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Experimentation with Two Formulas by Ramanujan

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In partial fulfillment of the requirements for the Master of Arts in Teaching with a Specialization in the Teaching of Middle Level Mathematics in the Department of Mathematics.

David Fowler, Advisor

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Srinivasa Ramanujan was a brilliant mathematician, considered by George Hardy to be in the same class as Euler, Gauss, and Jacobi. His short life, marred by illness and tragic educational events, was unique in the history of mathematics. Mathematical discoveries are still being gleaned from his personal notebooks. Paper was a hard commodity to come by so his notebooks were a cluttered mix of pen over pencil mathematical hieroglyphics. The following highlights Ramanujan's life in connection with Hardy, his work with ellipses, and his work with the partition function.

Biographical Sketch of Ramanujan and His Connection to Godfrey Hardy

If there is only one thing to learn from Srinivasa Ramanujan, both Hardy (1940) and Kanigel (1991) would agree that as math educators we should always be on the lookout for math genius and know what to do in order to nurture that genius. Hardy argued that Ramanujan's mathematical genius was grossly mishandled during his prime years of intellectual cultivation by the defunct education system of India at the time.

Ramanujan was born in 1887 surrounded by a very normal Indian world. He was raised in southern India in a relatively poor family. His education at the lower levels was for the most part normal, except for the unusual growth he had in mathematics. Before Ramanujan was 10 years old Hardy (1940) states, "he rediscovered Euler's theorems for sine and cosine and was later disappointed to find that they were already discovered." (pg 2). By the age of 13 he had completely mastered a book on Trigonometry. "In 1903 he passed the Matriculation Examination of the University of Madras" (pg 5) and received scholarships to study at a government college. Here is where normal development for Ramanujan ends and his romantic yet at the same time tragic life begins. At about the same time he was enrolled in college, Ramanujan's genius was awakened by a book called, *A Synopsis of Elementary Results in Pure and Applied Mathematics.* This book was in no way a masterpiece of mathematics. It was instead a book developed to aide students at Cambridge in the passing of the Tripos examination for mathematics. This book was not heavy in proof. It was information that was meant for memorization and regurgitation and much of the reason that Ramanujan did little in the way of proving any of his conjectures. In India at the time there was no one with the ability to tutor Ramanujan in mathematics and this was the only book that he could get a hold of given his limited resources. This is underlined by the fact that in 1912 Ramanujan wrote to the professor of mathematics at University College London, M C M Hill, who "replied in a fairly encouraging way but showed that he had failed to understand Ramanujan's results on divergent series." Hardy later states that, ""I have never met his equal, and can compare him only with Euler or Jacobi." Both according to The MacTutor History of Mathematics archive. (n.d)

Yet the book unlocked Ramanujan's mathematical powers, and here is where a man unknowingly gave his life in the pursuit of mathematics. In his college courses he feverishly studied math. Although he would attend courses outside of the realm of mathematics, he would not pay attention in these courses. Instead he would be completely engrossed in mathematics and oblivious to his surroundings during these mathematical meditations. Some of these would last for 20 hours. As a result he failed to earn a degree. He was unable to pass examinations outside of mathematics, and failed most of his coursework outside of mathematics. After loosing his scholarship and dropping out of college he pursued funding for independent research in mathematics. This funding was hard to come by for a college drop out in India and he spent much of his time on the brink of starvation.

Eventually, mathematicians in India with connections to England had Ramanujan send his findings to Cambridge. George H. Hardy was the only mathematician to respond. In fact the letters written to Hardy are considered some of the greatest in mathematical history due to the effect they had on both men's lives and the effect they had on the field of mathematics in general.

> "The great philosopher Bertrand Russell says that one evening in Trinity College he found the usually placid Hardy in a wild state of excitement talking about a new Euler or Jacobi from India! Hardy was convinced that Ramanujan was wasting his time in India rediscovering past work, and would profit immensely by coming into contact with professional mathematicians." The Hindu (2002)

From 1914 to 1919 Ramanujan collaborated with Hardy and his assistant John Littlewood in England. Here Ramanujan's lack of formal education in mathematics stood out and it was up to Littlewood to fill in the gaps. This was an exasperating assignment. During these lessons into formal mathematics Hardy later wrote, "*it was extremely difficult because every time some matter, which it was thought that Ramanujan needed to know, was mentioned, Ramanujan's response was an avalanche of original* *ideas which made it almost impossible for Littlewood to persist in his original intention."* The MacTutor History of Mathematics archive. (n.d)

These few short years in England at Trinity College became one of the most interesting collaborations in mathematical history. Ramanujan's abilities to intuitively arrive at a result and Hardy's formal mathematics training in rigorous proof made the pair an astounding mathematical match. Hardy did his best to understand and prove many of Ramanujan's ideas. However, the climate and cultural differences took their toll on Ramanujan and in 1918 he fell ill. In fact Kanigel (1991) suggests that Hardy fed Ramanujan's addiction for mathematics and unknowingly drove him to sickness. At this time Hardy used his influence at Cambridge to bring recognition to Ramanujan. Before Ramanujan left for India he was awarded a Bachelor of Science degree "by research" and elected a Fellow of the Royal Society. In 1919 Ramanujan returned to India in poor health and died the following year at the age of 32.

This partnership was a perfect match in mathematics, but that is where the match between Hardy and Ramanujan ends. Hardy was atheist while Ramanujan was an orthodox high-cast Hindu. Their differences in ideologies have spawned a recent revival about the lives of these two prominent historical figures. "A First Class Man" is a play that highlights the life of Ramanujan and the differences and challenges that he and Hardy faced, both mathematically and culturally. A partial reading and discussion of this play at the Tribeca Film Festival can be found at: http://www.brightcove.com/title.jsp?title=958499689&channel=1960916

It seems that Hollywood cannot get enough of Ramanujan, and production may already be underway for the movie titled "The Man Who Knew Infinity" based on the biography by Robert Kanigel. In March of 2006 Edward R. Pressman Film Corp purchased the rights to produce a film based on the book, and is slated to begin production some time in 2007.

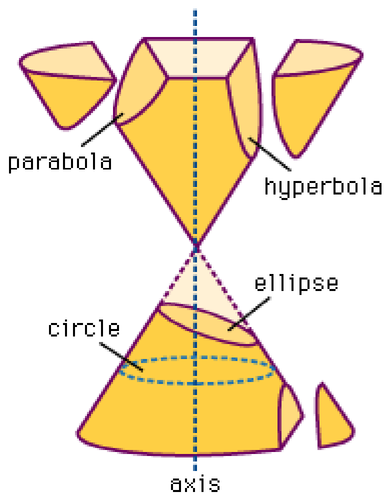


Figure 1 – Conic Sections Picture from http://www.bramboroson.com/cultural/jan26.html

The Ellipse

Before I bog down this paper in mathematical equations and definitions it might be pertinent to mention that the ellipse occurs naturally and unnaturally many places in our world. Any object in a stable orbit follows an elliptical path. Architects have used the properties of ellipses for centuries because of the acoustical value of the two foci. These two foci also allow the medical community to operate inside the human body without cutting open the patient. Sonic shock waves fired from one focus bounce and concentrate on the other focus breaking up kidney stones or cauterizing blood vessels. Additionally, ellipses have been useful in the study of aerodynamics.

The ellipse is a specific type of curve in the family of curves known as conic sections. A conic section can be thought of as placing two cones so that their bases are parallel with their vertices intersecting and then cutting these cones with a plane. The resulting shape traced on the plane at the location of the cut is called a conic section. The three basic conic sections are the ellipse, parabola, and hyperbola. This is represented in figure 1. An ellipse, according to Brown et al (2000), "is a set of all points P in the plane such that the sum of the distances from P to two fixed points is a given constant." (pg. 418) The two fixed points are referred to as foci. Using this definition it is simple to attach two ends of a string with tacks and then use the string to trace out the shape of the ellipse. Figure 2 is an example created in the web based mathematical sketchpad called Geogebra. Both line a and line b are the string with point P being the point of a pencil.

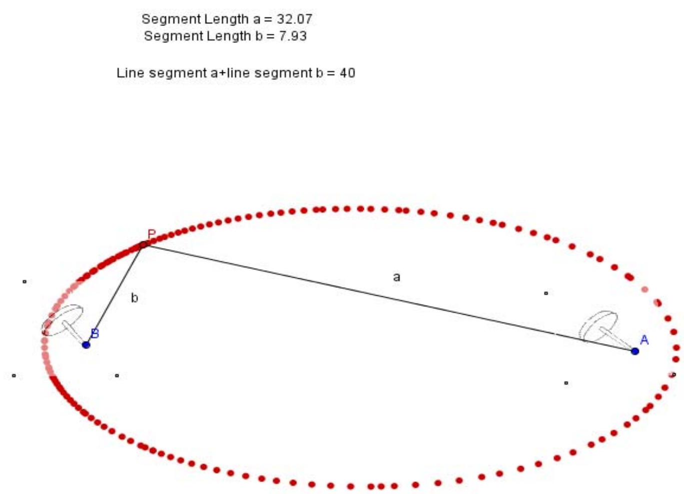
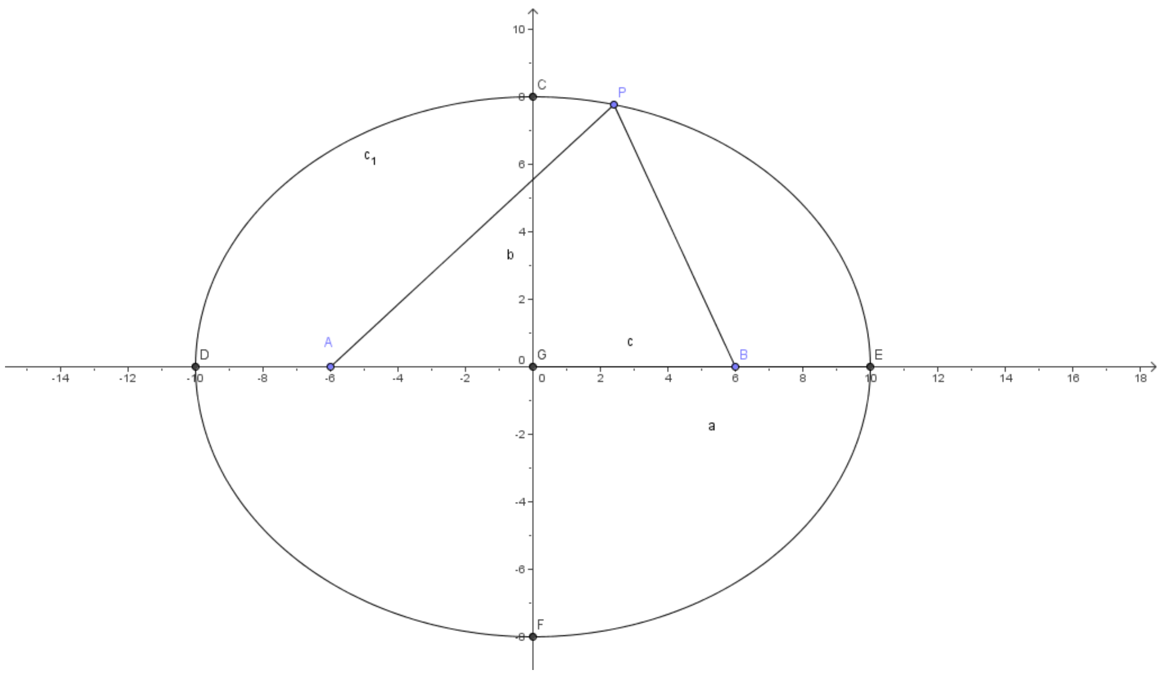


Figure 2 – Ellipse P created with string and tacks A and B.

Before we proceed with the creation of an ellipse on the rectangular coordinate system there needs to be some vocabulary defined that is special to the ellipse. Both A and B on the picture above are called the foci of the ellipse. In figure 3 line segment DE is called the major axis while line segment FC is called the minor axis. This means that line segment GE and line segment GD are called the semi-major axes. While line segment FG and line segment GC are called the semi-minor axes. To create a rectangular coordinate system equation of the ellipse we simply need to use the above picture and the distance equation of a line. We know that the string connecting points A, P and B is of fixed length. This length can be represented by the segment addition postulate as AP +

PB. If we were to trace the point P down to coincide with point A then the total string length could be represented as EA + EB. So we know that AP + PB = AE + EB. We know by symmetry that EB is the same length as DA and by substitution AP + PB = EA + AD. By segment addition again AP + PB = ED. If we let GE = a then we know that AP + PB = 2a. Figure 3 represents a specific example of this relationship.



Next, let the length of GC = b and the length of GA = c. In Figure 4 tracing P back to the point C, the lines AP and PB form the legs of an isosceles triangle APB with altitude GP. Because the triangle is isosceles AP = PB and the equation AP + PB = 2a can be written as PB + PB = 2a. Using addition 2PB = 2a and finally with division PB = a. With this relationship and the right triangle BGC, $b^2 + c^2 = a^2$ by the Pythagorean Theorem.

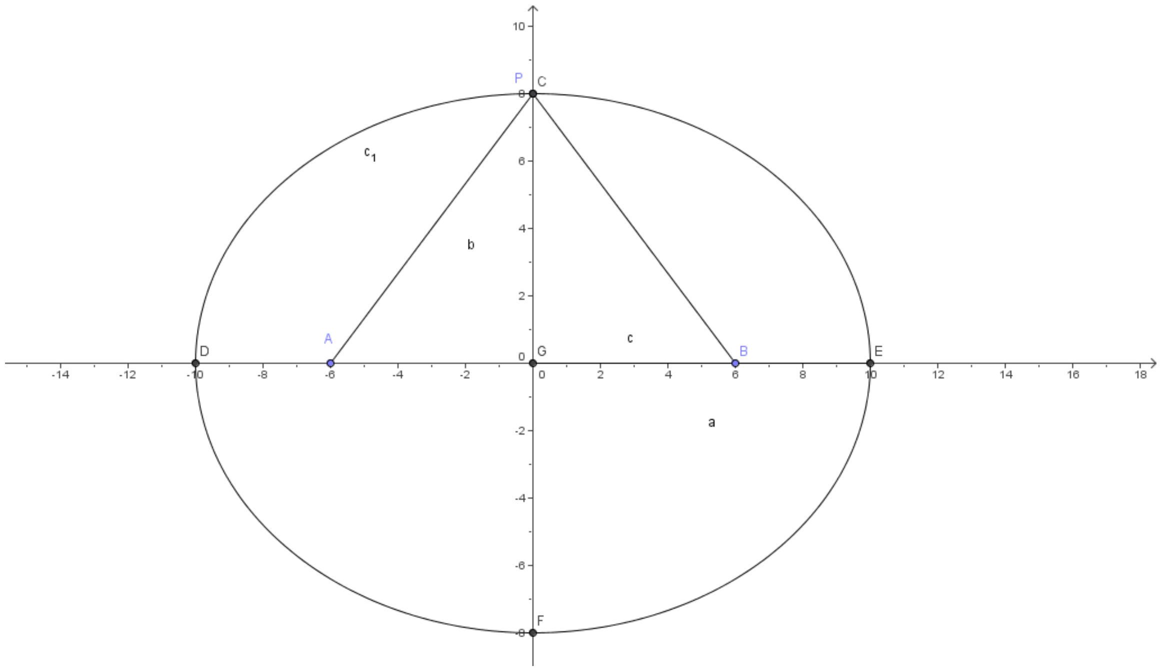


Figure 4 – Ellipse P created with string and tacks A and B on rectangular coordinate system.

We are now a little closer to the equation of the ellipse in the rectangular coordinate system. To finish this derivation out and find the rectangular equation we will follow the format of proof found in Gordon et al. (1997). Keeping the algebra simpler, let us look at the special case where the major and minor axes lie on the x and y axis respectively and use Figure 3 as our representation. If we let P be any point on the ellipse (x,y), and E(a,0), D(-a,0), A(-c,0), B(c,0), C(b,0), F(-b,0) . Form the previous derivation we know that 2a = AP + PB. The length of AP and PB can be found using the distance formula. $AP = \sqrt{(x+c)^2 + (y)^2}$

$$PB = \sqrt{(x-c)^2 + (y)^2}$$

By substitution $2a = \sqrt{(x+c)^2 + (y)^2} + \sqrt{(x-c)^2 + (y)^2}$

Addition property of equality $\sqrt{(x+c)^2 + (y)^2} = 2a - \sqrt{(x-c)^2 + (y)^2}$

Square both sides $(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + (y)^2} + (x-c)^2 + y^2$

Multiplication $x^2 + 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + (y)^2} + x^2 - 2xc + c^2 + y^2$

Addition property of equality $4xc = 4a^2 - 4a\sqrt{(x-c)^2 + (y)^2}$

Divide by 4 $xc = a^2 - a\sqrt{(x-c)^2 + (y)^2}$

Addition property of equality $a^2 - xc = a\sqrt{(x-c)^2 + (y)^2}$

Square both sides $a^4 - 2a^2xc + x^2c^2 = a^2(x-c)^2 + a^2(y)^2$

Multiplication $a^4 - 2a^2xc + x^2c^2 = a^2(x^2 - 2xc + c^2) + a^2(y)^2$

More multiplication $a^4 - 2a^2xc + x^2c^2 = a^2x^2 - 2a^2xc + a^2c^2 + a^2y^2$

Addition property of equality $a^4 + x^2c^2 = a^2x^2 + a^2c^2 + a^2y^2$

Addition property of equality $-x^2c^2 + a^2x^2 = a^4 - a^2c^2 - a^2y^2$

Commutative property of addition $a^2x^2 - x^2c^2 = a^4 - a^2c^2 - a^2y^2$

Distribution property $x^2(a^2 - c^2) = a^2(a^2 - c^2) - a^2y^2$

Divide by $a^2(a^2 - c^2)$ to get $\frac{x^2}{a^2} = 1 - \frac{y^2}{(a^2 - c^2)}$

Addition property of equality $\frac{x^2}{a^2} + \frac{y^2}{(a^2 - c^2)} = 1$

We know from before that $b^2 + c^2 = a^2$ or $b^2 = a^2 - c^2$ so by substitution

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Therefore this is the basic equation for an ellipse where the center is at the origin. If the center is translated to another point on the coordinate system, say (h,k) then the equation becomes: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

The center of the ellipse above is (h,k). The lengths of the semi-major and semi-minor axes are a and b, where a is the length of the horizontal axis and b is the length of the vertical axis. The foci can even be found with the equation we derived earlier. The distance from the center of the ellipse to the foci on the major axis is $\pm$ c. This could be found with $c^2 = a^2 - b^2$. Another piece of information that we can glean from this equation is the eccentricity which will be described below. Eccentricity (e) of an ellipse can be found with the formula e = c/a.

Eccentricity can also be found by the ratio of the length of the line from a point on the curve to a fixed line, called the directrix, and the distance from that point on the curve to the focus closest to the directrix. The eccentricity of an ellipse is greater than or equal to zero and less than 1. An ellipse with an eccentricity of about zero is almost a circle while an ellipse with an eccentricity of about 1 has a large major axis and a small minor axis. Figure 5 shows an ellipse with an eccentricity of 0.02. Figure 6 shows an ellipse with an eccentricity of 0.92.

Figure5

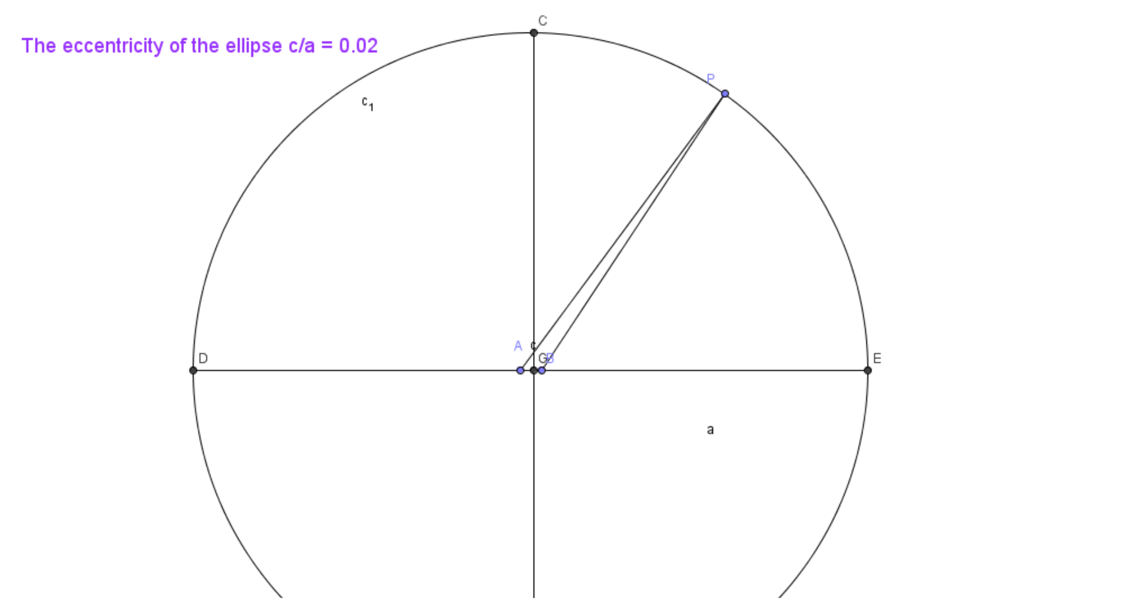
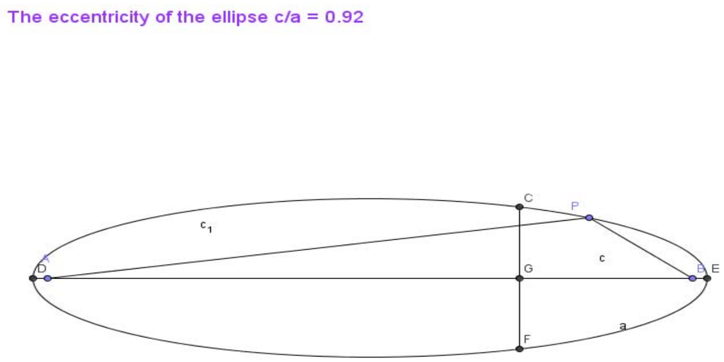
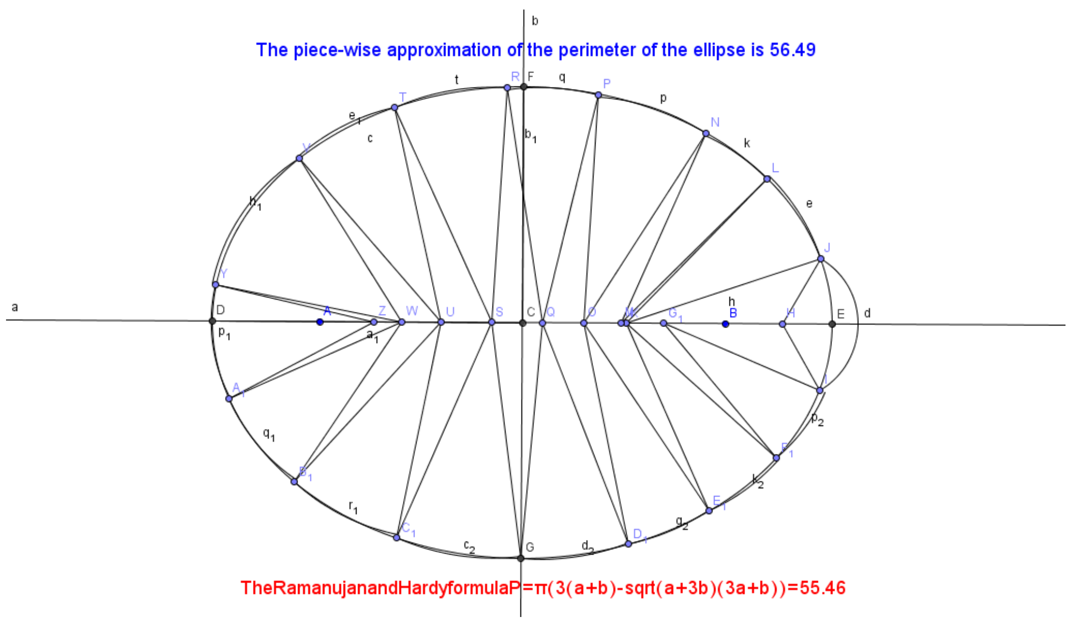


Figure6



Here it clearly shows that the ellipse will be a circle if the foci are allowed to coincide. The reason for this is that c is zero if the foci intersect the center of the ellipse. 0/a =0. The perimeter of this special ellipse is the equation for the circumference of a circle $C = 2\pi r$. The case where the eccentricity is one is a parabola. In this case the perimeter would be infinity unless bounded.

Figure7

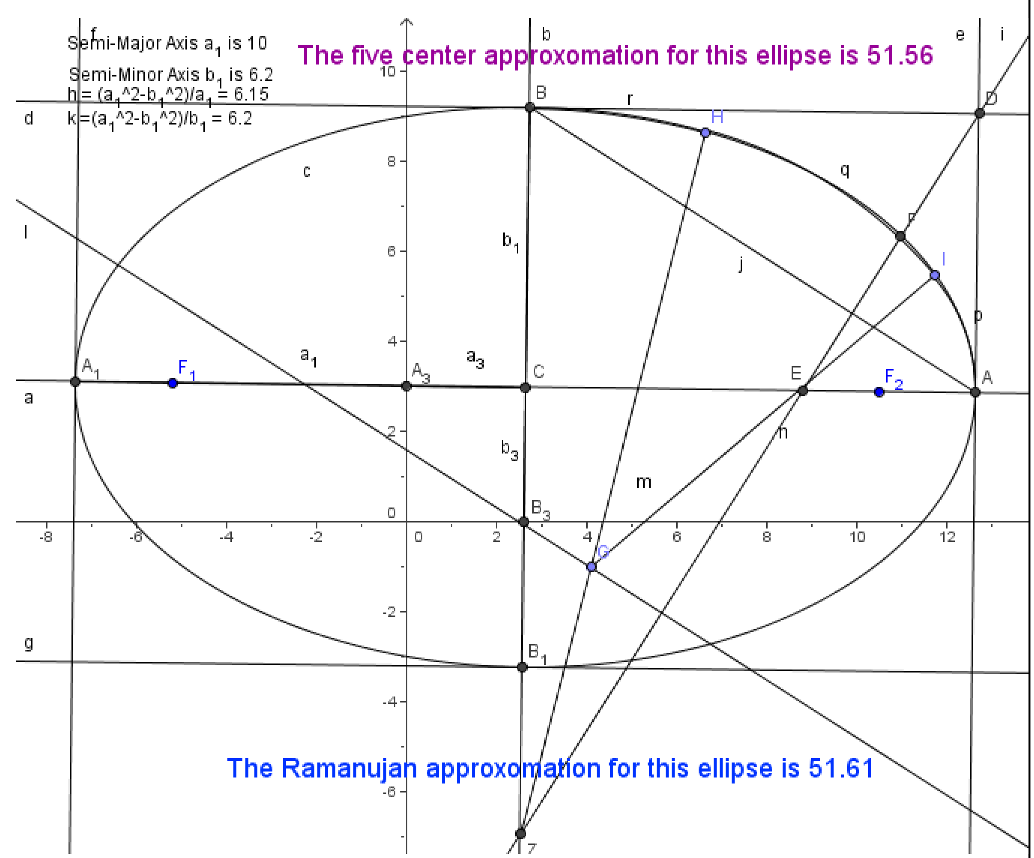


The Ellipse and Ramanujan

Ramanujan discovered an approximation of the perimeter of an ellipse. This equation is $P = \pi(3(a+b) - \sqrt{(a+3b)(3a+b)}$), where a is the length of the semi-major axis and b is the measure of the semi-minor axis. To approximate the perimeter of an ellipse, I first just divided the ellipse into many sectors, calculated the arc length of these, and then summed them up. My piece-wise approximation was 56.49. Figure 7 was my first attempt at this approximation. Arc d with center at point H has been exaggerated to show the arc of at least one of the sectors. Simply sliding H toward the center of the ellipse until the arc matches the curvature of the ellipse will give the answer of the Ramanujan approximation, in this case 55.46. There are many approximations for the perimeter of an ellipse. Some give better results than others.

In figure 7 there was no real method to my madness. I just plopped in a bunch of arcs. There is, however, a construction that requires only five centers to approximate the perimeter of an ellipse to a high degree of accuracy. My approximation above took a total of 16 separate arcs.

Figure8



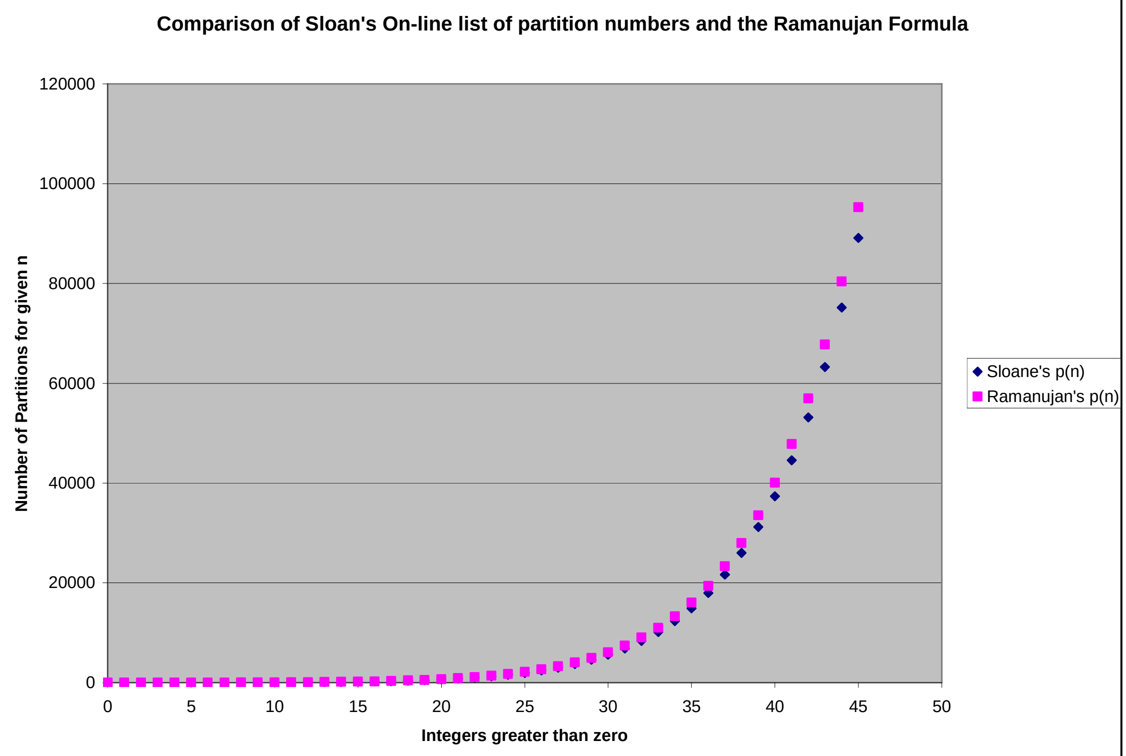
In the 5 centered arc construction from Rosin and Pitteway (2001) I used the symmetry of the ellipse and multiplied the sum of the three arcs in the first quadrant of the ellipse by 4 to get the approximation for the entire ellipse as 51.56. The Ramanujan approximation yields 51.61. Figure 8 is this piecewise approximation that yields a very accurate answer. Arcs r, q, and p are the arcs I used to approximate the entire ellipse. The construction of this 5 centered approach goes beyond the scope of this paper and so I will not explain how it was derived here.

Partitions of Numbers

The partition function p(n) is the number of ways that an integer n can be represented as the sum of two positive integers less than or equal n. Starting with zero, p(0) = 1 because only zero can be added to produce zero. The same goes for one. Two has two partitions: 2 and 1+1. Three has three partitions: 3, 2+1, 1+1+1. The integer 4 with five partitions can be represented as: 4, 3+1, 2+2, 2+1+1, 1+1+1+1. Five can be represented as: 5, 4+1, 3+2, 3+1+1,2+2+1, 2+1+1, 1+1+1+1+1, and so five has seven partitions. Six can be represented as: 6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1, and so six has eleven partitions. Sloane's On-line Encyclopedia of Integer Sequences gives the number of partitions for the integers greater than or equal to zero as: 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627.

Ramanujan created a formula for the asymptotic approximation of the partition numbers. The formula is $p(n) = \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}$. Partitions are discrete, in other words like the number of children one has. How long it takes to get to the market would be an example of a continuous function. The above formula is continuous, and since partitions are discrete this formula cannot give the exact value. In this case the term asymptotic needs to be addressed. An asymptote is a line or curve that a function approaches. The function can cross that line many times but it gets closer and closer to the asymptote each time, so asymptotic means to approach a value or curve closely. In our case Ramanujan's formula is approaching the partition values listed by Sloane. Figure 9 is the graphical representation of this relationship up to n = 45.

Figure9



Ramanujan came up with this formula to study very large values of the partition function. In fact if we look at p(1000). The actual partition for *p*(1000) = 24,061,467,864,032,622,473,692,149,727,991 ≈ 2.4×10^{31}. If we use 1000 for n in the asymptotic formula and figure that this exact value is beyond the reach of most hand held calculators because of the immense number of digits, then the TI-84 gives a function value of $2.44019963 \text{ X } 10^{31}$.

Here we have glimpsed the genius of Ramanujan. To fully understand him would be to delve into mathematics that has been mastered by but a handful of people on this planet. Kanigal's book, *The Man Who Knows Infinity*, comes closest to bringing us mere mortals into this man's world. A man so engrossed in mathematics that he neglected his own health to the point of death. A man who gave his life to understanding and developing mathematics that, at the time, had no purpose other than the mathematical realm. Hardy once stated that the mathematics should stay pure. The only mathematics worth knowing was that which has no application. Hardy wouldn't have cared in the least that the function, "known as the Hardy-Ramanujan asymptotic formula, has been widely applied in physics to find quantum partition functions of atomic nuclei (first used by Niels Bohr) and to derive thermodynamic functions of non-interacting Bose-Einstein systems." (From the Wikipedia) In fact he would have been appalled that his discovery indirectly led to nuclear weapons.

My parting thoughts are of Hardy, for without him we would not know of Ramanujan or his mathematics. These two men, although worlds apart culturally, were on the same page mathematically and worked together to bring our species to new heights of discovery.

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