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The structure of force free magnetic fields

E. A. Evangelidis*, L. L. Vaughan, G. J. J. Botha*
Centre for Physics and Mathematics, 2 Solonos St., Volos, Greece
Dept. Pure and Applied Physics, Queen's University, Belfast BT7 1NN, UK
Main author

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Abstract. Incontrovertible evidence is presented that the force free magnetic fields exhibit strong stochastic behaviour. Arnold's solution is given with the associated first integral of energy. A subset of the solution is shown to be non-ergodic whereas the full solution is shown to be ergodic. The first integral of energy is applied to the study of these fields to prove that the equilibrium points of such magnetic configurations are saddle points. Finally, the potential function of the first integral of energy is shown to be a member of the Helmholtz family of solutions. Numerical results corroborate the theoretical conclusions and demonstrate the robustness of the energy integral, which remains constant for arbitrarily long computing times.

Keywords: force free fields, energy integral, ergodicity, instability

1. Introduction

The concept of magnetic configurations producing no effect on ionised matter was introduced in astrophysics by Lüst and Schlüter (1954) in order to interpret an eruption on the sun. The explicit solution of the governing equation was given by Chandrasekhar (1956), Chandrasekhar and Kendall (1957) in terms of the Hansen (1935) solutions of the vectorial Helmholtz equation. The theoretical development of the issue led to Woltjer (1958) formulating the following result: *For any magnetic configuration with a magnetic field* $\mathbf{B}$ *derivable from a vector potential* $\mathbf{A}$ *constant on some boundary enclosing a volume* V*, the integral* $\int_V \mathbf{A} \cdot \bigtriangledown \wedge \mathbf{A} dV = \int_V \mathbf{A} \cdot \mathbf{B} dV$ *is an ideal MHD invariant.* This expression was then used by Chandrasekhar and Woltjer (1958) to show that the variation of the integral $\int_V [B^2/8\pi - (\alpha/8\pi)\mathbf{A} \cdot \bigtriangledown \wedge \mathbf{A}]dV$, with α a constant Lagrange multiplier and $\mathbf{A}$ constant on the boundary, leads to the force free field expression

$$\bigtriangledown \wedge \mathbf{B} = \alpha \mathbf{B}. \quad (1)$$

It is noticed that the only requirement on α is that it should be a constant. With the advent of projects aiming at the magnetic confinement of thermonuclear plasma to produce energy from fusion, the force free field configuration has been used extensively in spheromacs and reversed field pinches. An exhaustive review of this state of affairs can be found in Taylor (1986).

Today we know that the solar atmosphere is structured on a wide range of scales and that it is extremely dynamic (Golub and Pasachoff, 1997). The first evidence for this was obtained from ground-based observations during the 1950s and 1960s, but it was only with the advent of space-based observations that the true nature of the solar atmosphere was discovered. Observations from Skylab during the 1970s (Poletto *et al.*, 1975) and more recently from the Yohkoh, SoHO and TRACE missions show a solar plasma dominated by an ubiquitous magnetic field where force free fields play a major role (Priest, 1984; Zirker *et al.*, 1997; Démoulin, 1999). Since there is an extensive literature on this subject we concentrate on the discussion of the structure of the force free field equation.

The present analysis begins by presenting Arnold's solution of the force free field equation in Cartesian coordinates and the first integral of energy is derived, which has gone undetected so far. A subset of solutions, with one of the components of the magnetic field zero, is proved theoretically and confirmed numerically to be non-ergodic (Section 3). Then, in Sections 4 and 5, the equilibrium points of the full solution are shown to be saddle points of hyperboloidal surfaces in phase space, thus proving that the force free field configuration is unstable. Finally, the analytic solution is generalised and the associated potential function of the energy integral is shown to be an eigenfunction of the Helmholtz equation of eigenvalue $\sqrt{2}$.

2. The dynamical structure of the force free magnetic fields

The solution of the force free field equation in Cartesian coordinates offers an insight into the nature of such magnetic configurations. It is a simple matter to prove that the following system of equations

$$\begin{aligned} \frac{dx}{ds} &= B_x = A\sin z + C\cos y \\ \frac{dy}{ds} &= B_y = B\sin x + A\cos z \\ \frac{dz}{ds} &= B_z = C\sin y + B\cos x \end{aligned} \tag{2}$$

satisfies the force free field equation in the form $\mathbf{B} \wedge (\bigtriangledown \wedge \mathbf{B}) = 0$. In system (2) A, B and C are constants and x, y and z are some generalised angle coordinates. s is an affine parameter, for instance it could represent distance along a streamline, or time in the case of a velocity

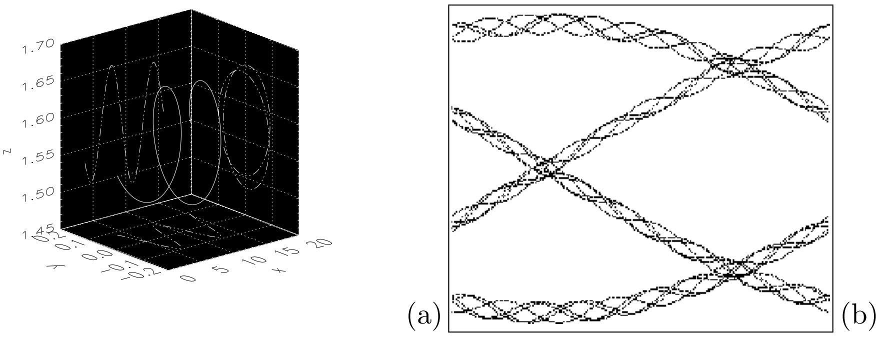

Figure 1. Streamlines of Arnold's solution with: (a) $A = 1.1$, $B = 0.3$, $C = 1.3$ and initial position $x = 3.78$, $y = 0.08$ and $z = 1.53$; (b) typical parameters and initial conditions.

field. This solution was first given by Arnold (1965) in the context of topology. Figure 1(a) follows one streamline in three dimensional Cartesian space, using a Runge-Kutta numerical integration of system (2). The pitch angle can be calculated from the expression dz/dy. The solution is extremely sensitive to the parameters and initial position, as shown by Figure 1(b). Figure 1(b) also shows that the streamlines twist around each other as they move through space. These numerical results were obtained with an energy conservation to machine accuracy. The general case of solution (2) with all constants nonzero, will be discussed first, since this will facilitate the understanding of the analysis of the special cases.

Taking the derivatives of expressions (2) the system follows

$$\begin{aligned} \frac{d^2x}{ds^2} &= \alpha \cos x \cos z - \beta \sin x \sin y \\ \frac{d^2y}{ds^2} &= \beta \cos y \cos x - \gamma \sin y \sin z \\ \frac{d^2z}{ds^2} &= \gamma \cos z \cos y - \alpha \sin z \sin x \end{aligned} \tag{3}$$

with $\alpha = AB$, $\beta = BC$ and $\gamma = CA$. These are the equations of motion of a three dimensional oscillator each component of which is under the influence of a combination of periodic forces described by the expressions on the right hand sides. It is remarkable that the simplest possible equation of magnetohydrodynamics has been transformed into a complicated problem of dynamical mechanics. These results can be written in the succinct form

$$\frac{d^2x_i}{ds^2} = -\frac{\partial U}{\partial x_i} \tag{4}$$

with

$$-U = \alpha \sin x \cos z + \beta \sin y \cos x + \gamma \sin z \cos y. \tag{5}$$

The existence of a potential function allows the derivation of the Hamiltonian of the system, which in this occasion reads

$$\begin{aligned} H &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &\quad -\alpha \sin x \cos z - \beta \sin y \cos x - \gamma \sin z \cos y \qquad (6) \\ &= \frac{1}{2}(A^2 + B^2 + C^2). \qquad (7) \end{aligned}$$

It is noticed that the results presented so far correspond to the special case of the force free field equation $\bigtriangledown \wedge \mathbf{B} = k\mathbf{B}$ when $k = 1$. The generalisation of Arnold's solution to cases when $k \neq 1$ will be given in section 6.

3. Particular cases of Arnold's solution

If Arnold's solution, system (2), is considered with any two of its constants equal to zero, then the motion it describes is non-ergodic. For instance, when $B = C = 0$ the motion is the straight line $y = \alpha_0 x + \alpha_1$ with α_0 and α_1 constant.

If one of the constants in Arnold's solution is zero (say $A = 0$), then the system of equations can be reduced to its normal form of two independent oscillators

$$\begin{aligned} \frac{d^2\xi}{dt^2} &= \beta \cos \xi \\ \frac{d^2\eta}{dt^2} &= \beta \cos \eta \end{aligned} \tag{8}$$

with

$$2x = \xi - \eta, \qquad 2y = \xi + \eta. \tag{9}$$

The corresponding energy levels are

$$\frac{1}{2}\dot{\xi}^2 - \beta \sin \xi = e_1, \qquad \frac{1}{2}\dot{\eta}^2 - \beta \sin \eta = e_2 \tag{10}$$

whereas the integral of energy is

$$E = \frac{1}{2}\left(\frac{\dot{\xi}^2 + \dot{\eta}^2}{2} + \dot{z}^2\right) - \beta \sin \frac{\xi + \eta}{2} \cos \frac{\xi - \eta}{2}. \tag{11}$$

The solution of equations (10) can be given in terms of elliptic functions and the problem is reduced to the motion of a point under the influence of two forces perpendicular to and independent of each other. Hence, the motion in the (ξ, η) plane is in general a complicated Lissajous figure. The z-component can be written now as

$$\dot{z} = \sqrt{2E - (e_1 + e_2)} \equiv D, \quad \text{constant.} \tag{12}$$

Equations (10) are valid for $e_1 - BC \geq 0$, $e_2 - BC \geq 0$, so that the motion in the (ξ, η) plane takes place inside the rectangle defined by the turning points of these equations. The decomposition (10) shows that the reduced problem with one of the constants A, B or C zero, is non-ergodic, since a point lying initially on manifold e_1 will never visit manifold e_2 and vice versa.

Another way of studying the reduced Arnold's solution is the following. The system of equations

$$\dot{x} = C \cos y, \qquad \dot{y} = B \sin x \tag{13}$$

has the particular integral

$$B \cos x + C \sin y = D, \tag{14}$$

a result directly derivable from the z-component of (3), which confirms that the velocity $\dot{z}$ is constant and therefore $z = Dt + z(0)$. The simplified form of Arnold's solution for $B = C$, $D = 0$ can be integrated immediately to give

$$\tan \frac{x}{2} = e^{Ct}, \qquad \tan \frac{y}{2} = \frac{e^{Ct} - 1}{e^{Ct} + 1} \tag{15}$$

which shows that the solution settles to the point attractor $x = \pi$, $y = \pi/2$ as $t \to \infty$. Arnold's solution for $A = 0$, B, $C \neq 0$ (see Figures 2 and 3) is reduced to quadrature upon deriving

$$\frac{\dot{x}^2}{C^2} + \frac{D^2}{C^2}\left(1 - \frac{B}{D}\cos x\right)^2 = 1 \tag{16}$$

$$\frac{\dot{y}^2}{B^2} + \frac{D^2}{B^2}\left(1 - \frac{C}{D}\sin y\right)^2 = 1. \tag{17}$$

These expressions are obtained by squaring each component of system (2) and substituting equation (14) into them. Equations (16) and (17) will be used to draw two and three dimensional phase portraits $(\dot{x}, x)$, $(\dot{y}, y)$ or $(x, y, \dot{x})$. It is rather obvious that they represent segments of circles in phase space of radii C and B respectively. For instance, the circle for (16) is defined by the transformation $X = D(1 - B\cos x/D)$.

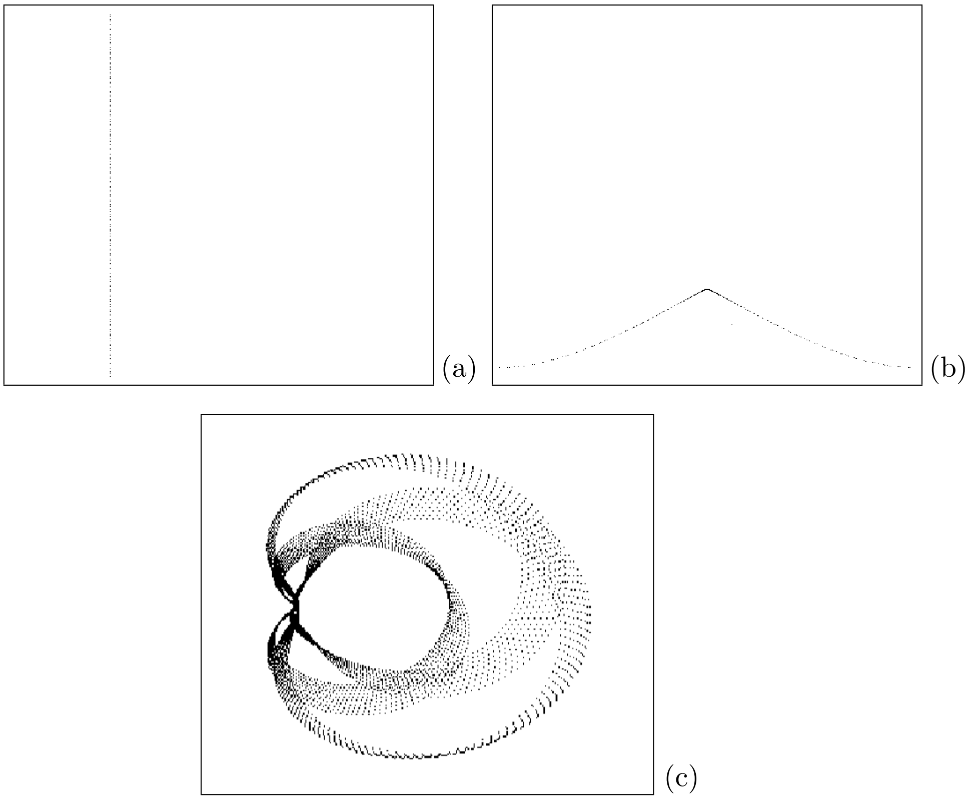

Figure 2. Non-ergodic Poincaré surface sections: (a) on the $x = 0$ plane and (b) on the $z = 0$ plane, with (c) the associated attractor, obtained by projecting the solution onto a periodic cylinder and plotting the (r, θ) plane with the z axis contracted. The parameters are $A = 0,\ \ B = 2,\ \ C = 5$ and the initial position is $x = 0,\ \ y = 0.201,$ $z = 3$.

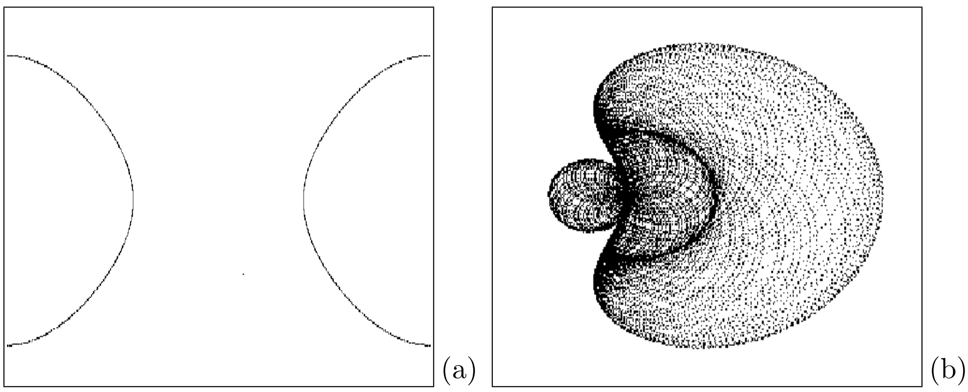

Figure 3. Non-ergodic (a) Poincaré surface section on the $z = 0$ plane with (b) the associated attractor. The parameters are $A = 0,\ \ B = 1.7321,\ \ C = 1.333$ and the initial position is $x = 4.867,\ \ y = 0.862,\ \ z = 0$.

The integrated form of (16) reveals a palindromic motion. In order to integrate (16) it is rewritten in the form

$$Cdt = \frac{dx}{\sqrt{1 - \frac{D^2}{C^2}\left(1 - \frac{B}{D}\cos x\right)^2}} \tag{18}$$

the denominator of which leads to the inequality

$$\frac{D-C}{B} < \cos x < \frac{D+C}{B} \tag{19}$$

which provides the upper and lower limits of the integration. By expanding the right hand side of (18) and applying the integration limits (19), the integrated form of (16) gives

$$\begin{aligned}\frac{T_x}{2} &= \frac{1}{C}\left\{\left[\cos^{-1}\left(\frac{D-C}{B}\right) - \cos^{-1}\left(\frac{D+C}{B}\right)\right]\left[1 + \frac{D^2}{2C^2} + \frac{B^2}{4C^2}\right]\right. \\ &\quad + \frac{BD}{4C^2}\left[\left(3 - \frac{C}{D}\right)\sqrt{1 - \frac{(D+C)^2}{B^2}}\right. \\ &\qquad\qquad \left.\left. - \left(3 + \frac{C}{D}\right)\sqrt{1 - \frac{(D-C)^2}{B^2}}\right]\right\}\end{aligned} \tag{20}$$

accurate to first order, where T_x is the period of the palindromic motion. Similarly the motion in $(\dot{y}, y)$ space has two turning points and the motion takes place within the limits

$$\frac{D-B}{C} < \sin y < \frac{D+B}{C}. \tag{21}$$

The period of the $(\dot{y}, y)$ motion is obtained from (20) by interchanging the constants B and C. Similar results hold true for the particular cases of Arnold's solution when one of B or C is zero and the relevant equations are summarised immediately below:

$$\begin{aligned}&B = 0: \\ &\dot{x} = C\cos y + A\sin z \equiv F, \\ &\frac{\dot{y}^2}{A^2} + \frac{F^2}{A^2}\left(1 - \frac{C}{F}\cos y\right)^2 = 1, \quad \frac{\dot{z}^2}{C^2} + \frac{F^2}{C^2}\left(1 - \frac{A}{F}\sin z\right)^2 = 1.\end{aligned} \tag{22}$$

$$\begin{aligned}&C = 0: \\ &\dot{y} = A\cos z + B\sin x \equiv G, \\ &\frac{\dot{z}^2}{B^2} + \frac{G^2}{B^2}\left(1 - \frac{A}{G}\cos z\right)^2 = 1, \quad \frac{\dot{x}^2}{A^2} + \frac{G^2}{A^2}\left(1 - \frac{B}{G}\sin x\right)^2 = 1.\end{aligned} \tag{23}$$

The corresponding integrals of energy, similar to (11), can be found by equating to zero the corresponding coefficient in (6). Therefore, the study of this subset of special force free configurations leads to the conclusion that in the cross section z = constant there develop areas in which motion along streamlines becomes trapped in "boxes", within which it performs a crescent like motion. On this motion is further superimposed the downstream motion. The similarity of this behaviour with that of the "banana" orbits of magnetohydrodynamics is striking. Since the period of the palindromic motion depends only on the constants B, C and D (for $A = 0$), there is an ∞^3 of points partaking in the motion for any given set. At the same time the downstream motion along the z-direction tends to produce thinner and thinner filaments of any initial magnetic field, due to the incompressibility of such fields. This behaviour corresponds to the classical description of an attractor and therefore, there are a total of ∞^5 such objects (Landau and Lifshitz, 1993).

4. Stability analysis of force free magnetic configurations

Figure 4 shows numerically the ergodic behaviour of force free magnetic fields when A, B, $C \neq 0$ for Arnold's solution (2). These numerical results were obtained with an energy conservation to machine accuracy. The ergodic behaviour was also shown by Hénon (1966) using a purely numerical approach. In section 2 it was shown that Arnold's solution leads naturally to the formulation of the potential function (5) and to the first integral of energy

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \alpha \sin x \cos z - \beta \sin y \cos x - \gamma \sin z \cos y. \quad (24)$$

The existence of this first integral allows the discussion of ergodicity to proceed in terms of the decomposability of the energy manifold (Birkhoff, 1966; Chintsin, 1964). The existence of a potential function reduces the discussion of stability of the dynamical problem to the discussion of its extrema (Lagrange criterion). Indeed, if $\overline{\mathbf{x}}$ is a maximum of the potential function then for a perturbation leading to movement away from the maximum, $\dot{\mathbf{x}}^2/2 = U(\overline{\mathbf{x}}) - U(\mathbf{x}) > 0$. For a real potential this relation can always be satisfied. For direction of motion towards a maximum the relation $\dot{\mathbf{x}}^2/2 = U(\mathbf{x}) - U(\overline{\mathbf{x}}) < 0$ shows that such motion is physically impossible for real potentials.

Let $(\overline{x}, \overline{y}, \overline{z})$ be such an equilibrium point of system (2). Then from the defining expressions

$$A \sin \overline{z} = -C \cos \overline{y}, \qquad B \sin \overline{x} = -A \cos \overline{z}, \qquad C \sin \overline{y} = -B \cos \overline{x} \quad (25)$$

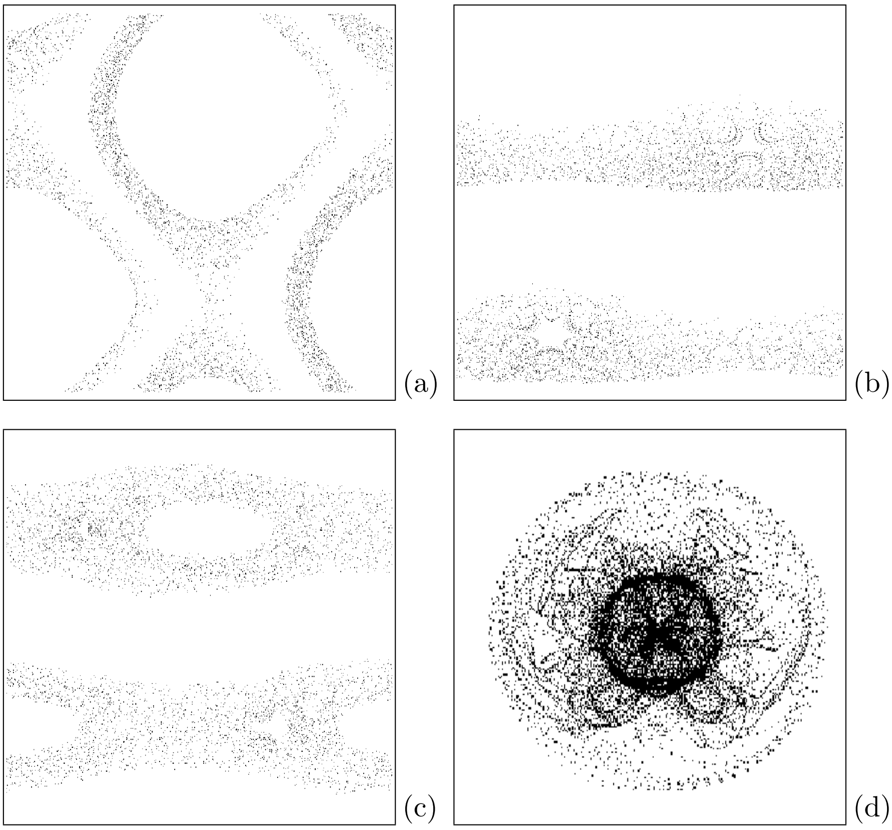

Figure 4. Ergodic Poincaré surface sections with their attractor. The parameters are $A = 1.111$, $B = 0.3115$, $C = 1.333$ and the initial position is $x = 1.57$, $y = 1.57$, $z = 0$. (a) The $x = 0$ plane. (b) The $y = 0$ plane. (c) The $z = 0$ plane. (d) The attractor.

the solution

$$\begin{aligned}
\cos 2\overline{x} &= \frac{C^2 - A^2}{B^2}, \quad \sin^2\overline{x} = \frac{A^2 + B^2 - C^2}{2B^2}, \quad \cos^2\overline{x} = \frac{B^2 + C^2 - A^2}{2B^2} \\
\cos 2\overline{y} &= \frac{A^2 - B^2}{C^2}, \quad \sin^2\overline{y} = \frac{B^2 + C^2 - A^2}{2C^2}, \quad \cos^2\overline{y} = \frac{C^2 + A^2 - B^2}{2C^2} \\
\cos 2\overline{z} &= \frac{B^2 - C^2}{A^2}, \quad \sin^2\overline{z} = \frac{C^2 + A^2 - B^2}{2A^2}, \quad \cos^2\overline{z} = \frac{A^2 + B^2 - C^2}{2A^2}
\end{aligned} \tag{26}$$

follows. One verifies easily that the derivatives of the potential function (5), that is the equations of motion (3), become zero at the stationary points (26). According to a theorem of real analysis:
A critical point c of a function f is a relative strict maximum, if the second derivative satisfies the relation $D_{ij}f(c)w^iw^j < 0$ for all direc-

tions w. (Bartle, 1976)
Indeed, let us expand the potential in the vicinity of the critical point $\overline{\mathbf{x}}$ as defined by (25). It is

$$U(\mathbf{x}) = U(\overline{\mathbf{x}}) + (\mathbf{x} - \overline{\mathbf{x}})_i D_i U(\overline{\mathbf{x}}) + \frac{1}{2}(\mathbf{x} - \overline{\mathbf{x}})_i (\mathbf{x} - \overline{\mathbf{x}})_j D_{ij} U(\overline{\mathbf{x}}). \quad (27)$$

We have already proved that the first derivatives $D_i U(\overline{\mathbf{x}})$, that is the equations of motion, are zero for the stationary points of the force free field equations. Recalling that at equilibrium points the velocity is zero, it follows from (24) that $U(\overline{\mathbf{x}}) = E$ and since $U(\mathbf{x}) - E = -u^2/2$, we are left with the equation

$$(\mathbf{x} - \overline{\mathbf{x}})_i (\mathbf{x} - \overline{\mathbf{x}})_j D_{ij} U(\overline{\mathbf{x}}) + u^2 = 0. \quad (28)$$

The second term can be calculated in the vicinity of the critical point from (2). It is

$$u^2 = A^2 + B^2 + C^2 + 2\alpha(x - \overline{x}) + 2\beta(y - \overline{y}) + 2\gamma(z - \overline{z}) \quad (29)$$

when terms up to first order are kept. Hence we find

$$\begin{aligned} &B^2(x - \overline{x})^2 + C^2(y - \overline{y})^2 + A^2(z - \overline{z})^2 \\ &-2(x - \overline{x})(y - \overline{y})\sqrt{(E_0 - C^2)(E_0 - B^2)} \\ &-2(y - \overline{y})(z - \overline{z})\sqrt{(E_0 - A^2)(E_0 - C^2)} \\ &-2(z - \overline{z})(x - \overline{x})\sqrt{(E_0 - B^2)(E_0 - A^2)} \\ &-2\alpha(x - \overline{x}) - 2\beta(y - \overline{y}) - 2\gamma(z - \overline{z}) - 2E_0 = 0 \end{aligned} \quad (30)$$

where $2E_0 = A^2 + B^2 + C^2$. Expression (30) can be further transformed into its canonical form upon using a well established algebraic theorem, namely that:
Every quadratic form $A(x, x) = \sum_{r=1}^{n} \sum_{s=1}^{n} \alpha_{rs} x_r x_s$ *in* n *variables and of rank* r*, with coefficients in a given field* F*, can be transformed by a non-singular transformation, with coefficients in* F*, into the form* $\alpha_1 X_1^2 + ... + \alpha_r X_r^2$ *(canonical form), where* $\alpha_1, ..., \alpha_r$ *are numbers in* F *and no one of them is equal to zero.* (Ferrar, 1957)
For $n = 3$ the six distinct quadratic surfaces of geometry are retrieved. The actual expressions effecting the canonical transformation can be found in the Appendix, but they are of no importance in the qualitative discussion of the stability here. Naturally, not all of these six surfaces are acceptable solutions. Since they represent potentials of the force free magnetic fields, their second derivatives must satisfy the constraints mentioned above. We summarise the situation in Table I.

The relief maps of the potential function in figures 5 and 6 demonstrate clearly that the equilibrium points are saddle points, confirming the results of the theoretical description.

Table I. Quadratic surfaces of geometry

1	$U = \alpha^2x^2 + \beta^2y^2 + \gamma^2z^2 - 1.$ Ellipsoid.	$D^2f(c)(w)^2 = 2(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2) > 0.$ Always positive. Physically unacceptable.
2	$U = \alpha^2x^2 + \beta^2y^2 - \gamma^2z^2 - 1.$ One-sheet hyperboloid.	$D^2f(c)(w)^2 = 2(\alpha^2x^2 + \beta^2y^2 - \gamma^2z^2).$ A saddle point in general. In the present context $D^2f(c)(w)^2 < 0$ for $\alpha^2x^2 + \beta^2y^2 < \gamma^2z^2$ is the only physically acceptable branch. Branch $\alpha^2x^2 + \beta^2y^2 > \gamma^2z^2$ is inadmissible.
3	$U = \alpha^2x^2 - \beta^2y^2 - \gamma^2z^2 - 1.$ Two-sheet hyperboloid.	Critical points are saddles. Only branch $\alpha^2x^2 < \beta^2y^2 + \gamma^2x^2$ is acceptable.
4	$U = \alpha^2x^2 + \beta^2y^2 - z.$ Elliptic hyperboloid.	No critical point. Therefore, it is not a solution of the present dynamical system.
5	$U = \alpha^2x^2 - \beta^2y^2 - z.$ Hyperbolic paraboloid.	No critical point. Inadmissible.
6	$U = \alpha^2x^2 + \beta^2y^2 - \gamma^2z^2.$ Quadratic cone.	Trivial case, realisable for $A = B = C$. ($U(0) = 0$)

5. Stability of the linear mapping

Similar conclusions can be reached upon examination of the eigenvalues of the linear mapping of Arnold's solution (2). Expanding the right hand sides of those expressions in the neighbourhood of the equilibrium point $(\overline{x}, \overline{y}, \overline{z})$, using double angle formulas and small angle approximations for the trigonometric functions, it is found that

$$\begin{aligned} \dot{X} &= ZA\cos\overline{z} - YC\sin\overline{y} \\ \dot{Y} &= XB\cos\overline{x} - ZA\sin\overline{z} \\ \dot{Z} &= YC\cos\overline{y} - XB\sin\overline{x} \end{aligned} \tag{31}$$

where $X = x - \overline{x}$, $Y = y - \overline{y}$ and $Z = z - \overline{z}$. Then the normal solution of the form

$$X = (X, Y, Z) = (X_i(0), Y_i(0), Z_i(0))e^{\lambda_i t} \tag{32}$$

can be found from system (31)'s characteristic equation (Dombre *et al.*, 1986)

$$\lambda^3 - \lambda\Lambda_1 - \Lambda_0 = 0 \tag{33}$$

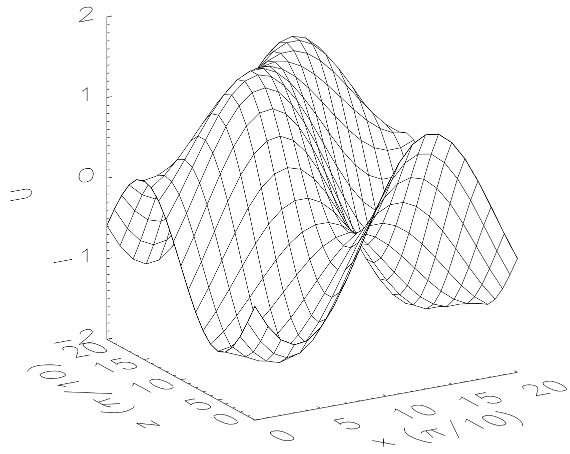

Figure 5. The potential function U on the (x, z) plane with $y = \pi/5$ and $\alpha = \beta = \gamma = 1$

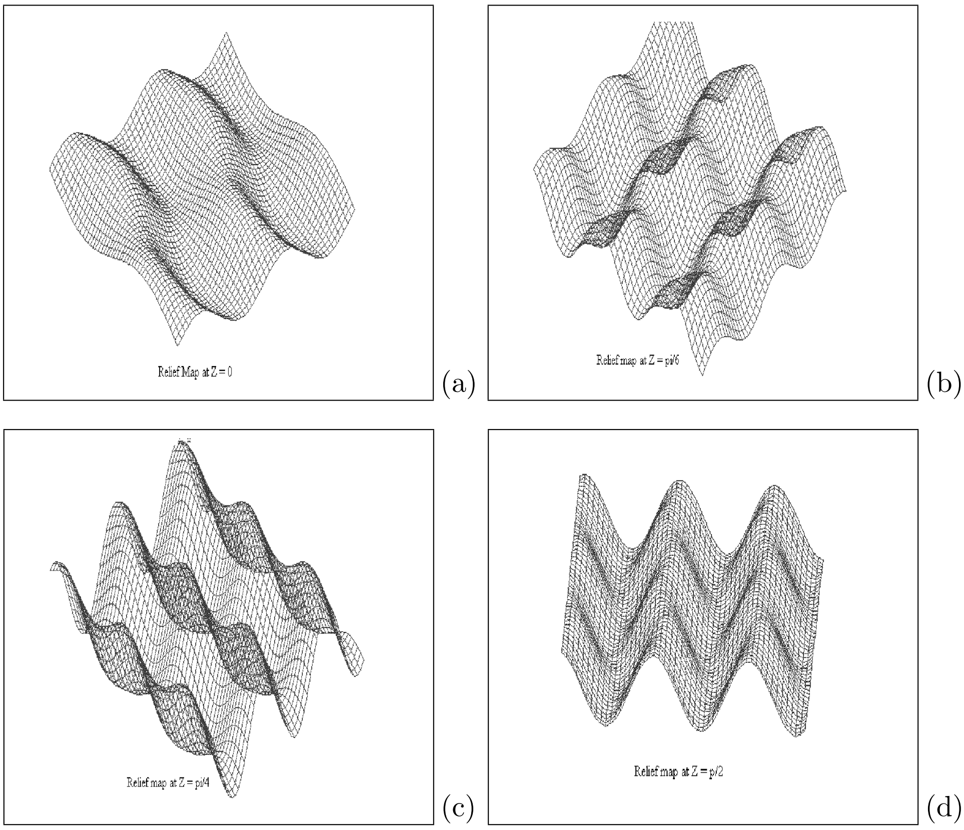

Figure 6. Relief maps of the potential U for typical values of α, β and γ on the (x, y) plane for (a) $z = 0$, (b) $z = \pi/6$, (c) $z = \pi/4$, (d) $z = \pi/2$.

with

$$\Lambda_1 = -A\cos\overline{z}B\sin\overline{x} - C\sin\overline{y}B\cos\overline{x} - A\sin\overline{z}C\cos\overline{y} \tag{34}$$
$$\Lambda_0 = ABC(\cos\overline{x}\cos\overline{y}\cos\overline{z} - \sin\overline{x}\sin\overline{y}\sin\overline{z}). \tag{35}$$

Employing further expressions (25) we find

$$\Lambda_1 = \frac{1}{2}(A^2 + B^2 + C^2) = E_0 \tag{36}$$
$$\Lambda_0 = 2ABC\cos\overline{x}\cos\overline{y}\cos\overline{z}. \tag{37}$$

The last expression can be further transformed by means of (26) and $2E_0 = A^2 + B^2 + C^2$ into the form

$$\Lambda_0 = \pm 2\sqrt{(E_0 - A^2)(E_0 - B^2)(E_0 - C^2)} \tag{38}$$

so that the eigenvalues are given by the roots of

$$\lambda^3 - E_0\lambda \mp 2\sqrt{(E_0 - A^2)(E_0 - B^2)(E_0 - C^2)} = 0. \tag{39}$$

The behaviour of the general solution depends on whether the discriminant

$$\frac{b^2}{4} - \frac{E_0^3}{27} \quad \text{with} \quad b \equiv 2\sqrt{(E_0 - A^2)(E_0 - B^2)(E_0 - C^2)} \tag{40}$$

is larger, equal or smaller than zero. The particular case where E_0 assumes the value of any amplitude A^2, B^2 or C^2 leads to the simplified equation

$$\lambda(\lambda^2 - E_0) = 0 \tag{41}$$

with eigenvalues

$$\lambda = 0, \quad \lambda = \pm\sqrt{E_0}. \tag{42}$$

In this case solution (32) is

$$X = X_1(0) + X_2(0)e^{+\sqrt{E_0}t} + X_3(0)e^{-\sqrt{E_0}t} \qquad (E_0 = A^2, B^2, C^2). \tag{43}$$

The existence of the positive eigenvalue shows clearly that these particular equilibrium points are unstable for all real values of the parameters A, B, C in this particular case, except when the initial value of $X_2(0)$ is zero. The special case of the subset of solutions with $A = 0$, say, leads to exactly the same expression with $E_0 = (B^2 + C^2)/2$.

6. A remark on the potential function of the force free magnetic fields

As can be seen from the rotation of the force free magnetic field equation (1),

$$\bigtriangledown^2 \mathbf{B} + k^2 \mathbf{B} = 0 \tag{44}$$

for any divergence free field, i.e. $\bigtriangledown \cdot \mathbf{B} = 0$. It follows that all solutions of equation (1) should be members of the family of solutions to the Helmholtz equation (44), although the reverse is not true. Therefore, the question arises as to whether Arnold's solution is a member of the Helmholtz family of solutions. To that effect Arnold's solution is generalised to cases with $k \neq 1$. The generalised solution is easily found to be

$$\dot{x} = A \sin kz + C \cos ky \tag{45}$$

$$\dot{y} = B \sin kx + A \cos kz \tag{46}$$

$$\dot{z} = C \sin ky + B \cos kx \tag{47}$$

$$-U = \alpha \sin kx \cos kz + \beta \sin ky \cos kx + \gamma \sin kz \cos ky \tag{48}$$

with the first integral of energy

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \alpha \sin kx \cos kz - \beta \sin ky \cos kx - \gamma \sin kz \cos ky. \tag{49}$$

The second partial derivatives of the generalised potential (48) give

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = -2k_m^2 U. \tag{50}$$

The subscript has been added to indicate the eigenvalue of the magnetostatic problem. The eigenvalue of Arnold's solution ($k = 1$) is $\sqrt{2}$. For the generalised case the relation $k_H = \sqrt{2} k_m$ holds true. Therefore, Arnold's solution is a member of the Helmholtz family of solutions.

7. Concluding remarks

Following Arnold's considerations about fields described by magnetic force free or vorticity free type of equations, that they are generally chaotic, we have been able to derive rigorously explicit expressions for the source of this chaos. That is, we have proved that the equations of motion of these fields are described by an autonomous (time independent) potential, whose equilibrium points are saddle points. The analysis has been particularly successful in defining explicitly ergodic and non-ergodic regions in parameter space and numerical integrations have borne out to any degree of accuracy the analytical proofs.

Force free magnetic fields have attracted interest in space plasma research, where it is thought that this type of magnetic configuration might hold the key to the interpretation of observations. The idea has evolved that force free fields represent steady state minima. This study shows that these fields are extremely unstable, which may play a significant part in explaining plasma behaviour in space.

Appendix

We proceed here with the detailed derivation of the canonical expressions for the potential describing the force free magnetic fields in the vicinity of their equilibrium point(s). For

$$\begin{aligned} b_{12} &= -\sqrt{(E_0 - C^2)(E_0 - B^2)}\,, & b_{11} &= B^2 \\ b_{23} &= -\sqrt{(E_0 - A^2)(E_0 - C^2)}\,, & b_{22} &= C^2 \\ b_{13} &= -\sqrt{(E_0 - B^2)(E_0 - A^2)}\,, & b_{33} &= A^2 \end{aligned} \tag{51}$$

equation (30) can be written as

$$\begin{aligned} B(x,x) &= \sum_{i,j=1}^{3} b_{ij} x_i x_j = \sum_{r=1}^{3} b_{rr} x_r^2 + 2 \sum_{r<s=1}^{3} b_{rs} x_r x_s \\ &= 2E_0 + 2\alpha x + 2\beta y + 2\gamma z \end{aligned} \tag{52}$$

with $x_1 = x$, $x_2 = y$ and $x_3 = z$. By making the transformation

$$X_1 = x_1 + \frac{b_{12}}{b_{11}} x_2 + \frac{b_{13}}{b_{11}} x_3, \qquad X_s = x_s, \quad s = 2,\ 3 \tag{53}$$

$$\beta_{22} = b_{22} - \frac{b_{12}^2}{b_{11}}, \qquad \beta_{33} = b_{33} - \frac{b_{13}^2}{b_{11}}, \qquad \beta_{23} = b_{23} - \frac{b_{12} b_{13}}{b_{11}} \tag{54}$$

followed by a second transformation

$$X = X_1, \qquad Y = X_2 + \frac{\beta_{23}}{\beta_{22}} X_3, \qquad Z = X_3 \tag{55}$$

$$\alpha_{11} = b_{11}, \qquad \alpha_{22} = \beta_{22}, \qquad \alpha_{33} = \beta_{33} - \frac{\beta_{23}^2}{\beta_{22}} \tag{56}$$

the bilinear form of equation (52) becomes

$$B(x,x) = \alpha_{11} X^2 + \alpha_{22} Y^2 + \alpha_{33} Z^2. \tag{57}$$

Using transformations (53) through to (56), the initial variables are now expressed in terms of the canonical variables of the quadratic form. They are

$$x = X - c_{12}Y + c_{13}Z, \qquad y = Y - c_{23}Z, \qquad z = Z \tag{58}$$

with

$$c_{12} = \frac{b_{12}}{b_{11}}, \qquad c_{13} = \frac{1}{b_{11}}\left(b_{12}\frac{\beta_{23}}{\beta_{22}} - b_{13}\right), \qquad c_{23} = \frac{\beta_{23}}{\beta_{22}}. \tag{59}$$

Substituting expressions (57) and (58) in (52), completing the squares and collecting terms produce the required expression

$$\delta_1 x_1^2 + \delta_2 x_2^2 + \delta_3 x_3^2 - 1 = 0 \tag{60}$$

used in the text. The symbols δ_i are abbreviations for the quantities

$$\delta_1 = \frac{\alpha_{11}}{\triangle}, \qquad \delta_2 = \frac{\alpha_{22}}{\triangle}, \qquad \delta_3 = \frac{\alpha_{33}}{\triangle} \tag{61}$$

where

$$\triangle = \frac{\alpha^2}{\alpha_{11}} + \frac{\gamma_{12}^2}{\alpha_{22}} + \frac{\gamma_{13}^2}{\alpha_{33}} + 2E_0 \tag{62}$$

$$\gamma_{12} = \beta - \alpha c_{12}, \qquad \gamma_{13} = \alpha c_{13} - \beta c_{23} + \gamma \tag{63}$$

and also

$$x_1 = X - \frac{\alpha}{\alpha_{11}}, \qquad x_2 = Y - \frac{\gamma_{12}}{\alpha_{22}}, \qquad x_3 = Z - \frac{\gamma_{13}}{\alpha_{33}}. \tag{64}$$

With these expressions we have completed our task of deriving the canonical magnetic surfaces of the force free fields in terms of the initial constant magnetic fields of the problem.

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References

V. I. Arnold, 1965, Comptes Rendues de l'Academie des Sciences, Paris, **261**, 17.

R. G. Bartle, 1976, The Elements of Real Analysis, John Wiley & Sons, New York, p. 379.

G. Birkhoff, 1966, Dynamical Systems, American Mathematical Society, vol.106.
S. Chandrasekhar, 1956, Proc. Nat. Acad. Sci., **42**, 1.
S. Chandrasekhar, 1956, ApJ, **126**, 232.
S. Chandrasekhar and P. C. Kendall, 1957, ApJ, **126**, 457.
S. Chandrasekhar and L. Woltjer, 1958, Proc. Nat. Acad. Sci., **44**, 285.
A. J. Chintsin, 1964, Mathematische Grundlagen der Statistischen Mechanik, Bibliographisches Institut, Manheim.
P. Démoulin, 1999, Journal of Atmospheric and Solar-Terrestrial Physics, **61**, 101.
T. Dombre, U. Frisch, J. M. Greene, M. Hénon, A. Mehr and A. M. Soward, 1986, J. Fluid Mech., **167**, 353. After submitting our paper we discovered this paper with results similar to equation (33).
W. L. Ferrar, 1957, Algebra, Oxford University Press, p.148.
L. Golub and J. M. Pasachoff, 1997, The Solar Corona, Cambridge University Press, Cambridge.
W. W. Hansen, 1935, Physical Reviews, **47**, 139.
M. Hénon, 1966, Comptes Rendues de l'Academie des Sciences, Paris, **262**, 312.
E. Jahnke and F. Emde, 1945, Tables of Functions, Dover Publications, New York, p. 168.
L. D. Landau and E. M. Lifshitz, 1993, Fluid Mechanics, Pergamon Press, Oxford, p.13.
R. Lüst and A. Schlüter, 1954, Zeitschrift für Astrophysik, **34**, 263.
G. Poletto, G. S. Vaiana, M. V. Zombeck, A. S. Krieger, A. F. Timothy, 1975, Solar Physics, **44**, 83.
E. R. Priest, 1984, Solar Magnetohydrodynamics, D. Reidel Publishing Company, Dordrecht.
J. B. Taylor, 1986, Rev. Mod. Phys., **58**, 741.
L. Woltjer, 1958, Proc. Nat. Acad. Sci., **44**, 489 and 833.
J. B. Zirker, S. F. Martin, K. Harvey, V. Gaizauskas, 1997, Solar Physics, **175**, 27.