Constraining Inverse-Curvature Gravity with Supernovae

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We show that models of generalized modified gravity, with inverse powers of the curvature, can explain the current accelerated expansion of the Universe without resorting to dark energy and without conflicting with solar system experiments. We have solved the Friedmann equations for the full dynamical range of the evolution of the Universe and performed a detailed analysis of supernovae data in the context of such models that results in an excellent fit. If we further include constraints on the current expansion of the Universe and on its age, we obtain that the matter content of the Universe is $0.07 \leq \omega_m \leq 0.21$ (95% C.L.). Hence the inverse-curvature gravity models considered *cannot* explain the dynamics of the Universe just with a baryonic matter component.

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It is now widely accepted that recent supernovae (SNe) observations imply that our Universe is currently experiencing a phase of accelerated expansion [1]. This seems to be independently confirmed by observations of clusters of galaxies [2] and the cosmic microwave background [3]. The accelerated expansion is usually explained through violations of the strong energy condition by introducing an extra component in the Einstein equations in the form of dark energy with an equation of state $w < -1/3$. However, such an explanation is plagued with theoretical and phenomenological problems, such as the extreme fine-tuning of initial conditions and the so-called coincidence problem [4], and it is therefore natural to seek alternatives to dark energy as the source of the acceleration. One possibility is an inhomogeneous Universe with only local acceleration; albeit it is hard to explain natural boundary conditions for such a local void [5]. The other, which we elaborate on in this Letter, is modifications of gravity that turn on only at very large distances [6] or small curvatures [7,8], therefore giving a geometrical origin to the accelerated expansion of the Universe.

It was shown in [7] that a simple modification of the gravitational action adding inverse of curvature invariants to the Einstein-Hilbert term would naturally have effects only at low curvatures and therefore at late cosmological times. The simplest of such modifications includes just one single inverse of the curvature scalar μ^4/R, with μ a parameter with dimensions of mass. This results in a model governed by the Einstein-Hilbert term, i.e., usual gravity, for curvatures $R \gg \mu^2$ but can lead to an accelerated expansion at curvatures $R \lesssim \mu^2$. This simple model is equivalent to a Brans-Dicke theory [7]. Based on this equivalence, it was subsequently proven by a number of authors that the model is in conflict with solar system data [9] (see, however, [10]) and is unstable when matter is introduced [11]. This conclusion naturally extends to generalizations of this action where the Einstein-Hilbert term is supplemented with an arbitrary function of R, except for particular cases that could still lead to viable models [12].

With this restriction in mind, the authors of [13] discussed a more general modification of gravity based on the following gravitational action:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - \frac{\mu^{4n+2}}{(aR^2 + bP + cQ)^n}\right] + \int d^4x\sqrt{-g}\,\mathcal{L}_M, \tag{1}$$

where $P \equiv R_{\mu\nu}R^{\mu\nu}$, $Q \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, G Newton's constant, $\mathcal{L}_M$ the matter Lagrangian, and g the determinant of the metric.

In this generalized case the equivalence with a Brans-Dicke theory is not clear, and a more detailed analysis of modifications of Newton's potential has to be done to compare with solar system data. The authors of [14] computed the corrections to Newton's law in these models as a perturbation around Schwarzschild geometry and found that as long as we include inverse powers of the Riemann tensor ($c \neq 0$), Newton's law is not modified in the solar system at distances shorter than $r_c \sim 10$ pc, and therefore all solar system experiments are well under control. Note that, as long as the Riemann tensor is present, this result is independent of whether we include or not inverse powers of the scalar curvature or the Ricci tensor squared, as they vanish in the background solution. This important result restricts the parameter space of phenomenologically relevant inverse-curvature gravity models to the ones with inverse powers of the Riemann tensor squared present. Other constraints come from the absence of ghosts in the spectrum, requiring specific relations between b and c [15]. Finally, we restrict our analysis in this Letter to models with $n = 1$.

Let us turn now to the cosmology of models governed by the gravitational action (1). Assuming a cosmological setup with a spatially flat Friedmann-Robertson-Walker metric, $ds^2 = -dt^2 + a(t)d\vec{x}^2$, all models with $n = 1$ can be characterized by just three parameters, α, $\hat{\mu}$, and σ, given in terms of the parameters in Eq. (1) by

$$\alpha \equiv \frac{12a + 4b + 4c}{12a + 3b + 2c}, \tag{2}$$

$$\hat{\mu} \equiv \mu/|12a + 3b + 2c|^{1/6}, \tag{3}$$

$$\sigma \equiv \mathrm{sgn}(12a + 3b + 2c). \tag{4}$$

In order to write the corresponding Friedmann equation in the simplest possible way we use logarithmic variables, $u \equiv \ln(H/\hat{\mu})$ and $N \equiv \ln a$, where as usual $H = \frac{\dot{a}}{a}$, with a dot denoting the time derivative. The generalized Friedmann equation in these variables reads

$$u''\mathcal{P}_1(u') + \mathcal{P}_2(u') + 18\sigma(\mathcal{P}_3(u'))^3 e^{6u}(e^{2(\bar{u}-u)} - 1) = 0, \tag{5}$$

where a prime denotes the derivative with respect to N and we have defined the following polynomials:

$$\mathcal{P}_1(y) = 6\alpha^2 y^2 + 24\alpha y + 32 - 8\alpha, \tag{6}$$

$$\mathcal{P}_2(y) = 15\alpha^2 y^4 + 2\alpha(50 - 3\alpha)y^3 + 4(40 + 11\alpha)y^2 + 24(8 - \alpha)y + 32, \tag{7}$$

$$\mathcal{P}_3(y) = \alpha y^2 + 4y + 4. \tag{8}$$

The source is $\bar{u} \equiv \ln[\bar{\omega}_r \exp(-4N) + \bar{\omega}_m \exp(-3N)]/2$, where we have defined the appropriately normalized values of the energy densities *today* as

$$\frac{8\pi G}{3}\frac{\rho_{r,m0}}{\hat{\mu}^2} \equiv \bar{\omega}_{r,m}, \tag{9}$$

with $\rho^0_{r,m}$ the present densities in matter and radiation, and we have exploited the fact that the energy-momentum tensor is still covariantly conserved. This means that the source in Eq. (5) corresponds to the standard one with no dark energy.

The new Friedmann equation is no longer algebraic but a second order nonlinear differential equation. Furthermore, it becomes nonautonomous in the presence of sources, making its dynamical study a formidable problem. The asymptotic behavior of the system in vacuum was carefully studied in [13], where it was found that, depending on the value of α, but irrespective of σ, the system has a number of attractors, including sometimes singularities. The same attractor and singular points are relevant when sources are present. In that case, however, both the value of σ and the fact that the Universe is in a matter dominated era before the new corrections become relevant are crucial to determine the fate of the Universe.

A careful analysis of the dynamical behavior of the system reveals that physically valid solutions exist only for certain combinations of α and σ. In order to classify the different regions, we define the following special values of α: $\alpha_1 \equiv 8/9$, $\alpha_2 \equiv 4(11 - \sqrt{13})/27 \approx 1.095$, $\alpha_3 \equiv 20(2 - \sqrt{3})/3 \approx 1.786$, and $\alpha_4 \equiv 4(11 + \sqrt{13})/27 \approx 2.164$. For $\alpha < \alpha_1$ both signs of σ result in an acceptable (nonsingular) dynamical evolution, but nevertheless in a bad fit to supernovae data. For $\alpha_1 < \alpha < \alpha_2$ only $\sigma = -1$ leads to an acceptable expansion history, since for $\sigma = +1$ a singular point is violently approached in the past. For $\alpha_2 < \alpha < \alpha_4$ the singular point is approached for $\sigma = -1$; hence $\sigma = +1$ is the only physically valid solution. In this latter case, when $\alpha_2 < \alpha < \alpha_3$, the system goes to a stable attractor that is decelerated, thus giving a bad fit to SNe data, for $\alpha < 32/21$ and gets accelerated for larger α. For $\alpha_3 < \alpha < \alpha_4$ there is no longer a stable attractor, and the system smoothly goes (through an accelerated phase) to a singularity in the future. For small enough $\bar{\omega}_m$ the singularity occurs in the past, that region being phenomenologically excluded. It is important to stress that this singularity is approached in a very smooth fashion, allowing for a phenomenologically viable behavior of the system, as opposed to the evolution when the *wrong* value of σ is chosen, where the singularity is hit almost instantaneously. Finally, for values $\alpha_4 < \alpha$, $\sigma = +1$ leads the system to a sudden singularity, whereas $\sigma = -1$ leads it (smoothly) to a singularity in the future (or a stable attractor for $\alpha \gtrsim 24.9$), but it is never accelerated, thus giving a very bad fit to SNe data. To summarize, there are two regions that give a dynamical evolution of the system compatible with SNe data, the *low* region with $\alpha_1 < \alpha < \alpha_2$, for which $\sigma = -1$, and the *high* region where $\alpha_2 < \alpha < \alpha_4$, for which $\sigma = +1$.

As we have emphasized, it is extremely difficult to solve the dynamics of the system analytically. To overcome this limitation, we have performed a comprehensive numerical study of the model resulting in the general behavior we have outlined above. To make things more complicated, the new Friedmann equation is extremely stiff, due to the exponentials in the last term. This stiffness is directly linked to the nature of the corrections that are negligibly small in the far past, where the curvature is much larger than the scale $\hat{\mu}^2$. It also makes it essentially impossible to numerically integrate it from a radiation dominated era all the way to the present. In order to circumvent this problem, we have matched a perturbative analytical solution that tracks the solution in standard Einstein gravity in the far past to the corresponding numerical one in the region $z \gtrsim 5$, where the analytical solution is still an extremely good approximation, and the numerical codes can cope with the integration. Although the matching at this point is accurate below the 1% level, we emphasize that it is safely above the redshift range probed by SNe. The approximate solution from the perturbation analysis, for $\alpha \neq 8/9$, is given by

$$H_{\mathrm{approx}} = \hat{\mu} e^{\bar{u}}\left(1 + \frac{e^{-6\bar{u}}}{36\sigma}\frac{\bar{u}''\mathcal{P}_1(\bar{u}') + \mathcal{P}_2(\bar{u}')}{(\mathcal{P}_3(\bar{u}'))^3}\right). \tag{10}$$

This is an extremely accurate solution to the full nonlinear equation as long as $z \gtrsim 5$, regardless of the values of α and $\bar{\omega}_m$. At the boundaries between regions with different dynamical behavior (including α_1) the sensitivity to initial conditions is large, and therefore nothing conclusive can be said at these points. The question of sensitivity to initial conditions is a relevant one due to the nonlinear nature of the Friedmann equation. However, because of the complication of any analytical study for non-negligible sources alluded to above, we will defer its study to a future publication. In the present Letter we content ourselves with the particular solution in Eq. (10) that we are guaranteed tracks the standard behavior in Einstein gravity in the past. We further explicitly confirmed, by a numerical analysis, that our conclusions are not sensitive to the exact position of the matching point in the past.

Once we have solved for the Hubble parameter as a function of the scale factor, we perform a fit to SNe data to get the allowed values of the different parameters defining our model. In principle, there is a total of five parameters defining our Universe in this framework, namely, the three parameters defining the model, $\hat{\mu}$, α, and σ, and the two parameters determining the sources, $\bar{\omega}_{r,m}$. The absolute value of the cosmic microwave background (CMB) temperature, however, fixes the total radiation content of the Universe, constraining $\bar{\omega}_r \hat{\mu}^2$. For relevant values of $\hat{\mu}$ this constraint makes radiation irrelevant in the analysis of SNe data. Since the intrinsic magnitude of SNe is a nuisance parameter in our analysis, it is not possible to determine $\hat{\mu}$ as an independent parameter with SNe only. For a standard ΛCDM (cold dark matter) Universe this corresponds to the inability of SNe data to independently determine the Hubble constant H_0. However, we will be able to determine the value of $\hat{\mu}$ once we impose other constraints, like the measurement of the Hubble constant by the Hubble Key Project, $H_0 = 72 \pm 8\ \mathrm{Km\,s^{-1}\,Mpc^{-1}}$ [16]. Hence, this leaves us with just three parameters, α, σ, and $\bar{\omega}_m$, relevant for the analysis of SNe data and an additional nuisance parameter in terms of the intrinsic magnitude.

The fits are performed using the recent gold SNe data set from the last reference in [1]. The apparent magnitude is given by $m(z) = \mathcal{M} + 5\log\mathcal{D}_L$ where $\mathcal{M} \equiv M - 5\log\hat{\mu} + 25$ and $\mathcal{D}_L \equiv \hat{\mu} d_L$ with $d_L \propto \int H^{-1}(z)dz$. Note that the parameter $\hat{\mu}$ appears in the definition of the magnitude compared to the usual definition involving H_0 [1]. The important point is that $\mathcal{D}_L(z)$ can now be computed solely in terms of $\bar{\omega}_m$ and α, where $\hat{\mu}$ and the intrinsic magnitude have been absorbed into the nuisance parameter $\mathcal{M}$ that can be marginalized analytically in the probability function.

We have performed independently two parameter fits to SNe data for each of the low and high regions. This results in the 1σ and 2σ joint likelihoods shown in Fig. 1, with best fit values given by

$$\mathrm{low}: \alpha = 0.9, \qquad \bar{\omega}_m = 0.105, \qquad \chi^2 = 184.9, \quad (11)$$

$$\mathrm{high}: \alpha = 2.15, \qquad \bar{\omega}_m = 0.085, \qquad \chi^2 = 185.2. \quad (12)$$

For comparison purposes, we have also performed the fit using the standard ΛCDM model for a spatially flat universe and absorbing H_0 as a nuisance parameter into $\mathcal{M}$, resulting in $\chi^2 = 183.3$ for 156 data points (the difference in the apparent magnitude is virtually indistinguishable given the present precision of the data, with a magnitude difference near 0.01). We further show in Fig. 1 the points α_i. The shaded area on the bottom right-hand corner is excluded from a singularity being hit in the past. Note that the contours have a sharp cutoff at α_1, α_2, and α_4. However, at α_3 there is no singularity hit violently and the 2σ contour of the high region extends below α_3. In the low region we obtain $\bar{\omega}_m = 0.122 \pm 0.034$ after marginalization over α and in the high region $\bar{\omega}_m = 0.075 \pm 0.031$. Note that our best fit points in both regions are close to the borders of the allowed region. This is because within the regions there is a smooth behavior of the likelihood, and only the dynamics of the system cuts off the likelihood space if certain parameter values are reached.

If we additionally apply the Hubble Space Telescope measurement of H_0 [16], we can determine $\hat{\mu}$ and the matter content $\omega_m = \Omega_m h^2$, with $H_0 = 100h$ km/s/Mpc. Finally, we can restrict the allowed region in ω_m-$\hat{\mu}$ a little bit more by imposing a prior on the age of the Universe with a mean of $t_0 = 13.4$ Gyrs and a 95% confidence lower limit of 11.2 Gyrs [17]. In Fig. 2 we show the joint 1σ and 2σ likelihoods in the ω_m-$\hat{\mu}$ plane, with both priors imposed (solid line) and without the age of the Universe prior imposed (dashed line), on the left for the low region and on the right for the high region. First, we recognize that $\hat{\mu}$ is roughly twice the size of the Hubble constant H_0. If we further marginalize over $\hat{\mu}$, the physical matter content in the Universe is $\omega_m = 0.14 \pm 0.03$ and

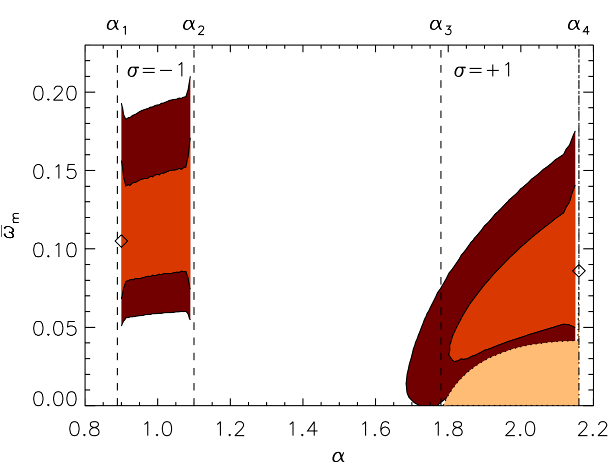

FIG. 1 (color online). The 1σ and 2σ joint likelihoods on $\bar{\omega}_m$ and α. In the *low* region $\sigma = -1$, whereas in the *high* region $\sigma = +1$. The shaded area in the bottom right-hand corner determines the region that is excluded because of a singularity being hit in the past. The diamonds denote the maximum likelihood points.

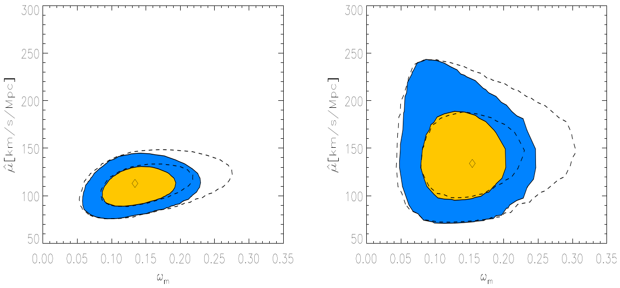

FIG. 2 (color online). The 1σ and 2σ joint likelihoods in the ω_m-$\hat{\mu}$ plane, when additional priors on H_0 and the age of the Universe are imposed. On the left for the *low* region and on the right for the *high* region. Diamonds are the maximum likelihoods. Further, the dashed contours are the joint likelihoods if we impose only the H_0 prior.

$\omega_m = 0.14 \pm 0.04$ in the low and high regions, respectively. Note that the matter content in the budget of the Universe is clearly higher than the measured baryonic content. Overall we find $0.07 \leq \omega_m \leq 0.21$ at the 95% confidence level. If we compare this number with the results from big bang nucleosynthesis $\omega_b = 0.0214 \pm 0.0020$ [18], it is clear that we require a dark matter component to explain the data.

Other cosmological probes such as clusters of galaxies and CMB could further constrain these models. However, such an analysis is beyond the scope of this Letter since it requires a detailed recalculation of, e.g., cluster potentials and CMB perturbations for the models discussed here.

We have studied the viability of a geometrical explanation for the present acceleration of the Universe. This is possible if the Einstein-Hilbert action is supplemented with new terms that are negligibly small at high cosmological curvatures but become relevant when the curvature of the Universe gets smaller. Despite the phenomenological problems of the simplest models, it has been shown that there exists a broad class of modifications of gravity that are phenomenologically viable and have accelerated attractors at late times. In this Letter we have performed a detailed numerical analysis of the dynamics of these models. We emphasize that this hard numerical problem has not been solved previously. The result of this analysis allowed us to compare inverse-curvature gravity with supernovae data. We found that SNe data can be fitted in our model without the need of any dark energy and getting meaningful constraints in the free parameters. We further have shown that these models still require a dark matter component. Of course, this latter conclusion does not need to hold for more general models, for instance, those with $n \neq 1$. We are planning to study more general models and their implications for dark matter in the near future. However, we emphasize the generality of our study. We have parametrized *all* models governed by Eq. (1) with $n = 1$. Finally, we are currently extending this analysis to CMB and cluster data sets, a nontrivial task. This will further constrain these models and maybe even distinguish them from dark energy.

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