3-2011

Quantifying Performance in Fading Channels Using the Sampling Property of a Delta Function

Won Mee Jang
University of Nebraska-Lincoln, wjang1@unl.edu

Quantifying Performance in Fading Channels Using the Sampling Property of a Delta Function

Won Mee Jang

***Abstract*—We apply the sampling property of a delta function to obtain the probability of error in fading channels. Our approach reduces the integration to a sampling. The sampling point is obtained in terms of fading parameters and the average signal-to-noise ratio (SNR) to provide the closed form solution of the performance.**

***Index Terms*—Fading channel, performance, delta function, sampling.**

I. Introduction

OUR sampling method is remarkably simple compared to other performance analysis methods for wireless communication in fading channels discussed in [1-8]. The bit error rate (BER) obtained from the proposed method, called delta approximation denoted by '$\doteq$', is graphically or numerically displayed and compared to the exact theoretical result for various fading channels. We extend the result to the integer power of the Q-function to exhibit the average symbol error probability (ASEP) of the differentially encoded quadrature phase-shift keying (DE-QPSK).

II. Derivation of Delta Approximation

We can express the probability density function (PDF) of the fading channel as

$$p(\beta) = K\exp\{-b\beta\}\beta^{c-1}f(\beta) \tag{1}$$

where β is the magnitude square of the fading gain. K is a constant, and $f(\beta)$ is an auxiliary function that depends on fading characteristics. For example, $K = 1$, $b = 1$, $c = 1$ and $f(\beta) = 1$ for Rayleigh fading channels. Then the integer power probability of error in fading channels can be obtained as

$$P_b^p(\bar{\gamma}) = \int_0^\infty Q^p(\sqrt{\bar{\gamma}\beta})p(\beta)d\beta \tag{2}$$

$$= K\int_0^\infty Q^p(\sqrt{\bar{\gamma}\beta})\exp\{-b\beta\}\beta^{c-1}f(\beta)d\beta \tag{3}$$

where $Q(\alpha) = \int_\alpha^\infty (2\pi)^{-1/2}e^{-y^2/2}dy$. $\bar{\gamma}$ is the average received SNR. For binary phase-shift keying (BPSK), $\bar{\gamma} = 2\bar{\gamma}_b$ where $\bar{\gamma}_b = E_b/N_o$. E_b is the average bit energy and N_o is the one-sided noise power spectral density (PSD).

Coherent detection: To find the sampling point, we introduce the Q-function approximation,

$$Q_1(\alpha) \approx \frac{1}{\sqrt{2\pi}\alpha}\exp\{-\alpha^2/2\} - \frac{1}{\sqrt{2\pi}(\alpha+1)}\exp\{-(\alpha+1)^2/2\} \tag{4}$$

where the first term (we call it $Q_o(\alpha)$) is the well-known upper bound of the Q-function. We show $Q_o(\alpha) \geq Q_1(\alpha) \geq Q(\alpha)$ in Appendix A. The integer power of the Q-function approximation can be found using the binomial expansion:

$$Q_1^p(\alpha) \approx \sum_{k=0}^{p}\binom{p}{k}(-1)^k\left(\frac{1}{\sqrt{2\pi}\alpha}\exp\{-\alpha^2/2\}\right)^{p-k} \left(\frac{1}{\sqrt{2\pi}(\alpha+1)}\exp\{-(\alpha+1)^2/2\}\right)^k. \tag{5}$$

With a change of variables ($\bar{\gamma}\beta = x$) and $Q_1^p(\alpha)$ into Eq. (3),

$$P_b^p(\bar{\gamma}) \approx \frac{K}{\bar{\gamma}^c}\int_0^\infty Q_1^p(\sqrt{x})\exp\{-b(x/\bar{\gamma})\}x^{c-1}f(x/\bar{\gamma})dx. \tag{6}$$

Applying Eq. (5) to Eq. (6) and with another change of variables ($x = y^N$),

$$P_b^p(\bar{\gamma}) \approx \left(\frac{1}{\sqrt{2\pi}}\right)^p\frac{K}{\bar{\gamma}^c}\sum_{k=0}^{p}\binom{p}{k}(-1)^k\exp\{-k/2\} \int_0^\infty f(y^N/\bar{\gamma})(\sqrt{y^N}+1)^{-k}(\sqrt{y^N})^{-(p-k)}\exp\{-k\sqrt{y^N}\} \exp\{-(p/2+b/\bar{\gamma})y^N\}y^{N(c-1)}Ny^{N-1}dy. \tag{7}$$

With $a = p/2 + b/\bar{\gamma}$, let us define

$$g(y^N) := \exp\{-ay^N\}Ny^{cN-1}. \tag{8}$$

Then, we find

$$\int_0^\infty g(y^N)dy = \frac{\Gamma(c)}{a^c} \tag{9}$$

with the Gamma function defined as $\Gamma(\alpha) := \int_0^\infty y^{\alpha-1}e^{-y}dy$ for $\alpha > 0$. We also show in Appendix B that

$$\lim_{N\to\infty} g(y^N) = \frac{\Gamma(c)}{a^c}\delta\left(y^N - \frac{c}{a}\right). \tag{10}$$

Now, applying the sampling property of a delta function to Eq. (7),

$$P_b^p(\bar{\gamma}) \doteq \left(\frac{1}{\sqrt{2\pi}}\right)^p\frac{K}{\bar{\gamma}^c}\frac{\Gamma(c)}{a^c}\sum_{k=0}^{p}\binom{p}{k}(-1)^k\exp\{-k/2\} \int_0^\infty f(y^N/\bar{\gamma})(\sqrt{y^N}+1)^{-k}(\sqrt{y^N})^{-(p-k)}\exp\{-k\sqrt{y^N}\} \delta\left(y^N - \frac{c}{a}\right)dy = \left(\frac{1}{\sqrt{2\pi}}\right)^p\frac{K\Gamma(c)}{(\bar{\gamma}a)^c}f\left(\frac{c}{a\bar{\gamma}}\right) \left[\sqrt{\frac{a}{c}} - \left(\sqrt{\frac{c}{a}}+1\right)^{-1}\exp\left\{-\left(\sqrt{\frac{c}{a}}+\frac{1}{2}\right)\right\}\right]^p. \tag{11}$$

Manuscript received October 15, 2010. The associate editor coordinating the review of this letter and approving it for publication was Y. Chen.

The author is with the University of Nebraska–Lincoln, Omaha, NE 68182 (e-mail: wjang1@unl.edu).

Digital Object Identifier 10.1109/LCOMM.2011.011011.101964

TABLE I
PARAMETERS FOR FADING CHANNELS

fading	K	a	c	$f\left(\frac{c}{a\bar{\gamma}}\right)$
Nakagami-m	$m^m/\Gamma(m)$	$p/2+m/\bar{\gamma}$	m	1
Nakagami-n	$(1+n^2)\exp\{-n^2\}$	$p/2+1/\bar{\gamma}$	1	$\exp\{-n^2/(a\bar{\gamma})\}I_o\left(2n\sqrt{(1+n^2)/(a\bar{\gamma})}\right)$
Nakagami-q	$(1+q^2)/(2q)$	$p/2+(1+q^2)^2/(4q^2\bar{\gamma})$	1	$I_o\left((1-q^4)/\{(4q^2)(a\bar{\gamma})\}\right)$

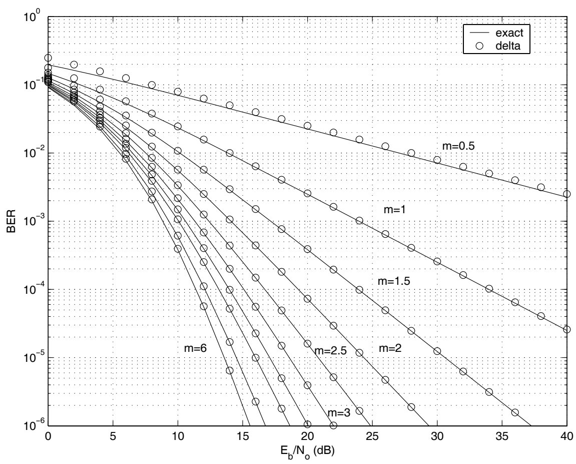

Fig. 1. Nakagami-m fading, BPSK, m=0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5, 6.

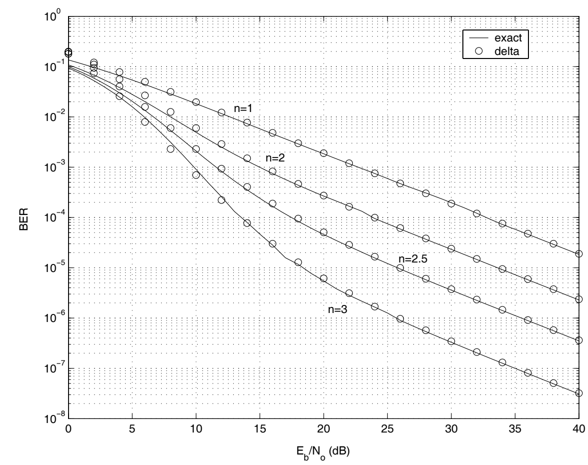

Fig. 2. Nakagami-n fading, BPSK, n=1, 2, 2.5, 3.

Noncoherent detection: The BER in a noncoherent additive white Gaussian noise (AWGN) channel can be expressed as $P_e(\bar{\gamma}) = 2^{-1}e^{-\bar{\gamma}/2}$ where $\bar{\gamma} = 2\bar{\gamma}_b$ for differential PSK (DPSK) [3, Eq. (14.3-9)]. Then, the integer power of the BER in fading channels can be expressed using Eq. (1) as

$$P_b^p(\bar{\gamma}) = \frac{K}{2^p}\int_0^\infty \exp\{-p\bar{\gamma}\beta/2\}\exp\{-b\beta\}\beta^{c-1}f(\beta)d\beta. \quad (12)$$

With a change of variables ($\bar{\gamma}\beta = x$),

$$P_b^p(\bar{\gamma}) = \frac{K}{2^p}\frac{1}{\bar{\gamma}^c}\int_0^\infty \exp\{-px/2\}\exp\{-bx/\bar{\gamma}\} x^{c-1}f(x/\bar{\gamma})dx. \quad (13)$$

With another change of variables ($x = y^N$),

$$P_b^p(\bar{\gamma}) = \frac{K}{2^p}\frac{1}{\bar{\gamma}^c}\int_0^\infty \exp\{-(p/2+b/\bar{\gamma})y^N\} y^{N(c-1)}f(y^N/\bar{\gamma})Ny^{N-1}dy. \quad (14)$$

Applying the same process in coherent detection, the delta approximation can be obtained as

$$P_b^p(\bar{\gamma}) \doteq \frac{K}{2^p}\frac{\Gamma(c)}{(\bar{\gamma}a)^c}f\left(c/a\bar{\gamma}\right). \quad (15)$$

III. NUMERICAL RESULTS

Nakagami-m fading: The PDF of the fading is [1]

$$p(\beta) = \frac{m^m\beta^{m-1}}{\Gamma(m)}\exp\{-m\beta\}, \qquad \beta \geq 0. \quad (16)$$

Making reference to Eq. (1), we choose $K = m^m/\Gamma(m)$, $c = m$, $a = p/2+m/\bar{\gamma}$ and $f(\frac{c}{a\bar{\gamma}})$=1. We summarize the parameter selection in different fadings in Table I. The result is compared to the exact performance of BPSK [2, Eq. (5.80)]

$$P_e = \frac{1}{\pi}\int_0^{\pi/2}\left(1+\frac{\bar{\gamma}_b}{m\sin^2\phi}\right)^{-m}d\phi. \quad (17)$$

In Fig. 1 we plot the exact BER and the delta approximation for m= 0.5 to 6. We can see that the delta approximation and the exact performance agree well.

Nakagami-n fading: For Nakagami-n fading channels, the fading PDF is shown as [1]

$$p(\beta) = (1+n^2)\exp\{-n^2\}\exp\{-(1+n^2)\beta\} I_o(2n\sqrt{(1+n^2)\beta}), \qquad \beta \geq 0 \quad (18)$$

and we choose $K = (1+n^2)\exp\{-n^2\}$, $c = 1$, $a = p/2 + 1/\bar{\gamma}$ and $f(\frac{c}{a\bar{\gamma}}) = \exp\{-n^2/(a\bar{\gamma})\}\ I_o(2n\sqrt{(1+n^2)/(a\bar{\gamma})})$. It is important to choose an auxiliary function $f(\cdot)$ as flat as possible with respect to (w.r.t.) the SNR to be robust against the sampling point error that may be introduced by the Q-function approximation. The modified Bessel function $I_o(\cdot)$ is a decreasing function w.r.t. the SNR while $\exp\{-n^2/(a\bar{\gamma})\}$ is an increasing function. As a result, the product of the two is rather flat w.r.t. the SNR and thus less sensitive to the sampling point error. The BER of Nakagami-n with BPSK is shown in Fig. 2 for n=1, 2, 2.5 and 3. The delta approximation agrees well with the exact BER at a moderate and high SNR. The exact performance is numerically obtained from Eq. (2) with $p = 1$ using a double integration with the modified Bessel function.

Nakagami-q fading: The PDF of Nakagami-q fading channels is presented as [1]

$$p(\beta) = \frac{1+q^2}{2q}\exp\left\{-\frac{(1+q^2)^2\beta}{4q^2}\right\}I_o\left(\frac{(1-q^4)\beta}{4q^2}\right), \ \beta \geq 0 \quad (19)$$

and we choose $K = (1+q^2)/(2q)$, $c = 1$, $a = p/2 + (1+q^2)^2/(4q^2\bar{\gamma})$, and $f(\frac{c}{a\bar{\gamma}}) = I_o((1-q^4)/\{(4q^2)(a\bar{\gamma})\})$. Since Nakagami-$q$ fading with $q = 1$ corresponds to the Rayleigh fading channel, we compare the delta approximation to the exact performance of BPSK [3, Eq. (14.3-7)]

$$P_e = (1/2)\left(1-\sqrt{\bar{\gamma}_b/(1+\bar{\gamma}_b)}\right). \quad (20)$$

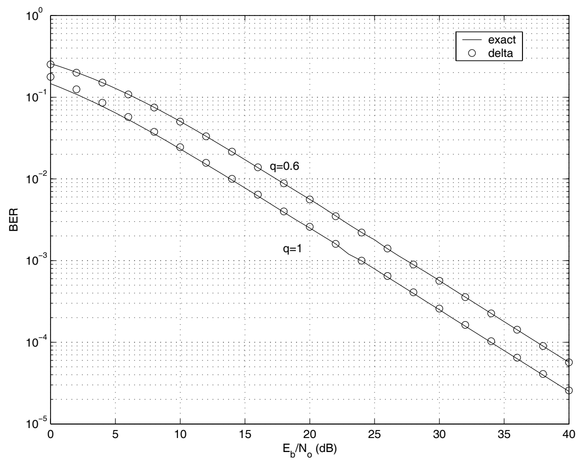

Fig. 3. Nakagami-q fading, q=0.6 (DPSK), q=1 (BPSK).

The delta approximation of DPSK is obtained from Eq. (15) with $p = 1$ and then compared to the exact performance [2, Eqs. (2.12) & (8.201)]

$$P_e = (1/2)\left\{1 + 2\bar{\gamma}_b + (2\bar{\gamma}_b q)^2/(1+q^2)^2\right\}^{-1/2}. \quad (21)$$

The delta approximation agrees well with the exact performance as shown in Fig. 3.

Differentially Encoded QPSK: The ASEP of the DE-QPSK in the AWGN channel is given by [4]

$$P_e = 4Q(\sqrt{\bar{\gamma}}) - 8Q^2(\sqrt{\bar{\gamma}}) + 8Q^3(\sqrt{\bar{\gamma}}) - 4Q^4(\sqrt{\bar{\gamma}}). \quad (22)$$

Therefore, the ASEP of the DE-QPSK in fading channels can be obtained as

$$P(\bar{\gamma}) \doteq 4P_b^1(\bar{\gamma}) - 8P_b^2(\bar{\gamma}) + 8P_b^3(\bar{\gamma}) - 4P_b^4(\bar{\gamma}) \quad (23)$$

with $P_b^p(\bar{\gamma})$ defined in Eq. (11). The ASEP of the DE-QPSK in Nakagami-m fading channels is shown in Tables II and III for m=2.5 and 3.5, respectively. The result is compared to approximations in Chiani [5, Eq. (14)], Börjesson [6, Eq. (9)], Karagiannidis [7, Eq. (12)] and Isukapalli [8, Eq. (11)]. The delta approximation provides a tighter approximation than Chiani and Börjesson for $\bar{\gamma} \geq 10$ dB for both m=2.5 and 3.5. Our method also provides a tighter approximation than Karagiannidis and Isukapalli for $\bar{\gamma} \geq 20$ dB for both m=2.5 and 3.5. Karagiannidis ASEP is obtained with parameter values specified in [7]. Isukapalli ASEP is obtained with the number of terms in Taylor series expansion, N_a=14. The order of its computational complexity is $O(\exp\{N_a \ln(N_a)\})$ that can make Isukapalli method infeasible for a large value of m or SNR.

TABLE II
ASEP OF DE-QPSK (m=2.5)

$\bar{\gamma}$	Exact	Chiani	Börj.	Karag.	Isukap.	Eq. (23)
0	4.814e-1	5.061e-1	5.000e-1	5.259e-1	4.841e-1	5.532e-1
5	2.133e-1	2.436e-1	2.250e-1	2.292e-1	2.130e-1	2.298e-1
10	4.326e-2	5.122e-2	4.570e-2	4.528e-2	4.270e-2	4.430e-2
15	4.410e-3	5.257e-3	4.653e-3	4.546e-3	4.318e-3	4.412e-3
20	3.098e-4	3.694e-4	3.266e-4	3.174e-4	3.022e-4	3.073e-4
25	1.877e-5	2.239e-5	1.978e-5	1.919e-5	1.828e-5	1.857e-5

TABLE III
ASEP OF DE-QPSK (m=3.5)

$\bar{\gamma}$	Exact	Chiani	Börj.	Karag.	Isukap.	Eq. (23)
0	4.763e-1	5.045e-1	4.964e-1	5.217e-1	4.793e-1	5.557e-1
5	1.950e-1	2.269e-1	2.063e-1	2.085e-1	1.944e-1	2.128e-1
10	2.889e-2	3.480e-2	3.053e-2	2.977e-2	2.832e-2	2.986e-2
15	1.502e-3	1.802e-3	1.580e-3	1.512e-3	1.452e-3	1.512e-3
20	4.081e-5	4.869e-5	4.285e-5	4.068e-5	3.909e-5	4.069e-5
25	8.398e-7	9.995e-7	8.811e-7	8.342e-7	8.009e-7	8.345e-7

APPENDIX A

$Q_o(\alpha) \geq Q_1(\alpha) \geq Q(\alpha)$*:* Let $e(\alpha) = Q_1(\alpha) - Q(\alpha)$. Then $e(0) = \infty$ and $e(\infty) = 0$. If we assume $de(\alpha)/d\alpha < 0, \forall\alpha \geq 0$,

$$de(\alpha)/d\alpha = (2\pi)^{-1/2}[-\alpha^{-2}\exp\{-\alpha^2/2\} \quad (24)$$

$$+(\alpha+1)^{-2}\exp\{-(\alpha+1)^2/2\} + \exp\{-(\alpha+1)^2/2\}] < 0, \text{ or} \quad (25)$$

$$\{\alpha^2/(\alpha+1)^2\}\exp\{-(\alpha+1/2)\} + \alpha^2\exp\{-(\alpha+1/2)\} < 1. \quad (26)$$

Let $y_1(\alpha)$ and $y_2(\alpha)$ be the first and second term of Eq. (26). Then,

$$y(\alpha) = y_1(\alpha) + y_2(\alpha) < 1. \quad (27)$$

$y_1(\alpha)$ and $y_2(\alpha)$ are unimodal functions for $\alpha \geq 0$ with their peaks at α=1 and α=2, respectively. Thus, $y(\alpha) < y_1(1) + y_2(2) < 1$. Indeed, $de(\alpha)/d\alpha < 0$ for $\alpha \geq 0$. In result, $Q_o(\alpha) \geq Q_1(\alpha) \geq Q(\alpha)$.

APPENDIX B

A limit impulse: To obtain the critical point y_*^N of $g(y^N)$:

$$\frac{dg(y^N)}{dy} = -aNy^{N-1}\exp\{-ay^N\}Ny^{cN-1} + \exp\{-ay^N\}N(cN-1)y^{cN-2} = 0, \quad (28)$$

or $y_*^N = \lim_{N\to\infty} y^N = \frac{c}{a}$. Applying the above result to Eq. (8), we find that

$$\lim_{N\to\infty} g(y_*^N) = \lim_{N\to\infty} \exp\{-c\}N(\frac{c}{a})^c = \infty. \quad (29)$$

From Eq. (28), we can see that $dg(y^N)/dy > 0$ for $0 < y^N < y_*^N$, and $dg(y^N)/dy < 0$ for $y^N > y_*^N$. Together with Eqs. (9) and (29), we obtain Eq. (10).

REFERENCES

[1] Z. Wang and G. B. Giannakis, "A simple and general parameterization quantifying performance in fading channels," *IEEE Trans. Commun.*, vol. 51, no. 8, pp. 1389-1398, Aug. 2003.
[2] M. K. Simon and M. S. Alouini, *Digital Communications over Fading Channels: A Unified Approach to Performance Analysis*, 2nd edition. Wiley, 2000.
[3] J. G. Proakis, *Digital Communications*, 4th edition. McGraw-Hill, 2001.
[4] M. K. Simon, "Integral representation of certain integer powers of the Gaussian Q-function and their applications," *IEEE Commun. Lett*, vol. 6, no. 12, pp. 532-534, Dec. 2002.
[5] M. Chiani, D. Dardari, and M. K. Simon, "New exponential bounds and approximations for the computation of error probability in fading channels," *IEEE Trans. Wireless Commun.*, vol. 2, no. 4, pp. 840-845, July 2003.
[6] P. O. Börjesson and C. E. Sundberg, "Simple approximations of the error function Q(x) for communications applications," *IEEE Trans. Commun.*, vol. COM-27, no. 3, pp. 639-643, Mar. 1979.
[7] G. K. Karagiannidis and A. S. Lioumpas, "An improved approximation for the Gaussian $Q-$function," *IEEE Commun. Lett.*, vol. 11, no. 8, pp. 644-646, Aug. 2007.
[8] Y. Isukapalli and B. D. Rao, "An analytically tractable approximation for Gaussian Q-function," *IEEE Commun. Lett.*, vol. 12, no. 9, pp. 669-671, Sep. 2008.