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# Possible conserved quantity for the Henon-Heiles problem

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# Possible conserved quantity for the Hénon-Heiles problem

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We study a power-series expansion for a conserved quantity $K$ in the case of the two-dimensional Hénon-Heiles potential. An alternative technique to that of Gustavson [Astron. J. **71**, 670 (1966)] is applied to find the coefficients in the expansion for $K$. The technique is used to determine twelve orders for the conserved quantity $K$, more than twice as many as that calculated by Gustavson. We investigate the degree of constancy of our truncated $K$ in regions where the motion is known to be chaotic and also where it is nonchaotic.

## I. INTRODUCTION

Some time ago, Hénon and Heiles[1] studied the following model Hamiltonian in two dimensions:

$$H=\tfrac{1}{2}(p_x^2+p_y^2+x^2+y^2)+x^2y-\tfrac{1}{3}y^3 . \qquad (1.1)$$

They were interested in finding if another integral of the motion (or conserved quantity) existed in addition to the total energy $H$. If such a quantity exists, the motion is not expected to be chaotic. Their numerical studies showed that both chaotic and nonchaotic motion existed for this system. For $H<0.11$ (approximately), the motion appeared to be nonchaotic. For $H>0.11$, the motion was in general chaotic, although for certain initial conditions some nonchaotic motion still existed at the higher energies.

Hénon and Heiles presented their numerical results as a two-dimensional graph of $\dot y$ versus $y$ showing the successive intersections of a trajectory with the $x=0$ plane (called a Poincaré section). Here and below, we use $\dot x \equiv p_x$ and $\dot y \equiv p_y$. By convention, only points are plotted which have a positive $\dot x$ as the trajectory crosses the $x=0$ plane. Examples of such graphs are shown in Figs. 1(a)–5(a).

The fact that the points on the section seem to lie on a closed curve in Fig. 1(a) suggests the presence of another conserved quantity in addition to the total energy $H$, whereas Fig. 3(a) would appear to indicate the absence of such a quantity. Usually, of course, a conserved quantity has a time derivative which vanishes identically, irrespective of the particular values of the energy or the initial conditions. More precisely, a nontrivial conserved quantity is a first integral, which is valid over some domain of parameter space, or a configurational invariant, which is defined for some specific value of a parameter.

It is conceivable that a conserved quantity $K$ might be expressed as a formal power series in the four variables consisting of the coordinates and velocities and that such a power series might converge on nonchaotic trajectories and diverge (or be asymptotic) on chaotic trajectories. Recent work by Ziglin[2] gives a formal proof that no meromorphic first integrals in addition to $H$ exist in te case of the Hénon-Heiles problem.[3] However, earlier work by Gustavson[4] shows that a truncated formal power-series expression for a conserved quantity clearly has relevance in the Hénon-Heiles problem and, in fact, can be used to approximately reproduce curves such as those shown in Fig. 1(a).

Gustavson developed a formal power series for a conserved quantity using an intricate technique of successive canonical transformations which progressively, order-by-order, brought the Hamiltonian $H$ into so-called normal form. From the fact that $H$ could be brought into normal form, a formal power series for a conserved quantity distinct from $H$ could be deduced. Gustavson expressed his series as a sum of multinomials in the coordinates and velocities for orders $n=4$–$8$. Gustavson verified that the quantity obtained by truncating his formal power series was indeed approximately constant over curves of the type shown in Fig. 1(a).

The technique used in this paper simply begins with a formal power series in the coordinates and velocities for a quantity $K$. By taking the time derivative of $K$ and inserting the equations of motion, a recursion relation involving the power-series coefficients is derived and the resulting equations are solved. (A somewhat similar approach was investigated by Leach,[5] but his calculations contain an error which we discuss later.) This technique is rather straightforward since it does not involve canonical transformations to other variables. This method, however, does require the treatment of certain subtle uniqueness issues in determining $K$.

## II. RECURSION RELATION FOR THE EXPANSION COEFFICIENTS

In this section we will derive and discuss the properties of the expansion coefficients for a conserved quantity $K$ for motion corresponding to the Hamiltonian $H$ of (1.1). It is useful to introduce the complex notation $z=x+iy$. In terms of $z$, the equations of motion implied by the $H$ of (1.1) are

$$\ddot z=-z-i\bar z^2 , \qquad (2.1)$$

where $\bar z$ is the complex conjugate of $z$. We now assume the existence of a conserved quantity $K$ which has the following expansion in terms of non-negative powers:

$$K=\sum_{p,q,r,s} i^n b_{pqrs} z^p \bar{z}^q \dot{z}^r \dot{\bar{z}}^s , \tag{2.2}$$

where $n=p+q+r+s$ is the order. Taking the time derivative of $K$, inserting (2.1), and requiring that the result be identically zero leads to the following recursion relations:

$$\begin{aligned}(p+1)b_{p+1,q,r-1,s}&+(q+1)b_{p,q+1,r,s,-1}\\ &-(r+1)b_{p-1,q,r+1,s}-(s+1)b_{p,q-1,r,s+1}\\ &=(r+1)b_{p,q-2,r+1,s}-(s+1)b_{p-2,q,r,s+1} .\end{aligned} \tag{2.3}$$

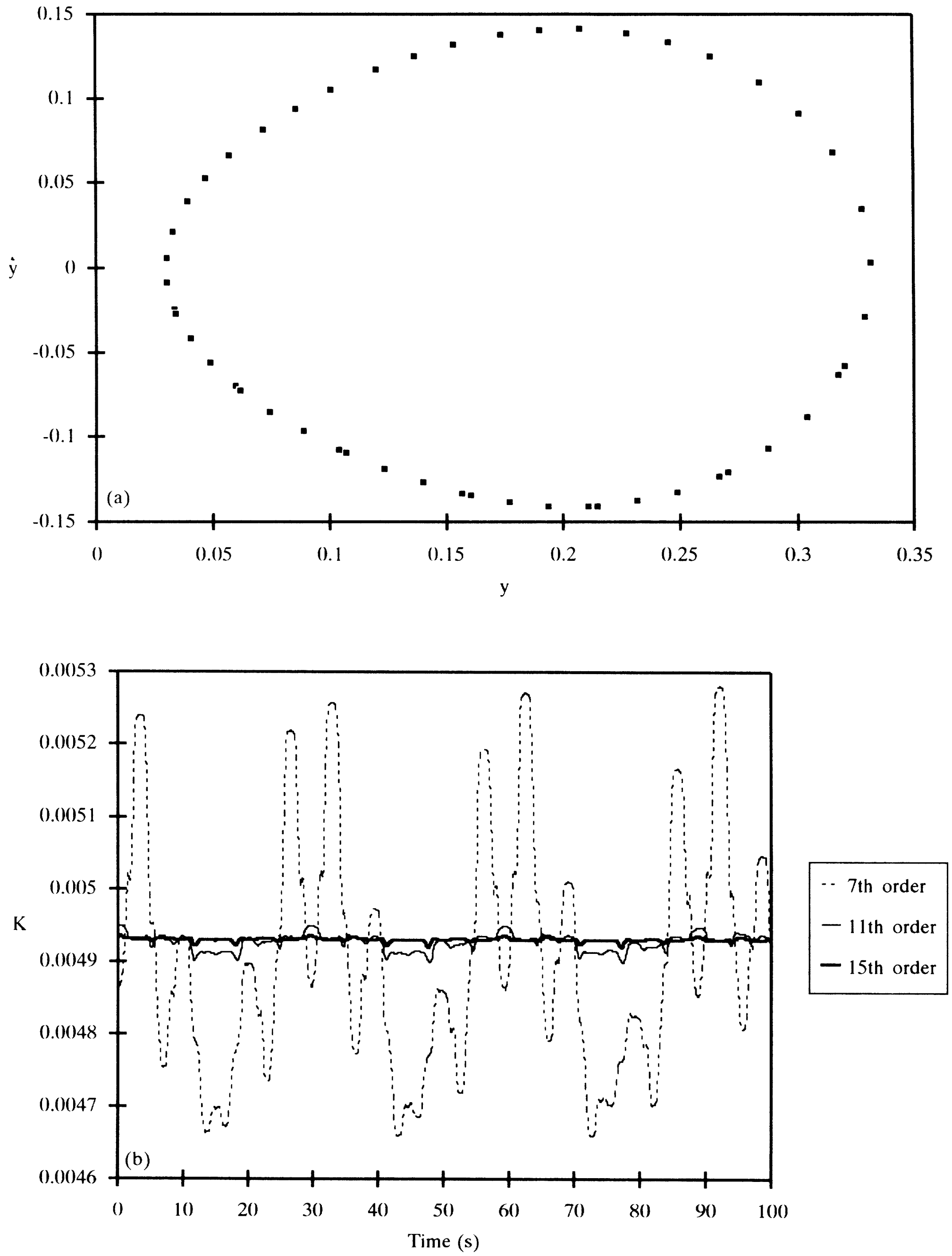



FIG. 1. (a) Graph of the Poincaré section for an energy of $H=0.05$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative. (b) Graph of the series $K$ through 7th, 11th, and 15th orders. The energy is $H=0.05$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative. (c) Graph of the series $K$ through 11th and 15th orders. The energy is $H=0.05$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative.

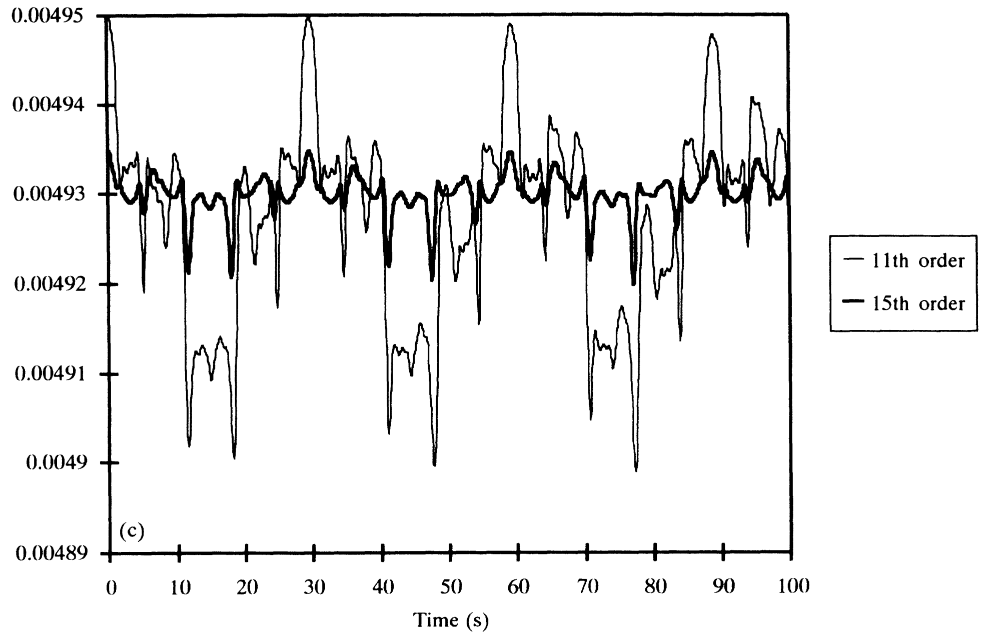



FIG. 1. (*Continued*).

Equation (2.3) relates coefficients of order $n$ on the left-hand side to those of order $n-1$ on the right-hand side. Note that coefficients with any negative index are zero by definition.

Although it is not obvious from Eqs. (2.2) and (2.3), the quantity $K$ so constructed will respect the symmetries of the Hamiltonian $H$ given in Eq. (1.1), as we discuss in Sec. V. In that section it will become clear that $K$ will satisfy the consequences of such theorems as that of Thompson,[6] which states that when $H$ is quadratic in the momenta, one need only look for autonomous first integrals which are either even or odd in the momenta.

Our program is to solve the Eq. (2.3) iteratively using a computer algebra system.[7] There are

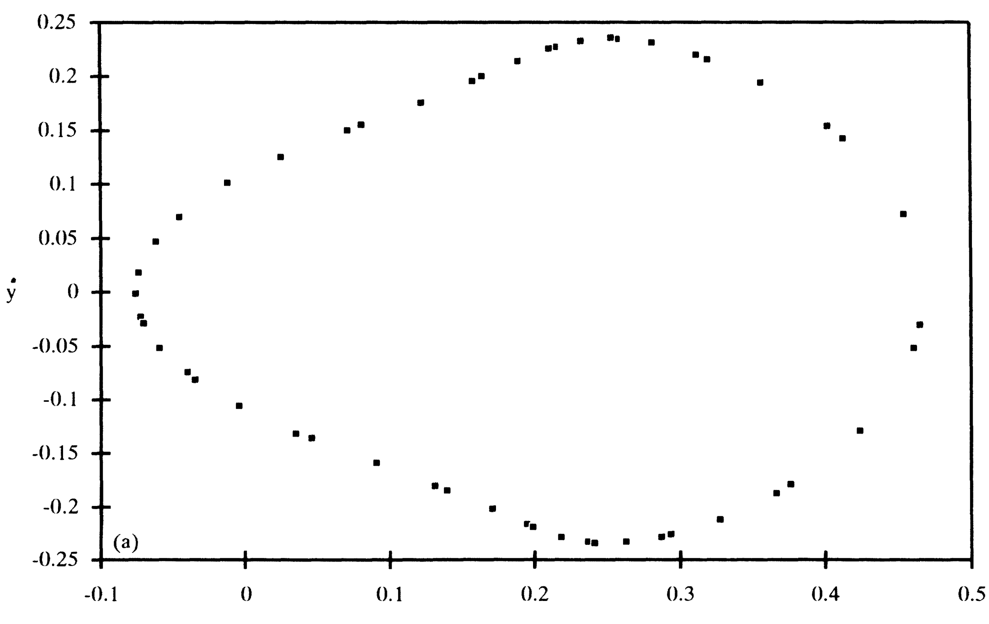



FIG. 2. (a) Graph of the Poincaré section for an energy of $H=0.08$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative. (b) Graph of the series $K$ through 7th, 11th, and 15th orders. The energy is $H=0.08$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative. (c) Graph of the series $K$ through 11th and 15th orders. The energy is $H=0.08$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative.

$N=(n+3)(n+2)(n+1)/6$ coefficients of order $n$ to be determined, and (2.3) represents the $N$ linear equations that are to be solved. Some of the properties of the solutions to (2.3) are discussed in Sec. III.

## III. IMPLICATIONS OF THE RECURSION RELATION

We begin the process of finding iterative solutions to (2.3) by solving for the first-order $b$ coefficients. (The irrelevant $b_{0000}$ has been set equal to zero.) Equations (2.3) for the four $n=1$ coefficients then have the unique solution

$$b_{1000}=b_{0100}=b_{0010}=b_{0001}=0 \ . \tag{3.1}$$

The first allowed nonzero coefficients are thus of second order, and the set of Eqs. (2.3) for these second-order $b$'s is clearly homogeneous since all the terms on the right-hand side of equality in (2.3) vanish by (3.1). We find that these equations have nontrivial solutions, and we deter-

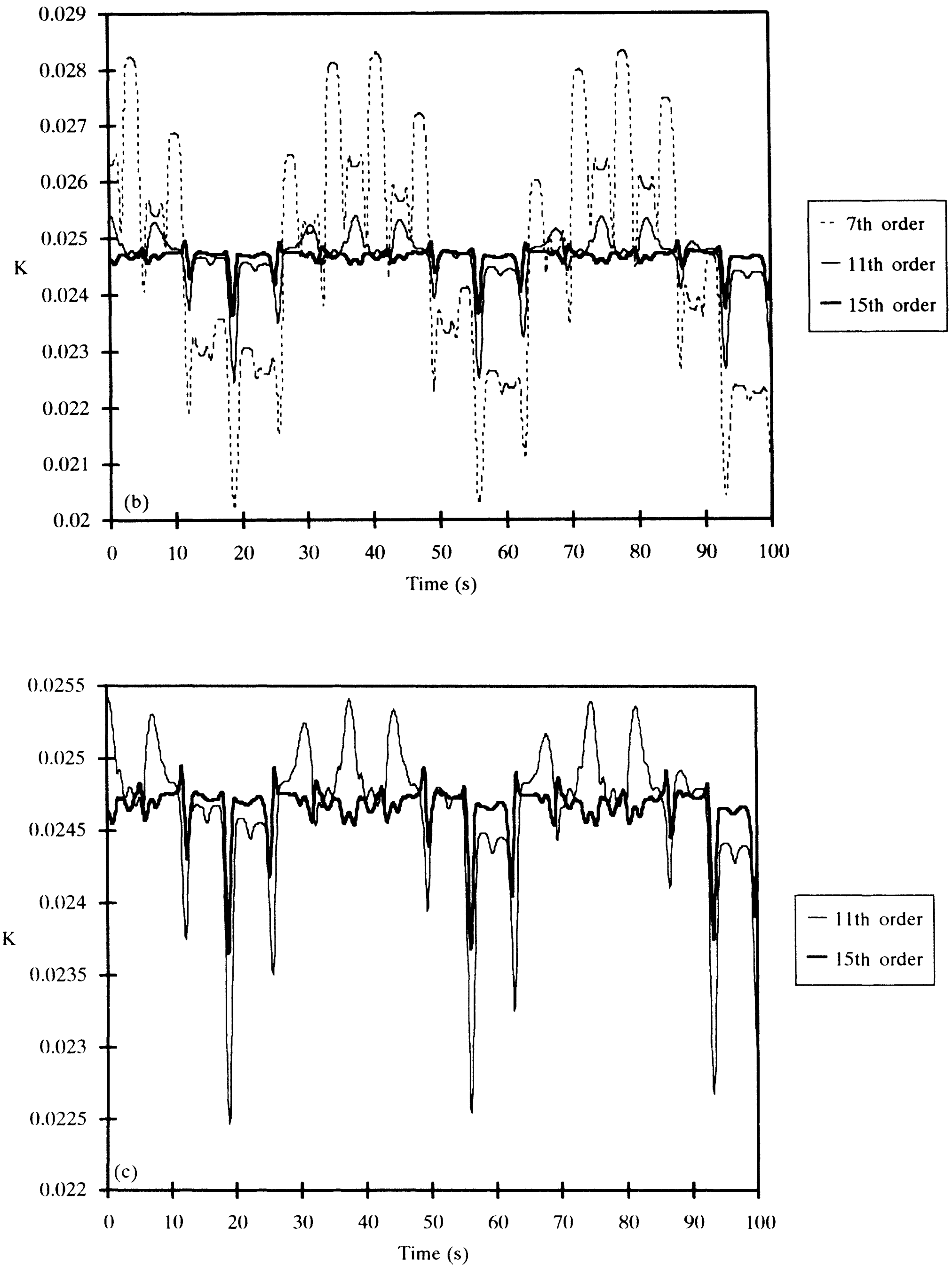



FIG. 2. (*Continued*).

mine the relative values of the $b$'s for these solutions. In second order, $K$ is thus made up of an arbitrary linear combination of the following linearly independent second-order terms (written here in terms of the original $x$ and $y$ variables):

$$h_1=\tfrac{1}{2}(\dot{x}^2+x^2),\quad h_2=\tfrac{1}{2}(\dot{y}^2+y^2)\ ,$$
$$h_3=\dot{x}\dot{y}+xy,\quad h_4=x\dot{y}-y\dot{x}\ . \tag{3.2}$$

We can now write $K$ in second order, which we designate $K_2$, as

$$K_2=r_1h_1+r_2h_2+r_3h_3+r_4h_4\ , \tag{3.3}$$

where the $r_i$ are arbitrary constants linearly related to the $b$ coefficients. Among the second-order terms in (3.2), we find the $x$ and $y$ subenergies $h_1$ and $h_2$ and the angular momentum $h_4$. These second-order terms have $b$

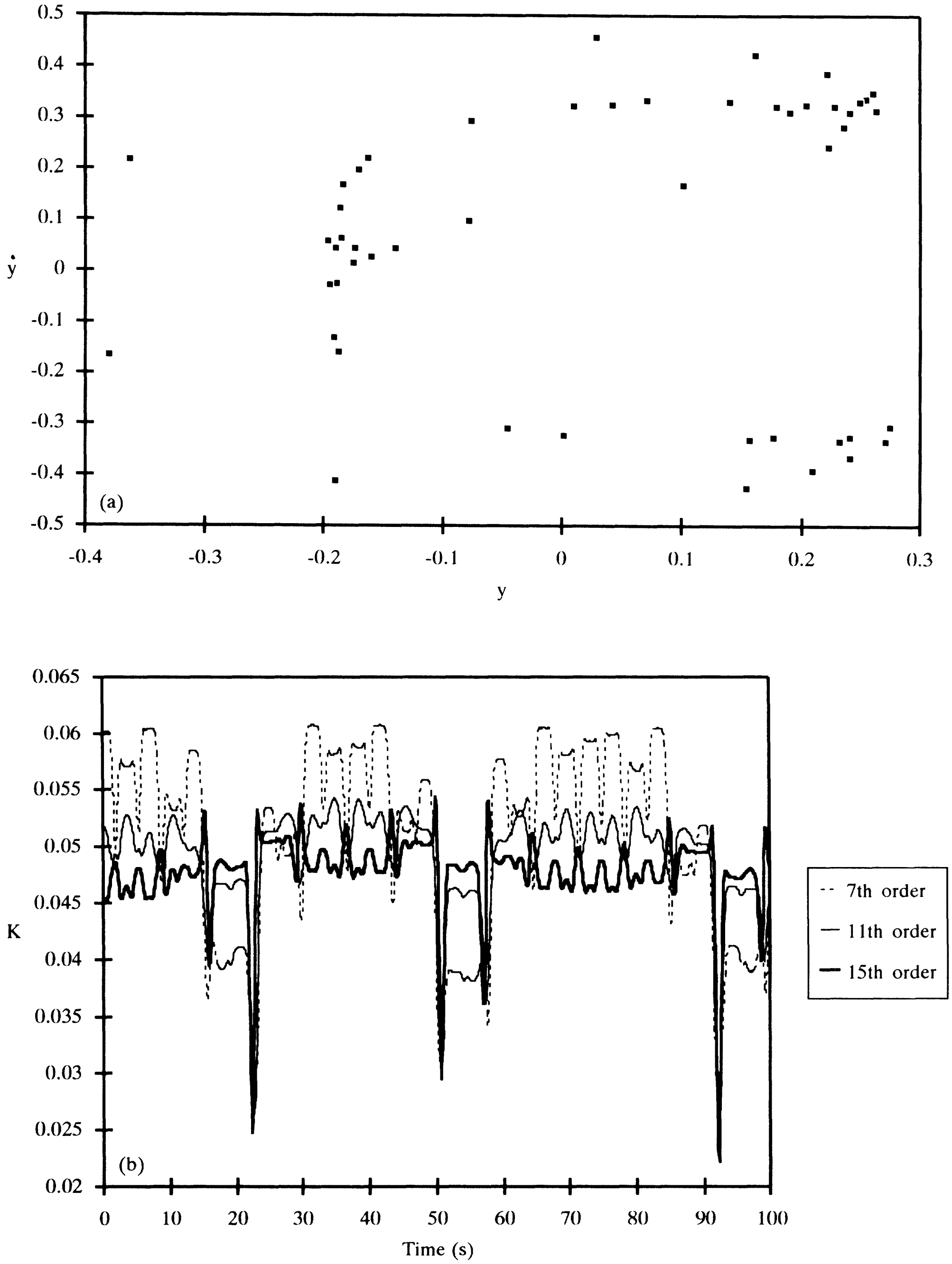



FIG. 3. (a) Graph of the Poincaré section for an energy of $H=0.11$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative. (b) Graph of the series $K$ through 7th, and 15th orders. The energy is $H=0.11$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative. (c) Graph of the series $K$ through 11th and 15th orders. The energy is $H=0.11$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative.

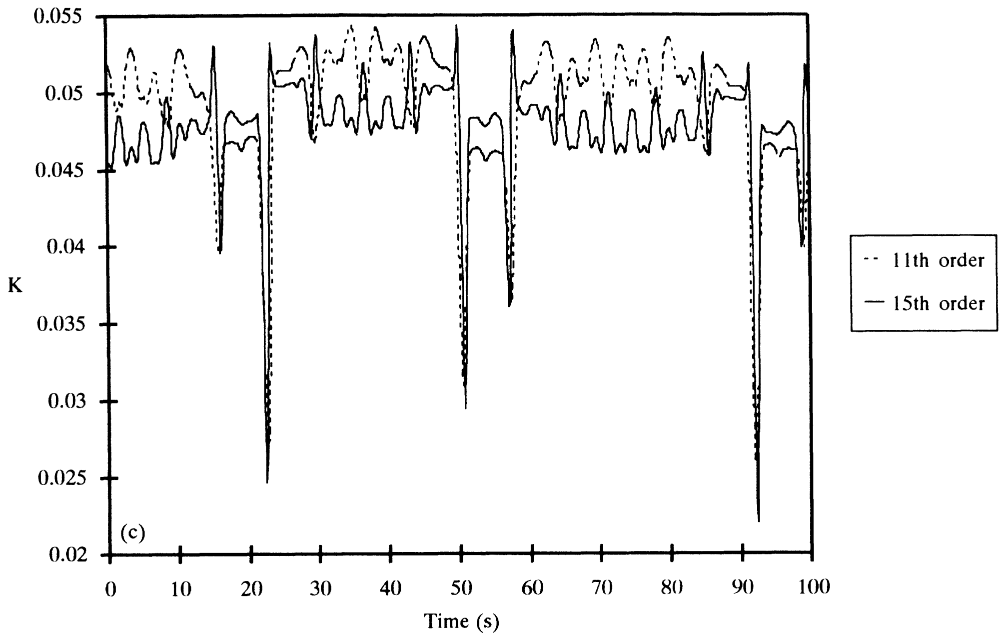



FIG. 3. (*Continued*).

coefficients which are solutions of the recursion-relation equations because if we take the time derivative of any of the $h_i$ in the (3.2) and insert the equations of motion [i.e., (2.1) written in the $x$ and $y$ variables], the resulting expressions are of third order. The next step is to find the third-order terms in $K$.

We find that Eqs. (2.3) determining the third-order $b$'s have a unique solution in terms of the second-order $b$'s. Each $h_i$ (3.3) in fact has a unique extension to third order. For example, to third order we have

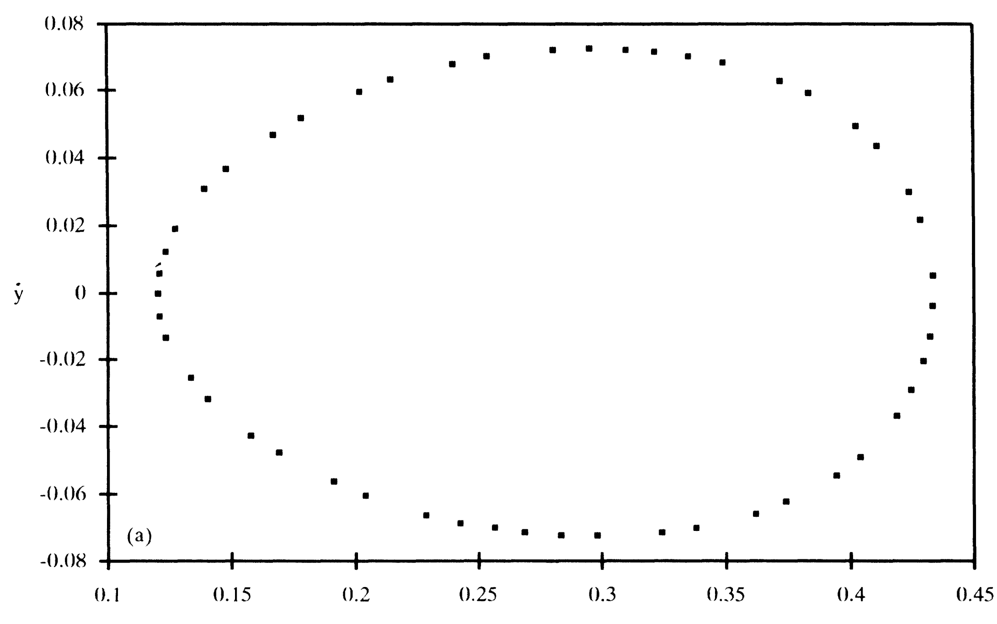


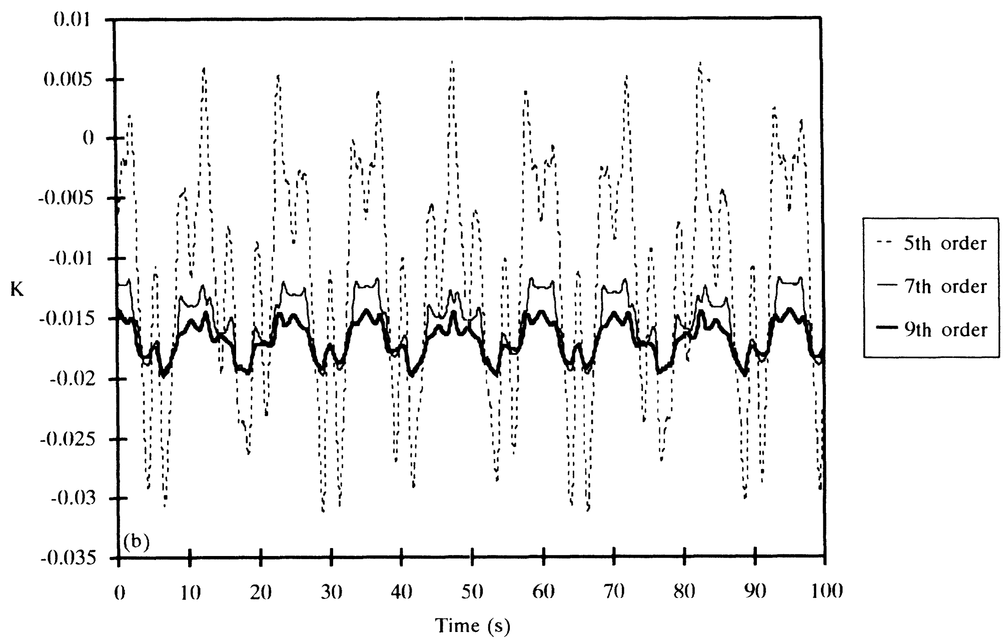

FIG. 4. (a) Graph of the Poincaré section for an energy of $H = 0.11$ with the initial conditions $x = \dot{y} = 0.0$, $y = 0.12$, and $\dot{x}$ positive. (b) Graph of the series $K$ through 5th, 7th, and 9th orders. The energy is $H = 0.11$ with the initial conditions $x = \dot{y} = 0.0$, $y = 0.12$, and $\dot{x}$ positive. (c) Graph of the series $K$ through 7th, 11th, and 15th orders. The energy is $H = 0.11$ with the initial conditions $x = \dot{y} = 0.0$, $y = 0.12$, and $\dot{x}$ positive. (d) Graph of the series $K$ through 11th and 15th orders. The energy is $H = 0.11$ with the initial conditions $x = \dot{y} = 0.0$, $y = 0.12$, and $\dot{x}$ positive.

$$h_4' = h_4 - [(x^2 - y^2)\dot{x} - 2xy\dot{y}] - \tfrac{2}{3}(\dot{x}^3 - 3\dot{x}\dot{y}^2) . \quad (3.4)$$

The time derivative of (3.4) gives now a quantity which is fourth order in the coordinates and velocities.

The extensions of the process to determine the fourth-order $b$'s presents a special problem because the equations determining these $b$'s have nontrivial homogeneous solutions. This situation can be simply understood from the fact that the time derivative of products of terms like $h_i h_j$, where $h_i$ and $h_j$ are two second-order terms from (3.2), are now fifth order in the coordinates and velocities. Therefore, the associated $b$ coefficients must satisfy (2.3) in homogeneous form. In fact, homogeneous solutions to the recursion relations (3.2) for the $b$'s always exist for any even-order $n$ due to the fact that products of $n/2$ factors of the second-order $h_i$ possess time derivatives of order $n+1$.

For a general even-order $n$, we write the $n$th-order $b$ coefficients as a column vector denoted $b^{(n)}$. The recursion relation (2.3) determining the $b$'s can then be written



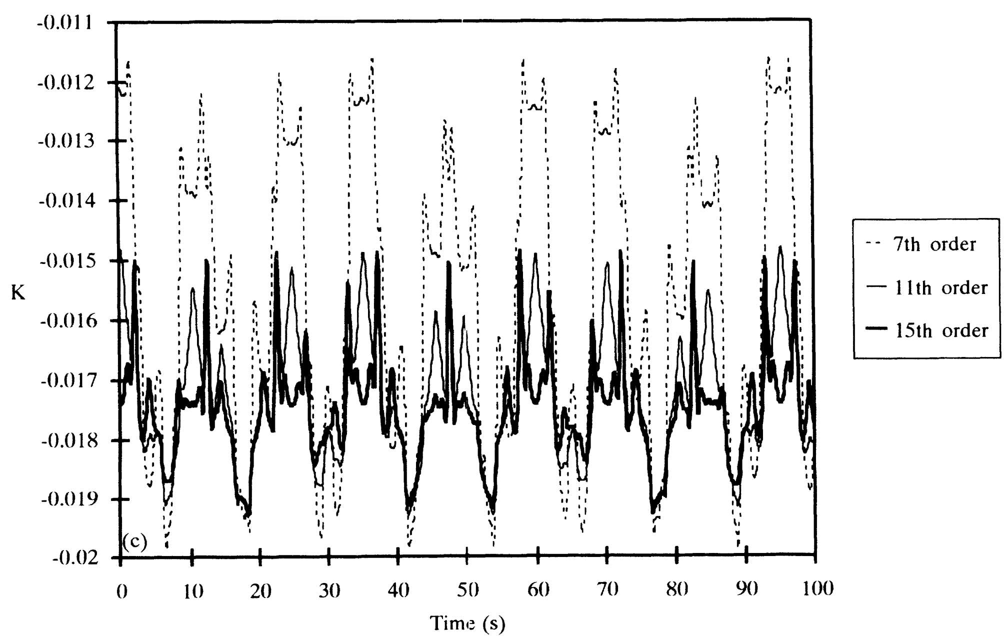




FIG. 4. (*Continued*).

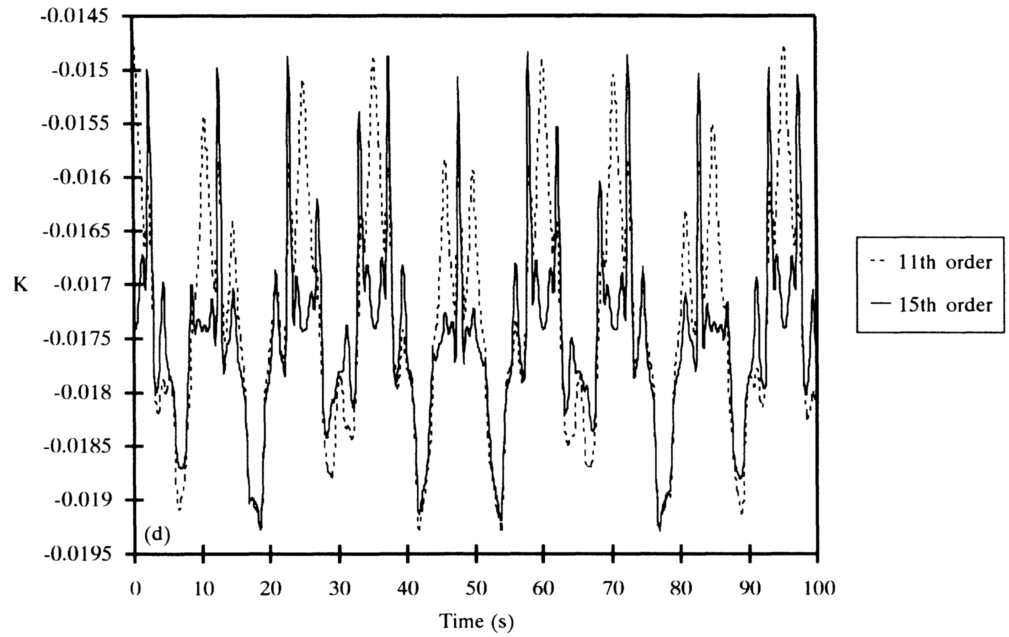



FIG. 4. (*Continued*).

as a matrix equation as follows:

$$Ab^{(n)}=c^{(n-1)} , \tag{3.5}$$

where $c^{(n-1)}$ is a column vector determined by the $b$ coefficients of order $n-1$. The matrix $A$, of course, depends upon the order $n$. The homogeneous solutions referred to above satisfy the equation

$$Ah^{(n)}=0 , \tag{3.6}$$

where $h^{(n)}$ is a column vector made up of $b$ coefficients that arise from products of $n/2$ factors of the $h_i$ mentioned above or linear combinations thereof. For the case $n=2$, as we saw above, there is no inhomogeneous term, and Eqs. (3.5) and (3.6) are the same.

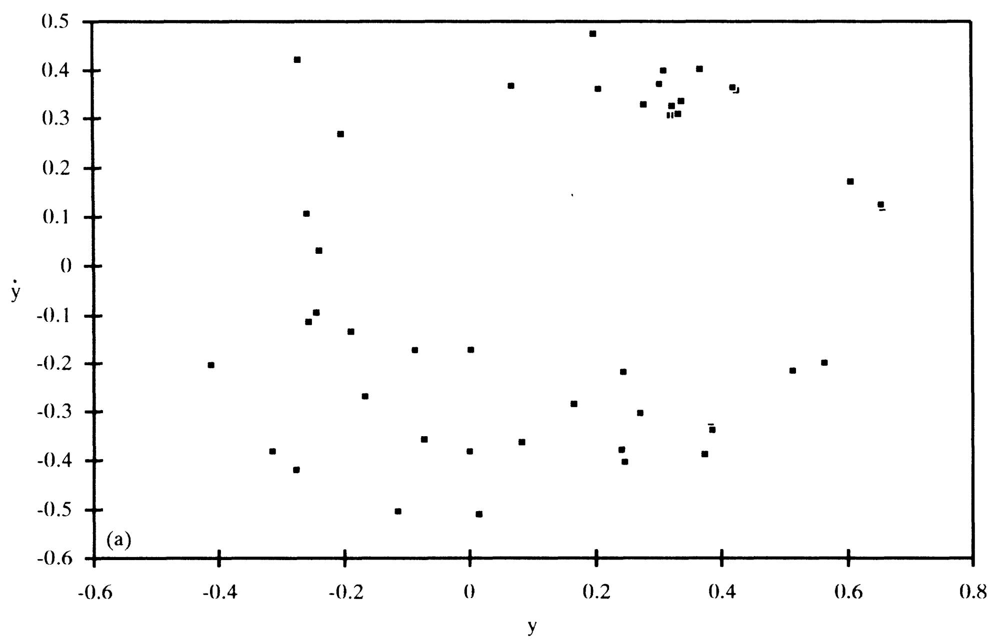



FIG. 5. (a) Graph of the Poincaré section for an energy of $H=0.14$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative. (b) Graph of the series $K$ through 7th, 11th, and 15th orders. The energy is $H=0.14$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative. (c) Graph of the series $K$ through 11th and 15th orders. The energy is $H=0.14$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative. (d) Graph of the series $K$ through 11th and 15th orders. The energy is $H=0.14$ with the initial conditions $x=\dot{y}=0.0$, $y=-0.15$, and $\dot{x}$ negative.

In the case $n=4$, we find that only certain linear combinations of the original second-order $h_i$ will lead to consistent solutions in fourth order. In fact, requiring the fourth-order equations to be consistent leads to further constraints on the *second-order* $b$'s. These constraints can be expressed as conditions on the $r_i$ in (3.3) as follows: For consistent solutions to the fourth-order equations to exist, we must have $r_1=r_2$ and $r_3=0$. This means that at second order, $K$ is restricted to be

$$K_2=r_1(h_1+h_2)+r_4h_4 \; , \tag{3.7}$$

where $r_1$ and $r_4$ are arbitrary. The term multiplying $r_1$ is just the second-order part of the total energy. This term just becomes the total energy when third-order corrections are made. Since we are interested in finding a $K$ conserved independently of the total energy, we shall take $r_1=0$. One choice of a fourth-order extension of $h_4$ that can now be constructed is given [using (3.2) and (3.4)] by[8]

$$h_4''=h_4'-(x\dot{y}-y\dot{x})(\dot{x}^2+\dot{y}^2-x^2-y^2) \; . \tag{3.8}$$

[Note that any multiple of $(h_1+h_2)h_4$ could be added to $h_4''$.] The time derivative of (3.8) gives a quantity which is

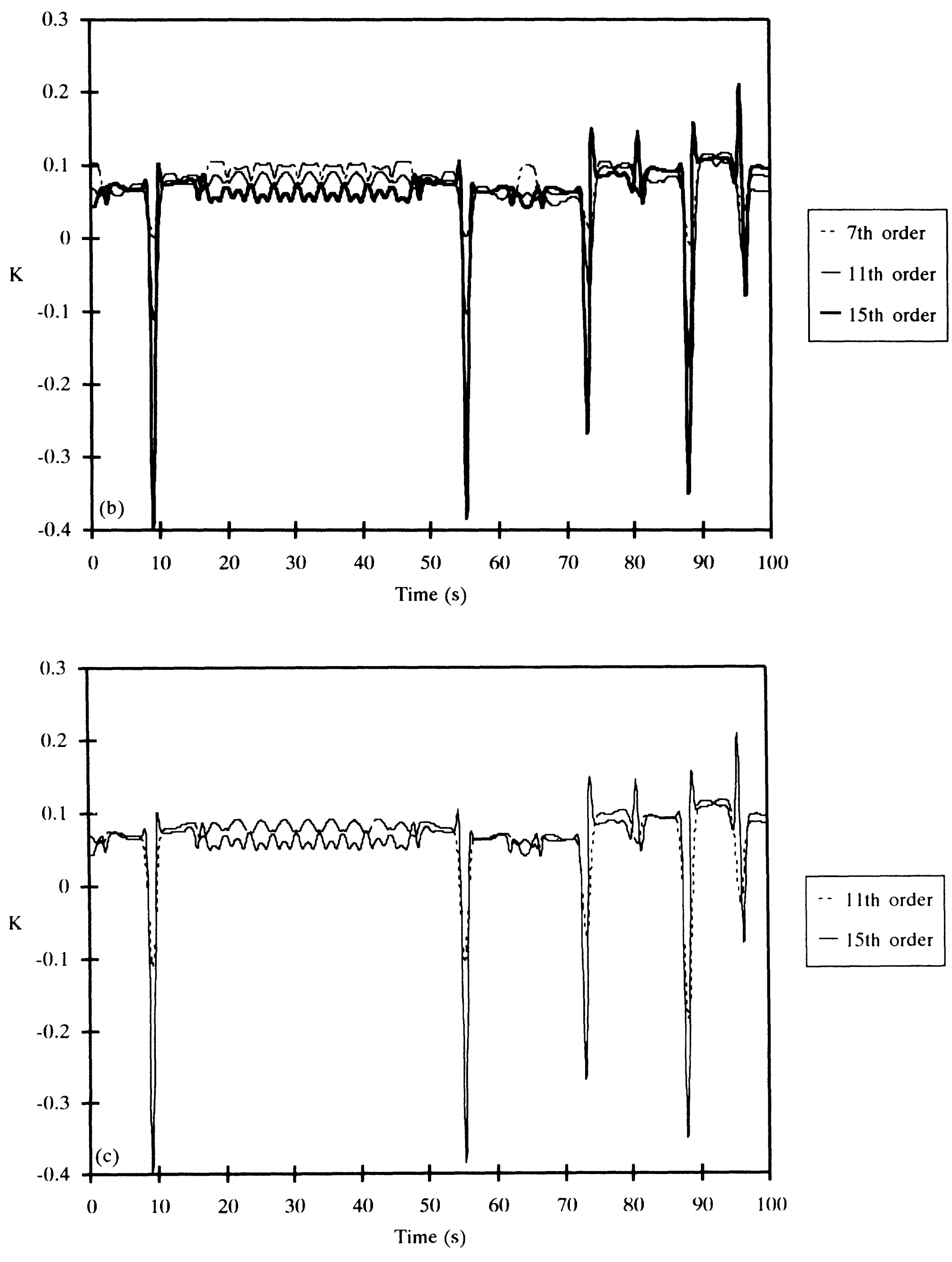



FIG. 5. (*Continued*).

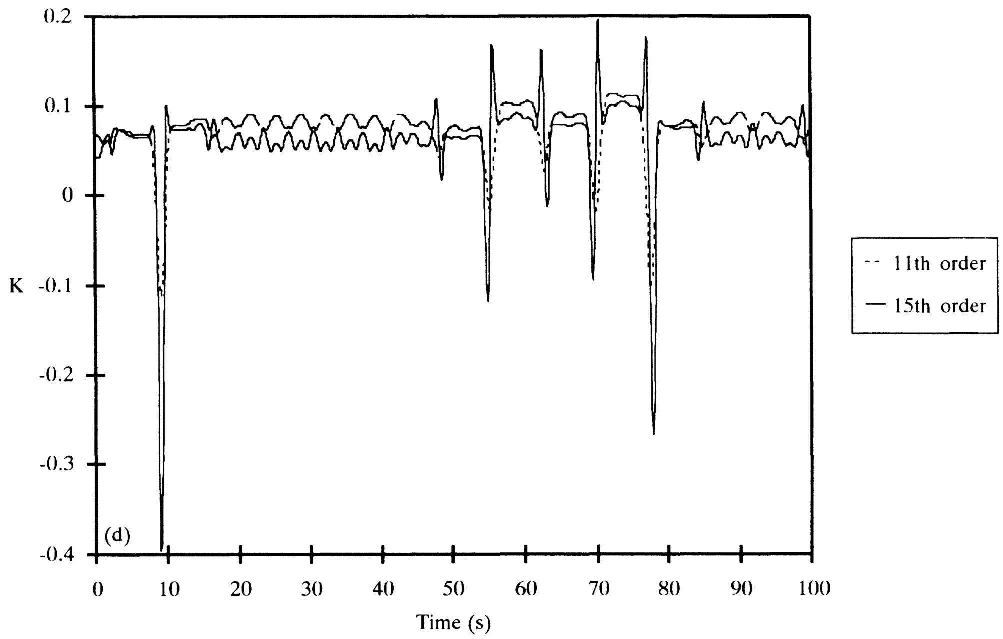



FIG. 5. (*Continued*).

fifth order in the coordinates and velocities.

We have now guaranteed that consistent solutions to the fourth-order equations exist. There is additional arbitrariness in the solutions, of course, due to the presence of solutions to the homogeneous equation (3.6) at this order. We can, therefore, introduce a general expression for $K$ through fourth order characterized by more $r$-type parameters, the number of such new parameters corresponding to the number of linearly independent solutions to the homogeneous equation.

Proceeding to fifth order, we find a unique solution to Eq. (3.5) in terms of $r_4$ and the other $r$ parameters introduced at fourth order as just discussed, It should be remarked that Eq. (2.4) determining the $b$ coefficients for odd-order $n$ always possess unique solutions regardless of the right-hand side of the equation. This is because there are no homogeneous solutions to the equations for odd order, even though homogeneous solutions always exist at even order. Thus matrix equations of theorm (3.5) also exist for odd $n$, but equations of the form (3.6) have no solution in this case.

When the procedure is carried to sixth order, it is found that there are no consistent solutions unless $r_4=0$. Thus we conclude that there are no conserved quantities $K$ of the form (2.2) with nonvanishing $b$ coefficients below fourth order (except, of course, for the total energy).

## IV. CONSERVED QUANTITY WITH NONVANISHING FOURTH-ORDER TERMS

In Sec. III we have seen that no conserved quantity $K$ of the form (2.2) exists with nonvanishing coefficients below fourth order if we take $r_1=0$, thereby eliminating a trivial addition of the total energy. It is natural to ask what happens if we continue the procedure begun in Sec. III, having now shown that $r_4$ and, hence, $c^{(3)}$ are both required to vanish in order to make the fourth-order equations consistent. Is there any hope that we can generate an infinite sequence of nonvanishing coefficients, or will consistency requirements force all of the coefficients to vanish for any $K$ which is not just a function of the total energy?

To answer these questions, we can turn to the work of Gustavson[4] mentioned in Sec. I. Using an approach which is totally different from the one given here, he developed a prescription for a power series of the form (3.2) whose first nonvanishing terms, excluding a term linear in $H$, were of order $n=4$. His program made it clear that there should be an infinite number of nonvanishing coefficients in the series. Thus we expect our procedure to produce an infinite series of nontrivial terms which meet the consistency requirements.

So far, we have explicitly verified this by using the recursion relation to determine nonvanishing coefficients for orders $n=4-15$ giving us over twice as many orders for the conserved quantity $K$ as that obtained by Gustavson. We discuss our process in detail in the following sections. Our approach, of course, does not require us to make the sequence of canonical transformations used in the approach of Gustavson, but enables us to solve for the $b$ coefficients, thus giving $K$ directly in terms of the original coordinates and velocities.

Furthermore, our approach enables us to focus on the important question of the ambiguities of the conserved quantity $K$. As we have indicated in Sec. III, at each even order there are homogeneous solutions to the recursion relation (2.3). Thus, at each even order, if consistent solutions to the recursion relation exist, there will be ambiguity in the solution for the $b$ coefficients since any combination of homogeneous solutions may be added. As we explain in detail in Sec. VII, part of this ambiguity is removed by the restrictions placed on the coefficients at one even order to ensure the existence of solutions at the next even order. Any remaining ambiguity is removed by requiring that the "norm" of the coefficients at each order be as small as possible, as we explain in Sec. VI. This last requirement should optimize the convergence possibilities of the series.

## V. SYMMETRIES OF $H$ AND THE $B$ COEFFICIENTS

In this section we discuss the symmetries of the $b$ coefficients. A knowledge of these symmetries simplifies the calculation of the coefficients, which is described in detail in Sec. VII. We find that the Hénon-Heiles Hamiltonian (1.1) is symmetric under the following discrete transformations:

$$z \to z, \quad \dot{z} \to -\dot{z} , \tag{5.1}$$

$$z \to -\bar{z}, \quad \dot{z} \to -\dot{\bar{z}} , \tag{5.2}$$

$$z \to e^{i\phi} z, \quad \dot{z} \to e^{i\phi} \dot{z}, \quad e^{3i\phi} = 1 . \tag{5.3}$$

The symmetry (5.1) is just time reversal; (5.2) corresponds to a reflection through the $x$ origin; (5.3) represents rotations in the $x$-$y$ plane by multiples of 120°. Thus, if a physical trajectory is subjected to any of the transformations (5.1)–(5.3), the transformed trajectory is also physical (i.e., is a solution to the equations of motion). It follows that if $K$ is a conserved quantity and if the variables of $K$ are transformed according to any of the transformations (5.1)–(5.3), then the resulting function must also be a conserved quantity, and one conserved quantity generates others. Thus, e.g., if $K(z,\bar{z},\dot{z},\dot{\bar{z}})$ is a conserved quantity, so also is $K(z,\bar{z},-\dot{z},-\dot{\bar{z}})$. Similar statements can be made about the other symmetries. It is then possible by taking sums and differences of conserved quantities to construct quantities which exhibit even or odd behavior under the symmetry transformations (5.1)–(5.3) of the Hamiltonian.

As indicated above in Sec. IV, the work of Gustavson[4] shows that a conserved quantity $K$ can be constructed whose lowest-order nonvanishing terms are $n=4$. His fourth-order terms consist of a linear combination of the square of the angular momentum $h_4$ and the fourth-order terms in the square of the energy $(h_1+h_2)^2$. Such terms are even (or invariant) under all the symmetries (5.1)–(5.3). The Poincaré section of Fig. 1 suggests that if a conserved quantity independent of the energy exists, there can be only one such quantity, since otherwise the curve in the section would degenerate to a single point. Since we are constructing a single conserved quantity $K$, we require it to have in all orders the even symmetry of the Hamiltonian under the above transformations.

This immediately leads to a set of restrictions on the $b$ coefficients. Assuming $K$ to be invariant at all orders under the transformations (5.1)–(5.3) leads to the following conditions on the $b_{pqrs}$:

$$b_{pqrs} = (-1)^{p+q} b_{qpsr} , \tag{5.4}$$

$$r+s = \text{even integer} , \tag{5.5}$$

$$p-q+r-s = 3m, \quad m = 0, \pm 1, \pm 2, \ldots . \tag{5.6}$$

The conditions (5.4)–(5.6) greatly reduce the number of unknown coefficients which must be determined, and these conditions are introduced into the computer program before the recursion relation for the $b$'s is solved at each order.

## VI. SPECIAL SUBSET OF HOMOGENEOUS SOLUTIONS

We now identify a special subset of the homogeneous solutions at even order, which all satisfies the symmetry conditions (5.4)–(5.6). As we shall see, these special solutions can be used to determine the power series (2.2), which has the smallest "norm" for the coefficients $b$.

First, we note that if $K$ is a conserved quantity independent of the energy $H$, then we can add to $K$ linear combinations of terms of the form $K^n H^m$ and still have a conserved quantity independent of $H$ and which has the same even symmetry. This fact points to a fundamental arbitrariness in the conserved quantity $K$ and shows that the recursion relations (2.3) cannot have at each order a unique solution. As we now explain, this arbitrariness is related to the presence of homogeneous solutions to the recursion-relation equations at even-order $n$ discussed in Sec. III.

At even order there are homogeneous solutions to the recursion relations which can be associated with terms of the form $K^n H^m$ in the following manner. Calling $K_4$ the fourth-order part of $K$ and $H_2$ the second-order part of $H$ [i.e., $(h_1+h_2)$], there are homogeneous solutions at various even orders associated with powers of $K_4$ and $H_2$ as follows:

$$\begin{aligned} &H_2^2, \quad n=4 \\ &H_2^3, \quad H_2 K_4, \quad n=6 \\ &H_2^4, \quad H_2^2 K_4, \quad K_4^2, \quad n=8 \end{aligned} \tag{6.1}$$

where the generalization to higher even orders is evident and each of these terms represents a particular $K^n H^m$ in lowest order. Note that the $b$ coefficients associated with terms in (6.1) all possess the even symmetries expressed in (5.4)–(5.6).

At even-order $n$, the special subsets of homogeneous solutions of the form (6.1) each give rise to column vectors which we designate $h_s^{(n)}$ satisfying (3.6). We shall insist that these column vectors $h_s^{(n)}$ be orthogonal to the column vector $b^{(n)}$, which is a solution to the recursion relation at this order thus satisfying (3.5). Thus we require

$$(h_s^{(n)})^T b^{(n)} = 0 . \tag{6.2}$$

In effect, our procedure throws out the uninteresting terms $K^n H^m$ by requiring that our solution to (3.5) be orthogonal to the $K^n H^m$ in the sense of (6.2) in the lowest order in which they appear. The requirement (6.2) thus wipes out the components of $b^{(n)}$ in the subspace spanned by $h_s^{(n)}$, which results in minimizing the norm of the vector $b^{(n)}$ at each even order. This process also eliminates some of the arbitrariness in $K$ introduced by the homogeneous solutions at even order. As we shall indicate in Sec. VII, arbitrariness introduced in the even-order solutions by the presence of homogeneous terms other than

(6.1) is eliminated by the requirement that consistent solutions exist at even orders. The end result is a completely unique formal power series for the conserved quantity $K$.

## VII. SYSTEMATIC GENERATION OF SOLUTIONS TO THE RECURSION RELATIONS

We now wish to summarize our discussion of the previous sections. Described here is the systematic process by which we generate consistent and unique solutions to the recursion relations (2.3) and deal with the ambiguities which arise due to the presence of homogeneous solutions to these equations at even order. All calculations are done using the program MACSYMA.[7] The following is an outline of the first few steps taken to produce a unique formal power series for the conserved quantity $K$, the generalization higher orders being straightforward.

(i) $n=4$. The solution for the $b$ coefficients is given by the homogeneous equation (3.5) with $n=4$, subject to the symmetry conditions (5.4)–(5.6). Two linearly independent solutions result, corresponding to $H_2^2$ and the square of the angular momentum $h_4^2$. We now minimize the norm of the fourth-order $b$ coefficients by requiring that our solution be orthogonal to the subspace spanned by $H_2^2$ as discussed in Sec. VI. This requirement can be expressed in the form (6.2), where the column vector $h_s^{(4)}$ consists of the $b$ coefficients of $H_2^2$. This now determines a unique fourth-order solution for the $b$'s, up to an allover multiplicative factor which we set equal to 1.

(ii) $n=5$. At this order we must solve a set of linear equations of the form (3.5) with $n=5$. The inhomogeneous term $c^{(4)}$ is determined by the fourth-order coefficients resulting from (i). In this case, as for every odd order, there is a unique solution for the fifth-order $b$ coefficients.

(iii) $n=6$. We again have homogeneous solutions. After requiring the symmetry conditions (5.4)–(5.6), we find that there are three linearly independent homogeneous solutions two of which are in the special subset (6.1). Consistent solutions to (3.5) can again be found in this case. By requiring that (6.2) be satisfied using the two special $n=6$ homogeneous solutions in (6.1), we minimize the norm of the sixth-order coefficients. The residual arbitrariness in the solution for the sixth-order $b$ coefficients is characterized by a single parameter associated with the third homogeneous solution.[9] This parameter is carried along until it is determined [see step (v)] at eight order.

(iv) $n=7$. As mentioned earlier, there are no homogeneous solutions at this or any other odd order. As in the case $n=5$, there is a unique solution to (3.5) for the seventh-order coefficients. The inhomogeneous term $c^{(6)}$ and, therefore, the solution for the seventh-order coefficients will depend upon the single arbitrary parameter introduced at sixth order.

(v) $n=8$. At this order there are four linearly independent homogeneous solutions satisfying the symmetry requirements, three of which are in the special set (6.1). There will be consistent solutions to (3.5) if and only if the arbitrary parameter introduced at order $n=6$ in step (iii) above takes on a specific value.[9] At this point the power series for $K$ is now uniquely specified through seventh order. The eighth-order coefficients are now determined to within four arbitrary parameters associated with the four homogeneous solutions. By requiring that (6.2) be satisfied with the three special $n=6$ homogeneous solutions in (6.1), the norm of coefficients is again minimized, and the resulting arbitrariness in the solution for the eighth-order order $b$ coefficients is again characterized by a single parameter. This parameter is now carried along until it is subsequently determined at tenth order.

The procedure outlined here for the first few orders is extended to determine a unique set of $b$ coefficients through $n=15$. At each even order, there is some residual arbitrariness after the orthogonalization requirements (6.2) have been imposed. This residual arbitrariness can be characterized by one or more parameters. These parameters are then completely determined at the next even order by the requirements of consistency.

Thus we generate by a well-defined sequence of steps a unique formal power-series expression for the conserved quantity $K$ in the Hénon-Heiles problem. We have relied on the results of Gustavson to motivate the existence of such a power series for $K$. The fact that arbitrary parameters present at one even order all become completely determined at the next even order once the orthogonalization requirement (6.2) is enforced is by no means obvious.

## VIII. CALCULATIONAL RESULTS FOR $K$

In this section we present our results for the form for the quantity $K$ calculated through 15th order. We also give preliminary results for the calculation of $K$ along selected representative trajectories with different energies and comparable initial conditions. We use the Verlet algorithm[10] for the numerical integration, and except where indicated otherwise, the mesh size used in the numerical computation of the trajectories is 0.01 s. We call our dimensionless time unit seconds (s) for convenience.

Table I gives our results for the $b$ coefficients from 4th to 15th order. The notation used in the table is

$$[pqrs]=b_{pqrs} . \tag{8.1}$$

Parentheses are used within the brackets to eliminate ambiguity when any of the subscripts are larger than 9. Also, when a set of $b$ coefficients is related through the symmetry condition (5.4), only one of these coefficients is given in Table I. For compactness we are presenting the coefficients as decimal fractions instead of the exact rational numbers which we calculated.

We have calculated the quantity $K$ along several trajectories with different energies and initial conditions. These, along with the corresponding Poincaré sections, are presented graphically in Figs. 1–5. A much more detailed analysis of the numerical properties of $K$ is being undertaken and will be discussed in a later paper. We present representative results for four energies, $H=0.05$, 0.08, 0.11, and 0.14. From Hénon-Heiles, we expect no chaotic trajectories to occur at the first two energies,

TABLE I. This table gives our decimal representations for the $b$ coefficients from 4th to 15th order. The notation used is $[pqrs]=b_{pqrs}$. Parentheses are used within the square brackets to eliminate ambiguity when any of the subscripts are larger than 9. Also, when a set of $b$ coefficients is related through the symmetry condition (5.4), only one of these coefficients is given in the table.

| | | | |
|---|---|---|---|
| | | Order 4 | |
| [0220]=1.5 | [1111]=-1.0 | [2200]=1.0 | [0022]=1.0 |
| | | Order 5 | |
| [0311]=−2.333 333 | [1202]=3.0 | [1400]=0.666 667 | [0113]=−2.0 |
| [0140]=−2.0 | | | |
| | | Order 6 | |
| [1320]=−3.847 222 | [2211]=7.652 778 | [3300]=−1.236 111 | [0231]=4.152 778 |
| [1122]=−7.347 222 | [0033]=0.319 444 | [0402]=2.25 | [0600]=0.361 111 |
| [0204]=2.75 | [0006]=0.916 667 | | |
| | | Order 7 | |
| [1411]=2.831 481 | [3202]=−6.776 852 | [2500]=−1.513 889 | [1213]=5.835 185 |
| [1240]=4.829 629 | [0520]=0.917 593 | [0322]=−2.519 444 | [0124]=−0.670 37 |
| [0151]=−0.670 37 | | | |
| | | Order 8 | |
| [2420]=3.841 354 | [3311]=−8.171 499 | [4400]=2.558 358 | [1331]=0.236 277 |
| [2222]=1.304 377 | [0440]=−2.968 022 | [0242]=−4.141 979 | [1133]=5.734 674 |
| [0044]=−0.637 321 | [0611]=−2.875 81 | [1502]=−3.620 023 | [1700]=−1.040 625 |
| [0413]=−5.278 665 | [1304]=−4.664 159 | [0215]=−4.887 616 | [1106]=−1.629 205 |
| [0017]=−1.629 205 | | | |
| | | Order 9 | |
| [2511]=3.359 273 | [3402]=7.861 303 | [3600]=3.025 071 | [2313]=19.601 97 |
| [2340]=1.762 42 | [1620]=−2.421 289 | [1422]=−18.4352 | [1224]=−24.2819 |
| [1251]=−17.185 67 | [0531]=9.450 799 | [0360]=5.080 957 | [0333]=15.1289 |
| [0135]=4.730 818 | [0162]=4.730 818 | [0702]=−1.685 069 | [0900]=−0.693 994 |
| [0504]=−1.629 205 | [0306]=−0.543 068 | [0108]=0.0 | |
| | | Order 10 | |
| [3520]=3.694 813 | [4411]=−10.804 08 | [5500]=−4.499 268 | [2431]=−59.276 13 |
| [3322]=87.510 83 | [1540]=18.190 42 | [1342]=59.9024 | [2233]=−88.640 85 |
| [0451]=−12.295 31 | [0253]=−5.887 928 | [1144]=13.879 81 | [0055]=0.420 791 |
| [1711]=5.009 957 | [2602]=11.733 59 | [2800]=4.400 819 | [1513]=28.0709 |
| [2404]=4.052 959 | [1315]=37.778 04 | [2206]=−7.001 715 | [0820]=4.317 128 |
| [0622]=5.597 295 | [0424]=4.066 992 | [0280]=−3.455 926 | [1117]=13.643 97 |
| [0226]=6.642 261 | [0028]=3.366 062 | | |
| | | Order 11 | |
| [3611]=−27.128 94 | [4502]=−0.403 579 | [4700]=−8.886 767 | [3413]=−136.4467 |
| [3440]=−52.479 21 | [2720]=−5.398 895 | [2522]=77.071 67 | [2324]=96.22961 |
| [2351]=91.8488 | [1631]=−28.099 81 | [1460]=−36.595 24 | [1433]=−42.23662 |
| [1235]=27.996 91 | [1262]=2.221 826 | [0740]=−13.864 91 | [0542]=−31.83882 |
| [0371]=−8.947 837 | [0344]=−42.997 77 | [0146]=−17.183 39 | [0173]=−17.18339 |
| [0911]=−5.448 897 | [1(10)00]=3.185 361 | [1802]=16.146 27 | [0713]=−14.69613 |
| [1604]=28.948 85 | [0515]=−16.411 61 | [1406]=25.387 77 | [0317]=−9.993542 |
| [1208]=12.237 58 | [01(10)0]=−2.719 463 | [0119]=−2.7194 63 | |
| | | Order 12 | |
| [4620]=−44.727 03 | [5511]=109.4422 | [6600]=7.987 373 | [3531]=230.0708 |
| [4422]=−333.28 03 | [2640]=−56.790 49 | [2442]=0.645 53 | [3333]=24.286 45 |
| [1551]=−34.948 62 | [1353]=−230.8863 | [2244]=299.4666 | [0660]=23.489 84 |
| [0462]=84.312 66 | [0264=56.662 01 | [1155]=−107.3596 | [0066]=0.994 068 |
| [2811]=−1.689 138 | [3702]=−25.067 83 | [3900]=−12.3607 | [2613]=−37.119 44 |
| [3504]=28.471 68 | [2415]=10.151 77 | [3306]=62.730 35 | [1920]=−15.022 03 |
| [1722]=−87.795 01 | [1524]=−231.2888 | [1380]=22.082 94 | [2217]=100.4803 |

TABLE I. (*Continued*)

| | | | |
|---|---|---|---|
| [1326]=−270.0377 | [0831]=40.437 63 | [0633]=121.0204 | [0435]=133.3649 |
| [0291]=49.277 57 | [1128]=−100.3819 | [0237]=47.450 82 | [0039]=−0.608 916 |
| [0(10)02]=13.213 76 | [0(12)00]=1.7482 19 | [0804]=36.708 43 | [0606]=51.679 85 |
| [0408]=40.009 08 | [020(10)]=16.275 57 | [000(12)]=2.712 597 | |

Order 13

| | | | |
|---|---|---|---|
| [4711]=183.9202 | [5602]=−131.8388 | [5800]=9.977 557 | [4513]=528.8384 |
| [4540]=359.0139 | [3820]=−23.133 77 | [3622]=−201.5325 | [3424]=519.1566 |
| [3451]=27.944 58 | [2731]=29.291 11 | [2560]=166.13 | [2533]=−722.8276 |
| [2335]=−1319.621 | [2362]=−832.6909 | [1840]=42.226 79 | [1642]=532.3505 |
| [1471]=292.1612 | [1444]=1002.599 | [1246]=392.1408 | [1273]=417.6925 |
| [0751]=−98.712 28 | [0580]=−17.070 58 | [0553]=−221.5038 | [0382]=−118.3449 |
| [0355]=−88.978 28 | [0157]=17.0345 | [0184]=17.0345 | [1(10)11]=−20.328 84 |
| [2(11)00]=−14.646 75 | [2902]=−56.3344 | [1813]=−38.220 72 | [2704]=−71.208 92 |
| [1615]=54.574 29 | [2506]=−42.184 89 | [1417]=171.4117 | [2308]=−20.342 98 |
| 12(10)0]=6.965.256 | [1219]=122.1018 | [0(11)20]=10.5397 | [0922]=−12.852 52 |
| [0724]=−129.2386 | [0526]=−208.3652 | [0328]=−126.5958 | [0(11)1]=−25.5859 |
| [012(10)]=−25.5859 | | | |

Order 14

| | | | |
|---|---|---|---|
| [5720]=231.7731 | [6611]=−473.7999 | [7700]=9.929 093 | [4631]=−205.0175 |
| [5522]=289.0572 | [3740]=−10.720 35 | [3542]=−2217.106 | [4433]=2997.25 |
| [2651]=892.8157 | [2453]=2227.853 | [3344]=−3205.876 | [1760]=−336.7448 |
| [1562]=−607.725 | [1364]=126.1688 | [2255]=88.944 38 | [0671]=−200.1801 |
| [0473]=−287.3489 | [0275]=−231.9005 | [1166]=383.1568 | [0077]=−11.520 61 |
| [3911]=277.9768 | [4(10)00]=23.095 97 | [4802]=−171.803 | [3713]=976.7071 |
| [4604]=−791.2013 | [3515]=523.6336 | [4406]=−916.001 | [2(10)20]=−79.8271 |
| [2822]=−413.5697 | [2624]=144.267 | [2480]=−381.968 | [3317]=−478.8429 |
| [2426]=920.2822 | [1931]=249.7401 | [1733]=232.0079 | [1535]=−125.5053 |
| [1391]=−291.8641 | [2228]=262.3014 | [1337]=128.8331 | [0(10)40]=−144.5753 |
| [0842]=−351.2261 | [0644]=−367.5704 | [04(100]=−54.172 44 | [0446]=−314.3394 |
| [02(10)2]=−113.5544 | [1139]=176.2321 | [0248]=−151.7121 | [004(10)]=−12.719 21 |
| [0(12)11]=−30.359 04 | [1(11)02]=−34.973 75 | [1(13)00]=−9.279 646 | [0(19)13]=−124.5257 |
| [1904]=−55.491 21 | [0815]=−208.4543 | [1706]=−57.590 14 | [0617]=−186.9277 |
| [1508]=−48.451 87 | [0419]=−101.8696 | [130(10)=−26.249 16 | [021(11)]=34.777 79 |
| [110(12)]=−5.7963 | [001913)]=−5.7963 | | |

Order 15

| | | | |
|---|---|---|---|
| [5811]=−766.296 | [6702]=556.8142 | [6900]=23.015 38 | [5613]=11.944 71 |
| [5640]=−231.4916 | [4920]=364.4094 | [4722]=−276.6554 | [4524]=−5035.76 |
| [4551]=−3716.904 | [3831]=−86.856 97 | [3660]=1491 377 | [3633]=3283.831 |
| [3435]=−1466.569 | [3462]=559.4066 | [2940]=459.2791 | [2742]=−522.1302 |
| [2571]=−700.2444 | [2544]=4772.193 | [2346]=10824.03 | [2373]=7286.126 |
| [1851]=−49.865 81 | [1680]=1003.742 | [1653]=−5100.347 | [1482]=−3330.23 |
| [1455]=−9562.744 | [1257]=−4661.177 | [1284]=−4233.177 | [0960]=498.1489 |
| [0762]=2917.367 | [0591]=1545.467 | [0564]=4333.167 | [0393]=1844.217 |
| [0366]=2287.732 | [0168]=285.3332 | [0195]=285.3332 | [2(11)11]=421.5545 |
| [3(10)02]=−171.9098 | [3(12)00]=35.4716 | [2913]=2350.75 | [3804]=−1317.13 |
| [2715]=4563.563 | [3606]=−2661.629 | [2517]=4257.682 | [3408]=−2517.029 |
| [23(10)0]=1213.19 | [2319]=2093.162 | [1(12)20]=−92.922 31 | [1(10)22]=−1238.658 |
| [1824]=−3471.631 | [1626]=−4496.391 | [1428]=−3271.283 | [12(11)1]=−455.0692 |
| [122(10)]=−1260.646 | [0(11)31]=485.3791 | [0933]=1835.139 | [0735]=2902.98 |
| [0537]=2440.261 | [03(12)0]=245.9171 | [0339]=1059.652 | [01(12)2]=179.0171 |
| [013(11)]=179.0171 | [0(13)02]=−23.537 29 | [0(15)00]=−5.162 242 | [0(11)04]=−45.364&75 |
| [0906]=−48.426 33 | [0708]=−31.113 65 | [050(10)]=−11.5926 | [030(12)]=−1.9321 |
| [010(14)]=0.0 | | | |

whereas chaotic trajectories are expected at the two higher energies.

In Fig. 1(a) we show the Poincaré section corresponding to the plane $x=0$, for a trajectory with $H=0.05$ and with initial conditions $x=0.0$, $y=-0.15$, $\dot{y}=0.0$, and the value of $\dot{x}$ which is then determined from energy conservation is taken to be negative. The Poincaré section Fig. 1(a) then shows the points where the trajectory intersects the $x=0$ plane with $\dot{x}$ positive. For this value of the energy, no chaos is expected, regardless of the initial conditions. This expectation is borne out in that the points on the section seem to lie on a smooth curve.

In Figs. 1(b) and 1(c) we display the quantity $K$ along the points of the trajectory for a total time of 100 s. The points on the Poincaré section cover a total time of about 314 s, and so the calculation of $K$ covers a time corresponding to about one-third of the points of Fig. 1(a). In Fig. 1(b) we have plotted $K$ evaluated as a summation from the 4th though the 7th, 11th, and 15th orders to give a feeling for the convergence of the series. In Fig. 1(c) we have plotted just the 11th- and 15th-order curves from Fig. 1(b) to show more detail. The difference in the vertical scales should be noted. Clearly, the fluctuations in $K$ are becoming smaller as more higher-order terms are included in the series. Based on this evidence, to this order the series for $K$ appears to be converging.

Figure 2 presents the corresponding series of graphs for the case $H=0.08$, again with initial conditions $x=0$, $y=-0.15$, $\dot{y}=0$, and $\dot{x}$ negative. Here, again, the points on the section plot in Fig. 2(a) lie on a smooth curve, and there appears to be convergence of the series for $K$ through 15th order. The series does not appear to converge as rapidly as for the case where $H=0.05$, but this is to be expected since the variables can take on larger values at this energy.

At energy $H=0.11$, we present two series of graphs, Figs. 3 and 4, for two trajectories. In Fig. 3 the initial conditions are the same as in the cases of the previous two energies. In this case we get chaotic behavior as is indicated by Fig. 3(a), where the points no longer lie on smooth curve. From Figs. 3(b) and 3(c) we see that the series for $K$ appears to be diverging or perhaps asymptotic at certain times. In Fig 4 we have used initial conditions $y=0.12$, $x=0.0$, $\dot{y}=0.0$, and $\dot{x}$ positive. We see from Fig. 4(a) that the trajectory appears to be nonchaotic. Figure 4(b) gives a graph of $K$ through orders 5, 7, and 9, showing a clear tendency for the series to improve up to this point. Figure 4(c) and 4(d) correspond to what we have been displaying for the smaller energies. The results clearly show that $K$ to these orders is more nearly constant here than in the case of the chaotic trajectory of Fig. 3. Figure 4(d), however, shows that $K$ through 15th order is only about as constant as $K$ through 13th order. Again, this may indicate either that the series may ultimately diverge or be asymptotic.

Finally, in Fig. 5 we show the results for energy $H=0.14$. At this higher energy the trajectory appears even more chaotic and the series for $K$ diverging. We see in Fig. 5(b), in fact, that at certain times, $K$ through 15th order is worse than $K$ though 7th order. Also, of particular interest is Fig. 5(d), where we have a plot of $K$ for the same trajectory and time period as Fig. 5(c), except that a mesh size of 0.001 s is used. Use of this smaller size produced no significant changes in the earlier figures. In this case the right half of the graph shows a markedly different structure than it did in Fig. 5(c). This is presumably due to the more chaotic nature of the trajectory and its greater sensitivity to numerical fluctuations.

The results presented here represent a preliminary discussion of the series for $K$ and its properties. Interesting questions are raised by the structure of the graphs for $K$. For example, what is the physical significance of the extreme variations of $K$ indicated by the spikes which are particularly noticeable in Figs. 3(c), 5(c), and 5(d)? Our preliminary study of this issue seems to indicate that these spikes are associated with regions of phase space in which the trajectory is the most sensitive to initial conditions. This study will be reported in a later paper.

Another program we are pursuing is the adaptation of our procedure to a new method which will allow us to calculate more orders in the series for $K$ with greater speed. This should make it possible to gain more information about the series in the delicate region where chaos sets in and also obtain more confirmation about whether the series is really converging at lower energies. This study is planned to be reported in a later paper.

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[8]Our result disagrees with that of leach (Ref. 5) who concluded that $h_4$ could not survive at fourth order. The first of Leach's Eq. (5.10) and the entries in Eq. (5.14) lead to the result $B_4(9A_3^2-3A_3A_1-3A_2A_0-2A_1^2-A_2^2)=0$, instead of his erroneous second line in Eq. (5.18).

[9]The third homogeneous solution is proportional to $\mathrm{Re}(h_1-h_2+ih_3)^3$, where the $h_i$'s are given in Eq. (3.2), and is essential for consistency at higher orders.

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