1976

Binary Numbers

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Comments to Instructors:

The Exploration Activity of this learning cycle on the use of binary numbers will take at least half of the class period. Most students will probably leave out the null set in Exploration 2 and 3. Students who don't know how to answer Exploration 4 could be told not to worry and just wait until the concept invention at the end of class.

Exploration 5 is done this way: The student thinks of an integer between 0 and 15 inclusive, but does not tell the instructor. The instructor (without looking at the sheet) asks, "Is your number in the column headed by 8?" After showing the student his closed left fist, the instructor lifts his index finger if the answer is yes, or leaves it down if the answer is no. The instructor then asks, "Is your number in the column headed by 4?" Then the middle finger is lifted on not depending on the answer. Likewise for the columns headed by 2 and by one. After these four questions the instructor looks at his left hand, and reading 1 for a raised finger and 0 for a non-raised finger, sees the binary representation of the student's number. It's important here to let the student know you don't have the table memorized, haven't looked at it, but you do know how the table was constructed. Here it is easier to keep a straight face if the student's chosen number is not four.

The card sort trick in Exploration 7 needs a deck of sixteen 3x5 cards prepared with the numbers 0 through 16. Four holes on each side are punched and some are cut out to the edges so the binary representation of number is indicated on the right and its complement is on the left edge. The deck is then put together, all facing up. Two nails are used in opposite holes and pulled apart so each card either goes to the left or two the right. Then these two stacks are put one atop the other and that gives one "move." Some study is necessary to discover which holes are used and in which order and which stack goes on top. Here it is fair (and necessary) to look through the stacks to figure this out.

Exploration 8 needs to be done in pairs. If the bills are put in a line in 8 4 2 1 order and just shoved a few inches toward the person receiving them, then there will be visual clues (away is 1, close is 0) as to the binary number of the number of times a dollar has been paid.

In the Concept Invention portion of this exercise, you may have to explain how the binary numbers are constructed. Here it may be helpful to consider a binary odometer where only zeros and ones can occur. Most of this portion will be spent going over the Exploration Activities in terms of the binary numbers.

Hopefully, the Application Activity will be fairly easy homework.

I. Exploration Activity

1. List all the possible outcomes, in terms of heads and tails, of simultaneously flipping a penny, nickel and dime.

2. List all possible flavors of milk-shakes the dairy can make if only apple, banana and cherry flavorings are available.

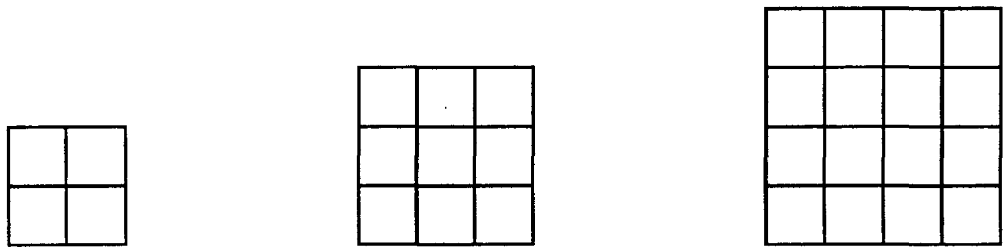
3. List the possible contributions you could make to the Salvation Army Christmas kettle if you just have a penny, dime and dollar.

4. Write the numbers zero to seven in the binary (base two) system.

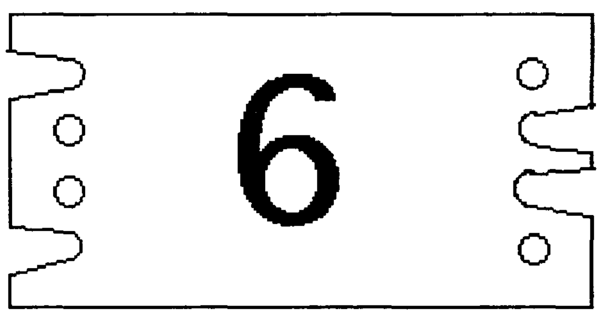
5. "Guess the number" is played with a 4 by 8 number array as shown below. Play it a couple of times to see what the trick is. The teacher will give you the rules.

8	4	2	1
9	5	3	3
10	6	6	5
11	7	7	7
12	12	10	9
13	13	11	11
14	14	14	13
15	15	15	15

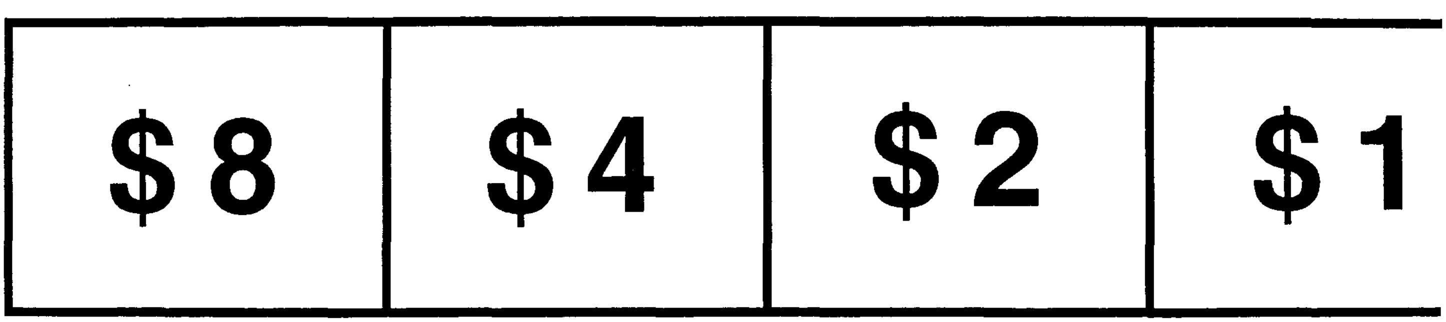
6. The King of Persia rewarded the inventor of chess by paying in grains of wheat piled on the playing board: one grain on the first square, twice that many on the second square, twice that many on the next square, etc. If chess is played on a 2×2 board, what would be the total number of grains of wheat? What about a 3×3 board? 4×4 board? Do you see any pattern developing?



7. The object of the "card sort" trick is to put the 16 cards in numerical order in the fewest number of moves. A move is done with two nails in opposite holes. Try your idea as to how this could be done.



8. Work in pairs. Just one of the pair should detach the bills below. Using just these four bills, see how many times you can pay your partner exactly one dollar. Once your partner has some of your bills, change can be made. Is your method of payment unique?



II. Concept Invention

The numbers 0 to 15 in the binary system:

8	4	2	1	
0	0	0	0	0
0	0	0	1	1
0	0	1	0	2
0	0	1	1	3
0	1	0	0	4
0	1	0	1	5
0	1	1	0	6
0	1	1	1	7
1	0	0	0	8
1	0	0	1	9
1	0	1	0	10
1	0	1	1	11
1	1	0	0	12
1	1	0	1	13
1	1	1	0	14
1	1	1	1	15

By considering 1 as meaning "is present" and 0 as meaning "is not present", the binary numbers with n digits will list all possible subsets of a set with n elements.

There are 2^n such subsets, numbered 0 through $2^n - 1$.

Any positive integer has a unique expression as a sum of single powers of two. That is to say any positive integer can be written in base two in exactly one way.

The sum of all powers of two up to a certain power is one less than the next power of two. For example,

$2^0 + 2^1 + \ldots + 2^8 = 2^9 - 1$. Likewise, $2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$.

Thus it follows that, using base two, $11 = 2^2 - 1$
$111 = 2^3 - 1$, $1111 = 2^4 - 1$, $1111111 = 2^7 - 1$, etc.

By considering 0 and 1 to denote two separate choices, say left and right, the binary numbers of n digits uniquely give all possible choices of directions to 2^n locations.

III. Application Activity

1. What is the maximum distance a "binary odometer" can measure if it looks like this at the start?

0	0	0	0	0	0

2. Design a five column table to do a "guess the number between 1 and 31" trick. Try it on someone.

3. Design a card sort system for one million cards. How many moves will it take to sort these cards?

4. Since chess is played on an 8×8 board, how many grains of wheat should the inventor have gotten?

5. If the fingers of one hand can be only up or down, how far can you count with that hand? Assume all digits move independently.

6. A rat running a maze must make four right/left decisions. Suppose that "rlrr" denotes the path of right turn, right turn, left turn, right turn. List all possible paths through the maze using this notation.