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Taxicab Geometry

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Expository Paper

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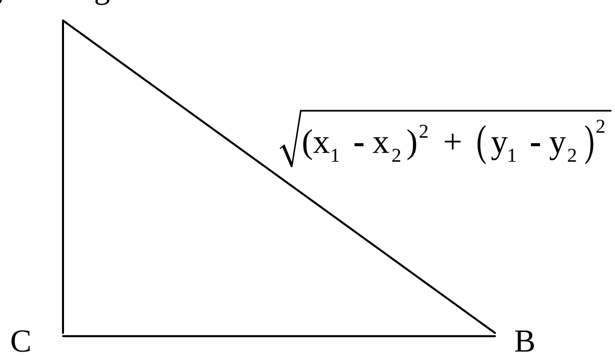
Taxicab geometry was founded by a gentleman named Hermann Minkowski. Mr. Minkowski was one of the developers in "non-Euclidean" geometry, which led into Einstein's theory of relativity. Minkowski and Einstein worked together a lot on this idea Mr. Minkowski wanted people to know that the side angle side axiom does not always hold true for all geometries. He wanted to prove this in the case that you can not always use the hypotenuse to find the shortest way from one spot to another. The best way to think of his idea is to think of a taxicab going from one place to another, thus the name *taxicab geometry*.

Now let's look at what the difference is between Euclidean approach and taxicab. For Euclidean we have seen quite a few times before the following definition…

Euclidean d_E (A, B)

$$= d_E \left[(x_1, y_1),(x_2, y_2)\right] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Then, the Pythagorean Theorem can be used to find the length of the hypotenuse for a right triangle.



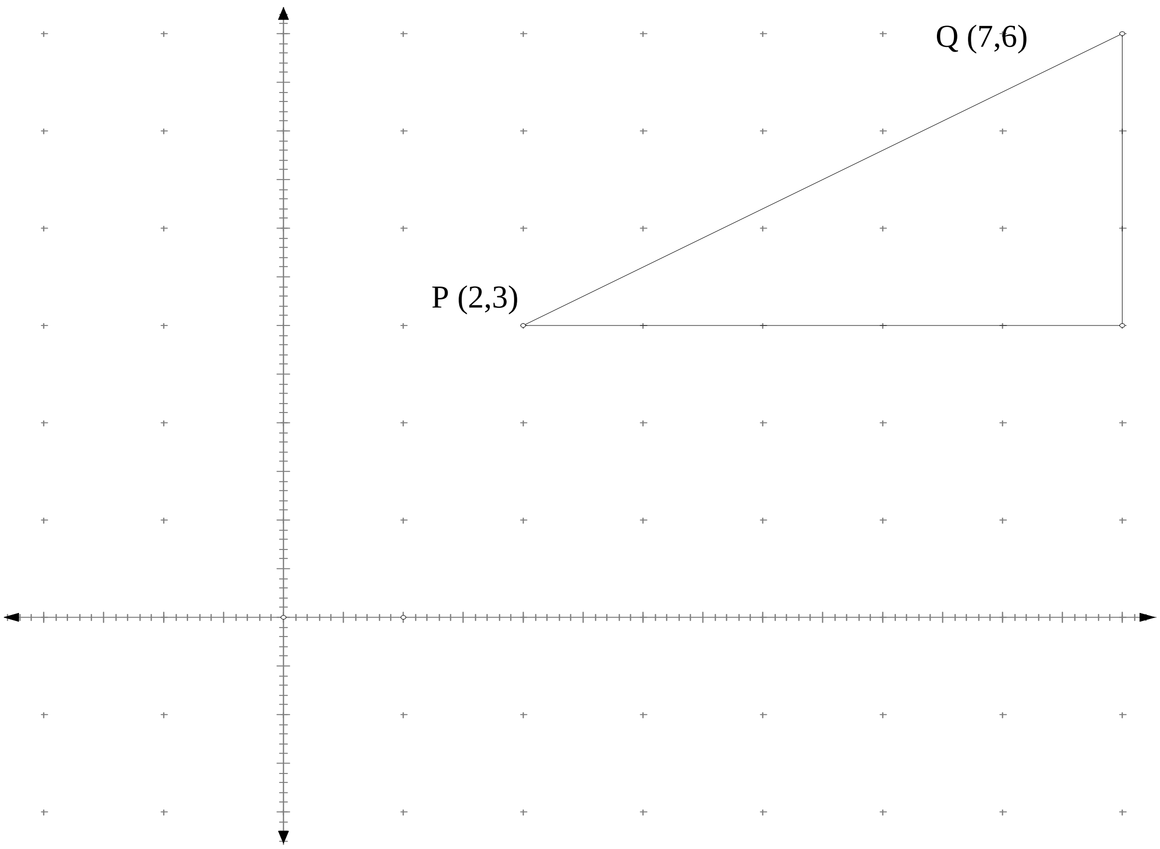
Now what we need to think about is what does taxicab geometry say?

Taxicab d_T (P, Q)

$$= d_T \left[(x_1, y_1),(x_2, y_2)\right] = |x_1 - x_2| + |y_1 - y_2|$$

This shows us what the lengths are for the legs (using coordinates on a planer graph) and we add them together.

Now let's do a specific example. If we have the point P (2, 3) and point Q (7, 6) we can find the distance first using taxicab geometry.



Taxicab d_T (P, Q)

$= d_T\left[(x_1, y_1), (x_2, y_2)\right] = |x_1 - x_2| + |y_1 - y_2|$

$d_T\left[(2,3),\ (7,6)\right]$

$= |7\text{-}2| + |6\text{-}3|$

$= |5| + |3|$

$= 5 + 3 = 8$

Using P, Q as your points, plug them in.
Subtract the x and y's and then take the absolute value.
Add them together and get 8 for your solution

Now, how about using the Euclidian approach?

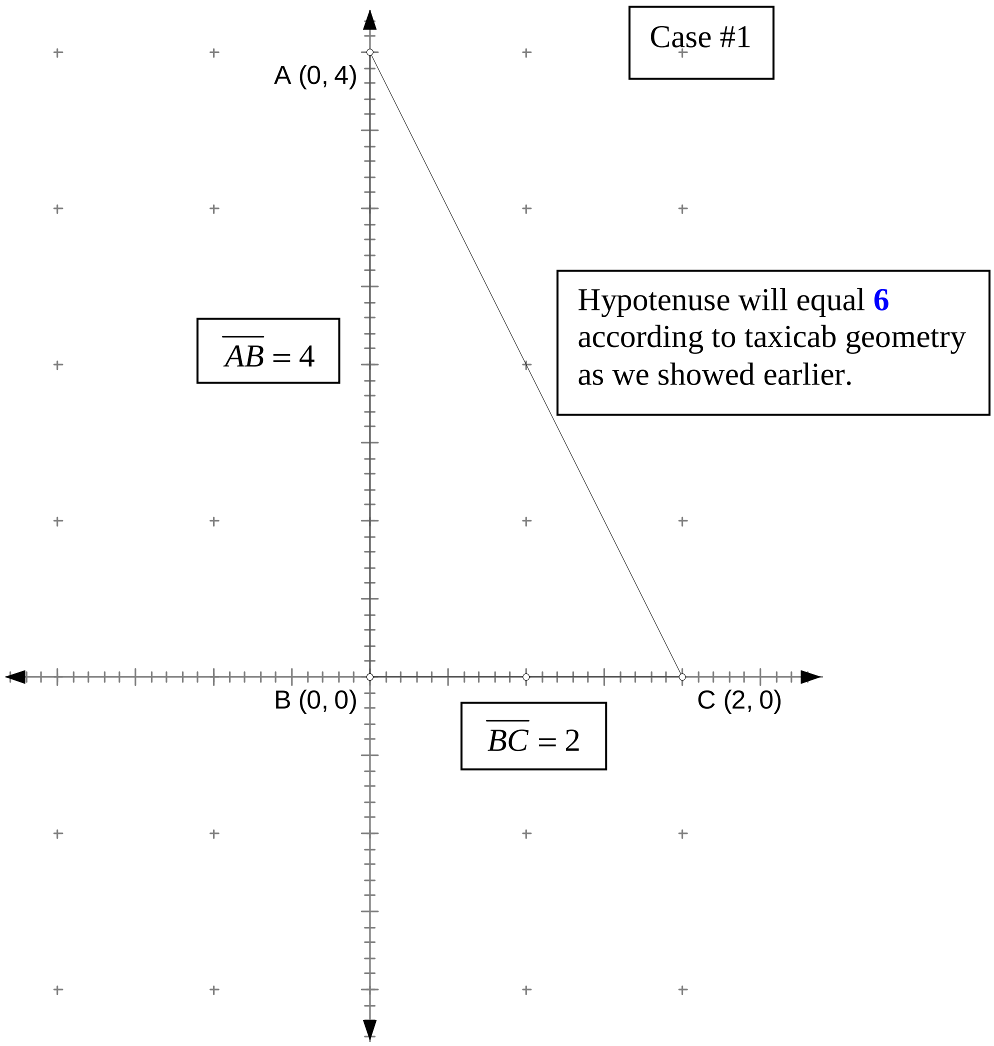
Euclidean d_E (A, B)

$= d_E\left[(x_1, y_1), (x_2, y_2)\right] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

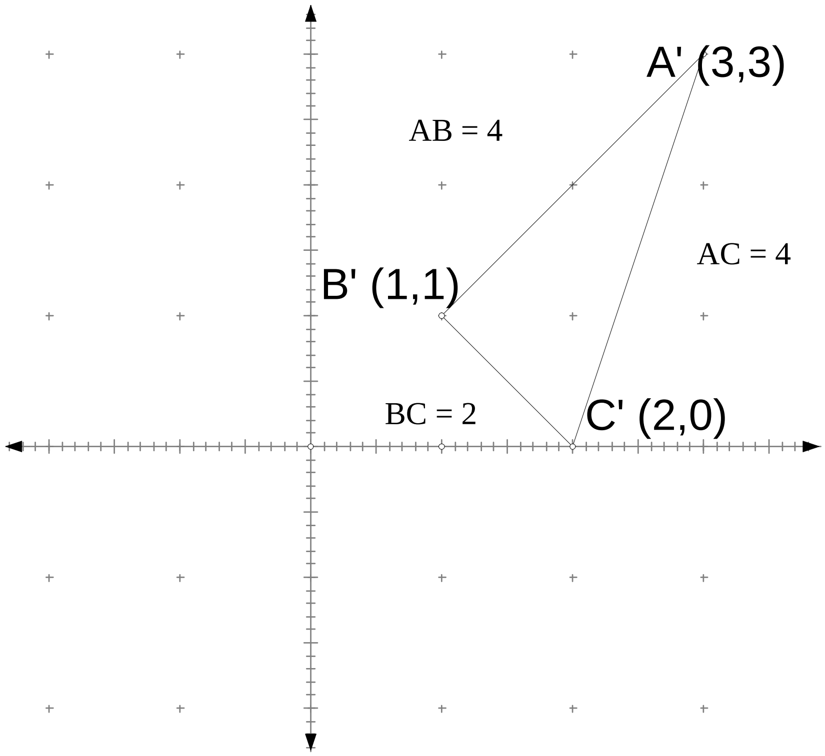
$$= d_E \left[(2,3),(7,6)\right] = \sqrt{(2 - 7)^2 + (3 - 6)^2}$$
$$= \sqrt{(-5)^2 + (-3)^2}$$
$$= \sqrt{25 + 9}$$
$$= \sqrt{34} \approx 5.83$$

We would have to use 8 if we have to stay on roads, but if "as the crow flies" it would only be 5.83

Now let's look at what the entire purpose of this taxicab geometry is, which is the fact that side-angle-side of the Euclidean sense does not hold true for some geometries. I will show two cases that will prove they are not equal and will fail to hold true for all occasions. We will show the side, angle and side are the same, but the hypotenuse will not be the same.



Now, let's consider something where we use different points to create a triangle with the same legs.

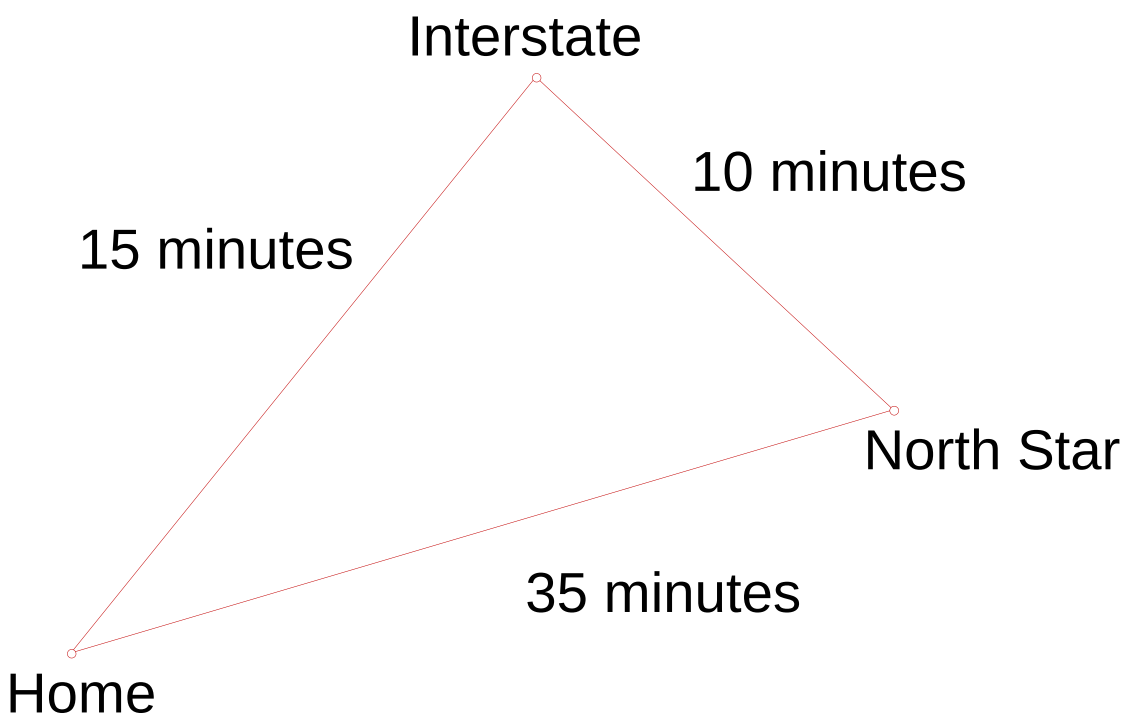


So, here we are using the same lengths of legs, but we have a different hypotenuse, which lets you know that you will have a different angle used between the two triangles, which then proves side, ANGLE, side will not hold true in taxicab geometry (Schattschneider, 1984 p. 2).

We have worked with taxicab geometry triangles so far, where our hypotenuse has always been the distance between two points. No matter how the triangle is shown, such as in the previous figure, we are still having the hypotenuse as the distance from A to C or A to B or even B to C. We always have to be on or parallel to the x or y-axis when we are looking for taxicab distance, so this is why it is always your hypotenuse.

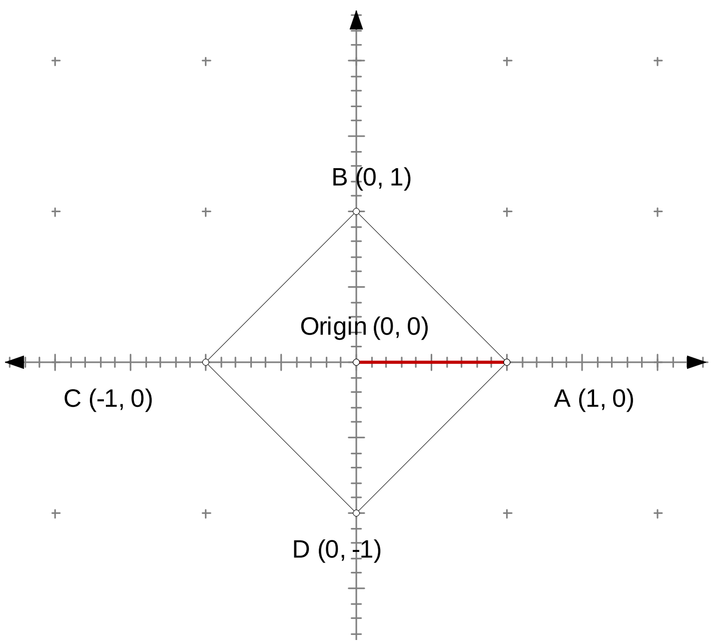
When we think of a hypotenuse we have to also think about what happens if we have a situation like we have in the previous figure, where we have sides of 2, 4, and 4? You would say 4 is your hypotenuse, but which 4 are we talking about? Are they the same length, such as in an isosceles triangle? We will get into this more later in this paper.

Sometimes we get caught up in thinking about taxicab geometry, or any geometry, and just think about distance. We should also think about time, for example to get from one place to another taxicab geometry might help us more than Euclidean.



Here I have given three points. If I try to drive across town to reach Northstar I have to deal with traffic, so it would take me 35 minutes to get from home to Northstar. Now if I go to the interstate and then take the interstate to Northstar I don't see as much traffic, so it only takes me 25 minutes. Yes the distance is longer in Euclidean, but if you think about taxicab and time if would be shorter to go to the interstate before going to Northstar. This is just something to think about while we are think on the lines of geometry and the differences from one geometry concept to another.

Now, let's talk about what a taxicab circle might look like. We have shown that we have to stay on the legs of a triangle to get from one spot to another. The abbreviation will look like this $C_T = \{\mathrm{P}|\mathrm{d}_\mathrm{T}(0, P) = r\}$. The C_T tells us a taxicab circle, with P as the set such that the distance from 0 to P is the radius of the circle. Let's see a picture of that this might look like using a radius of 1 about the origin.



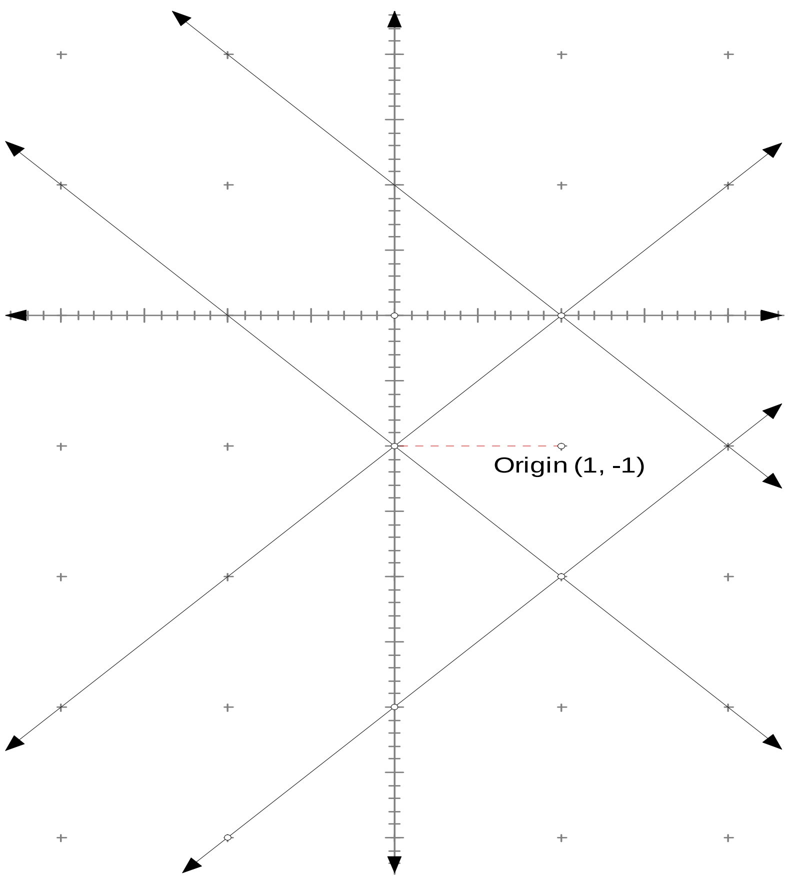
In this form we can pick any point P, which is A, B, C, or D, which is called a variable point. Using this we see that we can find the distance from (0, 0) to any point (x, y) using.

d_T (variable point, origin) = ((x, y), (0, 0) = $|x - 0| + |y - 0|$

Since, the origin is the center in this case when we look at the point A (1, 0) and plug this into the formula just given we have d_T ((1, 0), (0, 0) = $|1 - 0| + |0 - 0| = 1$. We have chosen to use radius of 1, so this is why we have set this distance in taxicab circles equal to 1.

In the previous figure we show what a circle might look like with a radius of 1 and centered around the origin. Now I would like to show how this works in an algebraic sense. If we want to center our circle around another point, let's use (1, -1) and still have a radius of one we can show the reasoning behind the circle looking like a diamond in taxicab geometry. Since we use the formula

d_T (variable point, origin) = ((x, y), (0, 0) = $|x - 0| + |y - 0|$, we see there can be equations of x + y =1, -x + y = 1, x + (-y) = 1, and (-x) + y = 1. The negatives are given from the fact that there will be positive and negative values for our x's and y's, therefore there are four.



This graph shows we have a circle of radius 1 centered around the point (1, -1), so we are going to have slopes of either 1 or -1.

x + y = 1

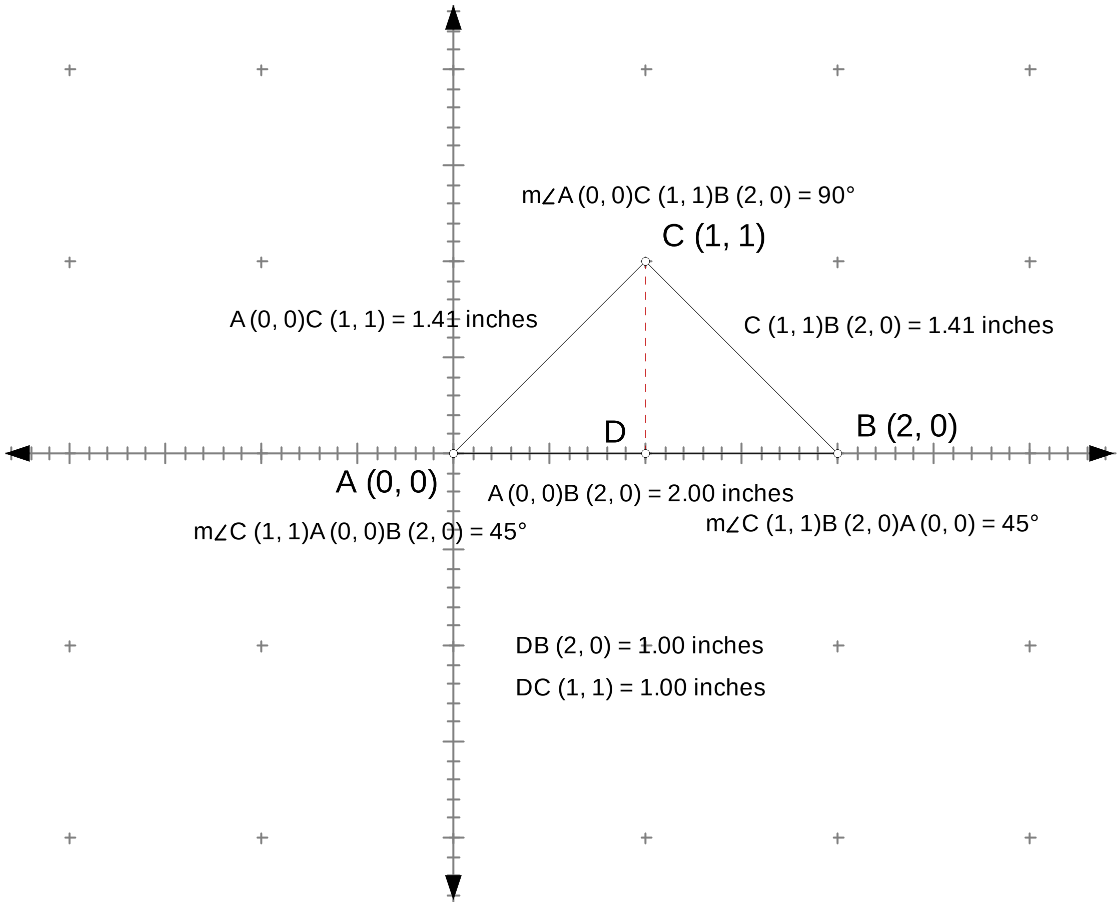
x - y = 3

(-x) + y = -1

(-x) + (-y) = 1

Based from
$|x-1| + |y+1| = 1$

Now lets work some more with applications.

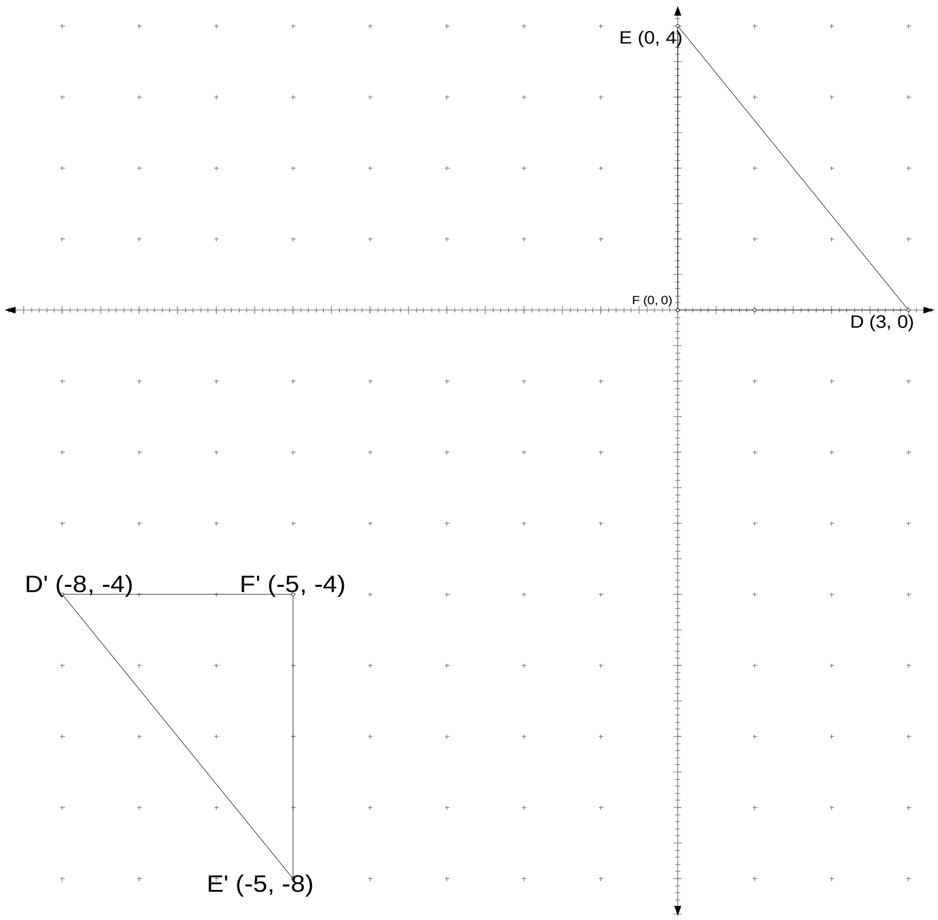


We will take this information (given above) to answer a few different questions. First of all we need to find the measures of the angles, which are given, in the figure above. Next, we are looking for the lengths of the sides using Euclidean geometry, and those lengths are shown in the figure above. Next we need to find the taxicab lengths of the sides, so they will be $\overline{AC} = 2, \overline{BC} = 2, \overline{AB} = 2$ (using the formula for taxicab geometry) for the length of the hypotenuse.

With the information given, I would say this is an equilateral triangle since the definition of an equilateral triangle has sides of equal length. This will not be an equiangular triangle, since we have angles of 45, 45, and 90. Taxicab geometry is a case where we don't worry about what the angles are, we just want to mainly know what the lengths are as we stay on the x and y-axis.

Now, let's look at a situation where we have a right triangle where the Pythagorean Theorem holds true for Euclidean, but does not hold true for taxicab. Lets us the triangle A(2, 2), B (2, 0), C (4, 0). The lengths of the legs are $\overline{AB} = 2$ and $\overline{\mathrm{BC}} = 2$, so for taxicab the length of the hypotenuse would be equal to 4, but for Euclidean we would have $\sqrt{(2-4)^2 + (2 - 0)^2} = \sqrt{8} = 2\sqrt{2}$. This is obviously not equal, so the Pythagorean Theorem is not true in taxicab geometry.

Now, I would like to show where we have two triangles that are congruent for taxicab geometry and also congruent for Euclidean Geometry. First let's talk about the taxicab triangles.

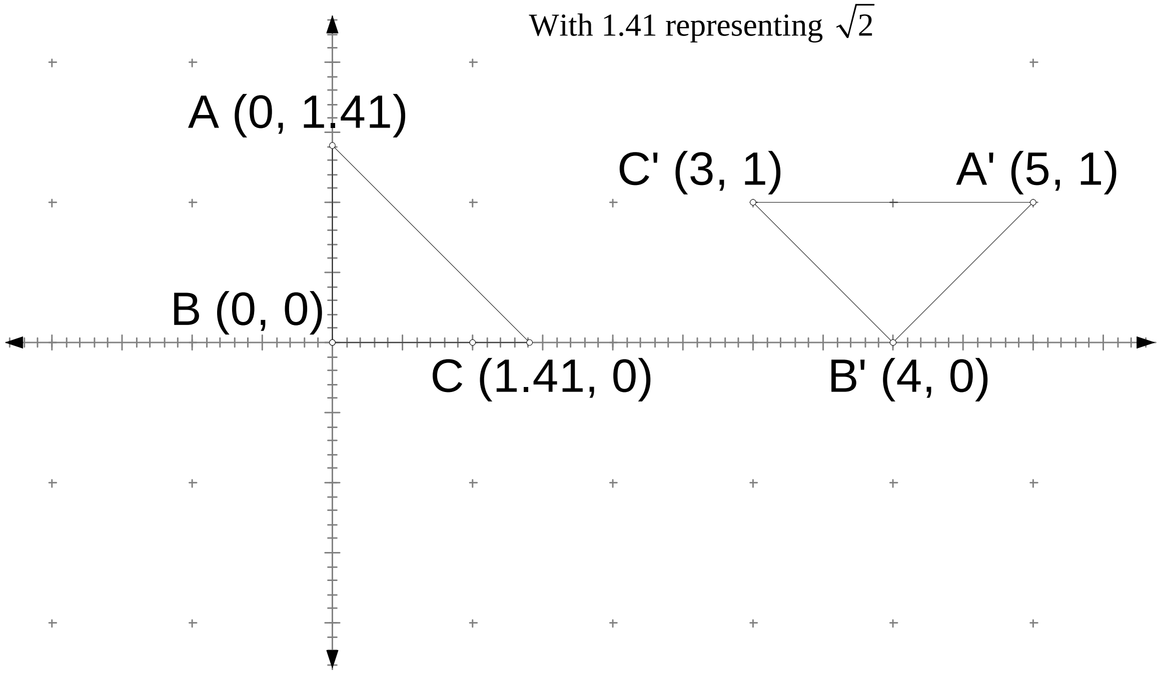


Taxicab	**Euclidean**
d_T (D,F) = $\lvert 3 - 0 \rvert + \lvert 0 - 0 \rvert = 3$	d_E (E, F) = 4
d_T (D',F') = $\lvert(-8) - (-5)\rvert + \lvert(-4) - (-4)\rvert = 3$	d_E (E', F') = 4
d_T (E,F) = $\lvert(0) - (0)\rvert + \lvert(4) - (0)\rvert = 4$	d_E (D, F) = 3
d_T (E',F') = $\lvert(-5) - (-5)\rvert + \lvert(-8) - (-4)\rvert = 4$	d_E (D', F') = 3
d_T (D,E) = $\lvert(3) - (0)\rvert + \lvert(0) - (4)\rvert = 7$	d_E (D,E) = $\sqrt{((3) - (0))^2 + ((0) - (4))^2} = 5$

d_T (D',E') = $\lvert(-8) - (-5)\rvert + \lvert(-4) - (-8)\rvert = 7$	d_E (D',E') = $\sqrt{((-8) - (-5))^2 + ((-4) - (-8))^2} = 5$

Now if I want to show two triangles are congruent in Euclidean Geometry, but not taxicab it might look like this.

With 1.41 representing $\sqrt{2}$

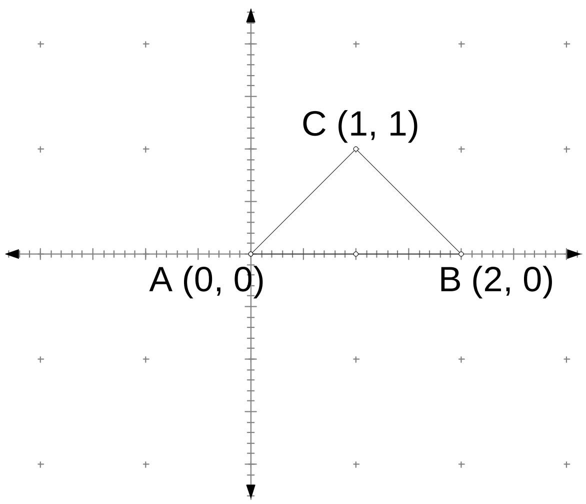


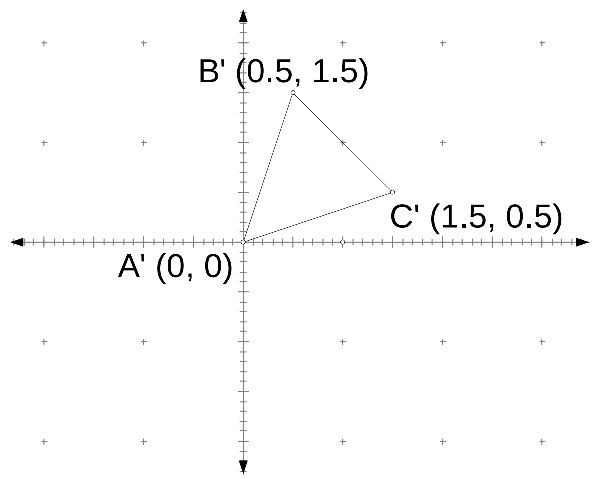
Taxicab	Euclidean
d_T (A, B) = $\lvert 0 - 0\rvert + \lvert 1.41 - 0\rvert = 1.41$	
d_T (A', B') = $\lvert 5 - 4\rvert + \lvert 1 - 0\rvert = 2$	
	d_E (A, C) = $\sqrt{((0) - (1.41))^2 + ((1.41) - (0))^2} \approx 1.99$

	d_E (A', C') = $\sqrt{((3) - (5))^2 + ((1) - (1))^2}$ = 2

We can already see that taxicab will not work in this situation as the first side we tested was not congruent with the side of the other triangle. Therefore I have given an example of two triangles, which for Euclidean are congruent and are not congruent for taxicab.

We have shown side, angle, side is not true for taxicab geometry, now how about side, side, side? Let’s look at these two triangles to start with.





The sides have distances of 2, 2, and 2 in the first triangle using taxicab geometry, and sides of the exact same in the second triangle. We have lengths that are the same, but $\triangle ABC$ has a right angle, where $\triangle A'B'C'$ does not, therefore they are not congruent. Side, side, side axiom tells us that we must have congruent corresponding parts for this to hold true, therefore this proves why these two triangles with equal distant sides are not congruent.

This is an interesting approach to geometry that I am surprised is not even talked about in geometry courses. It is interesting to give the students something more to think about and to probe into the higher level thinking “what if” statements. I personally tell my students there is more than one way to get from our middle school downtown, but along the ways we have rules, like not walking through backyards or others houses'. This shows a concrete thinking of what I tell my students all the time, like following the different streets downtown. This was a fun topic to think about and I look forward to using this idea in the fall.

References

Beem, John K. *Geometry Connections Mathematics for Middle School Teachers*.

(Most of all the problems used in this paper were from this reference)

Schattschneider, Doris J. *The Taxicab Group*.