ANALYSIS OF INTERFACIAL FORCES ON THE PHYSICS OF TWO-PHASE FLOW AND HYPERBOLICITY OF THE TWO-FLUID MODEL

BY

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THESIS

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Abstract

The scope of this Master Thesis is the two-fluid model. This study aims at analysing their ill-posedness, showing how it is related to non-hyperbolicity and giving a measurement that I call the numerical hyperbolicity. As a cure to this important issue, we will propose a combination of physical and mathematical solutions. The physical forces that we will focus on are two interfacial forces: the interfacial pressure and the virtual mass. We will quantify the minimum correction that is necessary to ensure hyperbolicity of the two-fluid model. We will then compare the two forces and combine them into an aggregated correction. This correction is proposed for use as a criteria in the next generation of nuclear reactor thermal-hydraulics codes.

Acknowledgments

Dear reader,

Before we dive into pure science and the main content of this Master Thesis, I would like to take a brief moment to mention part of what happened to me in the USA when I was not using my beloved office coffee machine in Illinois. Therefore, if you are not that keen on so personal matters and are more interested in two-phase flows, please do not hesitate to skip to the main part of this work. Otherwise, thank you for joining me in this little incomplete retrospective.

I would like to seize this opportunity to give some of the reasons that explain why I decided, one day when I was still at the École Polytechnique in France, to leave for the University of Illinois at Urbana-Champaign. I still often joke about my initial intention to see the well-known "American Dream" from my own eyes. What is sure is that I had the clear intention to know more about the people living on the American continent and, actually, to learn a little bit more about myself. I could talk here about my many trips, make a comparison of lifestyles on different shores of the Atlantic Ocean, or perhaps build a theory on US politics, but instead I would like to try to give some focus on the encounters I had the opportunity to make. I acknowledge this may sound very cliché, but it is undoubtedly true that the United States is synonym with a strong sense of community, with diversity and with openness of the inhabitants.

The thesis you are presently reading was certainly a studious personal work. Nevertheless I do not forget that I benefited from great support from the Nuclear, Plasma, and Radiological Engineering department faculty. I am a little hesitant to cite names in this foreword because of the fear of not mentioning somebody. But it is clear to me that I am thinking of my research advisor Pr. Tomasz Kozlowski, my second reader Pr. Rizwan Uddin and my final reader and head of the department Pr. James F. Stubbins. They have all expressed valued understanding of my sometimes unconventional needs and have always been confident in my ability to deliver. I personally also associate Gail, Becky and Idell to the faculty because of their help and friendliness in many cases. Working in the ARTS research group lead by Tomasz really was a pleasure, since it permitted me to become much closer to my friends Xu, Rabie, Rijan, Rui-Lin, Cem and Kuan-Che (no particularly important order). If I really wanted to make an insider joke, I would say that there exists a particular photo commemorating the spirit of the team, but I am of course not going to mention it, and I will rather keep that joke private.

I am forced to stop citing names from now on, because I would like to express my sincere gratitude towards the rather large group of friends I made in the United States. I spent extremely nice moments with two generations of the American Nuclear Society Student Chapter, with my church group the Grad Rosary, with the international community of the Urbana-Champaign campus (seemingly owning Murphy's), with the Zagloba Polish Club, with my courses classmates, with the members of different associations and in general with all the Undergrads, Grads, Professors and other non-students who shared with me part of their story. Many of those extraordinary memories were gathered in a rich "farewell poster". If you think recognize yourself in the previous paragraph, please do never hesitate to visit me one day wherever I live in the future.

You may wonder why all this matters so significantly. In fact, my journey and stay in America had a great influence on my knowing myself; for instance I am very thankful for all the spontaneous encounters which I made

and which made me reflect on myself. More simply, it also positively shaped how I imagine my future professional life. Participating in the ANS events in Nevada and California made me realize how strong, knowledgeable and welcoming the nuclear scientific community was. So I decided to go back to Europe, in order to pursue new challenges, and also because I was claimed back by family and old friends. I presently work in Germany for AREVA as a Research and Development Analyst, and starting in October 2013 I hope to start a PhD at the CEA, the French National Lab, in Saclay. I already met my future research advisor, Stéphane Dellacherie, and he agreed that I would set sail for the "Direct Numerical Simulation of bubbles in low Mach number conditions and with Adaptative Mesh Refinement". I have to admit that I am more and more excited to work together with him and my academic advisor, Grégoire Allaire.

I now start again in a new environment, with a new challenge, and will hopefully meet some new fascinating people. I definitely look back with joy at my American experience. I sincerely hope I have been able to share with others as much as they have shared with me. Everything I learned will be of great help in the future for new perspectives. I will remember all the people I met and I will hold them in my heart.

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Nomenclature

α	Void fraction
α_i	Volume fraction
ΔP	Interfacial pressure (coefficient)
ϵ	Small real number
η	Weight factor
γ	Calculation intermediary for simplification
λ_0	An eigenvalue of A_0
ΔP_c	Interfacial pressure as used in CATHARE
Δv	Relative velocity
μ	Calculation intermediary for simplification
ρ	Volumetric mass (or density) of fluid
σ	Surface tension
$\underline{\gamma}$	Ideal gas factor
$\vec{g}$	Gravitational acceleration
A	Matrix of dimension 4 representing the hyperbolicity problem
A_0	Matrix of dimension 4 representing the hyperbolicity problem in the two-fluid model in its commonly used form
AC	Aggregated correction
B	Matrix of dimension 4 computing $\frac{\partial U}{\partial t}$
C	Matrix of dimension 4 computing $\frac{\partial U}{\partial x}$
c	Speed of sound in the fluid
Cvm	Virtual mass coefficient
D	Dimension 4 matrix used in high order partial differential equations
e	Volumetric energy

e_{int}	Volumetric internal energy
F	Force applied on the system
f	Characteristic function of an equation of state: $f(p, e_{int}) = 0$
G	Calculation intermediary for simplification
H	Numerical hyperbolicity
L	Calculation intermediary for simplification
M	Matrix of dimension 4 in two-fluid model
N	Matrix of dimension 4 in source terms
p	Pressure
Q_{vol}	An arbitrary volumetric quantity
R	Calculation intermediary for simplification
S	Source term column vector
t	Time variable or date
U	Vector of the four variables p, α, u_g and u_l
u	Velocity of fluid
x	Space variable or position
z	Height
^{p}	As a superscript: relative to interfacial pressure
^{norm}	As a superscript: normalized
^{vm}	As a superscript: relative to virtual mass
$_{g}$	As a subscript: relative to the gaseous phase
$_{l}$	As a subscript: relative to the liquid phase
$_{t}$	As a subscript: relative to the time coordinate
$_{x}$	As a subscript: relative to the spatial coordinate
$_{optimal}$	As a subscript: calculated from a optimization problem

Summary

The study of two-phase flows is an essential discipline in Thermal-hydraulics Nuclear Engineering and is a core component of the design of next generation nuclear codes. Fundamentally, it relies on a few conservation laws and equations of state. With a view to simplifying the mathematical problem, we also have to choose a particular set of hypothesises. The assumptions that I will choose in this thesis lead to solving a system of Partial Differential Equations. The unknowns can be restricted to a family of only four, namely p (the pressure), α (the void fraction), u_g (the velocity of the gas) and u_l (the velocity of the liquid).

If we define the vector U as

$$U = \begin{pmatrix} p & \alpha & u_g & u_l \end{pmatrix}^t \tag{1}$$

then it can be shown that the problem can be written in a simple matrix form:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0 \tag{2}$$

The properties of matrix A have a large influence on how to solve the equation. A is said to be hyperbolic if all its eigenvalues are real. In that case, the equation with initial conditions is a well-posed Cauchy problem and has one existing and unique solution. In some specific cases, the model has one unique solution. However, by introducing H, a numerical measurement of the hyperbolicity, I will show that in most use-cases, the Cauchy problem is in fact ill-posed.

Because this is an important problem, the scientific community has searched for and found several different ways to make the two-fluid model be well-posed. Taking into account additional interfacial forces is a generally accepted way to make sure the problem is hyperbolic, so we will consider the interfacial pressure and the virtual mass force. This will modify the matrix A and its associated numerical hyperbolicity H. Nevertheless we will explain why current implementations are not as satisfying as they could be.

First with the interfacial pressure and then with the virtual mass, we will study what requirements those two interfacial forces should have, depending on the flow conditions p, α, u_g and u_l. We will then go further by finding a consistent way to compare the interfacial pressure and the virtual mass force; by normalizing their respective coefficients. This will permit me to combine the two corrections into what I will call the aggregated correction AC. This aggregated correction will be used to determine what the minimal correction should be to ensure hyperbolicity, also depending on (p, α, u_g, u_l).

In the end we will try to suggest ways of using the results of this work and examine what could be done in further work, with a view to including it into future thermal-hydraulics codes.

All the analytical results will be supported by consistent and comparable numerical examples, for pressures equal to those of a BWR reactor. The same results will be repeated in the appendices for atmospheric pressure and PWR reactor pressure. One may also find in the appendices some details of particular cases where the problem is already well-posed.

Chapter 1

Introduction to two-phase flow thermal-hydraulics

1.1 Physics of two-phase flows

As the science of the movement and transformation of fluids, Thermal-Hydraulics is a fundamental aspect of the broader topic of Nuclear Engineering. Water in particular, in all its different states, plays a core role for the nuclear engineer: water is the most used moderator and cooling fluid in today's nuclear power plants. It is responsible for appropriate neutronics and also for carrying the energy, which will later be transformed into energy.

Thermal-Hydraulics as a science uniquely merges the study of the mechanics of fluids and the study of their thermodynamic properties. Those two aspects have a large influence on each other. For instance, a warming fluid will see its density be reduced, so at constant speed the volumetric kinetic energy will decrease too. The equations that give a representation of these phenomenons are intrinsically coupled together.

Multi-phase flows are a broad subtopic of Thermal-Hydraulics. Those are flows of different fluids together. The fluids can obviously be different chemical species separated into different phases, like a mixture of water and oil. But they can also be different states of matter of the same chemical species, like liquid water and (gaseous) steam together. Sometimes, models even differentiate different phases depending on the physical structure and behavior: for water, for example, this would mean differentiating liquid water, steam, foam, fog, etc. Multi-phase flows can be found in a large amount of engineering disciplines, like in Aerospace Engineering or in the design process of oil refineries. However, Nuclear Engineering has the particularity of dealing with far larger amounts of fluid and energy.

In this study, we will focus on two-phase flows. The fluids will only be two and will typically be liquid water and steam. They will be referred as the liquid phase and the gaseous phase. This means that the results of this work are meant to be used primarily for the design of computational simulations of flows in nuclear power plants, that is to say for nuclear codes. As Boiling Water Reactors (BWRs) are, by design, reactors where liquid water is supposed to evaporate into steam, they are the perfect example of application. But this work may also apply to Pressurised Water Reactors (PWRs), where small and large bubbles do appear along the fuel rods. We will try to cover a complete range of pressures useful for the analysis of Loss Of Coolant Accidents (LOCAs), from atmospheric pressure to PWR pressure.

Liquid water and vapor mixtures can adopt different topologies. The easiest ones to think of are bubble flow (small bubbles in water) and its counterpart, droplets travelling through steam. When the fluids cohabit in comparable quantities, they can aggregate and form a slug flow. Other types of flow appear when there is an influence of the surrounding environment: churn flow, annular flow and wispy annular flow in a pipe for instance. Figure 1.1 shows a representation of these types of flow. To a very large extent, this present study is agnostic as far as the flow topology is concerned.

The physics of multiphase flows in general and two-phase flows in particular are studied in numerous books. For further reference, see for instance Kleinstreuer [11], Levy [12] or Crowe [2], to name a few.

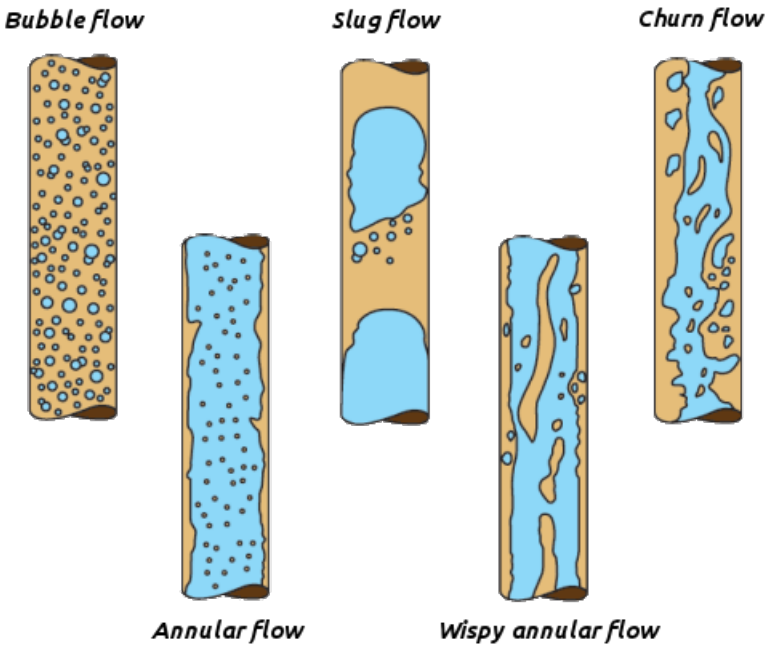

Figure 1.1: Different types of two-phase flow (http://dpwsd.waterworld.com)

1.2 Conservations laws and equations of state

The model that is most commonly used in two-phase thermal-hydraulics is the two-fluid model. It states the conservation of mass, momentum and energy in each phase of the two. That logically results in six equations. We will take the usual hypothesis of a flow without any shock, so derivatives with respect to time and with respect to position are well defined and can be used in equations.

In a geometry where $\vec{x}$ indicates the position and t indicates the time, the conservation of any arbitrary volumetric scalar quantity $Q_{vol}(\vec{x},t)$ in a medium going at a velocity $\vec{u}(\vec{x},t)$ is given by the following equation:

$$\frac{\partial Q_{vol}}{\partial t} + \vec{\nabla}.(Q_{vol}\vec{u}) = \{\text{Sources}\} - \{\text{Sinks}\} \tag{1.1}$$

Similarly, the conservation of any arbitrary volumetric vector quantity $\vec{Q}_{vol}(\vec{x},t)$ in a medium going at a velocity $\vec{u}(\vec{x},t)$ is given by the following equation:

$$\frac{\partial \vec{Q}_{vol}}{\partial t} + \nabla(\vec{Q}_{vol} \otimes \vec{u}) = \{\vec{\text{Sources}}\} - \{\vec{\text{Sinks}}\} \tag{1.2}$$

$\vec{Q}_{vol} \otimes \vec{u}$ is a matrix of dimension 3 which general expression is given by $(Q_{vol,i}u_j)_{i,j}$. So $\vec{Q}_{vol} \otimes \vec{u}$ is also a second order tensor. Therefore, $\nabla(\vec{Q}_{vol} \otimes \vec{u})$ is the divergence of a second order tensor, so it is a column vector.

In this work, we will subscript the symbols by either g (as "gas") or l (as "liquid") to specify to which phase the symbol is referring to. For instance, $\vec{u}_g$ is the velocity of the gas phase and $\vec{u}_l$ is the velocity of the liquid phase.

Let us call α_g the gas volume fraction and α_l the liquid volume fraction. α_g is also equivalent to the void fraction α. Let us also call ρ_g the volumetric mass (or density) of the gas and ρ_l the volumetric mass of the liquid. Then I can apply the conservation formula to the volumetric mass in the gas $\alpha_g\rho_g$ and to the volumetric mass in the liquid $\alpha_l\rho_l$:

$$\begin{aligned}\frac{\partial}{\partial t}(\alpha_g\rho_g) + \vec{\nabla}(\alpha_g\rho_g\vec{u}_g) = 0\\ \frac{\partial}{\partial t}(\alpha_l\rho_l) + \vec{\nabla}(\alpha_l\rho_l\vec{u}_l) = 0\end{aligned} \tag{1.3}$$

The pressure will be chosen as the same in the gas and in the liquid phase, which the most common hypothesis, and will be referred to as p. Let us call $\vec{F}_g$ the forces applied on the gas phase and finally $\vec{F}_l$ the forces applied on the liquid phase. Then we can apply the general formula again to state the conservation of the momentum of the gas phase $\alpha_g\rho_g\vec{u}_g$ and the conservation of the momentum of the liquid phase $\alpha_l\rho_l\vec{u}_l$.

$$\begin{aligned}\frac{\partial}{\partial t}(\alpha_g\rho_g\vec{u}_g) + \nabla(\alpha_g\rho_g\vec{u}_g\otimes\vec{u}_g) + \alpha_g\vec{\nabla}p = \vec{F}_g\\ \frac{\partial}{\partial t}(\alpha_l\rho_l\vec{u}_l) + \nabla(\alpha_l\rho_l\vec{u}_l\otimes\vec{u}_l) + \alpha_l\vec{\nabla}p = \vec{F}_l\end{aligned} \tag{1.4}$$

Finally, let us call e_{int} the volumetric internal energy of a phase and $e = \frac{1}{2}\rho|\vec{u}|^2 + e_{int}$ the volumetric energy of a fluid. Then the conservation of the volumetric energy in the gas $\alpha_g e_g$ and the volumetric energy in the liquid $\alpha_l e_l$ give us the two formulas as well.

$$\begin{aligned}\frac{\partial}{\partial t}(\alpha_g e_g) + \vec{\nabla}(\alpha_g e_g\vec{u}_g) + \alpha_g\vec{\nabla}(p\vec{u}_g) = \{\text{Net energy exchange with the environment}\}_g + \{\text{Net energy generation}\}_g\\ \frac{\partial}{\partial t}(\alpha_l e_l) + \vec{\nabla}(\alpha_l e_l\vec{u}_l) + \alpha_l\vec{\nabla}(p\vec{u}_l) = \{\text{Net energy exchange with the environment}\}_l + \{\text{Net energy generation}\}_l\end{aligned} \tag{1.5}$$

So the general conservation formulas establish six linear equations, as used for instance by Yeom and Chang in [20] and [21]:

$$\begin{aligned}\frac{\partial}{\partial t}(\alpha_g\rho_g) + \vec{\nabla}(\alpha_g\rho_g\vec{u}_g) = 0\\ \frac{\partial}{\partial t}(\alpha_l\rho_l) + \vec{\nabla}(\alpha_l\rho_l\vec{u}_l) = 0\\ \frac{\partial}{\partial t}(\alpha_g\rho_g\vec{u}_g) + \nabla(\alpha_g\rho_g\vec{u}_g\otimes\vec{u}_g) + \alpha_g\vec{\nabla}p = \vec{F}_g\\ \frac{\partial}{\partial t}(\alpha_l\rho_l\vec{u}_l) + \nabla(\alpha_l\rho_l\vec{u}_l\otimes\vec{u}_l) + \alpha_l\vec{\nabla}p = \vec{F}_l\\ \frac{\partial}{\partial t}(\alpha_g e_g) + \vec{\nabla}(\alpha_g e_g\vec{u}_g) + \alpha_g\vec{\nabla}(p\vec{u}_g) = \{\text{Net energy exchange with the environment}\}_g + \{\text{Net energy generation}\}_g\\ \frac{\partial}{\partial t}(\alpha_l e_l) + \vec{\nabla}(\alpha_l e_l\vec{u}_l) + \alpha_l\vec{\nabla}(p\vec{u}_l) = \{\text{Net energy exchange with the environment}\}_l + \{\text{Net energy generation}\}_l\end{aligned} \tag{1.6}$$

There are no sources or sinks for the conservation of mass equations, hence the zero right-hand side of the first two equations. The forces $\vec{F}_g$ and $\vec{F}_l$ account for several phenomena, like gravity, drag force, other interfacial forces. However, they do not include the force due to the pressure gradient, since this is accounted for separately in the left hand side of the equations by $\vec{\nabla}p$. Similarly, the terms {Net energy exchange with the environment} and {Net energy generation} account for many phenomena but not the energy due to the pressure gradient, which is accounted for by $\vec{\nabla}(p\vec{u})$.

In addition to those six conservation equations, we also have to consider some additional ones. The closure

relationship for volumetric fractions is expressed as

$$\alpha_g + \alpha_l = 1 \tag{1.7}$$

We have two equations of state as well: those are equations that establish the relationship between the pressure p and the internal energy e_{int}, using a determined function f.

$$\begin{aligned} f_g(p, e_{int}) &= 0 \\ f_l(p, e_{int}) &= 0 \end{aligned} \tag{1.8}$$

For instance, let us consider the gaseous phase as a Laplace ideal gas. Let us call $\underline{\gamma}$ the ideal gas factor, equal to the heat capacity at constant volume divided by the heat capacity at constant pressure. $\underline{\gamma}$ is equal to $\frac{5}{3}$ if the Laplace ideal gas is a monoatomic gas (like helium He) and $\underline{\gamma} = \frac{7}{5}$ if the gas is diatomic (like nitrogen N_2). In this case we can use the ideal gas law as the equation of state.

$$f_g(p, e_{int,g}) = p - \rho_g e_{int,g}(\underline{\gamma} - 1) = 0 \tag{1.9}$$

To summarize, we have 11 equations: two conservations of mass, two conservations of momentum, two links between energy and internal energy, two conservations of energy, one volume fraction closure relationship and finally two equations of state.

We have 11 unknowns: $\vec{u}_g$, $\vec{u}_l$, α_g, α_l, ρ_g, ρ_l, p, e_g, e_l, $e_{int,g}$ and $e_{int,l}$. That can be reordered as p plus twice α_i, $\vec{u}$, ρ, e, e_{int}.

And we have 6 external factors as source terms for the conservation laws: $\vec{F}_g$, $\vec{F}_l$, the net energy exchange between the gas phase and the environment, the net energy generation in the gas phase, the net energy exchange between the liquid phase and the environment, the net energy generation in the liquid phase. Their expression is given by the environment.

Aside from the external source terms, we thus have as many equations as unknowns; 11. The reason is that the system is mathematically and physically closed.

Finally, we will later also consider a last equation. Newton's third fundamental law tells us that if $\vec{F}_g$ and $\vec{F}_l$ are interfacial forces (not gravity for instance), then they have a reciprocal role:

$$\vec{F}_g + \vec{F}_l = \vec{0} \tag{1.10}$$

1.3 Two-phase flow approximations

In order to make theoretical derivations as well as computational applications, it is usual to make some approximations to simplify the model.

1.3.1 Homogeneous equilibrium model

The homogeneous equilibrium model makes the assumption that not only the pressure is identical in the two phases, but also the velocity and the internal energy. This means that whenever pressure, velocity and internal energy are different in the two phases, they balance each other out quickly enough, on a time scale that is negligible compared to the time scale of the other variables. So the pressure, the velocity and the internal energy of the two phases are considered as in a homogeneous equilibrium. As explained by Corradini in [1], this can be applied for instance when one phase is in minority and is dispersed in the other. In this case there exists a large exchange interface surface compared to the volume of the least present phase.

If we consider the hypothesis of the homogeneous equilibrium model, then the equations are significantly simplified. Let us call $\vec{u}$ the common velocity and e_{int} the common internal energy. The six conservation laws become

$$\begin{gathered}
\frac{\partial}{\partial t}(\alpha_g \rho_g) + \vec{\nabla}(\alpha_g \rho_g \vec{u}) = 0 \\
\frac{\partial}{\partial t}(\alpha_l \rho_l) + \vec{\nabla}(\alpha_l \rho_l \vec{u}) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g \vec{u}) + \nabla(\alpha_g \rho_g \vec{u} \otimes \vec{u}) + \alpha_g \vec{\nabla} p = \vec{F}_g \\
\frac{\partial}{\partial t}(\alpha_l \rho_l \vec{u}) + \nabla(\alpha_l \rho_l \vec{u} \otimes \vec{u}) + \alpha_l \vec{\nabla} p = \vec{F}_l \\
\frac{\partial}{\partial t}(\alpha_g e_g) + \vec{\nabla}(\alpha_g e_g \vec{u}) + \alpha_g \vec{\nabla}(p\vec{u}) = \{\text{Net energy exchange with the environment}\}_g + \{\text{Net energy generation}\}_g \\
\frac{\partial}{\partial t}(\alpha_l e_l) + \vec{\nabla}(\alpha_l e_l \vec{u}) + \alpha_l \vec{\nabla}(p\vec{u}) = \{\text{Net energy exchange with the environment}\}_l + \{\text{Net energy generation}\}_l
\end{gathered} \tag{1.11}$$

If we sum the equations for the two phases together, then we get set of three equations:

$$\begin{gathered}
\frac{\partial}{\partial t}(\alpha_g \rho_g + \alpha_l \rho_l) + \vec{\nabla}((\alpha_g \rho_g + \alpha_l \rho_l)\vec{u}) = 0 \\
\frac{\partial}{\partial t}((\alpha_g \rho_g + \alpha_l \rho_l)\vec{u}) + \nabla((\alpha_g \rho_g + \alpha_l \rho_l)\vec{u} \otimes \vec{u}) + (\alpha_g + \alpha_l)\vec{\nabla} p = \vec{F}_g + \vec{F}_l \\
\frac{\partial}{\partial t}(\alpha_g e_g + \alpha_l e_l) + \vec{\nabla}((\alpha_g e_g + \alpha_l e_l)\vec{u}) + (\alpha_g + \alpha_l)\vec{\nabla}(p\vec{u}) = \{\text{Energy exchange}\} + \{\text{Energy generation}\}
\end{gathered} \tag{1.12}$$

Let us write $\bar{\rho} = \alpha_g \rho_g + \alpha_l \rho_l$ for the volume-averaged volumetric mass and $\bar{e} = \alpha_g e_g + \alpha_l e_l$ for the volume-averaged energy. Then, using the closure relationship $\alpha_g + \alpha_l = 1$, I obtain a very interesting set of three equations:

$$\begin{gathered}
\frac{\partial}{\partial t}(\bar{\rho}) + \vec{\nabla}(\bar{\rho}\vec{u}) = 0 \\
\frac{\partial}{\partial t}(\bar{\rho}\vec{u}) + \nabla(\bar{\rho}\vec{u} \otimes \vec{u}) + \vec{\nabla} p = \vec{F}_g + \vec{F}_l \\
\frac{\partial}{\partial t}(\bar{e}) + \vec{\nabla}(\bar{e}\vec{u}) + \vec{\nabla}(p\vec{u}) = \{\text{Energy exchange}\} + \{\text{Energy generation}\}
\end{gathered} \tag{1.13}$$

This can be interpreted as the study of a single global fluid. This fluid would then be at pressure p, with a velocity $\vec{u}$ and internal energy e_{int}, but also with a volumetric mass $\bar{\rho}$ and a volumetric energy $\bar{e}$. It would be subject to the sum of all forces as well as to the sum of all the energy exchanges and generations. The three equations are then interpreted as the conservation laws applied to the global fluid.

The thorough study of the homogeneous equilibrium model is not the scope of this thesis, so the reader can get more information about the model and its eventual improvements by reading Kim and Dunsheath's work [10].

1.3.2 Drift flux model

The drift flux model is a refinement of the homogeneous equilibrium model. It still approximates the pressure p and the internal energy e_{int} as equal in both phases, but now it allows the velocities to be different one from another. This is why this model is also called the separated flow model. According to Corradini [1], this is a good approximation when buoyancy forces tend to induce a velocity difference between the lighter and the heavier phases. The model is said to be useful in particular for the calculation of pressure drops in pipes.

The conservation equations that result from the approximation are as follows:

$$\begin{gathered}
\frac{\partial}{\partial t}(\alpha_g \rho_g) + \vec{\nabla}(\alpha_g \rho_g \vec{u}_g) = 0 \\
\frac{\partial}{\partial t}(\alpha_l \rho_l) + \vec{\nabla}(\alpha_l \rho_l \vec{u}_l) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g \vec{u}_g) + \nabla(\alpha_g \rho_g \vec{u}_g \otimes \vec{u}_g) + \alpha_g \vec{\nabla} p = \vec{F}_g \\
\frac{\partial}{\partial t}(\alpha_l \rho_l \vec{u}_l) + \nabla(\alpha_l \rho_l \vec{u}_l \otimes \vec{u}_l) + \alpha_l \vec{\nabla} p = \vec{F}_l \\
\frac{\partial}{\partial t}(\alpha_g e_g) + \vec{\nabla}(\alpha_g e_g \vec{u}_g) + \alpha_g \vec{\nabla}(p \vec{u}_g) = \{\text{Net energy exchange with the environment}\}_g + \{\text{Net energy generation}\}_g \\
\frac{\partial}{\partial t}(\alpha_l e_l) + \vec{\nabla}(\alpha_l e_l \vec{u}_l) + \alpha_l \vec{\nabla}(p \vec{u}_l) = \{\text{Net energy exchange with the environment}\}_l + \{\text{Net energy generation}\}_l
\end{gathered} \tag{1.14}$$

We can sum up the equations of conservation of mass and of momentum to obtain the following four equations:

$$\begin{gathered}
\frac{\partial}{\partial t}(\alpha_g \rho_g + \alpha_l \rho_l) + \vec{\nabla}(\alpha_g \rho_g \vec{u}_g + \alpha_l \rho_l \vec{u}_l) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g \vec{u}_g) + \nabla(\alpha_g \rho_g \vec{u}_g \otimes \vec{u}_g) + \alpha_g \vec{\nabla} p = \vec{F}_g \\
\frac{\partial}{\partial t}(\alpha_l \rho_l \vec{u}_l) + \nabla(\alpha_l \rho_l \vec{u}_l \otimes \vec{u}_l) + \alpha_l \vec{\nabla} p = \vec{F}_l \\
\frac{\partial}{\partial t}(\alpha_g e_g + \alpha_l e_l) + \vec{\nabla}(\alpha_g e_g \vec{u}_g + \alpha_l e_l \vec{u}_l) + \alpha_g \vec{\nabla}(p \vec{u}_g) + \alpha_l \vec{\nabla}(p \vec{u}_l) = \{\text{Energy exchange}\} + \{\text{Energy generation}\}
\end{gathered} \tag{1.15}$$

As we can see, it is not as easy as for the homogeneous equilibrium model to formally introduce volume-averaged quantities and to obtain the equations for a global fluid. So what is usually done is to force the analysis of a fictional global fluid, which follows the same equations as in the previous subsection 1.3.1, and to add a model for the calculation of the difference in velocities $u_g - u_l$. That way the model is closed and we can get a finer representation than the homogeneous equilibrium model.

1.3.3 Two-fluid model

The model that we will focus on in this work is the two-fluid model. It aims at being the most complete approximation. The two-fluid model is often also referred to as the six-equation model because it keeps the six conservation laws that we saw earlier:

$$\begin{gathered}
\frac{\partial}{\partial t}(\alpha_g \rho_g) + \vec{\nabla}(\alpha_g \rho_g \vec{u}_g) = 0 \\
\frac{\partial}{\partial t}(\alpha_l \rho_l) + \vec{\nabla}(\alpha_l \rho_l \vec{u}_l) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g \vec{u}_g) + \nabla(\alpha_g \rho_g \vec{u}_g \otimes \vec{u}_g) + \alpha_g \vec{\nabla} p = \vec{F}_g \\
\frac{\partial}{\partial t}(\alpha_l \rho_l \vec{u}_l) + \nabla(\alpha_l \rho_l \vec{u}_l \otimes \vec{u}_l) + \alpha_l \vec{\nabla} p = \vec{F}_l \\
\frac{\partial}{\partial t}(\alpha_g e_g) + \vec{\nabla}(\alpha_g e_g \vec{u}_g) + \alpha_g \vec{\nabla}(p \vec{u}_g) = \{\text{Net energy exchange with the environment}\}_g + \{\text{Net energy generation}\}_g \\
\frac{\partial}{\partial t}(\alpha_l e_l) + \vec{\nabla}(\alpha_l e_l \vec{u}_l) + \alpha_l \vec{\nabla}(p \vec{u}_l) = \{\text{Net energy exchange with the environment}\}_l + \{\text{Net energy generation}\}_l
\end{gathered} \tag{1.16}$$

In the commonly used form of the six-equation model, the source terms are well-defined. They are functions of the other parameters p, α_g, α_l, $\vec{u}_g$, $\vec{u}_l$, ρ_g, ρ_l, etc. Furthermore, in the commonly used form, they are not functions of the derivatives of the other parameters. That is to say, their expression does not include $\frac{\partial \alpha_g}{\partial t}$ or $\vec{\nabla}\vec{u}_l$ for instance.

Gravity is a good example of a force that could be present in the two-fluid model in its commonly used form: $\vec{F}_{gravity} = \rho\vec{g}$, where $\vec{g}$ is the gravitational acceleration, equal to approximately $9.8\,\mathrm{m\ s^{-2}}$.

1.4 The one-dimensional barotropic two-fluid model

1.4.1 One-dimensional two-fluid model

In this work we are going to study the two-fluid model applied to a one-dimensional geometry. This is typically useful for the design of nuclear codes, since pipes and other constrained conducts can be represented in 1D. In the derivations, this transforms the spatial position variable $\vec{x}$ into just x and the spatial derivative ∇ into $\frac{\partial}{\partial x}$. Moreover the velocity $\vec{u}$ and the forces $\vec{F}$ can be rewritten as a scalar variables u and F. Let us see how this simplifies the 11 initial equations:

$$
\begin{aligned}
\frac{\partial}{\partial t}(\alpha_g\rho_g) + \frac{\partial}{\partial x}(\alpha_g\rho_g u_g) &= 0 \\
\frac{\partial}{\partial t}(\alpha_l\rho_l) + \frac{\partial}{\partial x}(\alpha_l\rho_l u_l) &= 0 \\
\frac{\partial}{\partial t}(\alpha_g\rho_g u_g) + \frac{\partial}{\partial x}(\alpha_g\rho_g u_g^2) + \alpha_g\frac{\partial p}{\partial x} &= F_g \\
\frac{\partial}{\partial t}(\alpha_l\rho_l u_l) + \frac{\partial}{\partial x}(\alpha_l\rho_l u_l^2) + \alpha_l\frac{\partial p}{\partial x} &= F_l \\
\frac{\partial}{\partial t}(\alpha_g e_g) + \frac{\partial}{\partial x}(\alpha_g e_g u_g) + \alpha_g\frac{\partial}{\partial x}(p u_g) &= \{\text{Net energy exchange with the environment}\}_g + \{\text{Net energy generation}\}_g \\
\frac{\partial}{\partial t}(\alpha_l e_l) + \frac{\partial}{\partial x}(\alpha_l e_l u_l) + \alpha_l\frac{\partial}{\partial x}(p u_l) &= \{\text{Net energy exchange with the environment}\}_l + \{\text{Net energy generation}\}_l \\
e_g &= \frac{1}{2}\rho_g u_g^2 + e_{int,g} \\
e_l &= \frac{1}{2}\rho_l u_l^2 + e_{int,l} \\
f_g(p, e_{int}) &= 0 \\
f_l(p, e_{int}) &= 0 \\
\alpha_g + \alpha_l &= 1
\end{aligned}
\tag{1.17}
$$

The 11 unknowns of the problem are p plus twice α_i, u_i, ρ_i, e_i and $e_{i,int}$.

1.4.2 Barotropic analysis

Throughout this analysis, we are going to consider the problem with a constant energy. That is to say, we will not be considering the equations of conservation of energy (fifth and sixth lines of 1.17), nor the equations linking the energy and the internal energy (seventh and eighth equations). As explained by Shames [17], this is similar to saying that we approximate the two fluids as being "barotropic".

The reason for this approximation is that the analysis we will carry on the mass and momentum will still be qualitatively correct when one adds the energy to the system. Solving the problems for mass and momentum will, by extension, also provide a solution for the similar problems for mass, momentum and energy. So here we will use

only the four first equations of 1.17 and the closure relationship $\alpha_g + \alpha_l = 1$.

$$\begin{gathered}
\frac{\partial}{\partial t}(\alpha_g \rho_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g) = 0 \\
\frac{\partial}{\partial t}(\alpha_l \rho_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g u_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g u_g) + \alpha_g \frac{\partial p}{\partial x} = F_g \\
\frac{\partial}{\partial t}(\alpha_l \rho_l u_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l u_l) + \alpha_l \frac{\partial p}{\partial x} = F_l \\
f_g(p, e_{int}) = 0 \\
f_l(p, e_{int}) = 0 \\
\alpha_g + \alpha_l = 1
\end{gathered} \tag{1.18}$$

If we substitute α_g and α_l in the four first lines of 1.18 thanks to $\alpha_g = \alpha$ and the closure relationship 1.7, we get

$$\begin{gathered}
\frac{\partial}{\partial t}(\alpha \rho_g) + \frac{\partial}{\partial x}(\alpha \rho_g u_g) = 0 \\
\frac{\partial}{\partial t}((1-\alpha) \rho_l) + \frac{\partial}{\partial x}((1-\alpha) \rho_l u_l) = 0 \\
\frac{\partial}{\partial t}(\alpha \rho_g u_g) + \frac{\partial}{\partial x}(\alpha \rho_g u_g u_g) + \alpha \frac{\partial p}{\partial x} = F_g \\
\frac{\partial}{\partial t}((1-\alpha) \rho_l u_l) + \frac{\partial}{\partial x}((1-\alpha) \rho_l u_l u_l) + (1-\alpha) \frac{\partial p}{\partial x} = F_l \\
f_g(p, e_{int}) = 0 \\
f_l(p, e_{int}) = 0
\end{gathered} \tag{1.19}$$

So if F_g and F_l are defined as expressions of the other variables, the unknowns of the problem are just p, α, ρ_g, ρ_l, u_g, u_l. This is six unknowns for six equations, since the system is still correctly closed.

ρ_g and ρ_l are functions of the pressure p and the temperature (the internal energy), thanks to the equations of state. The temperature depends only on the pressure, since we know we have two phases, so the temperature is equal to the saturation temperature associated with p where liquid and gas can coexist. The temperature is also known as the boiling temperature associated to p. Therefore ρ_g is also the saturated vapor density and ρ_l is the saturated liquid density. For example, in a BWR (Boiling Water Reactor) reactor with a pressure of 76 bar, the saturation temperature T is 291 °C, and $(\rho_g, \rho_l) = (40.1\,\text{kg m}^{-3}, 729\,\text{kg m}^{-3})$.

So this leads to a problem of four partial differential equations with four unknowns:

$$p, \alpha, u_g \text{ and } u_l \tag{1.20}$$

The final set of four partial differential equations is

$$\begin{gathered}
\frac{\partial}{\partial t}(\alpha \rho_g) + \frac{\partial}{\partial x}(\alpha \rho_g u_g) = 0 \\
\frac{\partial}{\partial t}((1-\alpha) \rho_l) + \frac{\partial}{\partial x}((1-\alpha) \rho_l u_l) = 0 \\
\frac{\partial}{\partial t}(\alpha \rho_g u_g) + \frac{\partial}{\partial x}(\alpha \rho_g u_g u_g) + \alpha \frac{\partial p}{\partial x} = F_g \\
\frac{\partial}{\partial t}((1-\alpha) \rho_l u_l) + \frac{\partial}{\partial x}((1-\alpha) \rho_l u_l u_l) + (1-\alpha) \frac{\partial p}{\partial x} = F_l
\end{gathered} \tag{1.21}$$

Now the point is to establish whether the two-fluid model has a unique solution (p, α, u_g, u_l).

Chapter 2

Ill-posedness of the two-fluid model in its commonly used form

2.1 From the conservation laws to a matrix form

2.1.1 Converting the system of equations into a matrix equation

In the last part, we saw how to reduce the initial problem to a set of four linear partial differential equations. We now want to express it as a matrix problem, because the tools we have for matrix linear algebra are easier to use and more powerful [7].

As a starting point, let us consider the mass and momentum conservation equations:

$$\begin{aligned}
\frac{\partial}{\partial t}(\alpha_g\rho_g) + \frac{\partial}{\partial x}(\alpha_g\rho_g u_g) &= 0 \\
\frac{\partial}{\partial t}(\alpha_l\rho_l) + \frac{\partial}{\partial x}(\alpha_l\rho_l u_l) &= 0 \\
\frac{\partial}{\partial t}(\alpha_g\rho_g u_g) + \frac{\partial}{\partial x}(\alpha_g\rho_g u_g u_g) + \alpha_g\frac{\partial p}{\partial x} &= F_g \\
\frac{\partial}{\partial t}(\alpha_l\rho_l u_l) + \frac{\partial}{\partial x}(\alpha_l\rho_l u_l u_l) + \alpha_l\frac{\partial p}{\partial x} &= F_l
\end{aligned} \tag{2.1}$$

Expanding all terms using the chain rule leads to

$$\begin{aligned}
\rho_g\frac{\partial\alpha_g}{\partial t} + \alpha_g\frac{\partial\rho_g}{\partial t} + \rho_g u_g\frac{\partial\alpha_g}{\partial x} + \alpha_g u_g\frac{\partial\rho_g}{\partial x} + \alpha_g\rho_g\frac{\partial u_g}{\partial x} &= 0 \\
\rho_l\frac{\partial\alpha_l}{\partial t} + \alpha_l\frac{\partial\rho_l}{\partial t} + \rho_l u_l\frac{\partial\alpha_l}{\partial x} + \alpha_l u_l\frac{\partial\rho_l}{\partial x} + \alpha_l\rho_l\frac{\partial u_l}{\partial x} &= 0 \\
\rho_g u_g\frac{\partial\alpha_g}{\partial t} + \alpha_g u_g\frac{\partial\rho_g}{\partial t} + \alpha_g\rho_g\frac{\partial u_g}{\partial t} + \rho_g u_g u_g\frac{\partial\alpha_g}{\partial x} + \alpha_g u_g u_g\frac{\partial\rho_g}{\partial x} + 2\alpha_g\rho_g u_g\frac{\partial u_g}{\partial x} + \alpha_g\frac{\partial p}{\partial x} &= F_g \\
\rho_l u_l\frac{\partial\alpha_l}{\partial t} + \alpha_l u_l\frac{\partial\rho_l}{\partial t} + \alpha_l\rho_l\frac{\partial u_l}{\partial t} + \rho_l u_l u_l\frac{\partial\alpha_l}{\partial x} + \alpha_l u_l u_l\frac{\partial\rho_l}{\partial x} + 2\alpha_l\rho_l u_l\frac{\partial u_l}{\partial x} + \alpha_l\frac{\partial p}{\partial x} &= F_l
\end{aligned} \tag{2.2}$$

To simplify the notations, we re-arrange the unknowns in the order of (p (or ρ) $\quad \alpha_g \quad u_g \quad u_l$).

$$\begin{aligned}
\alpha_g\frac{\partial\rho_g}{\partial t} + \rho_g\frac{\partial\alpha_g}{\partial t} + \alpha_g u_g\frac{\partial\rho_g}{\partial x} + \rho_g u_g\frac{\partial\alpha_g}{\partial x} + \alpha_g\rho_g\frac{\partial u_g}{\partial x} &= 0 \\
\alpha_l\frac{\partial\rho_l}{\partial t} + \rho_l\frac{\partial\alpha_l}{\partial t} + \alpha_l u_l\frac{\partial\rho_l}{\partial x} + \rho_l u_l\frac{\partial\alpha_l}{\partial x} + \alpha_l\rho_l\frac{\partial u_l}{\partial x} &= 0 \\
\alpha_g u_g\frac{\partial\rho_g}{\partial t} + \rho_g u_g\frac{\partial\alpha_g}{\partial t} + \alpha_g\rho_g\frac{\partial u_g}{\partial t} + \alpha_g\frac{\partial p}{\partial x} + \alpha_g u_g u_g\frac{\partial\rho_g}{\partial x} + \rho_g u_g u_g\frac{\partial\alpha_g}{\partial x} + 2\alpha_g\rho_g u_g\frac{\partial u_g}{\partial x} &= F_g \\
\alpha_l u_l\frac{\partial\rho_l}{\partial t} + \rho_l u_l\frac{\partial\alpha_l}{\partial t} + \alpha_l\rho_l\frac{\partial u_l}{\partial t} + \alpha_l\frac{\partial p}{\partial x} + \alpha_l u_l u_l\frac{\partial\rho_l}{\partial x} + \rho_l u_l u_l\frac{\partial\alpha_l}{\partial x} + 2\alpha_l\rho_l u_l\frac{\partial u_l}{\partial x} &= F_l
\end{aligned} \tag{2.3}$$

Because we consider the case of constant energy, the two fluids are considered barotropic and the pressure is a function of the volumetric mass only (and not the temperature for instance): $p = p(\rho)$. So I can derive p with respect to ρ. If $\frac{\partial p}{\partial\rho} = \frac{dp}{d\rho} > 0$, then I can write $p'(\rho) = c^2$, where c is the speed of sound in the considered fluid. As

a consequence, ρ_g, ρ_l and α_l can be eliminated using the following six substitutions:

$$\begin{aligned}\frac{\partial \rho_g}{\partial t} &= \frac{\partial \rho_g}{\partial p}\frac{\partial p}{\partial t} = \frac{1}{c_g^2}\frac{\partial p}{\partial t} \\ \frac{\partial \rho_l}{\partial t} &= \frac{\partial \rho_l}{\partial p}\frac{\partial p}{\partial t} = \frac{1}{c_l^2}\frac{\partial p}{\partial t}\end{aligned} \tag{2.4}$$

$$\begin{aligned}\frac{\partial \rho_g}{\partial x} &= \frac{\partial \rho_g}{\partial p}\frac{\partial p}{\partial x} = \frac{1}{c_g^2}\frac{\partial p}{\partial x} \\ \frac{\partial \rho_l}{\partial x} &= \frac{\partial \rho_l}{\partial p}\frac{\partial p}{\partial x} = \frac{1}{c_l^2}\frac{\partial p}{\partial x}\end{aligned} \tag{2.5}$$

$$\begin{aligned}\frac{\partial \alpha_l}{\partial t} &= \frac{\partial (1-\alpha_g)}{\partial t} = -\frac{\partial \alpha_g}{\partial t} \\ \frac{\partial \alpha_l}{\partial x} &= \frac{\partial (1-\alpha_g)}{\partial x} = -\frac{\partial \alpha_g}{\partial x}\end{aligned} \tag{2.6}$$

Using those substitution in equations 2.3 gives

$$\begin{aligned}\frac{\alpha_g}{c_g^2}\frac{\partial p}{\partial t} + \rho_g \frac{\partial \alpha_g}{\partial t} + \frac{\alpha_g u_g}{c_g^2}\frac{\partial p}{\partial x} + \rho_g u_g \frac{\partial \alpha_g}{\partial x} + \alpha_g \rho_g \frac{\partial u_g}{\partial x} &= 0 \\ \frac{\alpha_l}{c_l^2}\frac{\partial p}{\partial t} - \rho_l \frac{\partial \alpha_g}{\partial t} + \frac{\alpha_l u_l}{c_l^2}\frac{\partial p}{\partial x} - \rho_l u_l \frac{\partial \alpha_g}{\partial x} + \alpha_l \rho_l \frac{\partial u_l}{\partial x} &= 0 \\ \frac{\alpha_g u_g}{c_g^2}\frac{\partial p}{\partial t} + \rho_g u_g \frac{\partial \alpha_g}{\partial t} + \alpha_g \rho_g \frac{\partial u_g}{\partial t} + \alpha_g \frac{\partial p}{\partial x} + \frac{\alpha_g u_g u_g}{c_g^2}\frac{\partial p}{\partial x} + \rho_g u_g u_g \frac{\partial \alpha_g}{\partial x} + 2\alpha_g \rho_g u_g \frac{\partial u_g}{\partial x} &= F_g \\ \frac{\alpha_l u_l}{c_l^2}\frac{\partial p}{\partial t} - \rho_l u_l \frac{\partial \alpha_g}{\partial t} + \alpha_l \rho_l \frac{\partial u_l}{\partial t} + \alpha_l \frac{\partial p}{\partial x} + \frac{\alpha_l u_l u_l}{c_l^2}\frac{\partial p}{\partial x} - \rho_l u_l u_l \frac{\partial \alpha_g}{\partial x} + 2\alpha_l \rho_l u_l \frac{\partial u_l}{\partial x} &= F_l\end{aligned} \tag{2.7}$$

We can now write the equations in a matrix form. Let us define the vector U which components are the unknown parameters:

$$U = \begin{pmatrix} p \\ \alpha_g \\ u_g \\ u_l \end{pmatrix} \tag{2.8}$$

This leads to rewriting the four equations into a dimension 4 matrix problem:

$$\begin{pmatrix} \frac{\alpha_g}{c_g^2} & \rho_g & 0 & 0 \\ \frac{\alpha_l}{c_l^2} & -\rho_l & 0 & 0 \\ \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g \rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l \rho_l \end{pmatrix} \frac{\partial U}{\partial t} + \begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g \rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l \rho_l \\ \alpha_g + \frac{\alpha_g u_g u_g}{c_g^2} & \rho_g u_g u_g & 2\alpha_g \rho_g u_g & 0 \\ \alpha_l + \frac{\alpha_l u_l u_l}{c_l^2} & -\rho_l u_l u_l & 0 & 2\alpha_l \rho_l u_l \end{pmatrix} \frac{\partial U}{\partial x} = \begin{pmatrix} 0 \\ 0 \\ F_g \\ F_l \end{pmatrix} \tag{2.9}$$

Let us simplify the set of equations by using the elementary operations $L_3 \leftarrow L_3 - u_g \cdot L_1$ and $L_4 \leftarrow L_4 - u_l \cdot L_2$. $L_3 \leftarrow L_3 - u_g \cdot L_1$ simply means that we add the first line L_1 multiplied by the real number $-u_g$ to the third line L_3. Notice that this step does not preserve eigenvalues and eigenvectors of any of the matrices, but this has no impact on the final results.

$$\begin{pmatrix} \frac{\alpha_g}{c_g^2} & \rho_g & 0 & 0 \\ \frac{\alpha_l}{c_l^2} & -\rho_l & 0 & 0 \\ 0 & 0 & \alpha_g\rho_g & 0 \\ 0 & 0 & 0 & \alpha_l\rho_l \end{pmatrix} \frac{\partial U}{\partial t} + \begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g\rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l\rho_l \\ \alpha_g & 0 & \alpha_g\rho_g u_g & 0 \\ \alpha_l & 0 & 0 & \alpha_l\rho_l u_l \end{pmatrix} \frac{\partial U}{\partial x} = \begin{pmatrix} 0 \\ 0 \\ F_g \\ F_l \end{pmatrix} \tag{2.10}$$

The problem is now written in the following simplified form:

$$M_t \frac{\partial U}{\partial t} + M_x \frac{\partial U}{\partial x} = S \tag{2.11}$$

M_t and M_x are square matrices of dimension 4. U and S are column vectors of dimension 4.

Notice that the matrices M_t and M_x depend on p, α_g, u_g and u_l, so they depend on U. So equation 2.11 is not a linear equation. However we can consider the two matrices as constant in a first order approximation, so the equation is linear in a first order approximation. This will be important for what follows.

2.1.2 Representation of the problem by a matrix A_0

The matrix M_t is invertible. Therefore, we can multiply the equation 2.11 on the left by M_t^{-1}.

$$\frac{\partial U}{\partial t} + M_t^{-1} M_x \frac{\partial U}{\partial x} = M_t^{-1} S \tag{2.12}$$

Let us define $A_0 = M_t^{-1} M_x$ and $S_0 = M_t^{-1} S$. We get the final matrix problem for the two-fluid model in its commonly used form:

$$\frac{\partial U}{\partial t} + A_0 \frac{\partial U}{\partial x} = S_0 \tag{2.13}$$

We can try to give a developed theoretical expression of A_0:

$$A_0 = \begin{pmatrix} \frac{\rho_l c_g^2 c_l^2}{\alpha_g\rho_l c_l^2 + \alpha_l\rho_g c_g^2} & \frac{\rho_g c_g^2 c_l^2}{\alpha_g\rho_l c_l^2 + \alpha_l\rho_g c_g^2} & 0 & 0 \\ \frac{\alpha_l c_g^2}{\alpha_g\rho_l c_l^2 + \alpha_l\rho_g c_g^2} & \frac{-\alpha_g c_l^2}{\alpha_g\rho_l c_l^2 + \alpha_l\rho_g c_g^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_g\rho_g} & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha_l\rho_l} \end{pmatrix} \cdot \begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g\rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l\rho_l \\ \alpha_g & 0 & \alpha_g\rho_g u_g & 0 \\ \alpha_l & 0 & 0 & \alpha_l\rho_l u_l \end{pmatrix} \tag{2.14}$$

In order to get a more compact expression, let us introduce a subsidiary symbol, $\gamma^2 = \frac{c_g^2 c_l^2}{\alpha_g\rho_l c_l^2 + \alpha_l\rho_g c_g^2}$. This leads to the following simplification:

$$A_0 = \begin{pmatrix} \gamma^2\rho_l & \gamma^2\rho_g & 0 & 0 \\ \gamma^2\frac{\alpha_l}{c_l^2} & -\gamma^2\frac{\alpha_g}{c_g^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_g\rho_g} & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha_l\rho_l} \end{pmatrix} \cdot \begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g\rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l\rho_l \\ \alpha_g & 0 & \alpha_g\rho_g u_g & 0 \\ \alpha_l & 0 & 0 & \alpha_l\rho_l u_l \end{pmatrix} \tag{2.15}$$

The multiplication gives

$$A_0 = \begin{pmatrix} \gamma^2\frac{\alpha_g\rho_l}{c_g^2}u_g + \gamma^2\frac{\alpha_l\rho_g}{c_l^2}u_l & \gamma^2\rho_g\rho_l(u_g - u_l) & \gamma^2\alpha_g\rho_g\rho_l & \gamma^2\alpha_l\rho_l\rho_g \\ \gamma^2\frac{\alpha_g\alpha_l}{c_g^2 c_l^2}(u_g - u_l) & \gamma^2\frac{\alpha_l\rho_g}{c_l^2}u_g + \gamma^2\frac{\alpha_g\rho_l}{c_g^2}u_l & \gamma^2\frac{\alpha_l\alpha_g\rho_g}{c_l^2} & -\gamma^2\frac{\alpha_g\alpha_l\rho_l}{c_g^2} \\ \frac{1}{\rho_g} & 0 & u_g & 0 \\ \frac{1}{\rho_l} & 0 & 0 & u_l \end{pmatrix} \tag{2.16}$$

Substituting $G = \frac{\alpha_g \rho_l}{c_g^2}$, $L = \frac{\alpha_l \rho_g}{c_l^2}$, $R = \rho_g \rho_l$, I obtain an expression that is a little easier to read:

$$A_0 = \begin{pmatrix} \gamma^2(Gu_g + Lu_l) & \gamma^2 R(u_g - u_l) & \gamma^2 R\alpha_g & \gamma^2 R\alpha_l \\ \gamma^2 \frac{GL}{R}(u_g - u_l) & \gamma^2(Lu_g + Gu_l) & \gamma^2 L\alpha_g & -\gamma^2 G\alpha_l \\ \frac{1}{\rho_g} & 0 & u_g & 0 \\ \frac{1}{\rho_l} & 0 & 0 & u_l \end{pmatrix} \tag{2.17}$$

As a conclusion, we get the equation $\frac{\partial U}{\partial t} + A_0 \frac{\partial U}{\partial x} = S_0$ where $U = \begin{pmatrix} p & \alpha & u_g & u_l \end{pmatrix}^t$. Let us remember that the matrix A_0 is all but constant, since it depends on many variables. The model is not linear. Nonetheless, it can be assumed as locally linear to the first order.

2.2 Ill-posed Cauchy problem because of its non-hyperbolicity

In the last part we showed that the two-fluid model could be reformulated as

$$\frac{\partial U}{\partial t} + A_0 \frac{\partial U}{\partial x} = S_0 \tag{2.18}$$

This matrix formulation is used by a large panel of authors, including for instance Ndjinga et al. [13] and Dinh [7]. Let us associate this matrix equation with initial conditions, such as the following:

$$\forall x, U(x, t = 0) = U_0(x) \tag{2.19}$$

If we consider A_0 as locally constant, which is true as a first order approximation, I then have by definition what is called a *Cauchy problem*. We want to know whether a solution exists and whether it is unique. So we can use a property of Cauchy problems:

> A Cauchy problem offers a unique solution U if and only if the problem is well-posed in the sense of Hadamard, that is to say if and only if the eigenvalues of A_0 are real and distinct [9]. If this is the case, then the matrix A_0 is said to be *hyperbolic* and the Cauchy problem is said to be hyperbolic too.

We will use this property to study the two-fluid model in its commonly used form. The expression of A_0 can be used to calculate the numerical value of its eigenvalues:

$$A_0 = \begin{pmatrix} \gamma^2(Gu_g + Lu_l) & \gamma^2 R(u_g - u_l) & \gamma^2 R\alpha_g & \gamma^2 R\alpha_l \\ \gamma^2 \frac{GL}{R}(u_g - u_l) & \gamma^2(Lu_g + Gu_l) & \gamma^2 L\alpha_g & -\gamma^2 G\alpha_l \\ \frac{1}{\rho_g} & 0 & u_g & 0 \\ \frac{1}{\rho_l} & 0 & 0 & u_l \end{pmatrix} \tag{2.20}$$

Let us take the example of this two-fluid model in the conditions of a BWR pressure. We may choose $(p, \alpha_g, u_g, u_l) = \left(76\,\mathrm{bar}, 0.5, 500\,\mathrm{m\,s^{-1}}, 20\,\mathrm{m\,s^{-1}}\right)$. Notice that this set of conditions is not necessarily representative of the normal state of operation of a BWR, since here the velocities of the fluids are really high and very different one from another. In a real BWR, the velocities should be in the order of magnitude of $1\,\mathrm{m\,s^{-1}}$ and the slip ratio $\frac{u_g}{u_l}$ is close to 1 and not 25. However we intentionally chose this set of conditions because it is a good set that will result in very understandable properties, later in this thesis. The numerical calculation leads to the following four eigenvalues :

- $986.14\,\mathrm{m\,s^{-1}}$

- $-149.28\,\mathrm{m\,s^{-1}}$
- $(101.57 + 106.75\mathrm{i})\,\mathrm{m\,s^{-1}}$
- $(101.57 - 106.75\mathrm{i})\,\mathrm{m\,s^{-1}}$

We immediately notice that the two first eigenvalues are real but that the two last eigenvalues are complex (conjugated). Therefore our problem is not hyperbolic. It is possible to show that the matrix A_0 for the two-fluid model is not hyperbolic for most conditions of practical interest. Most cases have two real eigenvalues and two conjugated complex ones. So in general, the two-fluid model in its commonly used form is ill-posed.

There are two main implications. First, any computational code that relies on the unmodified two-fluid model may give multiple solutions. This is not only mathematically incorrect but it also leads to some uncertainty if we force one of the solutions. Second, when we have a code based on a non-hyperbolic problem, we cannot say anything about space convergence. Stronger: the code is very likely not to have space convergence. So it is possible to get the counter-intuitive result that the more we refine the mesh, the less precise the results are. As a result, we cannot use the two-fluid model "as is". We have to find a way to improve it.

As a side note, the formal ill-posedness is acknowledged by the developers of the major thermal-hydraulics nuclear codes. See TRACE's manual for instance [3].

2.3 Defining a measurement of hyperbolicity

In equation 2.16, we have given an expression of A_0 as a function of $(p, \alpha_g, \alpha_l, u_g, u_l, c_g, c_l, \rho_g, \rho_l)$. α_l is linked to α_g by the closure relationship $\alpha_l = 1 - \alpha_g$. Moreover $(c_g, c_l, \rho_g, \rho_l)$ are properties of the two materials (steam and liquid water); they are functions of the pressure p and the temperature. Under the assumption that two phases are coexisting, the temperature is the saturation temperature associated to p. So the material properties in our barotropic fluids depend only on the pressure p. Therefore we can now calculate A_0 as a function of (p, α_g, u_g, u_l).

$$A_0 = A_0\left(p, \alpha_g, u_g, u_l\right) \tag{2.21}$$

The spectrum of A_0, that is to say the ensemble of the 4 eigenvalues of A_0, is thus also a function of just (p, α_g, u_g, u_l). Let us write the different eigenvalues of A_0 as $\lambda_{0,i}$, knowing that they may be repeated.

$$sp(A_0) = \{\lambda_{0,1}, \lambda_{0,2}, \lambda_{0,3}, \lambda_{0,4}\} \tag{2.22}$$

At this point, it would be extremely useful to introduce a tool to calculate whether the matrix A_0 is hyperbolic. If this tool is a real number instead of just a Boolean ($True$ or $False$), it will have definitely more applications. As a consequence, in this study I introduce the new concept of the *numerical hyperbolicity* as a practical measurement of hyperbolicity.

I will refer to the numerical hyperbolicity with the letter H. H will be a function of the matrix A_0 and will be able to explicitly reveal whether the problem is hyperbolic or not. If no eigenvalue is zero, then the numerical hyperbolicity is given by the following formula:

$$H(A_0) = 1 - \frac{1}{4}\sum_{i=1}^{4}\frac{|\Im(\lambda_{0,i})|}{|\lambda_{0,i}|} \tag{2.23}$$

In this formula 2.23, $\Im(.)$ stands for the imaginary part and $|.|$ stands for the module. The function H is defined for all four-dimensional square matrices.

H is a real number between 0 and 1 and this is easily proven. First, because for all i, $\Im(\lambda_i)$ and $|\lambda_i|$ are real numbers, we find that H is real too. Additionally,

$$\forall i \in \{1,2,3,4\}, 0 \leq |\Im(\lambda_{0,i})| \leq |\lambda_{0,i}| \tag{2.24}$$

$$\forall i \in \{1,2,3,4\}, 0 \leq \frac{|\Im(\lambda_{0,i})|}{|\lambda_{0,i}|} \leq 1 \tag{2.25}$$

$$0 \leq \sum_i \frac{|\Im(\lambda_{0,i})|}{|\lambda_{0,i}|} \leq 4 \tag{2.26}$$

$$\frac{0}{4} = 0 \geq -\frac{1}{4}\sum_i \frac{|\Im(\lambda_{0,i})|}{|\lambda_{0,i}|} \geq -\frac{4}{4} = -1 \tag{2.27}$$

$$1 \geq 1 - \frac{1}{4}\sum_i \frac{|\Im(\lambda_{0,i})|}{|\lambda_{0,i}|} = H \geq 0 \tag{2.28}$$

Finally $H \in [0,1]$.

H reaches its maximum value 1 if and only if $\forall i \in \{1,2,3,4\}, \Im(\lambda_{0,i}) = 0$. This happens if and only if all the $\lambda_{0,i}$ eigenvalues of A_0 are real. So $H = 1$ is equivalent to the Cauchy problem being hyperbolic, which is what we desire. Similarly, $H = 0$ would be the worst case scenario, since it would mean that $\forall i \in \{1,2,3,4\}, \frac{\Im(\lambda_{0,i})}{\lambda_{0,i}} = 1$ and that all eigenvalues are purely imaginary.

2.4 Numerical values of the hyperbolicity

H is a real function of the matrix A_0, so it is a function of (p, α_g, u_g, u_l). Let us take the example of the two-fluid model in its commonly used form with a BWR pressure. I chose

$$(p, \alpha_g, u_g, u_l) = \left(76\,\mathrm{bar}, 0.5, 500\,\mathrm{m\,s^{-1}}, 20\,\mathrm{m\,s^{-1}}\right) \tag{2.29}$$

The calculation of H results in $H = 0.64$. As $H < 1$, the two-fluid model in its commonly used form is indeed not hyperbolic under those conditions.

If we keep every other variable constant, we can plot the numerical hyperbolicity H as a function of the void fraction α. So for $(p, u_g, u_l) = \left(76\,\mathrm{bar}, 500\,\mathrm{m\,s^{-1}}, 20\,\mathrm{m\,s^{-1}}\right)$, it results in figure 2.1.

What we can see from the plot is that the numerical hyperbolicity is never equal to 1 except for when $\alpha = 0$ (only liquid water) and when $\alpha = 1$ (only steam). In all other cases, that is to say when $\alpha \in]0;1[$, the hyperbolicity is strictly smaller than 1 and the problem is ill-posed.

Similarly, we can also plot the numerical hyperbolicity H as a function of one of the velocities, if we set every parameter, but this velocity, to be constant. In fact, the calculation shows that H depends directly on the relative velocity $(u_g - u_l)$ but not on any individual velocity u_g or u_l. Therefore, it is most convenient to plot H as a function of $\frac{(u_g-u_l)^2}{c_g^2}$. The parameter $\frac{(u_g-u_l)^2}{c_g^2}$ is non-dimensional and is equivalent to a normalized relative velocity. We will further refer to it as just the *relative velocity* and we will write it sometimes as $\frac{\Delta v^2}{c_g^2}$.

Let us keep our example of the two-fluid model in its commonly used form in BWR-pressure conditions with equal volumes of liquid water and steam. If we set $(p, \alpha, u_l) = \left(76\,\mathrm{bar}, 0.5, 20\,\mathrm{m\,s^{-1}}\right)$, then we get figure 2.2.

The majority of relative velocities gives $H < 1$, this is shown by the convex curve. The numerical hyperbolicity is equal to 1 only when the relative velocity is zero $(u_g = u_l)$ and beyond a certain value of $\frac{\Delta v^2}{c_g^2}$, 2.5928 here. This is why we obtain a plateau of $H = 1$ after this boundary value. But in the range between 0 and 2.5928, the two-fluid model is non-hyperbolic and would need a correction.

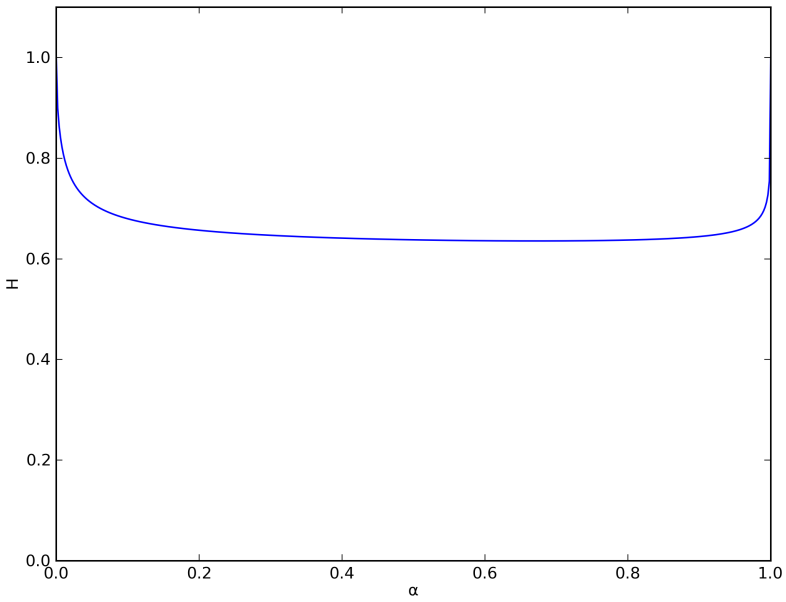

Figure 2.1: Numerical hyperbolicity as a function of void fraction, for the two-fluid model in its commonly used form

For a more complete visualization, we can also plot H as a function of two variables in the same time; the void fraction α and the relative velocity $\frac{(u_g-u_l)^2}{c_g^2}$. As a consequence, we obtain a 3D plot. Figure 2.3 represents the numerical hyperbolicity in the conditions of $(p, u_l) = \left(76\,\mathrm{bar}, 20\,\mathrm{m\,s^{-1}}\right)$.

This 3D plot summarizes what we have seen before: in most cases of practical interest, the hyperbolicity is different from 1 and the plot has the shape of a valley. This valley is surrounded by boundaries where $H = 1$, like an exception. This is the case for $\alpha = 0$, $\alpha = 1$ and $\frac{(u_g-u_l)^2}{c_g^2} = 0$. It is also the case when the relative velocity is large enough, hence the garnet-colored shape of a plateau.

As a side note, we can notice that more realistic conditions of $(p, \alpha_g, u_g, u_l) = \left(76\,\mathrm{bar}, 0.1, 3\,\mathrm{m\,s^{-1}}, 2\,\mathrm{m\,s^{-1}}\right)$ yield to a numerical hyperbolicity of $H = 0.9011$. So the problem is indeed non-hyperbolic. As we said before, in this thesis we will prefer to stay with the less common conditions of $(p, \alpha_g, u_g, u_l) = \left(76\,\mathrm{bar}, 0.5, 500\,\mathrm{m\,s^{-1}}, 20\,\mathrm{m\,s^{-1}}\right)$ because it leads to clearer results.

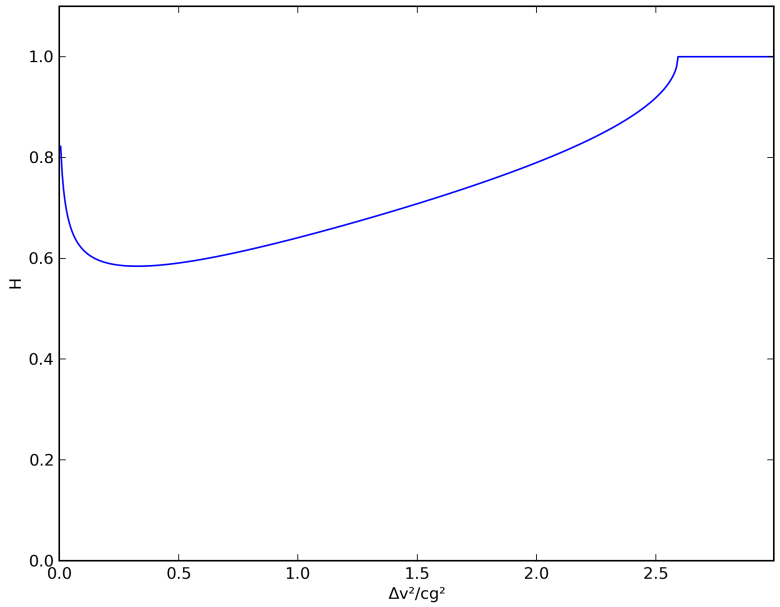

Figure 2.2: Numerical hyperbolicity as a function of relative velocity, for the two-fluid model in its commonly used form

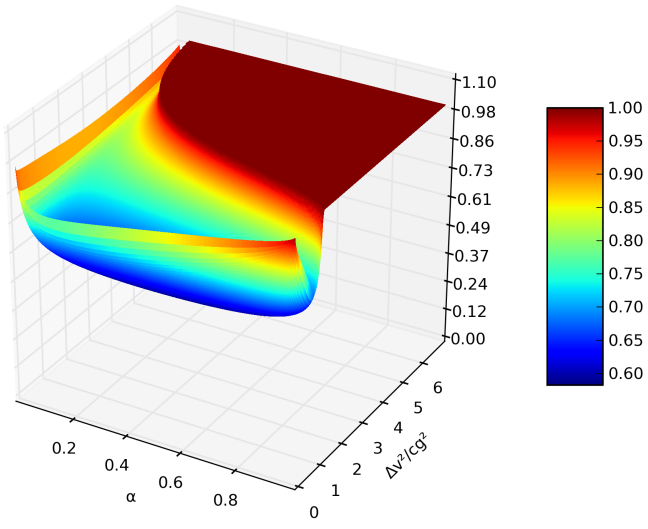

Figure 2.3: Numerical hyperbolicity as a function of void fraction and relative velocity, for the two-fluid model in its commonly used form

Chapter 3

Previous attempts to cure the ill-posedness with interfacial forces

3.1 Importance of curing the ill-posedness of the problem

There has been a vocal community pointing at the ill-posedness of the two-fluid model in its commonly used form, some notable examples being Stewart and Wendroff [18], or Wulff [19]. Nonetheless, there exists some controversy about the topic. Some still say it is not that of an issue to have the two-fluid model not hyperbolic in most cases. We know that we could have multiple solutions, but what is done in current generation nuclear codes is assuming that the multiple solutions are close enough to each other to introduce only a small uncertainty. This uncertainty is hopefully negligible compared to the uncertainty originating from other sources, like the uncertainty of the physical models or the numerical diffusion for instance. The TRACE Theory Manual [3] goes even beyond and says that because of the numerical discretization, some errors are introduced in the equations, which make them hyperbolic. If we hypothetically believed that this is true, then we would say that it is no use spending effort on ensuring hyperbolicity for so little reward. We would imitate those who call the ill-posedness a "moot point".

First of all, the non-hyperbolicity actually is an important issue. It is at least a conceptual issue, since getting multiple solutions does not seem very reassuring. We have seen in 2.2 that an ill-posed Cauchy problem implies that we cannot state anything about its eventual space convergence and that in most practical cases, the code has no space convergence. This means that we get the counter-intuitive property that the tighter the spatial grid of our model is, the less precise our results are. Therefore, in current codes manuals, it is stated that the user should not exceed a certain mesh refinement. It is true that that is acceptable for most study cases. However for a next generation code we should desire space convergence too.

Second, the non-hyperbolicity could be a far bigger issue than what we now think. The reason is that it is at the source of all models, it takes its origin in the fundamental equations which are the base layer for computation. So the uncertainty created by non-hyperbolicity, supposedly small, could in fact diffuse through all the steps of calculation (discretization, computation, etc) and then get bigger and bigger. There is little proof that the uncertainty is negligible in the final result.

Third, wishfully hoping that the numerical errors of solvers restore the well-posedness is not a sustainable position. Correcting an error with another error is avoiding the root cause of the issue. It also is not future-proof for the time that the numerical errors will be reduced thanks to the ongoing efforts made by the numerical methods community.

Fourth and finally, even if the uncertainty associated with non-hyperbolicity is very small, we still should have the ambition in next generation nuclear codes to reduce all uncertainties. Given the community's experience in codes and given the incredible development of computational power in recent years, the community should now try to tackle all the problems. This would be an improvement in precision, it would lead to models with far less uncertainty. The advantages for security and efficiency are then obvious.

3.2 Ways to avoid the ill-posedness, with and without interfacial forces

3.2.1 Differentiating the pressure in the two fluids

The derivation shows that the non-hyperbolicity and complex eigenvalue come from our initial hypothesis made in equation 1.4: the pressure p is similar in the liquid and the gas phases. This hypothesis is made by most scientists and engineers, and it is not an unreasonable one. This means that any eventual difference in pressure is quickly erased, in a time scale that is negligible compared to the time scale of the evolution of all the other parameters.

However, we may consider two different pressures, p_g and p_l, at least from a formal point of view. By adding a new term, the system of equations cannot be closed any more. This is why it is necessary to add an equation, and many [14] use the void fraction transport equation, here given for the 3D case:

$$\frac{\partial \alpha}{\partial t} + \vec{\nabla}(\alpha \vec{u_g}) = 0 \tag{3.1}$$

In a one-dimensional geometry, this equation is just

$$\frac{\partial}{\partial t}(\alpha) + \frac{\partial}{\partial x}(\alpha u_g) = 0 \tag{3.2}$$

So this leads to 12 equations with 12 unknowns:

$$\begin{gathered}
\alpha_g + \alpha_l = 1 \\
\frac{\partial}{\partial t}(\alpha_g) + \frac{\partial}{\partial x}(\alpha_g u_g) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g) = 0 \\
\frac{\partial}{\partial t}(\alpha_l \rho_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g u_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g^2) + \alpha_g \frac{\partial p_g}{\partial x} = F_g \\
\frac{\partial}{\partial t}(\alpha_l \rho_l u_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l^2) + \alpha_l \frac{\partial p_l}{\partial x} = F_l \\
\frac{\partial}{\partial t}(\alpha_g e_g) + \frac{\partial}{\partial x}(\alpha_g e_g u_g) + \alpha_g \frac{\partial}{\partial x}(p_g u_g) = \{\text{Net energy exchange}\}_g + \{\text{Net energy generation}\}_g \\
\frac{\partial}{\partial t}(\alpha_l e_l) + \frac{\partial}{\partial x}(\alpha_l e_l u_l) + \alpha_l \frac{\partial}{\partial x}(p_l u_l) = \{\text{Net energy exchange}\}_l + \{\text{Net energy generation}\}_l \\
e_g = \frac{1}{2}\rho_g u_g^2 + e_{int,g} \\
e_l = \frac{1}{2}\rho_l u_l^2 + e_{int,l} \\
f_g(p_g, e_{int}) = 0 \\
f_l(p_l, e_{int}) = 0
\end{gathered} \tag{3.3}$$

This idea could be promising. But for nearly incompressible flows, the second equation (void fraction transport) and the third equation (gas mass transport) become proportional one to another by a factor of ρ_g. In this case, we loose the advantage of having added a new equation. Anyway, the scope of this work is to use the conceptually simpler solution of interfacial forces.

3.2.2 Considering the interfacial pressure

Another option to solve the non-hyperbolicity issue is to consider the interfacial forces. Let us recall the first matrix equation we got in 2.11:

$$M_t \frac{\partial U}{\partial t} + M_x \frac{\partial U}{\partial x} = S \tag{3.4}$$

The source term is equal to $S = \begin{pmatrix} 0 & 0 & F_g & F_l \end{pmatrix}^t$ and the matrix M_t is invertible. So I went on to get

$$\frac{\partial U}{\partial t} + M_t^{-1} M_x \frac{\partial U}{\partial x} = M_t^{-1} S \tag{3.5}$$

$$\frac{\partial U}{\partial t} + A_0 \frac{\partial U}{\partial x} = S_0 \tag{3.6}$$

We explained that the previous chapter was dealing exclusively with the two-fluid model in its commonly used form. That means by definition that the two source terms F_g and F_l did not include any derivatives of the components of U. So studying the hyperbolicity of the main matrix A_0 was sufficient and led to prove that the problem was ill-posed.

Now it would be interesting to challenge the two-fluid model in its commonly used form. In other words, we could consider expressions for F_g and F_l that do contain $\frac{\partial}{\partial t}$ or $\frac{\partial}{\partial x}$ derivatives of $U = (p, \alpha, u_g, u_l)$. If this is the case, then those derivatives could be moved to the left-hand side of the equations, modify the matrices M_t and M_x and thus modify the main matrix of the problem [5]. As the eigenvalues may then be different, this means that the hyperbolicity could be restored, which translates into the numerical hyperbolicity being equal to 1.

The column-vector S can contain many different forces, like gravity or drag force, but most of them do not add components to the two matrices on the left hand side of the matrix equation 2.9. In the literature, two main forces are considered to solve that issue.

The first force we are going to consider here is the interfacial pressure force F^p. As explained by Ndjinga, Kumbaro, de Vuyst and Laurent-Gengoux [13], the interfacial pressure is a correction term added to the two-fluid model. It is used to take into account that the average pressure in the fluids and pressure in the interface between the fluids can be slightly different, but without having to explicitly define p_g and p_l. The interfacial pressure is an interfacial force that is commonly used to cure the non-hyberbolicity.

Many authors, including Theofanus, Chang, Nguyen, Sushchikh and Liou in [6], use the following expression for F_g^p:

$$F_g^p = -\Delta P \frac{\partial \alpha_g}{\partial x} \tag{3.7}$$

We call the ΔP factor the interfacial pressure. As F_g^p is a volumetric force, ΔP has the dimension of a pressure and has SI units of Pa or $\mathrm{kg\,m^{-1}\,s^{-2}}$. This will be the formula that we will use throughout this work.

As the F^p is an interfacial force that represent the action of the phases on each other, Newton's third law on reciprocity can be applied and leads to

$$F_l^p = -F_g^p \tag{3.8}$$

As we can see, the source S will have a term as $\begin{pmatrix} 0 & 0 & -\Delta P \frac{\partial \alpha_g}{\partial x} & \Delta P \frac{\partial \alpha_g}{\partial x} \end{pmatrix}^t$, which will have an influence on matrix M_x and therefore on the main matrix of the problem. We will study the precise influence in section 3.3.

Different analytical formulas have been suggested for the interfacial pressure ΔP. The article of Ndjinga et al. [13] recaps some of them:

- $\Delta P = 0$. This is the two-fluid model in its commonly used form, which we try to improve.
- $\Delta P = \rho_g (u_g - u_l)^2$

- $\Delta P = (1+\epsilon)\Delta P_c$, where $\epsilon > 0$ (like $\epsilon = 0.01$ for instance) and ΔP_c is given by

$$\Delta P_c = \frac{\alpha_g \alpha_l \rho_g \rho_l}{\alpha_g \rho_l + \alpha_l \rho_g}(u_l - u_g)^2 \tag{3.9}$$

  This is the formula used in nuclear code CATHARE.

3.2.3 Considering the virtual mass

The second interfacial force that is often considered in the literature to cure the hyperbolicity is the virtual mass force F^{vm}. The virtual mass force, also sometimes called added mass effect, is the inertia added to a body when it is moving in a fluid. The accelerating system must move the surrounding fluid as it evolves in it. This is a well-known phenomenon in the naval industry because that force must be accounted for when planning the quantity of fuel that a ship has to carry to arrive at destination. In our case of a two-phase flow, the virtual mass force typically applies to droplets moving through steam or to bubbles in water.

For F_g^{vm}, we will use the following expression, used for instance by Park, Drew and Lahey in [15]:

$$F_g^{vm} = -Cvm\left(\left(\frac{\partial u_g}{\partial t} + u_g\frac{\partial u_g}{\partial x}\right) - \left(\frac{\partial u_l}{\partial t} + u_l\frac{\partial u_l}{\partial x}\right)\right) \tag{3.10}$$

The Cvm factor is named the virtual mass coefficient. It is positive, it has the dimension of a volumetric mass (a density) and has SI units of $\mathrm{kg\,m^{-3}}$.

Because the virtual mass force is an interfacial force between the two phases, we can use Newton's third law on reciprocity:

$$F_l^{vm} = -F_g^{vm} \tag{3.11}$$

Since the expression of the virtual mass has both $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ terms, it will have an influence of both M_t and M_x, and therefore on the final matrix of the problem. The precise influence of F^{vm} on the eigenvalues and on the numerical hyperbolicity is studied in section 3.3.

Ndjinga et al. [13] recall some expressions used for the coefficient Cvm, which has the dimension of a volumetric mass (a density):

- $Cvm = 0$ is simply the two-fluid model in its commonly used form.
- $Cvm = \frac{1}{2}\alpha_g\alpha_l(\alpha_g\rho_g + \alpha_l\rho_l)$. This expression of Cvm is particularly adapted to flows of spherical bubbles as recalled by Ndjinga et al. [13].

Virtual mass is used in nuclear code RELAP5.

3.2.4 Considering higher order derivatives

Some authors, like Fullmer, Prabhudharwadkar, Vaidheeswaran , Ransom and Lopez de Bertodano explain in their article [8], try to cure the ill-posedness by modifying the order of the differential equation. This means that they get to an expression such as a third order partial differential equation.

$$D_t\frac{\partial U}{\partial t} + D_1\frac{\partial U}{\partial x} + D_2\frac{\partial^2 U}{\partial x^2} + D_3\frac{\partial^3 U}{\partial x^3} + D_0 = 0 \tag{3.12}$$

The way Fullmer gets to this kind of equation is considering different pressures in the two phases, but not letting both p_g and p_l as free variables. He explains that p_g is the reference pressure and is used as the variable. The

pressure in the liquid phase p_l is given by

$$p_l = p_g + \sigma z \frac{\partial^2 \alpha}{\partial x^2} \tag{3.13}$$

σ is the surface tension (in N m^{-1} for instance) and z is the channel height. Fullmer also accounts for the pressure variation due to gravity, but I did not include it here for simplicity.

Using the new definition of p_l, let us insert it into the following set of 11 equations:

$$\begin{gathered}
\alpha_g + \alpha_l = 1 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g) = 0 \\
\frac{\partial}{\partial t}(\alpha_l \rho_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g u_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g^2) + \alpha_g \frac{\partial p_g}{\partial x} = F_g \\
\frac{\partial}{\partial t}(\alpha_l \rho_l u_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l^2) + \alpha_l \frac{\partial p_l}{\partial x} = F_l \\
\frac{\partial}{\partial t}(\alpha_g e_g) + \frac{\partial}{\partial x}(\alpha_g e_g u_g) + \alpha_g \frac{\partial}{\partial x}(p_g u_g) = \{\text{Net energy exchange}\}_g + \{\text{Net energy generation}\}_g \\
\frac{\partial}{\partial t}(\alpha_l e_l) + \frac{\partial}{\partial x}(\alpha_l e_l u_l) + \alpha_l \frac{\partial}{\partial x}(p_l u_l) = \{\text{Net energy exchange}\}_l + \{\text{Net energy generation}\}_l \\
e_g = \frac{1}{2}\rho_g u_g^2 + e_{int,g} \\
e_l = \frac{1}{2}\rho_l u_l^2 + e_{int,l} \\
f_g(p_g, e_{int}) = 0 \\
f_l(p_l, e_{int}) = 0
\end{gathered} \tag{3.14}$$

For clarity, let us focus on the conservation laws within the barotropic hypothesis: conservation of mass and of momentum.

$$\begin{gathered}
\frac{\partial}{\partial t}(\alpha_g \rho_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g) = 0 \\
\frac{\partial}{\partial t}(\alpha_l \rho_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g u_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g^2) + \alpha_g \frac{\partial p_g}{\partial x} = F_g \\
\frac{\partial}{\partial t}(\alpha_l \rho_l u_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l^2) + \alpha_l \frac{\partial}{\partial x}(p_g + \sigma z \frac{\partial^2 \alpha_g}{\partial x^2}) = F_l
\end{gathered} \tag{3.15}$$

If the channel height z is constant, that is to say if the one-dimensional pipe is horizontal, and if the surface tension σ is uniform, then we can expand some terms:

$$\begin{gathered}
\frac{\partial}{\partial t}(\alpha_g \rho_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g) = 0 \\
\frac{\partial}{\partial t}(\alpha_l \rho_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g u_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g^2) + \alpha_g \frac{\partial p_g}{\partial x} = F_g \\
\frac{\partial}{\partial t}(\alpha_l \rho_l u_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l^2) + \alpha_l \frac{\partial p_g}{\partial x} + \alpha_l \sigma z \frac{\partial^3 \alpha_g}{\partial x^3} = F_l
\end{gathered} \tag{3.16}$$

The full development gives us

$$\begin{aligned}
\alpha_g\frac{\partial\rho_g}{\partial t}+\rho_g\frac{\partial\alpha_g}{\partial t}+\alpha_g u_g\frac{\partial\rho_g}{\partial x}+\rho_g u_g\frac{\partial\alpha_g}{\partial x}+\alpha_g\rho_g\frac{\partial u_g}{\partial x}&=0\\
\alpha_l\frac{\partial\rho_l}{\partial t}+\rho_l\frac{\partial\alpha_l}{\partial t}+\alpha_l u_l\frac{\partial\rho_l}{\partial x}+\rho_l u_l\frac{\partial\alpha_l}{\partial x}+\alpha_l\rho_l\frac{\partial u_l}{\partial x}&=0\\
\alpha_g u_g\frac{\partial\rho_g}{\partial t}+\rho_g u_g\frac{\partial\alpha_g}{\partial t}+\alpha_g\rho_g\frac{\partial u_g}{\partial t}+\alpha_g u_g^2\frac{\partial\rho_g}{\partial x}+\rho_g u_g^2\frac{\partial\alpha_g}{\partial x}+2\alpha_g\rho_g u_g\frac{\partial u_g}{\partial x}+\alpha_g\frac{\partial p_g}{\partial x}&=F_g\\
\alpha_l u_l\frac{\partial\rho_l}{\partial t}+\rho_l u_l\frac{\partial\alpha_l}{\partial t}+\alpha_l\rho_l\frac{\partial u_l}{\partial t}+\alpha_l u_l^2\frac{\partial\rho_l}{\partial x}+\rho_l u_l^2\frac{\partial\alpha_l}{\partial x}+2\alpha_l\rho_l u_l\frac{\partial u_l}{\partial x}+\alpha_l\frac{\partial p_g}{\partial x}+\alpha_l\sigma z\frac{\partial^3\alpha_g}{\partial x^3}&=F_l
\end{aligned}\tag{3.17}$$

As we did before, let us realize the elementary operations $L_3 \leftarrow L_3 - u_g \cdot L_1$ and $L_4 \leftarrow L_4 - u_l \cdot L_2$.

$$\begin{aligned}
\alpha_g\frac{\partial\rho_g}{\partial t}+\rho_g\frac{\partial\alpha_g}{\partial t}+\alpha_g u_g\frac{\partial\rho_g}{\partial x}+\rho_g u_g\frac{\partial\alpha_g}{\partial x}+\alpha_g\rho_g\frac{\partial u_g}{\partial x}&=0\\
\alpha_l\frac{\partial\rho_l}{\partial t}+\rho_l\frac{\partial\alpha_l}{\partial t}+\alpha_l u_l\frac{\partial\rho_l}{\partial x}+\rho_l u_l\frac{\partial\alpha_l}{\partial x}+\alpha_l\rho_l\frac{\partial u_l}{\partial x}&=0\\
\alpha_g\rho_g\frac{\partial u_g}{\partial t}+\alpha_g\rho_g u_g\frac{\partial u_g}{\partial x}+\alpha_g\frac{\partial p_g}{\partial x}&=F_g\\
\alpha_l\rho_l\frac{\partial u_l}{\partial t}+\alpha_l\rho_l u_l\frac{\partial u_l}{\partial x}+\alpha_l\frac{\partial p_g}{\partial x}+\alpha_l\sigma z\frac{\partial^3\alpha_g}{\partial x^3}&=F_l
\end{aligned}\tag{3.18}$$

Now let us use $\alpha = \alpha_g = 1 - \alpha_l$ and the barotropic result of $\partial\rho = \frac{1}{c^2}\partial p$.

$$\begin{aligned}
\frac{\alpha_g}{c_g^2}\frac{\partial p_g}{\partial t}+\rho_g\frac{\partial\alpha}{\partial t}+\frac{\alpha_g u_g}{c_g^2}\frac{\partial p_g}{\partial x}+\rho_g u_g\frac{\partial\alpha}{\partial x}+\alpha_g\rho_g\frac{\partial u_g}{\partial x}&=0\\
\frac{\alpha_l}{c_l^2}\frac{\partial p_l}{\partial t}-\rho_l\frac{\partial\alpha}{\partial t}+\frac{\alpha_l u_l}{c_l^2}\frac{\partial p_l}{\partial x}-\rho_l u_l\frac{\partial\alpha}{\partial x}+\alpha_l\rho_l\frac{\partial u_l}{\partial x}&=0\\
\alpha_g\rho_g\frac{\partial u_g}{\partial t}+\alpha_g\frac{\partial p_g}{\partial x}+\alpha_g\rho_g u_g\frac{\partial u_g}{\partial x}&=F_g\\
\alpha_l\rho_l\frac{\partial u_l}{\partial t}+\alpha_l\frac{\partial p_g}{\partial x}+\alpha_l\rho_l u_l\frac{\partial u_l}{\partial x}+\alpha_l\sigma z\frac{\partial^3\alpha}{\partial x^3}&=F_l
\end{aligned}\tag{3.19}$$

Because we do not want to obtain cross-derivatives as $\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}$, let us approximate $\frac{\partial p_l}{\partial t} = \frac{\partial p_g}{\partial t}$. In this case, if we have $U = \begin{pmatrix} p_g & \alpha & u_g & u_l \end{pmatrix}^t$, then we can write

$$\begin{pmatrix} \frac{\alpha_g}{c_g^2} & \rho_g & 0 & 0\\ \frac{\alpha_l}{c_l^2} & -\rho_l & 0 & 0\\ 0 & 0 & \alpha_g\rho_g & 0\\ 0 & 0 & 0 & \alpha_l\rho_l \end{pmatrix}\frac{\partial U}{\partial t}+\begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g\rho_g & 0\\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l\rho_l\\ \alpha_g & 0 & \alpha_g\rho_g u_g & 0\\ \alpha_l & 0 & 0 & \alpha_l\rho_l u_l \end{pmatrix}\frac{\partial U}{\partial x}+\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & \alpha_l\sigma z & 0 & 0 \end{pmatrix}\frac{\partial^3 U}{\partial x^3}=\begin{pmatrix} 0\\ 0\\ F_g\\ F_l \end{pmatrix}\tag{3.20}$$

So we can rewrite it in a compact matrix equation:

$$D_t\frac{\partial U}{\partial t}+D_1\frac{\partial U}{\partial x}+D_2\frac{\partial^2 U}{\partial x^2}+D_3\frac{\partial^3 U}{\partial x^3}+D_0=0\tag{3.21}$$

$$D_t=\begin{pmatrix} \frac{\alpha_g}{c_g^2} & \rho_g & 0 & 0\\ \frac{\alpha_l}{c_l^2} & -\rho_l & 0 & 0\\ 0 & 0 & \alpha_g\rho_g & 0\\ 0 & 0 & 0 & \alpha_l\rho_l \end{pmatrix}\tag{3.22}$$

$$D_1 = \begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g \rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l \rho_l \\ \alpha_g & 0 & \alpha_g \rho_g u_g & 0 \\ \alpha_l & 0 & 0 & \alpha_l \rho_l u_l \end{pmatrix} \tag{3.23}$$

$$D_2 = 0 \tag{3.24}$$

$$D_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_l \sigma z & 0 & 0 \end{pmatrix} \tag{3.25}$$

$$D_0 = \begin{pmatrix} 0 \\ 0 \\ F_g \\ F_l \end{pmatrix} \tag{3.26}$$

This is an example of how one gets a third order partial differential equation. This represents a whole new problem compared to what we had before. So the ill- or well-posedness rely on a different type of analysis. This type of analysis is not the scope of this thesis work, so the reader could refer to Fullmer et al. [8] for instance for further investigation. However, even if this technique formally brings well-posedness no matter how small the higher order derivatives value is, we have to recall that the ill-posedness is still a concern numerically. The approximations that transform the theoretical set of equations into a numerical system that gets solved may give a negligible value to the term that was supposed to cure the ill-posedness. The manual of nuclear code RELAP5 [4] recalls it well in its Semi-Implicit Scheme Difference Equations section.

3.3 Modification of matrix A by considering the interfacial pressure and the virtual mass

In this work, we will consider curing the ill-posedness of the two-fluid model by considering interfacial forces. So we will consider both the interfacial pressure and the virtual mass.

Let us start from the equation $M_t \frac{\partial U}{\partial t} + M_x \frac{\partial U}{\partial x} = S$ and let us use an extended expression for S, that is to say for F_g and F_l.

We explained that F_g and F_l contain many different forces, but the most interesting ones are those which contain derivatives of the variables. So we will split the interfacial forces F^p and F^{vm} out of the rest of the forces F':

$$\begin{aligned} F_g &= F_g^p + F_g^{vm} + F_g' \\ F_l &= F_l^p + F_l^{vm} + F_l' \end{aligned} \tag{3.27}$$

We already saw a full expression of the interfacial forces:

$$\begin{aligned} F_g^p &= -\Delta P \frac{\partial \alpha_g}{\partial x} \\ F_l^p &= -F_g^p \end{aligned} \tag{3.28}$$

$$\begin{aligned} F_g^{vm} &= -Cvm \left(\left(\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} \right) - \left(\frac{\partial u_l}{\partial t} + u_l \frac{\partial u_l}{\partial x} \right) \right) \\ F_l^{vm} &= -F_g^{vm} \end{aligned} \tag{3.29}$$

We can use these expressions in the equation $M_t \frac{\partial U}{\partial t} + M_x \frac{\partial U}{\partial x} = S$, where $U = \begin{pmatrix} p_g & \alpha & u_g & u_l \end{pmatrix}^t$. The source term column vector S of the equation can be rewritten as the following:

$$S = \begin{pmatrix} 0 \\ 0 \\ F_g \\ F_l \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -Cvm & Cvm \\ 0 & 0 & Cvm & -Cvm \end{pmatrix} \frac{\partial U}{\partial t} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\Delta P & -Cvmu_g & Cvmu_l \\ 0 & \Delta P & Cvmu_g & -Cvmu_l \end{pmatrix} \frac{\partial U}{\partial x} + \begin{pmatrix} 0 \\ 0 \\ F'_g \\ F'_l \end{pmatrix} \tag{3.30}$$

$$S = N_t \frac{\partial U}{\partial t} + N_x \frac{\partial U}{\partial x} + S' \tag{3.31}$$

So the matrix problem becomes

$$M_t \frac{\partial U}{\partial t} + M_x \frac{\partial U}{\partial x} = N_t \frac{\partial U}{\partial t} + N_x \frac{\partial U}{\partial x} + S' \tag{3.32}$$

$$(M_t - N_t) \frac{\partial U}{\partial t} + (M_x - N_x) \frac{\partial U}{\partial x} = S' \tag{3.33}$$

This is the fully developed equation:

$$\begin{pmatrix} \frac{\alpha_g}{c_g^2} & \rho_g & 0 & 0 \\ \frac{\alpha_l}{c_l^2} & -\rho_l & 0 & 0 \\ 0 & 0 & \alpha_g \rho_g + Cvm & -Cvm \\ 0 & 0 & -Cvm & \alpha_l \rho_l + Cvm \end{pmatrix} \frac{\partial U}{\partial t} + \begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g \rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l \rho_l \\ \alpha_g & \Delta P & \alpha_g \rho_g u_g + Cvmu_g & -Cvmu_l \\ \alpha_l & -\Delta P & -Cvmu_g & \alpha_l \rho_l u_l + Cvmu_l \end{pmatrix} \frac{\partial U}{\partial x} = \begin{pmatrix} 0 \\ 0 \\ F'_g \\ F'_l \end{pmatrix} \tag{3.34}$$

The problem is now written in the following simplified form:

$$B \frac{\partial U}{\partial t} + C \frac{\partial U}{\partial x} = S' \tag{3.35}$$

B and C are square matrices of dimension 4. U and S' are column vectors of dimension 4.

The matrix B is invertible. Therefore, we can multiply the equation 3.35 on the left by B^{-1}.

$$\frac{\partial U}{\partial t} + B^{-1} C \frac{\partial U}{\partial x} = B^{-1} S' \tag{3.36}$$

Let us define $A = B^{-1}C$ and $S'' = B^{-1}S'$. I get my very final matrix problem

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = S'' \tag{3.37}$$

We can try to give a developed theoretical expression of A:

$$A = \begin{pmatrix} \frac{\rho_l c_g^2 c_l^2}{\alpha_g \rho_l c_l^2 + \alpha_l \rho_g c_g^2} & \frac{\rho_g c_g^2 c_l^2}{\alpha_g \rho_l c_l^2 + \alpha_l \rho_g c_g^2} & 0 & 0 \\ \frac{\alpha_l c_g^2}{\alpha_g \rho_l c_l^2 + \alpha_l \rho_g c_g^2} & \frac{-\alpha_g c_l^2}{\alpha_g \rho_l c_l^2 + \alpha_l \rho_g c_g^2} & 0 & 0 \\ 0 & 0 & \frac{Cvm + \alpha_l \rho_l}{\alpha_g \rho_g \alpha_l \rho_l + Cvm(\alpha_g \rho_g + \alpha_l \rho_l)} & \frac{Cvm}{\alpha_g \rho_g \alpha_l \rho_l + Cvm(\alpha_g \rho_g + \alpha_l \rho_l)} \\ 0 & 0 & \frac{Cvm}{\alpha_g \rho_g \alpha_l \rho_l + Cvm(\alpha_g \rho_g + \alpha_l \rho_l)} & \frac{Cvm + \alpha_g \rho_g}{\alpha_g \rho_g \alpha_l \rho_l + Cvm(\alpha_g \rho_g + \alpha_l \rho_l)} \end{pmatrix}$$
$$\cdot \begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g \rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l \rho_l \\ \alpha_g & \Delta P & \alpha_g \rho_g u_g + Cvm u_g & -Cvm u_l \\ \alpha_l & -\Delta P & -Cvm u_g & \alpha_l \rho_l u_l + Cvm u_l \end{pmatrix} \tag{3.38}$$

In order to get a more compact expression, let us introduce two subsidiary symbols. Let $\gamma^2 = \frac{c_g^2 c_l^2}{\alpha_g \rho_l c_l^2 + \alpha_l \rho_g c_g^2}$ (in fact, γ^2 was already used in equation 2.15) and let $\mu = (\alpha_g \rho_g \alpha_l \rho_l + Cvm(\alpha_g \rho_g + \alpha_l \rho_l))^{-1}$. As $Cvm > 0$, μ does not diverge to $+\infty$ and I get the following simplification:

$$A = \begin{pmatrix} \gamma^2 \rho_l & \gamma^2 \rho_g & 0 & 0 \\ \gamma^2 \frac{\alpha_l}{c_l^2} & -\gamma^2 \frac{\alpha_g}{c_g^2} & 0 & 0 \\ 0 & 0 & \mu Cvm + \mu \alpha_l \rho_l & \mu Cvm \\ 0 & 0 & \mu Cvm & \mu Cvm + \mu \alpha_g \rho_g \end{pmatrix}$$
$$\cdot \begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g \rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l \rho_l \\ \alpha_g & \Delta P & \alpha_g \rho_g u_g + Cvm u_g & -Cvm u_l \\ \alpha_l & -\Delta P & -Cvm u_g & \alpha_l \rho_l u_l + Cvm u_l \end{pmatrix} \tag{3.39}$$

Multiplying out does not give an easier expression, however we can notice that when $\Delta P = 0$ and $Cvm = 0$, then we get the two-fluid model in its commonly used form again, and we conveniently have $A = A_0$.

As a conclusion, we get the equation $\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = S''$ where $U = \begin{pmatrix} p & \alpha & u_g & u_l \end{pmatrix}^t$. Let us remember that the matrix A is all but constant, since it depends on many variables. The model is not linear. Nonetheless, it can be assumed as locally linear to the first order.

We recognize here our Cauchy problem which may or may not be well-posed. A good way to measure the hyperbolicity is to use the numerical hyperbolicity we defined earlier. If we write the eigenvalues of A as $sp(A) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, then we can use the same definition for H as before:

$$H(A) = 1 - \frac{1}{4} \sum_{i=1}^{4} \frac{|\Im(\lambda_i)|}{|\lambda_i|} \tag{3.40}$$

The matrix A is not only a function of (p, α, u_g, u_l) as A_0 was, but also of the couple $(\Delta P, Cvm)$. So we will write $A = A\left((p, \alpha, u_g, u_l), (\Delta P, Cvm)\right)$. Similarly, the numerical hyperbolicity is also a function of the same variables.

$$H = H\left((p, \alpha, u_g, u_l), (\Delta P, Cvm)\right) \tag{3.41}$$

Here again it will be useful to see the value of H, since when it is equal to 1, it will mean that the problem is well-posed.

3.4 Limits of current implementations

In the literature, as Ndjinga et al. recall [13], some have already tried to solve the non-hyperbolicity problem by considering interfacial forces. The four cases that we will compare are the following:

- $\Delta P = 0$ and $Cvm = 0$, that is to say the two-fluid model in its commonly used form.
- $\Delta P = \rho_g(u_g - u_l)^2$ and $Cvm = 0$.
- $\Delta P = (1+\epsilon)\Delta P_c = (1+0.01)\frac{\alpha_g\alpha_l\rho_g\rho_l}{\alpha_g\rho_l+\alpha_l\rho_g}(u_l-u_g)^2$ and $Cvm = 0$.
- $\Delta P = 0$ and $Cvm = \frac{1}{2}\alpha_g\alpha_l(\alpha_g\rho_g + \alpha_l\rho_l)$, which we will refer to as the "bubble-type Cvm".

To make a consistent comparison, we will consider again our BWR pressure problem characterized by $(p, \alpha_g, u_g, u_l) = (76\,\text{bar}, 0.5, 500\,\text{m s}^{-1}, 20\,\text{m s}^{-1})$. I then get the following hyperbolicities:

- $\Delta P = 0$ and $Cvm = 0 \Rightarrow H = 0.64$
- $\Delta P = \rho_g(u_g - u_l)^2 = 92\,\text{bar}$ and $Cvm = 0 \Rightarrow H = 1$
- $\Delta P = (1+\epsilon)\Delta P_c = 44\,\text{bar}$ and $Cvm = 0 \Rightarrow H = 0.76$
- $\Delta P = 0$ and $Cvm = \frac{1}{2}\alpha_g\alpha_l(\alpha_g\rho_g + \alpha_l\rho_l) = 48.08\,\text{kg m}^{-3} \Rightarrow H = 0.79$

These results are gathered in Table 3.4, still for our conditions $(p, \alpha_g, u_g, u_l) = (76\,\text{bar}, 0.5, 500\,\text{m s}^{-1}, 20\,\text{m s}^{-1})$. We can notice that three corrections out of four do not restore the hyperbolicity. Only $\Delta P = \rho_g(u_g - u_l)^2$ ensures a hyperbolic problem.

ΔP	0	$\rho_g(u_g-u_l)^2 = 92\,\text{bar}$	$(1+\epsilon)\Delta P_c = 44\,\text{bar}$	0
Cvm	0	0	0	$\frac{1}{2}\alpha_g\alpha_l(\alpha_g\rho_g+\alpha_l\rho_l) = 48.08\,\text{kg m}^{-3}$
H	0.64	1	0.76	0.79

Table 3.1: Numerical hyperbolicity for different choices of $(\Delta P, Cvm)$

Like we did earlier in section 2.4, we can plot the numerical hyperbolicity H as a function of the void fraction α and as a function of the relative velocity $\frac{\Delta v^2}{c_g^2}$. This is shown in dark blue respectively in Figures 3.1 and 3.2. The label "Two-fluid model..." stands for "Two-fluid model in its commonly used form" and "bubble-type Cvm" stands for $Cvm = \frac{1}{2}\alpha_g\alpha_l(\alpha_g\rho_g + \alpha_l\rho_l)$.

With those two figures, we can see again that the two-fluid model in its commonly used form is ill-posed. Additionally, we can see that the correction

$$(\Delta P, Cvm) = \left(0, \frac{1}{2}\alpha_g\alpha_l(\alpha_g\rho_g + \alpha_l\rho_l)\right) \tag{3.42}$$

in cyan falls short to ensure a numerical hyperbolicity equal to 1 on the whole domain. Similarly, the correction

$$(\Delta P, Cvm) = \left((1+\epsilon)\Delta P_c = (1+0.01)\frac{\alpha_g\alpha_l\rho_g\rho_l}{\alpha_g\rho_l+\alpha_l\rho_g}(u_l-u_g)^2, 0\right) \tag{3.43}$$

in red also falls short for values of α close to 0 and $\alpha > 0.3$ (Figure 3.1). In other words, both corrections are insufficient.

On the contrary, the correction $(\Delta P, Cvm) = \left(\rho_g(u_g - u_l)^2, 0\right)$ in green seems sufficient to cure the ill-posedness of the problem. Indeed, Ndjinga et al. show in their article [13] that this is true. Nevertheless, the main problem is that this correction diverges to $+\infty$ when the relative velocity $\frac{\Delta v^2}{c_g^2}$ becomes larger. More precisely, it can be

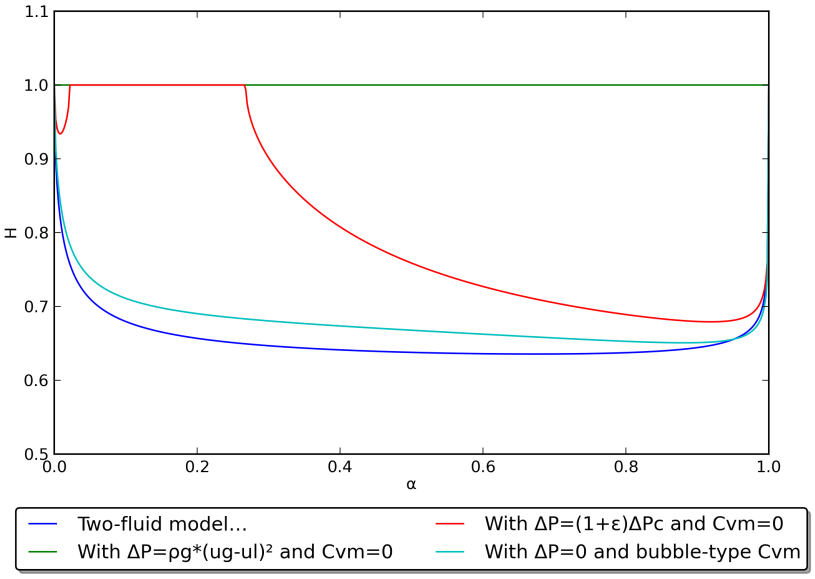

Figure 3.1: Numerical hyperbolicity as a function of void fraction, for different corrections

shown that this correction is, in most cases, unnecessarily too large. It is true that the expression of this correction is simple, however the numerical result is a number that is much larger than the physical pressure. As an example, when $|u_g - u_l| = c_g$ with a pressure of $p = 76\,\text{bar}$, we obtain $\Delta P = \rho_g(u_g - u_l)^2 = 94.740\,\text{bar}$ which is a high value compared to p. So this correction does work but is not necessarily just a small correction. It would be better to find a more appropriate one.

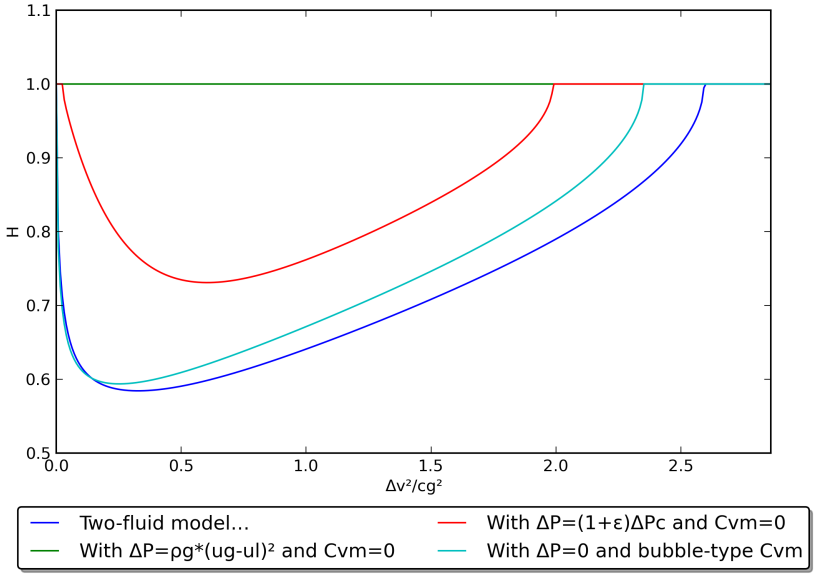

Figure 3.2: Numerical hyperbolicity as a function of relative velocities, for different corrections

Chapter 4

Analysis of differential terms in order to ensure hyperbolicity

4.1 Optimally restoring the hyperbolicity with interfacial pressure only

We see that the hyperbolicity depends largely on the inclusion in our model of the interfacial pressure force, proportional to the ΔP factor [6]. Let us study the influence of the interfacial pressure separately from the one of the virtual mass. For $Cvm = 0$, if $\Delta P = 0$, then the problem is not hyperbolic. On the contrary, for $\Delta P = \rho_g(u_g - u_l)^2$, Ndjinga et al. [13] showed that the problem is always hyperbolic, independently of the (p, α_g, u_g, u_l) conditions. We may be tempted in choosing this last value and generalizing it. However, this correction is too large. In other words, if we include a force that is excessively large, we may push ourselves away from physical reality.

Therefore, this is an optimization problem. We are searching here a value for ΔP that ensures hyperbolicity, but that in the same time is as small as possible, in the context of $Cvm = 0$. This value depends on the flow conditions, that's why we will call it $\Delta P_{optimal} = \Delta P_{optimal}(p, \alpha, u_g, u_l)$. In a logical language, we would write it like the following:

$$\Delta P_{optimal} = \min_{0 \leq \Delta P \leq \rho_g(u_g - u_l)^2} \{\Delta P | H(p, \alpha, u_g, u_l, \Delta P, Cvm = 0) = 1\} \tag{4.1}$$

The result of this optimization problem 4.1 is a numerical pressure value for each set of (p, α_g, u_g, u_l) conditions. For instance, $\Delta P_{optimal}(p = 76\,\text{bar}, \alpha = 0.5, u_g = 500\,\text{m}\,\text{s}^{-1}, u_l = 20\,\text{m}\,\text{s}^{-1}) = 59.830\,\text{bar}$.

We can plot the value of $\Delta P_{optimal}$ as a function of one of the parameters, considered as a variable. Figure 4.1 represents $\Delta P_{optimal}$ as a function of the void fraction α for a BWR pressure, $u_g = 500\,\text{m}\,\text{s}^{-1}$ and $u_l = 20\,\text{m}\,\text{s}^{-1}$. As expected, $\Delta P_{optimal}$ is equal to 0 in the cases of a single-fluid flow, that is to say when $\alpha = 0$ or $\alpha = 1$. No correction is needed in those cases. However, for all the other cases ($\alpha \in]0, 1[$), we need a correction to ensure hyperbolicity so $\Delta P_{optimal} > 0$.

Similarly, and as we did for H in 2.4, we can plot $\Delta P_{optimal}$ as a function of the relative velocity Δv. Here, for more readable results, we will represent $\Delta P_{optimal}$ as a function of $\frac{\Delta v^2}{c_g^2}$ in the usual set of conditions ($p = 76\,\text{bar}, \alpha = 0.5, u_l = 20\,\text{m}\,\text{s}^{-1}$) with figure 4.2.

Now, as we did for the numerical hyperbolicity H, we can try to have a better visualization of the dependency of $\Delta P_{optimal}$ on both the void fraction and the relative velocity. This gives a 3D plot of $\Delta P_{optimal}$ as a function of α and to $\frac{\Delta v^2}{c_g^2}$. The set of conditions for figure 4.3 is $(p, u_l) = (76\,\text{bar}, 20\,\text{m}\,\text{s}^{-1})$. Here again we can see that for large enough relative velocities, no correction is needed and hence the minimal interfacial pressure that ensures hyperbolicity is just simply 0.

It is important to stress the fact that this correction is the minimal and optimal one. It is not necessarily the most physical one or the easiest one to calculate. Especially, we can compare it to the expression $\Delta P = \rho_g(u_g - u_l)^2$. The latter expression always ensures the hyperbolicity of the problem, however it is probably unnecessarily large. In particular, for large relative velocities, $\Delta P = \rho_g \Delta v^2$ diverges to $+\infty$, whereas no correction is needed. Therefore it is possible to say that the values we found of $\Delta P_{optimal}$ may be more realistic in those cases. At least $\Delta P_{optimal}$

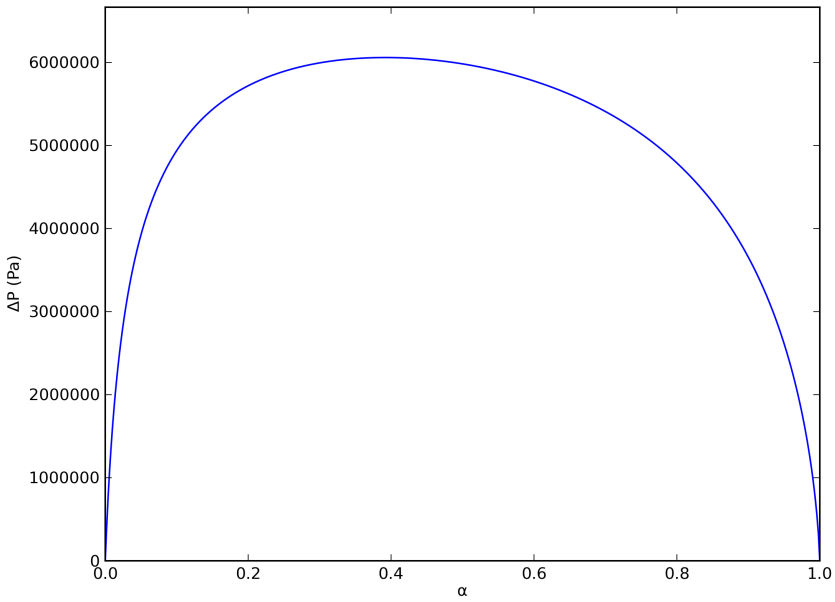

Figure 4.1: Minimal hyperbolicity-ensuring interfacial pressure as a function of void fraction

makes the corrected model as close to the commonly used two-fluid model as possible.

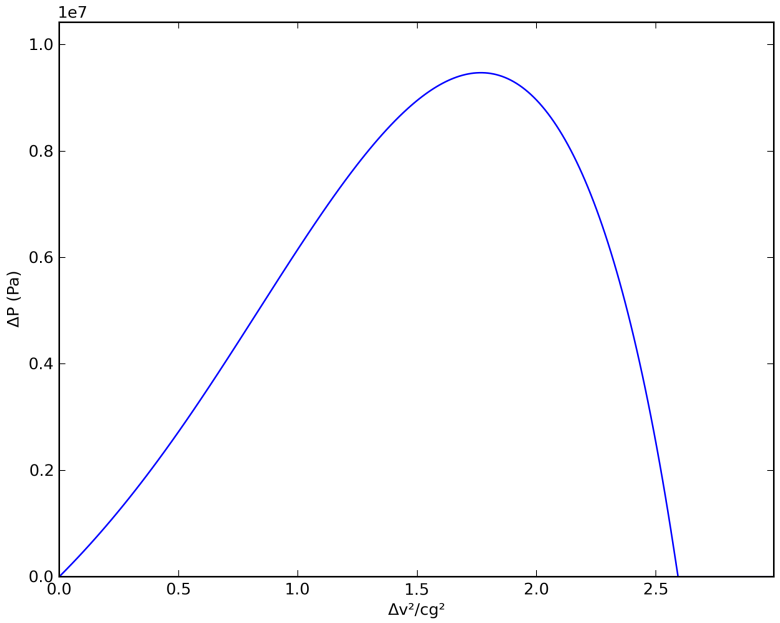

Figure 4.2: Minimal hyperbolicity-ensuring interfacial pressure as a function of relative velocity

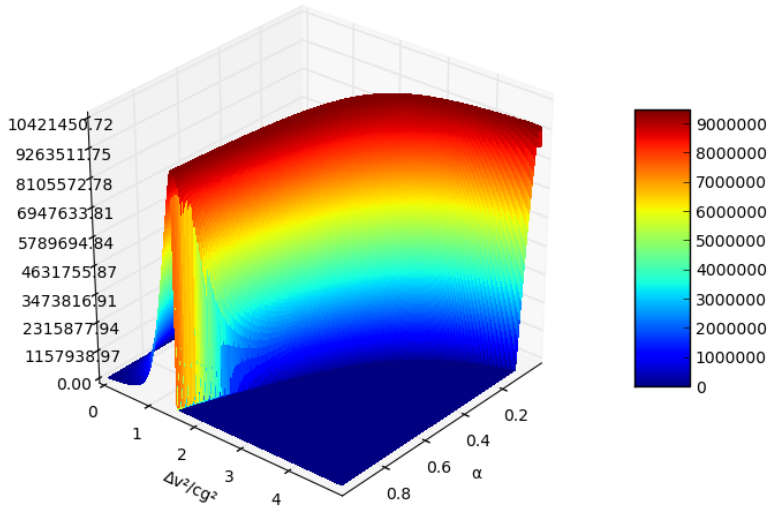

Figure 4.3: $\Delta P_{optimal}$ (in Pa) as a function of void fraction and relative velocity

4.2 Optimally restoring the hyperbolicity with virtual mass only

This time we want to study the influence of the virtual mass on the hyperbolicity, in the absence of any interfacial pressure. So here again, we have an optimization problem, as we already encountered for ΔP in 4.1. In the context of $\Delta P = 0$, we see that the bigger Cvm, the bigger the hyperbolicity H. At the same time, we would like to keep the virtual mass coefficient as small as possible, with a view to keeping the model as close to the common expression of the two-fluid model as possible. The best value depends on the flow conditions, so we will call it $Cvm_{optimal} = Cvm_{optimal}(p, \alpha, u_g, u_l)$. The expression in a logical language would be as follows:

$$Cvm_{optimal} = \min_{0 \leq Cvm} \{Cvm | H(p, \alpha, u_g, u_l, \Delta P = 0, Cvm) = 1\} \tag{4.2}$$

After calculation, the value of $Cvm_{optimal}$ in the case of $(p, \alpha_g, u_g, u_l) = (76\,\text{bar}, 0.5, 500\,\text{m s}^{-1}, 20\,\text{m s}^{-1})$ is $Cvm_{optimal} = 367.26\,\text{kg m}^{-3}$.

As we have done before, we can study the dependency of $Cvm_{optimal}$ as a function of the void fraction α. Figure 4.4 shows $Cvm_{optimal}$ for the set of conditions of a BWR pressure, $u_g = 500\,\text{m s}^{-1}$ and $u_l = 20\,\text{m s}^{-1}$. What we can

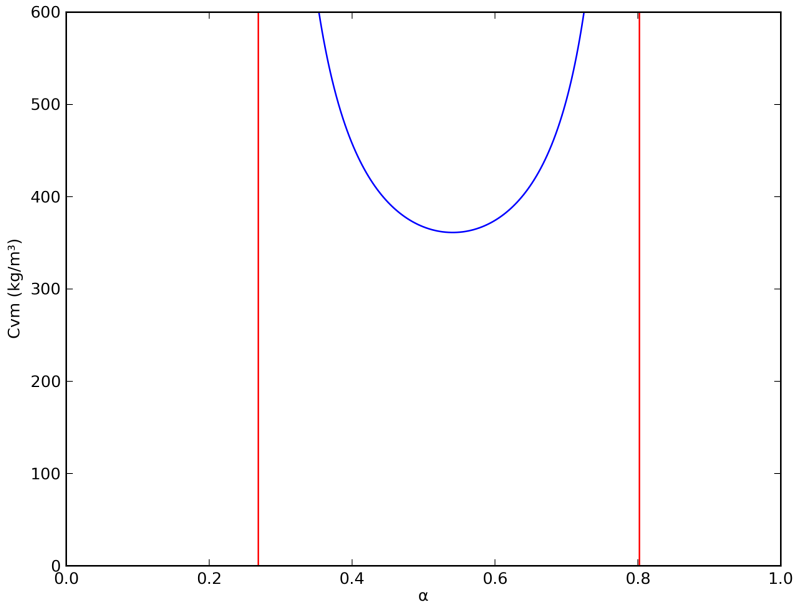

Figure 4.4: Minimal hyperbolicity-ensuring virtual mass as a function of void fraction

see in figure 4.4 is a little more complex than what we had found for $\Delta P_{optimal}$ in figure 4.1. Indeed, there exist two zones, marked by the red vertical lines, where it is not possible to find a value for $Cvm_{optimal}$. The formula 4.2 would give $Cvm_{optimal} = +\infty$, so this means that no matter how large Cvm is, if the void fraction is either too small or too large, then it is not possible to get a hyperbolic problem just by adding the virtual mass force to the two-fluid model. $Cvm_{optimal}$ exists only for α larger than 0.269 and smaller than 0.802. In the remaining cases, one has to use another correction than the virtual mass.

Similarly, we can plot $Cvm_{optimal}$ as a function of the relative velocity and get figure 4.5 for $(p = 76\,\text{bar}, \alpha = 0.5, u_l = 20\,\text{m s}^{-1})$. Here again, there is a zone marked by a red vertical line where it is not possible to find a non-infinite value for $Cvm_{optimal}$. If $\frac{\Delta v^2}{c_g^2}$ is smaller than 0.82036, then it is not possible to make this BWR-pressured problem hyperbolic with the help of virtual mass only.

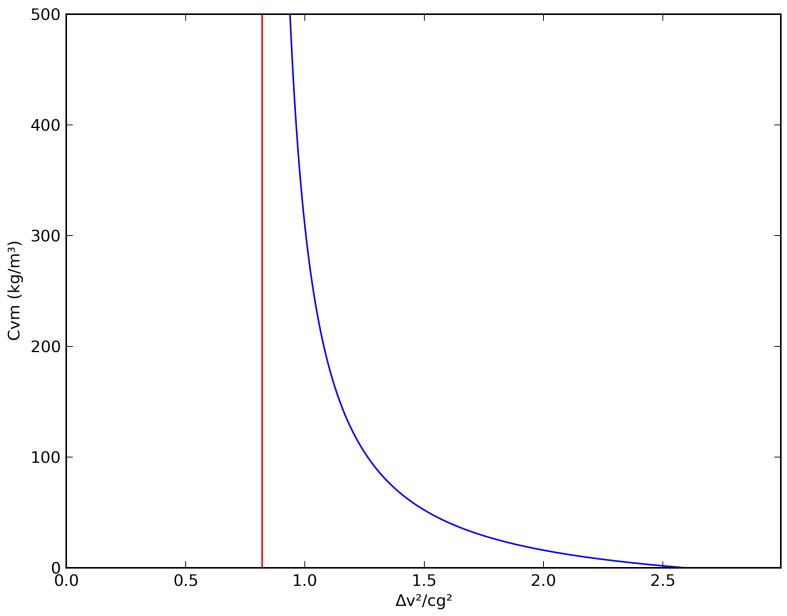

Figure 4.5: Minimal hyperbolicity-ensuring virtual mass as a function of relative velocity

For better visualization, we can show the 3D plot of $Cvm_{optimal}$ as a function of both the void fraction and the relative velocity in the case of ($p = 76\,\mathrm{bar}, u_l = 20\,\mathrm{m\,s^{-1}}$). This leads to figure 4.6. For visualization purposes, we chose to represent any value greater than or equal to $360\,\mathrm{kg\ m^{-3}}$ as equal to $360\,\mathrm{kg\ m^{-3}}$. This permits not to have a wall shaped plot but rather a plateau. We can clearly see on figure 4.6 that we get a garnet-colored plateau for the regions where the void fraction is close to 0 or 1 and/or where the relative velocity is too small. This region includes the area where virtual mass cannot ensure real and distinct eigenvalues, however large its value may be. But we can also see the deep blue-colored region, where $Cvm_{optimal}$ is equal to zero and where no correction is needed.

One may wonder why, on those graphs, $Cvm_{optimal}$ is not equal to 0 in the cases where the model is already supposed to be hyperbolic: when $\alpha = 0$, when $\alpha = 1$ and finally when $\frac{\Delta v^2}{c_g^2} = 0$. The explanation is not easy since it appeals to the notion of "evanescent phases". The evanescence of a phase is the phenomenon of disappearance of one phase out of the initial two, due to either total evaporation or total condensation. This phenomenon is out of the scope of the thesis, but the curious reader will be able to find more information by referring to the appendix E.1 analysing one-phase flows and also to Saleh's work [16].

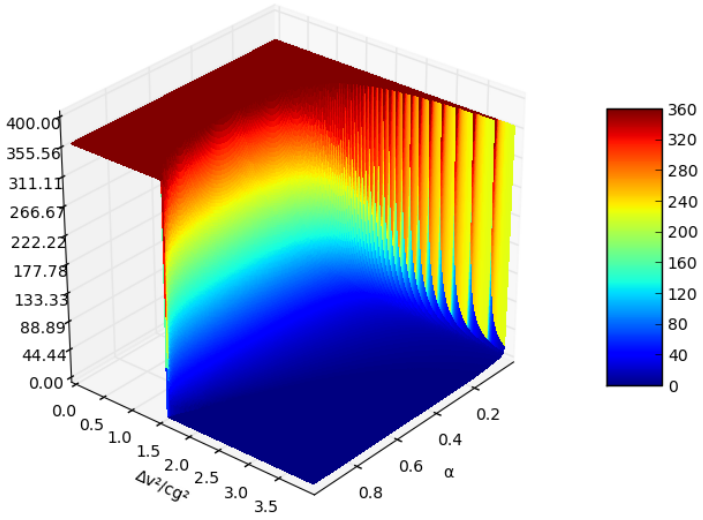

Figure 4.6: $Cvm_{optimal}$ (in kg m^{-3}) as a function of void fraction and relative velocity

4.3 Combination of corrections

4.3.1 Objective, comparison and normalization

In the previous section we have seen that introducing interfacial forces in the two-fluid model equations could help make them hyperbolic. Additionally, we have seen that I could find a minimum value for ΔP, called $\Delta P_{optimal}$, such that the problem is hyperbolic but ΔP is not unreasonnably large. $\Delta P_{optimal}$ is then a function of the flow conditions (p, α, u_g, u_l). We have also defined in the same manner $Cvm_{optimal}$ which is the minimum virtual mass coefficient to have a well-posed problem. However, we noticed that $Cvm_{optimal}$ was not always defined, depending on the conditions.

At this point, it would be interesting to have a method to know which of the two suggested corrections is the correct one. As a way of determining it, we can pick up the correction that is the smallest one, which can be interpreted as the most physical one. To do this we have to compare the numerical value of the two corrections.

Nonetheless it is not possible to compare the two forces F^p and F^{vm} directly because of their expression:

$$\begin{aligned} F_g^p &= -\Delta P \frac{\partial \alpha_g}{\partial x} \\ F_g^{vm} &= -Cvm \left(\left(\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} \right) - \left(\frac{\partial u_l}{\partial t} + u_l \frac{\partial u_l}{\partial x} \right) \right) \end{aligned} \tag{4.3}$$

We do have a numerical value for ΔP, Cvm, u_g and u_l, but we do not have a numerical value for the derivatives $\frac{\partial \alpha_g}{\partial x}$, $\frac{\partial u_g}{\partial t}$ and $\frac{\partial u_l}{\partial t}$ since we are considering only a time t_0 and a position x_0. So we do not have a value in N m^{-3} for the two forces; direct comparison is not possible.

As a consequence it would be interesting to compare the two coefficients we studied in the previous section, ΔP and Cvm. However, here direct comparison is not possible either. This is not because we do not have the numerical values, but rather because the two factors are of different dimensions (a volumetric pressure and a volumetric mass,

respectively).

In order to make a consistent comparison, I suggest to normalize them, to get non-dimensional values. Let us define the *normalized interfacial pressure*, ΔP^{norm}, as the interfacial pressure divided by the ambient pressure:

$$\Delta P^{norm} = \frac{\Delta P}{p} \tag{4.4}$$

And let us define the *normalized virtual mass coefficient*, noted Cvm^{norm}, as the ratio of the virtual mass coefficient by the volumetric masses of the fluids:

$$Cvm^{norm} = \frac{Cvm}{\alpha_g \rho_g + \alpha_l \rho_l} \tag{4.5}$$

In a similar manner, we can define $\Delta P^{norm}_{optimal}$ and $Cvm^{norm}_{optimal}$ so that they are both non-dimensional numbers, thus comparable. That way we can determine which force may be the smallest correction.

4.3.2 Choice of minimal optimal correction, definition of the aggregated correction

The idea is, for each set of (p, α, u_g, u_l) conditions, to identify the smallest correction between the interfacial pressure and the virtual mass. The choice is made by taking the force associated with the smallest factor between $\Delta P^{norm}_{optimal}$ and $Cvm^{norm}_{optimal}$. If $Cvm^{norm}_{optimal} = +\infty$ (or better said: is not defined), then we obviously take $\Delta P^{norm}_{optimal}$.

Figure 4.7 shows both $\Delta P^{norm}_{optimal}$ (in green) and $Cvm^{norm}_{optimal}$ (in blue) on the same graph, as functions of the relative velocity $\frac{\Delta v^2}{c_g^2}$. The conditions are those of a BWR pressure (76 bar), $\alpha = 0.5$ and $u_l = 20\,\mathrm{m\,s^{-1}}$.

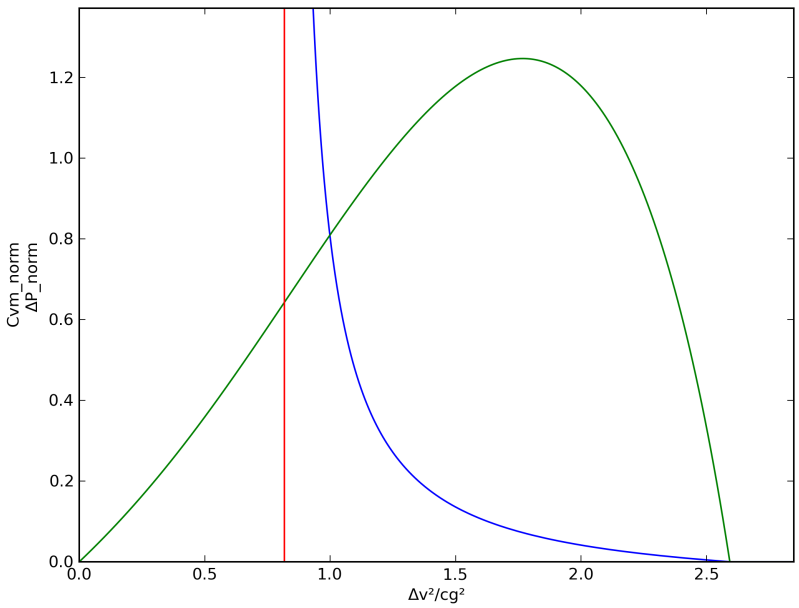

Figure 4.7: $\Delta P^{norm}_{optimal}$ and $Cvm^{norm}_{optimal}$ as a function of relative velocity

We can easily distinguish four zones, depending on the horizontal coordinate $\frac{\Delta v^2}{c_g^2}$:

1. From 0 to 0.82036, $Cvm^{norm}_{optimal}$ is infinite. So we have to choose the interfacial pressure as the minimal correction.

2. From 0.82036 to 1.0013, $Cvm^{norm}_{optimal}$ is defined but still larger than $\Delta P^{norm}_{optimal}$. So we still choose the interfacial pressure as the minimal correction.

3. From 1.0013 to 2.5928, $\Delta P^{norm}_{optimal}$ is actually larger than $Cvm^{norm}_{optimal}$. So we now choose the virtual mass as the minimal correction.

4. For any $\frac{\Delta v^2}{c_g^2}$ larger than 2.5928, both $\Delta P^{norm}_{optimal}$ and $Cvm^{norm}_{optimal}$ are equal to zero. No correction is needed at this point.

To summarize those results as a single number, it seemed useful to me to introduce an *aggregated correction* which I will write as AC. That would be a non-dimensional number equal to the minimum between $\Delta P^{norm}_{optimal}$ and $Cvm^{norm}_{optimal}$.

$$AC(p, \alpha, u_g, u_l) = min(\Delta P^{norm}_{optimal}(p, \alpha, u_g, u_l), Cvm^{norm}_{optimal}(p, \alpha, u_g, u_l), 0) \tag{4.6}$$

For further explanations, I find easier to introduce also a *weight factor* $\eta \in [0, 1]$ to give an equivalent definition of AC.

$$AC(\eta) = \eta \Delta P^{norm}_{optimal} + (1 - \eta) Cvm^{norm}_{optimal} \tag{4.7}$$

Then we minimize $AC(\eta)$ knowing that $\eta \in [0, 1]$. As a matter of fact, it results in only three cases.

- $\eta = 0$, which means that $AC = Cvm^{norm}_{optimal}$ and we choose the virtual mass.
- $\eta = 1$, which means that $AC = \Delta P^{norm}_{optimal}$ and we choose the interfacial pressure.
- η is not defined since we don't need any correction. The problem is already hyperbolic with $(\Delta P, Cvm) = (0, 0)$. By convention we will take $AC = 0$ in this particular case.

So if we fix two parameters (p and u_l for instance), we then can divide the $(\alpha, \frac{\Delta v^2}{c_g^2})$ space into three regions. Figure 4.8 represents this division into three regions for $p = 76\,\text{bar}$ and $u_l = 20\,\text{m}\,\text{s}^{-1}$.

The red region on the left is the one where $\eta = 1$ and the interfacial pressure is preferred. The blue region in the middle is where the weight factor η is equal to 0 and the virtual mass is the smaller correction. Finally the yellow region on the right is the region where no correction is needed at all, so the weight factor is not really defined.

4.3.3 The aggregated correction as a criteria rather than a tool for computation

I introduced earlier AC as $AC(\eta) = min(\Delta P^{norm}_{optimal}, Cvm^{norm}_{optimal}, 0)$. So it is alternatively equal to $\Delta P^{norm}_{optimal}$, to $Cvm^{norm}_{optimal}$ or to 0. When we plot it in 3D as a function of α and $\frac{\Delta v^2}{c_g^2}$, we see that this change from one value to another can be quite abrupt. Figure 4.9 shows this 3D plot with our usual values of $(p, u_l) = (76\,\text{bar}, 20\,\text{m}\,\text{s}^{-1})$.

The graph of AC is continuous everywhere, however it is not smooth at the junction lines. In particular the local maximums of the graph have the shape of a spike. Therefore it may bring calculation artefacts if used exactly that way in nuclear codes, because codes are not resistant to discontinuities in parameters. So the aggregated correction should not be used directly as a tool for computation.

In fact, the aggregated correction is to be interpreted as a boundary, as a criteria. If we decide to solve the non-hyperbolicity problem by adding interfacial forces, then the correction has to be at least greater than the aggregated correction given in figure 4.9. If the correction is smaller, then it does not fulfil its role and it should be rejected. Furthermore, a realistic correction, if it aims at being as little as possible, should be close to the aggregated correction we calculated here. This is why AC can play the role of a comparison criteria to characterize corrections.

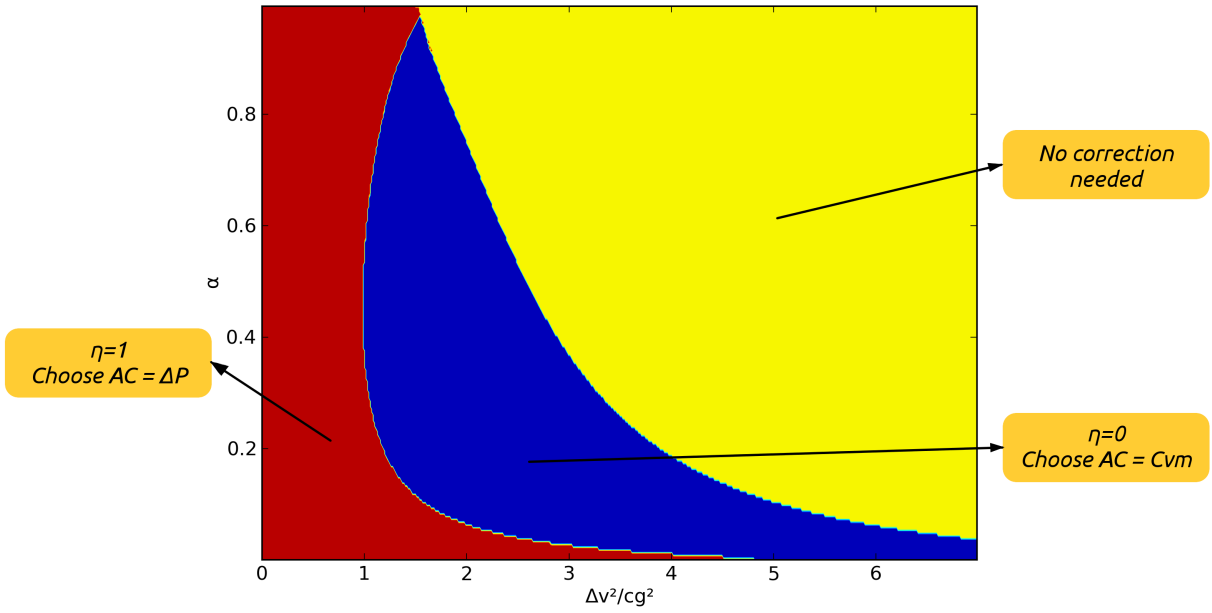

Figure 4.8: η zones as a function of void fraction and relative velocity

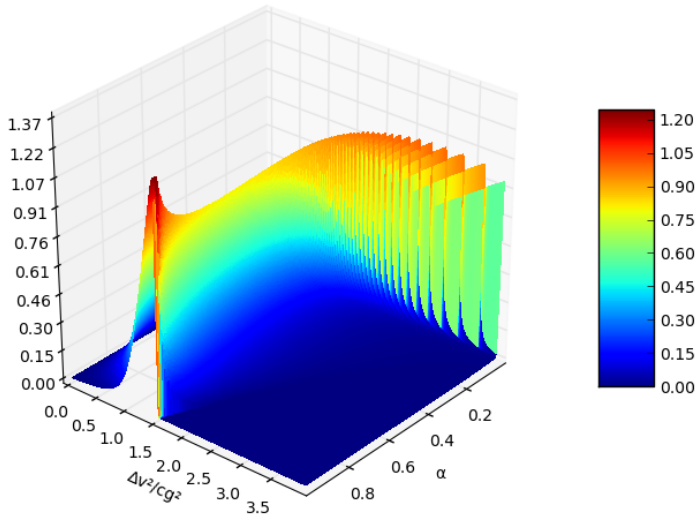

Figure 4.9: Aggregated correction as a function of void fraction and relative velocity

Chapter 5

Conclusions

5.1 Possible use of this work

In this work, I was able to introduce a few new tools, like the numerical hyperbolicity H, the normalized optimal interfacial pressure $\Delta P^{norm}_{optimal}$, the normalized optimal virtual mass coefficient $Cvm^{norm}_{optimal}$, the aggregated correction $AC(\eta)$ and its associated weight factor η. We have seen that they are useful to make the initial Cauchy problem of two-phase flows be well-posed.

Though, there may be some limits to these notions and some may question them. We will try to address the main one.

An eventual criticism would be that the proposed correction, especially the aggregated correction $AC(\eta)$, is not very physical. It does not represent a real force but is rather abruptly alternating between two interfacial forces. In nature, we would expect to have a mix of the two forces and not just the one that is the most mathematically convenient at that precise moment. As a consequence, this is why figure 4.9 representing $AC(\eta)$ as a function of both the void fraction and the relative velocity is such a non-smooth graph with very acute edges.

Let us start again from the beginning: the two-fluid model in its commonly used form. This model is by itself, before considering any additional forces, not physically accurate. As it is ill-posed, it may have multiple solutions and it is reasonable to say that this does not happen in nature. From one set of initial conditions and boundary conditions at time t, we should obtain a unique state at time $t + \delta t$ for our mechanical system. However, we should not totally reject the two-fluid model in its commonly used form simply because it is not physical. It just means that it is essential we try to improve it.

Furthermore, not only is the two-fluid model not physical, but also the currently-used corrections are not very physical. What is done nowadays in nuclear codes is adjusting the parameters of the codes in order to get results that are similar to previously made experiments for the most common conditions. Then we assume that the code is able to predict results for conditions we have not experimented yet. This is not an unreasonable reasoning as it gave us good models and then results. However, the means to get those models and results are not very physical; it would be preferable to have a good model in the beginning with physically realistic corrections.

Finally, we can acknowledge that the aggregated correction $AC(\eta)$ is not very physical. This eventual criticism is correct. That is why I introduced it not as the representation of an interfacial force, but rather as a criteria. It shows how large a corrective force should be to ensure hyperbolicity. Nonetheless, in further work, it should be possible to improve it to give more physical consistency based on this criteria.

5.2 Possible continuation for further work

It would be interesting to continue this work, for instance as a PhD thesis, to respond to the limits and improve the model.

First, there is one interrogation that did not really get answered. When we perform my analysis as a function of speeds, we see that it is better to analyse them as a function of relative velocities. We see that those results do

not depend on u_g or u_l individually but only on $(u_g - u_l)$. Hence the dependence on $\frac{\Delta v^2}{c_g^2}$. It would be a little more satisfying to not only give an experimental proof of that, which we already did, but also an analytical demonstration of the property.

Furthermore and more importantly, as we saw in the last section, $AC(\eta)$ is not physical by itself because it is a criteria. Let us recall its formula:

$$AC(\eta) = \eta \Delta P^{norm}_{optimal} + (1-\eta) Cvm^{norm}_{optimal} \tag{5.1}$$

Because of the definition of the aggregated correction as a minimal correction, η is either 0 or 1, and this is where the aggregated correction loses its physicality. So we would want to make a better and smoother mix between ΔP and Cvm, typically by giving η other values between 0 and 1. This could be done by not restraining ΔP^{norm} and Cvm^{norm} to their optimal value. So we could further study a new aggregated correction $AC'(\eta)$ which formula would be as follows:

$$AC'(\eta) = \eta \Delta P^{norm} + (1-\eta) Cvm^{norm} \tag{5.2}$$

$AC'(\eta)$ would be the result of an optimization problem and would be as small as possible. Here what remains to be done is give a proper way to determine the value of ΔP and Cvm, since they are not necessarily optimal. If we do this, we can get a real physical force that can ensure hyperbolicity.

$$F(\eta) = \eta F^p + (1-\eta) F^{vm} \tag{5.3}$$

Moreover, to better respond to the eventual criticism described in 3.1 about the importance of curing the ill-posedness of the Cauchy problem, it would be valuable to define a quantification of the uncertainty related to hyperbolicity. In particular, one could quantify how much uncertainty is kept in the two-fluid model when we do not ensure the hyperbolicity, and how much uncertainty is removed when taking into account the interfacial forces, like the interfacial pressure, the virtual mass force and any combination of both. That would probably be a good proof of how fundamental the well- or ill-posedness of the problem is.

Last, an ambitious continuation for this work would be to redesign a nuclear code by adding the results of this thesis. A possible work flow to design such a code would start with less complex systems already studied by experiments, and then continue with representing complex systems. Of course, the code should take the interfacial forces discussed above into account. The value of the new factors, namely ΔP, Cvm and/or η, would be either stored in a database with (p, α, u_g, u_l) as entries, or calculated numerically every time with an optimization problem, or calculated with a simple algebraic formula, comparable to those in 3.2.2 and 3.2.3.

5.3 Final conclusion

The two-phase flow models are Cauchy problems where, except for some specific cases, the numerical hyperbolicity H is different from 1. Therefore we have neither a unique solution to the problem nor definite space-convergence for current nuclear codes. For the next generation of codes, we have to take into account the interfacial forces to ensure hyperbolicity. A good criteria to determine what value those forces should be is the aggregated correction AC. This tool shows the regions in the (p, α, u_g, u_l) space where the interfacial pressure is preferred and where the virtual mass is preferred, or where a new combination could be designed.

Appendix A

Atmospheric and PWR pressure hyperbolicity

This appendix shows the same graphs as in the thesis, but for other pressure conditions: atmospheric pressure of 1.0132 bar and PWR pressure of 155 bar. The other conditions are still $(\alpha, u_g, u_l) = (0.5, 500\,\mathrm{m\,s^{-1}}, 20\,\mathrm{m\,s^{-1}})$ whenever that is possible.

A.1 Atmospheric pressure

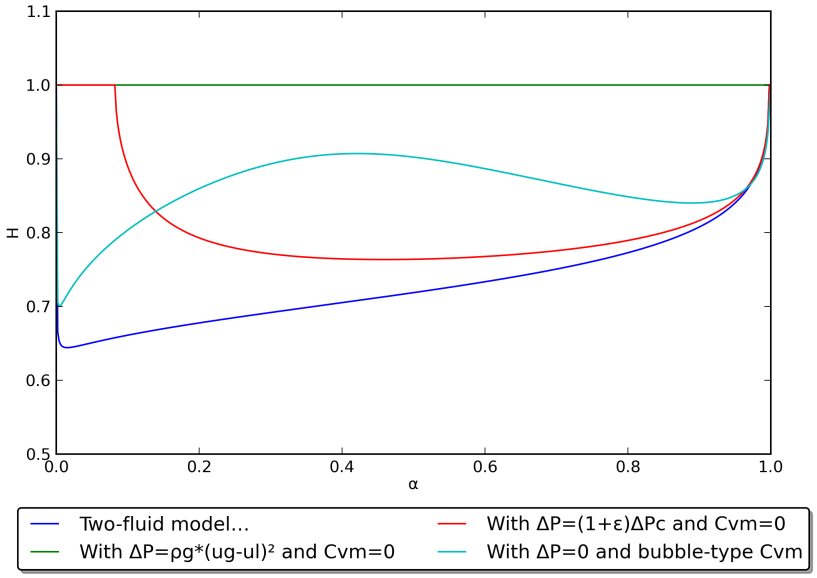

Figure A.1: Numerical hyperbolicity as a function of void fraction, for different corrections

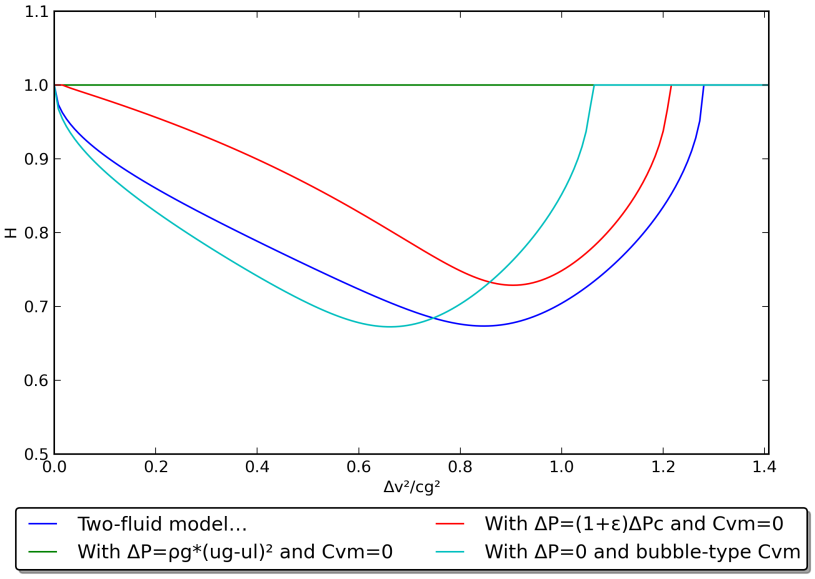

Figure A.2: Numerical hyperbolicity as a function of relative velocity, for different corrections

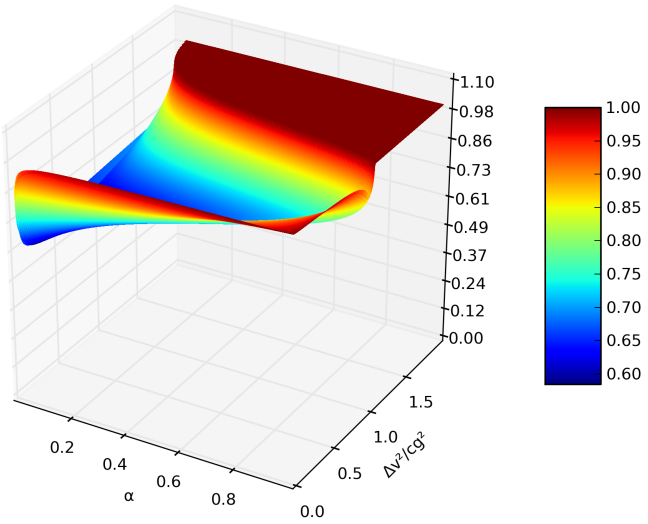

Figure A.3: Numerical hyperbolicity as a function of void fraction and relative velocity, for the two-fluid model in its commonly used form

A.2 PWR pressure

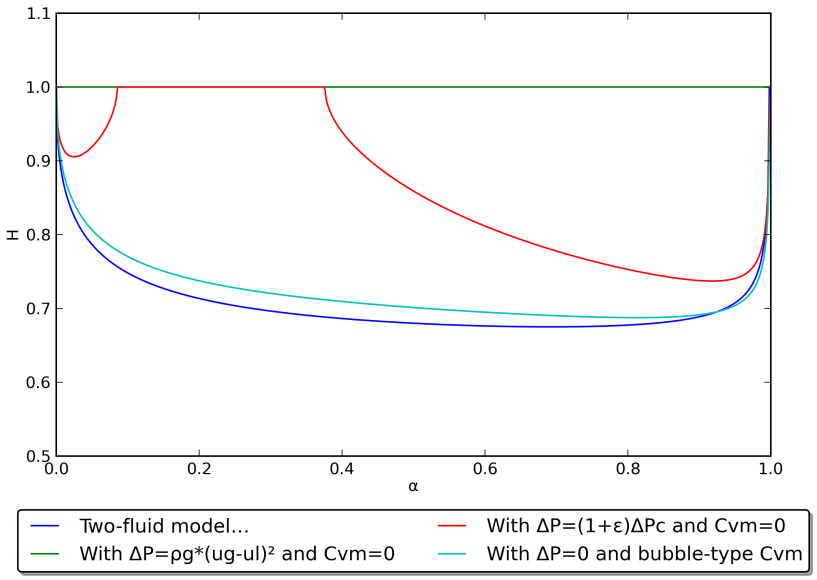

Figure A.4: Numerical hyperbolicity as a function of void fraction, for different corrections

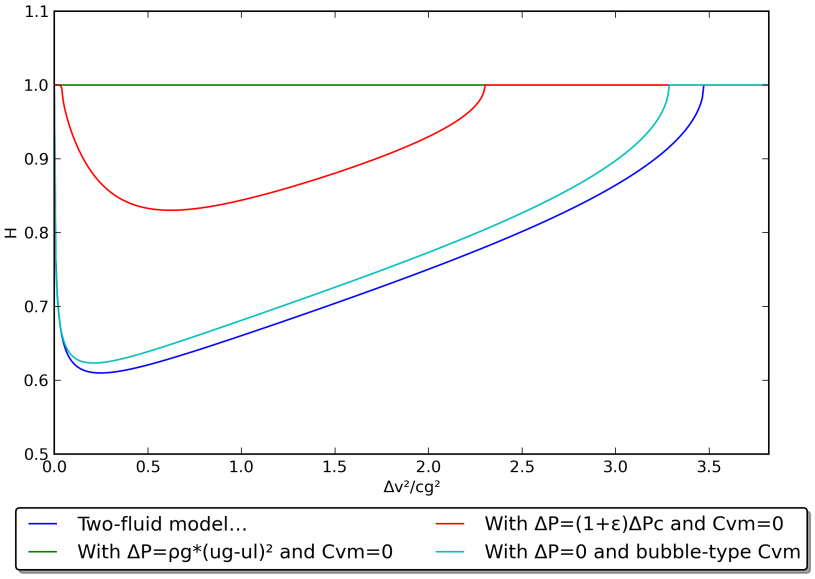

Figure A.5: Numerical hyperbolicity as a function of relative velocity, for different corrections

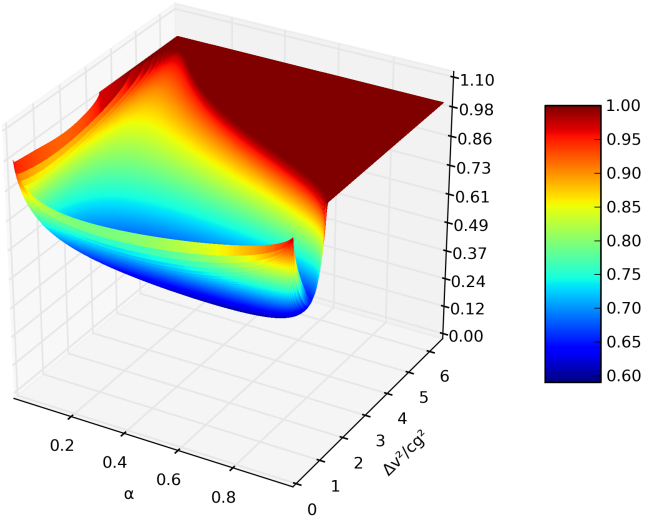

Figure A.6: Numerical hyperbolicity as a function of void fraction and relative velocity, for the two-fluid model in its commonly used form

Appendix B

Optimal interfacial pressure for atmospheric and PWR pressures

B.1 Atmospheric pressure

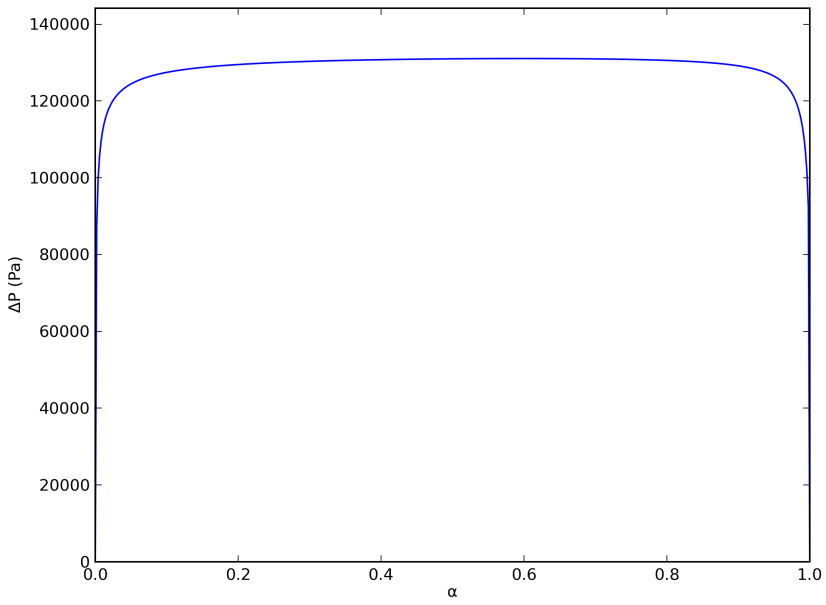

Figure B.1: Minimal hyperbolicity-ensuring interfacial pressure as a function of void fraction

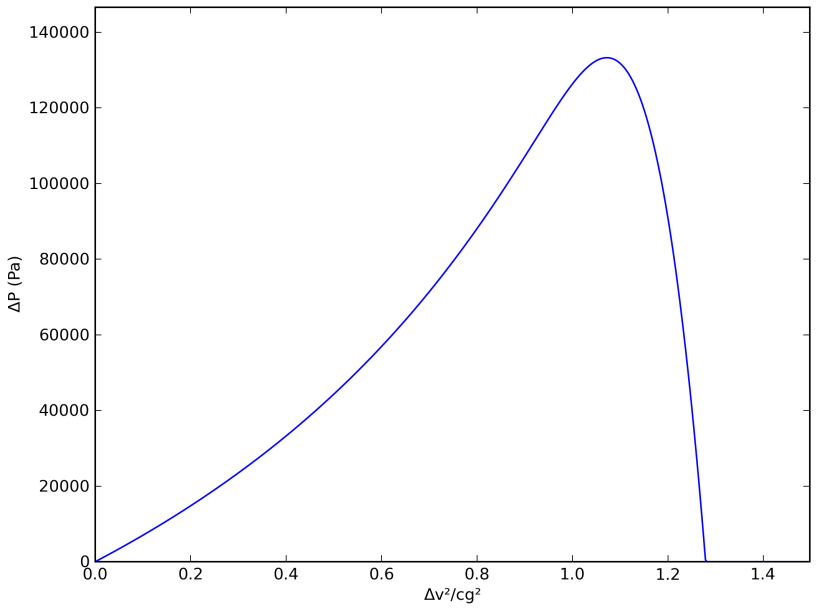

Figure B.2: Minimal hyperbolicity-ensuring interfacial pressure as a function of relative velocity

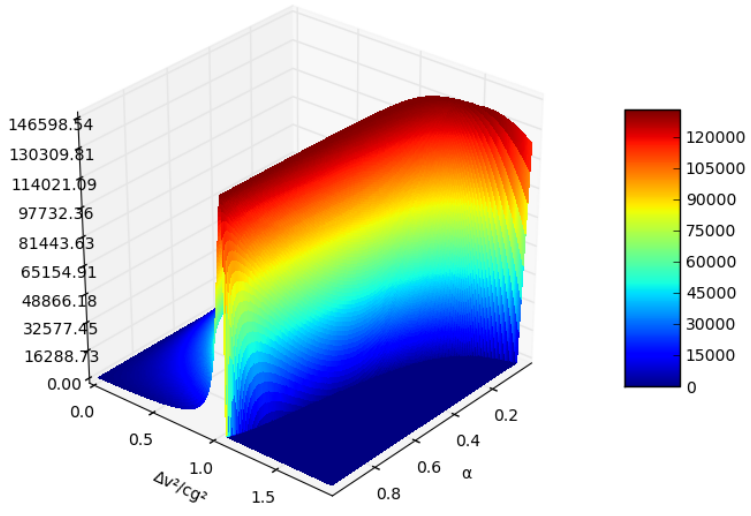

Figure B.3: $\Delta P_{optimal}$ (in Pa) as a function of void fraction and relative velocity

B.2 PWR pressure

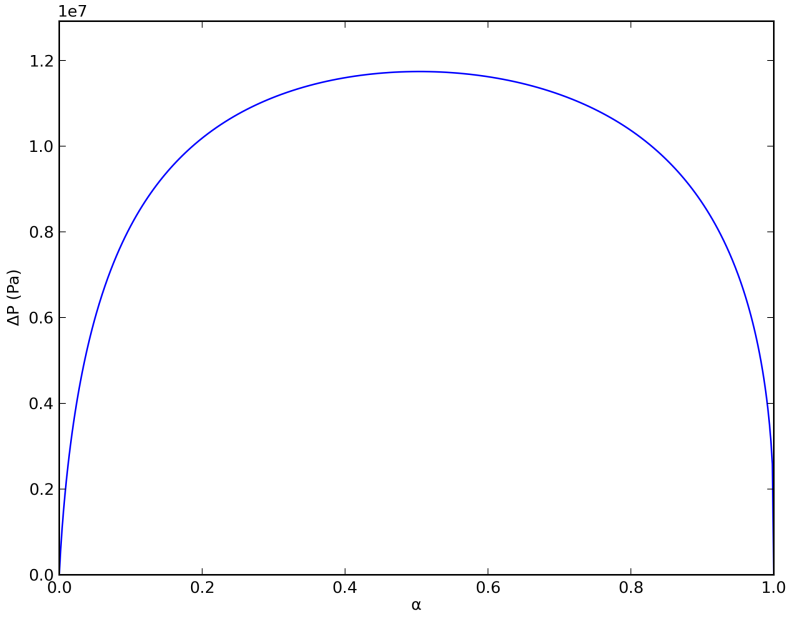

Figure B.4: Minimal hyperbolicity-ensuring interfacial pressure as a function of void fraction

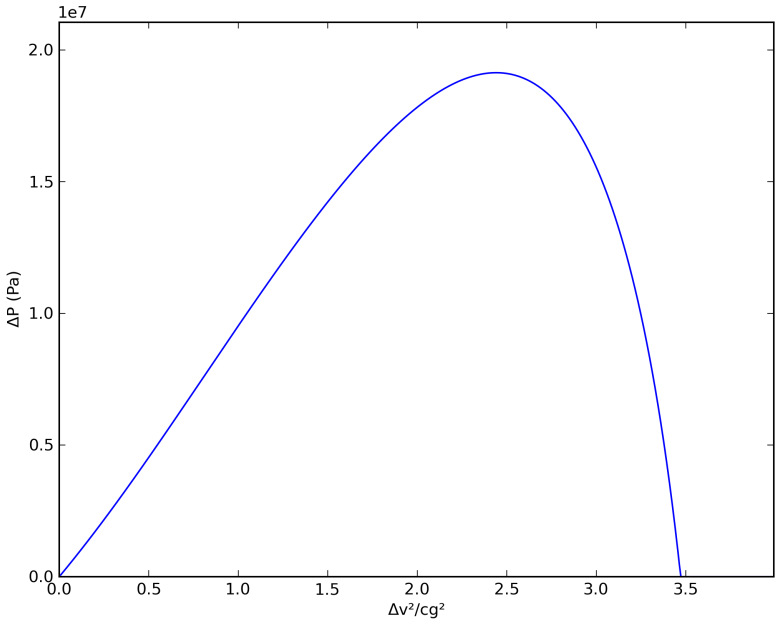

Figure B.5: Minimal hyperbolicity-ensuring interfacial pressure as a function of relative velocity

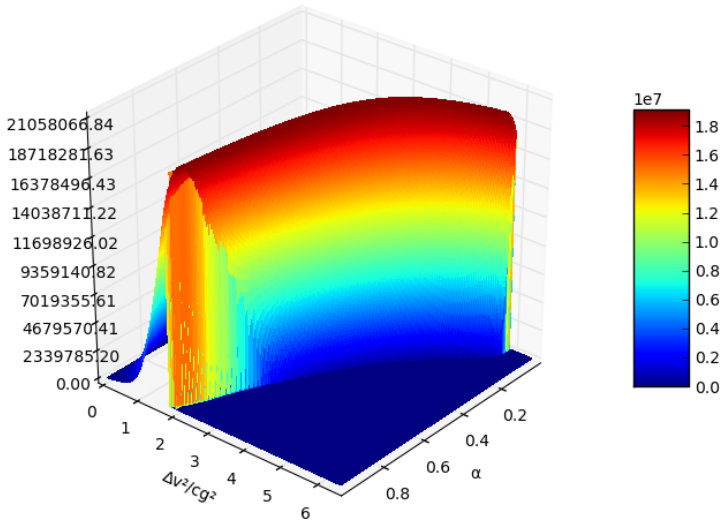

Figure B.6: $\Delta P_{optimal}$ (in Pa) as a function of void fraction and relative velocity

Appendix C

Optimal virtual mass for atmospheric and PWR pressures

C.1 Atmospheric pressure

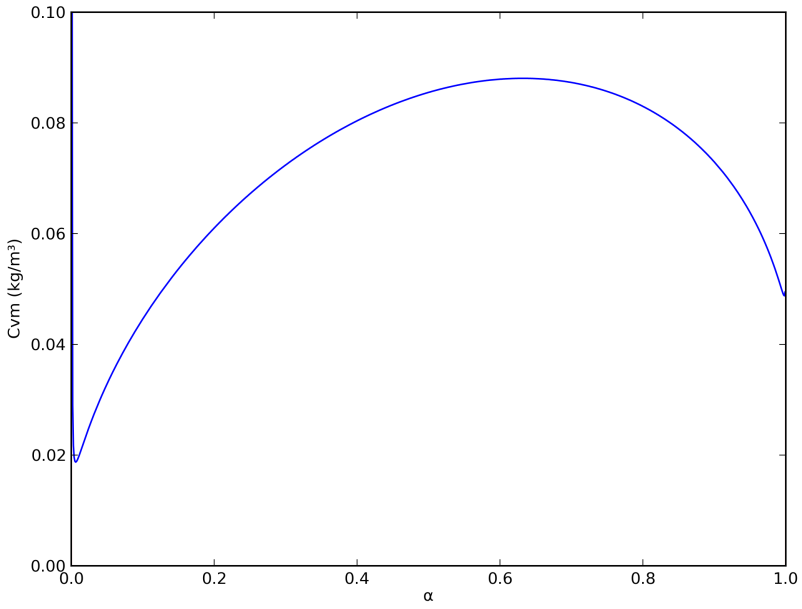

Figure C.1: Minimal hyperbolicity-ensuring virtual mass as a function of void fraction

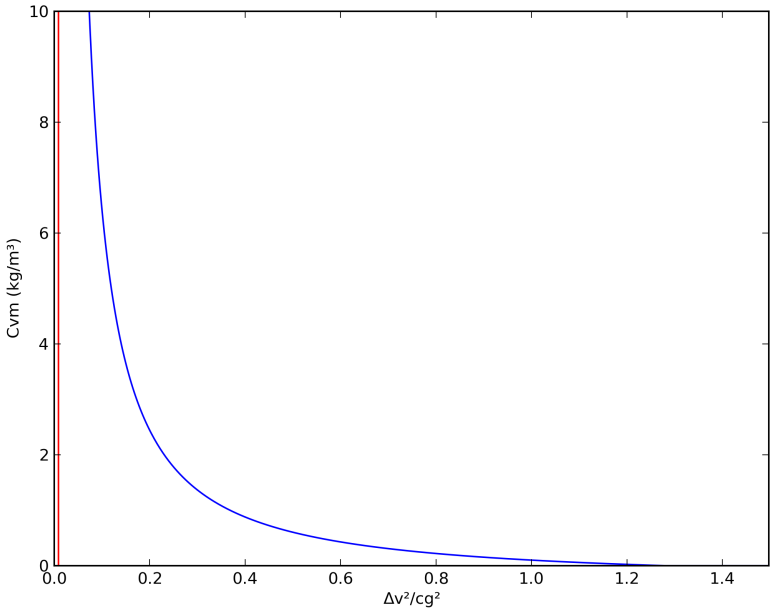

Figure C.2: Minimal hyperbolicity-ensuring virtual mass as a function of relative velocity

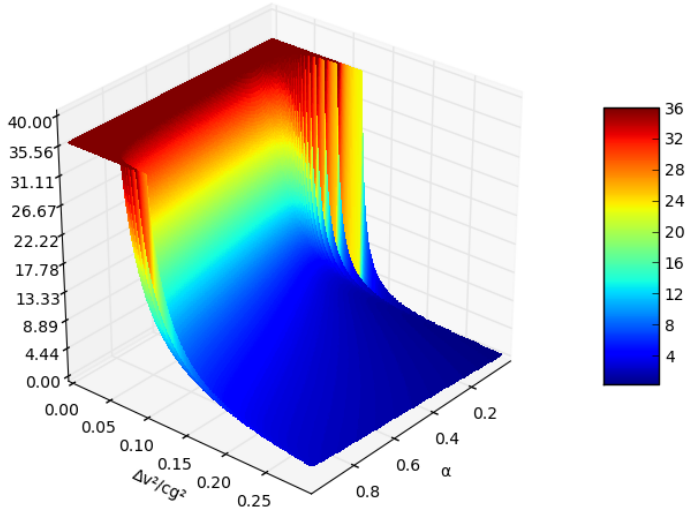

Figure C.3: $Cvm_{optimal}$ (in kg m^{-3}) as a function of void fraction and relative velocity

C.2 PWR pressure

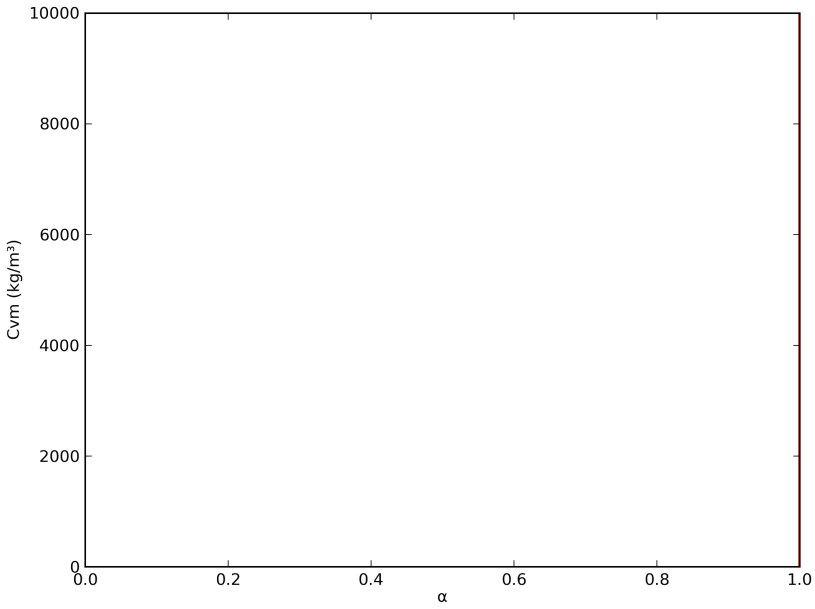

Figure C.4: For these conditions, Cvm cannot ensure hyperbolicity, whatever the value of α, hence a blank graph

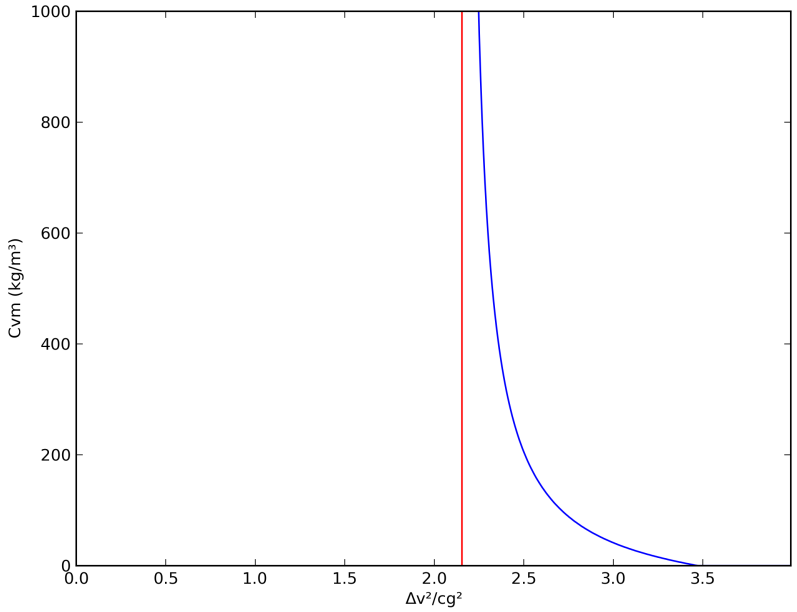

Figure C.5: Minimal hyperbolicity-ensuring virtual mass as a function of relative velocity

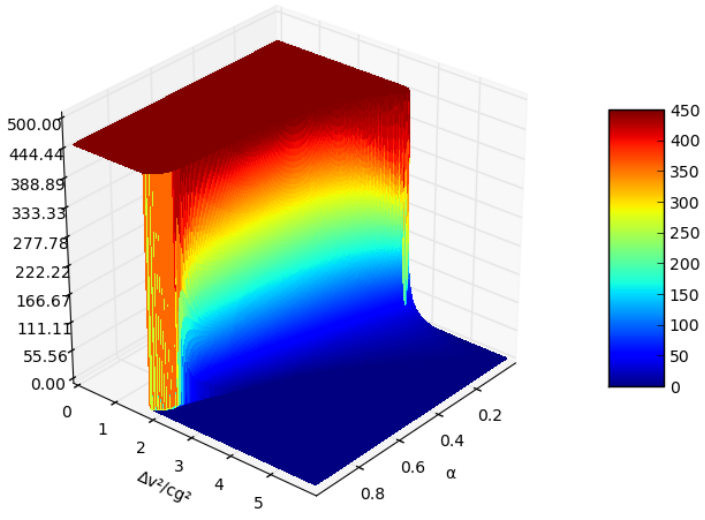

Figure C.6: $Cvm_{optimal}$ (in kg m^{-3}) as a function of void fraction and relative velocity

Appendix D

Aggregated correction for atmospheric and PWR pressures

D.1 Atmospheric pressure

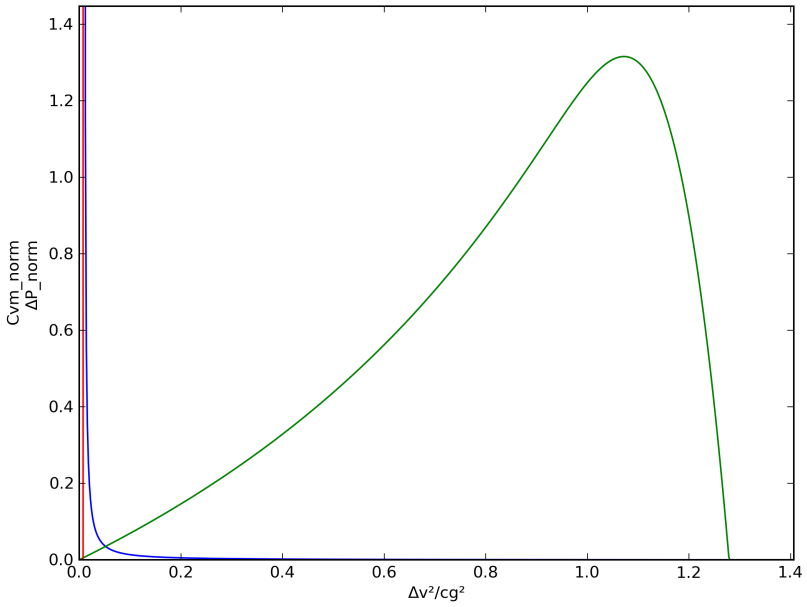

Figure D.1: $\Delta P^{norm}_{optimal}$ and $Cvm^{norm}_{optimal}$ as a function of relative velocity

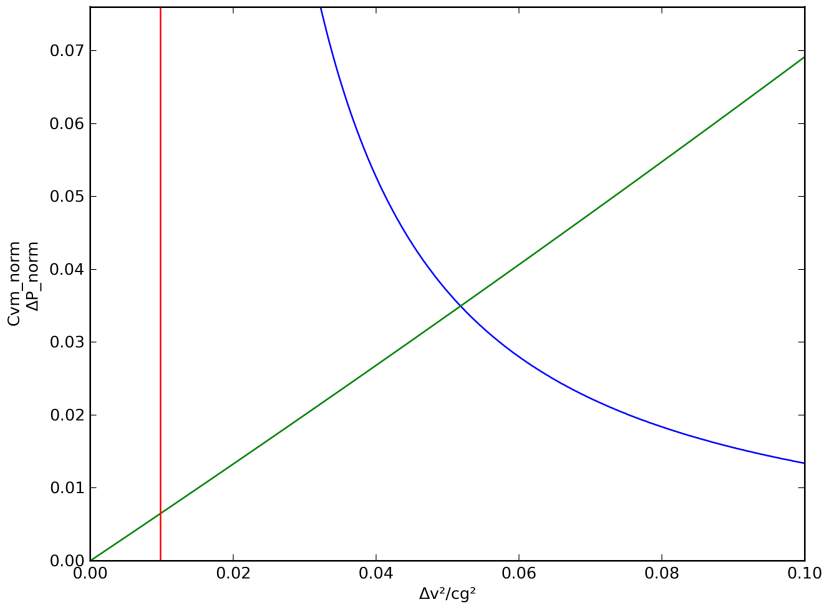

Figure D.2: $\Delta P^{norm}_{optimal}$ and $Cvm^{norm}_{optimal}$ as a function of relative velocity, zoom on the origin

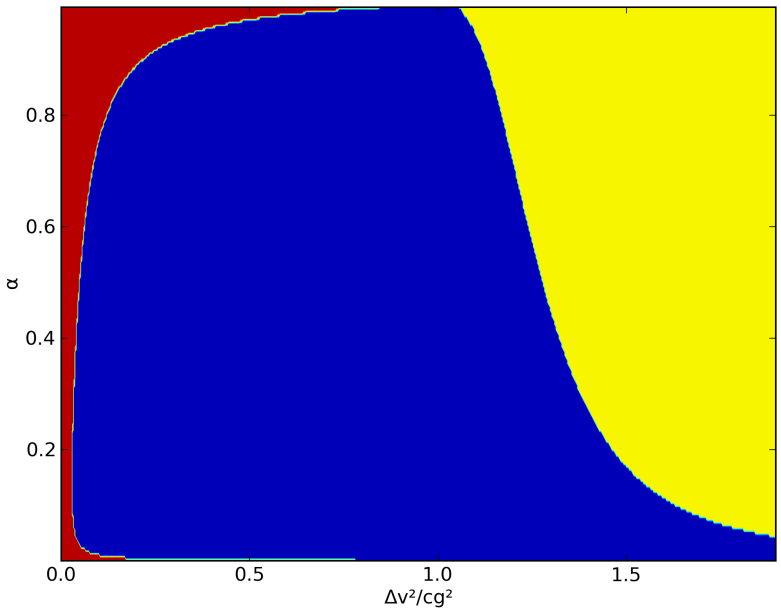

Figure D.3: η zones as a function of void fraction and relative velocity

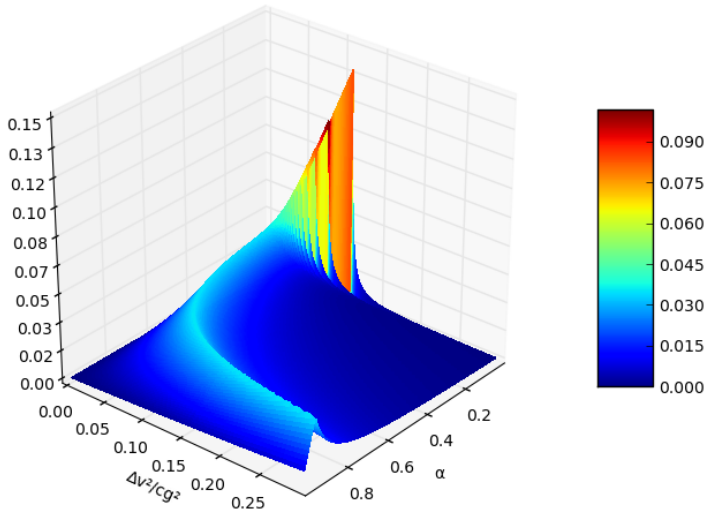

Figure D.4: Aggregated correction as a function of void fraction and relative velocity

D.2 PWR pressure

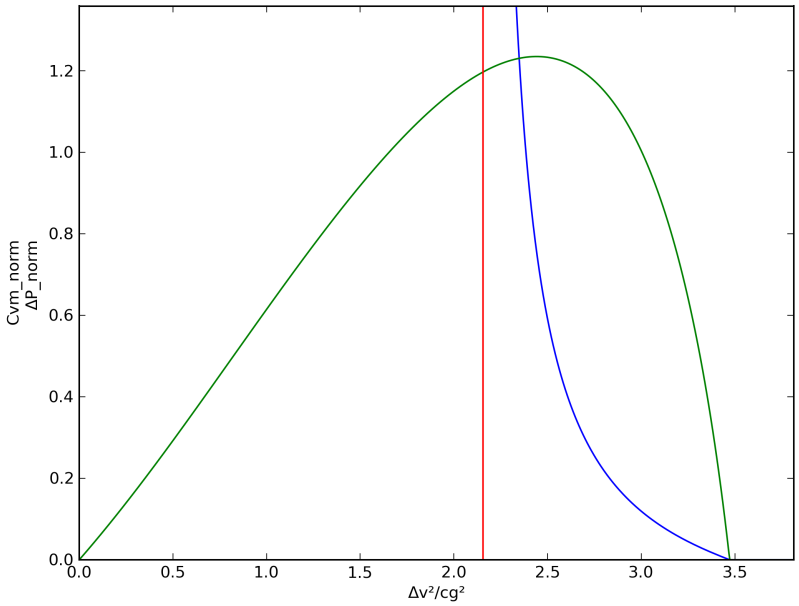

Figure D.5: $\Delta P^{norm}_{optimal}$ and $Cvm^{norm}_{optimal}$ as a function of relative velocity

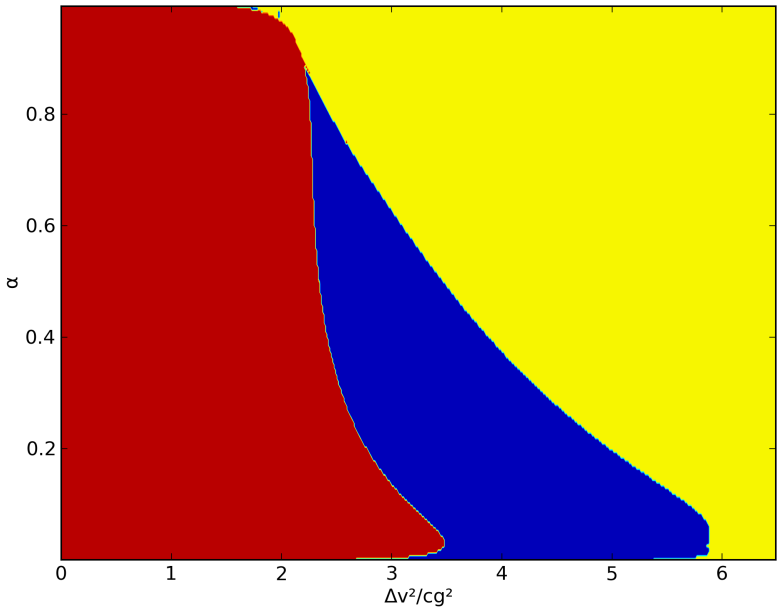

Figure D.6: η zones as a function of void fraction and relative velocity

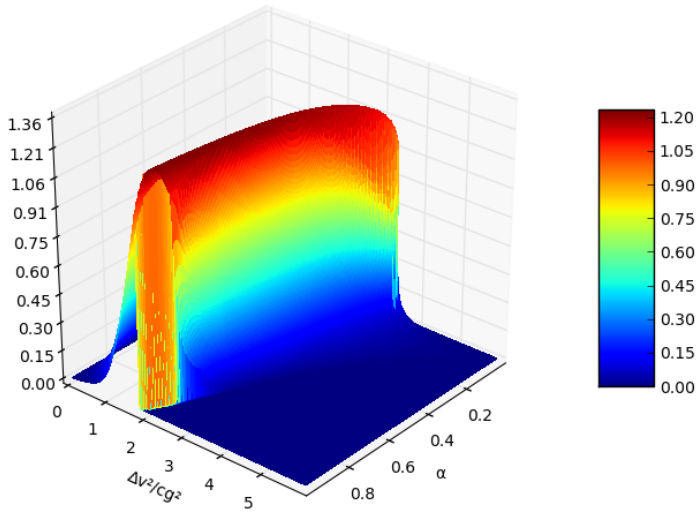

Figure D.7: Aggregated correction as a function of void fraction and relative velocity

Appendix E

Three particular cases with unique solution

The purpose of this appendix is to examine three simpler cases where we could have a unique solution:

1. One-phase flow, as a particular case of the two-fluid flow
2. Steady flow
3. α constant and uniform

In each one of those cases, we will set $F_g = F_l = 0$ to see if the case presents a unique solution without any additional force.

E.1 One-phase flow

A one-phase flow is a flow where there is either only gas ($\alpha = 1$ and $\alpha_l = 0$) or only liquid ($\alpha = \alpha_g = 0$). In those cases, two equations out of four become trivially $0 = 0$. So if i is the phase that is present and for which $\alpha_i = 1$, then we get the following symbolically equivalent form:

$$\begin{aligned} \frac{\partial}{\partial t}(\alpha_i \rho_i) + \frac{\partial}{\partial x}(\alpha_i \rho_i u_i) &= 0 \\ \frac{\partial}{\partial t}(\alpha_i \rho_i u_i) + \frac{\partial}{\partial x}(\alpha_i \rho_i u_i u_i) + \alpha_i \frac{\partial p}{\partial x} &= F_i = 0 \end{aligned} \tag{E.1}$$

So if we simply write $\rho = \rho_i$ and $u = u_i$, we get

$$\begin{aligned} \frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho u) &= 0 \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial p}{\partial x} &= 0 \end{aligned} \tag{E.2}$$

This model is well known and we can show that there is a unique solution (p, u). If we expand the derivatives, we get

$$\begin{aligned} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0 \\ u\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial t} + u^2\frac{\partial \rho}{\partial x} + 2u\rho\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} &= 0 \end{aligned} \tag{E.3}$$

If c is the speed of sound in the fluid, we can use the following substitutions:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} \\ \frac{\partial \rho}{\partial x} &= \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial p}{\partial x} \end{aligned} \tag{E.4}$$

The substitution leads to

$$\begin{aligned} \frac{1}{c^2}\frac{\partial p}{\partial t} + \frac{u}{c^2}\frac{\partial p}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0 \\ \frac{u}{c^2}\frac{\partial p}{\partial t} + \rho\frac{\partial u}{\partial t} + \frac{u^2}{c^2}\frac{\partial p}{\partial x} + 2u\rho\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} &= 0 \end{aligned} \tag{E.5}$$

This can be simplified into

$$\begin{aligned} \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho c^2\frac{\partial u}{\partial x} &= 0 \\ u\frac{\partial p}{\partial t} + \rho c^2\frac{\partial u}{\partial t} + (u^2+c^2)\frac{\partial p}{\partial x} + 2u\rho c^2\frac{\partial u}{\partial x} &= 0 \end{aligned} \tag{E.6}$$

Let us use the column vector $U = (p \quad u)^t$, which coordinates are the two unknowns of the problem.

$$\begin{pmatrix} 1 & 0 \\ u & \rho c^2 \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} p \\ u \end{pmatrix} + \begin{pmatrix} u & \rho c^2 \\ u^2+c^2 & 2u\rho c^2 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} p \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{E.7}$$

The first matrix is invertible, so we can multiply by that inverse:

$$\begin{pmatrix} 1 & 0 \\ \frac{-u}{\rho c^2} & \frac{1}{\rho c^2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & \rho c^2 \end{pmatrix} \frac{\partial U}{\partial t} + \begin{pmatrix} u & \rho c^2 \\ \frac{1}{\rho} & u \end{pmatrix} \frac{\partial U}{\partial x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{E.8}$$

We can rewrite that as

$$\frac{\partial U}{\partial t} + \begin{pmatrix} u & \rho c^2 \\ \frac{1}{\rho} & u \end{pmatrix} \frac{\partial U}{\partial x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{E.9}$$

The characteristic polynomial of $A = \begin{pmatrix} u & \rho c^2 \\ \frac{1}{\rho} & u \end{pmatrix}$ is $\chi_A(X) = X^2 - 2uX + (u^2 - c^2) = (X - (u-c))\,(X - (u+c))$. Hence the eigenvalues of A are $u - c$ and $u + c$.

As we saw in 2.2, this problem E.9 is a Cauchy problem as soon as we add initial conditions to it. It is well-posed if it is hyperbolic, that is to say if the eigenvalues of A are real and distinct. This is the case here, so we have proven that the one-phase flow model leads to a unique solution $(p \quad u)$.

As a side note, let us remember that this calculation is true for flows with a unique fluid, but the results may be different in the case of an "evanescent phase". The evanescence of a phase is the phenomenon of disappearance of one phase out of the initial two, due to either total evaporation or total condensation. In this case, it is hard to ignore the two $0 = 0$ equations because we lose all the information about the evanescent phase, for instance (p, u). As this is not the scope of this thesis, the reader may refer for example to Saleh's work [16] for further development.

E.2 Steady flow

A steady flow is a flow where the (p, α, u_g, u_l) state is constant, independent of time. In other words, $\frac{\partial}{\partial t} = 0$. This is different from the steady state because matter is still moving, the speed of each phase is not necessarily zero. But the velocity fields remain constant. In this subsection, we want to investigate on the existence and uniqueness of the steady flow.

In this case, considering $\frac{\partial}{\partial t} = 0$, the initial set for the two-fluid model

$$\begin{aligned}
\frac{\partial}{\partial t}(\alpha_g \rho_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g) = 0 \\
\frac{\partial}{\partial t}(\alpha_l \rho_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l) = 0 \\
\frac{\partial}{\partial t}(\alpha_g \rho_g u_g) + \frac{\partial}{\partial x}(\alpha_g \rho_g u_g u_g) + \alpha_g \frac{\partial p}{\partial x} = 0 \\
\frac{\partial}{\partial t}(\alpha_l \rho_l u_l) + \frac{\partial}{\partial x}(\alpha_l \rho_l u_l u_l) + \alpha_l \frac{\partial p}{\partial x} = 0
\end{aligned} \tag{E.10}$$

becomes

$$\begin{aligned}
\frac{\partial}{\partial x}(\alpha_g \rho_g u_g) = 0 \\
\frac{\partial}{\partial x}(\alpha_l \rho_l u_l) = 0 \\
\frac{\partial}{\partial x}(\alpha_g \rho_g u_g u_g) + \alpha_g \frac{\partial p}{\partial x} = 0 \\
\frac{\partial}{\partial x}(\alpha_l \rho_l u_l u_l) + \alpha_l \frac{\partial p}{\partial x} = 0
\end{aligned} \tag{E.11}$$

If we use the matrix notation with solution vector $U = (p \quad \alpha_g \quad u_g \quad u_l)^t$, then we can reuse the result of equation 2.10 from a previous derivation:

$$0\frac{\partial U}{\partial t} + \begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g \rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l \rho_l \\ \alpha_g & 0 & \alpha_g \rho_g u_g & 0 \\ \alpha_l & 0 & 0 & \alpha_l \rho_l u_l \end{pmatrix} \frac{\partial U}{\partial x} = 0 \tag{E.12}$$

Let us define

$$C = \begin{pmatrix} \frac{\alpha_g u_g}{c_g^2} & \rho_g u_g & \alpha_g \rho_g & 0 \\ \frac{\alpha_l u_l}{c_l^2} & -\rho_l u_l & 0 & \alpha_l \rho_l \\ \alpha_g & 0 & \alpha_g \rho_g u_g & 0 \\ \alpha_l & 0 & 0 & \alpha_l \rho_l u_l \end{pmatrix} \tag{E.13}$$

Then we have

$$C\frac{\partial U}{\partial x} = 0 \tag{E.14}$$

$\frac{\partial U}{\partial x} = 0$ is obviously a solution. What we want is this solution to be unique, up to a constant equal to $U(x = 0)$. This is the case if and only if C is invertible, if and only if its determinant is non-zero.

We can find the determinant of C, noted as $|C|$:

$$\begin{aligned}
|C| = \alpha \rho_l \rho_g^2 u_g^2 + \rho_g \alpha^2 \rho_l^2 u_l^2 + \rho_l \alpha^3 \rho_g^2 u_g^2 - \rho_g \alpha^3 \rho_l^2 u_l^2 - 2\rho_l \alpha^2 \rho_g^2 u_g^2 \\
+ \frac{\rho_g \alpha^3 \rho_l^2 u_g^2 u_l^2}{c_g^2} - \frac{\alpha \rho_l \rho_g^2 u_g^2 u_l^2}{c_l^2} - \frac{\rho_g \alpha^2 \rho_l^2 u_g^2 u_l^2}{c_g^2} - \frac{\rho_l \alpha^3 \rho_g^2 u_g^2 u_l^2}{c_l^2} + \frac{2\rho_l \alpha^2 \rho_g^2 u_g^2 u_l^2}{c_l^2}
\end{aligned} \tag{E.15}$$

We factorize a first time:

$$|C| = \alpha \rho_l \rho_g^2 u_g^2 \left(1 - \frac{u_l^2}{c_l^2}\right) + \rho_g \alpha^2 \rho_l^2 u_l^2 \left(1 - \frac{u_g^2}{c_g^2}\right) + \rho_l \alpha^3 \rho_g^2 u_g^2 \left(1 - \frac{u_l^2}{c_l^2}\right) - \rho_g \alpha^3 \rho_l^2 u_l^2 \left(1 - \frac{u_g^2}{c_g^2}\right) + 2\rho_l \alpha^2 \rho_g^2 u_g^2 \left(1 - \frac{u_l^2}{c_l^2}\right) \tag{E.16}$$

We factorize a second time:

$$|C| = \left(1 - \frac{u_l^2}{c_l^2}\right)(\alpha\rho_l\rho_g^2u_g^2 + \rho_l\alpha^3\rho_g^2u_g^2 - 2\rho_l\alpha^2\rho_g^2u_g^2) + \left(1 - \frac{u_g^2}{c_g^2}\right)(\rho_g\alpha^2\rho_l^2u_l^2 - \rho_g\alpha^3\rho_l^2u_l^2) \tag{E.17}$$

We factorize a third time:

$$|C| = \left(1 - \frac{u_l^2}{c_l^2}\right)\alpha\rho_l\rho_g^2u_g^2(1 + \alpha^2 - 2\alpha) + \left(1 - \frac{u_g^2}{c_g^2}\right)\rho_g\alpha^2\rho_l^2u_l^2(1 - \alpha) \tag{E.18}$$

We can simplify the determinant into a final expression:

$$|C| = \left(1 - \frac{u_l^2}{c_l^2}\right)\rho_l\rho_g^2u_g^2\alpha(1 - \alpha)^2 + \left(1 - \frac{u_g^2}{c_g^2}\right)\rho_g\rho_l^2u_l^2\alpha^2(1 - \alpha) \tag{E.19}$$

If $\alpha = 0$ or $\alpha = 1$, then we trivially have $|C| = 0$, which would imply the non-unicity. Let us exclude this case for what follows since it is the particular exception of evanescent phases, as explained in subsection E.1.

In normal conditions, we have $u_g < c_g$ and $u_l < c_l$. So $\left(1 - \frac{u_g^2}{c_g^2}\right) > 0$ and $\left(1 - \frac{u_l^2}{c_l^2}\right) > 0$. Therefore, as $0 < \alpha < 1$, we get $|C| > 0$. In particular, $|C| \neq 0$.

So we have proven that for a reasonable set of conditions, there is one unique solution for the steady flow in the two-fluid model. The equation becomes simply $\frac{\partial U}{\partial x} = 0$ and $U(x) = U(x = 0)$ is the solution.

E.3 α constant and uniform

As a final set of particular conditions, let us analyse the case where α is constant ($\frac{\partial\alpha}{\partial t} = 0$) and uniform ($\frac{\partial\alpha}{\partial x} = 0$).

Then the initial set of equations

$$\begin{aligned}
\frac{\partial}{\partial t}(\alpha_g\rho_g) + \frac{\partial}{\partial x}(\alpha_g\rho_g u_g) &= 0 \\
\frac{\partial}{\partial t}(\alpha_l\rho_l) + \frac{\partial}{\partial x}(\alpha_l\rho_l u_l) &= 0 \\
\frac{\partial}{\partial t}(\alpha_g\rho_g u_g) + \frac{\partial}{\partial x}(\alpha_g\rho_g u_g u_g) + \alpha_g\frac{\partial p}{\partial x} &= 0 \\
\frac{\partial}{\partial t}(\alpha_l\rho_l u_l) + \frac{\partial}{\partial x}(\alpha_l\rho_l u_l u_l) + \alpha_l\frac{\partial p}{\partial x} &= 0
\end{aligned} \tag{E.20}$$

becomes, after division by α_g or α_l,

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho_g) + \frac{\partial}{\partial x}(\rho_g u_g) &= 0 \\
\frac{\partial}{\partial t}(\rho_l) + \frac{\partial}{\partial x}(\rho_l u_l) &= 0 \\
\frac{\partial}{\partial t}(\rho_g u_g) + \frac{\partial}{\partial x}(\rho_g u_g u_g) + \frac{\partial p}{\partial x} &= 0 \\
\frac{\partial}{\partial t}(\rho_l u_l) + \frac{\partial}{\partial x}(\rho_l u_l u_l) + \frac{\partial p}{\partial x} &= 0
\end{aligned} \tag{E.21}$$

α, α_g and α_l are not present in the equations any more, so we have four equations for three unknowns: p, u_g and u_l. So the equations are not independent and could be inconsistent or redundant. Let us investigate whether this is a problem to have any solution.

If we expand the derivative operators, we obtain the following:

$$\begin{aligned}
\frac{\partial \rho_g}{\partial t} + u_g \frac{\partial \rho_g}{\partial x} + \rho_g \frac{\partial u_g}{\partial x} &= 0 \\
\frac{\partial \rho_l}{\partial t} + u_l \frac{\partial \rho_l}{\partial x} + \rho_l \frac{\partial u_l}{\partial x} &= 0 \\
u_g \frac{\partial \rho_g}{\partial t} + \rho_g \frac{\partial u_g}{\partial t} + u_g^2 \frac{\partial \rho_g}{\partial x} + \rho_g \frac{\partial u_g^2}{\partial x} + \frac{\partial p}{\partial x} &= 0 \\
u_l \frac{\partial \rho_l}{\partial t} + \rho_l \frac{\partial u_l}{\partial t} + u_l^2 \frac{\partial \rho_l}{\partial x} + \rho_l \frac{\partial u_l^2}{\partial x} + \frac{\partial p}{\partial x} &= 0
\end{aligned} \tag{E.22}$$

As $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t}$ and $\frac{\partial \rho}{\partial x} = \frac{1}{c^2}\frac{\partial p}{\partial x}$, we get

$$\begin{aligned}
\frac{1}{c_g^2}\frac{\partial p}{\partial t} + \frac{u_g}{c_g^2}\frac{\partial p}{\partial x} + \rho_g \frac{\partial u_g}{\partial x} &= 0 \\
\frac{1}{c_l^2}\frac{\partial p}{\partial t} + \frac{u_l}{c_l^2}\frac{\partial p}{\partial x} + \rho_l \frac{\partial u_l}{\partial x} &= 0 \\
\frac{u_g}{c_g^2}\frac{\partial p}{\partial t} + \rho_g \frac{\partial u_g}{\partial t} + \frac{u_g^2}{c_g^2}\frac{\partial p}{\partial x} + \frac{\partial p}{\partial x} + \rho_g \frac{\partial u_g^2}{\partial x} &= 0 \\
\frac{u_l}{c_l^2}\frac{\partial p}{\partial t} + \rho_l \frac{\partial u_l}{\partial t} + \frac{u_l^2}{c_l^2}\frac{\partial p}{\partial x} + \frac{\partial p}{\partial x} + \rho_l \frac{\partial u_l^2}{\partial x} &= 0
\end{aligned} \tag{E.23}$$

This is equivalent to

$$\begin{aligned}
\frac{\partial p}{\partial t} + u_g \frac{\partial p}{\partial x} + c_g^2 \rho_g \frac{\partial u_g}{\partial x} &= 0 \\
\frac{\partial p}{\partial t} + u_l \frac{\partial p}{\partial x} + c_l^2 \rho_l \frac{\partial u_l}{\partial x} &= 0 \\
\frac{u_g}{c_g^2}\frac{\partial p}{\partial t} + \rho_g \frac{\partial u_g}{\partial t} + \frac{u_g^2 + c_g^2}{c_g^2}\frac{\partial p}{\partial x} + 2u_g \rho_g \frac{\partial u_g}{\partial x} &= 0 \\
\frac{u_l}{c_l^2}\frac{\partial p}{\partial t} + \rho_l \frac{\partial u_l}{\partial t} + \frac{u_l^2 + c_l^2}{c_l^2}\frac{\partial p}{\partial x} + 2u_l \rho_l \frac{\partial u_l}{\partial x} &= 0
\end{aligned} \tag{E.24}$$

Let us apply three basic linear operations on the lines: $L_3 \leftarrow L_3 - \frac{u_g}{c_g^2} \cdot L_1$, $L_4 \leftarrow L_4 - \frac{u_l}{c_l^2} \cdot L_2$ and then $L_2 \leftarrow L_2 - L_1$.

$$\begin{aligned}
\frac{\partial p}{\partial t} + u_g \frac{\partial p}{\partial x} + c_g^2 \rho_g \frac{\partial u_g}{\partial x} &= 0 \\
(u_l - u_g)\frac{\partial p}{\partial x} + c_l^2 \rho_l \frac{\partial u_l}{\partial x} - c_g^2 \rho_g \frac{\partial u_g}{\partial x} &= 0 \\
\rho_g \frac{\partial u_g}{\partial t} + \frac{\partial p}{\partial x} + u_g \rho_g \frac{\partial u_g}{\partial x} &= 0 \\
\rho_l \frac{\partial u_l}{\partial t} + \frac{\partial p}{\partial x} + u_l \rho_l \frac{\partial u_l}{\partial x} &= 0
\end{aligned} \tag{E.25}$$

If we reorganize the lines, we get a clear set of equations with which we can easily work:

$$\begin{aligned}
\frac{\partial p}{\partial t} + u_g \frac{\partial p}{\partial x} + c_g^2 \rho_g \frac{\partial u_g}{\partial x} &= 0 \\
\frac{\partial u_g}{\partial t} + \frac{1}{\rho_g}\frac{\partial p}{\partial x} + u_g \frac{\partial u_g}{\partial x} &= 0 \\
\frac{\partial u_l}{\partial t} + \frac{1}{\rho_l}\frac{\partial p}{\partial x} + u_l \frac{\partial u_l}{\partial x} &= 0 \\
(u_l - u_g)\frac{\partial p}{\partial x} + c_l^2 \rho_l \frac{\partial u_l}{\partial x} - c_g^2 \rho_g \frac{\partial u_g}{\partial x} &= 0
\end{aligned} \tag{E.26}$$

If we define $U = \begin{pmatrix} p \\ u_g \\ u_l \end{pmatrix}$, then the three first lines can be rewritten in a matrix form:

$$\frac{\partial U}{\partial t} + \begin{pmatrix} u_g & c_g^2 \rho_g & 0 \\ \frac{1}{\rho_g} & u_g & 0 \\ \frac{1}{\rho_l} & 0 & u_l \end{pmatrix} \frac{\partial U}{\partial x} = 0 \tag{E.27}$$

Let us define matrix M as

$$M = \begin{pmatrix} u_g & c_g^2 \rho_g & 0 \\ \frac{1}{\rho_g} & u_g & 0 \\ \frac{1}{\rho_l} & 0 & u_l \end{pmatrix} \tag{E.28}$$

M is a dimension 3 matrix. We could calculate its three eigenvalues:

$$sp(M) = \{u_l, c_g + u_g, c_g - u_g\} \tag{E.29}$$

The eigenvalues are real and distinct, so as soon as we add initial conditions, this problem is a well-posed Cauchy problem. This 3-dimensional problem has a unique (p, u_g, u_l) solution.

However we have not proven that the existence of a solution to the 4-dimensional problem E.26. But we have proven that if the 4-dimensional problem has a solution, then the solution necessarily has to be the (p, u_g, u_l) we just found. So once we find it, we just have to check whether the (p, u_g, u_l) solution is compatible with

$$(u_l - u_g)\frac{\partial p}{\partial x} + c_l^2 \rho_l \frac{\partial u_l}{\partial x} - c_g^2 \rho_g \frac{\partial u_g}{\partial x} = 0 \tag{E.30}$$

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