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Amicable Pairs

Lexi Wichelt
University of Nebraska-Lincoln

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Amicable Pairs

Lexi Wichelt

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Dr. David Fowler, Advisor

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The ancient Greeks are often credited with making many new discoveries in the area of mathematics. Euclid, Aristotle, and Pythagoras are three such famous Greek mathematicians. One of their discoveries was the idea of an amicable pair. An **Amicable pair** is a pair of two whole numbers, each of which is the sum of the proper whole number divisors of the other.

The Greeks were aware of the smallest amicable pair as early as the 4th-century B.C.: "Iamblichus, in the fourth century BCE, wrote, 'The first two friendly numbers are these: *sigma pi delta* and *sigma kappa*' " (Sandifer, 2005, p. 1). In the Greek number system, *sigma*'s value was 200, *pi*'s value was 80, and *delta*'s value was 4. Thus, *sigma pi delta* equaled 284. Furthermore, *kappa*'s value was 20, which means that *sigma kappa* had a value of 220. Thus, the first amicable pair, or 'friendly numbers' as they were called, to be found was 220 and 284. This turns out to be the smallest amicable pair.

To show that the pair (220, 284) is an amicable pair, we must show that the proper divisors of each number adds up to the other number. By definition, a proper divisor of a number *n* is any positive divisor of *n*, excluding *n* itself. For example, the proper divisors of 6 are 1, 2, and 3, but not 6 itself. So, the first step in showing that the pair (220, 284) is amicable is to calculate the individual divisors of 220 and 284. We have that

220 is divisible by

1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 and 220.

284 is divisible by

1, 2, 4, 71, 142, and 284.

Hence, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110 and the proper divisors of 284 are 1, 2, 4, 71, and 142. Note that

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284, \text{ and}$$

$$1 + 2 + 4 + 71 + 142 = 220.$$

Therefore, (220, 284) is an amicable pair.

The first amicable pair is thought to have been discovered sometime during or before the 4th-century B.C. Another 1000 years passed before a second amicable pair was discovered. Interestingly there are many different stories about who found the second amicable pair. William Dunham states that in the 9th-century, Arab mathematician Thabit ibn Qurra (836-901 A.D.) probably discovered the next amicable pair: (17296, 18416) (Dunham, 2007, p. 2). According to M. Garcia, this second pair was found in the 14th-century by both Ibn al-Banna in Marakesh (1256-1321 A.D.) and Kamaladdin Farisi (1267-1318 A.D.) in Bagdad (Garcia, 2003, p. 2). Other reports state that the great French mathematician Pierre de Fermat (1601 – 1665 A.D.) discovered the pair (17296, 18416) in 1636. The third amicable pair to be discovered is (9363584, 9437056) and is also said to have been discovered by multiple people. Dunham states that the third pair was discovered in 1638 by Rene Descartes (1596-1650 A.D.), French mathematician and rival of Fermat. Costello states that this third pair was discovered by Muhammad Baqir Yazdi in Iran sometime during the 16th-century (Costello, 2002, p. 289).

These were the only three amicable pairs known to man until Swiss mathematician Leonhard Euler (1707-1783) published a paper entitled “De Numeris Amicabilibus” in 1750. In this paper, Euler described a method for finding new amicable pairs. During Euler’s 76 years of life, he was able to find over 50 new amicable pairs. This was quite an accomplishment for any mathematician since for 2000 years only three amicable pairs were known. “He single-handedly increased the world’s supply of amicable numbers twenty-fold.” (Dunham, 2007, p. 5).

To understand Euler’s method for finding amicable pairs, one must have a good understanding of basic algebra. For instance, in order to check that 220 and 284 is an amicable pair, one must find the divisors of 220 and 284 and then compare the sums of the proper divisors from each list.

Euler denoted the function that adds the divisors of a natural number ‘n’ by the symbol ‘∫’. According to Dunham this is, “..a choice that seems blasphemous … assigning a non-standard meaning to that most standard of symbols, the integral sign. Modern number theorists prefer to use ‘σ’.” (Dunham, 2007, p. 5). The symbol ‘σ’ is the Greek letter sigma, and this function is usually called the “Euler sigma function”: given a natural number n, $\sigma(n)$ is the sum of all the whole number divisors of n. For example,

$$\sigma(15) = 1 + 3 + 5 + 15 = 24.$$

To find amicable pairs, Euler used the following three important theorems.

Theorem 1. A natural number p is prime if and only if $\sigma(p) = p + 1$.

Proof: Assume that p is prime. Then the only numbers that divide p are one and itself. Hence, $\sigma(p) = p + 1$. Conversely, assume that $\sigma(p) = p + 1$. Since p and 1 are two divisors of p and since $\sigma(p)$ denotes the sum of the divisors of p, we conclude that the only divisors of p are p and 1. Therefore, p must be prime.

For example, consider the prime number 17. Since 17 is prime, the only divisors of 17 are 17 and 1. Hence, we have:

$$\sigma(17) = 17 + 1 = 18.$$

Theorem 2. If p and q are distinct primes, then $\sigma(pq) = \sigma(p) \cdot \sigma(q)$.

Proof: Let p and q be distinct prime numbers. Then the divisors of p are p and 1, and the divisors of q are q and 1. This means that the divisors of pq are 1, p, q, and pq. Hence

$$\sigma(pq) = 1 + p + q + pq = (1 + p) + q(1+p) = (1 + p)(1 + q) = \sigma(p) \cdot \sigma(q),$$

where the last equality follows from Theorem 1.

Theorem 2 tells us that given two distinct prime numbers, the sum of the divisors of one of the prime numbers times the sum of the divisors of the other prime number equals the sum of the divisor of the product of the two prime numbers.

For example, let $p = 3$ and $q = 7$, and note that p and q are distinct primes. Then,

$$\sigma(p) = \sigma(3) = 3 + 1 = 4, \text{ and}$$

$$\sigma(q) = \sigma(7) = 7 + 1 = 8.$$

The divisors of $pq = 21$ are 1, 3, 7, and 21, and by definition,

$$\sigma(21) = 1 + 3 + 7 + 21 = 32.$$

and since,

$$\sigma(21) = 32 = 4(8) = \sigma(3) \cdot \sigma(7),$$

We see that the result of Theorem 2 holds in this example.

Theorem 2 led Euler to recognize that this property works with all whole numbers whose greatest common factor is 1. This leads us into Theorem 3.

Theorem 3. Let a and b be two natural numbers with $\gcd(a,b) = 1$. Then $\sigma(ab) = \sigma(a) \cdot \sigma(b)$.

Proof: Let a and b be natural numbers such that their greatest common divisor is 1. Let a_1, $a_2,\ldots,a_m$ be the divisors of a, where $a_1 = 1$ and $a_m = a$, and let b_1, b_2, $\ldots,b_n$ be the divisors of b, where $b_1 = 1$ and $b_n = b$. If d is a divisor of ab, then $d = a_i \cdot b_k$ for some $I = 1,2,\ldots,m$ and $k = 1,2,\ldots,n$ because the greatest common divisor of a and b is 1. We have:

$$\sigma(a) = a_1 + a_2 + \ldots + a_m$$

$$\sigma(b) = b_1 + b_2 + \ldots + b_n.$$

So,

$$\begin{aligned}\sigma(a) \cdot \sigma(b) &= (a_1 + a_2 + \ldots + a_m)(b_1 + b_2 + \ldots + b_n) \\ &= a_1 (b_1 + b_2 + \ldots + b_n) + a_2 (b_1 + b_2 + \ldots + b_n) + \ldots + a_m(b_1 + b_2 + \ldots + b_n), \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j ,\end{aligned}$$

which is the sum of all the divisors of ab by the previous observation. Hence,

$$\sigma(ab) = \sigma(a) \cdot \sigma(b).$$

We see here that the sigma function is multiplicative.

Euler noticed that two given numbers M and N are amicable if and only if $\sigma(M) - M = N$ and $\sigma(N) - N = M$. Thus, M and N are amicable if and only if

$$\sigma (M) = M + N = \sigma (N). \tag{1}$$

With these three theorems and Euler's observation, we have enough tools to find new amicable pairs.

Euler found amicable pairs using many different methods of choosing the first numbers. One such method involved Euler's discovery of using (es, ep), where s is the product of distinct primes not dividing the common factor e and p is a single prime not dividing es (Costello, 1991, 859). We will study the following method.

Assume that M and N is and amicable pair given by $M = apq$ and $N = ar$, where p, q, and r are distinct primes and a is the greatest common divisor of M and N. Note that $M \neq N$ in this case. From equation (1), we have:

$$\sigma (apq) = \sigma (M) = \sigma (N) = \sigma (ar).$$

By Theorem 3, since M and N are numbers, we have

$$\sigma(apq) = \sigma(a) \cdot \sigma(p) \cdot \sigma(q) \qquad \text{and} \qquad \sigma(ar) = \sigma(a) \cdot \sigma(r).$$

So, $$\sigma(a) \cdot \sigma(p) \cdot \sigma(q) = \sigma(a) \cdot \sigma(r).$$

Dividing both sides by $\sigma(a)$, we obtain,

$$\sigma(p) \cdot \sigma(q) = \sigma(r). \tag{2}$$

Since p, q, and r are prime numbers, we know from Theorem 1 that

$$\sigma(p) = p + 1,$$

$$\sigma(q) = q + 1,$$

$$\sigma(r) = r + 1\ .$$

Thus, by equation (1) we see that:

$$(p + 1)(q + 1) = r + 1.$$

This is an equation in three variables, namely p, q, and r. Since there are three unknowns, Euler decided to use a substitution so that there would only be two variables to deal with. He let

$$x = p + 1,$$

$$y = q + 1,$$

$$xy = r + 1.$$

Then he solved for the prime numbers p, q, and r:

$$p = x - 1,$$

$$q = y - 1,$$

$$r = xy - 1.$$

Since

$$\sigma(apq) = \sigma(M) = M + N = apq + ar = a(pq + r),$$

we can write

$$\sigma(a) \cdot \sigma(p) \cdot \sigma(q) = a(pq + r),$$

and using the equations above,

$$\sigma(a) \cdot (p + 1)(q + 1) = a(pq + r).$$

Now, remembering Euler's substitution, we get

$$\sigma(a) \cdot xy = a([x - 1][y - 1] + [xy - 1])$$

$$= a(xy - x - y + 1 + xy - 1)$$

$$= a(2xy - x - y).$$

Solving for y, we have:

$$y = \frac{ax}{[2a - \sigma(a)]x - a}$$

Then Euler used the following substitution:

$$\frac{b}{c} = \frac{a}{2a - \sigma(a)} \tag{3}$$

Simplifying we have:

$$y = \frac{ax}{\left\{\frac{ac}{b}\right\}x - a} = \frac{bx}{cx - \mathrm{b}}$$

$$cy - b = c\left\{\frac{bx}{cx - b}\right\} - b = \frac{b^2}{cx - b}$$

Finally we get:

$$(cx - b)(cy - b) = b^2. \tag{4}$$

From this point, Euler had enough to find amicable pairs. There were just four steps to follow to find an amicable pair.

Step 1: Choose a value for a.

Step 2: Using (3), find values for b and c.

Step 3: Using (4), find values for $(cx - b)$ and $(cy - b)$, which will then be used to find values for x and y.

Step 4: Finally, if $p = x - 1$, $q = y - 1$, and $r = xy - 1$ are all prime, then $M = apq$ and $N = ar$ are the candidates for amicability.

The following is an example using these four steps.

Step 1: Let $a = 4$.

Step 2: Then,

$$\frac{b}{c} = \frac{a}{2a - \sigma(a)} = \frac{4}{2(4) - \sigma(4)} = \frac{4}{8 - (1 + 2 + 4)} = \frac{4}{8 - 7} = \frac{4}{1}$$

So, $b = 4$ and $c = 1$.

Step 3: From, $(cx - b) = (1x - 4)$ and $(cy - b) = (1y - 4)$, we have $(x - 4)(y - 4) = 4^2 = 16$.

Step 4: Lastly, we can set up a table where we know that $(x - 4)(y - 4)$ has to equal 16. So we can look for values of x and y.

$x - 4$	$y - 4$	x	y	$p = x - 1$	$q = y - 1$	$r = xy - 1$
16	1	20	5	19	4	99
8	2	12	6	11	5	71
4	4	8	8	7	7	63

We want p, q, and r to be prime, thus we need to look at the middle row because this is the only row that has a prime number for all three variables. From this row we can now see that

$$M = apq = 4(11)(5) = 220,$$

$$N = ar = 4(71) = 284,$$

which gives us the first amicable pair.

The above steps can be used to find other amicable pairs. Here is another amicable pair found by letting $a = 819$.

Step 1: Let $a = 819$.

Step 2: Then,

$$\frac{b}{c} = \frac{a}{2a - \sigma(a)} = \frac{819}{2(819) - \sigma(819)} = \frac{819}{1638 - (1456)} = \frac{819}{182} = \frac{9}{2}$$

So, $b = 9$ and $c = 2$.

Step 3: From, $(cx - b) = (2x - 9)$ and $(cy - b) = (2y - 9)$, we have

$$(2x - 9)(2y - 9) = 9^2 = 81.$$

Step 4: Lastly, we can set up a table where we know that $(2x - 9)(2y - 9)$ has to equal 81. So we can look for values of x and y.

$2x - 9$	$2y - 9$	x	y	$p = x - 1$	$q = y - 1$	$r = xy - 1$
81	1	45	5	44	4	224
27	3	18	6	17	5	107
9	9	9	9	8	8	80

We want p, q, and r to be prime, thus we need to look at the middle row because this is the only row that has a prime number for all three variables. From this row we can now see that

$$M = apq = 819(17)(5) = 69615,$$

$$N = ar = 819(107) = 87633,$$

which gives us another amicable pair. This method can be used again and again by choosing different values for a.

Many amicable pairs were found by using Euler's idea. Other amicable pairs were found using the idea of "daughter" pairs, "granddaughter" pairs, and "great-granddaughter" pairs. These pairs are new amicable pairs that are found by applying a few clever "tricks" to a known amicable pair; these "tricks" lead to new amicable pairs that have several factors in common with the known pair (Costello, 1991, p. 863). Today there are computer programs that are used to aid in the finding of these new pairs (see Costello 1991).

There are many methods to find "daughter" pairs. One method is based on the following theorem (Garcia, 2003, p. 9-10):

Theorem (te Riele's Rule): Let (au, ap) be a given amicable pair where p is a prime not dividing a. If a pair of distinct prime numbers r and s exists such that $\gcd(a, rs) = 1$, satisfying the bilinear equation

$$(r - p)(s - p) = (p + 1)(p + u),$$

and if a third prime q exists, with $\gcd(au, q) = 1$, such that

$$q = r + s + u,$$

then (auq, ars) is an amicable pair .

For example, consider the 106th known amicable pair, which can be written as a product of prime numbers as

$$(3^2 \cdot 5^3 \cdot 13 \cdot 11 \cdot 59, 3^2 \cdot 5 \cdot 13 \cdot 18719).$$

We can write this amicable pair in the form (au, ap), where

$$a = 3^2 \cdot 5 \cdot 13,$$

$$u = 5^2 \cdot 13 \cdot 11 \cdot 59,$$

$$p = 18719.$$

We can use p and u to substitute in to the equation

$$(r - p)(s - p) = (p + 1)(p + u)$$

to find values for r and s. Using this, we get that

$$r = 18719 + 2688 = 21407,$$

$$s = 18719 + 243360 = 262079.$$

Now we can use the values of r, s, and u to find q since

$$q = r + s + u.$$

We have

$$q = 21407 + 262079 + 16225 = 299711.$$

Since q, r, and s are prime the daughter pair of (au, ap) will be

$$(auq, ars) = (3^2 \cdot 5^3 \cdot 13 \cdot 11 \cdot 59 \cdot 299711, 3^2 \cdot 5 \cdot 13 \cdot 21407 \cdot 262079).$$

Patrick Costello, professor of mathematics at Eastern Kentucky University, is the discoverer of over 92000 amicable pairs (see http://math2.eku.edu/PJCostello/). Herman J. J. te Riele (1947 – present), a mathematician for the National Research Institute for Mathematics and Computer Science, was able to take Costello's idea combined with others' findings to look for "daughter" and "granddaughter" pairs. te Riele was able to find 1782 "daughter" pairs and 88 "granddaughter" pairs.

With Euler's idea and the aid of computers, many new amicable pairs have been found. According to Herman J. J. te Riele, "It is believed that there are infinitely many amicable pairs, although this has never been proved." (te Riele, 1984, p. 219). To date (June 21, 2009), there are a total of total 11,994,387 pairs known (see http://amicable.homepage.dk/knwnc2.htm).

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