$e(f(c))\int_{-\phi(a-c)}^{+\infty}e(\alpha y^2)\;dy=\frac{e(f(a))}{2\pi i f''(c)\phi(a-c)}+\frac{e(f(c))}{2\pi i f''(c)}\int_{-\phi(a-c)}^{+\infty}\frac{e(\alpha y^2)}{y^2}\;dy.$$

By the first derivative test (Lemma 5.1), we have

$$\left|\frac{e(f(c))}{2\pi i f''(c)}\int_{-\phi(a-c)}^{+\infty}\frac{e(\alpha y^2)}{y^2}\,dy\right| \le \frac{1}{4\pi^2 f''(c)\alpha|\phi(a-c)|^3} \le \frac{1}{8\pi^2\alpha^2(c-a)^3(1-\delta)^3}.$$

The proof of the second equality follows by the same argument. □

Proposition 6.6. *Suppose the hypotheses of Lemmas 6.5 and 6.4 hold and suppose $a \le c \le b$. Then*

$$\int_a^c g(x)e(f(x))\,dx = g(c)\frac{e(f(c)+1/8)}{2\sqrt{f''(c)}} - \frac{g(c)f^{(3)}(c)e(f(c))}{6\pi i f''(c)^2} - \frac{g(a)e(f(a))}{2\pi i f'(a)} + \frac{g'(c)}{2\pi i f''(c)}e(f(c)) + \mathcal{E}(a),$$

and

$$\int_c^b g(x)e(f(x))\,dx = g(c)\frac{e(f(c)+1/8)}{2\sqrt{f''(c)}} + \frac{g(c)f^{(3)}(c)e(f(c))}{6\pi i f''(c)^2} + \frac{g(b)e(f(b))}{2\pi i f'(b)} - \frac{g'(c)}{2\pi i f''(c)}e(f(c)) + \mathcal{E}(b),$$

where

$$\mathcal{E}(y) = O\left(\frac{g(c)}{2\pi^2 f''(c)^2|c-y|^3(1-\delta)^3}\right) + O\left(\frac{\theta_{32}\cdot g(c)}{f''(c)^{3/2}M^2}\right) + O\left(\frac{UM^2}{NT^{3/2}}\left(\theta_{33}+\theta_{34}\frac{M}{N}\right)\right),$$

with implicit constants all equal to 1.

Proof. We just apply Lemma 6.4 followed by Lemma 6.5. □

The advantage of our condition (M) is now the following.

Proposition 6.7. *Suppose condition (M) holds, that f has a stationary phase point at $x \in J$, and that $x \in [\alpha,\beta] \subset [x-M(x), x+M(x)] \cap J$. Then the conditions at the start of this section hold with*

$$M = M(x), \qquad N = M(x),$$

$$U = U(x), \qquad T = f''(x)M(x)^2,$$

and with C_{2^-}, C_2, C_3, C_4, D_0, D_1, D_2 and δ equal to the similarly named constants in condition (M). Since $\eta \ge 1$ in condition (M), this guarantees that $\beta - x \le \eta M$ and $x - \alpha \le \eta M$.

Moreover, with these definitions, the error terms become

$$\mathcal{E}(\alpha)+\mathcal{E}(\beta) = O\left(\theta_{35}\cdot\frac{U(x)}{f''(x)^2|x-\alpha|^3}\right) + O\left(\theta_{35}\cdot\frac{U(x)}{f''(x)^2|x-\beta|^3}\right) + O\left(2\theta_{36}\cdot\frac{U(x)}{f''(x)^{3/2}M(x)^2}\right),$$

where

$$\theta_{35} := \frac{D_0}{2\pi^2(1-\delta)^3}$$

$$\begin{aligned}\theta_{36} &:= D_0\theta_{32} + \theta_{33} + \theta_{34}\\ &= \frac{\sqrt{33}\cdot D_0\left(\frac{4}{3}C_{2-}^4 C_2^2 C_3^2 + (1+\delta)C_{2-}^3\left(\frac{1}{6}C_2C_3^2 + \frac{1}{4}C_2^2C_4\right)\right)}{\sqrt{2}\cdot\pi(1-\delta)^5}\\ &\quad + \frac{C_{2-}(C_2D_2 + C_3D_1)}{4\pi}\left(1 + \frac{2C_{2-}}{\pi}\right) + \frac{C_3C_{2-}^3D_0 + 3C_2C_{2-}^4C_3D_1 + C_{2-}C_2^2D_2 + C_{2-}^2D_2}{\pi^2}\end{aligned}$$

and if $x = \alpha$ (resp., $x = \beta$) then the first (resp., second) big-O term disappears and $2\theta_{36}$ may be replaced by θ_{36}.

6.4 The "short" stationary phase integral

Lemma 6.8 (Lemma 3 in [34])**.** *Assume all conditions from the start of this section.*

If $0 < c - a \leq \eta M$, then

$$\begin{aligned}\int_a^c e(f(x))\ dx = \frac{e(f(c)+1/8)}{2\sqrt{f''(c)}} - \frac{f^{(3)}(c)e(f(c))}{6\pi i f''(c)^2} + \frac{f^{(3)}(c)e(f(a))}{6\pi i f''(c)^2}\\ - e(f(c))\int_{-\phi(a-c)}^{+\infty} e(\alpha y^2)\ dy + O\left(\theta_{37}\cdot\frac{c-a}{T}\right) + O\left(\frac{\theta_{32}}{f''(c)^{3/2}M^2}\right),\end{aligned}$$

where

$$\theta_{37} := \frac{(9C_3^2 + 2C_2C_4)C_{2-}^3}{24\pi} + \frac{C_{2-}}{12\pi(1-\delta)^3}\left(2C_{2-}^4C_2^2C_3^2 + \left(1+\frac{\delta}{2}\right)C_{2-}^3\left(\frac{C_2C_3^2}{6} + \frac{C_2^2C_4}{4}\right)\right)$$

If $0 < b - c \leq \eta M$, then

$$\begin{aligned}\int_c^b e(f(x))\ dx = \frac{e(f(c)+1/8)}{2\sqrt{f''(c)}} + \frac{f^{(3)}(c)e(f(c))}{6\pi i f''(c)^2} - \frac{f^{(3)}(c)e(f(a))}{6\pi i f''(c)^2}\\ - e(f(c))\int_{\phi(b-c)}^{+\infty} e(\alpha y^2)\ dy + O\left(\theta_{37}\cdot\frac{b-c}{T}\right) + O\left(\frac{\theta_{32}}{f''(c)^{3/2}M^2}\right).\end{aligned}$$

The implicit constants are 1 uniformly in both cases.

Proof. By Lemma 6.3, we only need to show that

$$-\frac{e(f(a))}{2\pi i f'(a)} + \frac{e(f(a))}{4\pi i\alpha\phi(a-c)} = \frac{f^{(3)}(c)e(f(a))}{6\pi i f''(c)^2} + O\left(\theta_{37}\cdot\frac{c-a}{T}\right).$$

(The second case follows by the same method.)

First, consider the Taylor expansion of $(x-c)/f'(x)$ around $x=c$:

$$\frac{x-c}{f'(x)} = \frac{1}{f''(c)} - \frac{f^{(3)}(c)}{2f''(c)^2}(x-c) + O\left(\frac{(x-c)^2}{2!}\max_{y\in[a,c]}\left(\frac{y-c}{f'(y)}\right)''\right).$$

We have

$$\left(\frac{x-c}{f'(x)}\right)'' = \frac{2(x-c)f''(x)^2 - 2f'(x)f''(x) - (x-c)f'(x)f^{(3)}(x)}{f'(x)^3}.$$

If we take the derivative of the numerator, we obtain

$$\begin{aligned}
&\left|3f^{(3)}(x)\left((x-c)f''(x) - f'(x)\right) - (x-c)f'(x)f^{(4)}(x)\right| \\
&\qquad \le 3\frac{C_3T}{M^3}\cdot\frac{3C_3T}{2M^3}(x-c)^2 + |x-c|\cdot\frac{C_2T}{M^2}|x-c|\cdot\frac{C_4T}{M^4} \\
&\qquad = \frac{9C_3^2+2C_2C_4}{2}\cdot\frac{T^2}{M^6}(x-c)^2 \\
&\qquad = 3C_{2^-}^{-3}\mathcal{C}_4\frac{T^2}{M^6}(x-c)^2.
\end{aligned}$$

Thus on $[a,c]$, we have

$$\left|\left(\frac{x-c}{f'(x)}\right)''\right| \le \frac{C_{2^-}^{-3}\mathcal{C}_4\cdot\dfrac{T^2}{M^6}(x-c)^3}{\left(\dfrac{T}{C_{2^-}M^2}|x-c|\right)^3} = \mathcal{C}_4\cdot\frac{1}{T}.$$

Next, we consider a second Taylor expansion (expanding now about $x=0$):

$$\frac{x}{\phi(x)} = 1 - \frac{f^{(3)}(c)}{6f''(c)}x + O\left(\frac{x^2}{2!}\max_{y\in[0,c-a]}\left(\frac{y}{\phi(y)}\right)''\right).$$

Again, we have

$$\left(\frac{x}{\phi(x)}\right)'' = \frac{2x\phi'(x)^2 - 2\phi(x)\phi'(x) - x\phi(x)\phi''(x)}{\phi(x)^3},$$

and by applying 6.1, we see that the derivative of the numerator is bounded by

$$\begin{aligned}
&\left|3\phi''(x)\left(x\phi'(x) - \phi(x)\right) - x\phi(x)\phi^{(3)}\right| \\
&\qquad \le 3\cdot\frac{2C_{2^-}^2C_2C_3}{3M}\cdot\frac{C_{2^-}^2C_2C_3}{M}x^2 + x\cdot\left(1+\frac{\delta}{2}\right)x\cdot\frac{C_{2^-}^3}{M^2}\left(\frac{C_2C_3^2}{6}+\frac{C_2^2C_4}{4}\right) \\
&\qquad = \left(2C_{2^-}^4C_2^2C_3^2 + \left(1+\frac{\delta}{2}\right)C_{2^-}^3\left(\frac{C_2C_3^2}{6}+\frac{C_2^2C_4}{4}\right)\right)\frac{1}{M^2}x^2 \\
&\qquad =: 3(1-\delta)^3\mathcal{C}_5\frac{1}{M^2}x^2.
\end{aligned}$$

And thus

$$\left|\left(\frac{x}{\phi(x)}\right)''\right| \leq \frac{(1-\delta)^3\mathcal{C}_5\dfrac{1}{M^2}x^2}{(1-\delta)^3x^3} = \frac{\mathcal{C}_5}{M^2}.$$

Therefore, we have

$$\begin{aligned}
-\frac{e(f(a))}{2\pi i f'(a)} + \frac{e(f(a))}{4\pi i\alpha\phi(a-c)} &= -\frac{e(f(a))}{2\pi i}\left(\frac{1}{f''(c)(a-c)} - \frac{f^{(3)}(c)}{2f''(c)^2} + O\left(\frac{\mathcal{C}_4}{2}\frac{c-a}{T}\right)\right) \\
&\quad + \frac{e(f(a))}{2\pi i f''(c)}\left(\frac{1}{a-c} - \frac{f^{(3)}(c)}{6f''(c)} + O\left(\frac{\mathcal{C}_5}{2}\frac{c-a}{M^2}\right)\right) \\
&= \frac{f^{(3)}(c)e(f(a))}{6\pi i f''(c)^2} + O\left(\frac{\mathcal{C}_4+\mathcal{C}_5 C_{2-}}{4\pi}\cdot\frac{c-a}{T}\right).
\end{aligned}$$

This completes the proof. ☐

Proposition 6.9. *Suppose the hypotheses of Lemmas 6.5 and 6.4 hold and suppose $a \leq c \leq b$. Then*

$$\begin{aligned}
\int_a^c g(x)e(f(x))\,dx &= g(c)\frac{e(f(c)+1/8)}{2\sqrt{f''(c)}} - g(c)e(f(c))\int_{-\phi(a-c)}^{+\infty} e(\alpha y^2)\,dy \\
&\quad - \frac{g(c)f^{(3)}(c)}{6\pi i f''(c)^2}(e(f(c)) - e(f(a)) + \frac{g'(c)}{2\pi i f''(c)}(e(f(c)) - e(f(a))) + \mathcal{F}(a),
\end{aligned}$$

and

$$\begin{aligned}
\int_c^b g(x)e(f(x))\,dx &= g(c)\frac{e(f(c)+1/8)}{2\sqrt{f''(c)}} - g(c)e(f(c))\int_{\phi(b-c)}^{+\infty} e(\alpha y^2)\,dy \\
&\quad + \frac{g(c)f^{(3)}(c)}{6\pi i f''(c)^2}(e(f(c)) - e(f(b)) - \frac{g'(c)}{2\pi i f''(c)}(e(f(c)) - e(f(b))) + \mathcal{F}(b),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}(y) &= O\left(\left(g(c)\theta_{37} + \frac{C_{2-}^2C_3D_1}{4\pi}\cdot\frac{UM}{N} + \frac{C_{2-}^2C_2D_2}{4\pi}\cdot\frac{UM^2}{N^2}\right)\cdot\frac{|c-y|}{T}\right) \\
&\quad + O\left(\frac{\theta_{32}\cdot g(c)}{f''(c)^{3/2}M^2}\right) + O\left(\frac{UM^2}{NT^{3/2}}\left(\theta_{33} + \theta_{34}\frac{M}{N}\right)\right),
\end{aligned}$$

with implicit constants all equal to 1.

Proof. We apply Lemma 6.8 followed by Lemma 6.5. In this case, however, we also need to show that (in the first case)

$$\frac{g(c)-g(a)}{2\pi i f'(a)} = -\frac{g'(c)}{2\pi i f''(c)} + O\left(\frac{C_{2-}^2UM}{4\pi NT}\left(C_3D_1 + C_2D_2\frac{M}{N}\right)\cdot(c-a)\right).$$

Consider the derivative of the above quantity:

$$\left(\frac{g(c)-g(x)}{2\pi i f'(x)}\right)' = \frac{-f'(x)g'(x)-(g(c)-g(x))f''(x)}{2\pi i f'(x)^2}.$$

The derivative of the numerator can be bounded in the following way:

$$\begin{aligned}\left|-f'(x)g''(x)-(g(c)-g(x))f^{(3)}(x)\right| &\leq \frac{C_2T}{M^2}(x-c)\cdot\frac{D_2U}{N^2}+\frac{D_1U}{N}(x-c)\cdot\frac{C_3T}{M^3}\\ &= \frac{UT}{NM^3}(x-c)\left(C_3D_1+C_2D_2\frac{M}{N}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\left|\left(\frac{g(c)-g(x)}{2\pi i f'(x)}\right)'\right| &\leq \frac{\frac{UT}{2NM^3}(x-c)^2\left(C_3D_1+C_2D_2\frac{M}{N}\right)}{2\pi\left(\frac{T}{C_{2-}M^2}(x-c)\right)^2}\\ &= \frac{C_{2-}^2UM}{4\pi NT}\left(C_3D_1+C_2D_2\frac{M}{N}\right).\end{aligned}$$

The desired bound then follows from Taylor's theorem. □

Proposition 6.10. *Assume the same conditions as in Proposition 6.7. With the definitions there, the error terms in Proposition 6.9 become*

$$\begin{aligned}\mathcal{F}(\alpha)+\mathcal{F}(\beta) &= O\left(\theta_{38}\frac{U(x)}{f''(x)M(x)^2}\cdot|x-\alpha|\right)+O\left(\theta_{38}\frac{U(x)}{f''(x)M(x)^2}\cdot|x-\beta|\right)\\ &\quad+O\left(2\theta_{36}\cdot\frac{U(x)}{f''(x)^{3/2}M(x)^2}\right),\end{aligned}$$

where

$$\begin{aligned}\theta_{38} &:= D_0\theta_{37}+\frac{C_{2-}^2}{4\pi}(C_3D_1+C_2D_2)\\ &= \frac{(9C_3^2+2C_2C_4)C_{2-}^3D_0}{24\pi}+\frac{C_{2-}D_0}{12\pi(1-\delta)^3}\left(2C_{2-}^4C_2^2C_3^2+\left(1+\frac{\delta}{2}\right)C_{2-}^3\left(\frac{C_2C_3^2}{6}+\frac{C_2^2C_4}{4}\right)\right)\\ &\quad+\frac{C_{2-}^2}{4\pi}(C_3D_1+C_2D_2)\end{aligned}$$

and if x equals α or β, then the $2\theta_{36}$ may be replaced with θ_{36}.

Chapter 7

Proof of the main result: Theorem 4.1

7.1 The initial step

We begin by applying Poisson summation (Lemma 5.9). This gives

$$\sum_{a\le n\le b}^{*} g(n)e(f(n)) = \lim_{R\to\infty}\sum_{r=-R}^{R}\int_a^b g(x)e(f(x)-rx)\,dx.$$

We wish to alter the end-points of some of these integrals. With the function $M(x)$ as in Section 4.1, define a_r, the new left endpoint, by

$$a_r := \begin{cases} a+M(a), & \text{if } 0 < f'(a)-r < f''(a),\\ a-M(a), & \text{if } 0 > f'(a)-r > -f''(a),\\ a, & \text{otherwise,}\end{cases} \tag{7.1}$$

and b_r, the new right endpoint, by

$$b_r := \begin{cases} b+M(b), & \text{if } 0 < f'(b)-r < f''(b),\\ b-M(b), & \text{if } 0 > f'(b)-r > -f''(b),\\ b, & \text{otherwise.}\end{cases}$$

Note that $f'(a)-r$ is positive, if and only if x_r lies to the left of a and vice-versa. The transformation from $[a,b]$ to $[a_r,b_r]$ has the effect that if x_r is close to—but not equal to—an endpoint a or b, then we shift that endpoint away from x_r by a distance $M(a)$ or $M(b)$, respectively.

In the statement of Theorem 4.1, m_x is defined by

$$m_x := \left|\mathbb{Z}\cap\left(f'(x)-f''(x), f'(x)+f''(x)\right)\setminus\{f'(x)\}\right|.$$

The values m_a and m_b thus count the number of r such that $a_r \neq a$ and $b_r \neq b$, respectively.

After altering the end-points, we have

$$\begin{aligned} \sum_{a \le n \le b}^{*} g(n)e(f(n)) = \lim_{R\to\infty} \sum_{r=-R}^{R} \int_{a_r}^{b_r} g(x)e(f(x) - rx)\ dx \\ - \lim_{R\to\infty} \sum_{r=-R}^{R} \left(\int_{a_r}^{a} + \int_{b_r}^{b} \right) g(x)e(f(x) - rx)\ dx \end{aligned} \tag{7.2}$$

Since there are only finitely many values of r for which $a_r \neq a$ and $b_r \neq b$, the sum on line (7.2) is finite and so we may omit the limit and sum from $-\infty$ to ∞.

First, consider those integrals in the sum on line (7.2) arising from $0 < |f'(a) - r| < 1$. If $\langle f'(a) \rangle = 0$ or if $m_a = 0$, there are no such integrals, and so the total contribution of this (null) set of integrals is 0. Otherwise, there are at most two such integrals, which by Proposition 5.3 (our variant of the first and second derivative tests) are together bounded by

$$\min\left\{ \frac{2\theta_{25} \cdot U(a)}{\sqrt{f''(a)}}, \frac{2\theta_{24} \cdot U(a)}{\|f'(a)\|} \right\}. \tag{7.3}$$

Thus the contribution of these integrals is bounded by $\Delta_1^{(1)}(a)$, where

$$\Delta_1^{(1)}(x) := \begin{cases} \min\left\{ \dfrac{2\theta_{25} \cdot U(x)}{\sqrt{f''(x)}}, \dfrac{2\theta_{24} \cdot U(x)}{\|f'(x)\|} \right\}, & \text{if } \langle f'(x) \rangle = 0 \text{ or } m_a = 0 \\ 0, & \text{otherwise.} \end{cases}$$

There can only be additional integrals in line (7.2) if $f''(a) \geq 1$, so we will assume as such when bounding them.

Consider next those integrals from a_r to a arising from $1 \le |f'(a) - r| < \sqrt{f''(a)}$. There are $\le 2\sqrt{f''(a)}$ such integrals, each of which, by Proposition 5.3 is bounded by

$$\theta_{25} \cdot \frac{U(a)}{\sqrt{f''(a)}}.$$

So the total contribution of these integrals is bounded by

$$2\theta_{25} \cdot U(a).$$

Lastly, we consider the sum of those integrals in line (7.2) arising from $\sqrt{f''(a)} \le |f'(a) - r| < f''(a)$. In particular, let

$$S_1 := \sum_{\sqrt{f''(a)} \le |f'(a)-r| < f''(a)} \int_{a_r}^{a} g(x)e(f(x)-rx)\ dx$$

denote the sum in question. We apply Proposition 5.6 to each integral in S_1 to obtain

$$S_1 = \sum_r \frac{g(a)}{2\pi i(f'(a)-r)} e(f(a)-ra) \tag{7.4}$$

$$+ \sum_r O\left(\theta_{28} \cdot \frac{U(a)}{f''(a)M(a)}\right) \tag{7.5}$$

$$+ \sum_r \left(O\left(\theta_{27} \cdot \frac{U(a)}{M(a)(f'(a)-r)^2}\right) + O\left(\theta_{26} \cdot \frac{U(a)f''(a)}{|f'(a)-r|^3}\right)\right). \tag{7.6}$$

Here, each sum is over all r satisfying $\sqrt{f''(a)} \le |f'(a)-r| < f''(a)$.

For the sum on line (7.4), we apply a change of variables $m+\epsilon = f'(a)-r$, where $\epsilon = \langle f'(a) \rangle$, and then apply Proposition 5.10. The sum is then bounded by

$$\frac{21D_0}{2\pi} \cdot U(a).$$

The sum on line (7.5) has at most $2 \cdot f''(a)$ terms, so it is bounded by

$$2\theta_{28} \cdot \frac{U(a)}{M(a)}.$$

Since, for $x \ge 1$, we have

$$\sum_{n \ge x} \frac{1}{n^2} \le \frac{1}{x^2} + \int_x^\infty \frac{1}{y^2}\ dy \le \frac{2}{x}, \qquad \text{and} \qquad \sum_{n \ge x} \frac{1}{n^3} \le \frac{1}{x^3} + \int_x^\infty \frac{1}{y^3}\ dy \le \frac{3}{2x^2}, \tag{7.7}$$

the sum on line (7.6) is bounded by

$$4\theta_{27} \cdot \frac{U(a)}{M(a)\sqrt{f''(a)}} + 3\theta_{26} \cdot U(a).$$

Thus the total contribution of the integrals from a_r to a in line (7.2) arising from $1 \le |f'(a)-r| < f''(a)$ is bounded by $\mathcal{D}_1^{(1)}(a)$, where

$$\mathcal{D}_1^{(1)}(x) := \begin{cases} U(x)\left(\theta_{39} + \dfrac{\theta_{40}}{M(x)} + \dfrac{\theta_{41}}{\sqrt{f''(x)}M(x)}\right), & \text{if } f''(x) \ge 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\theta_{39} := 2\theta_{25} + \frac{21D_0}{2\pi} + 3\theta_{26}$$
$$= \frac{8C_{2^-}^{1/2}(D_0+2D_1)}{\sqrt{\pi}} + \frac{21D_0}{2\pi} + \frac{3(D_0+2D_0C_2+3D_1+C_3)}{4\pi^2},$$
$$\theta_{40} := 2\theta_{28} = \frac{2D_0}{\pi(2-C_3)},$$
$$\theta_{41} := 4\theta_{27} = \frac{2D_1+D_2}{\pi^2}.$$

Similarly the contribution of the integrals from b_r to b in line (7.2) is bounded by $\mathcal{D}_1^{(1)}(b) + \Delta_1^{(1)}(b)$ Therefore,

$$\sum_{a\le n\le b}^{*} g(n)e(f(n)) = \lim_{R\to\infty}\sum_{r=-R}^{R}\int_{a_r}^{b_r} g(x)e(f(x)-rx)\ dx + O\left(\Delta_1^{(1)}(a)+\Delta_1^{(1)}(b)\right)$$
$$+O\left(\mathcal{D}_1^{(1)}(a)\right) + O\left(\mathcal{D}_1^{(1)}(b)\right), \tag{7.8}$$

with implicit constants equal to 1 in all cases.

7.2 The half–stationary phase estimate

Suppose that there exists $r \in \mathbb{Z}$ with $f'(a) = r$. We will need to estimate the integral at this particular r in (7.8) separately.

We shall break the integral into a piece surrounding the stationary phase point and an integral away from the stationary phase point. In particular, we write

$$\int_a^{b_r} g(x)e(f(x)-rx)\ dx = \left(\int_a^{\beta_r} + \int_{\beta_r}^{b_r}\right) g(x)e(f(x)-rx)\ dx,$$

where $\beta_r = \min\{a+M(a), b_r\}$. Since $x_r = a$ is to the left of b, we have that b_r either equals b or $b+M(b)$.

By Propositions 6.6 and 6.7, we have

$$\int_a^{\beta_r} g(x)e(f(x)-rx)\ dx = \frac{g(a)e(f(a)-ra+1/8)}{2\sqrt{f''(a)}} + \frac{g(\beta_r)e(f(\beta_r)-r\beta_r)}{2\pi i(f'(\beta_r)-r)}$$
$$+\frac{g(a)f^{(3)}(a)e(f(a))}{6\pi i f''(a)^2} - \frac{g'(a)e(f(a))}{2\pi i f''(a)}$$
$$+O\left(\theta_{35}\cdot\frac{U(a)}{f''(a)^2}\left(\frac{1}{(b-a)^3}+\frac{1}{M(a)^3}\right)\right) \tag{7.9}$$

$$
\begin{aligned}
&+ O\left(\theta_{36}\frac{U(a)}{f''(a)^{3/2}M(a)^2}\right)\\
&= \frac{g(a)e(f(a)-ra+1/8)}{2\sqrt{f''(a)}} + \frac{g(\beta_r)e(f(\beta_r)-r\beta_r)}{2\pi i(f'(\beta_r)-r)} + \mathcal{D}_2(a) \qquad (7.10)\\
&+ O(\Delta_1^{(2)}(a) + \Delta_2^{(1)}(a)),
\end{aligned}
$$

where,

$$
\begin{aligned}
\Delta_1^{(2)}(x) &= \theta_{35}\frac{U(x)}{f''(x)^2(b-a)^3}\\
\Delta_2^{(1)}(x) &= \frac{U(x)}{f''(x)^2M(x)^3}\left(\theta_{35} + \theta_{36}\sqrt{f''(x)}M(x)\right).
\end{aligned}
$$

Here we used the bound

$$
\frac{1}{(\beta_r - a)^3} \leq \frac{1}{(b-a)^3} + \frac{1}{M(a)^3}.
$$

Note that $\Delta_1(x)$ given in Theorem 4.1 is given by

$$
\Delta_1(x) = \Delta_1^{(1)}(x) + \Delta_1^{(2)}(x).
$$

Similarly, if there exists $r \in \mathbb{Z}$ with $f'(b) = r$, then we dissect the integral from a_r to b in (7.8) in a similar way, obtaining,

$$
\begin{aligned}
\left(\int_{a_r}^{\alpha_r} + \int_{\alpha_r}^{b}\right) g(x)e(f(x)-rx)\,dx &= \frac{g(b)e(f(b)-rb+1/8)}{2\sqrt{f''(b)}} - \frac{g(\alpha_r)e(f(\alpha_r)-r\alpha_r)}{2\pi i(f'(\alpha_r)-r)} - \mathcal{D}_2(b) \qquad (7.11)\\
&+ O\left(\Delta_1(b)\right) + O\left(\Delta_2^{(1)}(b)\right) + \int_{a_r}^{\alpha_r} g(x)e(f(x)-rx)\,dx,
\end{aligned}
$$

where $\alpha_r = \max\{b - M(b), a_r\}$.

7.3 The full stationary phase estimates

For all remaining integrals with a stationary phase point in (7.8), we may assume $|f'(a)-r|, |f'(b)-r| > 0$. As before, we denote the stationary phase point corresponding to a given r by x_r. For each such $r \in (f'(a), f'(b))$, we write the corresponding integral in line (7.8) as

$$
\int_{a_r}^{b_r} g(x)e(f(x)-rx)\,dx = \left(\int_{a_r}^{\alpha_r} + \int_{\alpha_r}^{\beta_r} + \int_{\beta_r}^{b_r}\right) g(x)e(f(x)-rx)\,dx,
$$

where $\alpha_r = \max\{x_r - M(x_r), a_r\}$ and $\beta_r = \min\{x_r + M(x_r), b_r\}$. By construction, $[\alpha_r, \beta_r] = I_{x_r} \cap [a_r, b_r]$ with I_x as in the statement of condition (M).

For this section, we shall focus on the contribution of the middle terms. Consider

$$S_2 := \sum_{f'(a)<r<f'(b)} \int_{\alpha_r}^{\beta_r} g(x)e(f(x) - rx)\, dx.$$

Applying Propositions 6.6 and 6.7 with $c = x_r$ to S_2, this becomes

$$S_2 = \sum_{f'(a)<r<f'(b)} \frac{g(x_r)e(f(x_r) - rx_r + \frac{1}{8})}{\sqrt{f''(x_r)}} \tag{7.12}$$

$$+ \sum_{f'(a)<r<f'(b)} O\left(\frac{1}{2\pi^2(1-\delta)^3} \cdot \frac{g(x_r)}{f''(x_r)^2}\left(\frac{1}{(x_r - a_r)^3} + \frac{1}{(b_r - x_r)^3}\right)\right) \tag{7.13}$$

$$+ \sum_{f'(a)<r<f'(b)} O\left(\frac{U(x_r)}{f''(x_r)^2 M(x_r)^3}\left(2\theta_{35} + 2\theta_{36}\sqrt{f''(x_r)}M(x_r)\right)\right) \tag{7.14}$$

$$- \sum_{f'(a)<r<f'(b)} \frac{g(\alpha_r)e(f(\alpha_r) - r\alpha_r)}{2\pi i(f'(\alpha_r) - r)} + \sum_{f'(a)<r<f'(b)} \frac{g(\beta_r)e(f(\beta_r) - r\beta_r)}{2\pi i(f'(\beta_r) - r)}. \tag{7.15}$$

The sum in line (7.12) added to the first terms in lines (7.10) and (7.11)—if they exist—sum to

$$\sum_{f'(a)\le r\le f'(b)}^{*} \frac{g(x_r)e(f(x_r) - rx_r + 1/8)}{\sqrt{f''(x_r)}},$$

the main term in the van der Corput transform.

To begin evaluating the sum in line (7.13), consider values r such that $0 < x_r - a \le \min\{C_2^{-1}, M(a)\}$. By the Mean Value Theorem, we can write $x_r - a = (r - f'(a))/f''(\zeta)$ for some $\zeta \in (a, x_r) \subset (a, a + M(a))$. By the bounds of condition (M), the r under consideration must also satisfy

$$0 < r - f(a) = f''(\zeta)(x_r - a) \le C_2^{-1} f''(\zeta) \le f''(a),$$

which, in turn implies that $x_r - a_r = x_r - a + M(a) \ge M(a)$.

Therefore, we split the sum over

$$\frac{g(x_r)}{f''(x_r)^2(x_r - a_r)^3}$$

in line (7.13) into two pieces. The first piece, over all r such that $x_r - a \le \min\{C_2^{-1}, M(a)\}$, has at most

m_a terms, with each term bounded by

$$\theta_{42} \cdot \frac{U(a)}{f''(a)^2 M(a)^3}, \qquad \text{where} \qquad \theta_{42} := \frac{C_{2^-}^2 D_0}{2\pi^2(1-\delta)^3}. \tag{7.16}$$

Thus the total contribution of these terms is at most

$$\Delta_2^{(2)}(a) := \theta_{42} \cdot \frac{U(a) m_a}{f''(a)^2 M(a)^3}.$$

The second piece, over all remaining r between $f'(a)$ and $f'(b)$, is bounded by

$$\frac{1}{2\pi^2(1-\delta)^3} \sum_{\overline{a} \le x_r \le b} \frac{g(x_r)}{f''(x_r)^2 (x_r - a)^3}, \tag{7.17}$$

where $\overline{a}$ is the smallest value in the interval $[a + \min\{M(a), C_2^{-1}\}, b]$ such that $f'(\overline{a})$ is an integer. Note that if no such integer exists, then this sum is empty. We now apply Proposition 5.8 to (7.17) with

$$G(y) = \frac{g(y)}{f''(y)^2 (y-a)^3}$$

and $F(y) = f'(y)$ on the interval $[f'(\overline{a}), f'(b)]$. Using the bounds from condition (M) liberally, for example, to replace all instances of $f'''(x)$ with $O(C_3 f''(x)/M(x))$, we see that (7.17) is bounded by

$$\Delta_3(a) := \frac{1}{2\pi^2(1-\delta)^3} \int_{\overline{a}}^{b} \frac{U(x)}{f''(x)^2 (x-a)^3} \left(D_0 \cdot f''(x) + \frac{D_1 + 2C_3 D_0}{M(x)} + \frac{3D_0}{x-a} \right) dx \tag{7.18}$$

$$+ \frac{\theta_{35}}{2} \left(\frac{U(a)}{f''(a)^2 (\overline{a} - a)^3} + \frac{U(b)}{f''(b)^2 (b-a)^3} \right). \tag{7.19}$$

Since $f''(x) > 0$ on $[a, b]$, f' has no local extrema on this interval, so the final term from Proposition 5.8 does not appear.

We have a similar bound for the second sum in line (7.13):

$$\left| \frac{1}{2\pi^2(1-\delta)^3} \sum_{f'(a) < r < f'(b)} \frac{g(x_r)}{(f''(x_r))^2 (b_r - x_r)^3} \right| \le \Delta_2^{(2)}(b) + \Delta_3(b).$$

We apply Proposition 5.8 again to the sum on line (7.14) to get

$$\left| \sum_{f'(a) < r < f'(b)} \frac{U(x_r)}{f''(x_r)^2 M(x_r)^3} \left(2\theta_{35} + 2\theta_{36} \sqrt{f''(x_r)} M(x_r) \right) \right| \tag{7.20}$$

$$\leq \int_a^b \frac{U(x)}{f''(x)M(x)^3}\left(2\theta_{35}+2\theta_{36}\sqrt{f''(x)}M(x)\right)\left(1+\frac{|U'(x)|}{U(x)\cdot f''(x)}+\frac{2C_3+3|M'(x)|}{f''(x)M(x)}\right)\,dx \tag{7.21}$$

$$+\frac{U(a)}{f''(a)^2M(a)^3}\left(\theta_{35}+\theta_{36}\sqrt{f''(a)}M(a)\right) \tag{7.22}$$

$$+\frac{U(b)}{f''(b)^2M(b)^3}\left(\theta_{35}+\theta_{36}\sqrt{f''(b)}M(b)\right) \tag{7.23}$$

$$=: \Delta_4^{(1)}+\Delta_2^{(3)}(a)+\Delta_2^{(3)}(b).$$

Now consider

$$S_3 := \lim_{R\to\infty} \sum_{\substack{|r|\leq R\\ r<f'(a) \text{ or } r>f'(b)}} \int_{a_r}^{b_r} g(x)e(f(x)-rx)\,dx$$
$$+\sum_{f'(a)\leq r<f'(b)}\left(\frac{g(\beta_r)e(f(\beta_r)-r\beta_r)}{2\pi i(f'(\beta_r)-r)}+\int_{\beta_r}^{b_r} g(x)e(f(x)-rx)\,dx\right)$$
$$+\sum_{f'(a)<r\leq f'(b)}\left(\int_{a_r}^{\alpha_r} g(x)e(f(x)-rx)\,dx-\frac{g(\alpha_r)e(f(\alpha_r)-r\alpha_r)}{2\pi i(f'(\alpha_r)-r)}\right).$$

Note that the remaining terms of S_2 in line (7.15) as well as the second terms from lines (7.10) and (7.11)—if they exist—appear in S_3.

We have thus far shown that

$$\sum_{a\leq n\leq b}{}^{*} g(n)e(f(n)) = \sum_{f'(a)\leq r\leq f'(b)}{}^{*} \frac{g(x_r)e(f(x_r)-rx_r+1/8)}{\sqrt{f''(x_r)}}+S_3-\mathcal{D}_2(b)+\mathcal{D}_2(a)$$
$$+O\left(\mathcal{D}_1^{(1)}(a)+\mathcal{D}_1^{(1)}(b)+\Delta_1(a)+\Delta_1(b)+\sum_{i=1}^{3}\left(\Delta_2^{(i)}(a)+\Delta_2^{(i)}(b)\right)\right)$$
$$+O\left(\Delta_3(a)+\Delta_3(b)+\Delta_4^{(1)}\right).$$

7.4 The remaining integrals

For each integral in S_3, we apply Proposition 5.4. This gives

$$S_3 = -\lim_{R\to\infty}\sum_{r\neq f'(a)} \frac{g(a_r)e(f(a_r)-ra_r)}{2\pi i(f'(a_r)-a)}+\lim_{R\to\infty}\sum_{r\neq f'(b)} \frac{g(b_r)e(f(b_r)-rb_r)}{2\pi i(f'(b_r)-r)} \tag{7.24}$$

$$+\lim_{R\to\infty}\sum_{r\neq f'(a)} O\left(2h_r(a_r)\right)+\lim_{R\to\infty}\sum_{r\neq f'(b)} O\left(2h_r(b_r)\right) \tag{7.25}$$

$$+\sum_{f'(a)<r<f'(b)}{}' \left(O(2h_r(\alpha_r))+O(2h_r(\beta_r))\right) \tag{7.26}$$

$$+\sum_{f'(a)\le r\le f'(b)} (O(K_r(a_r,\alpha_r))+O(K_r(\beta_r,b_r)))+\lim_{R\to\infty}\sum_{r<f'(a)\text{ or }r>f'(b)} O(K_r(a_r,b_r)),$$

where in each sum it is assumed that $|r|\le R$ and K_r is defined as in Proposition 5.4. The primed sum in line (7.26) denotes that only the only terms $O(2h_r(\alpha_r))$ that appear are those where $\alpha_r\ne a_r$, and likewise for the terms $O(2h_r(\beta_r))$. This comes from the fact that if $\alpha_r=a_r$ or $\beta_r=b_r$, then there is no integral to apply Proposition 5.4 to. We could also remove these terms from (7.25), but the results are made simpler by including them.

We start estimating the first sum of (7.24) and write

$$-\lim_{R\to\infty}\sum_{r\ne f'(a)}\frac{g(a_r)e(f(a_r)-ra_r)}{2\pi i(f'(a_r)-r)}=\sum_{0<r-f'(a)\le f''(a)} O\left(\frac{g(a-M(a))}{2\pi(f'(a-M(a))-r)}\right) \tag{7.27}$$

$$+\sum_{0<-(r-f'(a))\le f''(a)} O\left(\frac{g(a+M(a))}{2\pi(f'(a+M(a))-r)}\right) \tag{7.28}$$

$$-\frac{g(a)e(f(a))}{2\pi i}\lim_{R\to\infty}\sum_{\substack{|r|\le R\\ |f'(a)-r|>f''(a)}}\frac{e(-ra)}{f'(a)-r} \tag{7.29}$$

There are at most m_a terms on lines (7.27) and (7.28), each with size bounded by

$$\theta_{28}\cdot\frac{U(a)}{f''(a)M(a)}$$

by Propositions 5.5 and 5.6, for a total contribution of at most

$$\Delta_2^{(4)}(a):=\theta_{28}\cdot\frac{U(a)m_a}{f''(a)M(a)}$$

We can evaluate the sum on line (7.29) explicitly in some cases. Suppose $f''(a)<1-\|f'(a)\|$. If $f''(a)<\|f'(a)\|$, then we pull out the term where $f'(a)-r=\langle f'(a)\rangle$, so that the remaining terms on line (7.29) form a modified sawtooth function. Otherwise, if $f''(a)\ge 1-\|f'(a)\|$, we may bound the sum on line (7.29) using Proposition 5.10. In particular, the sum on line (7.29) becomes

$$\mathcal{D}_1^{(2)}(a):=\begin{cases} g(a)e(f(a)+[\![f'(a)]\!]\,a)\left(-\frac{1}{2\pi i\langle f'(a)\rangle}+\psi\left(a,\langle f'(a)\rangle\right)\right), & \text{if } f''(a)<\|f'(a)\|,\\ g(a)e(f(a)+[\![f'(a)]\!]\,a)\psi\left(a,\langle f'(a)\rangle\right), & \text{if } \|f'(a)\|\le f''(a)<1-\|f'(a)\|,\\ O(21U(a)), & \text{if } 1-\|f'(a)\|\le f''(a).\end{cases}$$

We then let $\mathcal{D}_1(x)=\mathcal{D}_1^{(1)}(x)+\mathcal{D}_1^{(2)}(x)$.

By a similar argument, one can show that all the sums on line (7.24) are equal to

$$-\mathcal{D}_1^{(2)}(b) + \mathcal{D}_1^{(2)}(a) + O\left(\Delta_2^{(4)}(a) + \Delta_2^{(4)}(b)\right).$$

We break the sum from (7.25) into three smaller sums like so

$$\lim_{R\to\infty} \sum_{\substack{|r|\le R \\ r\neq f'(a)}} |2h_r(a_r)| = \lim_{R\to\infty} \sum_{\substack{|r|\le R \\ |f'(a)-r|\ge f''(a)}} |2h_r(a)| \tag{7.30}$$

$$+ \sum_{0<r-f'(a)<f''(a)} |2h_r(a-M(a))| \tag{7.31}$$

$$+ \sum_{0<f'(a)-r<f''(a)} |2h_r(a+M(a))|. \tag{7.32}$$

The sum on the right-hand side of line (7.30) is

$$\begin{aligned} &\lim_{R\to\infty} \sum_{\substack{|r|\le R \\ |f'(a)-r|\ge f''(a)}} \left| \frac{(f'(x)-r)g'(x) - g(x)f''(x)}{2\pi^2(f'(x)-r)^3} \right|_{x=a} \\ &\le \lim_{R\to\infty} \sum_{\substack{|r|\le R \\ |f'(a)-r|\ge f''(a)}} \left(\left| \frac{D_1U(a)}{2\pi^2 M(a)(f'(a)-r)^2} \right| + \left| \frac{D_0U(a)f''(a)}{2\pi^2(f'(a)-r)^3} \right| \right) \end{aligned} \tag{7.33}$$

If $\|f'(a)\| = 0$ or $m_a \ge 1$, then the smallest $|f'(a)-r|$ could be in this sum is $\max\{1/2, f''(a)\}$; otherwise, we will have a term where $|f'(a)-r| = \|f'(a)\|$. Thus by an argument similar to that of (7.7), this sum is bounded by

$$\Delta_2^{(5)}(a) := \begin{cases} \theta_{43}\cdot\dfrac{U(a)}{M(a)}\min\left\{2, \dfrac{1}{f''(a)}\right\} + \theta_{44}\cdot U(a)\min\left\{4f''(a), \dfrac{1}{f''(a)}\right\}, & \text{if } \|f'(a)\| = 0 \text{ or } m_a \ge 1, \\ \frac{\theta_{43}}{2}\cdot\dfrac{U(a)}{M(a)\|f'(a)\|^2} + \frac{\theta_{44}}{2}\cdot\dfrac{U(a)f''(a)}{\|f'(a)\|^3}, & \text{otherwise,} \end{cases}$$

where

$$\theta_{43} := \frac{3D_1}{\pi^2} \qquad \text{and} \qquad \theta_{44} := \frac{5D_0}{2\pi^2}$$

By using condition (M) and Proposition 5.5, each term on lines (7.31) and (7.32) has size at most

$$\theta_{45}\cdot\frac{U(a)}{f''(a)^2M(a)^3},$$

where

$$\theta_{45} := \frac{1}{2\pi^2}\left(D_1\left(\frac{2}{2-C_3}\right)^2 + D_0\left(\frac{2}{2-C_3}\right)^3\right),$$

and there are m_a such terms for a total contribution of at most

$$\Delta_2^{(6)}(a) := \theta_{45}\frac{U(a)m_a}{f''(a)^2M(a)^3}.$$

By a similar argument both sums on line (7.25) are bounded by $\Delta_2^{(5)}(b) + \Delta_2^{(6)}(b)$.

For the first sum on line (7.26), in all terms that appear, we have α_r equals $x_r - M(x_r)$. By Proposition 5.5 and the bounds of condition (M), each such term is bounded by

$$\theta_{45}\frac{U(x_r)}{f''(x_r)^2M(x_r)^3}.$$

We may therefore bound the sum over those r with $\alpha_r = x_r - M(x_r)$ by a sum of terms $g(x_r)f''(x_r)^{-2}M(x_r)^{-3}$ for all $r \in [f'(a), f'(b)]$ and obtain

$$\begin{aligned}\left|\sum_{x_r\in[a,b]}\theta_{45}\frac{U(x_r)}{f''(x_r)^2M(x_r)^3}\right| &\le \int_a^b \theta_{45}\frac{U(x)}{f''(x)M(x)^3}\left(1+\frac{|U'(x)|}{U(x)\cdot f''(x)}+\frac{2C_3+3|M'(x)|}{f''(x)M(x)}\right)\,dx\\ &\quad+\frac{\theta_{45}}{2}\frac{U(a)}{f''(a)^2M(a)^3}+\frac{\theta_{45}}{2}\frac{U(b)}{f''(b)^2M(b)^3}\\ &=: \Delta_4^{(2)} + \Delta_2^{(7)}(a) + \Delta_2^{(7)}(b)\end{aligned}$$

by the same argument as in line (7.20).

Therefore, we have shown that

$$\begin{aligned}\sum_{a\le n\le b}{}^{*} g(n)e(f(n)) = \sum_{f'(a)\le r\le f'(b)}{}^{*}\frac{g(x_r)e(f(x_r)-rx_r+\frac{1}{8})}{\sqrt{f''(x_r)}} - \mathcal{D}(b) + \mathcal{D}(a) + \sum_{i=1}^{3}O(\Delta_i(a)+\Delta_i(b)) + O(\Delta_4)\\ + \sum_{f'(a)\le r\le f'(b)}(O(K_r(a_r,\alpha_r)) + O(K_r(\beta_r,b_r))) + \sum_{r<f'(a)\text{ or }r>f'(b)}O(K_r(a_r,b_r)),\end{aligned}$$

where

$$\begin{aligned}\Delta_2(x) :&= \sum_{i=1}^{7}\Delta_2^{(i)}(x)\\ &= \frac{U(x)}{f''(x)^2M(x)^3}\left(2\theta_{35}+\frac{\theta_{45}}{2}+2\theta_{36}\sqrt{f''(x)}M(x)\right)+\frac{U(x)m_x}{f''(x)^2M(x)^3}\left(\theta_{42}+\theta_{45}+\theta_{28}f''(x)M(x)^2\right)\end{aligned}$$

$$
+\begin{cases} \theta_{43}\dfrac{U(x)}{M(x)}\min\left\{2, \dfrac{1}{f''(x)}\right\} \\ \quad + \theta_{44}U(x)\min\left\{4f''(x), \dfrac{1}{f''(x)}\right\}, & \text{if } \|f'(x)\| = 0 \text{ or } m_x \geq 1, \\ \theta_{43}\dfrac{U(x)}{M(x)\|f'(x)\|^2} + \theta_{44}\dfrac{U(x)f''(x)}{\|f'(x)\|^3}, & \text{otherwise,} \end{cases}
$$

$$
\begin{aligned}
\Delta_4 :&= \Delta_4^{(1)} + \Delta_4^{(2)} \\
&= \int_a^b \frac{U(x)}{f''(x)M(x)^3}\left(2\theta_{35} + \theta_{45} + 2\theta_{36}\sqrt{f''(x)}M(x)\right)\left(1 + \frac{|U'(x)|}{U(x)\cdot f''(x)} + \frac{2C_3 + 3|M'(x)|}{f''(x)M(x)}\right)\, dx.
\end{aligned}
$$

The proof will therefore be finished when we show that the sum of all the K_r terms is bounded by $O(\Delta_5)$.

7.5 Bounding the variation

In this section, K_r is defined as in Proposition 5.4 and h_r is defined as in (2.2). We may safely ignore points where the derivative does not exist, since all the intervals that give rise to the K_r terms do not contain stationary phase points.

The derivative of $h_r(x)$ is (using f_r' as shorthand for $f'(x) - r$)

$$
h_r' = \frac{-gf_r'f^{(3)} + g''(f_r')^2 + 3g(f'')^2 - 3g'f_r'f''}{4\pi^2(f_r')^4} = \frac{g''(f_r')^2 - Hf_r' + 3g(f'')^2}{4\pi^2(f_r')^4}.
$$

We set the numerator equal to 0 and solve for f_r'.

First, suppose $g(x) = 0$. If $g''(x)$ also equals 0, then the numerator of $h_r'(x)$ is 0 if and only if $g'(x)$ also equals 0 ($f''(x)$ is never 0, and we assumed $f'(x) \neq 0$ at all points in consideration); but if $g(x)$ and $g'(x)$ equal 0, then $h_r(x)$ also equals 0, so these points contribute nothing to K_r. Thus the only contribution from points x where $g(x) = 0$ comes when $g''(x) \neq 0$ and $g'(x) \neq 0$, i.e., from those points in J_{null} as defined in Section 4.2. At these points, solving for f_r' yields $3g'(x)f''(x)/g''(x)$, so the contribution to the K_r terms from these points is bounded by

$$
\sum_{x \in J_{\text{null}}} \frac{g''(x)^2}{36\pi^2 g'(x)f''(x)^2}.
$$

For the remainder of this section, we will suppose then that $g(x) \neq 0$. There is no way to make the numerator equal to zero if $g''(x) = H(x) = 0$, since both $g(x)$ and $f''(x)$ are assumed to be non-zero, or if $H(x)^2 - G(x) < 0$, since all the above functions are real-valued; otherwise, the solution is given by

$$
f_r' = \begin{cases} \dfrac{H \pm \sqrt{H^2 - G}}{2g''} & \text{on } J_\pm, \\ \dfrac{3g(f'')^2}{H} & \text{on } J_0 \end{cases}, \tag{7.34}
$$

where $J_\pm$ and J_0 are as defined in Section 4.2. Plugging this value of f_r' into h_r, we obtain

$$h_r(x) = \begin{cases} W_+(x) + W_-(x), & \text{if } x \in J_\pm, \\ W_0(x), & \text{if } x \in J_0, \end{cases}$$

where

$$\begin{aligned} W_\pm &= \frac{(2g'')^2 g'}{4\pi^2(H \pm \sqrt{H^2 - G})^2} - \frac{(2g'')^3 (f''g)}{4\pi^2(H \pm \sqrt{H^2 - G})^3}, \qquad \text{and} \\ W_0 &= \frac{H^2 g'}{4\pi^2(3g(f'')^2)^2} - \frac{H^3 f'' g}{4\pi^2(3g(f'')^2)^3} = -\frac{H^2 f^{(3)}}{108\pi^2 g (f'')^5}. \end{aligned}$$

If we ignore the constraint that r be an integer for the moment, then we can imagine that (7.34) determines functions

$$r_\pm = f' - \frac{H \pm \sqrt{H^2 - G}}{2g''} \quad \text{on } J_\pm \qquad \text{and} \qquad r_0 = f' - \frac{3g(f'')^2}{H} \quad \text{on } J_0.$$

The contribution of the K_r terms is then at most

$$\le \sum_{r_0^{-1}(n) \in J_0} W_0\left(r_0^{-1}(n)\right) + \sum_{r_+^{-1}(n) \in J_\pm} W_+\left(r_+^{-1}(n)\right) + \sum_{r_-^{-1}(n) \in J_\pm} W_-\left(r_-^{-1}(n)\right).$$

We now apply Proposition 5.8 to see that the contribution of the K_r terms is bounded by

$$\Delta_5 := \mathcal{K}(J_0, W_0, r_0) + \mathcal{K}(J_\pm, W_+, r_+) + \mathcal{K}(J_\pm, W_-, r_-)$$

This completes the proof of Theorem 4.1.

Chapter 8

Proofs of variants on the main theorem

8.1 Proof of Theorem 4.2

We will follow the proof of Theorem 4.1, except in place of $M(x)$, C_3, C_4, D_1, D_2, and η, we use $M^*(x) = \eta M(x)$, $C_3^* = \eta C_3$, $C_4^* = \eta^2 C_4$, $D_1^* = \eta D_1$, $D_2^* = \eta^2 D_2$, and $\eta^* = 1$, respectively. This guarantees that the restriction on η in condition (M) is satisfied. Otherwise the proof continues unaffected until Section 7.4.

Since $m_\mu = 0$ and $M(\mu) \geq b-a$ for μ equal to a and b, we may use Proposition 5.6 in place of Proposition 5.4 on the integrals in S_3. However, we will apply it with $M(x)$ and the unaltered assocatied constants. Thus, we obtain

$$S_3 = -\lim_{R\to\infty} \sum_{r\neq f'(a)} \frac{g(a)e(f(a)-ra)}{2\pi i(f'(a)-a)} + \lim_{R\to\infty} \sum_{r\neq f'(b)} \frac{g(b)e(f(b)-rb)}{2\pi i(f'(b)-r)} \tag{8.1}$$

$$+ \lim_{R\to\infty} \sum_{r\neq f'(a)} O\left(\theta_{26}\cdot \frac{U(a)f''(a)}{|f'(a)-r|^3} + \theta_{27}\cdot \frac{U(a)}{M(a)(f'(a)-r)^2}\right) \tag{8.2}$$

$$+ \lim_{R\to\infty} \sum_{r\neq f'(b)} O\left(\theta_{26}\cdot \frac{U(b)f''(b)}{|f'(b)-r|^3} + \theta_{27}\cdot \frac{U(b)}{M(b)(f'(b)-r)^2}\right) \tag{8.3}$$

We used a and b in place of a_r and b_r due to m_a and m_b being zero, and for each integral in S_3, we always apply Proposition 5.4 with x equal to either a or b.

Just as in the proof of Theorem 4.1, the terms in line (8.1) equal

$$-\mathcal{D}_1^{(2)}(b) + \mathcal{D}_1^{(2)}(a) + O(\Delta_2(a) + \Delta_2(b)).$$

By the same arguments that we applied to (7.33), the terms on lines (8.2) are bounded by

$$\Delta_2^{5'}(a) := 4\theta_{27}\cdot \frac{U(a)}{M(a)\|f'(a)\|^{*2}} + 3\theta_{26}\cdot \frac{U(a)f''(a)}{\|f'(a)\|^{*3}},$$

and likewise with (8.3).

Since we have no K_r terms, the Δ_5 term never exists. In addition, we only have

$$\begin{aligned}\Delta_2(\mu) &= \Delta_2^{(1)}(\mu) + \Delta_2^{(2)}(\mu) + \Delta_2^{(3)}(\mu) + \Delta_2^{(4)}(\mu) + \Delta_2^{(5')}(\mu) \\ &= \frac{U(x)}{f''(x)^2 M(x)^3}\left(2\theta_{35} + 2\theta_{36}\sqrt{f''(x)}M(x)\right) + 4\theta_{27}\cdot\frac{U(a)}{M(a)\|f'(a)\|^{*2}} + 3\theta_{26}\cdot\frac{U(a)f''(a)}{\|f'(a)\|^{*3}}\end{aligned}$$

and $\Delta_4 = \Delta_4^{(1)}$, which gives the corresponding new implicit constants. This completes the proof of Theorem 4.2 in this case.

8.2 Proof of Theorem 4.3

Recall from the statement of the theorem that we have r' such that $0 < |r' - f'(b)| < 1/2$ and $0 < |x_{r'} - b| \le M(x_{r'})$. We will follow the proof of Theorem 4.1, but make key changes in evaluating the integral $\int_a^b g(x)e(f(x) - r'x)\ dx$.

In Section 7.1, the only change we make is to have $b_{r'}$ equal b instead of $b \pm M(b)$. As a result, we may replace $\|f'(x)\|$ with $1 - \|f'(x)\|$ in the definition of $\Delta_1^{(1)}(x)$ in Section 7.1. Further, if $m_x = 1$ and $|r' - f'(b)| < f''(b)$, then no integrals get changed in Section 7.1, so we may take $\Delta_1^{(1)}(b) = 0$.

In Section 7.3, we let $\alpha_{r'} = \max\{x_r - M(x_r), a_r\}$ and take $\beta_{r'} = b$. If $r' > f'(b)$, then we extend the sum in the definition of S_2 to the range $f'(a) < r \le r'$.

First, suppose $r' < f'(b)$. In this case, we apply Propositions 6.6 and 6.7 (the "long" Redouaby-Sargos estimates) to $\int_{\alpha_{r'}}^{x_{r'}}$, and Propositions 6.9 and 6.10 (the "short" Redouaby-Sargos estimates) to $\int_{x_{r'}}^{\beta_{r'}}$. This yields

$$\begin{aligned}&\int_{\alpha_{r'}}^{\beta_{r'}} g(x)e(f(x) - r'x)\ dx \\ &\quad= g(x_{r'})\frac{e(f(x_{r'}) - r'x_{r'} + 1/8)}{\sqrt{f''(x_{r'})}} - g(x_{r'})e(f(x_{r'}) - r'x_{r'})\int_{\phi_{r'}(b - x_{r'})}^{+\infty} e\left(\frac{1}{2}f''(x_{r'})y^2\right)\ dy \\ &\qquad - \frac{g(x_{r'})f^{(3)}(x_{r'})e(f(b) - r'b)}{6\pi i f''(x_{r'})^2} + \frac{g'(x_{r'})e(f(b) - r'b)}{2\pi i f''(x_{r'})} - \frac{g(\alpha_{r'})e(f(\alpha_{r'}) - r'\alpha_{r'})}{2\pi i(f'(\alpha_{r'}) - r')} \\ &\qquad + O\left(\theta_{35}\cdot\frac{U(x_{r'})}{f''(x_{r'})^2|x_{r'} - \alpha_{r'}|^3}\right) + O\left(\theta_{38}\cdot\frac{U(x_{r'})}{f''(x_{r'})M(x_{r'})^2}\cdot|b - x_{r'}|\right) \\ &\qquad + O\left(\frac{2\theta_{36}U(x_{r'})}{f''(x_{r'})^{3/2}M(x_{r'})^2}\right) \\ &\quad= g(x_{r'})\frac{e(f(x_{r'}) - r'x_{r'} + 1/8)}{\sqrt{f''(x_{r'})}} - \mathcal{D}_3(b) \\ &\qquad + O\left(\theta_{35}\cdot\frac{U(x_{r'})}{f''(x_{r'})^2}\left(\frac{1}{M(x_{r'})^3} + \frac{1}{x_{r'} - a}\right)\right)\end{aligned}$$

$$+O\left(\frac{2\theta_{36}U(x_{r'})}{f''(x_{r'})^{3/2}M(x_{r'})^2}\right),$$

where

$$\begin{aligned}\mathcal{D}_3(b) = \operatorname{sgn}(b-x_{r'})g(x_{r'})e(f(x_{r'})-r'x_{r'})\int_{\phi_{r'}(b-x_{r'})}^{+\infty} e\left(\frac{1}{2}f''(x_{r'})y^2\right)\,dy \\ + \frac{g(x_{r'})f^{(3)}(x_{r'})e(f(b)-r'b)}{6\pi i f''(x_{r'})^2} - \frac{g'(x_{r'})e(f(b)-r'b)}{2\pi i f''(x_{r'})} \\ + O\left(\theta_{38}\cdot\frac{U(x_{r'})}{f''(x_{r'})M(x_{r'})^2}\cdot|b-x_{r'}|\right)\end{aligned}$$

and

$$\phi_{r'}(x) = \left(\frac{f(x_{r'}+x)-f(x_{r'})-r'\cdot x}{\frac{1}{2}f''(x_{r'})}\right)^{1/2}.$$

Note that this definition of ϕ lacks the signum function.

If $r' > f'(b)$, then recall that the condition (M) can extend to the larger interval, $[a, x_{r'}]$. So we likewise have

$$\begin{aligned}&\int_{\alpha_{r'}}^{\beta_{r'}} g(x)e(f(x)-r'x)\,dx \\ &= \int_{\alpha_{r'}}^{x_{r'}} g(x)e(f(x)-r'x)\,dx - \int_{\beta_{r'}}^{x_{r'}} g(x)e(f(x)-r'x)\,dx \\ &= -\mathcal{D}_3(b) + O\left(\theta_{35}\cdot\frac{U(x_{r'})}{f''(x_{r'})^2}\left(\frac{1}{M(x_{r'})^3}+\frac{1}{x_{r'}-a}\right)\right) \\ &\quad + O\left(\frac{2\theta_{36}U(x_{r'})}{f''(x_{r'})^{3/2}M(x_{r'})^2}\right) \\ &=: -\mathcal{D}_3(b) - \mathcal{D}_4(b).\end{aligned}$$

Note that if $x_r < b$, then the big-O terms

$$O\left(\theta_{35}\cdot\frac{U(x_{r'})}{f''(x_{r'})^2}\left(\frac{1}{M(x_{r'})^3}+\frac{1}{x_{r'}-a}\right)\right) + O\left(\frac{2\theta_{36}U(x_{r'})}{f''(x_{r'})^{3/2}M(x_{r'})^2}\right),$$

also appear in lines (7.13) and (7.14), and these terms can be bound as before. However, if $x_r > b$, then these terms do not appear in (7.13) and (7.14), hence why we created the separate term $\mathcal{D}_4(b)$.

There are only three remaining difference in the error terms. The first is that, since we no longer have a term

$$\frac{g(b)e(f(b)-r'b)}{2\pi i(f'(b)-r')},$$

we have that

$$-\mathcal{D}_1(b) = g(b)e\left(f(b) + [\![f'(b)]\!]\, b\right) \psi\left(b, \langle f'(b) \rangle\right)$$

for $f''(b) < 1 - \|f'(b)\|$. Second, since we do not have any term of the form $h_{r'}(b)$, we may replace the $\|f'(b)\|$ in $\Delta_2^{(5)}(b)$ with $1 - \|f'(b)\|$. Third, since the term $|x_{r'} - \beta_{r'}|^{-3}$ does not exist, if $\bar{b} = x_{r'}$, then we may replace $\bar{b}$ by the largest number in the interval $[a, x_{r'})$ such that $f'(\bar{b})$ is an integer.

This completes the proof of the theorem.

8.3 Proof of Theorem 4.4

The proof of this theorem mostly entails following the proof of Theorem 4.1 and simply being more careful with how certain error terms arise.

In Section 7.1, we want to show that we can replace $O(\mathcal{D}_1^{(1)}(a)) + O(\Delta_1^{(1)}(a))$ with some constant c_1. But these terms simply serve as a bound for the sum

$$-\sum_{r=-\infty}^{\infty} \int_{a_r}^{a} g(x)e(f(x) - rx)\; dx,$$

which is a constant. (By assumption, The interval $[a, b]$ is large enough so that $M(a)$ is fixed.)

Likewise, the terms $O(\Delta_1^{(2)}(a) + \Delta_2^{(1)}(a))$ that appear in Section 7.2 arise from $\mathcal{E}(\beta_r)$, which is the error in our estimate of the integral $\int_a^{\beta_r} g(x)e(f(x) - rx)\; dx$ in Proposition 6.6. In particular, since the interval $[a, \beta_r]$ is fixed, the term $\mathcal{E}(\beta_r)$ may be replaced by a constant c_2.

The error terms from Section 7.3 arise from bounding the sum

$$\sum_{a<x_r<b} \left(\mathcal{E}_r(\alpha_r) + \mathcal{E}_r(\beta_r)\right), \tag{8.4}$$

where $\mathcal{E}_r(\alpha_r)$ (respectively, $\mathcal{E}_r(\beta_r)$) is the error from Proposition 6.6 applied with a (resp., b) and c in the statement of the proposition equal to α_r (resp., β_r) and x_r.

Consider first the $\mathcal{E}_r(\alpha_r)$ terms. Each such term is fixed. Thus we write

$$\sum_{a<x_r<b} \mathcal{E}_r(\alpha_r) = \left(\sum_{a<x_r<b'} - \sum_{b\le x_r<b'} \right) \mathcal{E}_r(\alpha_r).$$

Note that we have, from Proposition 6.6 itself, that

$$\frac{1}{2}|\mathcal{E}_r(\alpha_r)| \le \Delta_6^{(1)}(x_r) := \theta_{17}\frac{U(x_r)}{f''(x_r)^2(x_r-a_r)^3} + \frac{U(x_r)}{f''(x_r)^2M(x_r)^3}\left(\theta_{17}+\frac{1}{2}\theta_{19}\sqrt{f''(x_r)}M(x_r)\right).$$

Suppose we have an increasing sequence $\{b^{(i)}\}$ such that $\Delta_6^{(1)}(b^{(i)})$ tends to 0. If $i<j$, then we have that

$$\begin{aligned}
&\sum_{b^{(i)}\le x_r<b^{(j)}} |\mathcal{E}_r(\alpha_r)| \\
&\quad \ll \int_{b^{(i)}}^{b^{(j)}} \frac{U(x)}{f''(x)(x-a)^3}\left(\theta_{14}+\frac{\theta_{15}}{f''(x)M(x)}+\frac{\theta_{16}}{f''(x)(x-a)}\right)\,dx \\
&\qquad + \int_{b^{(i)}}^{b^{(j)}} \frac{U(x)}{f''(x)M(x)^3}\left(2\theta_{17}+\theta_{19}\sqrt{f''(x)}M(x)\right)\left(1+\frac{|U'(x)|}{U(x)\cdot f''(x)}+\frac{\theta_{20}+4|M'(x)|}{f''(x)M(x)}\right)\,dx \\
&\qquad + \Delta_6^{(1)}(b^{(i)}) + \Delta_6^{(1)}(b^{(j)}) \\
&\quad =: \Delta_7([b^{(i)},b^{(j)}]) + \frac{1}{2}\Delta_8([b^{(i)},b^{(j)}]) + \Delta_6^{(1)}(b^{(i)}) + \Delta_6^{(1)}(b^{(j)})
\end{aligned}$$

by Proposition 5.8. By the assumptions of the theorem, the two integrals here can be bounded by an arbitrarily small constant, provided i and j are taken sufficiently large. Thus the sum

$$\sum_{a<x_r} \mathcal{E}_r(\alpha_r),$$

with no right endpoint, converges by Cauchy's criterion, and therefore

$$\sum_{a<x_r<b} \mathcal{E}_r(\alpha_r) = c_3 + O\left(\Delta_6^{(1)}(b)\right) + O\left(\Delta_7([b,\infty))\right) + O\left(\frac{1}{2}\Delta_8([b,\infty])\right).$$

We must apply an additional idea to perform the same technique to the terms $\mathcal{E}_r(\beta_r)$. Unlike $\mathcal{E}_r(\alpha_r)$, some of $\mathcal{E}_r(\beta_r)$ may change value as b changes. Namely, if $x_r+M(x_r)>b_r$ (i.e., $x_r\in K_b'$), then β_r changes as b grows. Let $\mathcal{E}_r^*(\beta_r)$ denote the corresponding value once b is larger than $x_r+M(x_r)$. Then we have

$$\sum_{a<x_r<b} \mathcal{E}_r(\beta_r) = \sum_{a<x_r<b} \mathcal{E}_r^*(\beta_r) + \sum_{x_r\in K_b} \left(\mathcal{E}_r(\beta_r)-\mathcal{E}_r^*(\beta_r)\right).$$

As above, we have that

$$\begin{aligned}
&\sum_{a<x_r<b} \mathcal{E}_r^*(\beta_r) \\
&\quad = c_4 + O\left(\frac{U(b)}{f''(b)^2M(b)^3}\left(\theta_{17}+\frac{1}{2}\theta_{19}\sqrt{f''(b)}M(b)\right)\right)
\end{aligned}$$

$$+ O\left(\int_b^\infty \frac{U(x)}{f''(x)M(x)^3}\left(2\theta_{17}+\theta_{19}\sqrt{f''(x)}M(x)\right)\left(1+\frac{|U'(x)|}{U(x)\cdot f''(x)}+\frac{\frac{5}{2}C_3+4|M'(x)|}{f''(x)M(x)}\right)\,dx\right)$$
$$= c_4 + O\left(\Delta_6^{(2)}(b)\right) + O\left(\frac{1}{2}\Delta_8([b,\infty))\right).$$

Moreover, we have

$$\sum_{x_r\in K_b} |\mathcal{E}_r(\beta_r)| + |\mathcal{E}_r^*(\beta_r)|$$
$$\le \int_{K_b'} \frac{U(x)}{f''(x)(b-x)^3}\left(\theta_{14}+\frac{\theta_{15}}{f''(x)M(x)}+\frac{\theta_{16}}{M(x)(b-x)}\right)\,dx$$
$$+ \sum_{x\in\partial K_b'\cup K_b'^*} \frac{D_0}{2\pi^2(1-\delta)^3}\cdot\frac{U(x)}{f''(x)^2(b-x)^3} + \Delta_2^{(2)}(b)$$
$$+ 2\int_{K_b} \frac{U(x)}{f''(x)M(x)^3}\left(2\theta_{17}+\theta_{19}\sqrt{f''(x)}M(x)\right)\left(1+\frac{|U'(x)|}{U(x)\cdot f''(x)}+\frac{\theta_{20}+4|M'(x)|}{f''(x)M(x)}\right)\,dx$$
$$+ 2\sum_{x\in\partial K_b\cup K_b^*} \frac{U(x)}{f''(x)^2M(x)^3}\left(2\theta_{17}+\theta_{19}\sqrt{f''(x)}M(x)\right)$$
$$=: \Delta_2^{(2)}(b) + \Delta_9(b).$$

The coefficient of 2 in the last sum and integral is an overestimation. We could replace $2\int_{K_b}$ with $\int_{K_b}+\int_{K_b'}$, and likewise in the sum. (Note that $\Delta_2^{(3)}(b)$ is implicitly contained in the last sum.)

In Section 7.4, the first sum on line (7.24) is constant for b sufficiently large, and the second sum is just $\mathcal{D}_1(b) + O(\Delta_2^{(4)}(b))$.

By following the details of Proposition 5.4 that we applied to S_3, we see that what we are left with at this point is

$$\sum_{f'(a)\le r\le f'(b)} \left(\int_{a_r}^{\alpha_r} + \int_{\beta_r}^{b_r}\right) h_r(x)e(f(x)-rx)\,dx \tag{8.5}$$
$$+ \sum_{r<f'(a)\text{ or }r>f'(b)} \int_{a_r}^{b_r} h_r(x)e(f(x))-rx)\,dx, \tag{8.6}$$

where in the latter sum we assume we are taking the limit as R tends to infinity of the sum with the additional restriction that $|r|\le R$. Let us, for ease of notation, rewrite the integrals as

$$\int_{[a_r,b_r]\setminus I_{x_r}} h_r(x)e(f(x)-rx)\,dx,$$

where as in condition (M) the set I_{x_r} equals $[x-M(x_r), x+M(x_r)]$.

Again, let us consider a (possibly different) increasing sequence $\{b^{(i)}\}$ such that $\Delta_2(b^{(i)})$ tends to 0 and so that $b^{(i+1)} - b^{(i)} \le \max\{M(b^{(i)}), M(b^{(i+1)})\}$. Then for $i < j$ we have

$$\left| \lim_{R\to\infty} \sum_{|r|\le R} \int_{[b_r^i, b_r^j]\setminus I_{x_r}} \frac{h_r(x)}{2\pi i} e(f(x) - rx)\ dx \right|$$
$$\le \sum_{n=5}^{7} O\left(\Delta_2^{(n)}(b^{(i)}) + \Delta_2^{(n)}(b^{(j)})\right) + O\left(\Delta_4^{(2)}\left([b^{(i)}, b^{(j)}]\right)\right) + O\left(\Delta_5\left([b^{(i)}, b^{(j)}]\right)\right).$$

Recall that by assumption, the integrals and sums in Δ_4 and Δ_5 converge—and since they are positive, converge uniformly—as b tends to 0, so the corresponding terms above must be arbitrarily small for sufficiently large i and j.

Thus we have that the terms on (8.5) and (8.6) are equal to

$$c_5 + \sum_{n=5}^{7} O\left(\Delta_2^{(n)}(b)\right) + O\left(\Delta_4^{(2)}\left([b,\infty)\right)\right) + O\left(\Delta_5([b,\infty))\right).$$

Setting $c = c_1 + c_2 + c_3 + c_4 + c_5$ completes the proof of Theorem 4.4.

8.4 Proof of Theorem 4.5

In the proof of Theorem 4.1, we gave only a coarse estimate of the size of $-\mathcal{D}(b) + \mathcal{D}(a)$—which roughly are the first-order endpoint contributions at b and a—when f'' was large. In this case, we included error terms of the size $O(U(a)) + O(U(b))$, which are at least the size of the first and last term of our initial sum. These terms can be improved, but this requires additional work.

But recall, at the beginning of the proof of Theorem 4.1, we bounded a sum of the type[1]

$$\sum_{r=-\infty}^{\infty} \int_{a_r}^{a} g(x) e(f(x) - rx)\ dx \tag{8.7}$$

in line (7.2), which along with the sum in line (7.29) contributed to the $O(U(a))$ error in $\mathcal{D}(a)$. In bounding this sum, we exploited the fact that the first-order endpoint contributions at a should cancel significantly (for example, in our estimation of line (7.4)). We expect, however, that more should be true, that not only should the first-order endpoint contributions display cancellations, but the integrals themselves should also display cancellation near a.

As in the conditions of Theorem 4.5, assume that $M(a), f''(a) \ge 1$, let C and L be real numbers satisfying

$$f''(a)^{-1/2} \ll C < M(a) \qquad \text{and} \qquad \sqrt{f''(a)} \ll L < \min\{f''(a), C\cdot f''(a), f''(a) - \|a\|\},$$

[1]Since $a_r = a$ for almost all r, the sum is in fact finite.

and let $\epsilon := \langle a \rangle$ and $\epsilon' := \langle f'(a) \rangle$. We now define a value a'_r similarly to how we defined a_r on line (7.1).

$$a'_r := \begin{cases} a + C, & \text{if } 1 \leq f'(a) - \epsilon' - r \leq L, \\ a - C, & \text{if } -1 \geq f'(a) - \epsilon' - r \geq -L, \\ a, & \text{otherwise.} \end{cases}$$

If $a'_r \neq a$, then a'_r is between a and a_r.

We now break the original sum (8.7) into several pieces:

$$\sum_{1 \leq |f'(a) - \epsilon' - r| \leq L} \int_{a'_r}^{a} + \sum_{1 \leq |f'(a) - \epsilon' - r| \leq L} \int_{a_r}^{a'_r} + \sum_{|f'(a) - \epsilon' - r| \geq L} \int_{a_r}^{a} + O(\Delta_1(a))$$

where the last term comes from the term $r = f'(a) - \epsilon'$ as in (7.3). In the sequel, we shall refer to these three groups of integrals as Type I, Type II, and Type III integrals, respectively.

We will postpone fully estimating the Type I integrals for now, and instead modify them into a more symmetric form. We first replace $g(x)$ by

$$g(a) + O\left(\max_{z \in [a'_r, a]} g'(z) \cdot |x - a| \right)$$

and $f(x) - rx$ by

$$(f(a) - ra) + (f'(a) - r)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + O\left(\frac{\max_{z \in [a'_r, a]} f^{(3)}(z)}{3!} |x - a|^3 \right).$$

Doing so gives

$$\begin{aligned}
&\int_{a'_r}^{a} g(x) e(f(x) - rx)\ dx \\
&\quad = g(a) \int_{a'_r}^{a} e(f(x) - rx)\ dx + O\left(\frac{U(a)}{M(a)} \int_{a'_r}^{a} |x - a|\ dx \right) \\
&\quad = g(a) \int_{a'_r}^{a} e(f(x) - rx)\ dx + O\left(\frac{U(a) C^2}{M(a)} \right) \\
&\quad = g(a) \int_{a'_r}^{a} e\left((f(a) - ra) + (f'(a) - r)(x - a) + \frac{1}{2} f''(a)(x - a)^2 \right) \left(1 + O\left(\frac{f''(a)}{M(a)} |x - a|^3 \right) \right)\ dx \\
&\qquad + O\left(\frac{U(a) C^2}{M(a)} \right) \\
&\quad = g(a) \int_{a'_r}^{a} e\left((f(a) - ra) + (f'(a) - r)(x - a) + \frac{1}{2} f''(a)(x - a)^2 \right)\ dx + O\left(\frac{U(a) f''(a) C^4}{M(a)} \right).
\end{aligned}$$

Shifting the bounds of integration and sending $f'(a) - r$ to $r + \epsilon'$, the Type I integrals can be rewritten as

$$\begin{aligned}
&\sum_{1 \le |f'(a)-\epsilon'-r| \le L} \int_{a'_r}^{a} g(x) e(f(x) - rx)\ dx \\
&\quad = g(a) e(f(a) + (\epsilon' - f'(a))a) \sum_{1 \le |r| \le L} \int_{a'_r - a}^{0} e\left(\frac{f''(a)}{2} x^2 + rx + ra + \epsilon' x\right)\ dx \\
&\quad\quad + O\left(\frac{U(a) f''(a) C^4 L}{M(a)}\right).
\end{aligned}$$

The Type II and Type III integrals we evaluate using Proposition 5.6. This gives us

$$\left(\sum_{1 \le |f'(a)-\epsilon'-r| \le L} \int_{a_r}^{a'_r} + \sum_{|f'(a)-\epsilon'-r| \ge L} \int_{a_r}^{a}\right) g(x) e(f(x) - rx)\ dx$$

$$= \sum_{|f'(a)-\epsilon'-r| < L} \frac{g(a'_r)}{2\pi i (f'(a'_r) - r)} e(f(a'_r) - r a'_r) \tag{8.8}$$

$$+ \sum_{\substack{L \le |f'(a)-\epsilon'-r| \\ |f'(a)-r| \le f''(a)}} \frac{g(a)}{2\pi i (f'(a) - r)} e(f(a) - ra)$$

$$+ O\left(\sum_{|f'(a)-\epsilon'-r| \le L} \left(\frac{U(a)}{M(a)(f'(a'_r) - r)^2} + \frac{U(a) f''(a)}{|f'(a'_r) - r|^3}\right)\right) \tag{8.9}$$

$$+ O\left(\sum_{\substack{L \le |f'(a)-\epsilon'-r| \\ |f'(a)-r| \le f''(a)}} \left(\frac{U(a)}{M(a) f''(a)} + \frac{U(a)}{M(a)^3 f''(a)^2}\right)\right) \tag{8.10}$$

$$+ O\left(\sum_{L \le |f'(a)-\epsilon'-r|} \left(\frac{U(a)}{M(a)(f'(a) - r)^2} + \frac{U(a) f''(a)}{|f'(a) - r|^3}\right)\right). \tag{8.11}$$

Similar to Proposition 5.5, we have that

$$f'(a \pm c) = f'(a) \pm f''(a) \cdot C \left(1 + O\left(\frac{\eta \cdot C}{2 \cdot M(a)}\right)\right)$$

with implicit constant 1, so that $|f'(a'_r) - r| \gg f''(a) C$ for $1 \le |f'(a) - \epsilon' - r| \le L$. Thus, the sum in line (8.8) is bounded by

$$\sum_{|f'(a)-\epsilon'-r| < L} O\left(\frac{U(a)}{f''(a) C}\right) = O\left(\frac{U(a) L}{f''(a) C}\right),$$

and the sum in line (8.9) is bounded by

$$O\left(\sum_{1\le r}\frac{U(a)}{M(a)(f''(a)C+r)^2}+\frac{U(a)f''(a)}{(f''(a)C+r)^3}\right)=O\left(\frac{U(a)}{M(a)f''(a)C}+\frac{U(a)}{f''(a)C^2}\right)$$
$$=O\left(\frac{U(a)}{f''(a)C^2}\right),$$

since $C\le M(a)$. The terms on line (8.10) are bounded by

$$O\left(\frac{U(a)}{M(a)}+\frac{U(a)}{f''(a)M(a)^3}\right)=O\left(\frac{U(a)}{M(a)}\right).$$

Finally, we apply Euler-Maclaurin summation to the sum on line (8.11) to bound it by

$$O\left(\frac{U(a)}{M(a)L}+\frac{U(a)f''(a)}{L^2}\right)=O\left(\frac{U(a)f''(a)}{L^2}\right),$$

since L is at most $f''(a)$.

Using all of these estimates for the integrals on line (8.7) (with the negative sign now, as they appeared in the proof of Theorem 4.1) and adding the terms from line (7.29), we obtain the following:

$$-\sum_{r=-\infty}^{\infty}\int_{a_r}^{a}g(x)e(f(x)-rx)\,dx-\lim_{R\to\infty}\sum_{\substack{|r|\le R\\|f'(a)-r|>f''(a)}}\frac{g(a)e(f(a)-ra)}{2\pi i(f'(a)-ra)}$$
$$=-g(a)e\left(f(a)+(\epsilon'-f'(a))a\right)\sum_{1\le|r|\le L}\int_{c\cdot\mathrm{sgn}(r)}^{0}e\left(\frac{f''(a)}{2}x^2+rx+ra+\epsilon'x\right)\,dx$$
$$-\lim_{R\to\infty}\sum_{\substack{|r|\le R\\L\le|f'(a)-\epsilon'-r|}}\frac{g(a)}{2\pi i(f'(a)-r)}e(f(a)-ra) \tag{8.12}$$
$$+O\left(\frac{U(a)f''(a)C^4L}{M(a)}+\frac{U(a)L}{f''(a)c}+\frac{U(a)f''(a)}{L^2}+\frac{U(a)}{f''(a)C^2}+\frac{U(a)}{M(a)}\right). \tag{8.13}$$

Now we estimate the Type I integrals and the sum on line (8.12) in three separate cases.

8.4.1 Case 1: a is an integer.

In this case, we begin by adding the r and $-r$ terms from the Type I integrals together, obtaining

$$-g(a)e(f(a)+(\epsilon'-f'(a))a)\sum_{1\le|r|\le L}\int_{C\cdot\mathrm{sgn}(r)}^{0}e\left(\frac{f''(a)}{2}x^2+rx+ra+\epsilon'x\right)\,dx$$

$$= -g(a)e(f(a)) \sum_{1\le r\le L} \int_C^0 e\left(\frac{f''(a)}{2}x^2 + rx\right) \cdot 2i\sin(2\pi\epsilon' x)\ dx. \tag{8.14}$$

We estimate each integral in (8.14) by an application of integration by parts.

$$\int_C^0 e\left(\frac{f''(a)}{2}x^2 + rx\right) \cdot 2i\sin(2\pi\epsilon x)\ dx = \frac{\sin(2\pi\epsilon' x)}{\pi(f''(a)x+r)} e\left(\frac{f''(a)}{2}x^2 + rx\right)\Bigg]_C^0 \tag{8.15}$$

$$+ \int_0^C \frac{2\epsilon'\cos(2\pi\epsilon' x)}{f''(a)x+r} e\left(\frac{f''(a)}{2}x^2 + rx\right)\ dx \tag{8.16}$$

$$- \int_0^C \frac{f''(a)\sin(2\pi\epsilon' x)}{\pi(f''(a)x+r)^2} e\left(\frac{f''(a)}{2}x^2 + rx\right)\ dx. \tag{8.17}$$

The term

$$\frac{\sin(2\pi\epsilon' x)}{\pi(f''(a)x+r)}$$

on line (8.15) is 0 when $x = 0$, and bounded by $O(\epsilon'/f''(a))$ if $x = C$.

We will then apply the first and second derivative tests (Lemmas 5.1 and 5.2) to the integral on line (8.16). The total variation may be bounded by

$$\begin{aligned}\int_0^C \left|\left(\frac{2\epsilon'\cos(2\pi\epsilon' x)}{f''(a)x+r}\right)'\right|\ dx &= O\left(\int_0^C \left|\frac{\epsilon'^2\sin(2\pi\epsilon' x)}{f''(a)x+r}\right|\ dx\right) + O\left(\int_0^C \left|\frac{\epsilon' f''(a)\cos(2\pi\epsilon' x)}{(f''(a)x+r)^2}\right|\ dx\right)\\ &= O\left(\frac{\epsilon'^2}{r}\int_0^C |\sin(2\pi\epsilon' x)|\ dx\right) + O\left(|\epsilon'|\int_0^C \frac{f''(a)}{(f''(a)x+r)^2}\ dx\right)\\ &= O\left(\frac{|\epsilon'|(1+|\epsilon'|C)}{r}\right) + O\left(\frac{|\epsilon'|}{r}\right).\end{aligned}$$

Since $|\cos(2\pi\epsilon x)| \le 1$ and $f''(a)x + r$ is at least r for $x \in [0, C]$, the maximum modulus on the interval is at most $|\epsilon'|/r$. Therefore the integral on line (8.16) is bounded by

$$O\left(\frac{|\epsilon'|(1+|\epsilon'|C)}{r}\min\left\{\frac{1}{\sqrt{f''(a)}}, \frac{1}{r}\right\}\right).$$

We will also apply the first and second derivative tests to the integral on line (8.17). Using $\sin(2\pi\epsilon' x) = 2\pi\epsilon' x + O(\epsilon^3 x^3)$, we have that

$$\begin{aligned}&\int_0^C \frac{f''(a)\sin(2\pi\epsilon' x)}{\pi(f''(a)x+r)^2} e\left(\frac{f''(a)}{2}x^2 + rx\right)\ dx\\ &= \int_0^C \frac{2\epsilon' f''(a)x}{(f''(a)x+r)^2} e\left(\frac{f''(a)}{2}x^2 + rx\right)\ dx + O\left(\int_0^C \frac{f''(a)\epsilon'^3 x^3}{r^2}\ dx\right)\\ &= O\left(\frac{|\epsilon'|}{r}\min\left\{\frac{1}{\sqrt{f''(a)}}, \frac{1}{r}\right\}\right) + O\left(\frac{f''(a)|\epsilon'|^3 C^4}{r^2}\right).\end{aligned}$$

Here we used that the function $ax/(ax+b)^2$ is 0 at $x=0$, increases monotonically to a maximum at $x=b/a$, and then decreases mononotonically to 0 as x tends to infinity.

Thus the Type I integrals (8.14) are bounded by

$$g(a)\left(\sum_{1\le r\le L} O\left(\frac{|\epsilon'|}{f''(a)}\right)+\sum_{1\le r\le L} O\left(\frac{|\epsilon'|(1+|\epsilon'|C)}{r}\min\left\{\frac{1}{\sqrt{f''(a)}},\frac{1}{r}\right\}\right)+\sum_{1\le r\le L} O\left(\frac{f''(a)|\epsilon'|^3C^4}{r^2}\right)\right)$$
$$=O\left(\frac{U(a)|\epsilon'|L}{f''(a)}\right)+O\left(\frac{U(a)|\epsilon'|(1+|\epsilon'|C)\log(1+f''(a))}{\sqrt{f''(a)}}\right)+O\left(U(a)f''(a)|\epsilon'|^3C^4\right).$$

Also, we bound the sum in line (8.12) by

$$O\left(\frac{U(a)}{L}\right)=O\left(\frac{U(a)f''(a)}{L^2}\right),$$

using Proposition 5.10.

This completes the proof in this case.

8.4.2 Case 2: $f'(a)$ is an integer, a is close to an integer.

As in Case 1, we sum r and $-r$ terms and use the first derivative test, obtaining

$$g(a)e(f(a)+(\epsilon'-f'(a))a)\sum_{1\le|r|\le L}\int_{C\cdot\mathrm{sgn}(r)}^{0} e\left(\frac{f''(a)}{2}x^2+rx+ra+\epsilon'x\right)\,dx$$
$$=g(a)e(f(a)-f'(a)a)\sum_{1\le r\le L} 2i\sin(2\pi\epsilon r)\int_C^0 e\left(\frac{f''(a)}{2}x^2+rx\right)\,dx$$
$$=O\left(U(a)\sum_{1\le r\le L}|\epsilon|r\cdot\frac{1}{r}\right)$$
$$=O\left(U(a)|\epsilon|L\right).$$

By the same argument, we may complete the Fourier series in (8.12).

$$-\lim_{R\to\infty}\sum_{\substack{|r|\le R\\ L\le|f'(a)-r|}}\frac{g(a)}{2\pi i(f'(a)-r)}e(f(a)-ra)=-\frac{g(a)e(f(a)-f'(a)a)}{2\pi i}\lim_{R\to\infty}\sum_{|r|\le R}\frac{e(\epsilon r)}{r}$$
$$+O\left(U(a)\sum_{r<L}\frac{sin(2\pi\epsilon r)}{r}\right)$$
$$=\psi(a)g(a)e(f(a)-f'(a)a)+O\left(U(a)|\epsilon|L\right).$$

Together these complete the proof in this case.

8.4.3 Case 3: $f'(a)$ is an integer, a is far from an integer.

In particular, by saying a is far from an integer, we want $|\epsilon| > C$. This implicitly requires that $C < 1/2$.

In this case, we sum over all positive values of r separately from all negative values of r. Ignoring the constant

$$g(a)e(f(a) + (\epsilon' - f'(a))a)$$

for the moment, the sum over positive r values give

$$\begin{aligned}
&\sum_{1\le r\le L}\int_C^0 e\left(\frac{f''(a)}{2}x^2 + rx + ra + \epsilon' x\right)\,dx \\
&\quad= \int_C^0 e\left(\frac{f''(a)}{2}x^2\right)\sum_{1\le r\le L} e(r(x+a))\,dx \\
&\quad= \int_C^0 e\left(\frac{f''(a)}{2}x^2 + \frac{1}{2}x\right)\frac{i}{2\sin(\pi(x+a))}\,dx \qquad (8.18)\\
&\qquad - \int_C^0 e\left(\frac{f''(a)}{2}x^2 + \left(L+\frac{1}{2}\right)x\right)\frac{i}{2\sin(\pi(x+a))}\,dx. \qquad (8.19)
\end{aligned}$$

Here we will apply the second derivative test to the integrals in lines (8.18)–(8.19).

The variation plus maximum modulus in both integrals is bounded by

$$O\left(\frac{1}{(|\epsilon| - C)}\right),$$

so that both integrals, as well as the corresponding integrals for the sum over negative r, are bounded by

$$O\left(\frac{1}{(|\epsilon| - C)\sqrt{f''(a)}}\right).$$

In this case, again, we bound the terms in line (8.12) by $O(g(a)/L|\epsilon|)$. But since $\sqrt{f''(a)} \ll L$, this term is dominated by

$$O\left(\frac{1}{(|\epsilon| - C)\sqrt{f''(a)}}\right).$$

This completes the proof in this case.

8.4.4 Combined case analysis

When $\epsilon' = 0$ and $M(a) \leq f''(a)^7$, the following choices of L and C optimize the error terms from line (8.13):

$$L = f''(a)^{8/15} M(a)^{1/15} \qquad \text{and} \qquad C = f''(a)^{-2/5} M(a)^{1/5},$$

when $|\epsilon| \leq f''(a)^{-3/5} M(a)^{-1/5}$ or $f''(a)^{-2/5} M(a)^{1/5} \leq |\epsilon|$;

$$L = f''(a)^{1/3} |\epsilon|^{-1/3} \qquad \text{and} \qquad C = f''(a)^{-1} |\epsilon|^{-1},$$

when $f''(a)^{-3/5} M(a)^{-1/5} \leq |\epsilon| \leq f''(a)^{-1/2}$; and,

$$L = f''(a)^{2/3} |\epsilon|^{1/3} \qquad \text{and} \qquad C = \epsilon/2,$$

when $f''(a)^{-1/2} \leq |\epsilon| \leq f''(a)^{-2/5} M(a)^{1/5}$.

Inserting these values into the error terms for Cases 2 and 3 gives the result at the end of Theorem 4.5.

These methods work similarly at b in place of a.

Chapter 9

Proofs of simple corollaries

9.1 Proof of Corollary 1.4

Recall that we are attempting to bound the error term in

$$\sum_{N\le n\le \nu N}^{*} \left(\frac{\alpha}{n}\right)^{\frac{1}{2}} e\left(\frac{X}{\alpha}\left(\frac{n}{N}\right)^{\alpha}\right) = \sum_{M\le m\le \mu M}^{*} \left(\frac{\beta}{m}\right)^{\frac{1}{2}} e\left(\frac{1}{8} - \frac{X}{\beta}\left(\frac{m}{M}\right)^{\beta}\right) + \Delta,$$

where $X > 0$, $N \gg 1$, $M \gg 1$, $\alpha > 1$, $\nu > 1$, $1/\alpha + 1/\beta = 1$, $\mu^\beta = \nu^\alpha$, and $MN = X$.

We will be applying Theorem 4.1. Without loss of generality, we may assume $N \le \sqrt{X}$ (otherwise, $M \le \sqrt{X}$ and we would instead consider the right-hand side of (1.5)).

Note that $f(x)$ and $g(x)$ are both power functions, so, as noted at the end of Section 4.1, condition (M) holds with $M(x)$ equal to ϵx for some small $\epsilon > 0$ dependent on α and the implicit constant in $N \gg 1$ only, and with $U(x)$ equal to $|g(x)|$.

The benefit of our conditions $1 \ll N \le \sqrt{X}$ is that on the interval $[N, \nu N]$, the functions $f''(x)$ and $M(x)$ are both bounded below by some constant (depending on α and the implicit constant in $1 \ll N$). This immediately gives that the $\mathcal{D}(x)$, $\Delta_1(x)$, and $\Delta_2(x)$ terms are all $O(U(a) + U(b)) = O(N^{-1/2})$, since each summand of these three terms contains a factor of $U(x)$ divided by some positive power of $f''(x)$ and $M(x)$, which are bounded from below. (Note that $\|f'(x)\|$ terms only appear if $\|f'(x)\|$ is bigger than $f''(x)$ to begin with.)

For the Δ_3 terms, we need to consider $\overline{a}$ and $\overline{b}$. It might be quite difficult to calculate these values explicitly, but we only make the Δ_3 terms bigger by assuming $\overline{a}$ is as small as possible (that is, $\overline{a} = a + \min\{M(a), C_2^{-1}\}$) and $\overline{b}$ is as large as possible. Since $M(x) \gg 1$ on $[N, \nu N]$, we have that $\overline{a} - a \gg 1$ and $b - \overline{b} \gg 1$. Thus, by first applying the fact that $f''(x) \gg 1$ and $M(x) \gg 1$ and then that $\overline{a} - a \gg 1$, the integral in $\Delta_3(a)$ may be bounded as

$$\int_{\overline{a}}^{b} \frac{U(x)}{f''(x)(x-a)^3}\left(1 + \frac{1}{f''(x)M(x)} + \frac{1}{f''(x)(x-a)}\right)\, dx$$

$$
\begin{aligned}
&\ll U(a) \int_{\overline{a}}^{b} \left(\frac{1}{(x-a)^3} + \frac{1}{(x-a)^4} \right) \, dx \\
&\ll U(a) \left(\frac{1}{(\overline{a}-a)^2} + \frac{1}{(\overline{a}-a)^3} \right) \\
&\ll U(a),
\end{aligned}
$$

which is $O(N^{-1/2})$ again. A similar argument shows that the remaining terms in $\Delta_3(a)$ and all the terms in $\Delta_3(b)$ are $O(N^{-1/2})$ as well.

To nicely bound the integral in Δ_4, however, we must use the fact that $M(x)$ and $f''(x)$ are not only bounded below, but also somewhat large. In particular, we have

$$
\begin{aligned}
&\int_a^b \frac{U(x)}{f''(x)M(x)^3} \left(1 + \sqrt{f''(x)}M(x)\right) \left(1 + \frac{|U'(x)|}{U(x) \cdot f''(x)} + \frac{1 + |M'(x)|}{f''(x)M(x)}\right) \, dx \\
&\quad \ll \int_a^b \frac{U(x)}{M(x)} \, dx \\
&\quad \ll \int_N^\infty x^{-3/2} \, dx \\
&\quad \ll N^{-1/2}.
\end{aligned}
$$

The greatest complication comes in evaluating the remaining condition from Section 4.2 dealing with the functions G and H and the remaining error terms in Δ_5. In this case, the function g'' is never zero on the interval $J = [N(1-\epsilon), \nu N(1+\epsilon)]$. Hence J_0 and J_{null} are empty, so both $\mathcal{K}(J_0, W_0, r_0)$ and the final sum of Δ_5 are 0. This also shows that the functions f and g satisfy the remaining condition from Section 4.2.

Since f and g are power functions, we can verify with simple hand calculations that the functions H, G, W_+, and W_- are all power functions as well; that is, each is of the form $c_1 \cdot x^{c_2}$. In particular, the functions $W_\pm(x)$ are equal to

$$
c_{\alpha,\pm} \left(\frac{N^\alpha}{X} \right)^2 x^{1/2-2\alpha},
$$

where $c_{\alpha,\pm}$ is a constant depending on α and the sign of $\pm$. The functions $r_\pm(x)$ are equal to

$$
c'_{\alpha,\pm} \frac{X}{N^\alpha} x^{\alpha-6} + c''_{\alpha,\pm} \frac{X}{N^\alpha} x^{\alpha-1},
$$

with new constants dependent on α and the sign of $\pm$.

Also, $H(x)^2 - G(x)$ is a power function; so, $J_\pm$ is the interval $[N(1-\epsilon), \nu N(1+\epsilon)]$ (or else, if $H(x)^2 - G(x) < 0$ for all x, $J_\pm$ is empty, in which case the remaining $\mathcal{K}$ terms equal 0 and we are finished).

Returning our attention to the remaining error terms of Δ_5, first consider the integrals. We have

$$\begin{aligned}
&\int_{J_\pm} \big(|W_-(x)||r_-'(x)| + |W_+(x)||r_+'(x)|\big) + \big(|W_+'(x)| + |W_-'(x)|\big)\ dx \\
&\quad \ll \int_{N(1-\epsilon)}^{\nu N(1+\epsilon)} \frac{N^\alpha}{X} x^{-1/2-\alpha} + \frac{N^{2\alpha}}{X^2} x^{-1/2-2\alpha}\ dx \\
&\quad \ll \frac{N^\alpha}{X} \cdot N^{1/2-\alpha} + \frac{N^{2\alpha}}{X^2} \cdot N^{1/2-2\alpha} \\
&\quad \ll \frac{N^{1/2}}{X} \\
&\quad \le N^{-1/2}
\end{aligned}$$

The functions r_+' and r_-' have at most one zero on $[a,b]$, and since the functions W_+ and W_- are decreasing and since $J_\pm^*$ is empty, the sum in Δ_5 are bounded by

$$\ll |W_-(N(1-\epsilon))| + |W_+(N(1-\epsilon))| \ll \left(\frac{N^\alpha}{X}\right)^2 N^{1/2-2\alpha} \ll N^{-1/2}.$$

(We implicitly used that $N \gg 1$ and $M \gg 1$ implies $X \gg 1$.)

Since we assumed $N \le \sqrt{X}$, we have $N^{-1/2} \ge M^{-1/2}$. This completes the proof.

9.2 Proof of Corollary 1.5

Recall that the conditions are that $f(x)$ and $g(x)$ are real-valued functions, $f \in C^4[a,b]$ and $g \in C^2[a,b]$. Also,

$$f''(x) \gg T/M^2, \qquad f^{(2+r)}(x) \ll T/M^{2+r}, \qquad g^{(r)}(x) \ll U/M^r,$$

for $r = 0,1,2$ on $[a,b]$, for constants $T \gg 1$, $M = b-a \gg 1$, and U.

We will use Theorem 4.2.

First, we wish to remove from consideration the case when

$$0 < \|f'(\mu)\| < \sqrt{f''(\mu)} \tag{9.1}$$

for μ equal to a or b.

Suppose that not only is (9.1) satisfied for both a and b, but that $[\![f'(a)]\!] = [\![f'(b)]\!]$—that is, the nearest integer to $f'(a)$ and $f'(b)$ are the same. In this case, we have by the Mean Value Theorem that $f'(b) - f'(a) = (b-a)f''(\xi)$ for some $\xi \in [a,b]$. But by our assumption on the size of $\|f'(a)\|$ and $\|f'(b)\|$, we have that

$f'(b) - f'(a) \ll \sqrt{T}/M$. Combining these we have that $b - a \ll f''(\xi)^{-1}\sqrt{T}/M \ll M/\sqrt{T}$. Thus we have

$$\sum_{a \le n \le b}^{*} g(n)e(f(n)) = O\left(\frac{UM}{\sqrt{T}}\right).$$

We can add in the sum

$$\sum_{f'(a) \le r \le f'(b)}^{*} \frac{g(x_r)}{\sqrt{f''(x_r)}} e(f(x_r) - rx_r + 1/8)$$

to the right-hand side, since this sum by assumption has at most one term, which is of size $UM/\sqrt{T}$. Thus the corollary is proved in this case.

If (9.1) is satisfied for some endpoint μ, but $[\![f'(a)]\!]$ does not equal $[\![f'(b)]\!]$, then we may replace μ with μ', the closest value to μ inside the interval $[a, b]$ where $\|f'(\mu')\| = \sqrt{f''(\mu')}$ or $\|f'(\mu')\| = 0$; by the same argument as in the previous paragraph, doing this generates an error of size $O(UM/\sqrt{T})$. The only way we could not do this is if $[\![f'(a)]\!] + 1 = [\![f'(b)]\!]$ and $f''(x)$ is at least $1/4$ at some point x on $[a, b]$, in which case, $f'(b) - f'(a) \ll \sqrt{T}/M$, so that $b - a \ll M/\sqrt{T}$ and the corollary follows in this case by the same argument as in the last paragraph.

Thus it suffices now to prove Corollary 1.5 in the case where $\|f'(\mu)\| \ge \sqrt{f''(\mu)}$ or $\|f'(\mu)\| = 0$ for μ equal to a and b. Note that, by assumption, we have $m_\mu = 0$ in both these cases.

The conditions of Theorem 4.2 hold with $M(x) = b - a$ on $[a, b]$. The size of the constants in condition (M) occur in the bounds on the derivatives of f and g in terms of the constants T, M, and U, and $U(x)$ just equals U.

Now, we go through the error terms of Theorems 4.1 and 4.2. First, we have that

$$\mathcal{D}(\mu) + O(\Delta_1(\mu)) = \begin{cases} g(\mu)e(f(\mu) + [\![f'(\mu)]\!]\,\mu)\left(-\dfrac{1}{2\pi i\langle f'(\mu)\rangle} + \psi\left(\mu, \langle f'(\mu)\rangle\right)\right), & \text{if } \|f'(\mu)\| \ge \sqrt{f''(\mu)}, \\ \dfrac{g(\mu)f^{(3)}(\mu)e(f(\mu))}{6\pi i f''(\mu)^2} - \dfrac{g'(\mu)e(f(\mu))}{2\pi i f''(\mu)} + O\left(U\left(1 + \dfrac{M}{T^2}\right)\right), & \text{if } \|f'(x)\| = 0. \end{cases}$$

For the $\Delta_2'(x)$ terms, we use that $M(x) = b - a = M$ and $f''(x) \asymp T/M^2$ to obtain

$$\Delta_2'(\mu) \ll \frac{UM}{T^2}\left(1 + \sqrt{T}\right) + \frac{U}{M\|f'(\mu)\|^{*2}} + \frac{UT}{M^2\|f'(\mu)\|^{*3}} \ll \frac{UM}{T^{3/2}} + \frac{U}{M\|f'(\mu)\|^{*2}} + \frac{UT}{M^2\|f'(\mu)\|^{*3}}.$$

Here we have combined both cases $\|f'(\mu)\| = 0$ and $\|f'(\mu)\| \ge \sqrt{f''(\mu)}$ through the use of the notation $\|\cdot\|^*$. Recall that $\|x\|^*$ equals 1 if x is an integer and equals $\|x\|$ otherwise.

For the $\Delta_3(x)$ terms, we again need to understand $\overline{a}$ and $\overline{b}$. By the Mean Value Theorem, for some $\xi \in [a, b]$ we have $\overline{a} - a = (f'(\overline{a}) - f'(a))/f''(\xi) \gg \|f'(a)\|^* M^2/T$ since $f'(\overline{a})$ must be an integer.

Thus for $\Delta_3(a)$, we have the following bound:

$$
\begin{aligned}
\Delta_3(a) &\ll \int_{\overline{a}}^{b} \frac{U}{f''(x)(x-a)^3}\left(1+\frac{1}{f''(x)M(x)}+\frac{1}{f''(x)(x-a)}\right)\,dx+\frac{g(\overline{a})}{f''(\overline{a})^2(\overline{a}-a)^3}+\frac{g(b)}{f''(b)^2(b-a)^3}\\
&\ll \frac{UM^2}{T(\overline{a}-a)^2}\left(1+\frac{M}{T}+\frac{M^2}{T(\overline{a}-a)}\right)+\frac{UM^4}{T^2(\overline{a}-a)^3}+\frac{UM}{T^2}\\
&\ll \frac{UT}{M^2\|f'(a)\|^{*2}}\left(1+\frac{M}{T}+\frac{1}{\|f'(a)\|^{*}}\right)+\frac{UM}{T^2}\\
&\ll \frac{U}{M\|f'(a)\|^{*2}}\left(1+\frac{T}{M}\right)+\frac{UT}{M^2\|f'(a)\|^{*3}}+\frac{UM}{T^2}.
\end{aligned}
$$

A similar bound holds for $\Delta_3(b)$.

Now for the last term Δ_4, we have the following much simpler calculation, since $U(x)$ and $M(x)$ are constant:

$$
\begin{aligned}
\Delta_4 &= \int_a^b \frac{U}{f''(x)M(x)^3}\left(1+\sqrt{f''(x)}M(x)\right)\left(1+\frac{1}{f''(x)M(x)}\right)\,dx\\
&\ll \int_a^b \frac{U}{TM}\left(1+\sqrt{T}\right)\left(1+\frac{M}{T}\right)\,dx\\
&\ll \frac{U}{\sqrt{T}}\left(1+\frac{M}{T}\right).
\end{aligned}
$$

This completes the proof of Corollary 1.5.

9.3 Proof of Corollary 1.10

Recall that we are attempting to bound the error in the following van der Corput transform.

$$
\sideset{}{^*}\sum_{1\le n\le N} e\left((n/3)^{3/2}\right) = \sideset{}{^*}\sum_{(1/12)^{1/2}\le r\le (N/12)^{1/2}} \sqrt{24r}\cdot e(-4r^3+1/8)+\Delta, \tag{9.2}
$$

with N a positive integer.

9.3.1 Calculating the function $M(x)$

In our case we have

$$
\begin{aligned}
f''(x) &= \frac{1}{2^2\cdot 3}\left(\frac{x}{3}\right)^{-1/2}, & f^{(3)}(x) &= -\frac{1}{2^3\cdot 3^2}\left(\frac{x}{3}\right)^{-3/2}, & f^{(4)}(x) &= -\frac{1}{2^4\cdot 3^2}\left(\frac{x}{3}\right)^{-5/2},\\
g(x) &= 1, & g'(x) &= 0, & g''(x) &= 0.
\end{aligned}
$$

In order to guarantee that $\delta < 1$ and $\eta \geq 1$, it is easiest to presume specific values for δ, C_{2^-}, C_2, and C_3 first and calculate $M(x)$ after. In particular, we let $\delta = (2/3)^{1/5}$ and $C_{2^-} = C_2 = C_3 = (3/2)^{1/5}$: this will guarantee that $\eta = 1$. Therefore, condition (M) part (I) holds.

We claim that the requisite bounds on $f''(z)$ and $f^{(3)}(z)$, for $z \in I_x$, hold for $M(x) = \frac{1}{7}x$. (Note that in this case we would have $I_x = [6x/7, 8x/7]$ for all x.) In particular, we have that

$$\max_{z \in I_x} f''(z) = f''\left(x - \frac{1}{7}x\right) = \frac{1}{\left(1-\frac{1}{7}\right)^{1/2}} \cdot \frac{1}{2^2 \cdot 3}\left(\frac{x}{3}\right)^{-1/2} = \frac{1}{\left(1-\frac{1}{7}\right)^{1/2}} \cdot f''(x) \leq C_2 \cdot f''(x).$$

Likewise,

$$\min_{z \in I_x} f''(z) = \frac{1}{\left(1+\frac{1}{7}\right)^{1/2}} \cdot f''(x) \geq \frac{1}{C_{2^-}} \cdot f''(x)$$
$$\max_{z \in I_x} f^{(3)}(z) = \frac{1}{14\left(1-\frac{1}{7}\right)^{3/2}} \cdot \frac{f''(x)}{M(x)} \leq C_3 \cdot \frac{f''(x)}{M(x)}.$$

One can also show that the remaining inequalities in condition (M) part (IV) hold with $U(x) = 1$, $D_0 = 1$, $D_1 = D_2 = 0$, and $C_4 = 3/100$.

Condition (M) part (II) clearly holds as soon as $N \geq 7/6$, since both $M(1) = 1/7$ and $M(N) = N/7$ are less than $N - 1$. (The statement of the corollary is trivially true for $1 \leq N < 7/6$, so we will assume $N \geq 7/6$ for the rest of the proof.)

Condition (M) part (III) also holds, since f and g are in fact $C^\infty[(0,\infty)]$.

9.3.2 The additional assumptions

First, we note that, as needed, f'' is a positive-valued function bounded away from 0 on any interval $[1, N]$. (It is not uniformly bounded for all N, but this is not necessary.) The function g is trivially real-valued.

We also have that

$$H(x) = f^{(3)}(x) = -\frac{1}{2^3 \cdot 3^2}\left(\frac{x}{3}\right)^{-3/2} \qquad \text{and} \qquad G(x) = 0.$$

By definition we have that $J_{\text{null}} = J_\pm = \varnothing$ and $J_0 = J$. Clearly $g(x)$ does not tend to 0 anywhere, so the conditions of Section 4.2 hold.

Note that we also have

$$W_0(x) = \frac{2}{9x^2}, \qquad r_0(x) = 2\sqrt{\frac{x}{3}}.$$

9.3.3 The proof proper

We will apply Theorem 4.4 together with the alternate $\mathcal{D}$ term, given by Theorem 4.3. We shall presume for now that $0 < |r' - f'(N)| < 1/2$ and consider the remaining cases at the end of the proof, using continuity.

Let us consider the additional conditions of Theorem 4.3. Here,

$$r' = \left[\left[(N/12)^{1/2}\right]\right],$$

where $[\![\cdot]\!]$ notation again denotes the nearest integer function. Thus we have that

$$x_{r'} = f'^{-1}(r') \geq 12\left(\frac{1}{2}\left(\frac{N}{3}\right)^{1/2} - \frac{1}{2}\right)^2 = N - 6\left(\frac{N}{3}\right)^{1/2} + 3$$

and

$$x_{r'} \leq N + 6\left(\frac{N}{3}\right)^{1/2} + 3.$$

Therefore, we have

$$|x_{r'} - N| \leq 6\left(\frac{N}{3}\right)^{1/2} + 3.$$

This implies, first, that $x_{r'} \asymp N$ and second that $|x_{r'} - N| \ll x_{r'}^{1/2}$ and thus is clearly less than $M(x_{r'})$ for sufficiently large N. Since we showed that condition (M) holds on all intervals $[1, N]$, all the conditions of Theorem 4.3 hold.

Now we consider the conditions of Theorem 4.4. First consider $\Delta_2(x)$. Recall that in Theorem 4.3, we replaced all incidences of $\|f'(x)\|$ with $1 - \|f'(x)\|$. Also, we have $m_x \ll 1$. Therefore, we have

$$\begin{aligned}\Delta_2(x) &\ll \frac{U(x)}{f''(x)^2 M(x)^3}\left(1 + \sqrt{f''(x)}M(x) + f''(x)M(x)^2\right) + \frac{U(x)}{M(x)} + U(x)f''(x)\\ &\ll x^{-2}(1 + x^{3/4} + x^{3/2}) + x^{-1} + x^{-1/2}\\ &\ll x^{-1/2}.\end{aligned}$$

We can clearly find an increasing sequence $\{b^{(i)}\}$ for with the necessary conditions for which $\Delta_2(b^{(i)})$ tends to 0.

Likewise we have

$$\begin{aligned}\Delta_6(x) &\ll \frac{U(x)}{f''(x)^2(x-1)^3} + \frac{U(x)}{f''(x)^2 M(x)^3}(1 + \sqrt{f''(x)}M(x))\\ &\ll x^{-2} + x^{-2}(1 + x^{3/4})\end{aligned}$$

$$\ll x^{-5/4},$$

and again we can find an increasing sequence with the desired properties.

We shall reserve the proof of convergence of the remaining terms for later.

Now let us consider the error terms. First we have $\mathcal{D}'(b)$. For all $N \geq 1$, we have $f''(N) < 1/2$, so that

$$\begin{aligned}\mathcal{D}'_1(N) &= e\left(f(N) + [\![f'(N)]\!]\, N\right) \psi\left(N, \langle f'(N)\rangle\right)\\ &= e\left((N/3)^{3/2}\right) \psi\left(N, \left\langle (N/12)^{1/2}\right\rangle\right).\end{aligned}$$

Here we used that N and $[\![f'(N)]\!]$ are both integers. We also have

$$\begin{aligned}\mathcal{D}'_3(N) &= \operatorname{sgn}(N - x_{r'}) e\left(f(x_{r'}) - r'x_{r'}\right) \int_{\phi_{r'}(N-x_{r'})}^{+\infty} e\left(\frac{1}{2} f''(x_{r'}) y^2\right) \, dy\\ &\quad + O\left(\frac{f^{(3)}(x_{r'})}{f''(x_{r'})^2}\right) + O\left(\frac{1}{f''(x_{r'})M(x_{r'})^2} \cdot |b - x_{r'}|\right)\\ &= \operatorname{sgn}\left(N - N'\right) e\left((N'/3)^{3/2}\right) \int_{\phi_{r'}(N-N')}^{+\infty} e\left(\frac{1}{24}\left(\frac{N'}{3}\right)^{-1/2} y^2\right) \, dy\\ &\quad + O(N^{-1/2}),\end{aligned}$$

where

$$N' = f'^{-1}(x_{r'}) = 12 \left[\!\!\left[(N/12)^{1/2}\right]\!\!\right]^2.$$

Here, we used the facts that $x_{r'} \asymp N$, that $|b - x_{r'}| \leq M(x_{r'})$, and that $x_{r'}$ and $f'^{-1}(x_{r'})$ are both integers. Finally we have

$$\begin{aligned}\mathcal{D}'_4(N) &\ll \frac{1}{f''(x_{r'})^2 M(x_{r'})^3}\left(1 + \sqrt{f''(x_{r'})}M(x_{r'})\right) + \frac{1}{f''(x_{r'})^2 (x_{r'} - 1)^3}\\ &\ll x_{r'}^{-2}(1 + x_{r'}^{3/4}) + x_{r'}^{-2}\\ &\ll N^{-1/2}.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}-\mathcal{D}'(N) &= \operatorname{sgn}\left(N' - N\right) e\left((N'/3)^{3/2}\right) \int_{\phi(N-N')}^{+\infty} e\left(\frac{1}{24}\left(\frac{N'}{3}\right)^{-1/2} y^2\right) \, dy\\ &\quad - e\left((N/3)^{3/2}\right) \psi\left(N, \left\langle (N/12)^{1/2}\right\rangle\right) + O(N^{-1/2}).\end{aligned}$$

Since $f''(x) < 1$ for sufficiently large x, we have that m_N is at most 1 for sufficiently large N, and since $f''(x) < M(x)$ for sufficiently large, we have that $\Delta_1(N) = 0$ for sufficiently large N.

As noted above $\Delta_2(N) \ll N^{-1/2}$.

For arbitrary $1 \leq a < b$, we have

$$\begin{aligned}
\Delta_4([a,b]) &\ll \int_a^b \frac{1}{f''(x)M(x)^3}\left(1+\sqrt{f''(x)}M(x)\right)\left(1+\frac{1+|M'(x)|}{f''(x)M(x)}\right)\,dx \\
&\ll \int_a^b x^{-5/2}(1+x^{3/4})(1+x^{-1/2})\,dx \\
&\ll \int_a^b x^{-7/4}\,dx \\
&\ll a^{-3/4},
\end{aligned}$$

and this clearly converges as b tends to infinity. Thus $\Delta_8([N,\infty)) = N^{-3/4}$.

For arbitrary $1 \leq a < b$, we have

$$\begin{aligned}
\Delta_5([a,b]) &\ll \int_{6a/7}^{8b/7} \left(|W_0(x)|\cdot|r_0'(x)| + |W_0'(x)|\right)\,dx + 2|W(6a/7)| + 2|W(8b/7)| \\
&\ll \int_{6a/7}^{8b/7} (x^{-5/2} + x^{-3})\,dx + a^{-2} \\
&\ll a^{-3/2}.
\end{aligned}$$

Here we used the fact that $r_0'(x)$ does not change sign on $(0,\infty)$. This again converges as b tends to infinity, so $\Delta_5([N,\infty)) \ll N^{-3/2}$.

As noted above $\Delta_6(N) \ll N^{-5/4}$.

We have

$$\begin{aligned}
\Delta_7([N,\infty)) &\ll \int_N^\infty \frac{1}{f''(x)(x-1)^3}\left(1+\frac{1}{f''(x)M(x)}+\frac{1}{f''(x)(x-a)}\right)\,dx \\
&\ll \int_N^\infty x^{-2}(1+x^{-1/2})\,dx \\
&\ll N^{-1}.
\end{aligned}$$

Finally, for Δ_9 we need to consider the set K_N. For large enough N, we have that $K_N = [7N/8, N]$. We will not calculate $\overline{N}$ explicitly, but let us consider for the moment that it is greater than $7N/8$ as otherwise the bound of the first integral and sum in Δ_9 is zero. Therefore, we have

$$\begin{aligned}
\int_{7N/8}^{\overline{N}} & \frac{1}{f''(x)(N-x)^3}\left(1+\frac{1}{f''(x)M(x)}+\frac{1}{M(x)(N-x)}\right)\,dx \\
&+\frac{1}{f''(7N/8)^2(N-7N/8)^3}+\frac{1}{f''(\overline{N})^2(N-\overline{N})^3} \\
&\ll \int_{7N/8}^{\overline{N}} \frac{1}{N^{-1/2}(N-x)^3}\left(1+N^{-1/2}+\frac{1}{N(N-x)}\right)\,dx+N^{-2}+\frac{N}{(N-\overline{N})^3} \\
&\ll \frac{N^{1/2}}{(N-\overline{N})^2}+\frac{N^{-1/2}}{(N-\overline{N})^3}+N^{-2}+\frac{N}{(N-\overline{N})^3}.
\end{aligned}$$

However, by the Mean Value Theorem, we have, for some $\eta \in [\overline{N}, N]$,

$$N-\overline{N}=\left(\left(\frac{N}{12}\right)^{1/2}-\left(\frac{\overline{N}}{12}\right)^{1/2}\right)\cdot 24\left(\frac{\eta}{12}\right)^{1/2} \gg N^{1/2},$$

since we have that $f'(N)-f'(\overline{N})$ is at least $1/2$. (If it were less than $1/2$ then $f'(\overline{N})=r'$, and hence $\overline{N}=x_{r'}$ contrary to hypothesis.) Therefore, we have

$$\begin{aligned}
\int_{7N/8}^{\overline{N}} & \frac{1}{f''(x)(N-x)^3}\left(1+\frac{1}{f''(x)M(x)}+\frac{1}{M(x)(N-x)}\right)\,dx \\
&+\frac{1}{f''(7N/8)^2(N-7N/8)^3}+\frac{1}{f''(\overline{N})^2(N-\overline{N})^3} \\
&\ll N^{-1/2}+N^{-2}+N^{-2}+N^{-1/2} \\
&\ll N^{-1/2}.
\end{aligned}$$

The remaining integral and sum in Δ_9 are bounded by $N^{-5/4}$ by our earlier calculation on Δ_4. This completes the proof in the case where $|r'-f'(N)|$ does not equal 0 or $1/2$. The case $|r'-f'(N)|=0$ can be seen to still hold since there is no integral term that arises in this case, and appropriately enough the signum function also vanishes. The case $|r'-f'(N)|=1/2$ can also be seen to hold by one-sided continuity (namely the direction that preserves continuity of $\langle f'(N)\rangle$).

Chapter 10

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