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BOUNDARY VALUE PROBLEMS FOR DISCRETE FRACTIONAL EQUATIONS

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by

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In this dissertation we develop certain aspects of the theory of discrete fractional calculus. The author begins with an introduction to the discrete delta calculus together with the fractional delta calculus which is used throughout this dissertation. The Cauchy function, the Green's function and some of their important properties for a ν^{th} order fractional boundary value problem for both the cases $0 < \nu \leq 1$ and $1 < \nu \leq 2$ are developed. This dissertation is comprised of four chapters. In the first chapter we introduce the delta fractional calculus. In the second chapter we give some preliminary definitions, properties and theorems for the fractional delta calculus and derive the appropriate Green's function and give some of its important properties. This allows us to prove some important theorems by using well-known fixed point theorems. In the third chapter we study and prove various results regarding the generalized fractional boundary value problem for the self-adjoint equation

$$\Delta^{\nu}_{\nu-1}(p\Delta x)(t) + q(t+\nu-1)x(t+\nu-1) = f(t),$$

where $0 < \nu \leq 1$ with Sturm-Liouville type boundary conditions. In the fourth chapter we prove some theorems regarding the existence and uniqueness of positive solution of a forced fractional equation with finite limit as t goes to ∞.

DEDICATION

This dissertation is dedicated to my late mom Tulsi Devi Awasthi, my wife Moyi Li Awasthi, my dad Madhawa Prashad Awasthi, my advisors Dr. Lynn Erbe and Dr. Allan C. Peterson, Dr. Judy Walker, Ms. Marilyn Johnson and my sisters and brothers. I would like to thank the Department of Mathematics and the University of Nebraska, Lincoln for their support in every aspect of my academic career. I would also like to mention the contribution of my professors Dr. Jamie Radcliffe and Dr. Petronela Radu who had faith in me that I could finish my dissertation. I am thankful to both Dr. Daniel Toundykov and Dr. Radu for proof reading my dissertation. I am thankful for Dr. Srikanth Iyengar, Dr. Vinodchandran N. Variam, for being on my supervisory committee. I must mention and respect the sacrifice and the patience of my wife and family when I was away from them. This dissertation is wholeheartedly devoted to all of the above mentioned people and the University of Nebraska.

Contents

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Chapter 1

Introduction

In this introductory chapter we introduce the basic delta calculus which will be useful for our later results. We refer to Kelley and Peterson [61] for more details. Frequently, the functions we consider will be defined on a set of the form

$$\mathbb{N}_a := \{a, a+1, a+2, \cdots\},$$

where $a \in \mathbb{R}$, or a set of the form

$$\mathbb{N}_a^b := \{a, a+1, a+2, \cdots, b\},$$

where $a, b \in \mathbb{R}$ and $b - a$ is a nonnegative integer.

Definition 1.0.1. *Assume $f : \mathbb{N}_a^b \to \mathbb{R}$, then if $b > a$ we define the forward difference operator Δ by*

$$\Delta f(t) := f(t+1) - f(t)$$

for $t \in N_a^{b-1}$.

Definition 1.0.2. *Assume* $f : \mathbb{N}_a^b \to \mathbb{R}$, *then if* $b > a$ *we define the forward jump operator* σ *by*

$$\sigma(t) = t + 1$$

for $t \in N_a^{b-1}$. *It is often convenient to use the notation* f^σ *to denote the function defined by the composition* $f \circ \sigma$, *that is*

$$f^\sigma(t) = (f \circ \sigma)(t) = f(\sigma(t)) = f(t+1),$$

for $t \in N_a^{b-1}$. *Also, the operator* Δ^n, $n = 1, 2, 3, \cdots$ *is defined recursively by* $\Delta^n f(t) = \Delta(\Delta^{n-1} f(t))$ *for* $t \in N_a^{b-n}$, *where we assume the integer* $b - n \geq a$. *Finally,* Δ^0 *denotes the identity operator, i.e.,* $\Delta^0 f(t) = f(t)$. *We will use the following well known properties of the difference operator [61] throughout this dissertation.*

Theorem 1.0.3. *Assume* $f, g : \mathbb{N}_a^b \to \mathbb{R}$ *and* $\alpha, \beta \in \mathbb{R}$, *then for* $t \in \mathbb{N}_a^{b-1}$

1. $\Delta\alpha = 0$;
2. $\Delta\alpha f(t) = \alpha\Delta f(t)$;
3. $\Delta\left(f(t) + g(t)\right) = \Delta f(t) + \Delta g(t)$;
4. $\Delta\alpha^{t+\beta} = (\alpha - 1)\alpha^{t+\beta}$;
5. $\Delta\left(f(t)g(t)\right) = f(\sigma(t))\Delta g(t) + \Delta f(t) g(t)$;
6. $\Delta\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)\Delta f(t) - \Delta f(t) g(t)}{g(t)g(\sigma(t))}$,

where in (6) *we assume* $g(t) \neq 0$, $t \in \mathbb{N}_a^{b-1}$

Next, we define the falling function.

Definition 1.0.4 (Falling Function). *For n a positive integer we define the falling function, $t^{\underline{n}}$, read t to the n falling, by*

$$t^{\underline{n}} := t(t-1)(t-2)\cdots(t-n+1).$$

Also we define $t^{\underline{0}} := 1$.

Theorem 1.0.5 (Power Rule). *The following formula holds*

$$\Delta t^{\underline{n}} = n\, t^{\underline{n-1}},$$

for $n = 0, 1, 2, \cdots$.

A very important (transendental) function in mathematics is the gamma function which is defined as follows.

Definition 1.0.6 (Gamma Function). *The gamma function is defined by*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

for those complex numbers for which the real part of z is positive (it can be shown that the above improper integral converges for all such z).

We will use the formula

$$\Gamma(z+1) = z\Gamma(z) \tag{1.0.1}$$

to extend the domain of the gamma function to all complex numbers $z \neq 0, -1, -2, \cdots$. On this domain the gamma function is an analytic function. Also note that since it can be shown that $\lim_{z\to 0}|\Gamma(z)| = \infty$ it follows from (1.0.1) that

$$\lim_{z\to -n}|\Gamma(z)| = \infty, \quad n = 0, 1, 2, \cdots,$$

which is a fundamental property of the gamma function which we will use from time to time. Another well known important consequence of (1.0.1) is that

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, \cdots.$$

Because of this, the gamma function is known as a generalization of the factorial function.

Note that for n a positive integer

$$\begin{aligned} t^{\underline{n}} &= t(t-1)\cdots(t-n+1) \\ &= \frac{t(t-1)\cdots(t-n+1)\Gamma(t-n+1)}{\Gamma(t-n+1)} \\ &= \frac{\Gamma(t+1)}{\Gamma(t-n+1)}. \end{aligned}$$

Motivated by this above calculation, we extend the domain of the falling function in the following definition.

Definition 1.0.7. *The (generalized) falling function is defined by*

$$t^{\underline{r}} := \frac{\Gamma(t+1)}{\Gamma(t-r+1)}$$

for those values of t and r such that the right hand side of this equation makes sense. We then extend this definition by making the common convention that $t^{\underline{r}} = 0$ when $t-r+1$ is a nonpositive integer and $t+1$ is not a nonpositive integer.

The motivation for the convention in Definition 1.0.7 is that whenever $t-r+1$ is a nonpositive integer and $t+1$ is not a nonpositive integer, then

$$\lim_{s\to t} t^{\underline{r}} = \lim_{s\to t} \frac{\Gamma(s+1)}{\Gamma(s-r+1)} = 0.$$

In the cases when we use this convention, one should always verify what we conclude by taking an appropriate limit. This step will usually not be included in our calculations.

Next we state and prove the generalized power rule.

Theorem 1.0.8 (Power Rules)**.** *The following power rules hold:*

$$\Delta(t+\alpha)^{\underline{r}} = r(t+\alpha)^{\underline{r-1}}, \tag{1.0.2}$$

and

$$\Delta(\alpha-t)^{\underline{r}} = -r(\alpha-\sigma(t))^{\underline{r-1}}, \tag{1.0.3}$$

whenever the expressions in these two formulas are well defined.

Note that when $n \geq k \geq 0$, the binomial coefficient is given by

$$\binom{n}{k} := \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{\Gamma(k+1)}.$$

Motivated by this we next define the (generalized) binomial coefficient as follows.

Definition 1.0.9. *The (generalized) binomial coefficient* $\binom{t}{r}$ *is defined by*

$$\binom{t}{r} := \frac{t^{\underline{r}}}{\Gamma(r+1)}$$

for those values of t and r so that the right hand side is well defined.

From the definition and the power rules, one can establish the following theorem

Theorem 1.0.10. *The following hold:*

1. $\Delta\binom{t}{r} = \binom{t}{r-1}, \quad r \neq 0;$

2. $\Delta\binom{r+t}{t} = \binom{r+t}{t+1}.$

1.1 Discrete Delta Integral

Next we define the discrete definite integral.

Definition 1.1.1. *Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $c, d \in \mathbb{N}_a$, then*

$$\int_c^d f(t)\Delta t := \begin{cases} \sum_{t=c}^{d-1} f(t), & \text{if} \quad d > c \\ 0, & \text{if} \quad d \leq c. \end{cases}$$

Defining this integral to be 0, when $d \leq c$ will be very beneficial when we study fractional calculus in Section 2. The following theorem gives some properties of this integral.

Theorem 1.1.2. *Let $f, g : \mathbb{N}_a \to \mathbb{R}$, $b, c, d \in \mathbb{N}_a$, $b < c < d$, and $\alpha \in \mathbb{R}$. Then*

1. $\int_b^c \alpha f(t)\Delta t = \alpha \int_b^c f(t)\Delta t;$

2. $\int_b^c (f(t) + g(t))\Delta t = \int_b^c f(t)\Delta t + \int_b^c g(t)\Delta t;$

3. $\int_b^b f(t)\Delta t = 0;$

4. $\int_b^d f(t)\Delta t = \int_b^c f(t)\Delta t + \int_c^d f(t)\Delta t;$

5. $|\int_b^c f(t)\Delta t| \leq \int_b^c |f(t)|\Delta t$;

6. *If* $F(t) := \int_b^t f(s)\Delta s$, *for* $t \in \mathbb{N}_b^c$, *then* $\Delta F(t) = f(t)$, $t \in \mathbb{N}_a^{c-1}$;

7. *If* $f(t) \geq g(t)$ *for* $t \in \{c, c+1, \cdots, d-1\}$, *then* $\int_b^c f(t)\Delta t \geq \int_b^c g(t)\Delta t$.

The following integration by parts formula follows in a standard way from the product rule.

Theorem 1.1.3 (Integration by Parts)**.** *Given two functions* $u, v : \mathbb{N}_a \to \mathbb{R}$ *and* $b, c \in \mathbb{N}_a$, $b < c$, *we have the integration by parts formulas:*

$$\int_b^c u(t)\Delta v(t)\Delta t = u(t)v(t)\Big|_b^c - \int_b^c v(\sigma(t))\Delta u(t)\Delta t, \tag{1.1.1}$$

$$\int_b^c u(\sigma(t))\Delta v(t)\Delta t = u(t)v(t)\Big|_b^c - \int_b^c v(t)\Delta u(t)\Delta t. \tag{1.1.2}$$

Definition 1.1.4. *Let* $f : \mathbb{N}_a^b \to \mathbb{R}$. *We say* $F(t)$ *is an antidifference of* $f(t)$ *on* $\mathbb{N}_a^b$ *provided*

$$\Delta F(t) = f(t), \quad t \in \mathbb{N}_a^{b-1}.$$

Theorem 1.1.5. *If* $f : \mathbb{N}_a^b \to \mathbb{R}$ *and* $G(t)$ *is an antidifference of* $f(t)$ *on* $\mathbb{N}_a^b$, *then* $F(t) = G(t) + C$ *is a general antidifference of* $f(t)$.

Definition 1.1.6. *If* $f : \mathbb{N}_a \to \mathbb{R}$, *then the delta indefinite integral of* f *is defined by*

$$\int f(t)\Delta t := F(t) + C,$$

where C *is an arbitrary constant.*

Any formula for a delta derivative gives us a formula for an indefinite integral, so we have the following theorem.

Theorem 1.1.7 (Fundamental Theorem for the Difference Calculus)**.** *Assume* $f : \mathbb{N}_a^b \to \mathbb{R}$ *and* $F(t)$ *is any antidifference of* $f(t)$ *on* $\mathbb{N}_a^b$*. Then*

$$\int_a^b f(t)\Delta t = \int_a^b \Delta F(t)\Delta t = F(t)|_a^b.$$

Proof. Assume $F(t)$ is any antidifference of $f(t)$ on $\mathbb{N}_a^b$. Let

$$G(t) := \int_a^t f(s)\Delta s, \quad t \in \mathbb{N}_a^b,$$

then by Theorem 1.1.2 (6), $G(t)$ is an antidifference of $f(t)$. Hence by Theorem 1.1.5, $F(t) = G(t) + C$, where C is a constant. Then

$$\begin{aligned} F(t)|_a^b &= F(b) - F(a) \\ &= [(G(b) + C) - (G(a) + C)] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\Delta t. \end{aligned}$$

□

1.2 Fractional Sums and Differences

Theorem 1.2.1 (Repeated Summation Rule)**.** *Let* $f : \mathbb{N}_a \to \mathbb{R}$ *be given, then*

$$\int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau_n)\Delta\tau_n \cdots \Delta\tau_2\Delta\tau_1 = \frac{1}{(n-1)!}\int_a^t (t-\sigma(\tau_1))^{\underline{n-1}} f(\tau_1)\Delta\tau_1.$$

Motivated by Theorem 1.2.1, we define the n-th fractional sum $\Delta^{-n} f(t)$ for positive integers n, by

$$\begin{aligned}\Delta_a^{-n} f(t) &= \frac{1}{(n-1)!}\int_a^t (t-\sigma(s))^{\underline{n-1}} f(s)\Delta s \\ &= \frac{1}{\Gamma(n)}\int_a^{t-n+1} (t-\sigma(s))^{\underline{n-1}} f(s)\Delta s,\end{aligned}$$

since

$$(t-\sigma(s))^{\underline{n-1}} = 0, \quad s = t-1 = t-2 = \cdots = t-n+1.$$

This, in turn, motivates the definition of the ν-th fractional sum.

Definition 1.2.2. *Let $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ with $N-1 < \nu \leq N$. Then the ν-th fractional sum of f (based at a) at the point $t \in \mathbb{N}_{a+\nu}$ is defined by*

$$\begin{aligned}\Delta_a^{-\nu} f(t) :=& \frac{1}{\Gamma(\nu)}\int_a^{t-\nu+1} (t-\sigma(s))^{\underline{\nu-1}} f(s)\Delta s \\ =& \frac{1}{\Gamma(\nu)}\sum_{k=a}^{t-\nu} (t-\sigma(k))^{\underline{\nu-1}} f(k).\end{aligned}$$

Note that by our convention on integrals (sums) we can extend the domain of $\Delta_a^{-\nu} f$ to $\mathbb{N}_{a+\nu-N}$ with the convention that

$$\Delta_a^{-\nu} f(t) = 0, \quad t \in \mathbb{N}_{a+\nu-N}^{a+\nu-2}$$

(note that the above formula also holds on $\mathbb{N}_{a+\nu-N}^{a+\nu-1}$).

Remark 1.2.3. *Note that the value of the ν-th fractional sum of f based at a is a linear combination of $f(a), f(a+1), \cdots, f(t-\nu)$, where the coefficient of $f(t-\nu)$ is one. In particular one can check that $\Delta_a^{-\nu} f(t)$ has the form*

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)}(t-\sigma(a))^{\underline{\nu-1}} f(a) + \cdots + \nu f(t-\nu-1) + f(t-\nu). \qquad (1.2.1)$$

Next we define the fractional difference in terms of the fractional sum.

Definition 1.2.4. *Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Choose a positive integer N such that $N-1 < \nu \leq N$. Then we define the ν-th fractional difference by*

$$\Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \quad t \in \mathbb{N}_{a+N-\nu}.$$

Note that our fractional difference agrees with our prior understanding of whole-order derivatives, that is, for any $\nu = N \in \mathbb{N}_0$

$$\Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t) = \Delta^N \Delta_a^{-0} f(t) = \Delta^N f(t), \text{ for } t \in \mathbb{N}_a. \qquad (1.2.2)$$

Theorem 1.2.5. *Assume $q : \mathbb{N}_0 \to \mathbb{R}$. Then the homogeneous fractional difference equation*

$$\Delta_{\nu-N}^{\nu} u(t) + q(t) u(t+\nu-N) = 0, \quad t \in \mathbb{N}_0 \qquad (1.2.3)$$

has N linearly independent solutions $u_i(t)$, $1 \leq i \leq N$, on $\mathbb{N}_0$ and

$$u(t) = c_1 u_1(t) + c_2 u_2(t) + \cdots + c_N u_N(t),$$

where $c_1, c_2, \cdots, c_N$ are arbitrary constants, is a general solution of this homogeneous fractional difference equation on N_0. Furthermore, if in addition, $y_p(t)$ is a particular *solution of the nonhomogeneous fractional difference equation*

$$\Delta^{\nu}_{\nu-N}u(t) + q(t)u(t+\nu-N) = h(t), \quad t \in \mathbb{N}_0 \tag{1.2.4}$$

on $\mathbb{N}_0$*, then*

$$y(t) = c_1u_1(t) + c_2u_2(t) + \cdots + c_Nu_N(t) + y_p(t),$$

where $c_1, c_2, \cdots, c_N$ *are arbitrary constants, is a general solution of the nonhomogeneous fractional difference equation* (1.2.4).

Remark 1.2.6. *From equation* (1.2.2) *we see that the value of the fractional sum* $\Delta^{\nu}_{a}f(t)$ *depends on the values of* f *on* $\mathbb{N}^{t+\nu}_{a+\nu-N}$*. This full history nature of the value of the* ν*-th fractional sum of* f *is one of the important features of this fractional sum. In contrast, if one is studying an* n*-th order difference equation, the term* $\Delta^n f(t)$ *only depends on the values of* f *at the* $n+1$ *points* $t, t+1, t+2, \cdots, t+n$.

Remark 1.2.7. *We could easily extend Theorem* 1.2.5 *to the case when* $f, q : \mathbb{N}_a \to \mathbb{R}$ *instead of the special case* $a = 0$ *that we considered in Theorem* 1.2.5*. Also, the term* $q(t)y(t+\nu-N)$ *in Equation* 1.2.4 *could be replaced by* $q(t)y(t+\nu-N+i)$ *for any* $0 \leq i \leq N-1$*. Note that if we picked the nice set* $\mathbb{N}_0$ *so that the fractional difference equation needs to be satisfied for all* $t \in \mathbb{N}_0$*, then the solutions are defined on the shifted set* $\mathbb{N}_{\nu-N}$*. By considering the fractional difference equation on a shifted set we could get our solutions are defined on the nicer set* $\mathbb{N}_0$*. In this dissertation we do the first case when considering fractional difference equations. Finally, one could give a version of this theorem for solutions on a finite set* $\mathbb{N}^{b}_{\nu-N}$.

1.3 Fractional Power Rules

The following Leibniz formula will be very useful.

Lemma 1.3.1 (Leibniz Formula)**.** *Assume* $f : \mathbb{N}_{a+\nu} \times \mathbb{N}_a \to \mathbb{R}$*. Then*

$$\Delta\left(\sum_{s=a}^{t-\nu} f(t,s)\right) = \sum_{s=a}^{t-\nu} \Delta f(t,s) + f(t+1, t+1-\nu) \tag{1.3.1}$$

for $t \in \mathbb{N}_{a+\nu}$*, where the* $\Delta f(t,s)$ *inside the sum means the difference with respect to* t*.*

Proof. Consider, for $t \in \mathbb{N}_{a+\nu}$,

$$\begin{aligned}\Delta\left(\sum_{s=a}^{t-\nu} f(t,s)\right) &= \sum_{s=a}^{t+1-\nu} f(t+1,s) - \sum_{s=a}^{t-\nu} f(t,s)\\ &= \sum_{s=a}^{t-\nu} \Delta_t f(t,s) + f(t+1, t+1-\nu).\end{aligned}$$

□

Using the Leibniz formula we will prove the following fractional sum power rule.

Theorem 1.3.2 (Fractional Sum Power Rule)**.** *Assume* $\mu \geq 0$ *and* $\nu > 0$*. If* $t \in \mathbb{N}_a$*, then*

$$\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\underline{\mu+\nu}} \tag{1.3.2}$$

for $t \in \mathbb{N}_{a+\mu+\nu}$.

Theorem 1.3.3 (Fractional Difference Power Rule)**.** *Assume $\mu > 0$ and $\nu \geq 0$. If $f : \mathbb{N}_a \to \mathbb{R}$, then*

$$\Delta^{\nu}_{a+\mu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)^{\underline{\mu-\nu}} \tag{1.3.3}$$

for $t \in \mathbb{N}_{a+\mu-\nu}$.

In the next theorem we give a formula for the fractional sum which we will call the summation definition of $\Delta^{\nu}_a f(t)$.

Theorem 1.3.4. *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given, with $N-1 < \nu \leq N$. Then*

$$\Delta^{\nu}_a f(t) := \begin{cases} \frac{1}{\Gamma(-\nu)}\displaystyle\sum_{s=a}^{t+\nu}(t-\sigma(s))^{\underline{-\nu-1}}f(s), & N-1 < \nu < N \\ \Delta^N f(t), & \nu = N \end{cases} \tag{1.3.4}$$

for $t \in \mathbb{N}_{a+N-\nu}$.

Remark 1.3.5. *By Theorems 1.3.3 and 1.3.4 we get for all $\mu > 0$ and all real numbers $\nu \notin \mathbb{N}_0$ that the formula for $\Delta^{\nu}_a f(t)$ can be obtained from the formula for $\Delta^{-\nu}_a f(t)$ in Definition 1.2.2 by replacing ν by $-\nu$ and vice-versa, but the domains are different.*

Theorem 1.3.6. *Assume $\mu > 0$ and N is a positive integer such that $N-1 < \mu \leq N$, then for any constant a*

$$x(t) = c_1(t-a)^{\underline{\mu-1}} + c_2(t-a)^{\underline{\mu-2}} + \cdots + c_N(t-a)^{\underline{\mu-N}}$$

for all constants $c_1, c_2, \cdots, c_N$, is a solution of the fractional difference equation $\Delta^{\mu}_{a+\mu-N}y(t) = 0$ on $\mathbb{N}_{a+\mu-N}$.

Theorem 1.3.7 (Continuity of Fractional Differences)**.** *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given. Then the fractional difference $\Delta_a^\nu f$ is continuous with respect to $\nu \geq 0$. More specifically, for every $\nu > 0$, let $t_{\nu,m} := a + \lceil \nu \rceil - \nu + m$ be a fixed but arbitrary point in $\mathcal{D}\left\{\Delta_a^\nu f\right\}$. Then for each $m \in \mathbb{N}_0$,*

$$\nu \mapsto \Delta_a^\nu f(t_{\nu,m}) \text{ is continuous on } [0, \infty).$$

Theorem 1.3.8 (Composition of Fractional Sums)**.** *Assume f is defined on $\mathbb{N}_a$ and μ, ν are positive numbers. Then*

$$\left[\Delta_{a+\nu}^{-\mu}\left(\Delta_a^{-\nu} f\right)\right](t) = \left(\Delta_a^{-(\mu+\nu)} f\right)(t) = \left[\Delta_{a+\mu}^{-\nu}\left(\Delta_a^{-\mu} f\right)\right](t)$$

for $t \in \mathbb{N}_{a+\mu+\nu}$.

Theorem 1.3.9 (Composition of an Integer Difference With a Fractional Sum)**.** *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\mu > 0$, and k is an integer satisfying $0 < k < \mu$. Then*

$$\left[\Delta^k\left(\Delta_a^{-\mu} f\right)\right](t) = \left(\Delta_a^{-(\mu-k)} f\right)(t)$$

for $t \in \mathbb{N}_{a+\mu}$.

Theorem 1.3.10 (Variation of Constants Formula)**.** *Assume $N \geq 1$ is an integer and $N - 1 < \nu \leq N$. If $f : \mathbb{N}_0 \to \mathbb{R}$, then the solution of the IVP*

$$\Delta_{\nu-N}^\nu y(t) = f(t), \quad t \in \mathbb{N}_0 \tag{1.3.5}$$

$$y(\nu - N + i) = 0, \quad 0 \leq i \leq N - 1 \tag{1.3.6}$$

is given by

$$y(t) = \Delta_0^{-\nu} f(t), \quad t \in \mathbb{N}_{\nu-N}.$$

Proof. Let

$$y(t) = \Delta_0^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s).$$

Then by our convention on sums

$$y(\nu - N + i) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{-N+i} (\nu - N + i - s - 1)^{\underline{\nu-1}} f(s) = 0$$

for $0 \leq i \leq N-1$, and hence the initial conditions (1.3.6) are satisfied.

Also, for $t \in \mathbb{N}_0$,

$$\begin{aligned}
\Delta_{\nu-N}^{\nu} y(t) &= \Delta^N \Delta_{\nu-N}^{-(N-\nu)} y(t) \\
&= \Delta^N \sum_{s=\nu-N}^{t-(N-\nu)} \frac{(t-\sigma(s))^{\underline{N-\nu-1}}}{\Gamma(N-\nu)} y(s) \\
&= \Delta^N \sum_{s=\nu}^{t-(N-\nu)} \frac{(t-\sigma(s))^{\underline{N-\nu-1}}}{\Gamma(N-\nu)} y(s)
\end{aligned}$$

where in the last step we used the initial conditions (1.3.6). Hence,

$$\begin{aligned}
\Delta_{\nu-N}^{\nu} y(t) &= \Delta^N \Delta_{\nu}^{-(N-\nu)} y(t) \\
&= \Delta^N \Delta_{0+\nu}^{-(N-\nu)} \Delta_0^{-\nu} f(t) \\
&= \Delta^N \Delta_0^{-N} f(t) \\
&= f(t).
\end{aligned}$$

Therefore y is a solution of fractional difference equation (1.3.5) on $\mathbb{N}_0$. $\square$

Theorem 1.3.11. *Assume* $f : \mathbb{N}_a \to \mathbb{R}$, $k \in \mathbb{N}_1$, *and* $0 < k \leq \nu$. *Then*

$$\Delta_a^{-\nu}\Delta^k f(t) = \Delta^k\Delta^{-\nu}f(t) - \sum_{i=0}^{k-1}\frac{(t-a)^{\underline{\nu-k+i}}}{\Gamma(\nu+i-k+1)}\Delta^i f(a) \tag{1.3.7}$$

for $t \in \mathbb{N}_{a+\nu}$.

In Chapter 2, we give results that generalize some recent results of Goodrich [57] and extend results in Kelley-Peterson [60] for the continuous case to the fractional case. In Chapter 3, for the first time in the literature, we introduce the so-called fractional self-adjoint equation. In Chapters 3 and 4, we extend several results in Kelley-Peterson [60] for the continuous case to the fractional case.

For other related papers of interest for discrete and continuous fractional calculus see the papers Ahrendt et at [1], Atici and Eloe [2], Atici and Eloe [3], Atici and Eloe [4], Oldham and Spanier [5], Podlubny [6], Agrawal [7], Nieto [8], Almeida [9], Arara [10], Eloe [11], Eloe [12], Eloe [13], Atici [14], Eloe [15], Babakhani [16], Bai [17], Bai [18], Bastos [19], Bastos [21], Bastos and Terris [22], Benchohra [23], Benchohra [24], Devi [25], Diethelm [26], Eidelman [27], Ferreira [28], Goodrich [29], Goodrich [30], Goodrich[31], Goodrich [32], Goodrich [33], Goodrich [34], Goodrich [35], Goodrich [36], Goodrich [37], Goodrich [38], Goodrich [39], Goodrich [40], Goodrich [41], Goodrich [42], Goodrich [43], Kirane and Malik [44], Lakshmikantham and Vatsala [45], Malinowka and Torres [46], Nieto [47], Su [48], Wang and Zhou [49], Wei, Li and Che [50], Xu, Jiang and Yuan [51], Zhang [52], Zhao and Ge [53], Zhou, Jiao and Li [54], Zhou, Jiao and Li [55], Zhou and Jiao [56].

Chapter 2

Existence And Uniqueness Of Solutions Of A Fractional Boundary Value Problem

2.1 Preliminaries

In this section we present a few basic definitions and lemmas which we will use in various sections of this chapter. We will be interested in defining the Cauchy function for the fractional difference equation $\Delta^{\nu}_{a+\nu-N}\, y(t) = 0$, where $a \in \mathbb{R}$, $N \in \mathbb{N}$ and $N-1 < \nu \leq N$. We then give a formula for the Green's function and some of its properties for the fractional boundary value problem (FBVP)

$$\begin{cases} \Delta^{\nu}_{\nu-2}\, y(t) = 0 \\ y(\nu-2) = 0 = y(\nu+b+1), \end{cases} \tag{2.1.1}$$

where $t \in [0, b+2]_{\mathbb{N}_0}$, $\nu \in (1, 2]$ and $b \in \mathbb{N}_0$. The properties which we will prove later will be helpful for proving some important results regarding the nonlinear FBVP

$$\begin{cases} \Delta^{\nu}_{\nu-2}\, y(t) = f(t, y(t+\nu-1)) \\ y(\nu-2) = A, \quad y(\nu+b+1) = B, \end{cases} \tag{2.1.2}$$

where $t \in [0, b+2]_{\mathbb{N}_0}$, $\nu \in (1, 2]$, $f : [0, b+1]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{N}_0$.

Remark 2.1.1. *If $f : \mathbb{N}_a \to \mathbb{R}$ and $c, d \in \mathbb{N}_a$, we use the standard convention that*

$$\sum_{t=c}^{d} f(t) = 0$$

whenever $d < c$.

Lemma 2.1.2. *[58] If $t \in \mathbb{N}_0$ and $N - 1 < \nu \leq N$, we have that*

$$\Delta_0^{-\nu} \Delta^{\nu}_{\nu-N}\, y(t) = y(t) + \sum_{i=0}^{N-1} C_i\, t^{\underline{i+\nu-N}},$$

for some constants C_i, $0 \leq i \leq N-1$.

Lemma 2.1.3. *If we let $\Delta_s f(t, s) := f(t, s+1) - f(t, s)$, then*

$$(t - \sigma(s))^{\underline{\nu-1}} = -\frac{1}{\nu} \Delta_s (t - s)^{\underline{\nu}}.$$

Proof. First notice that

$$\begin{aligned}
\Delta_s (t-s)^{\underline{\nu}} &= (t-s-1)^{\underline{\nu}} - (t-s)^{\underline{\nu}} \\
&= \frac{\Gamma(t-s)}{\Gamma(t-s-\nu)} - \frac{\Gamma(t-s+1)}{\Gamma(t-s+1-\nu)} \\
&= \frac{\Gamma(t-s)}{\Gamma(t-s-\nu)} \left[1 - \frac{(t-s)}{(t-s-\nu)} \right] \\
&= \frac{\Gamma(t-s)}{\Gamma(t-s-\nu)} \left[\frac{-\nu}{(t-s-\nu)} \right] \\
&= \frac{-(\nu)\Gamma(t-s)}{\Gamma(t-s+1-\nu)} \\
&= \frac{-(\nu)\Gamma(t-s-1+1)}{\Gamma(t-s-1+1+1-\nu)} \\
&= \frac{-(\nu)\Gamma(t-\sigma(s)+1)}{\Gamma(t-\sigma(s)+1+1-\nu)} \\
&= \frac{-(\nu)\Gamma(t-\sigma(s)+1)}{\Gamma(t-\sigma(s)+1+1-\nu)} \\
&= \frac{-(\nu)\Gamma(t-\sigma(s)+1)}{\Gamma(t-\sigma(s)+1-(\nu-1))} \\
&= -\nu(t-\sigma(s))^{\underline{\nu-1}}.
\end{aligned}$$

Thus we get that

$$(t-\sigma(s))^{\underline{\nu-1}} = -\frac{1}{\nu}\Delta_s (t-s)^{\underline{\nu}}.$$

□

Lemma 2.1.4. *[59] If $h : \mathbb{N}_a \to \mathbb{R}$, then the general solution to the equation*

$$\Delta^{\nu}_{a+\nu-N}\, y(t) = h(t), \quad t \in \mathbb{N}_a$$

is given by,

$$y(t) = \sum_{i=0}^{N-1} c_i\, (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu}\, h(t), \quad t \in \mathbb{N}_{a+\nu-N} \tag{2.1.3}$$

where c_i, $0 \le i \le N-1$, are arbitrary constants.

2.2 Green's Function And Some Of It's Properties

Theorem 2.2.1. *If $N \in \mathbb{N}$ and $N-1 < \nu \leq N$, then*

$$y(t) = \sum_{s=0}^{t-\nu+N-1} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} h(s) == \sum_{s=0}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} h(s)$$

is the solution for the fractional IVP

$$\begin{cases} \Delta^{\nu}_{\nu-N}\, y(t) = h(t), \quad t \in \mathbb{N}_0 \\ y(\nu-N) = y(\nu-N+1) = \cdots = y(\nu-1) = 0. \end{cases} \tag{2.2.1}$$

Proof. This result follows from formula (2.1.3) in Lemma (2.1.4) obtained by setting all constants to zero and using the definition of the ν^{th} fractional sum. □

The Green's function for the homogeneous FBVP (2.1.1) is the unique function with the property that the unique solution, $y(t)$, of the nonhomogeneous FBVP

$$\begin{cases} \Delta^{\nu}_{\nu-2}\, y(t) = h(t), \quad t \in \mathbb{N}_0 \\ y(\nu-2) = 0 = y(\nu+b+1) \end{cases} \tag{2.2.2}$$

is given by

$$y(t) = \int_0^{b+3} G(t,s)h(s)\Delta s = \sum_{s=0}^{b+2} G(t,s)h(s).$$

Next we will be interested in establishing a formula for the Green's function for the fractional conjugate BVP (2.1.1) and proving some of its properties. This will be helpful for presenting some new results regarding the existence and uniqueness of solutions to the conjugate FBVP via various fixed point theorems. Consider for

$t \in [0, b+2]_{\mathbb{N}_0}$

$$\Delta^{\nu}_{\nu-2}\ y(t) = h(t)$$

taking the fractional summation operator $\Delta_0^{-\nu}$ on both sides of the above equation and using Definition 1.2.2 together with Lemma 2.1.4 gives us that

$$y(t) = \sum_{s=0}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}\ h(s) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}}. \tag{2.2.3}$$

Using the first BC, i.e., $y(\nu-2) = 0$ gives

$$0 = \sum_{s=0}^{\nu-2-\nu} \frac{(\nu-2-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}\ h(s) + C_1(\nu-2)^{\underline{\nu-1}} + C_2(\nu-2)^{\underline{\nu-2}}.$$

Using the properties of the falling function and Remark 2.1.1 we have that $C_2 = 0$. Now using the second BC, $y(\nu+b+1) = 0$, we get that

$$0 = \sum_{s=0}^{\nu+b+1-\nu} \frac{(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}\ h(s) + C_1(\nu+b+1)^{\underline{\nu-1}}.$$

Solving for C_1 we have that

$$C_1 = -\frac{\displaystyle\sum_{s=0}^{b+1}(\nu+b+1-\sigma(s))^{\underline{\nu-1}}\ h(s)}{\Gamma(\nu)(\nu+b+1)^{\underline{\nu-1}}}.$$

Thus,

$$y(t) = \frac{1}{\Gamma(\nu)}\left[\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}h(s) - \frac{t^{\underline{\nu-1}}}{(\nu+b+1)^{\underline{\nu-1}}}\sum_{s=0}^{b+1}(\nu+b+1-\sigma(s))^{\underline{\nu-1}}h(s)\right].$$

Since $(\nu+b+1-\sigma(s))^{\underline{\nu-1}}=0$ for $s=b+2$, the above expression can be rewritten as

$$y(t)=\frac{1}{\Gamma(\nu)}\left[\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}h(s)-\frac{t^{\underline{\nu-1}}}{(\nu+b+1)^{\underline{\nu-1}}}\sum_{s=0}^{b+2}(\nu+b+1-\sigma(s))^{\underline{\nu-1}}h(s)\right]. \tag{2.2.4}$$

The Green's function $G(t,s)$ for the fractional conjugate BVP (2.1.1) is given by,

$$G(t,s)=:\begin{cases}\dfrac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}-\dfrac{t^{\underline{\nu-1}}\ (\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)\ (\nu+b+1)^{\underline{\nu-1}}}, & s\le t-\nu\\ -\dfrac{t^{\underline{\nu-1}}\ (\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)\ (\nu+b+1)^{\underline{\nu-1}}}, & t-\nu+1\le s\end{cases}$$

Proposition 2.2.2. *The Green's function for the FBVP* (2.1.1) *satisfies*

$$G(t,s)\le 0$$

for $\nu-2\le t\le \nu+b+1$, $0\le s\le b+2$.

Proof. First notice that for $0\le t-\nu+1\le s\le b+2$ we have that

$$\frac{-t^{\underline{\nu-1}}\ (\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)\ (\nu+b+1)^{\underline{\nu-1}}}\le 0.$$

Thus, it is sufficient to show that

$$\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}\le\frac{t^{\underline{\nu-1}}\ (\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)\ (\nu+b+1)^{\underline{\nu-1}}},\quad t-\nu+1\le s.$$

Equivalently, we can show that,

$$\frac{(t-\sigma(s))^{\underline{\nu-1}}\ (\nu+b+1)^{\underline{\nu-1}}}{(\nu+b+1-\sigma(s))^{\underline{\nu-1}}\ t^{\underline{\nu-1}}}\le 1.$$

Consider

$$\begin{aligned}
&\frac{(t-\sigma(s))^{\underline{\nu-1}}\ (\nu+b+1)^{\underline{\nu-1}}}{(\nu+b+1-\sigma(s))^{\underline{\nu-1}}\ t^{\underline{\nu-1}}}\\
=&\frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)}\frac{\Gamma(\nu+b+2)}{\Gamma(b+3)}\frac{\Gamma(t-\nu+2)}{\Gamma(t+1)}\frac{\Gamma(b+2-s)}{\Gamma(\nu+b-s+1)}.
\end{aligned}$$

But $t=\nu+s+k$ for some k,

$$\begin{aligned}
&=\left[\frac{\Gamma(\nu+k)}{\Gamma(k+1)}\right]\left[\frac{\Gamma(k+s+2)}{\Gamma(\nu+k+s+1)}\right]\left[\frac{\Gamma(b+2-s)}{\Gamma(b+3)}\right]\left[\frac{\Gamma(\nu+b+2)}{\Gamma(\nu+b+1-s)}\right]\\
&=\left[\frac{\Gamma(\nu+k)}{\Gamma(\nu+k+s+1)}\right]\left[\frac{\Gamma(k+s+2)}{\Gamma(k+1)}\right]\left[\frac{\Gamma(b+2-s)}{\Gamma(b+3)}\right]\left[\frac{\Gamma(\nu+b+2)}{\Gamma(nu+b+1-s)}\right].
\end{aligned}$$

Using the property of the Gamma function,we have that

$$\begin{aligned}
=&\left[\frac{1}{(\nu+k+s)\cdots(\nu+k)}\right]\left[\frac{(k+s+1)\cdots(k+1)}{1}\right]\\
&\left[\frac{1}{(b+2)\cdots(b+2-s)}\right]\left[\frac{(\nu+b+1)\cdots(\nu+b+1-s)}{1}\right].
\end{aligned}\tag{2.2.5}$$

Now choose appropriately one factor from each square bracket and we prove the product of those is less than or equal to one. Thus it would be enough to consider only a single set of such factors as the others are the same. Therefore, consider

$$\begin{aligned}
&\frac{(k+s+1)(\nu+b+1)}{(\nu+k+s)(b+2)}\\
&=\frac{(k+s)(b+1)+(k+s)\nu+\nu+(b+1)}{(k+s)(b+1)+(k+s)+\nu+\nu(b+1)}.
\end{aligned}$$

By observing the numerator and denominator it would be sufficient to show that

$$(k+s)+\nu(b+1)\geq(k+s)\nu+(b+1).$$

By assumption we have that

$$\begin{aligned}(k+s) \leq (b+1) &\Rightarrow (k+s)(\nu-1) \leq (b+1)(\nu-1)\\ &\Rightarrow (k+s)\nu + (b+1) \leq (k+s) + (b+1)\nu,\end{aligned}$$

which implies that

$$\frac{(k+s+1)(\nu+b+1)}{(\nu+k+s)(b+2)} \leq 1.$$

Therefore we have that $G(t,s) \leq 0$. $\square$

Theorem 2.2.3. *If* $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$*, then the Green's function satisfies*

$$\sum_{s=0}^{b+2} \big|G(t,s)\big| = \frac{t^{\underline{\nu-1}}\ (\nu+b+1-t)}{\Gamma(\nu+1)}.$$

Proof. By the definition of the Green's function and using the fact that for all $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$,

$$G(t, b+2) = 0.$$

We have that

$$\begin{aligned}\sum_{s=0}^{b+2} \big|G(t,s)\big| &= \sum_{s=0}^{b+1} \big|G(t,s)\big|\\ &= \sum_{s=0}^{t-\nu} \Big|\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} - \frac{t^{\underline{\nu-1}}\ (\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)\ (\nu+b+1)^{\underline{\nu-1}}}\Big|\\ &+ \sum_{s=t+\nu-1}^{b+1} \Big|-\frac{t^{\underline{\nu-1}}\ (\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)\ (\nu+b+1)^{\underline{\nu-1}}}\Big|.\end{aligned}$$

Using Proposition 2.2.2, i.e., $G(t,s) \leq 0$, and combining the sums, we have

$$\sum_{s=0}^{b+2} \left|G(t,s)\right| = \sum_{s=0}^{b+1} \frac{t^{\underline{\nu-1}}\ (\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)\ (\nu+b+1)^{\underline{\nu-1}}} - \sum_{s=0}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}.$$

Using the fundamental theorem of discrete calculus we have,

$$\sum_{s=0}^{b+2} \left|G(t,s)\right| = \frac{1}{\Gamma(\nu)} \left[\left[\frac{1}{-\nu} \frac{t^{\underline{\nu-1}}\ (\nu+b+1-s)^{\underline{\nu}}}{(\nu+b+1)^{\underline{\nu-1}}} \right]_0^{b+2} + \left[\frac{1}{\nu}(t-s)^{\underline{\nu}} \right]_0^{t-\nu+1} \right].$$

An easy calculation gives

$$\sum_{s=0}^{b+2} \left|G(t,s)\right| = \frac{t^{\underline{\nu-1}}\ (\nu+b+1-t)}{\Gamma(\nu+1)}.$$

$\square$

The following theorem generalizes a result due to Goodrich [57], since we find the exact value of the maximum of $\sum_{s=0}^{b+2} \left|G(t,s)\right|$.

Theorem 2.2.4. *The Green's function for the FBVP* (2.1.1) *satisfies*

$$\max_{t\in[\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b+2} \left|G(t,s)\right| = \frac{1}{\Gamma(\nu+1)} \left[\nu + \lceil b - \frac{b+2}{\nu} \rceil \right]^{\underline{\nu-1}} \left[b+1-\lceil b - \frac{b+2}{\nu} \rceil \right].$$

Proof. By Theorem 2.2.3 we have,

$$\sum_{s=0}^{b+2} |G(t,s)| = \frac{(\nu+b+1-t)\ t^{\underline{\nu-1}}}{\Gamma(\nu+1)}.$$

Let $F(t) = t^{\underline{\nu-1}}\ (\nu+b+1-t)$, then observe that $F(t) \geq 0$ with $F(\nu-2) = 0$ and

$F(\nu+b+1)=0$. So F has a nonnegative maximum and to find this maximum we consider,

$$\begin{aligned}\Delta_t F(t) &= (-1)\ t^{\underline{\nu-1}} + (\nu-1)(\nu+b-t)\ t^{\underline{\nu-2}}\\ &= t^{\underline{\nu-2}}\ \left[(-1)(t+\nu-2)+(\nu-1)(\nu+b-t)\right]\\ &= t^{\underline{\nu-2}}\ \left[\nu^2+b\nu-t\nu-b-2\right].\end{aligned}$$

Now we can treat the preceding expression inside the square brackets as a continuous function of t and by using calculus it turns out that $F(t)$ has a maximum on $[\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}}$ at $t=\nu+\lceil b-\frac{b+2}{\nu}\rceil$. Hence

$$\max_{t\in[\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}}}\sum_{s=0}^{b+2}\left|G(t,s)\right| = \frac{1}{\Gamma(\nu+1)}\left[\nu+\lceil b-\frac{b+2}{\nu}\rceil\right]^{\underline{\nu-1}}\left[b+1-\lceil b-\frac{b+2}{\nu}\rceil\right].$$

This completes the proof. $\square$

Remark 2.2.5. *It is very important to mention that we are able to calculate the actual maximum of the summation of the Green's function. One can check and compare the above result with the classical cases which are presented either in ordinary differential equation or difference equation text books simply by substituting $\nu=2$ and by choosing any nonnegative integer b.*

Lemma 2.2.6. *If $\nu\in(1,2]$ and $h:[0,b+2]_{\mathbb{N}_{\nu-2}}\to\mathbb{R}$, then the solution to the FBVP*

$$\begin{cases}\Delta^{\nu}_{\nu-2}\ y(t)=h(t)\\ y(\nu-2)=A,\quad y(\nu+b+1)=B\end{cases}\tag{2.2.6}$$

can be expressed as

$$y(t) = z(t) + \sum_{s=0}^{b+2} G(t,s)\ h(s),$$

where $z(t)$ is the unique solution to the FBVP

$$\begin{cases} \Delta^{\nu}_{\nu-2}\ z(t) = 0 \\ z(\nu-2) = A, \quad z(\nu+b+1) = B. \end{cases} \tag{2.2.7}$$

Proof. First we will solve

$$\begin{cases} \Delta^{\nu}_{\nu-2}\ z(t) = 0 \\ z(\nu-2) = A, \quad z(\nu+b+1) = B. \end{cases}$$

Applying the operator $\Delta_0^{-\nu}$ to both sides of the equation $\Delta^{\nu}_{\nu-2} z(t) = 0$ and using Lemma 2.1.4 we get that

$$z(t) \quad = C_1\ t^{\underline{\nu-1}} + C_2\ t^{\underline{\nu-2}}.$$

Using both boundary conditions $z(\nu-2) = A$ and $z(\nu+b+1) = B$, it turns out that

$$C_1 = \frac{1}{(\nu+b+1)^{\underline{\nu-1}}} \left[B - A \frac{(\nu+b+1)^{\underline{\nu-2}}}{\Gamma(\nu-1)} \right]$$

and

$$C_2 = \frac{A}{\Gamma(\nu-1)}.$$

Thus,

$$z(t) = \frac{1}{(\nu+b+1)^{\underline{\nu-1}}}\left[B - A\frac{(\nu+b+1)^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right] t^{\underline{\nu-1}} + \frac{A}{\Gamma(\nu-1)}\, t^{\underline{\nu-2}}. \tag{2.2.8}$$

Next we solve the FBVP

$$\begin{cases} \Delta^{\nu}_{\nu-2}\, y(t) = h(t) \\ y(\nu-2) = A, \quad y(\nu+b+1) = B. \end{cases}$$

Applying the operator $\Delta_0^{-\nu}$ to both sides of the equation $\Delta^{\nu}_{\nu-2}\, y(t) = h(t)$ and using Definition 1.2.2 together with Lemma 2.1.4 provides

$$y(t) = \sum_{0}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}\, h(s) + D_1\, t^{\underline{\nu-1}} + D_2\, t^{\underline{\nu-2}}. \tag{2.2.9}$$

Using the boundary conditions $y(\nu-2) = A$ and $y(\nu+b+1) = B$ yields

$$D_1 = \frac{1}{(\nu+b+1)^{\underline{\nu-1}}}\left[B - A\frac{(\nu+b+1)^{\underline{\nu-2}}}{\Gamma(\nu-1)} - \frac{1}{\Gamma(\nu)}\sum_{s=0}^{b+1}(\nu+b+1-\sigma(s))^{\underline{\nu-1}}\, h(s)\right]$$

and

$$D_2 = \frac{A}{\Gamma(\nu-1)},$$

but using the fact that for $s = b+2$, the term $(\nu+b+1-\sigma(s))^{\underline{\nu-1}} = 0$. Therefore, D_1 can be rewritten as

$$D_1 = \frac{1}{(\nu+b+1)^{\underline{\nu-1}}}\left[B - A\frac{(\nu+b+1)^{\underline{\nu-2}}}{\Gamma(\nu-1)} - \frac{1}{\Gamma(\nu)}\sum_{s=0}^{b+2}(\nu+b+1-\sigma(s))^{\underline{\nu-1}}\, h(s)\right].$$

By replacing the values of D_1 and D_2 in equation (2.2.9), we have that

$$
\begin{aligned}
&y(t)\\
&= \frac{1}{(\nu+b+1)^{\underline{\nu-1}}}\left[B - A\frac{(\nu+b+1)^{\underline{\nu-2}}}{\Gamma(\nu-1)} - \frac{1}{\Gamma(\nu)}\sum_{s=0}^{b+2}(\nu+b+1-\sigma(s))^{\underline{\nu-1}}\ h(s)\right] t^{\nu-1}\\
&+ \frac{A}{\Gamma(\nu-1)}t^{\nu-2} + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}\ h(s).
\end{aligned}
$$

Rewriting the above expression we obtain

$$
\begin{aligned}
y(t) &= \frac{1}{(\nu+b+1)^{\underline{\nu-1}}}\left[B - A\frac{(\nu+b+1)^{\underline{\nu-2}}}{\Gamma(\nu-1)}\right]\ t^{\underline{\nu-1}} + \frac{A}{\Gamma(\nu-1)}t^{\underline{\nu-2}}\\
&+ \frac{1}{\Gamma(\nu)}\left[\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}h(s) - \frac{t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)^{\underline{\nu-1}}}\sum_{s=0}^{b+2}(\nu+b+1-\sigma(s))^{\underline{\nu-1}}h(s)\right]\\
&= z(t) + \sum_{s=0}^{b+2} G(t,s)\ h(s).
\end{aligned}
$$

This completes the proof. $\square$

2.3 Various Fixed Point Theorems

Fixed point theorems are useful tools for guaranteeing the existence and uniqueness of solutions of nonlinear equations in ordinary differential equations, partial differential equations and many other areas of pure and applied mathematics. In this section, we will discuss the application of these theories to solve nonlinear fractional boundary value problems. In particular, conjugate discrete fractional boundary value problems will be our main interest. We start with the following well-known theorems.

Theorem 2.3.1 (Contraction Mapping Theorem). *[62] Let $(X, \|.\|)$ be a Banach space and $T : X \to X$ be a contraction mapping. Then T has a unique fixed point in*

X.

The following theorem is an application of the above theorem.

Theorem 2.3.2. *Assume that* $f : [0, b+2]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ *satisfies a uniform Lipschitz condition with respect to its second variable, i.e., there exists a* $k \geq 0$ *such that for all* $t \in [0, b+2]$ *and* $u, v \in \mathbb{R}$, $|f(t,u) - f(t,v)| \leq k\,|u - v|$. *If*

$$\left(\nu + \lceil b - \frac{b+2}{\nu} \rceil\right)^{\underline{\nu-1}} \left(b + 1 - \lceil b - \frac{b+2}{\nu} \rceil\right) < \frac{\Gamma(\nu+1)}{k}$$

then the nonlinear fractional boundary value problem

$$\begin{cases} \Delta^{\nu}_{\nu-2}\, y(t) = f(t, y(t+\nu-1), \quad t \in [0, b+2] \\ y(\nu-2) = A, \quad y(\nu+b+1) = B \end{cases} \tag{2.3.1}$$

has a unique solution.

Proof. Let ζ be the space of real valued functions defined on $[\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$. Then we define a norm $\|.\|$ on ζ by $\|x\| = \max\{|x(t)| : t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}\}$ so that the pair $(\zeta, \|.\|)$ is a Banach space. Now we define the map $T : \zeta \to \zeta$ by,

$$Tx(t) = z(t) + \sum_{s=0}^{b+2} G(t,s) f(s, x(s+\nu-1)\ , \quad t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}},$$

where z is the unique solution to the FBVP (2.2.7). Next, we will show that T defined as above is a contraction map. Observe for all $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$ and for all

$x, y \in \zeta$ that

$$
\begin{aligned}
&\|Tx(t) - Ty(t)\| \\
&\qquad = \max_{t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}} \Big|\sum_{s=0}^{b+2} G(t,s)[f(s, x(s+\nu-1)) - (f(s, y(s+\nu-1)]\Big| \\
&\qquad \leq \max_{t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b+2} |G(t,s)||f(s, x(s+\nu-1)) - (f(s, y(s+\nu-1)| \\
&\qquad \leq \max_{t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b+2} |G(t,s)|k\ |x(s+\nu-1) - y(s+\nu-1)| \\
&\qquad \leq k\|x-y\| \max_{t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b+2} |G(t,s)| \\
&\qquad = \frac{k\|x-y\|}{\Gamma(\nu+1)} \left(\nu + \lceil b - \frac{b+2}{\nu} \rceil\right)^{\underline{\nu-1}} \left(b+1 - \lceil b - \frac{b+2}{\nu} \rceil\right) \\
&\qquad \leq \alpha\|x-y\|,
\end{aligned}
$$

where

$$\alpha = \frac{k}{\Gamma(\nu+1)} \left(\nu + \lceil b - \frac{b+2}{\nu} \rceil\right)^{\underline{\nu-1}} \left(b+1 - \lceil b - \frac{b+2}{\nu} \rceil\right) < 1$$

by assumption.Therefore T is a contraction mapping on ζ. Hence T has a unique fixed point in ζ. Thus there exist a unique $\bar{x} \in \zeta$ such that $T(\bar{x}) = \bar{x}$. Next we will show that $\bar{x}$ is the unique solution of (2.3.1). Replacing $\bar{x}$ on the right hand side of the equation we get

$$
\begin{cases}
\Delta_{\nu-2}^{\nu}\ u(t) = f(t, \bar{x}(t+\nu-1)) \\
u(\nu-2) = A, \quad u(\nu+b+1) = B.
\end{cases}
\tag{2.3.2}
$$

Using lemma 2.2.6 we get that

$$u(t) = z(t) + \sum_{s=0}^{b+2} G(t,s)\ f(s, \bar{x}(s+\nu-1)).$$

But

$$u(t) = z(t) + \sum_{s=0}^{b+2} G(t,s)\ f(s, \bar{x}(s+\nu-1)) = T\bar{x}(t) = \bar{x}(t),$$

which implies that $\bar{x}$ is a solution to (2.1.2) since $u(t) = \bar{x}(t)$ for all $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$. In order to prove the uniqueness of $\bar{x}$ as a solution of (2.1.2), we assume the possible existence of another solution $\bar{y}$ of (2.1.2) and then solve

$$\begin{cases} \Delta^{\nu}_{\nu-2}\bar{y}(t) = f(t, \bar{x}(t+\nu-1)) \\ \bar{y}(\nu-2) = A, \quad \bar{y}(\nu+b+1) = B. \end{cases} \tag{2.3.3}$$

Again by using Lemma 2.2.6 we have that

$$\bar{y}(t) = z(t) + \sum_{s=0}^{b+2} G(t,s)\ f(s, \bar{x}(s+\nu-1)) = T\bar{x}(t) = \bar{x}(t)$$

proving $\bar{x}$ as a unique solution of (2.1.2). This completes the proof. □

Theorem 2.3.3 (Schauder's Theorem)**.** *[62] Every continuous function from a compact, convex subset of a topological vector space to itself has a fixed point.*

The following theorem is an application of Schauder's theorem.

Theorem 2.3.4. *Assume that $f : [0, b+2]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is continuous in its second variable and $M \geq \max_{t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}} |z(t)|$, where z is the unique solution to the FBVP*

$$\begin{cases} \Delta^{\nu}_{\nu-2}\ z(t) = 0 \\ z(\nu-2) = A, \quad z(\nu+b+1) = B. \end{cases}$$

Let $C = \max\{|f(t,u)| : 0 \leq t \leq b+2, u \in \mathbb{R}, |u| \leq 2M\}$, then the nonlinear FBVP

(2.1.2) *has a solution provided*

$$\left(\nu + \lceil b - \frac{b+2}{\nu}\rceil\right)^{\underline{\nu-1}}\left(b+1-\lceil b - \frac{b+2}{\nu}\rceil\right) \leq \frac{\Gamma(\nu+1)M}{C}.$$

Proof. Let ζ be the Banach space defined in the proof of Theorem 2.3.2. Thus ζ is a topological vector space. Let $K = \{y \in \zeta : \|y\| \leq 2M\}$, then K is a compact, convex subset of ζ. Next define the map $T : \zeta \to \zeta$ by

$$Ty(t) = z(t) + \sum_{s=0}^{b+2} G(t,s) f(s, y(s+\nu-1)).$$

We will first show that T maps K into K.

Observe that for $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$ and $y \in K$ we have

$$\begin{aligned}
|Ty(t)| &= |z(t) + \sum_{s=0}^{b+2} G(t,s)\ f(s, y(s+\nu-1))| \\
&\leq |z(t)| + \sum_{s=0}^{b+2} |G(t,s)||f(s, y(s+\nu-1))| \\
&\leq M + C\sum_{s=0}^{b+2} |G(t,s)| \\
&\leq M + C\frac{1}{\Gamma(\nu+1)}\left(\nu + \lceil b - \frac{b+2}{\nu}\rceil\right)^{\underline{\nu-1}}\left(b+1-\lceil b - \frac{b+2}{\nu}\rceil\right) \\
&\leq M + C\frac{1}{\Gamma(\nu+1)}\frac{\Gamma(\nu+1)M}{C} \\
&\leq 2M.
\end{aligned}$$

Since $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$ was arbitrary, we have that $\|Ty\| \leq 2M$. This proves that T maps K into K.

Next we will show that T is continuous on K. Let $\epsilon > 0$ be given and assume

$$l := \max_{t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b+2} |G(t,s)|.$$

Then by Theorem 2.2.4 we have that

$$l = \frac{1}{\Gamma(\nu+1)} \left(\nu + \lceil b - \frac{b+2}{\nu} \rceil\right)^{\underline{\nu-1}} \left(b + 1 - \lceil b - \frac{b+2}{\nu} \rceil\right).$$

Since f is continuous in its second variable on $\mathbb{R}$, f is uniformly continuous in its second variable on $[-2M, 2M]$. Therefore, there exist a $\delta > 0$ such that for all $t \in [0, b+2]_{\mathbb{N}_0}$ and for all $u, v \in [-2M, 2M]$ with $|(t,u) - (t,v)| < \delta$ we have,

$$|f(t,u) - f(t,v)| < \frac{\epsilon}{l}.$$

Thus for all $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$, we have that

$$\begin{aligned}
|Ty(t) - Tx(t)| &= |\sum_{s=0}^{b+2} G(t,s) f(s, y(s+\nu-1)) - \sum_{s=0}^{b+2} G(t,s) f(s, x(s+\nu-1)| \\
&\leq \sum_{s=0}^{b+2} |G(t,s)||f(s, y(s+\nu-1)) - f(s, x(s+\nu-1))| \\
&< \sum_{s=0}^{b+2} |G(t,s)| \frac{\epsilon}{l} \\
&\leq l \frac{\epsilon}{l} \\
&= \epsilon.
\end{aligned}$$

Now since $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$ was arbitrary we have that

$$\|T(x) - T(y)\| < \epsilon.$$

This establishes the continuity of T on K. Hence T is a continuous map from K into K, therefore by Theorem 2.3.4, T has a fixed point in K. Thus there exists a $\bar{x} \in K$ such that $T(\bar{x}) = \bar{x}$. This implies the existence of a solution to FBVP (2.1.2). This completes the proof. □

Remark 2.3.5. *The above theorem not only guarantees the existence of a solution $y(t)$ but also shows that the solution satisfies $|y(t)| \leq 2M$ for $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$.*

Theorem 2.3.6 (A specific case of the Browder's Theorem)**.** *[62] Let ζ be a Banach space and $T : \zeta \to \zeta$ be a compact operator. If $T(\zeta)$ is bounded, then T has a fixed point in ζ.*

As an application of this theorem we will prove a corollary regarding the existence of a solution of the FBVP (2.1.2) under a strong assumption on f.

Corollary 2.3.7. *Assume that $f : [0, b+2]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to its second variable and is bounded. Then the nonlinear FBVP* (2.1.2) *has a solution.*

Proof. Let $(\zeta, \|.\|)$ be the Banach space as defined earlier. Now we define the operator $T : \zeta \to \zeta$ by,

$$Ty(t) = z(t) + \sum_{s=0}^{b+2} G(t,s)\ f(s, y(s+\nu-1)), \quad t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$$

where z is the unique solution to the FBVP

$$\begin{cases} \Delta^{\nu}_{\nu-2}\ z(t) = 0 \\ z(\nu-2) = A, \quad z(\nu+b+1) = B. \end{cases} \tag{2.3.4}$$

It is easy to see that the operator T is compact. Next we will show that $T(\zeta)$ is bounded. Since f is bounded, there exist $m > 0$ such that for all $t \in [0, b+2]_{\mathbb{N}_0}$ and for all $u \in \mathbb{R}$, $|f(t,u)| \leq m$. Thus, for any $y \in \zeta$ and $t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$ we have that,

$$\begin{aligned} |Ty(t)| =& \left|z(t) + \sum_{s=0}^{b+2} G(t,s)\ f(s, y(s+\nu-1))\right| \\ \leq& |z(t)| + m \sum_{s=0}^{b+2} |G(t,s)| \\ \leq& \max_{t \in [\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}}} |z(t)| + m \max_{t \in [\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b+2} \big|G(t,s)\big|. \end{aligned}$$

Hence T is bounded on ζ and the conclusion follows as a result of Theorem 2.3.6. □

Theorem 2.3.8. *Assume that $f : [0, b+2]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ satisfies a uniform Lipschitz condition with respect to its second variable, i.e., there exists $k \geq 0$ such that for all $t \in [0, b+2]_{\mathbb{N}_0}$ and $u, v \in \mathbb{R}$, $|f(t,u) - f(t,v)| \leq k\,|u-v|$ and the equation $\Delta^{\nu}_{\nu-2}\, y(t) +\ k\ y(t+\nu-1) = 0$ has a positive solution u, it follows that the nonlinear fractional boundary value problem* (2.1.2) *has a unique solution.*

Proof. Since the equation

$$\Delta^{\nu}_{\nu-2}\ y(t) + k\ y(t+\nu-1) = 0$$

has a positive solution u on $[\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$, it follows that $u(t)$ is a solution of the FBVP

$$\begin{cases} \Delta^{\nu}_{\nu-2}\ y(t) = (-k)\ u(t+\nu-1) \\ y(\nu-2) = C, \quad y(\nu+b+1) = D, \end{cases} \tag{2.3.5}$$

where $C := u(\nu-2) > 0$ and $D := u(\nu+b+1) > 0$. By using the conclusion of

Lemma 2.2.6 we have that,

$$u(t) = z(t) + \sum_{s=0}^{b+2} G(t,s)\ (-k)\ u(s+\nu-1), \tag{2.3.6}$$

where z is the unique solution to the FBVP

$$\begin{cases} \Delta^{\nu}_{\nu-2}\ z(t) = 0 \\ z(\nu-2) = C, \quad z(\nu+b+1) = D. \end{cases}$$

Again by using Lemma 2.2.6, z is given by

$$z(t) = \frac{1}{(\nu+b+1)^{\underline{\nu-1}}} \left[u(\nu+b+1) - u(\nu-2) \frac{(\nu+b+1)^{\underline{\nu-2}}}{\Gamma(\nu-1)} \right] t^{\underline{\nu-1}} + \frac{u(\nu-2)}{\Gamma(\nu-1)}\ t^{\underline{\nu-2}}.$$

We now show that $z(t) > 0$ on $[\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}$. Since $t^{\underline{\nu-1}} = t^{\underline{\nu-2}}\ (t-\nu+2)$, then by replacing $t^{\underline{\nu-1}}$ with $t^{\underline{\nu-2}}\ (t-\nu+2)$ and rearranging the terms gives us that

$$\begin{aligned} z(t) &= t^{\underline{\nu-2}} \left[\frac{(t-\nu+2)}{(\nu+b+1)^{\underline{\nu-1}}} \left(u(\nu+b+1) - \frac{u(\nu-2)(\nu+b+1)^{\underline{\nu-2}}}{\Gamma(\nu-1)} \right) + \frac{u(\nu-2)}{\Gamma(\nu-1)} \right] \\ &= t^{\underline{\nu-2}} \left[\frac{(t-\nu+2)u(\nu+b+1)}{(\nu+b+1)^{\underline{\nu-1}}} + \frac{u(\nu-2)}{\Gamma(\nu-1)} \left(1 - \frac{(t-\nu+2)(\nu+b+1)^{\underline{\nu-2}}}{(\nu+b+1)^{\underline{\nu-1}}} \right) \right] \\ &= t^{\underline{\nu-2}} \left[\frac{(t-\nu+2)u(\nu+b+1)}{(\nu+b+1)^{\underline{\nu-1}}} + \frac{u(\nu-2)}{\Gamma(\nu-1)} \left(1 - \frac{(t-\nu+2)}{b+3} \right) \right] \\ &= t^{\underline{\nu-2}} \left[\frac{(t-\nu+2)u(\nu+b+1)}{(\nu+b+1)^{\underline{\nu-1}}} + \frac{(\nu+b+1-t)u(\nu-2)}{\Gamma(\nu-1)(b+3)} \right] \\ &= t^{\underline{\nu-2}} \quad h(t), \end{aligned}$$

where

$$h(t) = \frac{(t-\nu+2)u(\nu+b+1)}{(\nu+b+1)^{\underline{\nu-1}}} + \frac{(\nu+b+1-t)u(\nu-2)}{\Gamma(\nu-1)(b+3)}.$$

Since $t^{\underline{\nu-2}}$ is a decreasing function of t and $(\nu+b+1)^{\underline{\nu-2}} = \frac{\Gamma(\nu+b+2)}{\Gamma(b+4)} > 0$, we get that

$t^{\underline{\nu-2}} > 0$ and in order to show that $z(t)$ is positive we just need to show that $h(t)$ is positive. We note on the right hand side of the expression for $h(t)$ that the first term is zero only at the left end point $t = \nu - 2$ and is positive elsewhere. Also the second term is zero only at the right end point $t = \nu + b + 1$ and positive elsewhere. Therefore, combining these arguments we conclude that $z(t) > 0$ for all $t \in [\nu - 2, \nu + b + 1]_{\mathbb{N}_{\nu-2}}$. Thus, by (2.3.5) for all $t \in [\nu - 2, \nu + b + 1]_{\mathbb{N}_{\nu-2}}$ we have that

$$u(t) > \sum_{s=0}^{b+2} G(t, s)\ (-k)\ u(s + \nu - 1).$$

Hence,

$$\alpha = \max_{t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}} \frac{1}{u(t)} \sum_{s=0}^{b+2} G(t, s)(-k)\ u(s + \nu - 1) < 1.$$

Let ζ be the space of real valued functions defined on $[\nu - 2, \nu + b + 1]_{\mathbb{N}_{\nu-2}}$. Consider the (weighted) norm $\|.\|$ defined by $\|x\| = \max\{\frac{|x(t)|}{u(t)} : t \in [\nu - 2, \nu + b + 1]_{\mathbb{N}_{\nu-2}}\}$. Then the pair $(\zeta, \|.\|)$ is a complete normed space. Define T on ζ by

$$Tx(t) = z(t) + \sum_{s=0}^{b+2} G(t, s) f(s, x(s + \nu - 1)). \quad t \in [\nu - 2, \nu + b + 1]_{\mathbb{N}_{\nu-2}}.$$

Then for all $t \in [\nu - 2, \nu + b + 1]_{\mathbb{N}_{\nu-2}}$, we have that

$$\begin{aligned}
\frac{|Tx(t) - Ty(t)|}{u(t)} &= \frac{1}{u(t)} |\sum_{s=0}^{b+2} G(t,s)[f(s, x(s+\nu-1)) - (f(s, y(s+\nu-1)]| \\
&\leq \frac{1}{u(t)} \sum_{s=0}^{b+2} |G(t,s)| \; |f(s, x(s+\nu-1)) - (f(s, y(s+\nu-1)| \\
&\leq \frac{1}{u(t)} \sum_{s=0}^{b+2} |G(t,s)| \; k \; |x(s+\nu-1) - y(s+\nu-1)| \\
&= \frac{1}{u(t)} \sum_{s=0}^{b+2} |G(t,s)| \; k \; u(s+\nu-1) \frac{|x(s+\nu-1) - y(s+\nu-1)|}{u(s+\nu-1)} \\
&\leq \frac{\|x-y\|}{u(t)} \sum_{s=0}^{b+2} |G(t,s)| \; k \; u(s+\nu-1) \\
&= \frac{\|x-y\|}{u(t)} \sum_{s=0}^{b+2} G(t,s)(-k) \; u(s+\nu-1) \\
&\leq \frac{\|x-y\|}{u(t)} \max_{t \in [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b+2} G(t,s)(-k) \; u(s+\nu-1) \\
&= \alpha \|x-y\|.
\end{aligned}$$

Since $\alpha < 1$, T is a contraction mapping on ζ. Therefore, T has a unique fixed point in ζ by the Contraction Mapping Theorem. This implies the existence of a unique solution to the nonlinear FBVP (2.1.2). This completes the proof. $\square$

Chapter 3

Fractional Self-Adjoint Difference Equation

3.1 Introduction

In this section of Chapter 3 we introduce the special self-adjoint linear fractional difference equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + q(t+\mu-1)x(t+\mu-1) = f(t), \tag{3.1.1}$$

where $0 < \mu \leq 1$, $b \in \mathbb{N}_1$, $t \in \mathbb{N}_0^b$, $p : \mathbb{N}_{\mu-1}^{\mu+b} \to (0, \infty)$, $f : \mathbb{N}_0^b \to \mathbb{R}$ and $q : \mathbb{N}_{\mu-1}^{\mu+b} \to \mathbb{R}$. Note if $\mu = 1$ we get the standard self-adjoint difference equation

$$\Delta(p\Delta x)(t) + q(t)x(t) = f(t), \quad t \in \mathbb{N}_0^b.$$

and it is for this reason that we call (3.1.1) a fractional self-adjoint equation. Further motivation for this is that many of the results for the self-adjoint difference equation have analogues for the self-adjoint fractional equation (3.1.1).

In Section 3.2 we will prove that the solutions of equation (3.1.1) with appropriate initial conditions exist and are unique. In Section 3.3 we will give the variation of constants formula regarding equation (3.1.1). In Section 3.4 we will introduce the Cauchy function and the Green's function for

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0$$

with appropriate boundary conditions and in Section 3.5 we will give the generalized version of the Green's function for (3.1.1) with appropriate boundary conditions. In Section 3.6 we will give the Green's function when $p = 1$ and derive several properties of the Green's function for the conjugate boundary value problem. Finally, in Section 3.7 we will use the Banach Fixed Point Theorem to prove a theorem that ensures that a forced self-adjoint equation has a solution that approaches zero as t goes to ∞. An example illustrating this theorem will be given.

3.2 Existence And Uniqueness Theorem

In this section we will prove an existence and uniqueness theorem for the following fractional initial value problem

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) + q(t+\mu-1)x(t+\mu-1) = h(t), \quad t \in \mathbb{N}_0^b \\ x(\mu-1) = A, \quad x(\mu) = B, \end{cases} \tag{3.2.1}$$

where $h : \mathbb{N}_0^b \to \mathbb{R}$.

Theorem 3.2.1. *Assume* $h : \mathbb{N}_0^b \to \mathbb{R}$, $p : \mathbb{N}_{\mu-1}^{\mu+b} \to \mathbb{R}$ *with* $p(t) > 0$, $A, B \in \mathbb{R}$ *then* *the fractional initial value problem*

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) + q(t+\mu-1)x(t+\mu-1) = h(t), \quad t \in \mathbb{N}_0^b \\ x(\mu-1) = A, \quad x(\mu) = B, \end{cases} \tag{3.2.2}$$

has a unique solution that exists on $\mathbb{N}_{\mu-1}^{\mu+b+1}$.

Proof. Consider the self-adjoint equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + q(t+\mu-1)x(t+\mu-1) = h(t), \qquad t \in \mathbb{N}_0^b.$$

By applying the summation formula as given in Theorem 1.3.4 for the fractional difference of a function we have this fractional difference equation can be written in the form (for $t \in \mathbb{N}_0^b$)

$$\frac{1}{\Gamma(-\mu)} \sum_{s=\mu-1}^{t+\mu} (t-\sigma(s))^{\underline{-\mu-1}} p(s)\Delta x(s) + q(t+\mu-1)x(t+\mu-1) = h(t). \tag{3.2.3}$$

Letting $t = 0$ in Equation 3.2.3, we get

$$\begin{aligned} h(0) =& \frac{1}{\Gamma(-\mu)} \sum_{s=\mu-1}^{\mu} (-\sigma(s))^{\underline{-\mu-1}} p(s)\Delta x(s) + q(\mu-1)x(\mu-1) \\ =& \frac{1}{\Gamma(-\mu)} \left[(-\mu)^{\underline{-\mu-1}} p(\mu-1)(x(\mu) - x(\mu-1))\right] \\ &+ \frac{1}{\Gamma(-\mu)} \left[(-\mu-1)^{\underline{-\mu-1}} p(\mu)(x(\mu+1) - x(\mu))\right] + Aq(\mu-1). \end{aligned}$$

Hence

$$h(0) = (-\mu)p(\mu-1)(B-A) + p(\mu)(x(\mu+1) - B) + Aq(\mu-1).$$

Solving for $x(\mu+1)$ we have that

$$x(\mu+1) = \frac{h(0) + \mu p(\mu-1)(B-A) - Aq(\mu-1)}{p(\mu)} + B. \tag{3.2.4}$$

Thus we see that the value of $x(t)$ at $t = \mu+1$ is uniquely determined by the two initial values of $x(t)$ at $t = \mu - 1$ and $t = \mu$. Hence we get the existence and uniqueness of the solution of the FIVP (3.2.1) on $\mathbb{N}_{\mu-1}^{\mu+1}$. We now show the existence and uniqueness of our solution on $\mathbb{N}_{\mu-1}^{\mu+b+1}$ by induction. To this end assume there is a unique solution $x(t)$ on $\mathbb{N}_{\mu-1}^{t_0}$, where $t_0 \in \mathbb{N}_{\mu+1}^{\mu+b}$. We now show that the values of the solution $x(t)$ on $\mathbb{N}_{\mu-1}^{t_0}$, uniquely determine the value of the solution at $t_0 + 1$. To prove this, we first substitute $t = t_0 - \mu$ in (3.2.3) to get

$$\begin{aligned}
h(t_0-\mu) =& \frac{1}{\Gamma(-\mu)} \sum_{s=\mu-1}^{t_0} (t_0-\mu-\sigma(s))^{\underline{-\mu-1}} p(s)\Delta x(s) + q(t_0-1)x(t_0-1) \\
=& \frac{1}{\Gamma(-\mu)} \left[\sum_{s=\mu-1}^{t_0-1} (t_0-\mu-\sigma(s))^{\underline{-\mu-1}} p(s)\Delta x(s) \right] \\
+& \frac{1}{\Gamma(-\mu)} \left[(-\mu-1)^{\underline{-\mu-1}} p(t_0)[x(t_0+1)-x(t_0)] \right] + q(t_0-1)x(t_0-1) \\
=& \frac{1}{\Gamma(-\mu)} \left[\sum_{s=\mu-1}^{t_0-1} (t_0-\mu-\sigma(s))^{\underline{-\mu-1}} p(s)\Delta x(s) \right] + p(t_0)[x(t_0+1)-x(t_0)] \\
&+ q(t_0-1)x(t_0-1).
\end{aligned}$$

We can uniquely solve the above equation for $x(t_0+1)$ to get that

$$\begin{aligned}
x(t_0+1) =& x(t_0) + \frac{1}{p(t_0)} \left[h(t_0-\mu) - q(t_0-1)x(t_0-1) \right] \\
&- \frac{1}{p(t_0)} \left[\frac{1}{\Gamma(-\mu)} \sum_{s=\mu-1}^{t_0-1} (t_0-\mu-\sigma(s))^{\underline{-\mu-1}} p(s)\Delta x(s) \right].
\end{aligned}$$

Since, by the induction assumption, all the values of $x(t)$ in the expression

$$\frac{1}{\Gamma(-\mu)}\sum_{s=\mu-1}^{t_0-1}(t_0-\mu-\sigma(s))^{\underline{-\mu-1}}p(s)\Delta x(s)$$

are known, it follows that $x(t_0+1)$ is uniquely determined and hence we have that $x(t)$ is the unique solution of (3.2.2) on $\mathbb{N}_{\mu-1}^{t_0+1}$. Hence by mathematical induction, the fractional IVP (3.2.1) has a unique solution that exists on $\mathbb{N}_{\mu-1}^{\mu+b+1}$. □

Remark 3.2.2. *If* $\mu=1$ *(non-fractional case) in Theorem* 3.2.1, *it can be shown that for any* $t_0\in\mathbb{N}_{\mu-1}^{\mu+b}$ *the initial conditions* $x(t_0)=A$ *and* $x(t_0+1)=B$ *determine a unique solution of IVP* (3.2.1) *if* $q(t)\neq 0$. *Note that in the fractional case* $0<\mu<1$, *we just get the existence and uniqueness of the solution of* (3.2.1) *for the case* $t_0=\mu-1$. *The reason for this is in the true fractional case (i.e.,* $0<\mu<1$*) the fractional difference depends on all of its values back to its value at* $\mu-1$.

3.3 Variation Of Constants Formula

In this section we are interested in establishing the variation of constants formula for the self-adjoint FIVP

$$\begin{cases}\Delta_{\mu-1}^{\mu}(p\Delta x)(t)=h(t), \quad t\in\mathbb{N}_0^b\\ x(\mu-1)=\Delta x(\mu-1)=0\end{cases} \tag{3.3.1}$$

where $0<\mu\leq 1$, $b\in\mathbb{N}_1$, $p:\mathbb{N}_{\mu-1}^{b+\mu}\to\mathbb{R}$ with $p(t)>0$. Our variation of constants formula will involve the Cauchy function, which we now define.

Definition 3.3.1. *We define the Cauchy function* $x(.,.)$ *for the homogeneous frac-*

tional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0$$

to be the function $x : \mathbb{N}^{\mu+b+1}_{\mu-1} \times \mathbb{N}^b_0 \to \mathbb{R}$ *such that for each fixed* $s \in \mathbb{N}^b_0$, $x(.,s)$ *is the solution of the fractional initial value problem*

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0, \quad t \in \mathbb{N}^b_0 \\ x(s+\mu) = 0, \quad \Delta x(s+\mu) = \frac{1}{p(s+\mu)} \end{cases} \tag{3.3.2}$$

and is given by the formula

$$x(t,s) = \sum_{\tau=s+\mu}^{t-1} \left[\frac{1}{p(\tau)} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} \right], \quad t \in \mathbb{N}^{\mu+b+1}_{\mu-1}.$$

Note that by our convention on sums $x(t,s) = 0$ *for* $t \leq s + \mu$.

Theorem 3.3.2. *Let* $h : \mathbb{N}^b_0 \to \mathbb{R}$ *and* $p : \mathbb{N}^{\mu+b}_{\mu-1} \to \mathbb{R}$ *with* $p(t) > 0$ *then the solution to the initial value problem*

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) = h(t), \quad t \in \mathbb{N}^b_0 \\ x(\mu-1) = \Delta x(\mu-1) = 0 \end{cases} \tag{3.3.3}$$

is given by

$$x(t) = \sum_{s=0}^{t-\mu-1} x(t,s)h(s), \qquad t \in \mathbb{N}^{\mu+b+1}_{\mu-1}$$

where $x(t,s)$ *is the Cauchy function for* $\Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0$.

Proof. Let $x(t)$ be a solution of $\Delta^{\mu}_{\mu-1}(p\Delta x)(t) = h(t), \quad t \in \mathbb{N}^b_0$. Then

$$\Delta^{\mu}_{0+\mu-1}(p\Delta x)(t) = h(t), \quad t \in \mathbb{N}^b_0$$

Let $y(t) = (p\Delta x)(t)$ then $y(t)$ is a solution of

$$\Delta^{\mu}_{0+\mu-1}(y)(t) \;\; = h(t), \quad t \in \mathbb{N}_0^b$$

and hence is given by

$$y(t) = \sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) + c_0 t^{\underline{\mu-1}}, \quad t \in \mathbb{N}_{\mu-1}^{\mu+b}. \tag{3.3.4}$$

Dividing both sides by $p(t)$ we get that

$$\Delta x(t) = \frac{1}{p(t)} \left[\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) + c_0 t^{\underline{\mu-1}} \right], \quad t \in \mathbb{N}_{\mu-1}^{\mu+b}. \tag{3.3.5}$$

Using the second initial condition we get

$$0 = \Delta x(\mu-1) = \frac{1}{p(\mu-1)} \left[\sum_{s=0}^{\mu-1-\mu} \frac{(\mu-1-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) + c_0 (\mu-1)^{\underline{\mu-1}} \right]. \tag{3.3.6}$$

Note that the first term in the sum on the right hand side is zero by our convention on sums and therefore we are left with

$$0 = \Delta x(\mu-1) = \frac{1}{p(\mu-1)} \left[c_0 (\mu-1)^{\underline{\mu-1}} \right], \tag{3.3.7}$$

which implies that $c_0 = 0$. Thus

$$\Delta x(t) = \frac{1}{p(t)} \left[\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) \right], \quad t \in \mathbb{N}_{\mu-1}^{\mu+b}. \tag{3.3.8}$$

Summing both sides from $\tau = \mu - 1$ to $\tau = t - 1$ to get

$$\sum_{\tau=\mu-1}^{t-1} \Delta x(\tau) = \sum_{\tau=\mu-1}^{t-1} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) \right], \quad t \in \mathbb{N}_{\mu-1}^{\mu+b}. \tag{3.3.9}$$

Interchanging the order of the summations we have that

$$x(t) - x(\mu - 1) = \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-1-\mu} \sum_{\tau=s+\mu}^{t-1} \left[\frac{1}{p(\tau)} (\tau - \sigma(s))^{\underline{\mu-1}} h(s) \right]. \tag{3.3.10}$$

Using the first initial condition we have

$$x(t) = \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-1-\mu} \sum_{\tau=s+\mu}^{t-1} \left[\frac{1}{p(\tau)} (\tau - \sigma(s))^{\underline{\mu-1}} h(s) \right]. \tag{3.3.11}$$

It follows that

$$\begin{aligned} x(t) &= \sum_{s=0}^{t-1-\mu} \sum_{\tau=s+\mu}^{t-1} \left[\frac{1}{p(\tau)} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) \right] \\ &= \sum_{s=0}^{t-\mu-1} x(t,s) h(s), \quad t \in \mathbb{N}_{\mu-1}^{\mu+b+1}. \end{aligned}$$

This completes the proof. $\square$

Theorem 3.3.3. *Let $p : \mathbb{N}_{\mu-1}^{\mu+b} \to \mathbb{R}$ with $p(t) > 0$ and $b \in \mathbb{N}_1$, $0 < \mu \leq 1$ and assume that*

$$\rho = \alpha\gamma \sum_{\tau=\mu-1}^{\mu+b-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)} + \frac{\beta\gamma\Gamma(\mu)}{p(\mu-1)} + \frac{\alpha\delta(\mu+b)^{\underline{\mu-1}}}{p(\mu+b)}.$$

Then the homogeneous fractional boundary value problem (FBVP)

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0, \quad t \in \mathbb{N}^b_0 \\ \alpha x(\mu-1) - \beta\Delta x(\mu-1) = 0 \\ \gamma x(\mu+b) + \delta\Delta x(\mu+b) = 0 \end{cases} \tag{3.3.12}$$

has only the trivial solution if and only if $\rho \neq 0$.

Proof. Consider

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0, \quad t \in \mathbb{N}^b_0.$$

Then

$$(p\Delta x)(t) = c_0 t^{\underline{\mu-1}}, \quad t \in \mathbb{N}^{\mu+b}_{\mu-1},$$

and hence

$$\Delta x(t) = \frac{c_0 t^{\underline{\mu-1}}}{p(t)}, \quad t \in \mathbb{N}^{\mu+b}_{\mu-1}.$$

Summing both sides from $\tau = \mu - 1$ to $\tau = t - 1$ to get

$$x(t) - x(\mu-1) = \sum_{\tau=\mu-1}^{t-1} \frac{c_0 \tau^{\underline{\mu-1}}}{p(\tau)}, \quad t \in \mathbb{N}^{\mu+b+1}_{\mu-1}.$$

Let $c_1 = x(\mu-1)$, then the general solution of $\Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0$ is given by,

$$x(t) = \sum_{\tau=\mu-1}^{t-1} \frac{c_0 \tau^{\underline{\mu-1}}}{p(\tau)} + c_1, \quad t \in \mathbb{N}^{\mu+b+1}_{\mu-1}.$$

Now by using both the boundary conditions we get

$$-c_0 \frac{\beta\Gamma(\mu)}{p(\mu-1)} + c_1\alpha = 0$$

$$c_0\left(\gamma\sum_{\tau=\mu-1}^{\mu+b-1}\frac{\tau^{\underline{\mu-1}}}{p(\tau)}+\delta\frac{(\mu+b)^{\underline{\mu-1}}}{p(\mu+b)}\right)+c_1\gamma=0.$$

This system of equations in c_0 and c_1 has only the trivial solution if and only if the following determinant is not equal to zero

$$\rho=\begin{vmatrix} -\frac{\beta\Gamma(\mu)}{p(\mu-1)} & \alpha \\ \left(\gamma\sum_{\tau=\mu-1}^{\mu+b-1}\frac{\tau^{\underline{\mu-1}}}{p(\tau)}+\delta\frac{(\mu+b)^{\underline{\mu-1}}}{p(\mu+b)}\right) & \gamma \end{vmatrix}\neq 0.$$

Which implies that the self-adjoint fractional boundary valued problem has only trivial solution if and only if

$$\rho=\alpha\gamma\sum_{\tau=\mu-1}^{\mu+b-1}\frac{\tau^{\underline{\mu-1}}}{p(\tau)}+\frac{\beta\gamma\Gamma(\mu)}{p(\mu-1)}+\frac{\alpha\delta(\mu+b)^{\underline{\mu-1}}}{p(\mu+b)}\neq 0.$$

This completes the proof. □

Remark 3.3.4. *Letting $\mu=1$ in the above theorem gives us the result (see Kelley-Peterson [60] for a proof of this remark for the continuous case) that the BVP*

$$\begin{cases}\Delta(p\Delta x)(t)=0, \quad t\in\mathbb{N}_0^b\\ \alpha x(0)-\beta\Delta x(0)=0\\ \gamma x(b+1)+\delta\Delta x(b+1)=0\end{cases}$$

has only the trivial solution if and only if

$$\rho=\alpha\gamma\sum_{\tau=0}^{b}\frac{1}{p(\tau)}+\frac{\beta\gamma}{p(0)}+\frac{\alpha\delta}{p(b+1)}\neq 0.$$

3.4 Green's Function For A Two Point FBVP

In this section we will derive a formula for the Green's function for the following self-adjoint fractional boundary value problem

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0, \quad t \in \mathbb{N}_0^b \\ x(\mu-1) = 0, x(\mu+b+1) = 0 \end{cases} \tag{3.4.1}$$

where $h : \mathbb{N}_0^b \to \mathbb{R}$ and $p : \mathbb{N}_{\mu-1}^{\mu+b} \to \mathbb{R}$ with $p(t) > 0$.

Theorem 3.4.1. *Let $h : \mathbb{N}_0^b \to \mathbb{R}$ and $p : \mathbb{N}_{\mu-1}^{\mu+b} \to \mathbb{R}$ with $p(t) > 0$. Then the Green's function for the FBVP*

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0, \quad t \in \mathbb{N}_0^b \\ x(\mu-1) = 0, \quad x(\mu+b+1) = 0 \end{cases} \tag{3.4.2}$$

is given by

$$G(t,s) = \begin{cases} -\frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right) x(\mu+b+1,s), \quad t \le s+\mu \\ x(t,s) - \frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right) x(\mu+b+1,s), \quad s \le t-\mu-1. \end{cases} \tag{3.4.3}$$

Proof. In order to find the Green's function for the above homogenous FBVP, we first consider the following nonhomogeneous FBVP

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) = h(t), \quad t \in \mathbb{N}_0^b \\ x(\mu-1) = 0, \quad x(\mu+b+1) = 0. \end{cases}$$

Since we have already derived the expression for Δx in the preceding theorems we have,

$$\Delta x(t) = \frac{1}{p(t)}\left[\sum_{s=0}^{t-\mu}\frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}h(s) + c_0(t)^{\underline{\mu-1}}\right], \quad t\in\mathbb{N}_{\mu-1}^{\mu+b}.$$

Now summing from $\tau=\mu-1$ to $\tau=t-1$ and interchanging the order of summations we have that

$$x(t) = \sum_{s=0}^{t-1-\mu}\sum_{\tau=s+\mu}^{t-1}\left[\frac{1}{p(\tau)}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}h(s)}{\Gamma(\mu)}\right] + \sum_{\tau=\mu-1}^{t-1}\frac{c_0\tau^{\underline{\mu-1}}}{p(\tau)} + x(\mu-1).$$

By letting $c_1 = x(\mu-1)$, the above expression for the general solution can be rewritten as

$$x(t) = \sum_{s=0}^{t-1-\mu}\sum_{\tau=s+\mu}^{t-1}\left[\frac{1}{p(\tau)}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}h(s)}{\Gamma(\mu)}\right] + \sum_{\tau=\mu-1}^{t-1}\frac{c_0\tau^{\underline{\mu-1}}}{p(\tau)} + c_1.$$

Now if we represent the term

$$\sum_{\tau=s+\mu}^{t-1}\left[\frac{1}{p(\tau)}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}\right]$$

by $x(t,s)$, the Cauchy function, then the above expression for the general solution can be rewritten as

$$x(t) = \sum_{s=0}^{t-1-\mu}x(t,s)h(s) + \sum_{\tau=\mu-1}^{t-1}\frac{c_0\tau^{\underline{\mu-1}}}{p(\tau)} + c_1.$$

By using the first boundary condition $x(\mu-1)= 0$ we get

$$x(\mu-1) = \sum_{s=0}^{\mu-2-\mu}x(\mu-1,s)h(s) + \sum_{\tau=\mu-1}^{\mu-2}\frac{c_0\tau^{\underline{\mu-1}}}{p(\tau)} + c_1.$$

Notice that the first two sums in the preceding expression are zero by convention as the upper limit of summations is smaller than the lower limit, therefore we have that

$$c_1 = 0.$$

Thus

$$x(t) = \sum_{s=0}^{t-\mu-1} x(t,s)h(s) + \sum_{\tau=\mu-1}^{t-1} \frac{c_0 \tau^{\underline{\mu-1}}}{p(\tau)}.$$

By using the second boundary condition $x(\mu+b+1)$= 0 we have that

$$x(\mu+b+1) = \sum_{s=0}^{b} x(\mu+b,s)h(s) + \sum_{\tau=\mu-1}^{\mu+b} \frac{c_0 \tau^{\underline{\mu-1}}}{p(\tau)}. \tag{3.4.4}$$

Solving for c_0, we get that

$$c_0 = -\frac{\displaystyle\sum_{s=0}^{b} x(\mu+b+1,s)}{\displaystyle\sum_{\tau=\mu-1}^{\mu+b} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}} h(s).$$

Moreover, if we let $\rho = \displaystyle\sum_{\tau=\mu-1}^{\mu+b} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}$ and substitute the values of c_0 and c_1 to get the solution of the FBVP as

$$x(t) = \sum_{s=0}^{t-1-\mu} x(t,s)h(s) - \frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right) \sum_{s=0}^{b} x(\mu+b+1,s)h(s),$$

i.e.

$$\begin{aligned}
x(t) &= \left[\sum_{s=0}^{t-1-\mu} x(t,s)h(s) - \frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right)\sum_{s=0}^{b} x(\mu+b+1,s)h(s)\right] \\
&= \left[\sum_{s=t-\mu}^{b}\left(-\frac{1}{\rho}\right)\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right) x(\mu+b+1,s)h(s)\right] \\
&\quad + \left[\sum_{s=0}^{t-1-\mu}\left(x(t,s) - \frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right) x(\mu+b+1,s)\right)h(s)\right].
\end{aligned}$$

Thus the Green's function for the FBVP can be rewritten as

$$G(t,s) = \begin{cases} -\frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right) x(\mu+b+1,s), & t \le s+\mu \\ x(t,s) - \frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right) x(\mu+b+1,s), & s \le t-\mu-1. \end{cases}$$

This completes the proof. □

3.5 Green's Function For The General Two Point FBVP

In this section we will derive a formula for the Green's function for the general self-adjoint fractional boundary value problem

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0, & t \in \mathbb{N}_0^b \\ \alpha x(\mu-1) - \beta \Delta x(\mu-1) = 0 \\ \gamma x(\mu+b) + \delta \Delta x(\mu+b) = 0, \end{cases} \tag{3.5.1}$$

where $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$.

Theorem 3.5.1. *Assume* $p : \mathbb{N}_{\mu-1}^{\mu+b} \to \mathbb{R}$ *with* $p(t) > 0$ *and*

$$\rho := \alpha\gamma \sum_{\tau=\mu-1}^{\mu+b-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)} + \frac{\beta\gamma\Gamma(\mu)}{p(\mu-1)} + \frac{\alpha\delta(\mu+b)^{\underline{\mu-1}}}{p(\mu+b)} \neq 0.$$

Then the Green's function for the FBVP (3.5.1) *is given by*

$$G(t,s) = \begin{cases} u(t,s), & t \leq s+\mu \\ v(t,s), & s \leq t-\mu-1, \end{cases}$$

where

$$\begin{aligned} u(t,s) = -\frac{1}{\rho}\Big[& \left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right) \Big(\alpha\gamma\, x(\mu+b,s) \\ & + \frac{\alpha\delta}{p(\mu+b)} \frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}\Big) + \frac{\beta\,\gamma\Gamma(\mu)}{p(\mu-1)} x(\mu+b,s) \\ & + \frac{\beta\,\delta}{p(\mu-1)p(\mu+b)} (\mu+b-\sigma(s))^{\underline{\mu-1}}\Big], \end{aligned}$$

for $t \leq s+\mu$, *and*

$$v(t,s) := u(t,s) + x(t,s),$$

for $s \leq t-\mu-1$, *where* $x(t,s)$ *is the Cauchy function for*

$$\Delta_{\mu-1}^{\mu}(p\Delta x)(t) = 0.$$

Proof. In order to find the Green's function for the above homogeneous FBVP (3.5.1), we consider the following nonhomogeneous FBVP

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) = h(t), \quad t \in \mathbb{N}^b_0 \\ \alpha x(\mu-1) - \beta\Delta x(\mu-1) = 0 \\ \gamma x(\mu+b) + \delta\Delta x(\mu+b) = 0, \end{cases} \tag{3.5.2}$$

where $h : \mathbb{N}^b_0 \to \mathbb{R}$ is a given function. Since $\rho \neq 0$, we have by Theorem 3.3.3 that the corresponding homogeneous FBVP (3.5.1) has only the trivial solution. It is a standard argument that this implies that the homogeneous FBVP (3.5.2) has a unique solution. Let $x(t)$ be the solution of non-homogeneous FBVP (3.5.2). As in the proof of Theorem 3.2.2, we get that since $x(t)$ is a solution of the fractional difference equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) = h(t), \quad t \in \mathbb{N}^b_0,$$

then

$$\Delta x(t) = \frac{1}{p(t)}\left[\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) + c_0 (t)^{\underline{\mu-1}}\right], \quad t \in \mathbb{N}^{\mu+b}_{\mu-1}. \tag{3.5.3}$$

Now summing both sides from $\tau = \mu - 1$ to $\tau = t - 1$ and interchanging the order of the summations we have that

$$x(t) = \sum_{s=0}^{t-1-\mu} \sum_{\tau=s+\mu}^{t-1} \left[\frac{1}{p(\tau)} \frac{(\tau-\sigma(s))^{\underline{\mu-1}} h(s)}{\Gamma(\mu)}\right] + \sum_{\tau=\mu-1}^{t-1} \frac{c_0 \tau^{\underline{\mu-1}}}{p(\tau)} + x(\mu-1). \tag{3.5.4}$$

By letting $c_1 = x(\mu-1)$, the above expression for the general solution can be rewritten as,

$$\begin{aligned} x(t) &= \sum_{s=0}^{t-1-\mu} \sum_{\tau=s+\mu}^{t-1} \left[\frac{1}{p(\tau)} \frac{(\tau-\sigma(s))^{\underline{\mu-1}} h(s)}{\Gamma(\mu)} \right] + \sum_{\tau=\mu-1}^{t-1} \frac{c_0 \tau^{\underline{\mu-1}}}{p(\tau)} + c_1 \\ &= \sum_{s=0}^{t-1-\mu} x(t,s)h(s) + \sum_{\tau=\mu-1}^{t-1} \frac{c_0 \tau^{\underline{\mu-1}}}{p(\tau)} + c_1. \end{aligned} \tag{3.5.5}$$

Applying the two boundary conditions we get the following equations,

$$c_1 \alpha - c_0 \frac{\beta \Gamma(\mu)}{p(\mu-1)} = 0 \tag{3.5.6}$$

and

$$\begin{aligned} &\gamma \left(\sum_{s=0}^{b-1} x(\mu+b,s)h(s) + \sum_{\tau=\mu-1}^{\mu+b-1} \frac{c_0 \tau^{\underline{\mu-1}}}{p(\tau)} + c_1 \right) + \\ &\qquad \delta \left[\frac{1}{p(\mu+b)} \left(\sum_{s=0}^{b} \frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) + c_0 (\mu+b)^{\underline{\mu-1}} \right) \right] = 0 \end{aligned} \tag{3.5.7}$$

i.e.

$$\begin{aligned} c_1 \gamma + c_0 \left(\gamma \sum_{\tau=\mu-1}^{\mu+b-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)} + \delta \frac{(\mu+b)^{\underline{\mu-1}}}{p(\mu+b)} \right) &= -\gamma \sum_{s=0}^{b-1} x(\mu+b,s)h(s) \\ &\quad - \frac{\delta}{p(\mu+b)} \sum_{s=0}^{b} \frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) \end{aligned} \tag{3.5.8}$$

Now if we let

$$\rho = \alpha\gamma \sum_{\tau=\mu-1}^{\mu+b-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)} + \frac{\beta\gamma\Gamma(\mu)}{p(\mu-1)} + \frac{\alpha\delta(\mu+b)^{\underline{\mu-1}}}{p(\mu+b)}.$$

and solve the above system for c_0 and c_1 we have that

$$c_0 = -\frac{1}{\rho} \left(\alpha\gamma \sum_{s=0}^{b-1} x(\mu+b,s)h(s) + \frac{\alpha\delta}{p(\mu+b)} \sum_{s=0}^{b} \frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) \right)$$

$$c_1 = -\frac{1}{\rho}\frac{\beta\Gamma(\mu)}{\alpha p(\mu-1)}\left(\alpha\gamma\sum_{s=0}^{b-1} x(\mu+b,s)h(s) + \frac{\alpha\delta}{p(\mu+b)}\sum_{s=0}^{b}\frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}h(s)\right)$$

i.e.

$$c_1 = -\frac{1}{\rho}\Big(\frac{\beta\Gamma(\mu)}{p(\mu-1)}\gamma\sum_{s=0}^{b-1} x(\mu+b,s)h(s) + \frac{\beta\delta}{p(\mu-1)p(\mu+b)}\sum_{s=0}^{b}(\mu+b-\sigma(s))^{\underline{\mu-1}}h(s)\Big).$$

Now substituting the above values of c_0 and c_1 in the equation (3.5.5) we get that

$$x(t) = \sum_{s=0}^{t-1-\mu} x(t,s)h(s) - \frac{1}{\rho}\sum_{\tau=\mu-1}^{t-1}\frac{\tau^{\underline{\mu-1}}}{p(\tau)}\Big(\alpha\gamma\sum_{s=0}^{b-1}x(\mu+b,s)h(s) + \frac{\alpha\delta}{p(\mu+b)}\sum_{s=0}^{b}\frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}h(s)\Big) - \frac{1}{\rho}\Big(\frac{\beta\,\gamma\Gamma(\mu)}{p(\mu-1)}\sum_{s=0}^{b-1}x(\mu+b,s)h(s) + \frac{\beta\,\delta}{p(\mu-1)p(\mu+b)}\sum_{s=0}^{b}(\mu+b-\sigma(s))^{\underline{\mu-1}}h(s)\Big).$$

Using $x(\mu+b,b)=0$, we get that

$$\sum_{s=0}^{b-1} x(\mu+b,s) = \sum_{s=0}^{b} x(\mu+b,s).$$

Thus, we have that

$$x(t) = \sum_{s=0}^{t-1-\mu} x(t,s)h(s) - \frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right)\Big(\alpha\gamma \sum_{s=0}^{b} x(\mu+b,s)h(s)$$
$$+ \frac{\alpha\delta}{p(\mu+b)} \sum_{s=0}^{b} \frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s)\Big)$$
$$- \frac{1}{\rho}\Big(\frac{\beta\,\gamma\Gamma(\mu)}{p(\mu-1)} \sum_{s=0}^{b} x(\mu+b,s)h(s)$$
$$+ \frac{\beta\,\delta}{p(\mu-1)p(\mu+b)} \sum_{s=0}^{b} (\mu+b-\sigma(s))^{\underline{\mu-1}} h(s)\Big).$$

And hence we have that

$$x(t) = \sum_{s=0}^{t-1-\mu} x(t,s)h(s) - \frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right)\Big(\alpha\gamma\Big(\sum_{s=0}^{t-\mu-1} x(\mu+b,s)h(s)$$
$$+ \sum_{s=t-\mu}^{b} x(\mu+b,s)h(s)\Big)$$
$$+ \frac{\alpha\delta}{p(\mu+b)}\Big(\sum_{s=0}^{t-\mu-1} \frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s) + \sum_{s=t-\mu}^{b} \frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} h(s)\Big)\Big)$$
$$- \frac{1}{\rho}\Big(\frac{\beta\,\gamma\Gamma(\mu)}{p(\mu-1)}\Big(\sum_{s=0}^{t-\mu-1} x(\mu+b,s)h(s) + \sum_{s=t-\mu}^{b} x(\mu+b,s)h(s)\Big)$$
$$+ \frac{\beta\,\delta}{p(\mu-1)p(\mu+b)}\Big(\sum_{s=0}^{t-\mu-1} (\mu+b-\sigma(s))^{\underline{\mu-1}} h(s)$$
$$+ \sum_{s=t-\mu}^{b} (\mu+b-\sigma(s))^{\underline{\mu-1}} h(s)\Big)\Big).$$

Thus the solution to the given FBVP is given by

$$
\begin{aligned}
x(t) = \sum_{s=0}^{t-1-\mu} \Big[x(t,s) &- \frac{1}{\rho}\Big(\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)} \right) \Big(\alpha\gamma\, x(\mu+b,s) \\
&+ \frac{\alpha\delta}{p(\mu+b)} \frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} \Big) + \frac{\beta\,\gamma\Gamma(\mu)}{p(\mu-1)} x(\mu+b,s) \\
&+ \frac{\beta\,\delta}{p(\mu-1)p(\mu+b)} (\mu+b-\sigma(s))^{\underline{\mu-1}} \Big)\Big) \Big] h(s) \\
&+ \sum_{s=t-\mu}^{b} \Big[-\frac{1}{\rho}\Big(\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)} \right) \Big(\alpha\gamma\, x(\mu+b,s) \\
&+ \frac{\alpha\delta}{p(\mu+b)} \frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} \Big) + \frac{\beta\,\gamma\Gamma(\mu)}{p(\mu-1)} x(\mu+b,s) \\
&+ \frac{\beta\,\delta}{p(\mu-1)p(\mu+b)} (\mu+b-\sigma(s))^{\underline{\mu-1}} \Big)\Big) \Big] h(s).
\end{aligned}
$$

Hence the Green's function for the given FBVP is given by

$$
G(t,s) = \begin{cases} u(t,s), & t \le s+\mu \\ \\ v(t,s), & s \le t-\mu-1, \end{cases}
$$

where

$$
\begin{aligned}
u(t,s) = -\frac{1}{\rho}\Big[&\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)} \right) \Big(\alpha\gamma\, x(\mu+b,s) \\
&+ \frac{\alpha\delta}{p(\mu+b)} \frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} \Big) + \frac{\beta\,\gamma\Gamma(\mu)}{p(\mu-1)} x(\mu+b,s) \\
&+ \frac{\beta\,\delta}{p(\mu-1)p(\mu+b)} (\mu+b-\sigma(s))^{\underline{\mu-1}} \Big]
\end{aligned}
$$

for $t \leq s+\mu$, and

$$v(t,s) = x(t,s) - \frac{1}{\rho}\Big[\left(\sum_{\tau=\mu-1}^{t-1} \frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right)\Big(\alpha\gamma\, x(\mu+b,s) \\ + \frac{\alpha\delta}{p(\mu+b)}\frac{(\mu+b-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}\Big) + \frac{\beta\,\gamma\Gamma(\mu)}{p(\mu-1)}x(\mu+b,s) \\ + \frac{\beta\,\delta}{p(\mu-1)p(\mu+b)}(\mu+b-\sigma(s))^{\underline{\mu-1}}\Big]$$

for $s \leq t-\mu-1$. This completes the proof. □

Remark 3.5.2. *If $\alpha=\gamma=\delta=1$ and $\beta=0$, then we get the known formula for the Green's function for the conjugate case*

$$\begin{cases} \Delta^{\mu}_{\mu-1}(p\Delta x)(t) = 0, & t \in \mathbb{N}_0^b \\ x(\mu-1)=0, & x(\mu+b+1)=0. \end{cases} \tag{3.5.9}$$

as

$$G(t,s) = \begin{cases} -\frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1}\frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right)x(\mu+b+1,s), & t \leq s+\mu \\ x(t,s) - \frac{1}{\rho}\left(\sum_{\tau=\mu-1}^{t-1}\frac{\tau^{\underline{\mu-1}}}{p(\tau)}\right)x(\mu+b+1,s), & s \leq t-\mu-1. \end{cases} \tag{3.5.10}$$

3.6 Green's Function When p=1

Our goal in this section is to deduce some important properties of the Green's function for the conjugate case when $p(t)=1$ for $t \in \mathbb{N}_{\mu-1}^{\mu+b}$, which will be useful later in this dissertation. In order to do that, first we will explicitly give a formula for the Green's function when $p(t)=1$ in the following proposition.

Proposition 3.6.1. *Letting $p(t)=1$ in Remark 3.5.2 we get that the Green's function* *for the conjugate FBVP*

$$\begin{cases} \Delta^{\mu}_{\mu-1}(\Delta x)(t) = 0, \quad t \in \mathbb{N}_0^b \\ x(\mu-1) = 0, \quad x(\mu+b+1) = 0 \end{cases} \tag{3.6.1}$$

is given by

$$G(t,s) = \begin{cases} u(t,s) := -\frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu+1)}\right), \quad t \le s+\mu \\ v(t,s) := \frac{(t-\sigma(s))^{\underline{\mu}}}{\Gamma(\mu+1)} - \frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu+1)}\right), \quad s \le t-\mu-1, \end{cases} \tag{3.6.2}$$

where $\rho = \frac{1}{\mu}(\mu+b+1)^{\underline{\mu}}$.

Proof. First we observe that with $p(t) = 1$ for $t \in \mathbb{N}_{\mu-1}^{\mu+b}$, the Cauchy function $x(t,s)$ and ρ as mentioned in Theorem 3.2.1 takes the form

$$\begin{aligned} x(t,s) &= \sum_{\tau=s+\mu}^{t-1}\left[\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}\right] \\ &= \frac{1}{\Gamma(\mu)}\sum_{\tau=s+\mu}^{t-1}(\tau-\sigma(s))^{\underline{\mu-1}} \\ &= \frac{1}{\Gamma(\mu)}\sum_{\tau=s+\mu}^{t-1}\Delta_\tau\frac{(\tau-\sigma(s))^{\underline{\mu}}}{\mu} \\ &= \frac{1}{\Gamma(\mu+1)}\sum_{\tau=s+\mu}^{t-1}\Delta_\tau(\tau-\sigma(s))^{\underline{\mu}} \\ &= \frac{1}{\Gamma(\mu+1)}\left[(t-\sigma(s))^{\underline{\mu}} - (\mu-1)^{\underline{\mu}}\right] \\ &= \frac{(t-\sigma(s))^{\underline{\mu}}}{\Gamma(\mu+1)}. \end{aligned}$$

Thus $x(\mu+b+1,s) = \frac{(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\Gamma(\mu+1)}$ and

$$\begin{aligned}\rho =& \sum_{\tau=\mu-1}^{\mu+b} \tau^{\underline{\mu-1}}\\ =& \sum_{\tau=\mu-1}^{\mu+b} \Delta_\tau\big[\frac{\tau^{\underline{\mu}}}{\mu}\big]\\ =&\frac{1}{\mu}\big[(\mu+b+1)^{\underline{\mu}} - (\mu-1)^{\underline{\mu}}\big]\\ =&\frac{1}{\mu}(\mu+b+1)^{\underline{\mu}}.\end{aligned}$$

In the above, we have used the fact that $(\mu-1)^{\underline{\mu}} = 0$. Moreover, for $p(t) = 1$ we have

$$\sum_{\tau=\mu-1}^{t-1} \frac{t^{\underline{\mu-1}}}{p(\tau)} = \sum_{\tau=\mu-1}^{t-1} \tau^{\underline{\mu-1}}.$$

So we can write

$$\begin{aligned}\sum_{\tau=\mu-1}^{t-1} \tau^{\underline{\mu-1}} =& \sum_{\tau=\mu-1}^{t-1} \Delta_\tau\big(\frac{\tau^{\underline{\mu}}}{\mu}\big)\\ =&\frac{t^{\underline{\mu}}}{\mu}.\end{aligned}$$

Thus with $p(t) = 1$ and with these modified values of the Cauchy function $x(t,s)$, ρ, the preceding sum and the Green's function as derived in Theorem 3.2.1 can be rewritten as

$$G(t,s) = \begin{cases} u(t,s) := -\frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu+1)}\right), & t \le s+\mu \\ v(t,s) := \frac{(t-\sigma(s))^{\underline{\mu}}}{\Gamma(\mu+1)} - \frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu+1)}\right), & s \le t-\mu-1, \end{cases} \tag{3.6.3}$$

where $\rho = \frac{1}{\mu}(\mu+b+1)^{\underline{\mu}}$.

This completes the proof. $\square$

In the following four theorems we will derive some properties of this Green's function in Proposition 3.6.1. First we prove that this Green's function is of constant sign.

Theorem 3.6.2. *The Green's function for the FBVP* (3.6.1) *in Proposition* 3.6.1 *satisfies*

$$G(t, s) \leq 0.$$

Proof. We will show that each component, i.e., $u(t, s)$ and $v(t, s)$ of this Green's function is non-positive. First consider $u(t, s)$. Since $\rho > 0$ by its definition, $\Gamma(\mu+1) > 0$ since $0 < \mu \leq 1$, $t^{\underline{\mu}} \geq 0$ as $\mu - 1 \leq t \leq \mu + b + 1$ and $(\mu + b + 1 - \sigma(s))^{\underline{\mu}} \geq 0$ as $s \in \mathbb{N}_0^b$. This implies that for $t \leq s + \mu$ we have that

$$u(t, s) = -\frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu + b + 1 - \sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu + 1)}\right) \leq 0.$$

Next we will show that $v(t, s)$ of the Green's function is non-positive, i.e. we will show that for $0 \leq s \leq t - \mu - 1 \leq b$,

$$\frac{(t - \sigma(s))^{\underline{\mu}}}{\Gamma(\mu + 1)} - \frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu + b + 1 - \sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu + 1)}\right) \leq 0.$$

After substituting the value of $\rho = \frac{1}{\mu}(\mu+b+1)^{\underline{\mu}}$ and simplifying the above inequality, we get that

$$\frac{1}{(\mu + b + 1)^{\underline{\mu}}\Gamma(\mu + 1)}\left[(t - \sigma(s))^{\underline{\mu}}(\mu + b + 1)^{\underline{\mu}} - t^{\underline{\mu}}(\mu + b + 1 - \sigma(s))^{\underline{\mu}} \leq 0.\right.$$

Since $(\mu + b + 1)^{\underline{\mu}}$ and $\Gamma(\mu + 2) > 0$ it is sufficient to show that

$$\frac{(t - \sigma(s))^{\underline{\mu}}(\mu + b + 1)^{\underline{\mu}}}{t^{\underline{\mu}}(\mu + b + 1 - \sigma(s))^{\underline{\mu}}} \leq 1.$$

Thus for $0 \leq s \leq t-\mu-1 \leq b$, we consider

$$\begin{aligned}
&\frac{(t-\sigma(s))^{\underline{\mu}}(\mu+b+1)^{\underline{\mu}}}{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}\\
&= \frac{\Gamma(t-s)\Gamma(t+1-\mu)}{\Gamma(t-s-\mu)\Gamma(t+1)}\frac{\Gamma(\mu+b+2)\Gamma(b+2-\sigma(s))}{\Gamma(b+2)\Gamma(\mu+b+2-\sigma(s))}\\
&= \frac{\Gamma(t-s)}{\Gamma(t+1)}\frac{\Gamma(t+1-\mu)}{\Gamma(t-s-\mu)}\frac{\Gamma(\mu+b+2)}{\Gamma(\mu+b+2-\sigma(s))}\frac{\Gamma(b+2-\sigma(s))}{\Gamma(b+2)}.
\end{aligned}$$

Using the property of the gamma function, i.e., $\Gamma(r+1) = r\Gamma(r)$, for all reals r except for non-positive integers, we get

$$\begin{aligned}
&= \frac{\Gamma(t-s)}{t\cdots(t-s)\Gamma(t-s)}\frac{(t-\mu)\cdots(t-s-\mu)\Gamma(t-s-\mu)}{\Gamma(t-s-\mu)}\\
&\frac{(\mu+b+1)\cdots(\mu+b+2-\sigma(s))\Gamma(\mu+b+2-\sigma(s))}{\Gamma(\mu+b+2-\sigma(s))}\\
&\frac{\Gamma(b+2-\sigma(s))}{(b+1)\cdots(b+2-\sigma(s))\Gamma(b+2-\sigma(s))}.
\end{aligned}$$

Simplifying and rearranging the factors from the numerator and denominator we get,

$$= \Big[\frac{(t-\mu)(t-\mu-1)\cdots(t-\mu-s)}{t(t-1)\cdots(t-s)}\Big]\Big[\frac{(\mu+b+1)(\mu+b)\cdots(\mu+b+1-s)}{(b+1)b\cdots(b+1-s)}\Big].$$

A further rearrangement of the factors gives us

$$\Big[\frac{(t-\mu)(\mu+b+1)}{t(b+1)}\Big]\Big[\frac{(t-\mu-1)(\mu+b)}{(t-1)b}\Big]\cdots\Big[\frac{(t-\mu-s)(\mu+b+1-s)}{(t-s)(b+1-s)}\Big]. \quad (3.6.4)$$

Now in order to show that the above product in (3.6.4) is less than or equal to one, we will show that each factor within the square brackets is less than or equal to one. To do that, we can pick an appropriate general factor that represents all these factors from each of the square brackets and we show that is less or equal to one. Thus for

$0 \le k \le s$, we consider

$$\frac{(t-\mu-k)(\mu+b+1-k)}{(t-k)(b+1-k)}$$

and show it is less than or equal to one. Notice that the condition $s \le t-\mu-1 \le b$ implies that

$$t-\mu-1 \le b.$$

Multiplying both sides by $\mu > 0$ and simplifying we get

$$t\mu - \mu^2 - \mu b - \mu \le 0.$$

Adding to both sides the term $tb + t - tk - kb - k + k^2$ we get

$$t\mu + tb + t - tk - \mu^2 - \mu b - \mu + \mu k - \mu k - kb - k + k^2 \le tb + t - tk - kb - k + k^2,$$

i.e.

$$(t-\mu-k)(\mu+b+1-k) \le (t-k)(b+1-k)$$

This implies that

$$\frac{(t-\mu-k)(\mu+b+1-k)}{(t-k)(b+1-k)} \le 1.$$

Thus we have shown that a general factor in the above product given by (3.6.4) is less than or equal to one, therefore all the factors are less than or equal to one. This implies that the product given by (3.6.4) is less than or equal to one. Thus,

$$\frac{(t-\sigma(s))^{\underline{\mu}}}{\Gamma(\mu+1)} - \frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu+1)}\right) \le 0.$$

Moreover, since each expression for the Green's functions $G(t,s)$ is less than or equal to zero we conclude that $G(t,s) \le 0$. This completes the proof. $\square$

Next we find a formula for $\sum_{s=0}^{b}\left|G(t,s)\right|$ in the following proposition.

Proposition 3.6.3. *If* $t \in \mathbb{N}_{\mu-1}^{\mu+b+1}$, *then the Green's function for* (3.6.1) *satisfies*

$$\sum_{s=0}^{b}\left|G(t,s)\right| = \frac{t^{\underline{\mu}}}{\Gamma(\mu+2)}(\mu+b+1-t). \tag{3.6.5}$$

Proof.

$$\begin{aligned}\sum_{s=0}^{b}\left|G(t,s)\right| = &\sum_{s=0}^{t-\mu-1}\left|\frac{(t-\sigma(s))^{\underline{\mu}}}{\Gamma(\mu+1)} - \frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu+1)}\right)\right| \\ &+ \sum_{s=t-\mu}^{b}\left|-\frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu+1)}\right)\right|.\end{aligned}$$

In Proposition 3.6.2 we proved that $G(t,s) \leq 0$, which implies that $u(t,s) \leq 0$ for $t \leq s+\mu$ and $v(t,s) \leq 0$ for $s \leq t-\mu-1$. Thus the above expression can be rewritten as

$$\begin{aligned}&= \sum_{s=0}^{t-\mu-1}\left[\frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu+1)}\right) - \frac{(t-\sigma(s))^{\underline{\mu}}}{\Gamma(\mu+1)}\right] \\ &+ \sum_{s=t-\mu}^{b}\frac{1}{\rho}\left(\frac{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu+1)}\right) \\ &=\frac{1}{\rho}\left(\sum_{s=0}^{b}\frac{t^{\underline{\mu}}(\mu+b+1-\sigma(s))^{\underline{\mu}}}{\mu\Gamma(\mu+1)}\right) - \sum_{s=0}^{t-\mu-1}\frac{(t-\sigma(s))^{\underline{\mu}}}{\Gamma(\mu+1)} \\ &=\frac{t^{\underline{\mu}}}{\rho\mu\Gamma(\mu+1)}\sum_{s=0}^{b}\frac{\Delta_s(\mu+b+1-s)^{\underline{\mu+1}}}{-(\mu+1)} - \frac{1}{\Gamma(\mu+1)}\sum_{s=0}^{t-\mu-1}\frac{\Delta_s(t-s)^{\underline{\mu+1}}}{-(\mu+1)}.\end{aligned}$$

Using the Fundamental Theorem of Discrete Calculus, we get that

$$\begin{aligned}\sum_{s=0}^{b}\left|G(t,s)\right| =& \frac{1}{\Gamma(\mu+2)}\left[\left[(t-s)^{\underline{\mu+1}}\right]_{s=0}^{t-\mu} - \frac{t^{\underline{\mu}}}{\rho\mu}\left[(\mu+b+1-s)^{\underline{\mu+1}}\right]_{s=0}^{b+1}\right]\\ =& \frac{1}{\Gamma(\mu+2)}\left[\left(\mu^{\underline{\mu+1}} - t^{\underline{\mu+1}}\right) - \frac{t^{\underline{\mu}}}{\rho\mu}\left(\mu^{\underline{\mu+1}} - (\mu+b+1)^{\underline{\mu+1}}\right)\right].\end{aligned}$$

Since $\rho = \frac{1}{\mu}(\mu+b+1)^{\underline{\mu}}$ and using the fact that $\mu^{\underline{\mu+1}} = 0$, we get that

$$\sum_{s=0}^{b}\left|G(t,s)\right| = \frac{1}{\Gamma(\mu+2)}\left[\frac{t^{\underline{\mu}}(\mu+b+1)^{\underline{\mu+1}}}{(\mu+b+1)^{\underline{\mu}}} - t^{\underline{\mu+1}}\right].$$

Since $t^{\underline{\mu+1}} = t^{\underline{\mu}}(t-\mu)$, we have that

$$\sum_{s=0}^{b}\left|G(t,s)\right| = \frac{t^{\underline{\mu}}}{\Gamma(\mu+2)}(\mu+b+1-t), \quad t \in \mathbb{N}_{\mu-1}^{\mu+b+1}.$$

This completes the proof. $\square$

Proposition 3.6.4. *The Green's function for* (3.6.1) *satisfies*

$$\max_{t\in\mathbb{N}_{\mu-1}^{\mu+b+1}}\sum_{s=0}^{b+2}\left|G(t,s)\right| = \frac{1}{\Gamma(\mu+2)}\left[\mu+b-\lceil\frac{b+1}{\mu+1}\rceil\right]^{\underline{\mu-1}}\left[1+\lceil\frac{b+1}{\mu+1}\rceil\right],$$

where $\lceil x\rceil$ *denotes the ceiling of* x.

Proof. In the previous theorem we proved that

$$\sum_{s=0}^{b}\left|G(t,s)\right| = \frac{t^{\underline{\mu}}}{\Gamma(\mu+2)}(\mu+b+1-t), \quad t \in \mathbb{N}_{\mu-1}^{\mu+b+1}.$$

Thus

$$\max_{t\in\mathbb{N}_{\mu-1}^{\mu+b+1}} \sum_{s=0}^{b} \big|G(t,s)\big| = \frac{1}{\Gamma(\mu+2)} \max_{t\in\mathbb{N}_{\mu-1}^{\mu+b+1}} t^{\underline{\mu}}(\mu+b+1-t), \quad t\in\mathbb{N}_{\mu-1}^{\mu+b+1}.$$

Let $F(t) = t^{\underline{\mu}}(\mu+b+1-t)$, then observe that $F(t) \geq 0$ for $t \in \mathbb{N}_{\mu-1}^{\mu+b+1}$ with $F(\mu-1) = 0$ and $F(\mu+b+1) = 0$. So F has a nonnegative maximum and to find this maximum we consider,

$$\begin{aligned}
\Delta_t F(t) &= (-1)\ t^{\underline{\mu}} + (\mu)(\mu+b-t)\ t^{\underline{\mu-1}} \\
&= t^{\underline{\mu-1}}\ [(-1)(t+1-\mu) + (\mu)(\mu+b-t)] \\
&= t^{\underline{\mu-1}}\ \left[\mu^2 + b\mu + \mu - t\mu - t - 1\right].
\end{aligned}$$

Now by setting $\Delta_t F(t) = 0$ and using the fact that $t^{\underline{\mu-1}} > 0$, whenever $t \in \mathbb{N}_{\mu-1}^{\mu+b+1}$, we have that

$$\left(\mu^2 + b\mu + \mu - t\mu - t - 1\right) = 0.$$

Using standard calculus arguments, it turns out that $F(t)$ has a maximum at $t = \mu + b - \lceil \dfrac{b+1}{\mu+1} \rceil$. Thus

$$\max_{t\in\mathbb{N}_{\mu-1}^{\mu+b+1}} \sum_{s=0}^{b} \big|G(t,s)\big| = \frac{1}{\Gamma(\mu+2)} \left[\mu + b - \lceil \frac{b+1}{\mu+1} \rceil\right]^{\underline{\mu-1}} \left[1 + \lceil \frac{b+1}{\mu+1} \rceil\right].$$

This completes the proof. □

3.7 Zero-Convergent Solutions

In this section we will prove the existence of a zero tending solution as t goes to ∞ of the forced self-adjoint fractional difference equation (3.1.1) using the Banach Fixed

Point Theorem (Contraction Mapping Theorem). Also an example illustrating this result will be given.

Theorem 3.7.1. *Let* $p : \mathbb{N}_{\mu-1} \to \mathbb{R}$, $f : \mathbb{N}_0 \to \mathbb{R}$, $q : \mathbb{N}_{\mu-1} \to \mathbb{R}$ *and Assume*

(1) $p(t) > 0$ *and* $q(t) \geq 0$, *for all* $t \in \mathbb{N}_{\mu-1}$

(2) $\displaystyle\sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} < \infty$

(3) $\displaystyle\sum_{\tau=0}^{\infty} f(\tau) < \infty$

(4) $\displaystyle\sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \left(\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} q(s+\mu-1) \right) < \infty$

hold; then the forced self-adjoint fractional difference equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + q(t+\mu-1)x(t+\mu-1) = f(t), \quad t \in \mathbb{N}_0 \tag{3.7.1}$$

has a solution x *which satisfies* $\lim\limits_{t\to\infty} x(t) = 0$.

Proof. In order to prove this theorem we will use the Banach Fixed Point Theorem. Since $\displaystyle\sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \left(\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} q(s+\mu-1) \right) < \infty$ we can choose $a \in \mathbb{N}_{\mu}$ such that

$$\alpha := \sum_{\tau=a}^{\infty} \frac{1}{p(\tau)} \left(\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} q(s+\mu-1) \right) < 1.$$

Let ζ be the space of all real valued functions $x : \mathbb{N}_a \to \mathbb{R}$ such that $\lim\limits_{t\to\infty} x(t) = 0$ with norm $\|.\|$ on ζ defined by

$$\|x(t)\| = \max_{t \in \mathbb{N}_a} |x(t)|.$$

Define the operator T on ζ by

$$Tx(t) = \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} [q(s+\mu-1)x(s+\mu-1) - f(s)] \right],$$

for $t \in \mathbb{N}_a$. Now we will show that $T : \zeta \to \zeta$. To do this we will first show that T is a real valued function. Let $x \in \zeta$ be arbitrary but fixed, then this implies that x is a real valued function satisfying $\lim_{t\to\infty} x(t) = 0$. This implies that for some real number $M > 0$, $|x(t)| \leq M$ for all $t \in \mathbb{N}_a$, and since $\sum_{\tau=0}^{\infty} f(\tau)$ converges this implies that for some real number $N > 0$, $f(\tau) \leq N$ for all $t \in \mathbb{N}_{a-\mu}$. Thus

$$\begin{aligned}
Tx(t) =& \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} [x(s+\mu-1)q(s+\mu-1) - f(s)] \right] \\
\leq& \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} [Mq(s+\mu-1) + N] \right] \\
=& M \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} q(s+\mu-1) \right] \\
&+ N \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} \right] \\
<& \infty + \infty = \infty.
\end{aligned}$$

Thus T is well defined on ζ.

Next we will show that $T : \zeta \to \zeta$. We will use an argument similar to above. Let

$x \in \zeta$ is arbitrary. Notice that

$$\begin{aligned}
|Tx(t)| =& \left| \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} [x(s+\mu-1)q(s+\mu-1) - f(s)] \right] \right| \\
\leq& \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} [Mq(s+\mu-1) + N] \right] \\
=& M \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} q(s+\mu-1) \right] \\
&+ N \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} \right].
\end{aligned} \tag{3.7.2}$$

Again, since

$$\sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left(\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} q(s+\mu-1) \right) < \infty,$$

we have that

$$\lim_{t\to\infty} \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left(\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} q(s+\mu-1) \right) = 0,$$

for any real valued function $q(t) \geq 0$ defined on $\mathbb{N}_a$. Thus by applying the limit as $t \to \infty$ on both sides of the (3.7.2) we get that $\lim_{t\to\infty} Tx(t) = 0$. Thus $T : \zeta \to \zeta$. Next we will show that T is a contraction mapping on ζ. Let x, $y \in \zeta$ and $t \in \mathbb{N}_a$ be arbitrary but fixed. Consider

$$\begin{aligned}
|Tx(t)-Ty(t)| =&\left|\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left(\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}[(x-y)q(s+\mu-1)-f(s)+f(s)]\right)\right| \\
=&\left|\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left(\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}(x-y)q(s+\mu-1)\right)\right| \\
\leq&\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left(\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}|x-y|q(s+\mu-1)\right) \\
\leq&\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left(\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}q(s+\mu-1)\right)\|x-y\| \\
\leq&\sum_{\tau=a}^{\infty}\frac{1}{p(\tau)}\left(\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}q(s+\mu-1)\right)\|x-y\| \\
=&\alpha\|x-y\|,
\end{aligned}$$

for $t\in\mathbb{N}_a$. Since $t\in\mathbb{N}_a$ is arbitrary and $\alpha<1$ we conclude that T is a contraction mapping on ζ. Therefore by the Banach fixed point theorem there exists a unique fixed point $z\in\zeta$ such that $T(z)=z$, which implies that for all $t\in\mathbb{N}_a$

$$z(t)=\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}[q(s+\mu-1)z(s+\mu-1)-f(s)]\right].$$

We will now show that the unique fixed z is a solution to the forced self-adjoint fractional difference equation (3.1.1). To show this, consider

$$z(t)=\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}[q(s+\mu-1)z(s+\mu-1)-f(s)]\right].$$

Taking the difference operator Δ on both sides of the preceding equation we get

$$\begin{aligned}
&\Delta z(t)\\
&\quad=\Delta\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}[q(s+\mu-1)x(s+\mu-1)(s-\mu+1)-f(s)]\right]\\
&\quad=-\frac{1}{p(t)}\left[\sum_{s=0}^{t-\mu}\frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}[q(s+\mu-1)z(s-\mu+1)-f(s)]\right].
\end{aligned}$$

Thus

$$\begin{aligned}
p(t)\Delta z(t) &= -\left[\sum_{s=0}^{t-\mu}\frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}[q(s+\mu-1)z(s-\mu+1)-f(s)]\right]\\
(p\Delta z)(t) &= \left[\sum_{s=0}^{t-\mu}\frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}[f(s)-q(s+\mu-1)z(s-\mu+1)]\right]\\
(p\Delta z)(t) &= \Delta_0^{-\mu}[f(.)-q(.+\mu-1)z(.+\mu-1)](t), \quad t\in\mathbb{N}_\mu.
\end{aligned}$$

It follows that

$$\Delta_{\mu-1}^{\mu}\big(p\Delta z)\big)(t)=\Delta_{\mu-1}^{\mu}(\Delta_0^{-\mu}(f(.)-q(.+\mu-1)z(.+\mu-1)))(t), \quad t\in\mathbb{N}_0.$$

Now we use the fractional composition rule on the right hand side of the above equation to get

$$\begin{aligned}
\Delta_{\mu-1}^{\mu}(p\Delta z)(t) &= \Delta_{-1}^{0}[f(t)-q(t+\mu-1)z(t+\mu-1)], \quad t\in\mathbb{N}_0\\
&= f(t)-q(t+\mu-1)z(t+\mu-1), \quad t\in\mathbb{N}_0
\end{aligned}$$

i.e.

$$\Delta_{\mu-1}^{\mu}(p\Delta z)(t)+q(t+\mu-1)z(t+\mu-1)=f(t), \quad t\in\mathbb{N}_0.$$

Thus z satisfies the self-adjoint equation (3.1.1) and is therefore a solution to that equation. Since $z \in \zeta$ we conclude that self-adjoint equation (3.1.1) has a solution z satisfying $\lim_{t\to\infty} z(t) = 0$.
This completes the proof. $\square$

Next we will present an application the above theorem.

Example 3.7.2. *As an example if we let* $f(t) = \frac{1}{(t+1)^2}$ *for* $t \in \mathbb{N}_0$, $q(t) \equiv 1$ *for* $t \in \mathbb{N}_{\mu-1}$, $p(t) = (t+1)^2(t+1)^{\underline{\mu}}$ *for* $t \in \mathbb{N}_{\mu-1}$ *in Theorem* 3.7, *then all the assumptions of the theorem are satisfied as we can notice from the following facts:*

(1) $p(t) > 0$ *and* $q(t) \geq 0$, *for all* $t \in \mathbb{N}_{\mu-1}$

(2) $\displaystyle\sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} = \sum_{\tau=\mu}^{\infty} \frac{1}{(\tau+1)^2(\tau+1)^{\underline{\mu}}} < \infty$

(3) $\displaystyle\sum_{\tau=0}^{\infty} f(\tau) = \sum_{\tau=0}^{\infty} \frac{1}{(\tau+1)^2} < \infty$

(4) $\displaystyle\sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \left(\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} q(s+\mu-1) \right)$
$\displaystyle = \sum_{\tau=\mu}^{\infty} \frac{(\tau)^{\underline{\mu}}}{\Gamma(\mu+1)(\tau+1)^2(\tau+1)^{\underline{\mu}}} < \infty.$

Thus the theorem guarantees there exists a solution to

$$\Delta_{\mu-1}^{\mu}(p\Delta x)(t) + q(t+\mu-1)z(t+\mu-1) = f(t), \quad t \in \mathbb{N}_0$$

that goes to zero as t *goes to infinity.*

Chapter 4

Existence And Uniqueness Of A Positive Solution

4.1 Introduction

In this chapter we will focus on the quantitative and qualitative features of the solutions to the fractional self-adjoint equation mentioned in the previous chapter. Under certain conditions, we will show that the equation has a unique positive solution with unbounded domain. Our approach will involve the Contraction Mapping Theorem using a weighted norm and "Picard" iteration in a discrete time scale setting. Surprisingly, the results which we will discuss in this chapter will not only generalize some results in differential equations and difference equations but they will establish a new foundation for the research for the time scale community. In the second section we will prove and give a foundational theorem and lemma which we will use in future sections. In the third section we will prove one of our main results and give an example. Finally, in the fourth section we will give another important result concerning solutions of a forced equation with a positive horizontal asymptote.

4.2 Preliminary Theorems And Lemmas

We now give conditions under which the forced fractional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + F(t, x(t+\mu-1)) = 0, \quad t \in \mathbb{N}_0$$

has a solution with positive limit as t goes to ∞.

Theorem 4.2.1. *Let $p : \mathbb{N}_{\mu-1} \to (0,\infty)$ and $F : \mathbb{N}_0 \times \mathbb{R} \to [0,\infty)$. Let $M > 0$ and define*

$$\zeta_M = \Big\{ x : \mathbb{N}_{\mu-1} \to [M,\infty) : \Delta x(t) \leq 0, \Delta x(\mu-1) = 0 \Big\}.$$

Suppose for all the functions x defined on $\mathbb{N}_{\mu-1}$, the following series

$$\sum_{\tau=\mu-1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right]$$

is convergent. Then the fractional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + F(t, x(t+\mu-1)) = 0 \tag{4.2.1}$$

has a positive solution $x \in \zeta_M$ such that $\lim_{t\to\infty} x(t) = M$ if and only if the summation equation

$$x(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right] \tag{4.2.2}$$

has a solution x on $\mathbb{N}_{\mu-1}$.

Proof. Suppose the fractional equation

$$\Delta^{\mu}_{\mu-1}(p\Delta x(t)) + F(t, x(t+\mu-1)) = 0 \tag{4.2.3}$$

has a positive solution $x \in \zeta_M$ such that $\lim_{t\to\infty} x(t) = M$. First we let $y(t) = (p\Delta x)(t)$. Then applying the fractional sum operator on both sides of equation (4.2.3) and using the fractional composition rule given in Lemma 2.1.5 we get that

$$\begin{aligned} y(t) &= -\Delta_0^{-\mu} F(t, x(t+\mu-1)) + ct^{\underline{\mu-1}} \\ &= -\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) + ct^{\underline{\mu-1}}. \end{aligned}$$

It follows that

$$\Delta x(t) = -\frac{1}{p(t)}\left[\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\right].$$

Now summing from $\tau = t$ to ∞ we get that

$$M - x(t) = -\sum_{\tau=t}^{\infty} \frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\right].$$

Hence,

$$x(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\right].$$

Thus x is a solution to the summation equation (4.2.2).

On the other hand, if the summation equation given by (4.2.2) has a solution x on $\mathbb{N}_{\mu-1}$, then

$$x(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\right]. \tag{4.2.4}$$

Now by taking the delta difference on both sides of the last equation, we get that

$$\Delta x(t) = -\frac{1}{p(t)}\left[\sum_{s=0}^{t-\mu}\frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right]. \tag{4.2.5}$$

Hence,

$$(p\Delta x)(t) = -\left[\Delta_0^{-\mu}F(\cdot,x(\cdot-\mu+1)\right](t).$$

Taking the fractional difference of both sides of the last equation, we get that

$$\begin{aligned}\Delta_{\mu-1}^{\mu}(p\Delta x)(t) &= -\Delta_{\mu-1}^{\mu}\Delta_0^{-\mu}(F(\cdot,x(\cdot+\mu-1))(t)\\ &= -F(t,x(t+\mu-1)), \quad t\in\mathbb{N}_0.\end{aligned}$$

Which implies that

$$\Delta_{\mu-1}^{\mu}(p\Delta x)(t) + F(t,x(t+\mu-1)) = 0$$

Hence x is a solution of the fractional equation (4.2.1). We also observe that $x(t)\geq M$, since $p(t)>0$ for all $t\in\mathbb{N}_{\mu-1}$ and $F(t,u)\geq 0$ for all $(t,u)\in\mathbb{N}_0\times\mathbb{R}$. Moreover, notice that

$$\Delta x(\mu-1) = -\frac{1}{p(\mu-1)}\left[\sum_{s=0}^{-1}\frac{(\mu-1-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right] = 0$$

by the convention given in Remark 2.1.4. From the expression for $\Delta x(t)$ given by equation (4.2.5), we see that $\Delta x(t)\leq 0$ for all $t\in\mathbb{N}_{\mu-1}$. Thus $x\in\zeta_M$. Furthermore, since the series

$$\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right]$$

is convergent it follows from equation (4.2.4) that $\lim_{t\to\infty} x(t) = M$.

This completes the proof. $\square$

Remark 4.2.2. *If $\mu = 1$(non-fractional case) in the preceding theorem, the fractional summation equation reduces to the summation equation*

$$x(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-1} F(s, x(s)) \right].$$

Lemma 4.2.3. *Assume $M > 0$ and*

$$\zeta_M = \Big\{ x : \mathbb{N}_{\mu-1} \to [M, \infty) : \Delta x(t) \le 0, \Delta x(\mu - 1) = 0 \Big\},$$

where $F : \mathbb{N}_0 \times \mathbb{R} \to [0, \infty)$. Assume $p : \mathbb{N}_{\mu-1} \to (0, \infty)$ satisfies $\sum_{\tau=\mu-1}^{\infty} \ln(1 + \frac{1}{p(\tau)}) < \infty$ and define $d : \zeta_M \times \zeta_M \to [0, \infty)$ by $d(x, y) = \sup_{t\in\mathbb{N}_{\mu-1}} \frac{|x(t) - y(t)|}{w(t)}$, where $w(t) = e^{-[\sum_{\tau=\mu-1}^{t} ln(1+\frac{1}{p(\tau)})]}$. Note that $0 < L := \lim_{t\to\infty} w(t) \le 1$. Then the pair (ζ_M, d) is a complete metric space.

It is straight forward to show that (ζ_M, d) is a complete metric space.

Next we prove the existence and uniqueness of the solution of the fractional equation (4.2.1) tending to M as t goes to ∞ by using the Contraction Mapping Theorem.

4.3 Main Theorem And Example

Theorem 4.3.1. *Assume $F : N_0 \times \mathbb{R} \to [0, \infty)$ satisfies a uniform Lipschitz condition with respect to the second variable, i.e. if $u, v \in \mathbb{R}$ and $t \in \mathbb{N}_0$ then $\big|F(t, u) - F(t, v)\big| \le$*

$K\big|u-v\big|$ and assume $p:\mathbb{N}_{\mu-1}\to(0,\infty)$ and let (ζ_M,d) be the complete metric space as defined in Lemma 4.2.2.

If the following hypotheses $(H1)$ and $(H2)$

H1) The series $\displaystyle\sum_{\tau=\mu}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right]$ is convergent for each $x\in\zeta_M$,

H2) $\frac{1}{\Gamma(\mu+1)}\frac{K}{L}\left[\displaystyle\sum_{\tau=\mu}^{\infty}\frac{\tau^{\underline{\mu}}}{p(t)}\right]=\alpha<1$

are satisfied. Then there exist a unique positive solution of the fractional equation (4.2.1). Moreover $\lim\limits_{t\to\infty}x(t)=M$.

Proof. Let (ξ_M,d) be the complete metric space as defined in Lemma 4.2.2. Consider the map T on ζ_M defined by

$$Tx(t)=M+\sum_{\tau=t}^{\infty}\frac{1}{p(\tau)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\right].$$

First we will show that $T:\zeta_M\to\zeta_M$. First note that the above expression for $Tx(t)$ guarantees that $Tx(t)\geq M$ since $F(t,u)\geq 0$ for all $(t,u)\in N_0\times\mathbb{R}$ and $p(t)>0$ for all $t\in\mathbb{N}_{\mu-1}$. Next note that

$$\Delta Tx(t)=-\frac{1}{p(t)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}F(s,x(s+\mu-1))}{\Gamma(\mu)}\right]\leq 0.$$

Also it can be easily verified that $(\Delta Tx)(\mu-1)=0$ by our convention as mentioned in Remark 2.1.4. Thus $T:\xi_M\to\xi_M$. Moreover we will show that T is a contraction mapping on ζ_M. Let $t \in \mathbb{N}_{\mu-1}$ be arbitrary, then

$$\begin{aligned}
\left|\frac{Tx(t) - Ty(t)}{w(t)}\right| \qquad &= \tfrac{1}{w(t)} \sum_{\tau=t}^{\infty} \frac{1}{p(t)} \Big[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} |F(s, x(s+\mu-1)) \\
&\qquad\qquad -F(s, y(s+\mu-1)| \Big]. \\
&\leq \frac{K}{w(t)} \sum_{\tau=t}^{\infty} \frac{1}{p(t)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} w(s+\mu-1) \right] d(x,y).
\end{aligned}$$

$$\begin{aligned}
d(T(x), T(y)) &\leq \frac{K}{L} \sum_{\tau=t}^{\infty} \frac{1}{p(t)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} w(s+\mu-1) \right] d(x,y) \\
&\leq \frac{K}{L} \sum_{\tau=t}^{\infty} \frac{1}{p(t)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} \right] d(x,y) \\
&= \frac{K}{L} \sum_{\tau=t}^{\infty} \frac{1}{p(t)} \left[\frac{\tau^{\underline{\mu}}}{\Gamma(\mu+1)} \right] d(x,y) \\
&\leq \frac{1}{\Gamma(\mu+1)} \frac{K}{L} \left[\sum_{\tau=\mu-1}^{\infty} \frac{\tau^{\underline{\mu}}}{p(t)} \right] d(x,y) \\
&= \alpha \quad d(x,y).
\end{aligned}$$

Since $\alpha < 1$, T is a contraction mapping on ζ_M. Then it follows from the Contraction Mapping Theorem that there exists a unique fixed point x of T in ζ_M such that $T(x) = x$ and therefore x is the unique positive solution to the summation equation (4.2.2). Hence, by Theorem 4.2.1, x is the unique positive solution to the fractional equation (4.2.1). Moreover, since the series

$$\sum_{\tau=\mu-1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x_i(s+\mu-1)) \right]$$

is convergent,

$$\lim_{t\to\infty} x(t) = M + \lim_{t\to\infty} \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x_i(s+\mu-1)) \right]$$
$$= M.$$

This completes the proof. □

Next we will present an example to illustrate the above theorem.

Example 4.3.2. *As an example let us choose* $F(t,x) = K|x| - M$ *for* $t \in \mathbb{N}_0$. *Then* F *clearly satisfies a uniform Lipschitz condition with respect to the second variable* x *with Lipscitz constant* K. *Let* $p : \mathbb{N}_{\mu-1} \to (0,\infty)$ *be defined by*

$$p(t) = \begin{cases} 4, & t = \mu - 1 \\ \frac{K2^{(t-\mu+3)} t^{\underline{\mu}}}{\Gamma(\mu+1)L}, & t \in \mathbb{N}_\mu \end{cases} \tag{4.3.1}$$

in Theorem 4.3.1, *then all the hypotheses of Theorem* 4.3.1 *are satisfied as verified in the following arguments. First notice that*

$$\ln\left(1 + \frac{1}{p(t)}\right) \le \frac{1}{p(t)},$$

for $t \in \mathbb{N}_\mu$. *Therefore,*

$$\begin{aligned}\sum_{\tau=\mu-1}^{\infty} \frac{1}{p(t)} &= \frac{1}{4} + \sum_{\tau=\mu}^{\infty} \frac{1}{p(t)} \\ &= \frac{1}{4} + \frac{L}{K} \sum_{\tau=\mu}^{\infty} \frac{\Gamma(\mu+1)}{2^{(t-\mu+3)} t^{\underline{\mu}}} \\ &\leq \frac{1}{4} + \frac{L}{K} \sum_{\tau=\mu}^{\infty} \frac{1}{2^{(t-\mu+3)}} \\ &= \frac{1}{4} + \frac{L}{4K} \\ &< \infty.\end{aligned}$$

Hence, by the comparison thorem,

$$\sum_{\tau=\mu-1}^{\infty} \ln\left(1 + \frac{1}{p(t)}\right) < \infty.$$

Next we will show that $(H1)$ *is satisfied. Let* $x \in \zeta_M$ *be arbitrary but fixed. Consider the following series*

$$\sum_{\tau=\mu-1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right].$$

The following calculations show $(H1)$ *is satisfied for each such* $x \in \zeta_M$. *Note that*

$$
\begin{aligned}
&\sum_{\tau=\mu-1}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right] \\
&= \sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right] \\
&\le K|x(\mu-1)| \sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} \right] \\
&= \frac{1}{4} + Kx(\mu-1) \sum_{\tau=\mu}^{\infty} \frac{1}{p(\tau)} \frac{\tau^{\underline{\mu}}}{\Gamma(\mu+1)} \\
&= \frac{1}{4} + Kx(\mu-1)(\frac{1}{4}) \\
&< \infty.
\end{aligned}
$$

Next we will show that the second hypothesis $(H2)$ *also is satisfied. Notice that*

$$
\begin{aligned}
\frac{1}{\Gamma(\mu+1)} \frac{K}{L} \left[\sum_{\tau=\mu-1}^{\infty} \frac{\tau^{\underline{\mu}}}{p(t)} \right] &= \frac{1}{\Gamma(\mu+1)} \left[\sum_{\tau=\mu}^{\infty} \frac{\tau^{\underline{\mu}} \Gamma(\mu+1)}{\tau^{\underline{\mu}} 2^{(\tau-\mu+3)}} \right] \\
&= \sum_{\tau=\mu}^{\infty} \frac{1}{2^{(\tau-\mu+3)}} \\
&= \frac{1}{4} \\
&< 1.
\end{aligned}
$$

Thus the second hypothesis $(H2)$ *is also satisfied. Hence, Theorem* 4.3.1 *implies that with the above defined functions* F, p *with their respective domains*

$$
\Delta^{\mu}_{\mu-1}(p\Delta x(t)) + F(t, x(t+\mu-1)) = 0 \tag{4.3.2}
$$

has a unique positive solution that converges to M *as* t *goes to* ∞.

4.4 Preliminary Theorems

In this section, we first show the relationship between the existence of solutions of

$$\Delta^{\mu}_{\mu-1}(p\Delta x)(t) + F(t, x(t+\mu-1)) = 0$$

and the existence of solutions of a fractional summation equation in the following theorem.

Theorem 4.4.1. *Assume* $p : \mathbb{N}_{\mu-1} \to (0,\infty)$ *and* $F : \mathbb{N}_0 \times \mathbb{R} \to [0,\infty)$. *Let* $M > 0$ *and define*

$$\zeta_M = \Big\{ x : \mathbb{N}_{\mu-1} \to [M,\infty) : \Delta x(t) \leq 0, \Delta x(\mu-1) = 0 \Big\}.$$

Let $P(\tau,t) := \sum_{u=t}^{\tau} \frac{1}{p(u)}$, *where* $t \in \mathbb{N}_{\mu-1}$. *Suppose for all the functions* x *defined on* $\mathbb{N}_{\mu-1}$, *the following two series*

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1) \left[\sum_{s=0}^{\tau-\mu+1} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right], \tag{4.4.1}$$

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1) \big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \big]$$

are convergent and moreover, the later series satisfies the condition that

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1) \big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \big] \leq 0. \tag{4.4.2}$$

Then the fractional equation

$$\Delta_{\mu-1}^{\mu}(p\Delta x)(t) + F(t, x(t+\mu-1)) = 0$$

has a positive solution $x \in \zeta_M$ *such that* $\lim_{t\to\infty} x(t) = M$ *if and only if the summation equation*

$$x(t) = M - \sum_{\tau=t}^{\infty} P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\Big] \tag{4.4.3}$$

has a solution $x \in \mathbb{N}_{\mu-1}$.

Proof. Suppose the fractional equation

$$\Delta_{\mu-1}^{\mu}(p\Delta x(t)) + F(t, x(t+\mu-1)) = 0 \tag{4.4.4}$$

has a positive solution $x \in \zeta_M$ such that $\lim_{t\to\infty} x(t) = M$. We let $y(t) = (p\Delta x)(t)$ in equation (4.4.4). Then by applying the fractional sum operator on the both sides of equation (4.4.4) and using the fractional composition rule given in Lemma 2.1.5 we get that

$$\begin{aligned} y(t) &= -\Delta_0^{-\mu} F(t, x(t+\mu-1)) + ct^{\underline{\mu-1}} \\ &= -\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) + ct^{\underline{\mu-1}}. \end{aligned}$$

This implies that

$$\Delta x(t) = -\frac{1}{p(t)}\left[\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\right].$$

Now summing from $\tau = t$ to ∞ we get that

$$M - x(t) = -\sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right].$$

Therefore,

$$x(t) = M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right].$$

Now by using the definition of $P(\tau, t)$ as defined in the statement of the theorem we can rewrite the preceding equation as

$$x(t) = M + \sum_{\tau=t}^{\infty} \left[\Delta_\tau (P(\tau-1, t)) \right] \Big[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \Big],$$

and then by applying the summation by parts formula and the convergence of the series in (4.4.1) we get that

$$\begin{aligned} x(t) =& M + P(\tau-1, t) \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \right] \Big|_{\tau=t}^{\infty} \\ & - \sum_{\tau=t}^{\infty} P(\tau, t) \Big[\sum_{s=0}^{\tau-\mu} \frac{(\mu-1)(\tau - \sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \\ & + F(\tau-\mu+1, x(\tau)) \Big] \\ & = M - \sum_{\tau=t}^{\infty} P(\tau, t) \Big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau - \sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1)) \Big]. \end{aligned}$$

On the other hand, if the summation equation

$$y(t) = M - \sum_{\tau=t}^{\infty} P(\tau, t) \Big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau - \sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, y(s+\mu-1)) \Big]$$

has a solution x on $\mathbb{N}_{\mu-1}$, then

$$x(t) = M - \sum_{\tau=t}^{\infty} P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s,x(s+\mu-1))\Big].$$

Note that since the series

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s,x(s+\mu-1)\right]$$

is convergent and $\big(\sum_{u=t}^{\tau-1} \frac{1}{p(u)}\big)|_{\tau=t} = 0$, we have that

$$\Big[\big(\sum_{u=t}^{\tau-1} \frac{1}{p(u)}\big) \sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s,x(s+\mu-1))\Big]|_{\tau=t}^{\infty} = 0$$

and hence the expression for $x(t)$ as mentioned above can be rewritten as

$$\begin{aligned}
x(t) &= M + \big(\sum_{u=t}^{\tau-1} \frac{1}{p(u)}\big) \sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s,x(s+\mu-1))|_{\tau=t}^{\infty} \\
&\quad - \sum_{\tau=t}^{\infty} P(\tau,t)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s,x(s+\mu-1))\right] \\
&= M + \sum_{\tau=t}^{\infty} \big[\Delta_\tau P(\tau-1,t)\big] \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s,x(s+\mu-1))\right] \\
&= M + \sum_{\tau=t}^{\infty} \frac{1}{p(\tau)} \left[\sum_{s=0}^{\tau-\mu} \frac{(\tau-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s,x(s+\mu-1))\right].
\end{aligned}$$

Now by taking the delta difference of both sides of the last equation, we get that

$$\Delta x(t) = -\frac{1}{p(t)}\left[\sum_{s=0}^{t-\mu} \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s,x(s+\mu-1))\right]. \tag{4.4.5}$$

Hence,

$$(p\Delta x)(t) = -\left[\Delta_0^{-\mu} F(.,x(.-\mu+1)\right](t).$$

Taking the fractional difference of both sides of the last equation, we get that

$$\begin{aligned}\Delta_{\mu-1}^{\mu}(p\Delta x)(t) &= -\Delta_{\mu-1}^{\mu}\Delta_0^{-\mu}(F(.,x(.+\mu-1))(t)\\ &= -F(t,x(t+\mu-1)), \quad t\in\mathbb{N}_0.\end{aligned}$$

Therefore,

$$\Delta_{\mu-1}^{\mu}(p\Delta x)(t) + F(t,x(t+\mu-1)) = 0, \quad t\in N_0.$$

Hence, x is the solution of equation (4.4.4). Moreover it is not hard to see from the expression for Δx as given by equation (4.4.5), that $\Delta x(t) \leq 0$ and $\Delta x(\mu-1) = 0$. Also, since the series

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\Big[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\Big]$$

is convergent we have that $\lim_{t\to\infty} x(t) = M$. Furthermore, since

$$\sum_{\tau=\mu-1}^{\infty} P(\tau,\mu-1)\Big[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)}F(s,x(s+\mu-1))\Big] \leq 0, \tag{4.4.6}$$

we have that

$$M - \sum_{\tau=t}^{\infty} P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1}\frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)}F(s,y(s+\mu-1))\Big] \geq M,$$

i.e. $x(t) \geq M$. Hence, we conclude that the fractional equation (4.4.4) has a positive solution $x \in \zeta_M$. $\square$

4.5 Another Main Result

Next we will prove the following theorem which is an application of the Contraction Mapping Theorem.

Theorem 4.5.1. *Assume $F : N_0 \times \mathbb{R} \to [0, \infty)$ satisfies a uniform Lipschitz condition with respect to the second variable, i.e. if $u, v \in \mathbb{R}$ and $t \in \mathbb{N}_0$ then $\left|F(t,u) - F(t,v)\right| \leq K\left|u - v\right|$ and assume $p : \mathbb{N}_{\mu-1} \to (0, \infty)$, $P(\tau, t) := \sum_{u=t}^{\tau} \frac{1}{p(u)}$ and let (ζ_M, d) be the complete metric space as defined in Lemma 4.2.2. If the following hypotheses $(H1)$, $(H2)$, $(H3)$ and $(H4)$ are satisfied.*

H1) The series $\sum_{\tau=\mu-1}^{\infty} P(\tau, \mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\tau - \sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} F(s, x(s+\mu-1)\right]$ is convergent for each $x \in \zeta_M$.

H2) The series $\sum_{\tau=\mu-1}^{\infty} P(\tau, \mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau - \sigma(s))^{\underline{\mu-2}} F(s, x(s+\mu-1)}{\Gamma(\mu)}\right]$ is convergent for each $x \in \zeta_M$.

H3) $\sum_{\tau=\mu-1}^{\infty} P(\tau, \mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau - \sigma(s))^{\underline{\mu-2}} F(s, x(s+\mu-1)}{\Gamma(\mu)}\right] \leq 0$ for each $x \in \zeta_M$.

H4) $\frac{K}{L} \sum_{\tau=\mu-1}^{\infty} P(\tau, \mu-1)\left[\sum_{s=0}^{\tau-\mu+1} \frac{|\mu-1||(\tau - \sigma(s))^{\underline{\mu-2}}| w(s+\mu-1)}{\Gamma(\mu)}\right] = \alpha < 1$, where in hypothesis $(H4)$ we defined $L =: \lim_{t\to\infty} w(t) > 0$ as mentioned in Lemma 4.2.2. Then there exist a unique positive solution x of (4.4.4) such that $\lim_{t\to\infty} x(t) = M$.

Proof. Let (ζ_M, d) be the complete metric space as defined in Lemma 4.2.2 and consider the map T on ζ_M defined by

$$Tx(t) = M - \sum_{\tau=t}^{\infty} P(\tau, t)\Big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau - \sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1)), x(\tau))\Big].$$

First, we will show that $T : \zeta_M \to \zeta_M$. Let $x \in \zeta_M$ be arbitrary. Then similar to the derivation of (4.4.5) in Theorem 4.4.1, we have

$$\Delta Tx(t) = -\frac{1}{p(t)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}F(s,x(s+\mu-1))}{\Gamma(\mu)}\right].$$

Also since $p(t) > 0$ for all $t \in \mathbb{N}_{\mu-1}$ and $F(t,u) \geq 0$ for all $(t,u) \in \mathbb{N}_0 \times \mathbb{R}$, we have that

$$\Delta Tx(t) = -\frac{1}{p(t)}\left[\sum_{s=0}^{\tau-\mu}\frac{(\tau-\sigma(s))^{\underline{\mu-1}}F(s,x(s+\mu-1))}{\Gamma(\mu)}\right] \leq 0.$$

Also it is not hard to see that

$$\Delta Tx(\mu-1) = 0.$$

Moreover, by using hypotheses $(H2)$ and $(H3)$ we conclude that $T(x)(t) \geq M$ for $t \in \mathbb{N}_{\mu-1}$. Hence, we proved that $T : \zeta_M \to \zeta_M$. Next we will show that T is a contraction mapping on ζ_M. Let $t \in \mathbb{N}_{\mu-1}$ be arbitrary. Then notice that

$$\begin{aligned}
\left|\frac{Tx(t)-Ty(t)}{w(t)}\right| &\leq \frac{1}{w(t)}\sum_{\tau=t}^{\infty}P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1}\frac{|(\mu-1)||(\tau-\sigma(s))^{\underline{\mu-2}}|}{\Gamma(\mu)}|F(s,x(s+\mu-1)) \\
&- F(s,y(s+\mu-1)|\Big]. \\
&\leq \frac{K}{w(t)}\left[\sum_{\tau=t}^{\infty}P(\tau,t)\left(\sum_{s=0}^{\tau-\mu+1}\frac{|(\mu-1)||(\tau-\sigma(s))^{\underline{\mu-2}}|}{\Gamma(\mu)}w(s+\mu-1)\right)\right]d(x,y).
\end{aligned}$$

Thus

$$\begin{aligned}
&d(T(x),T(y)) \\
&\leq K/L\left[\sum_{\tau=\mu-1}^{\infty}P(\tau,\mu-1)\left(\sum_{s=0}^{\tau-\mu+1}\frac{|(\mu-1)||(\tau-\sigma(s))^{\underline{\mu-2}}|}{\Gamma(\mu)}w(s+\mu-1)\right)\right]d(x,y) \\
&= \alpha d(x,y).
\end{aligned}$$

Since $\alpha < 1$, by the hypothesis $(H4)$, T is a contraction mapping on ζ_M. Hence, by the Contraction Mapping Theorem there exist a unique positive fixed point x of T in ζ_M such that $T(x) = x$. Therefore, Theorem 4.4.1 gurantees that x is the unique positive solution to the summation equation

$$x(t) = M - \sum_{\tau=t}^{\infty} P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1)), x(\tau))\Big].$$

Moreover, by using hypothesis $(H2)$, we observe

$$\begin{aligned}\lim_{t\to\infty} x(t) &= M - \lim_{t\to\infty} \sum_{\tau=t}^{\infty} P(\tau,t)\Big[\sum_{s=0}^{\tau-\mu+1} \frac{(\mu-1)(\tau-\sigma(s))^{\underline{\mu-2}}}{\Gamma(\mu)} F(s, x(s+\mu-1))\Big] \\ &= M.\end{aligned}$$

This completes the proof. $\square$

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