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Abstract
An analysis is done of a relativistic paradox posed in the Feynman Lectures of Physics involving two interacting charges. The physical system presented is compared with similar systems that also lead to relativistic paradoxes. The momentum conservation problem for these systems is presented. The relation between the presented analysis and the ongoing debates on momentum conservation in the Aharonov-Bohm problem is discussed.

Keywords: Quantum mechanics, Aharonov-Bohm, Electromagnetic momentum, Vector potential

1 Introduction

It is well known that applying the laws of electromagnetism to the classical equations of motion leads to paradoxical situations. The appearance of paradoxes is due to the use of relativistic Maxwell's equations and Lorentz's force law combined with classical Newton's laws of motion. Even in the low velocity limit, this mismatch makes it hard to explicitly show the law of momentum conservation [1, 2]. The appearance of momentum carried by the electromagnetic fields turns out to be essential for this discussion.

Many paradoxes exist that belong in this category [3–9]. The archetypical paradox that can be posed with a minimum amount of physics knowledge has been discussed in the Feynman Lectures of Physics [3]. A charged particle is instantaneously found on the x-axis with its velocity pointing along the x-axis. Another charged particle is also found on the x-axis, but with its velocity pointing along the y-axis (Figure 1a).

Using the familiar electric and magnetic fields of the point charges in the low velocity limit, momentum appears not to be conserved. The electric forces on both particles form a Newton action-reaction pair and seem to present no problem. The magnetic forces, on the other hand, appear peculiar. One particle (q_1) generates a magnetic field at the position of the other (q_2), perpendicular to its motion and exerts a force, while the other does not. The magnetic force changes the momentum of the two-particle system in the y-direction. Feynman et al. suggest correctly that this apparent violation of Newton's third law can be resolved by including the electromagnetic momentum [10]. This resolution makes use of the non-relativistic fields. We show below that the use of the relativistic fields in the low velocity limit gives a field momentum change that is not solely in the y-direction. This relativistic extension of Feynman's argumentation exemplifies that electric forces of the same magnitude as the magnetic forces restore the momentum conservation.

The analysis of two interacting particles can be extended by the superposition principle to systems of many interacting particles. A particularly interesting phenomenon that occurs in a system of this type is the Aharonov-Bohm (AB) effect [11]. This effect has been fully analyzed quantum mechanically and observed for a charge passing a magnetic flux tube [12–14]. Recently, Caprez et al. observed the absence of force for a charge passing a solenoid [15], confirming an important feature of the AB-effect. It is exactly this decades-old prediction that classical forces are absent, and the observation of this quantum mechanical effect, that made the AB-effect famous.

Surprisingly, the absence of force and the issue of momentum conservation are still debated. Although experiments have been proposed, no experimental evidence other than the recent work by Caprez et al. appears to settle the issues concerning classical force. Also, a complete relativistic classical analysis of the forces on a charge passing a solenoid has never been done. Simplified systems, where the charge is at rest [16], or the solenoid is replaced by a current loop [17], are analyzed instead. The present work is considered to be a step towards a full relativistic treatment for the moving charge-solenoid system. Such a treatment, together with further experiments, will complement our understanding of the classical part of the AB-effect. As Aharonov and Rohrlich point out [16], this type of classical paradox is crucial to the understanding of the entirely quantum interactions of the AB-effect.

2 Relation to Previous Work

An excellent and extensive study (before Feynman's paradox was posed and before the discovery of the Aharonov-Bohm effect) that deals with action and reaction between uniformly moving point charges is that of Page and Adams in 1945 [18]. Their work was done in terms of the electromagnetic fields to second order in $\frac{1}{c}$. In this paper an alternate approach based on vector potentials is given, that is valid to all orders in $\frac{1}{c}$, and the approach is applied to Feynman's paradox. To gain insight into the issues of field momentum and Newton pairs, it is interesting to compare the Feynman case **a** (Figure 1a) to the work by Scanio [5] and Jefimenko [6] on related but simpler physical systems. In his work (case **b**), Scanio considered a charged particle that resides on the x-axis, while another charged particle is also found on the x-axis with its

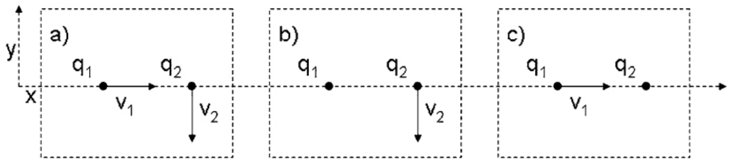

Figure 1 *Three related electrodynamics paradoxes*. In case (**a**), a particle with charge q_1 moves with velocity v_1 along the *x*-axis. It interacts with another charged particle q_2 that moves in the negative y-direction with velocity v_2. We refer to case (**a**) as "Feynman's paradox", which relies on an imbalance in the magnetic forces. Cases (**b**) and (**c**) are physically similar, but one of the charged particles is at rest and their associated paradoxes rely on an imbalance in the electric forces

velocity pointing along the *y*-axis (Figure 1b). Jefimenko (case c) considered a charged particle that again resides on the *x*-axis, while another charged particle is found on the *x*-axis, with its velocity pointing along the *x*-axis (Figure 1c). In both papers the apparent paradoxes, which are special cases of the work of Page and Adams, were clearly pointed out and resolved. The paradoxes in these two latter scenarios require the use of relativistically correct expressions for the electric field of a moving charge. Because the relativistic expression is needed to pose these two paradoxes, they are not as straightforward to state as Feynman's paradox.

The similarities in the three physical systems (Figure 1) are striking. The paradoxes in scenarios **b** and **c** are made clear by considering the relativistic electric field [19]. In the lab frame, the field strength of a moving point charge is increased by a factor of γ orthogonal to its motion as compared to its rest frame. Additionally, the field parallel to its motion is reduced by γ^2 as compared to that at rest (see equation (11.152) in [20, 21]). (The distance in the lab frame is multiplied by γ to compensate for the Lorentz contraction. For the electric force, this results in a factor of γ^2.) In paradox **b**, the moving charge exerts an electric force on the stationary charge that is stronger by γ as compared to the electric force from the charge at rest on the moving charge. No magnetic forces are present. In paradox **c**, the moving charge exerts an electric force weaker by γ^2 than the electric force exerted by the charge at rest on the moving charge. Again, no magnetic forces are present. For both scenarios, Newton's third law appears violated. Comparing Figures 1a, b and c it is clear that the electric field imbalances are also present in Feynman's case (Figure 1a).

In this paper it is shown that it is exactly this electric field imbalance, which in addition to the relativistic electromagnetic momentum, restores the total momentum conservation. Pointing out the relation between the three scenarios, and the complete answer to "Feynman's paradox", serves to illustrate that relativistic effects must be considered to properly describe even such simple systems as two point charges. In Sections 3 and 4, the three paradoxes are stated. In Sections 5 and 6, a consistent resolution is provided for all three. In Section 7, the non-relativistic approximation and the connection to the problem as stated in the Feynman Lectures of Physics is given. Finally, in Section 8, the relation to the AB-effect is discussed.

3 Statement of the "Feynman Paradox"

The classical electric field of particle 1 (Figure 1a) at the position of particle 2 is given by

$$\vec{E}_1(x_2) = \frac{kq_1}{(x_2 - x_1)^2}\hat{x}, \quad (1)$$

while the electric field of particle 2 at the position of particle 1 is given by

$$\vec{E}_2(x_1) = \frac{-kq_2}{(x_2 - x_1)^2}\hat{x}, \quad (2)$$

Note that these fields are only correct in the low velocity limit. The electric forces $q_1\vec{E}_2$ and $q_2\vec{E}_1$ form a Newton pair. The magnetic field of particle 1 (Figure 1a) at the position of particle 2 is zero, while the magnetic field of particle 2 at the position of particle 1 is given by

$$\vec{B}_2(x_1) = \vec{v}_2 \times \vec{E}_2(x_1) = \frac{-vkq_2}{c^2(x_2 - x_1)^2}\hat{z}, \quad (3)$$

where the z-axis points out of the page. The magnetic force

$$\vec{F} = q_1 v_1 \times \vec{B}_2 \quad (4)$$

has no partner to complete a Newton pair, and thus momentum conservation appears to be violated.

4 Statement of Paradoxes (b) and (c)

For case **b** (Figure 1b), the relativistic electric field of particle 1 at the position of particle 2 is given by

$$\vec{E}_1(x_2) = \frac{kq_1}{(x_2 - x_1)^2}\hat{x}, \quad (5)$$

while the electric field of particle 2 at the position of particle 1 is given by

$$\vec{E}_2(x_1) = \frac{-kq_2\gamma_2}{(x_2 - x_1)^2}\hat{x}, \quad (6)$$

where $\gamma_{1,2} = \frac{1}{\sqrt{1-\frac{v_{1,2}^2}{c^2}}}$. These two forces do not form a Newton pair. Because the equation of motion

$$\frac{d\vec{p}}{dt} = \vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B}) \quad (7)$$

(where all vectors and the time are defined in the laboratory frame) holds relativistically, momentum conservation appears to be violated. For case **c** (Figure 1c), the relativistic electric field of particle 1 at the position of particle 2 is given by

$$\vec{E}_1(x_2) = \frac{kq_1}{\gamma_1^2(x_2 - x_1)^2}\hat{x}, \quad (8)$$

while the electric field of particle 2 at the position of particle 1 is given by

$$\vec{E}_2(x_1) = \frac{-kq_2\gamma_2}{(x_2 - x_1)^2}\hat{x}. \quad (9)$$

And again the two forces do not form a Newton pair.

5 Resolution of Paradoxes (b) and (c)

To resolve paradoxes **b** and **c**, the central idea is to include the electromagnetic field momentum. Relativistic momentum conservation is obtained by demonstrating that [2]

$$\frac{d(\vec{P}mech + \vec{P}field)}{dt} = 0, \quad (10)$$

where the change in mechanical momentum is given by the relativistic equation

$$\frac{d\vec{P}mech}{dt} = \vec{F}_L. \quad (11)$$

As pointed out in a comment by Labarthe [22], working in the Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$) requires a minimum of computation to calculate the electromagnetic momentum. The electromagnetic momentum in this gauge is given by

$$\vec{P}field = q_1\vec{A}_{C2} + q_2\vec{A}_{C1}. \quad (12)$$

The vector potential, $\vec{A}_C$ at a field location $\vec{r}$ due to a particle with charge q at the origin is given by [22]

$$\vec{A}_C = \frac{q\vec{v}}{4\pi\varepsilon_0 c^2 R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}} + \frac{q}{4\pi\varepsilon_0 v R\sin\theta}\left(\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}-1\right)\hat{e}_\theta, \quad (13)$$

where θ is the angle between the $\vec{r}$ and $\vec{v}$, and $R = |\vec{r}|$.

The vector potential can be obtained by evaluating

$$A_C(t) = \int_{-\infty}^{t}\left[\hat{E}(\tau) + \vec{\nabla}\varphi_C(\tau)\right]dt \quad (14)$$

where

$$\vec{E} = \frac{kq\left(1-\frac{v^2}{c^2}\right)}{r^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\hat{r} \quad (15)$$

and

$$\varphi_C = \frac{kq}{R}. \quad (16)$$

As emphasized by Labarthe, in the more commonly used Lorentz gauge ($\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$), the vector potential is a simpler expression but the field momentum involves an integral over all space,

$$\vec{P}_{em} = \frac{-1}{c^2}\int\hat{E}\frac{\partial\varphi}{\partial t}dV + q\hat{A}. \quad (17)$$

The analysis of paradox c has been performed both in the Coulomb gauge [22] and in the Lorentz gauge [6]. Additionally, Scanio has analyzed paradox **b** using a fully gauge-independent approach in terms of the electric and magnetic field [5].

In the present notation, the resolution of case **b** can be given as follows. Working in the Coulomb gauge, the change of the field momentum is given by $q\frac{dA_C}{dt}$. Using

$$|\vec{x}_2 - \vec{x}_1| = \left\{(x_1(t=0) - x_2(t=0))^2 + (v_2 t)^2\right\}^{1/2} \quad (18)$$

and

$$\hat{e}_\theta(t) = \left(\sin\frac{v_2}{r_0}t\right)\hat{x} - \cos\left(\frac{v_2}{r_0}t\right)\hat{y}, \quad (19)$$

the implicit time dependence in (13) is rewritten explicitly, and thus the total time derivative becomes a partial time derivative. Taking the derivative leads to

$$q\frac{dA_C}{dt} = q\frac{\partial A_C(t)}{\partial t}\bigg|_{t=0} = \left[\frac{kq_2q_1}{(x_2-x_1)^2}(\gamma_2-1)\right]\hat{x}. \quad (20)$$

This restores the momentum conservation of paradox b. Summarizing the analysis of Jefimenko for paradox c, the field momentum change is given by

$$q\frac{dA_C}{dt} = q\frac{\partial A_C(t)}{\partial t}\bigg|_{t=0} = \left[\frac{kq_1q_2v_1^2}{c^2(x_2-x_1)^2}\right]\hat{x}. \quad (21)$$

This restores momentum conservation, where use has been made of

$$|\hat{x}_2 - \hat{x}_1| = x_1(t=0) + v_1 t - x_2(t=0), \quad (22)$$

and the time independence of $\hat{e}_\theta$

6 Resolution of Feynman's Paradox

Following Feynman's suggestion, the momentum in the field needs to be computed. In the Coulomb gauge, the change of the field momentum is given by $q\frac{dA_C}{dt}$. With

$$|\vec{x}_2 - \vec{x}_1| = \left\{(x_1(t=0) + v_1 t - x_2(t=0))^2 + (v_2 t)^2\right\}^{1/2} \quad (23)$$

$\Delta(q_1A_2) + \Delta(q_2A_1)$ in the resolution of "Feynman's paradox," highlighting that the sum of the field momenta for the first two examples does not add to that for the last example

and

$$\hat{e}_\theta(t) = \sin\left(\frac{v_2}{r_0}t\right)\hat{x} - \cos\left(\frac{v_2}{r_0}t\right)\hat{y}, \quad (24)$$

the field momentum is obtained:

$$q\frac{dA_C}{dt} = q\frac{dA_C(t)}{\partial t}\bigg|_{t=0}$$
$$= \left[\frac{kq_1q_2v_1^2}{c^2(x_2-x_1)^2} + \frac{kq_2q_1}{(x_2-x_1)^2}(\gamma_2-1)\right]\hat{x} + \frac{-kq_2q_1v_2v_1\gamma_2}{c^2(x_2-x_1)^2}\hat{y} \quad (25)$$

The momentum in the field (right hand side of (25)) has three terms. The first term is identical to the one that resolved paradox **c** ((21), Figure 2 top). The second term is identical to the one that resolved paradox **b** ((20), Figure 2 middle). The third term is equal in magnitude and directly opposed to the magnetic force of particle 2 on 1 (Figure 2 bottom)

$$F_B = \frac{v_1v_2kq_2q_1\gamma_2}{c^2(x_2-x_1)^2}\hat{y} \quad (26)$$

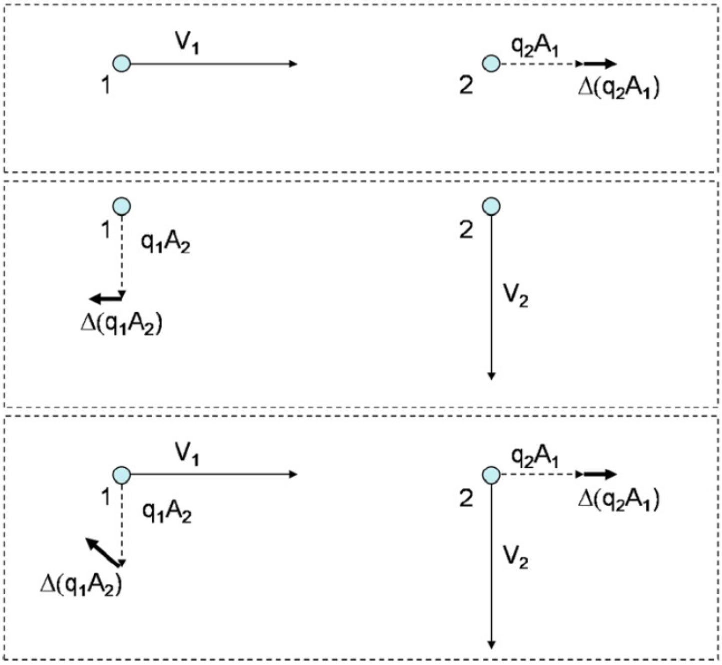

Figure 2 *Field momentum change*.The *top* figure indicates the field momentum change $\Delta(q_2A_1)$ in the resolution of "Jefimenko's paradox." The *middle* figure indicates the field momentum change $\Delta(q_1A_2)$ in the resolution of "Scanio's paradox." The *bottom* figure indicates the field momentum change

wherein (3) the relativistic expression for the electric field (6) is used. The comparison to paradoxes **b** and **c** shows that the electric force imbalance, which was not mentioned in the Feynman Lectures of Physics, is present. The imbalance in the electrical forces is given by

$$\Delta F_E = q_2\vec{E}_1(x_2) + q_1\vec{E}_2(x_1) = \left(\frac{1}{\gamma_1^2} - \gamma_2\right)\frac{kq_1q_2}{(x_2-x_1)^2}\hat{x}. \quad (27)$$

This expression is indeed the missing term that restores momentum conservation (the x-component in (25). It is interesting to note that the field momentum terms in paradoxes **b** and **c** do not add to that in paradox **a**; the y-component only appears when both particles are moving. To our knowledge, the above analysis has not been given previously.

7 The "Non-Relativistic" Limit

Using non-relativistic fields and potentials, the resolution of Feynman's paradox can also be attempted. In the limit that $v \ll c$ (13) becomes $\vec{A}_C = \frac{qv}{4\pi\varepsilon_0 c^2 R}$. Using (23), both in the Lorentz gauge and Coulomb gauge, $q\frac{dA}{dt}$ compensates the magnetic part of the Lorentz force. The non-relativistic approach can exemplify the need for electromagnetic field momentum. However, it is incomplete and can lead to difficulties in the explanation. First, even in the non-relativistic approach an x-component is found (first term in (25)). Second, the fully relativistic approach is needed to provide a meaningful relation between cases **a**, **b** and **c**. Third, the electric force imbalance and the magnetic force are of equal magnitude (for the case $v_1 = v_2$) and as such, one can not be considered a small perturbation as compared to the other.

8 Relation to the Aharonov-Bohm Effect

A decades-old and still ongoing discussion exists concerning momentum conservation for the Aharonov-Bohm (AB) effect [11]. For example Boyer [23], Spavieri and Cavalleri [24] and Hegerfeldt and Neumann [25] have all presented force explanations for the AB-effect, while Aharonov and D. Rohrlich [16] claim that no forces act, thereby representing the concensus as expressed in textbooks [26–28]. Boyer's controversial force explanation predicts measurable time delays, while Hegerfeldt's force description refutes the non-local action of the electromagnetic fields. Our recent experiment appears to rule out time delays in favor of the standard opinion [15], but a counter argument has already been made [29].

Momentum conservation arguments, which are the focus of this paper, are affected by the absence or presence of forces. The force controversy centers on the appropriate equation of motion for a magnetic dipole in the external field of a charge (and its Newton pair). This equation of motion determines the interpretation of both the AB and Aharonov-Casher (AC) effect. The AC-effect occurs when a neutron (magnetic dipole) passes a line of charge. Previously, there had been debate over whether the magnetic charge model or current loop model was correct for a neutron [30]. The issue was settled experimentally in 1951 in favor of the current loop model [31, 32], and it is now accepted to be correct for all intrinsic magnetic moments [33]. Boyer contends [34] the force on such a model in an external magnetic field $\vec{B}$ is given by

$$\vec{F}_{CL} = \nabla (\vec{m} \cdot \vec{B}) \quad (28)$$

where mis the magnetic dipole moment. The general consensus, as endorsed by Aharonov [35], Vaidman [17], and Hnidzo [36, 37], is that (28) is not correct if one wishes to define the force on the loop as the product of mass and acceleration. They argue that an additional term representing the "hidden momentum" of the current loop must be included, and thus the correct expression for the force is not (28), but rather

$$\vec{F}_{CL} = \nabla (\vec{m} \cdot \vec{B}) - \frac{d}{dt}\frac{\vec{m} \times \vec{E}}{c} \quad (29)$$

The additional term in (29) cancels the Lorentz force of the charge distribution on the magnetic dipole, and therefore the net force is zero. The source of the additional term is due to the internal mechanical structure of the current loop. Aharonov et al. [35] point out the requirement that, in order to be relativistically invariant, the classical electron model must include Poincaré stresses. In the classical electron case, the Poincaré stresses [38–42] are essentially a force of constraint that affects the equation of motion. If the classical neutron model (current loop) is analogous, then associated with the hidden momentum is a force of constraint affecting the equation of motion as described by (29). We propose that the fundamental difference between (28) and (29) may then be viewed as unconstrained versus constrained motion. It must be noted, however, that no experimental test has been conducted which would establish the validity of either (28)or(29). In the Caprez et al. experiment only the Newton pair of (28) (the inferred back-action on the electron [43]) was ruled out. Thus it seems the question is still open to theoretical debate.

One method to determine which regime (constrained or unconstrained) is correct is to examine the interaction time of the charge and the dipole as compared to the speed at which the constraint forces act. For a current loop dipole, this speed is essentially the inverse plasmon frequency. As Boyer points out [34], "In the experiments of Moellenstedt and Bayh [13], where the Aharonov-Bohm phase shift is clearly present, the passage times of electrons past the solenoids is of the order of 10^{-13} s (for a 40 keV electron passing 20 microns from the center of the solenoid). This time is not much longer than the collision time 10^{-14} s in the Drude model for conductivity of a metal. Indeed, Jackson [44] gives γ as of the order of 10^{13} inverse seconds where $-\gamma mv$ is the resistive damping of a particle of mass m and speed v." If the time in which the constraint forces act is indeed similar to the interaction time in experiments demonstrating the Aharonov-Bohm phase shift, then it is unclear in which regime these experiments occur. Thus it would seem the choice of the appropriate equation of motion is also unknown. Note that it would be highly unlikely that all the experiments confirming the AB-effect would give the same predicted AB-phase shift if the details of the transients mattered. Thus it must be possible to show theoretically that transients do not matter. The issue of transient fields needs to be addressed, because the charges composing the current loop are to some extent free during the typical interaction time.

Within the present context, our work on Feynman's paradox is relevant for the understanding of the AB-effect. The authors' study of the AB-effect led to a search for the simplest physical system which contains all the relevant momentum terms typically referred to in the discussions of the AB-effect. The Feynman case serves as that simplest system. To illustrate that Newton's third law and field momentum are a central part of the discussion on the AB-effect, consider Boyer's explanation [23]. He calculates the magnetic force on the current-carrying solenoid due to the electron. Newton's third law is then invoked to predict a force on the electron. The spatial shift Δx that the force causes can, in a semiclassical theory, be related to the phase shift by $\Delta\varphi = 2\pi\Delta x/\lambda_{dB}$. The surprising part of this calculation is that the exact value of the Aharonov-Bohm phase shift is obtained.

It is clear that the crux of Boyer's argument lies in two important issues. First, what is the correct force expression for the force on the solenoid and second, the assumption that Newton's third law holds in the sense that the change of the solenoid's momentum is compensated by the change of the electron's momentum. The discussion of "Feynman's paradox" shows that the latter is not always the case. It is possible that a change in field momentum is an essential part of the Aharonov-Bohm discussion, which is exactly what Aharonov and Casher claim in 1984 [45]. Many theoretical papers have discussed this issue [16, 17, 36, 37]. These discussions involve imbalanced forces, field momentum and relativistic terms, all of which are present in our above discussion. However, none of the discussions gives an explicit and exact derivation of the delicate balance of all the momentum terms, but often resort to a treatment of simplified systems. For example, Aharonov and D. Rohrlich [16] discuss a flux tube with a radially moving charge, instead of a charge passing by the flux tube. While the issue of whether the charge distribution of the solenoid is perturbed has been addressed [17, 36, 46], none of the discussions mention the relativistic electric field imbalance.

As it is possible to describe a solenoid as a collection of moving charged particles, the above treatment of the Feynman paradox provides hope to settle the theoretical discussion on forces. Integration over a solenoidal current distribution would provide an exact derivation of momentum conservation for the Aharonov-Bohm case.

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