# Towards Linear Modal Analysis for an L-Shaped Beam: Equations of Motion

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**Keywords**: L-shaped beam; linear equations of motion; modal analysis.

## 1. Introduction

Since the 1960s there has been continuous interest in the dynamics of L-shaped coupled structures exhibiting certain nonlinearities. Roberts and Cartmell in [Roberts and Cartmell 1984, Cartmell and Roberts 1987] studied autoparametric and combination resonances within an L-shaped beam structure, and Balachandran and Nayfeh performed nonlinear modal analysis considering only in-plane motions [Balachandran and Nayfeh 1990]. Warminski *et al* in [Warminski *et al* 2008], formulated the third order partial differential nonlinear equations of an L-shaped beam structure with different flexibilities in the two orthogonal directions, without taking into account rotary inertia effects. Ozonato *et al* studied post-buckled chaotic vibrations of an L-shaped beam structure considering only the in-plane bending nonlinear motions [Ozonato *et al* 2012]. In considering single beams Barbero and Raftoyiannis in [Barbero and Raftoyiannis 1994] studied the buckling modes and their coupling for pultruded I-beams subjected to various loading conditions. Fratenali in [Fratenali 1996] studied the formulation of models of layered composite, considering delamination effects using interfacial constitutive laws and delamination growth. In [Fratenali and Bilotti 1997], the one-dimensional theory was derived and a finite element model was given for the stress analysis of laminated curved composite beams, considering moderate large rotations, moderate large shear strains and different elastic behaviour of material in tension and in compression. In [Barbero 2000] the theoretical buckling mode interaction constant is considered, for pultruded structural shapes, using stability theory which demonstrated the existence of such buckling mode interaction. In [Fraternali and Feo 2000] the Vlasov theory of sectorial areas was used to formulate small strain and moderate rotations to model laminate composite thin walled beams. A finite element approximation of the theory was also carried out and several numerical applications were developed with reference to lateral buckling of the thin-walled members. Finally it should mentioned that modelling of composite beams with warping functions was studied by Librescu in [Librescu 2006].

In this article we consider Euler-Bernoulli beams made by isotropic material and we derive the first order approximation, the linear equations of motion for an L-shaped beam considering also rotary inertia effects. Examination of the equations of motion indicates that in-plane bending and other motions are well separated at the first order approximation. We show that in the absence of rotary inertia when considering the out-of-plane bending, the torsional equation of motion of the secondary beam is fully decoupled from the other equations of motion. We perform, numerically, linear modal analysis of two models for the L-shaped beam structure and we confirm that the modes are well separated in the two kinds of motions – in-plane and out-of-plane. Also, we compare the theoretical natural frequencies for decoupled torsional motion of the secondary beam with these obtained from finite element analysis, which shows that they are in disagreement, therefore the rotary inertia terms for out-of-plane bending should necessarily be considered in the equations of motions. Also, examination of the mode shapes corresponding to torsional motion of the secondary beam shows that torsion is coupled with the rest of the out-of-plane motions. This work is essential in order to perform accurate linear modal analysis of an L-shaped beam, and for the development in the near future of a new nonlinear model for L-shaped beam structures.

# 2. Theory

## *2.1 Equations of motion*

We consider Euler-Bernoulli beams made up from an isotropic material and with constant cross-section with respect to the longitudinal direction ($x_1$ for the primary beam and $x_2$ for the secondary beam). According to the Vlasov theory [Librescu 2006], when considering that the beam's a wall thickness is h, d is any characteristic dimension, and l is its length, and then the beam can be considered as a thin-walled beam when,

$$h_{max}/l \leq 0.1,\ d/l \leq 0, \tag{1a,b}$$

and therefore in this case, the shear forces can be neglected.

In advance of any consideration of a variational formulation for the system shown in Figure 1, we define the curvatures and rotary terms using local displacements, and so for the primary beam these are given by [Nayfeh and Pai 2004, Warminski *et al* 2008],

$$\rho_{\xi_1} = \varphi_1',\ \rho_{\eta_1} = -w_1'',\ \rho_{\zeta_1} = v_1'',\ \omega_{\xi_1} = \dot{\varphi}_1,\ \omega_{\eta_1} = -\delta_1 \dot{w}_1',\ \omega_{\zeta_1} = \delta_4 \dot{v}_1', \tag{2a-f}$$

and for secondary beam they are given by,

$$\rho_{\xi_2} = \varphi_2',\ \rho_{\eta_2} = -w_2'',\ \rho_{\zeta_2} = v_2'',\ \omega_{\xi_2} = \dot{\varphi}_2 + \delta_3 \omega_{\eta_{1,c}},\ \omega_{\eta_2} = \delta_5\left(-\dot{w}_2' + \omega_{\zeta_{1,c}}\right), \omega_{\zeta_2} = \delta_2\left(\dot{v}_2' + \omega_{\xi_{1,c}}\right), \tag{3a-f}$$

In order to follow the effect of rotary inertia terms at the end of the formulation a switching function in the form of the Kronecker delta ($\delta_i$ ) is used with ($i$=1-5).

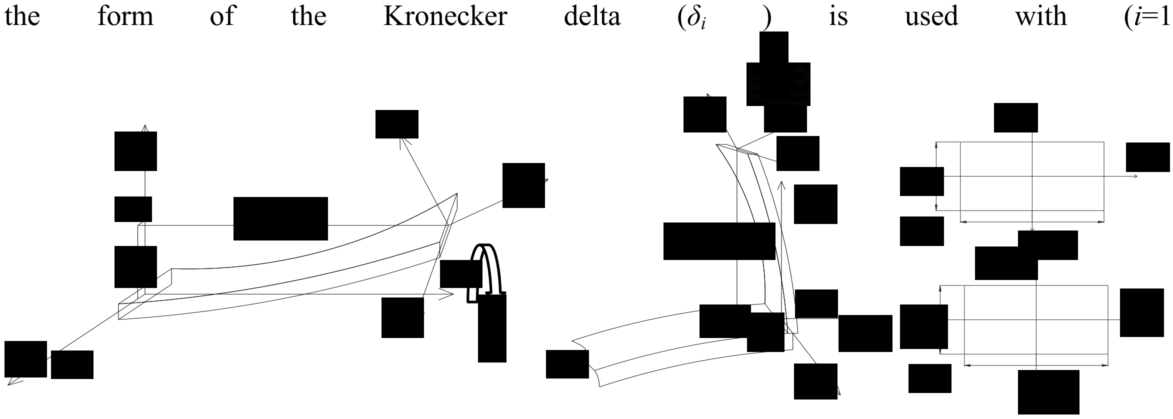

**Figure 1**. Indication of axis orientations and displacements for (a) the primary beam, (b) the secondary beam, (c) cross section of primary beam, and (d) cross section of secondary beam.

Translational velocities for the primary beam are given by,

$$V_{x_1} = \dot{u}_1,\ V_{y_1} = \dot{v}_1,\ V_{z_1} = \dot{w}_1. \tag{4a-c}$$

In the secondary beam the translational velocity $V_2$ of the centre of the cross section, due to the motion of point C, (clamped end, in local displacements) can be determined by considering the relative velocity $V_r$, and also the translational and angular motions of the origin of $X_2Y_2Z_2$ at point C. Therefore velocity $V_2$ in vector form is given by,

$$\vec{V}_2 = \vec{V}_C + \vec{\omega}_c \times \vec{r}_2 + \vec{V}_r, \tag{5}$$

with,

$$\vec{V}_C = \left[\dot{v}_{1,c}\ ,\ \dot{w}_{1,c}\ ,\ \dot{u}_{1,c}\right]^T, \vec{\omega}_c = [-\dot{w}'_{1C}\ , \dot{v}'_{1C}\ ,\ \dot{\varphi}_{1C}\ ]^T,$$

$$\vec{r}_2 = [s_2 + u_2\ , v_2\ , w_2\ ]^T, \qquad \vec{V}_r = [\dot{u}_2, \dot{v}_2,\ \dot{w}_2]^T. \tag{6a-d}$$

where T denotes the transpose of the matrix.
Substitution of equations 6a-d, into eq. 5, and retaining the first order terms, leads to the following definition of translational velocities for the secondary beam,

$$V_{x_2} = \dot{u}_2 + \dot{v}_{1,c}\ , V_{y_2} = \dot{v}_2 + \dot{w}_{1,c} + s_2\dot{\varphi}_{1,c}\ , V_{z_2} = \dot{w}_2 + \dot{u}_{1,c} - s_2\dot{v}'_{1,c}. \tag{7a-c}$$

The kinetic and potential energies are given by,

$$T_i = \tfrac{1}{2}\int_0^{l_i}\left[m_i\left(\left(V_{x_i}\right)^2 + \left(V_{y_i}\right)^2 + \left(V_{z_i}\right)^2\right) + I_{\xi_i}\left(\omega_{\xi_i}\right)^2 + I_{\zeta_i}\left(\omega_{\zeta_i}\right)^2 + I_{\eta_i}\left(\omega_{\eta_i}\right)^2\right]ds_i = \tfrac{1}{2}\int_0^{l_i} h_{T_i} ds_i, \tag{8}$$

$$V_i = \tfrac{1}{2}\int_0^{l_i}\left[D_{\xi_i}\left(\rho_{\xi_i}\right)^2 + D_{\eta_i}\left(\rho_{\eta_i}\right)^2 + D_{\zeta_i}\left(\rho_{\zeta_i}\right)^2\right]ds_i = \tfrac{1}{2}\int_0^{l_i} h_{V_i} ds_i\ , \tag{9}$$

with i=1,2 indicating the primary or secondary beams respectively.
In this work we assume that the beams are inextensional, therefore the following constraints are also imposed,

$$F_i = \tfrac{1}{2}\int_0^{l_i}\lambda_i\{1 - [(1+u'_i)^2 + (w'_i)^2 + (v'_i)^2]\}ds_i = \tfrac{1}{2}\int_0^{l_i} h_{F_i} ds_i. \tag{10}$$

It should be noted that $\lambda_i$ is the Lagrange multiplier, and that once again i=1,2 indicates the primary and secondary beams respectively.
The linear equations of motion are derived using Hamilton's principle of least action, such that,

$$\delta\int_{t_1}^{t_2}(T_1 - V_1 + F_1 + T_2 - V_2 + F_2)\,dt = 0 \Leftrightarrow$$

$$\Leftrightarrow \int_{t_1}^{t_2}\left\{\int_0^{l_1}\left(\delta h_{T_1} - \delta h_{V_1} + \delta h_{F_1}\right)ds_1 + \int_0^{l_2}\left(\delta h_{T_2} - \delta h_{V_2} + h_{F_2}\right)ds_2\right\}dt = 0, \tag{11}$$

taking into consideration the right hand sides of eq. (8-10).
The variations are defined, as follows,

$$\delta h_{T_1} = \textstyle\sum_{i=1}^{6}\frac{\partial h_{T_1}}{\partial p_i}\delta p_i, \qquad p = [\dot{u}_1, \dot{v}_1, \dot{w}_1, \dot{\varphi}_1, \dot{w}'_1, \dot{v}'_1\ ]^T, \tag{12a}$$

$$\delta h_{V_1} = \textstyle\sum_{i=1}^{3}\frac{\partial h_{V_1}}{\partial q_i}\delta q_i, \qquad q = [\varphi'_1, w''_1, v''_1\ ]^T, \tag{12b}$$

$$\delta h_{F_1} = \textstyle\sum_{i=1}^{4}\frac{\partial h_{F_1}}{\partial r_i}\delta r_i, \qquad r = [\lambda_1, u'_1, v'_1, w'_1\ ]^T, \tag{12c}$$

$$\delta h_{T_2} = \textstyle\sum_{i=1}^{12}\frac{\partial h_{T_2}}{\partial P_i}\delta P_i, \quad P = [\dot{u}_2, \dot{v}_2, \dot{w}_2, \dot{\varphi}_2, \dot{w}'_2, \dot{v}'_2, \dot{u}_{1C}, \dot{v}_{1C}, \dot{w}_{1C}, \dot{\varphi}_{1C}, \dot{w}'_{1C}, \dot{v}'_{1C}\ ]^T, \tag{12d}$$

$$\delta h_{V_2} = \textstyle\sum_{i=1}^{3}\frac{\partial h_{V_2}}{\partial Q_i}\delta Q_i, \quad Q = [\varphi'_2, w''_2, v''_2\ ]^T, \tag{12e}$$

$$\delta h_{F_2} = \textstyle\sum_{i=1}^{4}\frac{\partial h_{F_2}}{\partial R_i}\delta R_i, \quad R = [\lambda_2, u'_2, v'_2, w'_2\ ]^T\quad . \tag{12f}$$

Therefore by considering equations (12a-f) as substitutions within equation (11), and using integration by parts, then this leads to the following equation,

$$\int_{t_1}^{t_2}\left\{\left\{\int_0^{l_1}\left(-\frac{\partial^2 h_{T_1}}{\partial\dot{u}_1\partial t} - \frac{\partial^2 h_{F_1}}{\partial u'_1\partial s_1}\right)\delta u_1 ds_1 + \left(\frac{\partial h_{F_1}}{\partial u'_1}\delta u_1\right)\right|_{s_1=0}^{s_1=l_1} - \int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial\dot{u}_{1C}\partial t}ds_2\delta u_{1C} +\right.$$

$$-\int_0^{l_1}\left(\frac{\partial^2 h_{T_1}}{\partial\dot{v}_1\partial t}-\frac{\partial^3 h_{T_1}}{\partial\dot{v}_1'\partial s_1\partial t}+\frac{\partial^2 h_{F_1}}{\partial v_1'\partial s_1}+\frac{\partial^3 h_{V_1}}{\partial v_1''\partial s_1^2}\right)\delta v_1 ds_1+\left[\left(\frac{\partial h_{F_1}}{\partial v_1'}+\frac{\partial^2 h_{V_1}}{\partial v_1''\partial s_1}-\frac{\partial^2 h_{T_1}}{\partial\dot{v}_1'\partial t}\right)\delta v_1\right]\Bigg|_{s_1=0}^{s_1=l_1}-$$

$$-\left[\frac{\partial h_{V_1}}{\partial v_1''}\delta v_1'\right]\Bigg|_{s_1=0}^{s_1=l_1}-\int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial\dot{v}_{1C}\partial t}ds_2\delta v_{1C}-\int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial\dot{v}_{1C}'\partial t}ds_2\delta v_{1C}'+$$

$$+\int_0^{l_1}\left(-\frac{\partial^2 h_{T_1}}{\partial\dot{w}_1\partial t}+\frac{\partial^3 h_{T_1}}{\partial\dot{w}_1'\partial s_1\partial t}-\frac{\partial^2 h_{F_1}}{\partial w_1'\partial s_1}-\frac{\partial^3 h_{V_1}}{\partial w_1''\partial s_1^2}\right)\delta w_1 ds_1-\left[\frac{\partial h_{V_1}}{\partial w_1''}\delta w_1'\right]\Bigg|_{s_1=0}^{s_1=l_1}+$$

$$+\left[\left(-\frac{\partial^2 h_{T_1}}{\partial\dot{w}_1'\partial t}+\frac{\partial h_{F_1}}{\partial w_1'}+\frac{\partial^2 h_{V_1}}{\partial w_1''\partial s_1}\right)\delta w_1\right]\Bigg|_{s_1=0}^{s_1=l_1}-\int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial\dot{w}_{1C}\partial t}ds_2\delta w_{1C}-\int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial\dot{w}_{1C}'\partial t}ds_2\delta w_{1C}'+$$

$$+\int_0^{l_1}\left(-\frac{\partial^2 h_{T_1}}{\partial\dot{\varphi}_1\partial t}+\frac{\partial^2 h_{V_1}}{\partial\varphi_1'\partial s_1}\right)\delta\varphi_1 ds_1-\left(\frac{\partial h_{V_1}}{\partial\varphi_1'}\delta\varphi_1\right)\Bigg|_{s_1=0}^{s_1=l_1}-\int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial\dot{\varphi}_{1C}\partial t}ds_2\delta\varphi_{1C}+$$

$$+\int_0^{l_1}\frac{\partial h_{F_1}}{\partial\lambda_1}\delta\lambda_1 ds_1+\int_0^{l_2}\left(-\frac{\partial^2 h_{T_2}}{\partial\dot{u}_2\partial t}-\frac{\partial^2 h_{F_2}}{\partial u_2'\partial s_2}\right)\delta u_2 ds_2+$$

$$+\left(\frac{\partial h_{F_2}}{\partial u_2'}\delta u_2\right)\Bigg|_{s_2=0}^{s_2=l_2}+\int_0^{l_2}\left(-\frac{\partial^2 h_{T_2}}{\partial\dot{v}_2\partial t}+\frac{\partial^3 h_{T_2}}{\partial\dot{v}_2'\partial s_2\partial t}-\frac{\partial^2 h_{F_2}}{\partial v_2'\partial s_2}-\frac{\partial^3 h_{V_2}}{\partial v_2''\partial s_2^2}\right)\delta v_2 ds_2+$$

$$-\left[\frac{\partial h_{V_2}}{\partial v_2''}\delta v_2'\right]\Bigg|_{s_2=0}^{s_2=l_2}+\left[\left(-\frac{\partial^2 h_{T_2}}{\partial\dot{v}_2'\partial t}+\frac{\partial h_{F_2}}{\partial v_2'}+\frac{\partial^2 h_{V_2}}{\partial v_2''\partial s_2}\right)\delta v_2\right]\Bigg|_{s_2=0}^{s_2=l_2}-$$

$$-\int_0^{l_2}\left(\frac{\partial^2 h_{T_2}}{\partial\dot{w}_2\partial t}-\frac{\partial^3 h_{T_2}}{\partial\dot{w}_2'\partial s_2\partial t}+\frac{\partial^2 h_{F_2}}{\partial w_2'\partial s_2}+\frac{\partial^3 h_{V_2}}{\partial w_2''\partial s_2^2}\right)\delta w_2 ds_2-\left[\frac{\partial h_{V_2}}{\partial w_2''}\delta w_2'\right]\Bigg|_{s_2=0}^{s_2=l_2}+$$

$$+\left[\left(\frac{\partial h_{F_2}}{\partial w_2'}-\frac{\partial^2 h_{T_2}}{\partial\dot{w}_2'\partial t}+\frac{\partial^2 h_{V_2}}{\partial w_2''\partial s_2}\right)\delta w_2\right]\Bigg|_{s_2=0}^{s_2=l_2}+\int_0^{l_2}\left(-\frac{\partial^2 h_{T_2}}{\partial\dot{\varphi}_2\partial t}+\frac{\partial^2 h_{V_2}}{\partial\varphi_2'\partial s_2}\right)\delta\varphi_2 ds_2-$$

$$-\left(\frac{\partial h_{V_2}}{\partial\varphi_2'}\delta\varphi_2\right)\Big|_{s_2=0}^{s_2=l_2}+\int_0^{l_2}\frac{\partial h_{F_2}}{\partial\lambda_2}\delta\lambda_2 ds_2\Bigg\}dt=0. \tag{13}$$

Evaluation of the partial derivatives of equation (13), and using equations (2-4, 7-10) leads to the following equations of motion:

-for the primary beam,

a) Axial motion,

$$-m_1\ddot{u}_1+[(1+u_1')\lambda_1]'=0, \tag{14}$$

b) In-plane bending,

$$-m_1\ddot{v}_1+[\lambda_1 v_1']'-D_{\zeta_1}v_1^{IV}+\delta_4 I_{\zeta_1}\ddot{v}_1''=0, \tag{15}$$

c) Out-of-plane bending,

$$-m_1\ddot{w}_1+\delta_1 I_{\eta_1}\ddot{w}_1''+[\lambda_1 w_1']'-D_{\eta_1}w_1^{IV}=0, \tag{16}$$

d) Torsional motion,

$$-I_{\xi_1}\ddot{\varphi}_1+D_{\xi_1}\varphi_1''=0, \tag{17}$$

-for the secondary beam,

a) Axial motion,

$$-m_2(\ddot{u}_2+\ddot{v}_{1C})+[(1+u_2')\lambda_2]'=0, \tag{18}$$

b) In-plane bending,

$$-m_2(\ddot{w}_2 - s_2\ddot{v}_{1C}' + \ddot{u}_{1C}) + [\lambda_2 w_2']' - D_{\eta_2} w_2^{IV} + \delta_5 I_{\eta_2}\ddot{w}_2'' = 0, \quad (19)$$

c) Out-of-plane bending,

$$-m_2(\ddot{v}_2 + s_2\ddot{\varphi}_{1C} + \ddot{w}_{1C}) + \delta_2 I_{\zeta_2}\ddot{v}_2'' + [\lambda_2 v_2']' - D_{\zeta_2} v_2^{IV} = 0, \quad (20)$$

d) Torsional motion,

$$-I_{\xi_2}(\ddot{\varphi}_2 - \delta_3\delta_1\ddot{w}_{1C}') + D_{\xi_2}\varphi_2'' = 0, \quad (21)$$

and the inextensionality conditions, are given by,

$$(1+u_1')^2 + (w_1')^2 + (v_1')^2 = 1, \; (1+u_2')^2 + (w_2')^2 + (v_2')^2 = 1. \quad (22a,b)$$

The boundary conditions for equations (14-21), as shown in Appendix-A, arise from equation (13), from equations (2-4, 7-10), and also noting that the local displacements and rotations at the clamped ends for both beams are zero. Therefore the boundary conditions are,

-for the primary beam,

a) Axial motion,

$$u_1(0,t) = 0, \quad -[\lambda_1(l_1,t) + \lambda_1(l_1,t)u_1'(l_1,t)] - D_{\eta_2} w_2'''(0,t) - \delta_4\delta_5 I_{\eta_2}\ddot{v}_1'(l_1,t) = 0, \quad (23a,b)$$

b) In-plane bending,

$$v_1(0,t) = 0, \qquad v_1'(0,t) = 0,$$
$$-\delta_4 I_{\zeta_1}\ddot{v}_1'(l_1,t) - \lambda_1(l_1,t)v_1'(l_1,t) + D_{\zeta_1}v_1'''(l_1,t) + \big(1 + u_2'(0,t)\big)\lambda_2(0,t) = 0,$$
$$-D_{\zeta_1}v_1''(l_1,t) - D_{\eta_2}w_2''(0,t) - \int_0^{l_2}\lambda_2 w_2' ds_2 + I_{\eta_2}\delta_4\delta_5 l_2\ddot{v}_1'(l_1,t) - I_{\eta_2}\delta_5\ddot{w}_2(l_2,t) +$$
$$+\frac{I_{\eta_2}}{m_2}\delta_4\delta_5\{\lambda_2'(l_2,t)w_2'(l_2,t) - \lambda_2(0,t)w_2'(0,t) - D_{\eta_2}[w_2^{IV}(l_2,t) - w_2^{IV}(0,t)] - I_{\eta_2}\delta_5\ddot{w}_2''(0,t)\} = 0, \quad (24a\text{-}d)$$

c) Out-of-plane bending,

$$w_1(0,t) = 0, \qquad w_1'(0,t) = 0,$$
$$-\delta_1 I_{\eta_1}\ddot{w}_1'(l_1,t) - \lambda_1(l_1,t)w_1'(l_1,t) + D_{\eta_1}w_1'''(l_1,t) + \delta_2 I_{\zeta_2}\ddot{\varphi}_1(l_1,t) - D_{\zeta_2}v_2'''(0,t) = 0,$$
$$-D_{\eta_1}w_1''(l_1,t) - \delta_1\delta_3 D_{\xi_2}\varphi_2'(0,t) = 0, \quad (25a\text{-}d)$$

d) Torsional motion,

$$\varphi_1(0,t) = 0,$$
$$-D_{\xi_1}\varphi_1'(l_1,t) + \delta_2 l_2 I_{\zeta_2}\ddot{\varphi}_1(l_1,t) + \delta_2 I_{\zeta_2}\ddot{v}_2(l_2,t) + \int_0^{l_2}\lambda_2 v_2' ds_2 + D_{\zeta_2}v_2''(0,t) + \frac{\delta_2 I_{\zeta_2}^2}{m_2}\ddot{v}_2''(0,t) +$$
$$+\frac{\delta_2 I_{\zeta_2} D_{\zeta_2}}{m_2}\left(v_2^{IV}(l_2,t) - v_2^{IV}(0,t)\right) - \frac{\delta_2 I_{\zeta_2}}{m_2}\left(\lambda_2'(l_2,t)v_2'(l_2,t) - \lambda_2(0,t)v_2''(0,t)\right) = 0. \quad (26a,b)$$

-for the secondary beam,

a) Axial motion,

$$u_2(0,t) = 0, \qquad \lambda_2(l_2,t) + \lambda_2(l_2,t)u_2'(l_2,t) = 0, \quad (27a,b)$$

b) In-plane bending,

$$w_2(0,t) = 0, \qquad w_2'(0,t) = 0,$$
$$-\lambda_2(l_2,t)w_2'(l_2,t) + D_{\eta_2}w_2'''(l_2,t) + \delta_5 I_{\eta_2}\left(-\ddot{w}_2' + \ddot{v}_1'(l_1,t)\right) = 0, w_2''(l_2,t) = 0, \quad (28a\text{-}d)$$

c) Out-of-plane bending,

$$v_2(0,t) = 0, \qquad v_2'(0,t) = 0,$$
$$-\delta_2 I_{\zeta_2}\left(\ddot{v}_2'(l_2,t) + \ddot{\varphi}_1(l_1,t)\right) - \lambda_2(l_2,t)v_2'(l_2,t) + D_{\zeta_2}v_2'''(l_2,t) = 0,$$

$$v_2''(l_2,t)=0, \tag{29a-d}$$

d) Torsional motion,

$$\varphi_2(0,t)=0, \qquad \varphi_2'(l_2,t)=0. \tag{30a,b}$$

Taking into account the inextensionality conditions as shown in [Warminski *et al* 2008] then the axial displacements and accelerations are of second order and can be neglected since here we only consider a first order approximation, which is essentially the linear problem.
Using the axial equations of motion and the boundary conditions, then after integration, as shown in Appendix-B, the Lagrange multipliers are given by,

$$\lambda_1=\lambda_1^{(0)}+\lambda_1^{(1)}=0-D_{\eta_2}w_2'''(0,t)-I_{\eta_2}\delta_4\delta_5\ddot{v}_1'(l_1,t),\ \lambda_2=\lambda_2^{(0)}+\lambda_2^{(1)}=0+(s_2-l_2)m_2\ddot{v}_1(l_1,t), \tag{31a,b}$$

where the superscripts denote the order of the nonlinearity, a identification which is useful in order to derive the final equations of motion and the associated boundary conditions.
Finally, by considering the Langrange multipliers the linear equations of motion (neglecting axial motion since we just consider an inextensional beam), using eq. (14-22) and the boundary conditions eq. (23-30) take their final form,

In-plane motions (bending),

$$-m_1\ddot{v}_1-D_{\zeta_1}v_1^{IV}+\delta_4 I_{\zeta_1}\ddot{v}_1''=0, \tag{32}$$
$$-m_2(\ddot{w}_2-s_2\ddot{v}_{1C}'+\ddot{u}_{1C})-D_{\eta_2}w_2^{IV}+\delta_5 I_{\eta_2}\ddot{w}_2''=0, \tag{33}$$

boundary conditions,

$$\begin{gathered}v_1(0,t)=0, \qquad v_1'(0,t)=0,\\ -\delta_4 I_{\zeta_1}\ddot{v}_1'(l_1,t)+D_{\zeta_1}v_1'''(l_1,t)-l_2m_2\ddot{v}_1(l_1,t)=0,\\ -D_{\zeta_1}v_1''(l_1,t)-D_{\eta_2}w_2''(0,t)+I_{\eta_2}\delta_4\delta_5 l_2\ddot{v}_1'(l_1,t)-I_{\eta_2}\delta_5\ddot{w}_2(l_2,t)+\\ +\frac{I_{\eta 2}}{m_2}\delta_4\delta_5\{-D_{\eta_2}[w_2^{IV}(l_2,t)-w_2^{IV}(0,t)]-I_{\eta_2}\delta_5\ddot{w}_2''(0,t)\}=0,\end{gathered} \tag{34a-d}$$

$$\begin{gathered}w_2(0,t)=0, \qquad w_2'(0,t)=0,\\ D_{\eta_2}w_2'''(l_2,t)+\delta_5 I_{\eta_2}\left(-\ddot{w}_2'+\ddot{v}_1'(l_1,t)\right)=0, \qquad w_2''(l_2,t)=0.\end{gathered} \tag{35a-d}$$

Out-of-plane motions,

i) bending,

$$-m_1\ddot{w}_1+\delta_1 I_{\eta_1}\ddot{w}_1''-D_{\eta_1}w_1^{IV}=0, \tag{36}$$
$$-m_2(\ddot{v}_2+s_2\ddot{\varphi}_{1C}+\ddot{w}_{1C})+\delta_2 I_{\zeta_2}\ddot{v}_2''-D_{\zeta_2}v_2^{IV}=0, \tag{37}$$

ii) torsion,

$$-I_{\xi_1}\ddot{\varphi}_1+D_{\xi_1}\varphi_1''=0, \tag{38}$$
$$-I_{\xi_2}(\ddot{\varphi}_2-\delta_1\delta_3\ddot{w}_{1C}')+D_{\xi_2}\varphi_2''=0, \tag{39}$$

iii) boundary conditions,

$$\begin{gathered}w_1(0,t)=0, \qquad w_1'(0,t)=0,\\ -\delta_1 I_{\eta_1}\ddot{w}_1'(l_1,t)+D_{\eta_1}w_1'''(l_1,t)+\delta_2 I_{\zeta_2}\ddot{\varphi}_1(l_1,t)-D_{\zeta_2}v_2'''(0,t)=0,\\ -D_{\eta_1}w_1''(l_1,t)-\delta_1\delta_3 D_{\xi_2}\varphi_2'(0,t)=0,\\ \varphi_1(0,t)=0,\end{gathered} \tag{40a-d}$$

$$\begin{gathered}-D_{\xi_1}\varphi_1'(l_1,t)+\delta_2 l_2 I_{\zeta_2}\ddot{\varphi}_1(l_1,t)+\delta_2 I_{\zeta_2}\ddot{v}_2(l_2,t)+D_{\zeta_2}v_2''(0,t)+\frac{I_{\zeta_2}^2}{m_2}\delta_2\ddot{v}_2''(0,t)+\\ +\delta_2\frac{I_{\zeta_2}D_{\zeta_2}}{m_2}\left(v_2^{IV}(l_2,t)-v_2^{IV}(0,t)\right)=0,\end{gathered} \tag{41a,b}$$

$$\begin{gathered}v_2(0,t)=0, \qquad v_2'(0,t)=0,\\ -\delta_2 I_{\zeta_2}\left(\ddot{v}_2'(l_2,t)+\ddot{\varphi}_1(l_1,t)\right)+D_{\zeta_2}v_2'''(l_2,t)=0, \qquad v_2''(l_2,t)=0,\end{gathered} \tag{42a-d}$$

$$\varphi_2(0,t) = 0, \qquad \varphi_2'(l_2,t) = 0. \tag{43a,b}$$

It should be noted that by neglecting the rotary inertia term in the out-of-plane bending of the primary beam then $\delta_1 = \delta_3 = 0$, and the equation for torsion of the secondary beam is completely uncoupled from the out-of-plane motions, and is given by,

$$-I_{\xi_2}\ddot{\varphi}_2 + D_{\xi_2}\varphi_2'' = 0, \tag{44}$$

with boundary conditions, as follows,

$$\varphi_2(0,t) = 0, \qquad \varphi_2'(l_2,t) = 0. \tag{45a,b}$$

This can be explained by the fact that the rotary inertia terms are related to the curvature, therefore also with the angle of rotation at the end of the primary beam (point C), this being the initial rotation of the secondary beam. In this case (eq. 44,45) the theoretical natural frequency for torsion is trivial in form and is given by,

$$\omega_n = (2n-1)\frac{\pi}{2L_2}\sqrt{\frac{D_{\xi_2}}{I_{\xi_2}}}, \tag{46}.$$

It should be noted that in case of a *composite* thin-walled beam the torsional equations (eq. 38,39) are no longer of second order but are of fourth order due to the bimoment of warping [Librescu 2006], therefore our model is not valid for such beams.

# 3. Physical Models

We consider two physical models, with the same beam dimensions but with different secondary beam orientations. Configuration 1 is shown in Figures 2a and 2b , and configuration 2 is given in Figures 2c and 2d. The material used for both beams is aluminium, with density ρ = 2800 kg/$m^3$, Young's Modulus E = 70 MPa, Poisson's ratio ν = 0.33, and Shear Modulus, $G_{12}$ = 26.32 MPa. The dimensions of the beams are,

$L_1 \times b_1 \times h_1$= 0.18 m×0.00216 m×0.01295 m, $L_2 \times b_2 \times h_2$= 0.21 m×0.00216 m×0.01295 m.

The inertia term and torsional rigidity for the secondary beam, and also considering the warping effect by using Timoshenko's correction coefficient [Nayfeh and Pai 2004], are given by

$$I_{\xi_2} = \frac{\rho\left(b_2^3 h_2 + b_2 h_2^3\right)}{12} = 1.1250 \times 10^{-6} \text{ Kg m},$$

$$D_{\xi_2} = G_{12}\frac{1}{3}b_2 h_2^3\left(1 - \frac{192 h_2}{\pi^5 b_2}\sum_{n=1,3,\ldots}^{\infty}\left(\frac{1}{n^5}\tanh\left(\frac{n\pi b_2}{2h_2}\right)\right)\right) = 1.02445 \text{ Pa m}^4. \tag{47a,b}$$

In the case that the torsional motion is given by equation (44) (without consideration of the rotary inertia terms) the natural frequencies are given by equation (46), and by considering the parameters defined in eq. (47) the first 3 modes are as shown in Table 1.

**Table 1**. Theoretical natural frequencies of the secondary beam in torsion as simple cantilever beam.

| Mode | Freq. Theory (Hz) | Freq. Finite Element (Hz) | % Relative Difference |
|---|---|---|---|
| 1 | 1136.02 | 1131.40 | 0.4 |
| 2 | 3408.07 | 3394.08 | 0.4 |
| 3 | 5680.11 | 5656.38 | 0.4 |

**Table 2**. Natural frequencies for in plane motions.

| **Model-1** | **Model-2** |
|---|---|

| Mode | FE-Mode | Freq. (S4R el.) (Hz) | Freq. (B31- el.) (Hz) | % relative diff. | FE-Mode | Freq. (S4R el.) (Hz) | Freq. (B31 el.) (Hz) | % relative diff. |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 15.3 | 15.2 | 0 | 1 | 16.0 | 15.9 | 0 |
| 2 | 3 | 41.6 | 41.4 | 0 | 3 | 56.9 | 55.4 | 3 |
| 3 | 4 | 192.6 | 192.3 | 0 | 6 | 372.8 | 360.1 | 3 |
| 4 | 6 | 323.1 | 321.1 | 1 | 8 | 944.9 | 927.4 | 2 |
| 5 | 7 | 613.8 | 612.2 | 0 | 9 | 1088.3 | 1070.6 | 2 |
| 6 | 8 | 884.6 | 878.9 | 1 | 13 | 1928.6 | 1857.1 | 4 |
| 7 | 11 | 1263.8 | 1258.6 | 0 | 16 | 3070.7 | 3005.0 | 2 |
| 8 | 12 | 1734.5 | 1722.6 | 1 | 18 | 3333.9 | 3278.5 | 2 |
| 9 | 14 | 2144.8 | 2132.2 | 1 | 21 | 4714.6 | 4526.9 | 4 |
| 10 | 16 | 2864.6 | 2842.5 | 1 | 25 | 5961.6 | 5965.4 | 0 |

Abaqus software was used for the modal analysis and for each model two elements were taken for the modelling a) wire elements (B31) and b) shell elements (S4R). In the case of the wire elements and also the shell elements for model-1, the mesh comprised 150 elements for each beam. In the case of model-2 using shell elements 180 elements were used for the primary beam and 150 elements for the secondary beam. Also, in order to confirm the theoretical natural frequencies for the secondary beam in torsion a numerical modal analysis was performed using Abaqus for the clamped beam (Table 1). Comparison of the theoretical values for the secondary beam in torsion with those from the finite element simulations are in good agreement (Table 1).
Examination of the mode shapes verified that each mode is for in-plane motion or out-of-plane motion. In Table 2(3) it is shown that the natural frequencies for both models in the case of in-plane (out-of-plane) mode shapes and for the wire and shell models have no significant differences (less than 5%). Figure 2a (c), depict a representative mode shape for in-plane motion for model 1 (2). Figure 2b (d) shows a representative mode shape for out-of-plane motion for model 1 (2). The coupling between torsion of the secondary beam with out-of-plane bending is clearly demonstrated. Also, comparison of Table 1 (theoretical results without the rotary inertia effect) with the out-of-plane frequencies of the finite element model (Table 2) indicates that there is no proper correlation of the theoretical results with the finite element simulations, therefore neglecting the inertia terms for out-of-plane bending lead to completely different results for the torsional modes.

# 4. Conclusions

The linear equations of motion were derived for an L-shaped beam considering the rotary inertia terms.

**Table 3**. Natural frequencies for out-of-plane motions.

| | Model-1 | | | | Model-2 | | | |
|---|---|---|---|---|---|---|---|---|
| **Mode** | **FE-Mode** | **Freq. (S4R el.) (Hz)** | **Freq. (B31 el.) (Hz)** | **% relative diff.** | **FE-Mode** | **Freq. (S4R el.) (Hz)** | **Freq. (B31 el.) (Hz)** | **% relative diff.** |
| 1 | 2 | 24.3 | 23.9 | 2 | 2 | 20.8 | 20.6 | 1 |
| 2 | 5 | 217.4 | 217.6 | 0 | 4 | 157.5 | 157.5 | 0 |
| 3 | 9 | 1120.6 | 1111.1 | 1 | 5 | 297.4 | 297.4 | 0 |
| 4 | 10 | 1161.0 | 1160.3 | 0 | 7 | 625.7 | 624.6 | 0 |
| 5 | 13 | 1781.4 | 1782.8 | 0 | 10 | 1101.7 | 1115.2 | -1 |
| 6 | 15 | 2695.6 | 2621.1 | 3 | 11 | 1200.6 | 1197.0 | 0 |
| 7 | 18 | 3422.0 | 3377.7 | 1 | 12 | 1761.3 | 1760.4 | 0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 19 | 3552.7 | 3553.7 | 0 | 14 | 2065.1 | 2053.9 | 1 |
| 9 | 22 | 4875.4 | 4871.7 | 0 | 15 | 2557.8 | 2528.8 | 1 |
| 10 | 23 | 5428.4 | 5263.6 | 3 | 17 | 3228.6 | 3208.8 | 1 |

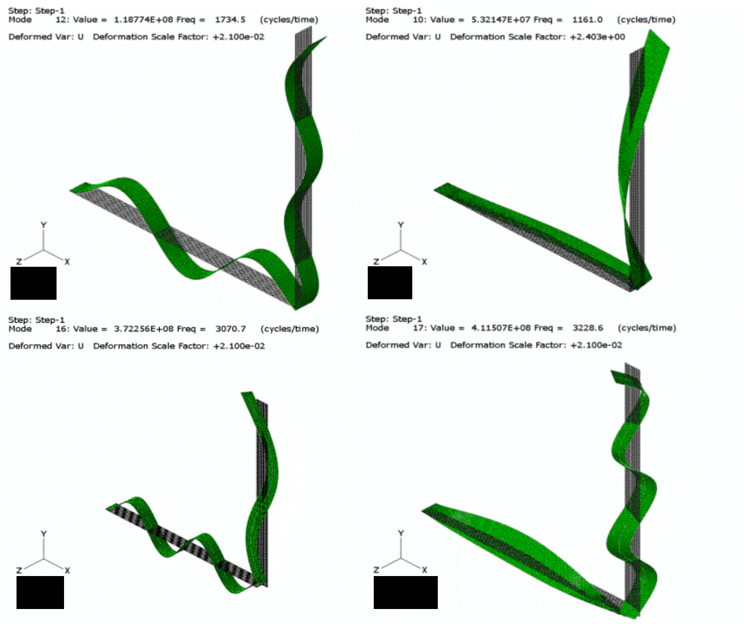


Figure 2. (a) Configuration 1, in-plane motions, (b) configuration 1, out-of-plane motions, (c) configuration 2, in-plane motions, (d) configuration 2, out-of-plane motions.

The equations of motion indicate that for the first order approximation the in-plane bending motions are coupled together, and fully uncoupled from out-of-plane motions, whereas all the other motions are coupled together. When neglecting the rotary inertia terms in out-of-plane bending, the equation for torsion of the secondary beam becomes uncoupled from the other out-of-plane motions. Numerical modal analysis was performed for two configurations of the L-shaped beam, and it was shown that the mode shapes can be distinguished for the in-plane and out-of-plane motions. Also the theoretical modal analysis for torsion, in the absence of rotary inertia terms, leads to completely different results to those from finite element analysis. Examination of the mode shapes shows that the out-of-plane bending is coupled to torsion of the secondary beam. Therefore it is necessary to consider rotary inertia for out-of-plane bending. The next step in this work will be to perform theoretical modal analysis of an L-shaped beam by solving the equations of motion and then comparing the obtained solutions with those from the finite element models.

## 5. Acknowledgements

The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007-2013), FP7 - REGPOT - 2009 - 1, under grant agreement No:245479. The support by the Polish Ministry of Science and Higher Education, Grant No 1471-1/7.PR UE/2010/7, is also acknowledged by the second author.

## 6. References

Barbero E. J. (2000). Prediction of Buckling-Mode Interaction in Composite Columns, Mechanics of Composite Materials and Structures, 7, 269-284.
Barbero E.J., Raftoyiannis I.G. (1994). Lateral and distortional buckling of pultruded I-beam, Composite Structures, 27, 261-268.
Cartmell, M.P., Roberts, J.W., 1987, Simultaneous combination resonances in a parametrically excited cantilever beam, Strain, 23, 117–126.

Balachandran, B., Nayfeh A.H., 1990, Nonlinear motion of beam-mass structure, Nonlinear Dynamics, 1, 39–61.
Fraternali, F. (1996) Energy Release Rates for Delamination of Composite Beams. Theoretical and Applied Fracture Mechanics, 25, 225-232, 1996.
Fraternali, F., Bilotti, G (1997). Non-Linear Elastic Stress Analysis in Curved Composite Beams. Computers & Structures, 62, 837-869.
Fraternali F., Feo L. (2000). On a Moderate Rotation Theory of Thin-Walled Composite Beams, Composites Part B: Engineering, 31, 141.
Librescu, L., Song, O., 2006, Thin-Walled Composite Beams (Theory and Application), Springer.
Nayfeh, A.H., Pai, F., 2004, Linear and Nonlinear Structural Mechanics, first edition, John Willey and sons, New Jersey.
Ozonato, N., Nagai, K.-I., Maruyama, S., Yamaguchi, T., 2012, Chaotic Vibrations of A Post-Buckled L-Shape Beam with an Axial Constraint, Nonlinear Dynamics, 67, 2363-2379.

Roberts, J.W., Cartmell, M.P., 1984, Forced vibration of a beam system with autoparametric coupling effects, Strain, 20, 123-131.
Warminski, J., Cartmell, M.P., Bochenski, M., Ivanov, I., 2008, Analytical and experimental investigations of an autoparametric beam structure, Journal of Sound and Vibration, 315, 486-508.

## 7. Appendix-A

The boundary conditions are shown here for the primary beam, since it is not a straightforward analysis from the variational equation (13).

-Axial motion,

$$\left(\frac{\partial h_{F_1}}{\partial u_1'}\delta u_1\right)\Big|_{s_1=0}^{s_1=l_1} - \int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial \dot{u}_{1C}\partial t}ds_2\delta u_{1C} = 0, \quad (A.1)$$

leads to,

$$u_1(0,t) = 0, \quad (23a)$$

$$-\left(\lambda_1(l_1,t) + \lambda_1(l_1,t)u_1'(l_1,t)\right) - \int_0^{l_2} m_2\left(\ddot{w}_2 - s_2\ddot{v}_1'(l_1,t) + \ddot{u}_1(l_1,t)\right)ds_2 = 0. \quad (A.2)$$

Using eq. (19) then (A.2) yields,

$$-\left(\lambda_1(l_1,t) + \lambda_1(l_1,t)u_1'(l_1,t)\right) - \lambda_2 w_2'\big|_{s_2=0}^{s_2=l_2} + D_{\eta_2}w_2'''\big|_{s_2=0}^{s_2=l_2} - \delta_5 I_{\eta_2}\ddot{w}_2'\big|_{s_2=0}^{s_2=l_2} = 0. \quad (A.3)$$

By using the boundary conditions for the secondary beam (eq.28), then the final equation is obtained,

$$-\left(\lambda_1(l_1,t) + \lambda_1(l_1,t)u_1'(l_1,t)\right) - D_{\eta_2}w_2'''(0,t) - \delta_4\delta_5 I_{\eta_2}\ddot{v}_1'(l_1,t) = 0. \quad (23b)$$

-In-plane bending motion,

$$\left[\left(\frac{\partial h_{F_1}}{\partial v_1'} + \frac{\partial^2 h_{V_1}}{\partial v_1''\partial s_1} - \frac{\partial^2 h_{T_1}}{\partial \dot{v}_1'\partial t}\right)\delta v_1\right]\Big|_{s_1=0}^{s_1=l_1} - \int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial \dot{v}_{1C}\partial t}ds_2\delta v_{1C} - \left[\frac{\partial h_{V_1}}{\partial v_1''}\delta v_1'\right]\Big|_{s_1=0}^{s_1=l_1} - \int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial \dot{v}_{1C}'\partial t}ds_2\delta v_{1C}' = 0. \quad (A.4)$$

Leads to,

$$v_1(0,t) = 0, \qquad v_1'(0,t) = 0, \quad (24a,b)$$

$$-\delta_4 I_{\zeta_1}\ddot{v}_1'(l_1,t) - \lambda_1(l_1,t)v_1'(l_1,t) + D_{\zeta_1}v_1'''(l_1,t) - \int_0^{l_2} m_2\left(\ddot{u}_2 + \ddot{v}_1(l_1,t)\right)ds_2 = 0,$$
$$-D_{\zeta_1}v_1''(l_1,t) + \int_0^{l_2}\left\{s_2 m_2\left[\ddot{w}_2 - s_2\ddot{v}_1'(l_1,t) + \ddot{u}_1(l_1,t)\right] + I_{\eta_2}\delta_4\delta_5\left[\ddot{w}_2' - \ddot{v}_1'(l_1,t)\right]\right\}ds_2 = 0. \quad (A.5a,b)$$

By using equation (18), equation (A.5a), takes the form,

$$-\delta_4 I_{\zeta_1}\ddot{v}_1'(l_1,t)-\lambda_1(l_1,t)v_1'(l_1,t)+D_{\zeta_1}v_1'''(l_1,t)-[(1+u_2')\lambda_2]\Big|_{s_2=0}^{s_2=l_2}=0. \qquad (A.6)$$

Considering also the boundary conditions for the secondary beam (eq. 27b) then the final form becomes,

$$-\delta_4 I_{\zeta_1}\ddot{v}_1'(l_1,t)-\lambda_1(l_1,t)v_1'(l_1,t)+D_{\zeta_1}v_1'''(l_1,t)+\big(1+u_2'(0,t)\big)\lambda_2(0,t)=0. \qquad (24c)$$

Similarly, for equation (A.5b) using eq. (19) and its derivative with respect to $s_2$, leads to,

$$-D_{\zeta_1}v_1''(l_1,t)+\int_0^{l_2}s_2\{[\lambda_2 w_2']'-D_{\eta_2}w_2^{IV}+I_{\eta_2}\delta_5\ddot{w}_2''\}ds_2+$$
$$+\tfrac{I_{\eta_2}}{m_2}\delta_4\delta_5\big[[\lambda_2 w_2']'-D_{\eta_2}w_2^{IV}+I_{\eta_2}\delta_5\ddot{w}_2''\big]\Big|_{s_2=0}^{s_2=l_2}=0,$$

then with integration by parts one obtains,

$$[s_2\lambda_2 w_2']\Big|_{s_2=0}^{s_2=l_2}-\big[s_2 D_{\eta_2}w_2'''\big]\Big|_{s_2=0}^{s_2=l_2}-\int_0^{l_2}\lambda_2 w_2' ds_2+\big[D_{\eta_2}w_2''\big]\Big|_{s_2=0}^{s_2=l_2}+\big[\delta_5 I_{\eta_2}s_2\ddot{w}_2'\big]\Big|_{s_2=0}^{s_2=l_2}$$
$$-D_{\zeta_1}v_1''(l_1,t)-\big[\delta_5 I_{\eta_2}\ddot{w}_2\big]\Big|_{s_2=0}^{s_2=l_2}+\tfrac{I_{\eta_2}}{m_2}\delta_4\delta_5\big[[\lambda_2 w_2']'-D_{\eta_2}w_2^{IV}+I_{\eta_2}\delta_5\ddot{w}_2''\big]\Big|_{s_2=0}^{s_2=l_2}=0. \qquad (A.7)$$

Considering the boundary conditions for the secondary beam (eq.28) leads to,

$$-D_{\zeta_1}v_1''(l_1,t)-D_{\eta_2}w_2''(0,t)-\int_0^{l_2}\lambda_2 w_2' ds_2+I_{\eta_2}\delta_4\delta_5 l_2\ddot{v}_1'(l_1,t)-I_{\eta_2}\delta_5\ddot{w}_2(l_2,t)+$$
$$+\tfrac{I_{\eta_2}}{m_2}\delta_4\delta_5\{\lambda_2'(l_2,t)w_2'(l_2,t)-\lambda_2(0,t)w_2'(0,t)-D_{\eta_2}[w_2^{IV}(l_2,t)-w_2^{IV}(0,t)]-I_{\eta_2}\delta_5\ddot{w}_2''(0,t)\}=0. \qquad (24d)$$

-Out-of-plane bending motion,

$$\left[\left(-\frac{\partial^2 h_{T_1}}{\partial\dot{w}_1'\partial t}+\frac{\partial h_{F_1}}{\partial w_1'}+\frac{\partial^2 h_{V_1}}{\partial w_1''\partial s_1}\right)\delta w_1\right]\Bigg|_{s_1=0}^{s_1=l_1}-\int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial\dot{w}_{1C}\partial t}ds_2\delta w_{1C}-$$
$$-\left[\frac{\partial h_{V_1}}{\partial w_1''}\delta w_1'\right]\Bigg|_{s_1=0}^{s_1=l_1}-\int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial\dot{w}_{1C}'\partial t}ds_2\delta w_{1C}'=0, \qquad (A.8)$$

then we get

$$w_1(0,t)=0, \qquad w_1'(0,t)=0, \qquad (25a,b)$$

$$-\delta_1 I_{\eta_1}\ddot{w}_1'(l_1,t)-\lambda_1(l_1,t)w_1'(l_1,t)+D_{\eta_1}w_1'''(l_1,t)-\int_0^{l_2}m_2\big(\ddot{v}_2+s_2\ddot{\varphi}_1(l_1,t)+\ddot{w}_1(l_1,t)\big)ds_2=0, \qquad (A.9a)$$

and,

$$-D_{\eta_1}w_1''(l_1,t)+\int_0^{l_2}\delta_1\delta_3 I_{\xi_2}\big(\ddot{\varphi}_2-\delta_1\delta_3\ddot{w}_1'(l_1,t)\big)ds_2=0. \qquad (A.9b)$$

Applying equation (20), means that equation (A.9a), takes the form,

$$-\delta_1 I_{\eta_1}\ddot{w}_1'(l_1,t)-\lambda_1(l_1,t)w_1'(l_1,t)+D_{\eta_1}w_1'''(l_1,t)-\big[\delta_2 I_{\zeta_2}\ddot{v}_2'\big]\Big|_{s_2=0}^{s_2=l_2}-[\lambda_2 v_2']\Big|_{s_2=0}^{s_2=l_2}+\big[D_{\zeta_2}v_2'''\big]\Big|_{s_2=0}^{s_2=l_2}=$$
$$=0. \qquad (A.10)$$

By considering also the boundary conditions of the secondary beam (eq.29) then it is possible to show that,

$$-\delta_1 I_{\eta_1}\ddot{w}_1'(l_1,t)-\lambda_1(l_1,t)w_1'(l_1,t)+D_{\eta_1}w_1'''(l_1,t)+\delta_2 I_{\zeta_2}\ddot{\varphi}_1(l_1,t)-D_{\zeta_2}v_2'''(0,t)=0. \qquad (25c)$$

Similarly, for equation (A.9b) when taking into account eq. (21), we get ,

$$-D_{\eta_1} w_1''(l_1,t) + \delta_1\delta_3\left[D_{\xi_2}\varphi_2'\right]\Big|_{s_2=0}^{s_2=l_2} = 0. \tag{A.11}$$

considering also the boundary conditions of the secondary beam (eq.30) leads to,

$$-D_{\eta_1} w_1''(l_1,t) - \delta_1\delta_3 D_{\xi_2}\varphi_2'(0,t) = 0. \tag{25d}$$

-Torsional motion,

$$-\left(\frac{\partial h_{V_1}}{\partial \varphi_1'}\delta\varphi_1\right)\Big|_{s_1=0}^{s_1=l_1} - \int_0^{l_2}\frac{\partial^2 h_{T_2}}{\partial\dot{\varphi}_{1C}\partial t}ds_2\delta\varphi_{1C} = 0,$$

and,

$$\varphi_1(0,t) = 0, \tag{26a}$$

$$-D_{\xi_1}\varphi_1'(l_1,t) - \int_0^{l_2}\left\{s_2 m_2\left(\ddot{v}_2 + s_2\ddot{\varphi}_1(l_1,t) + \ddot{w}_1(l_1,t)\right) + \delta_2 I_{\zeta_2}\left(\ddot{v}_2' + \ddot{\varphi}_1(l_1,t)\right)\right\}ds_2 = 0. \tag{A.12}$$

Using equation (20) and its derivative with respect to space, then equation (A.12), after integration by parts, takes the form,

$$-D_{\xi_1}\varphi_1'(l_1,t) - \left[\delta_2 s_2 I_{\zeta_2}\ddot{v}_2'\right]\Big|_{s_2=0}^{s_2=l_2} + \left[\delta_2 I_{\zeta_2}\ddot{v}_2\ \right]\Big|_{s_2=0}^{s_2=l_2} - \left[s_2\lambda_2 v_2'\right]\Big|_{s_2=0}^{s_2=l_2} + \int_0^{l_2}\lambda_2 v_2' ds_2 + $$
$$+\left[s_2 D_{\zeta_2} v_2'''\right]\Big|_{s_2=0}^{s_2=l_2}$$
$$-\left[D_{\zeta_2} v_2''\right]\Big|_{s_2=0}^{s_2=l_2} - \left[\delta_2\frac{I_{\zeta_2}^2}{m_2}\ddot{v}_2''\right]\Bigg|_{s_2=0}^{s_2=l_2} - \left[\frac{\delta_2 I_{\zeta_2}}{m_2}(\lambda_2 v_2')'\right]\Bigg|_{s_2=0}^{s_2=l_2} + \left[\frac{\delta_2 I_{\zeta_2}}{m_2}D_{\zeta_2}v_2^{IV}\right]\Bigg|_{s_2=0}^{s_2=l_2} = 0. \tag{A.13}$$

Finally, by considering the boundary conditions of the secondary beam eq.(29), we get

$$-D_{\xi_1}\varphi_1'(l_1,t) + \delta_2 l_2 I_{\zeta_2}\ddot{\varphi}_1(l_1,t) + \delta_2 I_{\zeta_2}\ddot{v}_2(l_2,t) + \int_0^{l_2}\lambda_2 v_2' ds_2 + D_{\zeta_2}v_2''(0,t) + \frac{\delta_2 I_{\zeta_2}^2}{m_2}\ddot{v}_2''(0,t) +$$
$$+\frac{\delta_2 I_{\zeta_2} D_{\zeta_2}}{m_2}\left(v_2^{IV}(l_2,t) - v_2^{IV}(0,t)\right) - \frac{\delta_2 I_{\zeta_2}}{m_2}\left(\lambda_2'(l_2,t)v_2'(l_2,t) - \lambda_2(0,t)v_2''(0,t)\right) = 0. \tag{26b}$$

## 8. Appendix-B

In this section the Lagrange multipliers are determined. Integration of the axial equation of motion for the primary beam (eq.14) leads to,

$$(1+u_1')\lambda_1 + c_{\lambda_1} = \int_0^{s_1} m_1\ddot{u}_1 ds_1, \tag{B.1}$$

applying the boundary conditions for the axial equation of the primary beam (eq.29b) we get,

$$c_{\lambda_1} = \int_0^{l_1} m_1\ddot{u}_1 ds_1 + D_{\eta_2}w_2'''(0,t) + I_{\eta_2}\delta_4\delta_5\ddot{v}_1'(l_1,t). \tag{B.2}$$

Therefore, by using (B.2), equation (B.1) takes the form,

$$\lambda_1 = \frac{1}{(1+u_1')}\left\{\int_0^{s_1} m_1\ddot{u}_1 ds_1 - \int_0^{l_1} m_1\ddot{u}_1 ds_1 - D_{\eta_2}w_2'''(0,t) - I_{\eta_2}\delta_4\delta_5\ddot{v}_1'(l_1,t)\right\},$$

or, by considering also the expansion of the fraction in series form,

$$\lambda_1 = \lambda_1^{(0)} + \lambda_1^{(1)} + HO = 0 - D_{\eta_2}w_2'''(0,t) - I_{\eta_2}\delta_4\delta_5\ddot{v}_1'(l_1,t) + HO. \tag{31a}$$

In the case of the secondary beam, we use a transformation by expressing the displacement in global coordinates,

$$U_2 = u_2 + v_1(l_1,t), \tag{B.3}$$

therefore the equation for axial motion (eq. 19) takes the form,

$$-m_2\ddot{U}_2 + [(1 + U_2')\lambda_2]' = 0, \tag{B.4}$$

with the boundary condition (eq. 28b)

$$\lambda_2(l_2,t)[1 + U_2'(l_2,t)] = 0. \tag{B.5}$$

Integration of equation (B.4) leads to,

$$[(1 + U_2')\lambda_2] + c_{\lambda_2} = \int_0^{s_2} m_2\ddot{U}_2 ds_2, \tag{B.6}$$

using also the boundary condition equation (B.5) we get,

$$c_{\lambda_2} = \int_0^{l_2} m_2\ddot{U}_2 ds_2. \tag{B.7}$$

Therefore the second Lagrange multiplier ($\lambda_2$) is given by,

$$\lambda_2 = \frac{1}{(1+U_2')}\left\{\int_0^{s_2} m_2\ddot{U}_2 ds_2 - \int_0^{l_2} m_2\ddot{U}_2 ds_2\right\}, \tag{B.8}$$

or, by using local displacement,

$$\lambda_2 = \frac{1}{(1+u_2')}\left\{\int_{l_2}^{s_2} m_2\ddot{u}_2 ds_2 + m_2(s_2 - l_2)\ddot{v}_1(l_1,t)\right\}, \tag{B.9}$$

and also the expansion of the fraction in series form, then this leads to the final form,

$$\lambda_2 = \lambda_2^{(0)} + \lambda_2^{(1)} + H.O.T. = 0 + (s_2 - l_2)m_2\ddot{v}_1(l_1,t) + H.O.\text{T}. \tag{31b}$$