3503

CLASSIFICATION OF REGULAR HOLONOMIC $\mathcal{D}$-MODULES

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CONTENTS

PREFACE

This thesis is about research that has not been finished yet. It does not present a rounded off study. This has to do with the subject of the matter. The title says "Classification of regular holonomic $\mathcal{D}$-modules" and we can merely dream of a title like "The regular holonomic $\mathcal{D}$-modules classified".

About four and a half years ago when we started our investigations little or nothing was known about the problem of classifying regular holonomic $\mathcal{D}$-modules. By that time it had become clear that the theory of $\mathcal{D}$-modules is a very useful tool in various parts of mathematics. In particular the so-called regular holonomic $\mathcal{D}$-modules played a major role. One of the highlights of the theory of $\mathcal{D}$-modules is the Riemann-Hilbert correspondence. It establishes a one-to-one correspondence between regular holonomic $\mathcal{D}_X$-modules and perverse sheaves on X (where X denotes a complex manifold). Now a perverse sheaf isn't really a sheaf but rather a complex of sheaves. More precise it is an object of a derived category. Also the notion of a morphism between perverse sheaves is difficult to handle. But on the other side it is clear what is meant by a $\mathcal{D}_X$-module and a $\mathcal{D}_X$-linear morphism. So if one wants to understand the structure of the perverse sheaves on X it is certainly worth-while to take advantage of the Riemann-Hilbert correspondence and to study the category of regular holonomic $\mathcal{D}_X$-modules. These objects are perhaps more accessible.

An example of a perverse sheaf is the intersection cohomology sheaf $\mathcal{IC}^{\cdot}_X$, which was expected to carry a pure Hodge structure. And indeed, quite recently, this was shown to be true by M. Saito using $\mathcal{D}$-module theoretic methods.

Our philosophy is to get a better understanding of e.g., the perverse sheaves by means of a good knowledge of the regular holonomic $\mathcal{D}$-modules. Besides the regular holonomic $\mathcal{D}$-modules are important in themselves. Another point is that the Riemann-Hilbert correspondence is complicated and therefore one would like to understand both sides.

Regular holonomic $\mathcal{D}$-modules have been the subject of our investigations for the past few years. A number of articles appeared on the subject of classifying regular holonomic $\mathcal{D}$-modules or perverse sheaves. In fact most authors classified perverse sheaves. Just a few cases are understood thus far; there is still a lot to be done. We briefly discuss the known cases in the last part of Chapter I. In that chapter we also provide some examples to motivate the study of regular holonomic $\mathcal{D}$-modules. For the reader's convenience the first part of Chapter I contains a survey of the theory of $\mathcal{D}$-modules. It is illustrated by a number of examples.

The organization of the rest of the material is more or less reflected by our approach to the classification problem. We study regular holonomic $\mathcal{D}_X$-modules whose singular loci are contained in a fixed hypersurface

X_0 given by $f \in \Gamma(X, \mathcal{O}_X)$. Let us assume that the situation is local i.e., $f: X \to S$ is a good representative of a germ $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. In case $X_0 = f^{-1}(0)$ is a divisor with normal crossings ($f = x_1 \ldots x_n$) the classification is well-known by now. Chapter II deals with it.

But how to proceed in more general situations? We use an idea inspired by Verdier's extension theorem (cf. §2.2.1.6). In fact in Chapter IV we prove a $\mathcal{D}$-module theoretic analogue of this theorem. It states that a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ such that $\mathcal{M}|_{X-X_0}$ is "without singularities" is determined by the data $(\mathcal{M}[f^{-1}], \phi\mathcal{M} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathcal{M})$. Here $\phi\mathcal{M}$ and $\psi\mathcal{M}$ are regular holonomic $\mathcal{D}_X$-modules supported on the hypersurface X_0. The arrows represent certain $\mathcal{D}_X$-morphisms. Now $\mathcal{M}[f^{-1}]$ is just the Deligne extension of the $\mathcal{D}_{X-X_0}$-module $\mathcal{M}|_{X-X_0}$. Saying that $\mathcal{M}|_{X-X_0}$ is "without singularities" means that it is a vector bundle with an integrable connection. Hence $\mathcal{M}|_{X-X_0}$ is determined by a finite dimensional representation of the fundamental group $\pi_1(X - X_0)$.

Let us turn to the modules $\phi\mathcal{M}$ and $\psi\mathcal{M}$. As we already mentioned these are regular holonomic $\mathcal{D}_X$-modules with support contained in X_0. Since $\dim X_0 = \dim X - 1$ we have, in principal, reduced the problem of describing them to a lower dimensional case. If X_0 is smooth then by Kashiwara's equivalence (cf. §1.6.2.1) $\phi\mathcal{M}$ and $\psi\mathcal{M}$ correspond to regular holonomic $\mathcal{D}_{X_0}$-modules. However in general X_0 will have singularities. Here we meet a drawback of the theory of $\mathcal{D}$-modules: on a singular variety the notion of $\mathcal{D}$-modules is of no use in general.

Let us assume that X_0 is a hypersurface of dimension 1, thus a plane curve. Assume furthermore that $f: (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is an irreducible germ with an isolated singularity at 0. Let $\pi: (\mathbb{C}, 0) \to (X_0, 0)$ be the normalization of $(X_0, 0)$ and put $\tilde{\pi}: (\mathbb{C}, 0) \to (X_0, 0) \hookrightarrow (\mathbb{C}^2, 0)$. In Chapter III we derive a modification of Kashiwara's equivalence which yields that the direct image functor $\tilde{\pi}_+$ establishes an equivalence between the categories of $\mathcal{D}_{\mathbb{C},0}$-modules and the category of $\mathcal{D}_{\mathbb{C}^2,0}$-modules with support contained in X_0. Hence the pair $\phi\mathcal{M} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathcal{M}$ is identified as a pair of regular holonomic $\mathcal{D}_{\mathbb{C},0}$-modules. These have been dealt with in Chapter II. Thus far we haven't obtained more general results e.g., when X_0 is of dimension 2.

We should like to add a remark. We started our investigations by looking at the normal crossings case. We found an elegant way to describe holonomic $\mathcal{D}$-modules with regular singularities along normal crossings. To be more concrete let us consider the one dimensional case $X = \mathbb{C}$, $f = x$. We introduced a pair of $\mathcal{D}_{\mathbb{C},0}$-modules $\mathcal{F} \rightleftarrows \mathcal{F}/\mathcal{O}$. To any regular holonomic $\mathcal{D}_{\mathbb{C},0}$-module M is associated the pair of finite dimensional vector spaces

$$S(M) := \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{F}) \rightleftarrows \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{F}/\mathcal{O}).$$

To a pair $E \rightleftarrows F$ of finite dimensional vector spaces we associated the $\mathcal{D}_{\mathbb{C},0}$-module

$$\mathrm{Hom}(E \rightleftarrows F, \mathcal{F} \rightleftarrows \mathcal{F}/\mathcal{O})$$

and we showed that this yields an inverse functor to S. A similar discussion applies in higher dimensions.

For a while we tried to generalize this to situations with arbitrary singularities. However we did not obtain satisfactory results. It is easy to give an intrinsic definition of $\mathcal{F}$, namely $\mathcal{F}$ is the sheaf on X_0 of the Nilsson class functions. It follows that

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})|_{X_0} \cong \Psi_f(\mathcal{L})$$

and

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}/\mathcal{O})|_{X_0} \cong \Phi_f(\mathcal{L}),$$

where $\mathcal{L} = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Ψ_f and Φ_f are functors introduced by Deligne (cf. Ch. IV). But $\Psi_f(\mathcal{L})$ and $\Phi_f(\mathcal{L})$ are perverse sheaves on X_0; we are out of the framework of $\mathcal{D}$-modules, something which we didn't want to do.

Remark.
The chapters II, III and IV appeared earlier as reports 8504, 8608 and 8713 of the Department of Mathematics of the University of Nijmegen under the same title as the chapter in question.
Chapter II has been published in Comp. Math. **60** (1986), 19–32.
The chapters III and IV are submitted for publication to RIMS.

Chapter I

BRIEF ENCOUNTERS

0 Introduction

This chapter consist roughly of two parts. In the first part—§1—we give a review of the theory of $\mathcal{D}$-modules. Throughout the text we supply some examples to illustrate the theory. We have included a subsection on derived categories because the theory of $\mathcal{D}$-modules most naturally fits into this framework. Also the so-called perverse sheaves are objects of a derived category. The last subsection contains the definition of regular holonomic $\mathcal{D}$-modules and the statement of the celebrated "Riemann-Hilbert correspondence". It sets up a dictionary between regular holonomic $\mathcal{D}_X$-modules and perverse sheaves on X (where X denotes a complex manifold).

In the second part—§2—we want to give some kind of a motivation for studying regular holonomic $\mathcal{D}$-modules. We mention three examples which should convince the reader. Next we concentrate on the problem of classifying regular holonomic $\mathcal{D}$-modules. Equivalently there is the problem of classifying perverse sheaves. We summarize the results obtained up to now. (As far as they are known to the author!)

1 Review of the theory of $\mathcal{D}$-modules

There are at least two survey articles on the theory of $\mathcal{D}$-modules, namely [Lê and Mebkhout, 1983] and [Oda, 1983]. Of course we profited from these. We also learned a lot from several fabulous talks of Björk (see e.g., [Björk, 1984]). Furthermore there are the papers [Brylinski, 1982a, 1983] on $\mathcal{D}$-modules and related topics. Certainly these provide a nice and sometimes speculative picture of the interplay between regular holonomic $\mathcal{D}$-modules, intersection cohomology and Hodge structures.

Perhaps the first systematic use of $\mathcal{D}$-modules appeared in [Sato, Kashiwara, Kawai, 1973]. Since then there have appeared several articles by Kashiwara and others. We should also mention the contributions of Malgrange. Furthermore Mebkhout used the theory of $\mathcal{D}$-modules to study the topology of singular varieties. Last but not least we mention the work of Beilinson and Bernstein regarding the algebraic aspect of the theory.

The picture we present is complex analytic i.e., concerns differential operators on complex manifolds. We do not deal with the algebraic—meaning differential operators on algebraic varieties—theory of $\mathcal{D}$-modules. In the algebraic theory the notions of holonomicity and regularity behave well with respect to direct and inverse images. This in contrast to the analytic theory (compare §1.6). Recently there has appeared the very nice book [Borel, 1987] that gives a good account of the algebraic theory. We refer the reader also to [Bernstein, 1983].

We also bypass the microlocal aspect of the theory. This is a powerful machinery that has been used to attack several problems in the theory of $\mathcal{D}$-modules. The reader may consult e.g., [Schapira, 1985] for a nice account of this aspect or [Pham, 1979]. For the algebraic counterpart see e.g., [Laumon, 1983, 1985] and also [van den Essen, 1986].

The plan of the exposition is as follows. First of all some generalities on $\mathcal{D}$-modules. Next comes a subsection on derived categories and derived functors. After this we are sufficiently armed to enter the scene of $\mathcal{D}$-modules. We introduce the reader to the functorial aspect i.e., direct and inverse images. After that it is about time for a closer examination of the $\mathcal{D}$-modules themselves. This culminates in a class of $\mathcal{D}$-modules of utmost importance: the regular holonomic $\mathcal{D}$-modules. These are the subject of our study.

In the sequel X denotes a smooth complex analytic variety of dimension n. $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X.

1.1 Differential operators on X
The sheaf of differential operators, denoted $\mathcal{D}_X$, is the subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ defined as follows:

Let $U \subset X$ be open and suppose $x_1, \dots, x_n$ are coordinates on U. Put $\partial_i = \frac{\partial}{\partial x_i}$, for all $i \in \{1, \dots, n\}$. Sections of $\mathcal{D}_X$ above U can be written as

$$\sum_{|\alpha| \leq m} a_\alpha \partial^\alpha \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_U)$$

where $m \in \mathrm{N}$, $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, $a_\alpha \in \Gamma(U, \mathcal{O}_X)$, all $\alpha \in \mathrm{N}^n, |\alpha| \leq m$.

Composition of operators gives $\mathcal{D}_X$ the structure of a sheaf of noncommutative algebras. It is the subalgebra of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X)$; $\mathcal{D}_X$ is a coherent sheaf of rings. It is a quasi-coherent $\mathcal{O}_X$-module. The structure sheaf $\mathcal{O}_X$ becomes in an obvious way a coherent left $\mathcal{D}_X$-module.

1.1.1 Example Let $X = \mathbb{C}^n$. Then $\mathcal{D}_X = \mathcal{O}_X[\partial_1, \dots, \partial_n]$. Furthermore $\mathcal{O}_X \cong \mathcal{D}_X / \mathcal{D}_X(\partial_1, \dots, \partial_n)$ is a coherent $\mathcal{D}_X$-module.

1.1.2 Kashiwara notes the following. Consider a system of linear homogeneous partial differential equations

$$(*) \qquad \sum_{j=1}^{p} P_{ij} u_j = 0, \qquad i = 1, \dots, q$$

with $P_{ij} \in \Gamma(X, \mathcal{D}_X)$. Letting the matrix (P_{ij}) act from the right on $\mathcal{D}_X$ yields

$$\mathcal{D}_X^q \xrightarrow{(P_{ij})} \mathcal{D}_X^p \twoheadrightarrow \mathcal{M}.$$

So $\mathcal{M} := \mathrm{Coker}(P_{ij})$ is a left $\mathcal{D}_X$-module of finite presentation. The holomorphic solutions of our system (*) can be interpreted as elements of the $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ and vice versa. This leads one to consider coherent left $\mathcal{D}_X$-modules and the sheaf of $\mathbb{C}$-vector spaces $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$; more generally $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$ for coherent left $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$.

1.2 Left versus right

1.2.1 Consider the (locally free of rank 1) $\mathcal{O}_X$-module $\Omega^n_X =: \omega_X$. It has in a natural way the structure of a right $\mathcal{D}_X$-module. Namely for all $\omega \in \omega_X$ put

$$\omega\xi := -L_\xi \omega \in \omega_X, \quad \text{for all } \xi \in \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X),$$

where L_ξ denotes the Lie derivative taken in the direction of the vectorfield ξ.

1.2.2 Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module and consider the $\mathcal{O}_X$-module

$$\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M}.$$

It has in a natural way a right $\mathcal{D}_X$-structure given by, for all $\xi \in \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X)$

$$(\omega \otimes m)\xi := \omega\xi \otimes m - \omega \otimes \xi m, \quad \text{for all } \omega \in \omega_X, m \in \mathcal{M}.$$

1.2.3 Let $\mathcal{N}$ be a right $\mathcal{D}_X$-module. The $\mathcal{O}_X$-module

$$\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N})$$

has in a natural way a left $\mathcal{D}_X$-structure given as follows: for all $\xi \in \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X)$, all $\varphi \in \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N})$

$$(\xi\varphi)(\omega) := \varphi(\omega\xi) - \varphi(\omega)\xi, \quad \text{for all } \omega \in \omega_X.$$

1.2.4 Clearly the second application is an inverse for the first one. Hence this sets up an equivalence between the category of left $\mathcal{D}_X$-modules and the category of right $\mathcal{D}_X$-modules.

It has become customary to restrict attention to the left $\mathcal{D}_X$-modules. As we can move freely from left modules to right modules and back, no harm is done. One often omits the word "left" and just writes "$\mathcal{D}_X$-module" instead of "left $\mathcal{D}_X$-module". We'll do likewise.

Notation $\mathcal{M}od(\mathcal{D}_X)$ denotes the category of $\mathcal{D}_X$-modules. $\mathcal{C}oh(\mathcal{D}_X)$ denotes the category of coherent $\mathcal{D}_X$-modules.

1.3 Examples of $\mathcal{D}_X$-modules

1.3.1 Let $P \in \Gamma(X, \mathcal{D}_X)$. Then $\mathcal{D}_X/\mathcal{D}_X P$ is a coherent $\mathcal{D}_X$-module. (Compare with Example 1.1.1.)

1.3.2 The structure sheaf $\mathcal{O}_X$ is a coherent $\mathcal{D}_X$-module.

1.3.3 More general: Let $\mathcal{V}$ be a vector bundle on X (i.e., a locally free $\mathcal{O}_X$-module of finite rank) with an integrable connection ∇. Recall that this is a $\mathbb{C}$-linear mapping

$$\nabla: \mathcal{V} \to \Omega^1_X \underset{\mathcal{O}_X}{\otimes} \mathcal{V}$$

satisfying the Leibniz-rule $\nabla(av) = da \otimes v + a\nabla v$, for all $a \in \mathcal{O}_X$ and $v \in \mathcal{V}$. ∇ is called integrable if its curvature is zero i.e.,

$$\nabla \circ \nabla: \mathcal{V} \to \Omega^2_X \underset{\mathcal{O}_X}{\otimes} \mathcal{V}$$

is the zero map. Equivalently one can say that

$$\nabla_{[\xi,\eta]} = [\nabla_\xi, \nabla_\eta], \qquad \text{for all } \xi, \eta \in \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X).$$

Now $\mathcal{V}$ can be given the structure of a coherent $\mathcal{D}_X$-module by defining for all $\xi \in \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X)$

$$\xi v := \langle \nabla_\xi, v \rangle, \qquad \text{for all } v \in \mathcal{V}.$$

Conversely suppose $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module such that the underlying $\mathcal{O}_X$-structure is coherent, then one can show—which is not difficult—that $\mathcal{M}$ is in fact a vector bundle with connection.

1.4 Derived categories and derived functors

For an adequate setting of the theory of $\mathcal{D}$-modules the machinery of derived categories is indispensable. We will give a simplified description of the derived category in a particular case which applies to almost all situations we have in mind. A similar treatment may be found in [Iversen, 1986]. For a detailed treatment we refer the reader to [Hartshorne, 1966], [Verdier, 1977] or [Borel, 1987], Ch. 1. Of course the ultimate test to try one's hand on is [Beilinson, Bernstein, Deligne, 1982].

1.4.1 Derived categories

Let $\mathcal{A}$ be an abelian category. $\mathrm{C}(\mathcal{A})$ denotes the abelian category of complexes of objects of $\mathcal{A}$. $\mathrm{K}(\mathcal{A})$ denotes the additive category consisting of the objects of $\mathrm{C}(\mathcal{A})$ but the morphisms are homotopy classes of morphisms in $\mathrm{C}(\mathcal{A})$. Note that $\mathcal{A}$ may be identified as the full subcategory of $\mathrm{K}(\mathcal{A})$ consisting of objects $A^\cdot$ with $A^n = 0$ for all $n \neq 0$. A morphism in $\mathrm{K}(\mathcal{A})$ $f: A^\cdot \to B^\cdot$ is called a quasi-isomorphism—abbreviated as q.i.—if $H^k(f)$ is an isomorphism for all $k \in \mathbf{Z}$; notation $f: A^\cdot \xrightarrow{\sim} B^\cdot$.

Denote by $\mathrm{K}^+(\mathcal{A})$ (resp. $\mathrm{K}^-(\mathcal{A})$, resp. $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$) the full subcategory of $\mathrm{K}(\mathcal{A})$ consisting of complexes $A^\cdot \in \mathrm{K}(\mathcal{A})$ which are bounded below i.e., $A^n = 0$ for $n \ll 0$ (resp. bounded above, resp. bounded).

Assume that $\mathcal{A}$ has enough injectives. Denote by $\mathcal{I}$ the additive subcategory of injective objects. $\mathrm{K}^+(\mathcal{I}) \subset \mathrm{K}^+(\mathcal{A})$ is the additive subcategory of bounded below complexes of injective objects. $I^\cdot \in \mathrm{K}^+(\mathcal{I})$ is called an injective resolution of $A^\cdot \in \mathrm{K}^+(\mathcal{A})$ if there exists a quasi-isomorphism $A^\cdot \xrightarrow{\sim} I^\cdot$. Every object in $\mathrm{K}^+(\mathcal{A})$ has up to isomorphism a unique injective resolution i.e., there exists a functor

$$\rho: \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{I})$$

sending $A^\cdot$ to its injective resolution.

Define $\mathrm{D}^+(\mathcal{A}) := \mathrm{K}^+(\mathcal{I})$. This is called "the derived category of bounded below complexes of objects of $\mathcal{A}$". It comes together with a functor $\rho: \mathrm{K}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{A})$. Often one identifies $\mathcal{A}$ with the full subcategory of $\mathrm{D}(\mathcal{A})$ consisting of complexes $I^\cdot$ with $H^k(I^\cdot) = 0$, for all $k \neq 0$. If moreover any object in $\mathcal{A}$ has a bounded resolution by injectives, one may define $\mathrm{D}^{\mathrm{b}}(\mathcal{A}) := \mathrm{K}^{\mathrm{b}}(\mathcal{I})$. It is the full subcategory of $\mathrm{D}^+(\mathcal{A})$ consisting of objects $I^\cdot$ such that $H^k(I^\cdot) = 0$ for $k \gg 0$.

A similar discussion applies when $\mathcal{A}$ has enough projectives. Then $\mathrm{D}^-(\mathcal{A}) := \mathrm{K}^-(\mathcal{P})$ where $\mathcal{P} \subset \mathcal{A}$ denotes the class of projective objects.

1.4.1.1 Examples and notations We give some notations that will be used later.

$$\mathrm{D}^+(\mathcal{D}_X) := \mathrm{D}^+(\mathcal{M}od(\mathcal{D}_X))$$
$$\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) := \mathrm{D}^{\mathrm{b}}(\mathcal{M}od(\mathcal{D}_X))$$
$$\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathcal{M}od(\mathbb{C}_X)),$$

where $\mathcal{M}od(\mathbb{C}_X)$ denotes the category of sheaves of $\mathbb{C}$-vector spaces.

1.4.1.2 $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ denotes the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting of objects whose cohomology modules are coherent $\mathcal{D}_X$-modules.

1.4.1.3 In general, for any thick abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ one defines $\mathrm{D}^+_{\mathcal{A}'}(\mathcal{A})$ as the full subcategory of $\mathrm{D}^+(\mathcal{A})$ consisting of objects $A^\cdot$ such that $H^k(A^\cdot) \in \mathcal{A}'$ for all $k \in \mathbf{Z}$. Similarly for $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ and $\mathrm{D}^-_{\mathcal{A}'}(\mathcal{A})$. Recall that "thick" means the following: any extension of objects in $\mathcal{A}'$ is again in $\mathcal{A}'$.

1.4.2 Derived functors

Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories with enough injectives. Denote $\mathcal{I} \subset \mathcal{A}$ (resp. $\mathcal{J} \subset \mathcal{B}$) the class of injective objects in $\mathcal{A}$ (resp. $\mathcal{B}$). Denote by

$$\rho\colon \mathrm{K}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{A}), \qquad \rho'\colon \mathrm{K}^+(\mathcal{B}) \to \mathrm{D}^+(\mathcal{B})$$

the canonical functors.

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. It induces a functor on complexes $F^+\colon \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{B})$. The "right derived functor"

$$\mathrm{R}F\colon \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$$

is defined by

$$\mathrm{R}F(I^\cdot) := \rho' F^+(I^\cdot), \quad \text{for all } I^\cdot \in \mathrm{K}^+(\mathcal{I}) = \mathrm{D}^+(\mathcal{A}).$$

For all $n \in \mathbf{Z}$ define $\mathrm{R}^n F := H^n \circ \mathrm{R}F$. Abusing notation we write $\mathrm{R}F(A^\cdot)$ instead of $\mathrm{R}F(\rho A^\cdot)$ for all $A^\cdot \in \mathrm{K}^+(\mathcal{A})$. Even more one often omits the "R" and denotes the derived functor also by F.

1.4.2.1 Let $A \in \mathcal{A}$. Then $\mathrm{R}^n F(A) = H^n\big(F(I^\cdot)\big)$ where $I^\cdot = \rho A$ is an injective resolution of A. Hence $\mathrm{R}^n F$ coincides with the usual n-th derived functor.

1.4.2.2 Let $A^\cdot \in \mathrm{K}^+(\mathcal{A})$. Let $I^\cdot = \rho A^\cdot$ be an injective resolution. Then

$$\mathrm{R}^n F(A^\cdot) = H^n\big(\rho' F(I^\cdot)\big) = H^n\big(F(I^\cdot)\big),$$

Thus $\mathrm{R}^n F$ is nothing else but the n-th hypercohomology of F.

1.4.2.3 An object $C \in \mathcal{A}$ is called F-acyclic if $\mathrm{R}^n F(C) = 0$ for all $n \neq 0$. If $A^\cdot \in \mathrm{K}^+(\mathcal{A})$ and $A^\cdot \xrightarrow{\sim} C^\cdot$ where $C^\cdot$ is a complex of F-acyclic objects, then $\mathrm{R}F(A^\cdot) \cong \rho' F^+(C^\cdot)$. So in order to calculate $\mathrm{R}F$ it suffices to construct F-acyclic resolutions.

1.4.2.4 A nice and important property is the following. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three abelian categories with enough injectives. Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume that F sends injective objects to G-acyclic objects. Then $\mathrm{R}(G \circ F) = \mathrm{R}G \circ \mathrm{R}G$.

1.4.2.5 Similarly in case $\mathcal{A}$ and $\mathcal{B}$ have enough projectives one defines the "left derived functor" $\mathrm{L}F$ for any right exact functor $F\colon \mathcal{A} \to \mathcal{B}$.

1.4.2.6 Examples (i) We have the bifunctors

$$\mathrm{RHom}_{\mathcal{D}_X}(\,\cdot\,,\,\cdot\,), \qquad \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\,\cdot\,,\,\cdot\,).$$

The k-th derived functors are denoted $\mathrm{Ext}^k_{\mathcal{D}_X}(\,\cdot\,,\,\cdot\,)$, resp. $\mathcal{E}xt^k_{\mathcal{D}_X}(\,\cdot\,,\,\cdot\,)$.

(ii) The bifunctor $\cdot \underset{\mathcal{O}}{\otimes} \cdot$ has a left derived functor denoted

$$\cdot \overset{\mathrm{L}}{\underset{\mathcal{O}}{\otimes}} \cdot .$$

The $(-k)$-th derived functor is $\mathcal{T}or_k^{\mathcal{O}}(\cdot, \cdot)$.
(iii) The global section functor has a right derived functor $\mathrm{R}\Gamma(X, \cdot)$. Its k-th derived functor is denoted $\mathcal{H}^k(X, \cdot)$; this yields the hypercohomology.
(iv)

$$\mathrm{R}\Gamma(X, \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})) = \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}).$$

1.4.3 Distinguished triangles

In general the categories $\mathrm{K}^+(\mathcal{A})$ and $\mathrm{D}^+(\mathcal{A})$ are not abelian anymore. Therefore one has introduced the concept of "triangulated category" (cf. [Beilinson, Bernstein, Deligne, 1982]). Let $A^\cdot \in \mathrm{K}(\mathcal{A})$ be given and denote by $d_{A^\cdot}$ the differential of $A^\cdot$. For any $n \in \mathbf{Z}$ the complex $A^\cdot[n] \in \mathrm{K}(\mathcal{A})$ is given by

$$(A^\cdot[n])^k = A^{n+k}, \quad \text{for all } k \in \mathbf{Z}; \quad d_{A^\cdot[n]} = (-1)^n d_{A^\cdot}.$$

A triangle in $\mathrm{K}(\mathcal{A})$ is a sextuple $(A^\cdot, B^\cdot, C^\cdot, u, v, w)$ of objects and morphisms $u: A^\cdot \to B^\cdot$, $v: B^\cdot \to C^\cdot$, $w: C^\cdot \to A^\cdot[1]$ in $\mathrm{K}(\mathcal{A})$.

For any morphism $u: A^\cdot \to B^\cdot$ in $\mathrm{K}(\mathcal{A})$ denote by $C_u^\cdot$ the mapping cone of u. It comes equipped with natural morphisms $p: C_u^\cdot \to A^\cdot[1]$ and $i: B^\cdot \to C_u^\cdot$. The sextuple $(A^\cdot, B^\cdot, C_u^\cdot, u, i, p)$ is called a standard triangle. A distinguished triangle in $\mathrm{K}(\mathcal{A})$ (resp. $\mathrm{K}^+(\mathcal{A})$, resp. $\mathrm{K}^+(\mathcal{I})$) is a triangle isomorphic to a standard triangle in $\mathrm{K}(\mathcal{A})$ (resp. $\mathrm{K}^+(\mathcal{A})$, resp. $\mathrm{K}^+(\mathcal{I})$).

The functor ρ sends distinguished triangles to distinguished ones. The derived functor $\mathrm{R}F$ transforms distinguished triangles in $\mathrm{D}^+(\mathcal{A})$ into distinguished ones in $\mathrm{D}^+(\mathcal{B})$.

A distinguished triangle is often written as

$$\begin{tikzcd}[column sep=small] & C^\cdot \arrow[dl, "+"'] & \\ A^\cdot \arrow[rr] & & B^\cdot \arrow[ul] \end{tikzcd}$$

where $C^\cdot \xrightarrow{+} A^\cdot$ denotes a morphism $C^\cdot \longrightarrow A^\cdot[1]$. Applying the cohomology functor H to this triangle yields the long exact sequence

$$\cdots \to H^k(A^\cdot) \to H^k(B^\cdot) \to H^k(C^\cdot) \to H^{k+1}(A^\cdot) \to \cdots .$$

1.4.3.1 Example Every short exact sequence $A \hookrightarrow B \twoheadrightarrow C$ in $\mathcal{A}$ gives rise to a distinguished triangle in $\mathrm{D}^+(\mathcal{A})$

$$\begin{tikzcd}[column sep=small] & C \arrow[dl, "+"'] & \\ A \arrow[rr] & & B. \arrow[ul] \end{tikzcd}$$

1.4.3.2 Example Let $Z \subset X$ be a closed subset. Put $U = X - Z$, $j: U \hookrightarrow X$ the inclusion. Let $\mathcal{F}$ be a sheaf of $\mathbb{C}$-vector spaces on X. There exists a distinguished triangle in $\mathrm{D^b}(X)$

$$\begin{tikzcd} & \mathrm{R}j_*j^{-1}\mathcal{F} \arrow[dl, "+"'] & \\ \mathrm{R}\Gamma_Z\mathcal{F} \arrow[rr] & & \mathcal{F}. \arrow[ul] \end{tikzcd}$$

The same holds for a complex $\mathcal{F}^\cdot \in \mathrm{D^b}(X)$.

1.5 Solution functor, de Rham complex

1.5.1 We have already seen that it makes sense to consider the sheaves of vector spaces $\mathcal{E}xt_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ for any $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)$. These are the derived functors of

$$S: \mathrm{D^b}(\mathcal{D}_X) \to \mathrm{D^b}(X)$$

defined by

$$\mathcal{M}^\cdot \mapsto \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\cdot, \mathcal{O}_X).$$

Thus S maps each $\mathcal{D}_X$-module $\mathcal{M}$ to its complex of solutions.

1.5.2 To any $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)$ there is associated another complex in $\mathrm{D^b}(X)$ called the de Rham complex $\mathrm{DR}(\mathcal{M})$ of $\mathcal{M}$,

$$\cdots \to 0 \to \mathcal{M} \to \Omega_X^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \to \cdots \to \Omega_X^n \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \to 0 \to \cdots$$

where $\mathcal{M}$ is placed in degree 0. For $\mathcal{M} = \mathcal{O}_X$ this yields the usual de Rham complex $\Omega_X^\cdot$.

1.5.3 Example Let $P \in \Gamma(X, \mathcal{D}_X)$. Put $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$. Then

$$S(\mathcal{M}) = \quad \mathcal{O}_X \xrightarrow{P\cdot} \mathcal{O}_X \quad .$$

Consequently

$$\mathrm{Ker}(P, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

$$\mathrm{Coker}(P, \mathcal{O}_X) = \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

1.5.4 Example Let $\mathcal{V}$ be a vector bundle with an integrable connnection ∇. Then

$$\mathrm{DR}(\mathcal{V}) = \Omega_X^\cdot \underset{\mathcal{O}_X}{\otimes} \mathcal{V} = \Omega_X^\cdot(\mathcal{V}) = \mathrm{Ker}(\nabla, \mathcal{V})$$

by the Poincaré lemma.

$\mathrm{Ker}(\nabla, \mathcal{V})$ is called the sheaf of horizontal sections of $(\mathcal{V}, \nabla)$. It is a local system on X i.e., a locally constant sheaf of complex vector spaces of finite dimension.

In particular $\mathrm{DR}(\mathcal{O}_X) = \Omega_X^\cdot = \mathbb{C}_X$.

1.6 Construction of inverse, direct images and $R\Gamma_{[Z]}$.

In this subsection we discuss three operations on $\mathcal{D}$-modules. Let Y be a complex manifold of dimension m and suppose we are given a holomorphic map $f: X \to Y$.

1.6.1 Inverse images

Let $\mathcal{N}$ be a left $\mathcal{D}_Y$-module. We get a quasi-coherent $\mathcal{O}_X$-module by putting

$$f^*\mathcal{N} = \mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{N}.$$

Here f^{-1} denotes the usual inverse images of sheaves; $f^*\mathcal{N}$ is the inverse images in the category of $\mathcal{O}$-modules. $f^*\mathcal{N}$ is in a natural way a left $\mathcal{D}_X$-module. In local coordinates $y_1, \ldots, y_m$ on Y and $x_1, \ldots, x_n$ on X this structure is given by

$$\frac{\partial}{\partial x_i}(a \otimes s) = \frac{\partial a}{\partial x_i} \otimes s + \sum_{j=1}^{m} \frac{\partial f_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} s,$$

for all $i \in \{1, \ldots, n\}$, all $a \in \mathcal{O}_X$, all $s \in \mathcal{N}$.

One usually writes

$$f^*\mathcal{N} = (\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{D}_Y) \underset{f^{-1}\mathcal{D}_Y}{\otimes} f^{-1}\mathcal{N} = \mathcal{D}_{X\to Y} \underset{f^{-1}\mathcal{D}_Y}{\otimes} f^{-1}\mathcal{N},$$

where

$$\mathcal{D}_{X\to Y} := \mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{D}_Y = f^*\mathcal{D}_Y$$

is a left $\mathcal{D}_X$-, right $f^{-1}\mathcal{D}_Y$-bimodule. f^* is right exact and its left derived functor $\mathrm{L}f^*$ is given by

$$\mathrm{L}f^*\mathcal{N}^{\cdot} = \mathcal{D}_{X\to Y} \underset{f^{-1}\mathcal{D}_Y}{\overset{\mathrm{L}}{\otimes}} f^{-1}\mathcal{N}^{\cdot}$$

because f^{-1} is exact.

1.6.1.1 Remark Alternatively one may view $\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{D}_Y$ as the module of differential operators from $f^{-1}\mathcal{O}_Y$ to $\mathcal{O}_X$. This module has the structure of a left $\mathcal{D}_X$-module by composition of operators. This structure coincides with the above one.

1.6.1.2 Example Let $j: U \hookrightarrow X$ be an open embedding. Then $\mathcal{D}_{U\to X} = \mathcal{D}_U$ and $\mathrm{L}j^* = j^* = j^{-1}$ is just the restriction to U.

1.6.1.3 Example Let $X = \mathbb{C}^n \overset{i}{\hookrightarrow} \mathbb{C}^{n+1} = Y$ be the closed submanifold given by $x_1 = 0$. Then

$$i^*\mathcal{D}_Y = \mathcal{D}_{X\to Y} = \mathcal{D}_Y / x_1\mathcal{D}_Y \cong \mathcal{D}_X[\partial_1].$$

In particular $i^*\mathcal{D}_Y$ is not of finite type over $\mathcal{D}_X$.

1.6.1.4 Example If $i: X \to Y$ is a closed embedding, then the previous example shows that $\mathcal{D}_{X\to Y}$ is a locally free left $\mathcal{D}_X$-module. Furthermore it is coherent as a right $\mathcal{D}_Y$-module. In fact

$$\mathcal{D}_{X\to Y} = \mathcal{D}_Y/\mathcal{I}_X\mathcal{D}_Y$$

if $\mathcal{I}_X$ denotes the ideal sheaf of the submanifold X.

1.6.1.5 Example Let $f: X = Y \times Z \to Y$ be a submersion, say the projection onto the first factor. Then f^* is exact because $\mathcal{O}_X$ is a flat $f^{-1}\mathcal{O}_Y$-module. Moreover $\mathcal{D}_{X\to Y}$ is a coherent $\mathcal{D}_X$-module.

1.6.2 Direct images

The construction of the direct image is more involved. For any right $\mathcal{D}_X$-module $\mathcal{N}$ one defines $f_*(\mathcal{N} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$ to be the direct image. Here f_* denotes the direct image functor in the category of sheaves. Notice that the right $f^{-1}\mathcal{D}_Y$-structure on $\mathcal{D}_{X\to Y}$ gives rise to a right $\mathcal{D}_Y$-structure on the direct image.

But we prefer to work with left modules. Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. By §1.2.2 $\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a right $\mathcal{D}_X$-module. Taking the direct image and tensoring with ω_Y^{-1} yields a left $\mathcal{D}_Y$-module (cf. §1.2.3)

$$\omega_Y^{-1} \underset{\mathcal{O}_Y}{\otimes} f_*\big((\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M}) \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y}\big).$$

This can be rewritten as

$$f_*\Big((f^{-1}\omega_Y^{-1} \underset{f^{-1}\mathcal{O}_Y}{\otimes} (\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_{X\to Y})) \underset{\mathcal{D}_X}{\otimes} \mathcal{M}\Big).$$

Put

$$\mathcal{D}_{Y\leftarrow X} := f^{-1}\omega_Y^{-1} \underset{f^{-1}\mathcal{O}_Y}{\otimes} (\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_{X\to Y}).$$

This has the structure of a left $f^{-1}\mathcal{D}_Y$-, right $\mathcal{D}_X$-bimodule. We define the direct image functor

$$f_+: \mathrm{D^b}(\mathcal{D}_X) \to \mathrm{D^b}(\mathcal{D}_Y)$$

by

$$f_+(\mathcal{M}^\cdot) := \mathrm{R}f_*(\mathcal{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}^\cdot).$$

If instead we had put $f_0\mathcal{M} = f_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M})$, we run into trouble because it turns out that in general $(g \circ f)_0 \neq g_0 \circ f_0$ for $g: Y \to Z$.

Since every holomorphic map $f: X \to Y$ splits as a composition of a closed embedding followed by a submersion, it is worth-while to study the complicated direct image functor f_+ in these two cases.

Remark 1. In case $j: U \hookrightarrow X$ is an open embedding then $\mathcal{D}_{X\leftarrow U} = \mathcal{D}_U$ and $j_+ = \mathrm{R}j_*$ the ordinary direct image.

2. Other notations for the direct image found in the literature are f_* and $\int_f$.

1.6.2.1 Case of a closed embedding
Let $i: X \hookrightarrow Y$ be a closed submanifold. $\mathcal{D}_{Y \leftarrow X}$ is a locally free right $\mathcal{D}_X$-module. It is coherent as a left $i^{-1}\mathcal{D}_Y$-module (cf. §1.6.1.4). Moreover i_* is exact, hence

$$i_+(\cdot) = i_*(\mathcal{D}_{Y \leftarrow X} \underset{\mathcal{D}_X}{\otimes} \cdot)$$

is an exact functor from $\mathcal{M}od(\mathcal{D}_X)$ to $\mathcal{M}od(\mathcal{D}_Y)$ that preserves coherency.

Theorem. (Kashiwara's equivalence, cf. [Kashiwara, 1978], Prop. 4.2). *The functor i_+ establishes an equivalence between $Coh(\mathcal{D}_X)$ and the category of coherent $\mathcal{D}_Y$-modules with support contained in X.*

The inverse to i_+ is the restriction of the functor

$$\begin{aligned} i^+ : \mathcal{M}od(\mathcal{D}_Y) &\to \mathcal{M}od(\mathcal{D}_X) \\ \mathcal{N} &\mapsto \mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{N}) \end{aligned}$$

to the category of coherent $\mathcal{D}_Y$-modules with support contained in X.

Note. i^+ and i^* are related to each other by the formula

$$\mathrm{R}i^+[d] = \mathrm{L}i^*,$$

where $d = \operatorname{codim}(X, Y)$.

Example (Cf. §1.6.1.3). Let $X = \mathbb{C}^n \xhookrightarrow{i} \mathbb{C}^{n+1} = Y$ be the closed submanifold given by $x_1 = 0$. Then $i_+\mathcal{M} = \mathcal{M}[\partial_1]$, $i^+\mathcal{N} = \operatorname{Ker}(x_1, \mathcal{N})$.

Case of a submersion
Let $f: X \to Y$ be a submersion. In this situation there exists a well-known short exact sequence

$$f^*\Omega^1_Y \hookrightarrow \Omega^1_X \twoheadrightarrow \Omega^1_{X/Y}.$$

$\Omega^1_{X/Y}$—the sheaf on X of relative differential forms—is locally free of rank $d = n - m = \dim X - \dim Y$ and gives rise to the relative de Rham complex $\Omega^{\cdot}_{X/Y}$ (cf. Example 1.6.2.3). It follows

$$\Omega^d_{X/Y} = \omega_X \underset{\mathcal{O}_X}{\otimes} f^*\omega_Y^{-1} = \omega_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\omega_Y^{-1}$$

yielding that

$$\mathcal{D}_{Y \leftarrow X} = \Omega^d_{X/Y} \underset{\mathcal{O}_X}{\otimes} f^*\mathcal{D}_Y.$$

We have a surjection $\mathcal{D}_X \twoheadrightarrow f^*\mathcal{D}_Y$, which gives rise to a surjection $\Omega^d_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{D}_X \twoheadrightarrow \mathcal{D}_{Y \leftarrow X}$.

We obtain a relative de Rham complex for $\mathcal{D}_X$

$$\mathrm{DR}_{X/Y}(\mathcal{D}_X) = \Omega^{\cdot}_{X/Y} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$$

and a verification in local coordinates (cf. [Pham, 1979], Ch. II, 14.3.5) yields that the relative de Rham complex $\mathrm{DR}_{X/Y}(\mathcal{D}_X)[d]$ is a locally free resolution of $\mathcal{D}_{Y\leftarrow X}$ in the category of right $\mathcal{D}_X$-modules. The differential maps in the relative de Rham complex $\mathrm{DR}_{X/Y}(\mathcal{D}_X)[d]$ are linear for the left $f^{-1}\mathcal{D}_Y$-structure. Hence we obtain that for all left $\mathcal{D}_X$-modules $\mathcal{M}$

$$f_+\mathcal{M} = \mathrm{R}f_*(\Omega^{\cdot}_{X/Y}(\mathcal{M}))[d].$$

1.6.2.3 Example Let $X = \mathbb{C}^m \times \mathbb{C}^d$, $Y = \mathbb{C}^m$. Let $f: X \to Y$ be the projection $(y_1, \ldots, y_m, z_1, \ldots, z_d) \mapsto (y_1, \ldots, y_m)$. Then an element $\omega \in \Omega^k_{X/Y}$ can be written as

$$\omega = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} dz_{i_1} \wedge \ldots \wedge dz_{i_k}, \quad \text{with } a_{i_1 \cdots i_k} \in \mathcal{O}_X.$$

The differential d is given by

$$d\omega = \sum_i \sum_{i_1 < \cdots < i_k} \frac{\partial a_{i_1 \cdots i_k}}{\partial z_i} dz_i \wedge dz_{i_1} \wedge \ldots \wedge dz_{i_k}.$$

We have the identification $\mathcal{D}_{Y\leftarrow X} = \mathcal{D}_X/(\partial_{z_1}, \ldots, \partial_{z_d})\mathcal{D}_X$.

1.6.2.4 Example Let $f: X = \mathbb{C}^{n-1} \times \mathbb{C} \to \mathbb{C}^{n-1} = Y$ be the projection. Then $\mathcal{D}_{Y\leftarrow X} = \mathcal{D}_X/\partial_n\mathcal{D}_X$. Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. The direct image is given by the complex

$$f_+\mathcal{M} = \quad \mathcal{M} \xrightarrow{\partial_n \cdot} \mathcal{M} \quad .$$

Hence

$$H^0 f_+\mathcal{M} = \mathcal{M}/\partial_n\mathcal{M}, \qquad H^{-1} f_+\mathcal{M} = \mathrm{Ker}(\partial_n, \mathcal{M}).$$

1.6.2.5 Example (Cf. [Pham, 1979], Ch. II, §15). Let us denote by $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ a map-germ with an isolated singularity at 0. Let $f: \mathcal{X} \to \Delta$ be a good representative. Consider the submersion $f: X \to S$, where $X = \mathcal{X} - f^{-1}(0)$, $S = \Delta - \{0\}$. By the relative Poincaré lemma $f^{-1}\mathcal{O}_S \hookrightarrow \Omega^{\cdot}_{X/S}$ is a quasi-isomorphism. By [Deligne, 1970], Ch. I, Prop. 2.28

$$\mathrm{R}f_* f^{-1}\mathcal{O}_S = \mathrm{R}f_* \mathbb{C}_X \underset{\mathbb{C}}{\otimes} \mathcal{O}_S.$$

It follows that

$$f_+\mathcal{O}_X[-n] = \mathrm{R}f_*(\Omega^{\cdot}_{X/S}) = \mathrm{R}f_*\mathbb{C}_X \underset{\mathbb{C}}{\otimes} \mathcal{O}_S.$$

In particular

$$H^0 f_+\mathcal{O}_X = \mathrm{R}^n f_*\mathbb{C}_X \underset{\mathbb{C}}{\otimes} \mathcal{O}_S$$

at least as $\mathcal{O}_S$-modules. Hence the $\mathcal{D}_S$-module $H^0 f_+\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module (cf. §1.3.3) i.e., a vector bundle with an integrable connection.

On the other hand $\mathrm{R}^n f_*\mathbb{C}_X$ is a local system on S (of rank μ the Milnor number of f) and $\mathrm{R}^n f_*\mathbb{C}_X \otimes_{\mathbb{C}} \mathcal{O}_S$ is the corresponding vector bundle with connection. This connection coincides with the one arising from the left $\mathcal{D}_S$-structure. It is known as the Gauss-Manin connection.

1.6.3 Algebraic local cohomology

Finally let us define a third operation on $\mathcal{D}$-modules. We are aiming at an analogue for the usual functor "sections with support".

Let $Z \subset X$ be a closed subvariety defined by an ideal $\mathcal{I} \subset \mathcal{O}_X$. For any $\mathcal{D}_X$-module $\mathcal{M}$ define

$$\Gamma_{[Z]}\mathcal{M} := \varinjlim \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k, \mathcal{M}).$$

It is the subsheaf of $\mathcal{M}$ of sections annihilated by some power of $\mathcal{I}$. Moreover it is a $\mathcal{D}_X$-submodule of $\mathcal{M}$ because $\xi\mathcal{I}^k \subset \mathcal{I}^{k-1}$ for every derivation ξ. $\Gamma_{[Z]}(\cdot)$ is a left exact functor on $\mathcal{M}od(\mathcal{D}_X)$. Its k-th derived functor is often denoted by $\mathcal{H}^k_{[Z]}(\cdot)$.

Let $i: Y \hookrightarrow X$ be a closed submanifold of codimension d. Then one has (cf. §1.6.2.1)

$$\mathrm{R}\Gamma_{[Y]} = i_+\mathrm{R}i^+.$$

Furthermore one puts $\mathcal{B}_{Y|X} := \mathcal{H}^d_{[Y]}\mathcal{O}_X$.

Closely related is the following

$$\Gamma_{[X|Z]}\mathcal{M} := \varinjlim \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^k, \mathcal{M}).$$

Again this has the structure of a $\mathcal{D}_X$-module. It is a $\mathcal{D}_X$-submodule of $j_*j^{-1}\mathcal{M}$, where $j: U \hookrightarrow X$ denotes the inclusion of $U = X - Z$ in X. In $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ one obtains a distinguished triangle

$$\begin{tikzcd} & \mathrm{R}\Gamma_{[X|Z]}\mathcal{M} \arrow[dl, "+1"'] & \\ \mathrm{R}\Gamma_{[Z]}\mathcal{M} \arrow[rr] & & \mathcal{M} \arrow[ul] \end{tikzcd}.$$

1.6.3.1 Remark
(i) $\mathrm{R}\Gamma_{[Z]}\mathcal{M} = \mathcal{M}$ implies $\mathrm{supp}(\mathcal{M}) \subset Z$.
(ii) If $\mathcal{M} \in \mathcal{C}oh(\mathcal{D}_X)$ and $\mathrm{supp}(\mathcal{M}) \subset Z$, then $\mathrm{R}\Gamma_{[Z]}\mathcal{M} = \mathcal{M}$.

1.6.3.2 Example Suppose $Z \subset X$ is a hypersurface given by $f = 0$, where $f \in \Gamma(X, \mathcal{O}_X)$. In that case

$$\mathrm{R}\Gamma_{[X|Z]}\mathcal{M} = \mathcal{M}[f^{-1}].$$

1.6.3.3 Example Suppose $Z \subset X$ is a divisor with normal crossings. Then

$$\mathrm{R}\Gamma_{[X|Z]}\mathcal{O}_X = \mathcal{O}_X[*Z]$$

the sheaf of holomorphic functions on $X - Z$ with poles along Z. There exists an exact sequence

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X[*Z] \twoheadrightarrow \mathcal{H}^1_{[Z]}\mathcal{O}_X.$$

1.6.3.4 Example Let $X = \mathbb{C}$, $Z = \{0\} \overset{i}{\hookrightarrow} X$, $\mathcal{I} = (x) \subset \mathcal{O}_X$, $j\colon U = \mathbb{C} - \{0\} \hookrightarrow X$. Put

$$\mathcal{M} := j_*\mathcal{O}_U/\mathcal{O}_X = \mathrm{R}^1\Gamma_Z\mathcal{O}_X.$$

This is a $\mathcal{D}_X$-module with support the origin. Clearly $i^+\mathcal{M} \cong \mathbb{C}$. Thus

$$i_+i^+\mathcal{M} = \mathrm{R}^1\Gamma_{[Z]}\mathcal{O}_X = \mathcal{O}_X[x^{-1}]/\mathcal{O}_X,$$

which is a proper subsheaf of $\mathcal{M}$. Hence $\mathcal{M}$ cannot be a coherent $\mathcal{D}_X$-module (Kashiwara's equivalence). In particular $j_+\mathcal{O}_U = j_*\mathcal{O}_U$ is not coherent as a $\mathcal{D}_X$-module i.e., the direct image functor does not preserve coherency.

Note further that $\mathcal{N} := \mathcal{M}/\mathrm{R}^1\Gamma_{[Z]}\mathcal{O}_X$ is a non-zero $\mathcal{D}_X$-module supported at the origin. However $\mathrm{R}\Gamma_{[Z]}\mathcal{N} = 0$, so the implication in the previous remark §1.6.3.1(i) cannot be reversed.

1.6.3.5 Example (Cf. 1.6.1.3.) Let $X = \mathbb{C}^n \overset{i}{\hookrightarrow} \mathbb{C}^{n+1} = Y$ be the closed submanifold given by $x_1 = 0$. By Kashiwara's equivalence

$$\mathcal{H}^1_{[X]}\mathcal{D}_Y = i_+\mathrm{R}^1i^+\mathcal{D}_Y$$

is a coherent $\mathcal{D}_Y$-module iff $\mathrm{R}^1i^+\mathcal{D}_Y$ is a coherent $\mathcal{D}_X$-module. But $\mathrm{R}^1i^+\mathcal{D}_Y \cong \mathcal{D}_X[\partial_1]$ which is not coherent. Hence $\mathrm{R}\Gamma_{[X]}$ does not preserve coherency.

There exists an exact sequence

$$\mathcal{D}_Y \hookrightarrow \mathcal{D}_Y[x_1^{-1}] \twoheadrightarrow \mathcal{H}^1_{[X]}\mathcal{D}_Y.$$

It follows that $\mathcal{D}_Y[x_1^{-1}]$ is not coherent over $\mathcal{D}_Y$. Later (§1.8.10) we will see that $\mathcal{O}_Y[x_1^{-1}]$ is a coherent $\mathcal{D}_Y$-module.

1.7 The characteristic variety

Now we are acquainted with various operations on $\mathcal{D}$-modules, let us take a closer look at the modules themselves. First we introduce the important notion of the characteristic variety or singular support of a coherent $\mathcal{D}_X$-module. It is defined by means of so-called "good filtrations". Next we go into the geometry of the characteristic variety. This leads to the important result in §1.7.8.

1.7.1 The sheaf $\mathcal{D}_X$ is filtered according to the degree of a differential operator. In local coordinates $x_1, \dots, x_n$ on an open $U \subset X$

$$\Gamma(U, \mathcal{D}_X(m)) = \{\, P \in \Gamma(U, \mathcal{D}_X) \mid P = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha, \quad a_\alpha \in \mathcal{O}_U \,\},$$

for all $m \in \mathbf{N}$. This yields an increasing filtration on $\mathcal{D}_X$ by coherent $\mathcal{O}_X$-submodules.

Let $\pi: T^*X \to X$ denote the canonical projection. The quotient $\mathcal{D}_X(m)/\mathcal{D}_X(m-1)$ can be identified with $\pi_*(\mathcal{O}_{T^*X}(m))$, where we denote by $\mathcal{O}_{T^*X}(m)$ the subsheaf of $\mathcal{O}_{T^*X}$ of sections homogeneous of degree m in the fibres. Furthermore

$$\operatorname{gr} \mathcal{D}_X = \bigoplus_{m \in \mathbf{N}} \mathcal{D}_X(m)/\mathcal{D}_X(m-1)$$

identifies with the sheaf of holomorphic functions on T^*X which are polynomial in the fibres.

Example Put $X = \mathbb{C}^n$, $T^*X = \mathbb{C}^{2n}$. Then

$$\operatorname{gr} \mathcal{D}_X \xrightarrow{\simeq} \mathcal{O}_X[\xi_1, \dots, \xi_n]$$

by sending $\frac{\partial}{\partial x_i} \in \mathcal{D}_X(1)$ to the indeterminate ξ_i.

1.7.2 An increasing filtration $(\mathcal{M}_k)_{k \in \mathbf{Z}}$ by coherent $\mathcal{O}_X$-submodules of a $\mathcal{D}_X$-module $\mathcal{M}$ is called a good filtration if

(i) $\mathcal{D}_X(m)\mathcal{M}_k \subset \mathcal{M}_{k+m}$, for all $m \in \mathbf{N}$, $k \in \mathbf{Z}$;
(ii) $\bigcup_{k \in \mathbf{Z}} \mathcal{M}_k = \mathcal{M}$;
(iii) locally: if $k \gg 0$, then $\mathcal{D}_X(m)\mathcal{M}_k = \mathcal{M}_{k+m}$, for all $m \in \mathbf{N}$;
if $k \ll 0$, then $\mathcal{M}_k = 0$.

In that case

$$\operatorname{gr} \mathcal{M} = \bigoplus_{k \in \mathbf{Z}} \mathcal{M}_k/\mathcal{M}_{k-1}$$

is a coherent $\operatorname{gr} \mathcal{D}_X$-module. Locally every coherent $\mathcal{D}_X$-module $\mathcal{M}$ has a good filtration, induced by the local presentation

$$\mathcal{D}_X^q \longrightarrow \mathcal{D}_X^p \twoheadrightarrow \mathcal{M}.$$

In fact any good filtration locally arises in this way, up to some shift in degree. In particular this implies that a $\mathcal{D}_X$-module carrying a good filtration is coherent.

1.7.3 Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Locally this gives rise to a coherent ideal in $\operatorname{gr}\mathcal{D}_X$ namely the annihilator of $\operatorname{gr}\mathcal{M}$. It turns out that its radical does not depend on the good filtration. So these locally defined radical ideals patch together and yield a coherent homogeneous radical ideal in $\mathcal{O}_{T^*X}$. This defines a closed conic subvariety $\mathrm{SS}(\mathcal{M})$, called the singular support of $\mathcal{M}$ or the characteristic variety of $\mathcal{M}$.

1.7.3.1 Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Closely related to the singular support $\mathrm{SS}(\mathcal{M})$ is the characteristic cycle $\mathrm{char}(\mathcal{M})$. This is the formal linear combination of the irreducible components of $\mathrm{SS}(\mathcal{M})$ counted with their multiplicities. This notion has been introduced in [Kashiwara, 1983b] (cf. also [Schapira, 1985]) and has many important applications.

1.7.4 Example Let $X = \mathbb{C}$. Let $P = \sum_{i=0}^{m} a_i\partial^i \in \mathcal{D}_X$, $a_m \neq 0$. Then

$$\sigma(P) = a_m\xi^m \in \operatorname{gr}\mathcal{D}_X = \mathcal{O}_X[\xi] \subset \mathcal{O}_{T^*X} = \mathcal{O}_{\mathbb{C}^2}$$

and $\mathrm{SS}(\mathcal{D}_X/\mathcal{D}_X P) = \mathrm{V}\big(\sigma(P)\big)$ a hypersurface in T^*X. Traditionally $\sigma(P)$ is called the principal symbol of P. Suppose that $a_m = x^n\varphi(x)$ with $\varphi(0) \neq 0$. Then $\mathrm{char}(\mathcal{D}_X/\mathcal{D}_X P) = n[\{x = 0\}] + m[\{\xi = 0\}]$ (cf. [Kashiwara, 1983b], §2.6.15).

1.7.5 Example Let $(\mathcal{V}, \nabla)$ be a vector bundle on X with an integrable connection. Then $\mathrm{SS}(\mathcal{V}) = T^*_X X$ the zero section of π. Note that

$$0 \subset \mathcal{V} \subset \mathcal{V} \subset \cdots$$

is a good filtration on $\mathcal{V}$.

1.7.6 Example (Cf. [Brylinski, 1982], §2.) (i) Let $f: X \to S$ be as in Example 1.6.2.5. Let $\mathcal{V} = H^0 f_+\mathcal{O}_X = \mathrm{R}^n f_*\mathbb{C}_X \otimes_{\mathbb{C}} \mathcal{O}_S$ with the Gauss-Manin connection ∇. $\mathcal{V}$ is equipped with a descending filtration—the Hodge filtration—$\{\mathcal{F}^p\}$ of subbundles of $\mathcal{V}$. ∇ satisfies the transversality condition of Griffiths

$$\nabla(\mathcal{F}^p) \subset \Omega^1_S \underset{\mathcal{O}_S}{\otimes} \mathcal{F}^{p-1}$$

Consider the increasing filtration given by $\mathcal{V}_p = \mathcal{F}^{-p}$ on $\mathcal{V}$. This yields a good filtration on the $\mathcal{D}_S$-module $\mathcal{V}$ because $\nabla_\xi(\mathcal{V}_p) \subset \mathcal{V}_{p+1}$ for every vector field ξ.

(ii) In general, let $\mathcal{V}$ be a vector bundle with a connection ∇. Assume $\mathcal{V}$ carries a descending filtration by subbundles $\{\mathcal{F}^p\}$. Then the transversality condition is equivalent to the fact that the $\mathcal{V}_k = \mathcal{F}^{-k}$ define a good filtration on $\mathcal{V}$ (cf. [loc. cit.]).

1.7.7 In general the structure of the characteristic variety is very complicated. The geometry of the characteristic variety can be studied in the cotangent bundle T^*X. The locally defined 1-form $\sum_{i=1}^{n} \xi_i dx_i$ does not depend on the choice of coordinates and defines a global 1-form θ—the canonical 1-form—on T^*X. It makes T^*X into a symplectic manifold. $\omega = d\theta$ is called the canonical 2-form. For any point $\xi \in T^*X$, $T_\xi T^*X$ is equipped with a bilinear, antisymmetric, nondegenerate form ω_ξ.

Let $V \subset T^*X$ be a closed subvariety. V is called involutive if for any smooth point $\xi \in V$

$$(T_\xi V)^\perp \subset T_\xi V.$$

Involutivity of V implies that $\dim V \geq \dim X$.

V is called Lagrangian—or holonomic—if for any smooth point $\xi \in V$

$$(T_\xi V)^\perp = T_\xi V.$$

If $V \subset T^*X$ is an irreducible conic Lagrangian subvariety of T^*X, then $\pi(V)$ is an irreducible subvariety of X and $V = T^*_{\pi(V)}X$ which is by definition the closure in T^*X of $T^*_{\pi(V)_{reg}}X$.

1.7.8 We are arrived at a deep theorem, proved microlocally in [Sato, Kashiwara, Kawai, 1973] (see also [Malgrange, 1979]). Later Gabber [Gabber, 1981] found a purely algebraic proof.

Theorem. *The characteristic variety* $\mathrm{SS}(\mathcal{M})$ *of a coherent* $\mathcal{D}_X$*-module* $\mathcal{M} \neq 0$ *is involutive.*

This implies $\dim \mathrm{SS}(\mathcal{M}) \geq \dim X$. This is known as Bernstein's inequality (cf. [Bernstein, 1972]). Next we single out the important class of modules whose characteristic varieties have the minimal possible dimension (i.e., equal to $\dim X$). The so-called maximally overdetermined or holonomic systems.

1.8 Holonomic modules

A $\mathcal{D}_X$-module is said to be holonomic if it is coherent and its characteristic variety is Langrangian. Equivalently the characteristic variety is of dimension equal to $\dim X$ since it is always involutive.

1.8.1 Example Let $X = \mathbb{C}$, $0 \neq P \in \mathcal{D}_X$. Then $\mathcal{D}_X/\mathcal{D}_X P$ is holonomic (cf. §1.7.4).

1.8.2 Example $\mathcal{D}_X$ is never holonomic (unless $\dim X = 0$) because $\mathrm{SS}(\mathcal{D}_X) = T^*X$.

1.8.3 Example Assume $\dim X > 1$. Let $P \in \mathcal{D}_X$ and assume that $\mathcal{D}_X/\mathcal{D}_X P \neq 0$. Then it is not holonomic since $\dim \mathrm{SS}(\mathcal{D}_X/\mathcal{D}_X P) = 2n - 1$ (cf. 1.7.4).

1.8.4 Example Any vector bundle with an integrable connection is holonomic.

1.8.5 Let $\mathcal{M}' \hookrightarrow \mathcal{M} \twoheadrightarrow \mathcal{M}''$ be an exact sequence of $\mathcal{D}_X$-modules. If two of them are holonomic so is the third. Moreover a coherent submodule of a holonomic module is again holonomic. So the category of holonomic $\mathcal{D}_X$-modules forms a thick abelian subcategory of $\mathcal{M}od(\mathcal{D}_X)$ (cf. §1.4.1.3). Denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)$ the derived category of bounded complexes of $\mathcal{D}_X$-modules with holonomic cohomology.

1.8.6 Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. The characteristic variety is of the form (cf. 1.7.7)

$$\mathrm{SS}(\mathcal{M}) = \bigcup_{\alpha} T^*_{S_\alpha} X$$

where $S_\alpha = \pi(V_\alpha)$ and the V_α are the irreducible components of $\mathrm{SS}(\mathcal{M})$.
The set

$$\bigcup_{\dim S_\alpha < n} S_\alpha$$

is called the singular locus of $\mathcal{M}$ (cf. [Pham, 1979]). Outside the singular locus $\mathcal{M}$ is a vector bundle with an integrable connection or zero. This follows from the fact that any holonomic module whose characteristic variety is the zero section of T^*X, is a coherent $\mathcal{O}_X$-module (cf. 1.3.3).
The characteristic cycle of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is of the form

$$\mathrm{char}(\mathcal{M}) = \sum_{\alpha} m_\alpha T^*_{S_\alpha} X$$

for certain $m_\alpha \in \mathbf{N}$.

1.8.7 Let us recall the notion of constructibility. (See e.g., [Borel, 1984].) A sheaf $\mathcal{F}$ on X of complex vector spaces is called constructible if there exists a stratification $X = \bigcup_\alpha S_\alpha$ such that the restriction of $\mathcal{F}$ to each stratum S_α is a local system i.e., locally constant with finite dimensional stalks. A complex $\mathcal{F}^{\cdot} \in \mathrm{D}^{\mathrm{b}}(X)$ is called constructible if its cohomology is constructible. Denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ the derived category of complexes with constructible cohomology.

1.8.8 The importance of the holonomic modules stems from

Theorem. (Cf. [Kashiwara, 1975]). *The solution complex* $\mathrm{S}(\mathcal{M})$ *of a holonomic* $\mathcal{D}_X$*-module* $\mathcal{M}$ *is constructible.*

Kashiwara even proves that there exists a Whitney stratification $X = \bigcup_\alpha S_\alpha$ such that $\mathrm{SS}(\mathcal{M}) \subset \bigcup_\alpha T^*_{S_\alpha} X$ and $\mathrm{S}(\mathcal{M})$ is constructible with respect to this stratification.

Example Let $X = \mathbf{C}$. Put $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X(x\partial - \alpha)$, $\alpha \in \mathbf{C} - \mathbf{Z}$. Then $\mathrm{S}(\mathcal{M})|_{\mathbf{C}-\{0\}}$ is a local system; its monodromy is given by $\exp(2\pi i \alpha)$. $\mathrm{S}(\mathcal{M})_{\{0\}} = 0$.

1.8.9 The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ is equipped with a notion of duality—called Verdier duality—defined by

$$(\mathcal{F}^{\cdot})^{*} = \mathrm{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{F}^{\cdot}, \mathbb{C}_X).$$

A holonomic $\mathcal{D}_X$-module has the remarkable property that for all $k \neq n = \dim X$

$$\mathcal{E}xt^{k}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0.$$

In fact this characterizes the holonomic modules. The left $\mathcal{D}_X$-module

$$\mathcal{E}xt^{n}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1}$$

is again holonomic (if $\mathcal{M}$ is so). The contravariant functor on $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)$ defined by

$$\mathcal{M} \mapsto \mathcal{E}xt^{n}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1} =: \mathcal{M}^{*}$$

establishes an equivalence of $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)$ with itself (cf. [Kashiwara, 1975]). One has $\mathcal{M}^{**} \cong \mathcal{M}$ and $\mathrm{SS}(\mathcal{M}^{*}) = \mathrm{SS}(\mathcal{M})$. For every holonomic $\mathcal{D}_X$-module $\mathcal{M}$ there are natural isomorphisms in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ (cf. [Mebkhout, 1981])

$$\mathrm{S}(\mathcal{M}^{*}) \cong \mathrm{DR}(\mathcal{M}) \cong \mathrm{S}(\mathcal{M})^{*}.$$

It follows that the de Rham complex $\mathrm{DR}(\mathcal{M})$ of a holonomic $\mathcal{D}_X$-module is constructible.

Example (i) Let $X = \mathbb{C}$. Put $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X(\partial x) \cong \mathcal{O}_X[x^{-1}]$. Then $\mathcal{M}^{*} \cong \mathcal{D}_X/\mathcal{D}_X(x\partial)$. Furthermore

$$\mathrm{S}(\mathcal{M})|_{\mathbb{C}-\{0\}} \cong \mathbb{C}_{\mathbb{C}-\{0\}}, \quad \mathrm{S}(\mathcal{M})_0 = 0;$$
$$H^0\mathrm{DR}(\mathcal{M}) = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{*}, \mathcal{O}_X) \cong \mathbb{C}_X;$$
$$H^1\mathrm{DR}(\mathcal{M}) = \mathcal{E}xt^{1}_{\mathcal{D}_X}(\mathcal{M}^{*}, \mathcal{O}_X) \cong \mathbb{C}_{\{0\}}.$$

(ii) Let $\mathcal{V}$ be a vector bundle with connection ∇. Then $\mathrm{DR}(\mathcal{V})$ is a local system on X (cf. §1.5.4), hence constructible.

1.8.10 So far we have seen just a few examples of holonomic $\mathcal{D}_X$-modules. The following result changes this. Cf. [Kashiwara, 1978] and in the particular case of $\mathcal{O}_X$ also [Mebkhout, 1977].

Theorem. *Let $Z \subset X$ be a closed subvariety and $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module. Then $\mathrm{R}\Gamma_{[Z]}\mathcal{M}$ has holonomic cohomology.*

Using the distinguished triangle of §1.6.3 it follows that we have also $\mathrm{R}\Gamma_{[X|Z]}\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)$.

The proof of the theorem is reduced to the case of a hypersurface; say Z is defined by $f \in \Gamma(X, \mathcal{O}_X)$. It suffices to show that $\mathcal{M}[f^{-1}]$

is holonomic for any holonomic $\mathcal{D}_X$-module $\mathcal{M}$. The difficult part is the assertion that $\mathcal{M}[f^{-1}]$ is coherent as a $\mathcal{D}_X$-module. (Compare with Example 1.6.3.5.)

In particular one has to show that $\mathcal{O}_X[f^{-1}]$ is coherent as a $\mathcal{D}_X$-module. Since the ascending chain of coherent submodules

$$\mathcal{D}_X \tfrac{1}{f} \subset \mathcal{D}_X \tfrac{1}{f^2} \subset \cdots \subset \mathcal{O}[f^{-1}]$$

is exhaustive, it is necessary and sufficient to see that the chain is stationary. This means: find $N \in \mathbf{N}$ such that

$$\frac{1}{f^k} \in \mathcal{D}_X \frac{1}{f^N}, \qquad \text{for all } k \geq N.$$

The key to solve this is the existence of the Bernstein-Sato polynomial.

1.8.11 Theorem. *There exists a non-zero polynomial $b(s)$ and a differential operator $P(s) \in \mathcal{D}_X[s]$ such that*

$$P(s)f^{s+1} = b(s)f^s.$$

The unitary polynomial of lowest degree satisfying the theorem is called the Bernstein-Sato polynomial or the b-function of f. The name "b-function" originates independently from Sato and from Bernstein (cf. [Kashiwara, 1976]). The existence in the polynomial case was proved by Bernstein [Bernstein, 1972]. Björk [Björk, 1979] generalized this to analytic functions. Kashiwara [Kashiwara, 1976] proved the rationality of the roots of the b-function. In case f has an isolated singularity Malgrange [Malgrange, 1976] also showed this; moreover he shows that $\{\exp(2\pi i\alpha) \mid \alpha \text{ is a root of the b-function of } f\}$ is the collection of the eigenvalues of the monodromy. The rationality of the roots implies also that the monodromy is quasi-unipotent.

Although the origins of the b-function are seemingly unrelated to $\mathcal{D}$-modules, the existence of b-functions has become one of the cornerstones of the theory of $\mathcal{D}$-modules.

1.8.12 Example (i) Let $X = \mathbb{C}^n$. Let $f = \sum_{i=1}^n x_i^2$. Then one has

$$\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f^{s+1} = 2(s+1)(2s+n)f^s.$$

For more examples and relations with the singularity of f see [Yano, 1983].

(ii) Let $Y \xhookrightarrow{i} X$ be a closed submanifold. Then $\mathcal{B}_{Y|X}$ is holonomic. $\mathrm{SS}(\mathcal{B}_{Y|X}) = T_Y^*X$. In particular $\mathcal{B}_{Y|X}$ is coherent over $\mathcal{D}_X$. Furthermore

$$i^+(\mathcal{B}_{Y|X}) = \mathrm{R}^d i^+ \mathcal{O}_X = i^* \mathcal{O}_X = \mathcal{O}_Y.$$

Thus by Kashiwara's equivalence $i_+\mathcal{O}_Y = \mathcal{B}_{Y|X}$ (cf. §1.6.2.1).

1.8.13 Functorial behaviour of holonomicity.

As we have seen the notion of coherency behaves very bad under the operations of inverse images (Ex. 1.6.1.3), direct images (Ex. 1.6.3.4) and local cohomology (Ex. 1.6.3.5). Let us investigate the holonomic modules in this aspect. The algebraic local cohomology has already been dealt with. It is not difficult to infer from this

Theorem. (Cf. [Kashiwara, 1978], Thm 4.4.) *Let $f: X \to Y$ be a morphism of complex manifolds. Then for any holonomic $\mathcal{D}_Y$-module $\mathcal{N}$, the complex $\mathrm{L}f^*\mathcal{N}$ has holonomic cohomology.*

The direct images are nastier. The following result has been obtained by Kashiwara ([Kashiwara, 1976], Thm 4.2).

Theorem. *Let $f: X \to Y$ be a projective morphism. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and assume that $\mathcal{M}$ carries a global good filtration. Then the direct image $f_+\mathcal{M}$ has holonomic cohomology. Moreover*

$$\mathrm{R}f_*\mathrm{S}(\mathcal{M})[\dim X] = \mathrm{S}(f_+\mathcal{M})[\dim Y]$$

(cf. [Mebkhout, 1984], Thm 3.1.1.)

Remark If $i: Y \hookrightarrow X$ is a closed embedding then i_+ preserves holonomicity. (Compare with Kashiwara's equivalence in §1.6.2.1.)

Remark An important ingredient to obtain results like those above is of course a study of the behaviour of the singular support—or better the characteristic cycle (cf. §1.7.3.1)—of the module in question. (See e.g., [Ginzburg, 1986] or [Malgrange, 1985].)

1.9 Regular holonomic $\mathcal{D}$-modules

At last we encounter the main topics: regular holonomic $\mathcal{D}$-modules, the Riemann-Hilbert correspondence, perverse sheaves. We will treat these successively in the remaining subsections.

The notion of regular singularities is classical. For an overview see for instance [Bertrand, 1980]. We also refer the reader to the chapters III and IV of [Borel, 1987] for a nice account of the theory. One studies systems of linear differential equations in the complex plane in the neighbourhood of a singular point. There are various ways to decide wether or not a system has a regular singularity (cf. [Gérard and Levelt, 1973]). The notion of regular singularity has been successfully generalized to higher dimensions by Deligne [Deligne, 1970]. Generalizations to $\mathcal{D}$-modules may be found in [Kashiwara and Kawai, 1981], [Mebkhout, 1979] and [Ramis, 1978]. We ignore here systems with irregular singularities. These have been studied mainly in the one dimensional case (see e.g., [Levelt, 1973]). A complete classification is due to Malgrange (cf. [Malgrange, 1983c]).

One meets several equivalent definitions of the notion of regularity in the literature. Let us say a few words on the one dimensional case in order to motivate some of the definitions in higher dimensions.

1.9.1 Let $X = \Delta \subset \mathbb{C}$ be a disc around 0. Put $U = \Delta^* = \Delta - \{0\}$. Let $P = a_m\partial^m + \cdots + a_0$ be a differential operator with $a_i \in \Gamma(X, \mathcal{O}_X)$, $a_m \neq 0$. Suppose furthermore that $a_m^{-1}(0) \cap \Delta^* = \emptyset$.

(a) P is said to have regular singularities at 0 if the (multivalued) solutions of $Pu = 0$ have a moderate growth near 0. Equivalently all (multivalued) solutions of P are of the form

$$\sum_{\alpha,i} a_{\alpha,i} x^\alpha (\log x)^i, \qquad \text{with } a_{\alpha,i} \in \mathcal{O}_X.$$

Now consider the situation at the stalk in 0 i.e.,

$$\mathcal{O}_{X,0} \xrightarrow{P\cdot} \mathcal{O}_{X,0}.$$

In [Malgrange, 1974] Malgrange shows that the complex vector spaces $\mathrm{Ker}(P, \mathcal{O}_{X,0})$ and $\mathrm{Coker}(P, \mathcal{O}_{X,0})$ are finite dimensional and that

$$\chi(P, \mathcal{O}_{X,0}) = \dim_{\mathbb{C}} \mathrm{Ker}\, P - \dim_{\mathbb{C}} \mathrm{Coker}\, P = m - v(a_m)$$

(where $v(g)$ denotes the valuation of $g \in \mathcal{O}_X$ at 0). If we replace $\mathcal{O}_{X,0}$ by its completion $\widehat{\mathcal{O}} = \mathbb{C}[[x]]$, then

$$\chi(P, \widehat{\mathcal{O}}) = \sup\{\, p - v(a_p) \mid 0 \leq p \leq m \,\}$$

(b) Moreover he proved: the operator P has regular singularities at 0 iff $\chi(P, \mathcal{O}_{X,0}) = \chi(P, \widehat{\mathcal{O}})$. In particular this gives a classical result of Fuchs that says: P has regular singularities at 0 iff $v(a_p) - v(a_m) \geq p - m$ for all p i.e., $\frac{a_p}{a_m}$ has a pole of order at most $m - p$.

(c) In terms of complexes: P has regular singularities at 0 iff the complex " $\mathcal{O}_{X,0} \xrightarrow{P} \mathcal{O}_{X,0}$ " is quasi-isomorphic to " $\widehat{\mathcal{O}} \xrightarrow{P} \widehat{\mathcal{O}}$ " .

1.9.2 Suppose X is an n-dimensional manifold. We will write down some equivalent definitions of regular holonomic $\mathcal{D}_X$-modules.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. The following statements are equivalent.

(i) For any smooth curve C and any morphism $f: C \to X$, $\mathrm{L}f^*\mathcal{M}$ is a complex whose cohomology modules have regular singularities.

(ii) (Compare with (b).) For every $x \in X$

$$\sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i_{\mathcal{D}_{X,x}}(\mathcal{M}_x, \mathcal{O}_{X,x})$$

$$= \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i_{\mathcal{D}_{X,x}}(\mathcal{M}_x, \widehat{\mathcal{O}_{X,x}}),$$

where $\widehat{\mathcal{O}_{X,x}}$ denotes the completion of $\mathcal{O}_{X,x}$.

(iii) (Cf. [Mebkhout, 1984] and also [Ramis, 1978].) For every closed subspace $Z \subset X$

$$\mathrm{DR}(\mathrm{R}\Gamma_{[Z]}\mathcal{M}) = \mathrm{R}\Gamma_Z(\mathrm{DR}(\mathcal{M})).$$

Or equivalently

$$\begin{aligned} \mathrm{S}(\mathcal{M})_Z &= \mathrm{S}(\mathrm{R}\Gamma_{[Z]}\mathcal{M}) \\ &= \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{\widehat{X|Y}}), \end{aligned}$$

where $\mathcal{O}_{\widehat{X|Y}} = \varprojlim \mathcal{O}_X/\mathcal{I}_Z^k$ is the formal completion of $\mathcal{O}_X$ along Z. (Compare with (c).)

(iv) (Cf. [Kashiwara and Kawai, 1981], Cor. 5.1.11.) There exists a global good filtration on $\mathcal{M}$ such that the annihilator of $\mathrm{gr}\,\mathcal{M}$ is a radical ideal in $\mathrm{gr}\,\mathcal{D}_X$.

If one of these equivalent conditions is satisfied, $\mathcal{M}$ is said to be regular holonomic. A complex $\mathcal{M}^\cdot \in \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)$ is called regular holonomic if its cohomology is regular holonomic. $\mathrm{D}^{\mathrm{b}}_{\mathrm{hr}}(\mathcal{D}_X)$ denotes the corresponding category.

1.9.3 Remarks

(1) Regularity can be checked locally on X.

(2) Conditions (i), ..., (iv) remain equivalent if one replaces $\mathcal{M}$ by a bounded complex $\mathcal{M}^\cdot$ with holonomic cohomology.

1.9.4 Example $\mathcal{O}_X$ is a regular holonomic. To see this we may assume that $X = \mathbb{C}^n$. Thus $\mathrm{gr}\,\mathcal{D}_X = \mathcal{O}_X[\xi_1, \ldots, \xi_n]$. Take the good filtration $0 \subset \mathcal{O}_X \subset \mathcal{O}_X \subset \cdots$. It follows that the annihilator of $\mathrm{gr}\,\mathcal{O}_X$ is the ideal generated by $\xi_1, \ldots, \xi_n$, which is a radical ideal.

1.9.5 Example If $\mathcal{V}$ is a vector bundle with an integrable connection, then $\mathcal{V}$ is regular holonomic (on X!).

1.9.6 Example Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module. Let $Z \subset X$ be a closed subspace. Then $\mathrm{R}\Gamma_{[Z]}\mathcal{M}$ and $\mathrm{R}\Gamma_{[X|Z]}\mathcal{M}$ are again regular holonomic. This follows immediately from 1.9.2(iii).

1.9.7 Example If $\mathcal{M}$ is regular holonomic, then its dual $\mathcal{M}^*$ is regular holonomic (cf. [Mebkhout, 1984]).

1.9.8 Example Let $X = \mathbb{C}$. Put $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X(x^2\partial - 1)$. Then $\mathcal{M}$ is holonomic, but not regular.

1.9.9 Inverse images of regular holonomic modules

Theorem. (Cf. [Mebkhout, 1984], Thm 3.2.1 and also [Kashiwara and Kawai, 1981], Cor. 5.4.8.) *Let $f: X \to Y$ be a holomorphic map. Let $\mathcal{N}$ be a regular holonomic $\mathcal{D}_Y$-module. Then $\mathrm{L}f^*\mathcal{N}$ has regular holonomic cohomology and $\mathrm{S}(\mathrm{L}f^*\mathcal{N}) = f^{-1}\mathrm{S}(\mathcal{N})$.*

1.9.10 Direct images of regular holonomic modules

Theorem. *Let $f: X \to Y$ be a proper morphism. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module. Then $f_+\mathcal{M}$ has regular holonomic cohomology.*

(Cf. [Kashiwara and Kawai, 1981], Thm 6.2.1 or [Mebkhout, 1984] in case f is projective.) Generalizations of this result are obtained in [Malgrange, 1985], [Houzel and Schapira, 1984] and [Kashiwara and Schapira, 1985].

1.9.11 Remark If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module one defines the local index of $\mathcal{M}$ at x as

$$\chi(\mathcal{M}, x) = \sum_i (-1)^i \dim \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x .$$

If $X = \mathbb{C}$ and $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ (notations as in §1.9.1), then $\chi(\mathcal{M}, 0) = \chi(P, \mathcal{O}_{X,0})$. The index theorem due to Malgrange $\chi(P, \mathcal{O}_{X,0}) = m - v(a_m)$ (cf. §1.9.1) relates the local index with the multiplicities of the characteristic cycle of $\mathcal{M}$, $\mathrm{char}(\mathcal{M}) = v(a_m)[\{x = 0\}] + m[\{\xi = 0\}]$ (cf. Example 1.7.4). This generalizes to a general index formula due to Dubson (cf. [Brylinski, Dubson, Kashiwara, 1981]) relating the local index at x, the multiplicities m_α of the irreducible components $T^*_{S_\alpha}X$ (compare §1.8.6) and a topological invariant—the Euler obstruction—of S_α at x.

1.9.12 Example Let $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an isolated singularity at 0. Let $f: X \to \Delta$ be a good representative. Then $f_+\mathcal{O}_X$ is a regular holonomic $\mathcal{D}_\Delta$-module. In particular the Gauss-Manin system $\mathcal{G} = H^0 f_+\mathcal{O}_X$ is regular holonomic. Cf. [Brieskorn, 1970] and §2.1.2.

1.10 The Riemann-Hilbert correspondence

Theorem. *The solution functor* S *restricted to* $\mathrm{D}^{\mathrm{b}}_{\mathrm{hr}}(\mathcal{D}_X)$ *establishes an equivalence of categories*

$$\mathrm{S}: \mathrm{D}^{\mathrm{b}}_{\mathrm{hr}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X).$$

The Riemann-Hilbert correspondence is one of the highlights in the theory of $\mathcal{D}$-modules. It establishes a bridge between analytic objects (regular holonomic modules) and geometric ones (constructible sheaves). It has been proved independently by Mebkhout (cf. [Mebkhout, 1984a,b]) and Kashiwara (cf. [Kashiwara and Kawai, 1981]). Its proof uses the whole machinery of $\mathcal{D}$-modules, Hironaka's resolution of singularities and Deligne's canonical extension of connections. In [Kashiwara, 1984] Kashiwara constructs an inverse functor using the sheaf of distributions on X. One interprets the Riemann-Hilbert correspondence as a generalization of Deligne's solution of the 21-st problem of Hilbert (cf. [Katz, 1976] and also [Mebkhout, 1980]).

1.10.1 Example (i) $\mathrm{S}(\mathcal{O}_X) = \mathbb{C}_X$.
(ii) Let $Z \subset X$ be a closed subvariety. Then

$$\mathrm{S}(\mathrm{R}\Gamma_{[Z]}\mathcal{O}_X) = \mathrm{S}(\mathcal{O}_X)_Z = \mathbb{C}_Z \qquad \text{(cf. 1.9.2(iii))}.$$

Note that in general $\mathrm{R}\Gamma_{[Z]}\mathcal{O}_X$ is a complex, however $\mathbb{C}_Z$ is a single sheaf. Example 1.8.9(i) shows an occurrence of the other extreme.

1.10.2 Example If $\mathcal{V}$ is a vector bundle with connection ∇, then $\mathrm{S}(\mathcal{V}) = \mathrm{DR}(\mathcal{V})^*$ is the Verdier dual of the local system $\mathrm{Ker}(\nabla, \mathcal{V})$. So in a sense the Riemann-Hilbert correspondence generalizes the well-known one-to-one correspondence between vector bundles with integrable connection and local systems.

1.11 Perverse sheaves
Recall that we have identified $\mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ as the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{hr}}(\mathcal{D}_X)$ consisting of complexes $\mathcal{M}^\cdot$ satisfying $H^k(\mathcal{M}^\cdot) = 0$ for all $k \neq 0$. The constructible complexes of sheaves that correspond via S to an object in $\mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ can be nicely characterized.

An object $\mathcal{F}^\cdot \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ is called a perverse sheaf if it satifies:

(i) $H^i(\mathcal{F}^\cdot) = 0$, for all $i < 0$;
(ii) $\operatorname{codim}\operatorname{supp}\big(H^i(\mathcal{F}^\cdot)\big) \geq i$, for all $i \in \mathrm{N}$;
(iii) the Verdier dual $(\mathcal{F}^\cdot)^*$ satifies (i) and (ii).

The category of perverse sheaves is denoted $\mathrm{Perv}(X)$. Note that a perverse sheaf is in general a complex of sheaves. For many reasons the category of perverse sheaves is important. See e.g., the nice survey article [MacPherson, 1984]. (Cf. also §2 and [Beilinson, Bernstein, Deligne, 1982].) We have (cf. [Brylinski, 1982a] for a prove)

Theorem. *The Riemann-Hilbert correspondence induces an equivalence of categories*

$$\mathrm{S}\colon \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}} \to \mathrm{Perv}(X).$$

1.11.1 Example Let $Y \overset{i}{\hookrightarrow} X$ be a closed submanifold of codimension d. Then

$$\mathrm{S}(\mathcal{B}_{Y|X}) = \mathrm{S}(\mathrm{R}\Gamma_{[Y]}\mathcal{O}_X[d]) = \mathbb{C}_Y[-d]$$

is a perverse sheaf on X.

1.11.2 Example Let us mention a nice application of the above theorem. Let $Z \subset X$ be a closed subvariety of codimension d. Assume that Z is not locally a complete intersection. Then, in general, $\mathrm{R}\Gamma_{[Z]}\mathcal{O}_X$ is a complex and thus $\mathbb{C}_Z[-d] = \mathrm{S}(\mathrm{R}\Gamma_{[Z]}\mathcal{O}_X[d])$ is not perverse.

1.11.3 The intersection complex

Examples of perverse sheaves are the intersection complexes (see §2.1.3). This leads to a formulation in terms of $\mathcal{D}$-modules. Let $Y \subset X$ be a closed subvariety of codimension d. Let $\Sigma \subset Y$ be a closed subspace containing Y_{sing}. Finally let V be a local system on $Y - \Sigma$.

Theorem. (Cf. [Brylinski and Kashiwara, 1981], Prop. 8.5.) *There exists a unique regular holonomic $\mathcal{D}_X$-module, denoted by $\mathcal{L}(Y,X,V)$, such that*

(i) $\mathcal{L}(Y,X,V)|_{X-\Sigma} \cong V \underset{\mathbb{C}}{\otimes} \mathcal{B}_{Y-\Sigma|X-\Sigma}$

(ii) $\mathcal{H}^0_{[\Sigma]}(\mathcal{L}(Y,X,V)) = \mathcal{H}^0_{[\Sigma]}(\mathcal{L}(Y,X,V)^*) = 0.$

The module $\mathcal{L}(Y,X,V)$ corresponds to the intersection complex (see §2.1.3 for notation)

$$\mathrm{DR}(\mathcal{L}(Y,X,V)) = \mathcal{IC}^{\cdot}_Y(V)[-d]$$

1.11.4 Example Let V be the constant sheaf $\mathbb{C}_{Y-\Sigma}$. Put $\mathcal{M} = \mathcal{H}^d_{[Y-\Sigma]}\mathcal{O}_X$. Then

$$\mathcal{L}(Y,X) := \mathcal{L}(Y,X,V) = \left(\mathcal{M}^*/\mathcal{H}^0_{[\Sigma]}(\mathcal{M}^*)\right)^*$$

1.11.5 Example Let Y be a submanifold of X of codimension d and denote $i\colon Y \hookrightarrow X$ the inclusion. Then

$$\mathcal{B}_{Y|X} \underset{\mathcal{O}_X}{\otimes} \Omega^{\cdot}_X = \mathrm{DR}(\mathcal{B}_{Y|X}) = \mathbb{C}_Y[-d]$$

which implies that

$$\mathcal{H}^{i+d}(X, \mathrm{DR}(\mathcal{B}_{Y|X})) = H^i(Y,\mathbb{C}).$$

Now $\mathcal{B}_{Y|X} = \varinjlim \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k, \mathcal{O}_X)$ is filtered by the coherent $\mathcal{O}_X$-modules $\mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k, \mathcal{O}_X)$. Give $\Omega^{\cdot}_X$ the stupid filtration. This yields a filtration on $\mathrm{DR}(\mathcal{B}_{Y|X})$ and hence a filtration on $H^i(Y,\mathbb{C})$. If Y is projective this is the Hodge filtration (cf. [Brylinski, 1982a], §3).

1.11.6 Example Suppose that X is a projective manifold and let Y be a divisor with normal crossings. Note $j\colon U = X - Y \hookrightarrow X$ the open embedding. Note that (by 1.9.2(iii))

$$\mathrm{DR}(\mathrm{R}\Gamma_{[X|Y]}\mathcal{O}_X) = \mathrm{R}j_*j^{-1}\mathbb{C}_X = \mathrm{R}j_*\mathbb{C}_U.$$

Now $\mathrm{R}\Gamma_{[X|Y]}\mathcal{O}_X = \mathcal{O}_X[*Y]$ (cf. Example 1.6.3.3) is filtered by pole order. Again this yields a filtration on $\mathrm{DR}(\mathcal{O}_X[*Y])$ and thus a filtration on

$$\mathcal{H}^i(X, \mathrm{DR}(\mathcal{O}_X[*Y])) = \mathcal{H}^i(X, \mathrm{R}j_*\mathbb{C}_U) = H^i(U,\mathbb{C}),$$

the Hodge filtration (cf. [loc. cit.]).

2 Motivation and summary

2.1 Motivation

The interest in the theory of $\mathcal{D}$-modules is due to its applications in various parts of mathematics. We briefly discuss three examples.

2.1.1 Representation theory

2.1.1.1 Let G be a semi-simple, connected linear algebraic group over $\mathbb{C}$. Let $B \subset G$ be a Borel subgroup. Let $X = G/B$ be the flag variety of G. This is a projective manifold. One may identify the Lie algebra $L = L(G)$ of G with the right invariant vector fields on G. This gives rise to a homomorphism

$$\varphi: \mathcal{U}(L) \to \Gamma(X, \mathcal{D}_X).$$

Let $\mathcal{Z}$ be the centre of $\mathcal{U}(L)$ and let I be the ideal in $\mathcal{U}(L)$ generated by $\mathcal{Z} \cap \mathcal{U}(L)L$. One can show that $\operatorname{Ker} \varphi = I$ and even that φ is surjective (cf. [Brylinski, 1981]).

Example. Let $G = SL_2$ and B the subgroup of the upper triangular matrices. Then $X = G/B \cong \mathbf{P}^1$, the projective line. Let $(z_0 : z_1)$ be homogeneous coordinates on $\mathbf{P}^1$, then

$$\varphi: \mathcal{U}(sl_2) \to \mathcal{D}(\mathbf{P}^1)$$

is given by

$$e_{ij} \mapsto z_i \frac{\partial}{\partial z_j}, \quad 0 \le i, j \le 1.$$

2.1.1.2 Define $\tilde{\mathcal{O}}$ to be the category of $\mathcal{U}(L)$-modules M that satisfy:

(i) M is a finitely generated $\mathcal{U}(L)$-module;
(ii) for any $m \in M$, $\dim_{\mathbb{C}} \mathcal{U}(L(B))m < \infty$, (where $L(B)$ denotes the Lie algebra of B).

One can prove that any object of $\tilde{\mathcal{O}}$ has a composition series of finite length. Furthermore this category contains all the Verma modules. The multiplicities of the simple modules appearing in a composition series of a Verma module have been conjectured by Kazhdan and Lusztig.

Finally let $\tilde{\mathcal{O}}_{\text{triv}}$ be the subcategory of $\tilde{\mathcal{O}}$ consisting of objects $M \in \tilde{\mathcal{O}}$ that satisfy $IM = 0$. Now we are in a position to state (cf. [Brylinski and Kashiwara, 1981] and [Beilinson and Bernstein, 1981])

Theorem. *The functor $F: \mathcal{M} \mapsto \Gamma(X, \mathcal{M})$ establishes an equivalence between the category of regular holonomic $\mathcal{D}_X$-modules whose characteristic varieties are contained in $\bigcup_{w \in W} T^*_{X_w} X$ and the category $\tilde{\mathcal{O}}_{\text{triv}}$.*

Here W denotes the Weyl group and $X = \bigcup_{w \in W} X_w$ is the stratification of X by the Bruhat cells X_w.

2.1.1.3 In [loc. cit.] this result is used to prove the Kazhdan-Lusztig conjecture about the multiplicities of the simple modules appearing in a composition series of a Verma module. (Cf. also [Springer, 1982].) One proceeds by invoking the Riemann-Hilbert correspondence. This yields an interpretation of the Kazhdan-Lusztig conjecture in topological terms, which then could be solved.

The Verma modules correspond to $\mathcal{H}^{n-l(w)}_{[X_w]}\mathcal{O}_X$, $w \in W$ (where $n = \dim X$) whereas the simple modules correspond to (see § 1.11 for notation) $\mathcal{L}(\overline{X}_w, X)$, $w \in W$. Via the Riemann-Hilbert correspondence the Verma modules agree with $\mathbb{C}_{X_w}[l(w)-n]$ and the simple modules agree with the intersection cohomology complexes $\mathcal{IC}^{\cdot}_{\overline{X}_w}[l(w)-n]$.

2.1.2 Singularity theory

2.1.2.1 Suppose $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a germ of a holomorphic function with an isolated singularity. Let $f: X \to S$ be a good representative. Now f defines a submersion $f': X' = X - \{0\} \to S' = S - \{0\}$. One obtains a local system $\mathrm{R}^n f'_* \mathbb{C}_{X'}$ on S' whose stalk above $s \in S'$ equals $H^n(X_s, \mathbb{C}) =: H$, the cohomology of the Milnor fibre. Equivalently one has the associated vector bundle $\mathcal{H} := \mathrm{R}^n f'_* \mathbb{C}_{X'} \otimes \mathcal{O}_{S'}$ with an integrable connection ∇, which is traditionally called the Gauss-Manin connection (cf. [Brieskorn, 1970]). The vector bundle $\mathcal{H}$ extends uniquely to a vector bundle $\mathcal{H}_X$ on S with a logarithmic connection ∇ such that the eigenvalues of the residue of ∇ are contained in $\{ z \in \mathbb{C} \mid -1 < \operatorname{Re} z \leq 0 \}$.

2.1.2.2 The projective case. Assume f is projective. In that case $H^n(X_s, \mathbb{C})$ carries a pure Hodge structure. This gives rise to a Hodge filtration on the vector bundle $\mathcal{H}$ satisfying Griffiths' transversality. According to Schmid (cf. [Schmid, 1973]) this filtration extends to $\mathcal{H}_X$ and yields a mixed Hodge structure on $\mathcal{H}_{X,0}/t\mathcal{H}_{X,0} \cong H$.

2.1.2.3 The general case. In [Steenbrink, 1976] Steenbrink constructed a mixed Hodge structure on H by using a resolution of singularities of f. Varchenko [Varchenko, 1980, 1982] defined a mixed Hodge structure on H by using the asymptotic expansion of period integrals. One is interested in these because the mixed Hodge structure gives rise to useful invariants of the singularity of f. Pham [Pham, 1983] (cf. also [Pham, 1979]) advocated the use of $\mathcal{D}$-modules in these. In [Scherk and Steenbrink, 1985] Steenbrink's filtration is described using the theory of regular holonomic $\mathcal{D}$-modules. Similar results are obtained by M. Saito (cf. [Saito, 1982, 1984]).

2.1.2.4 Let us briefly indicate how one proceeds. The vector bundle $\mathcal{H}$ with connection ∇ may be viewed as a $\mathcal{D}_{S'}$-module (cf. §1.3.3). As such it coincides with $H^0(f'_+\mathcal{O}_{X'})$, where the subscript "+" indicates the direct image in the theory of $\mathcal{D}$-modules (see §1.6.2 and example 1.6.2.5).

Put $\mathcal{G} := H^0(f_+\mathcal{O}_X)$, then $\mathcal{G}|_{S'} = \mathcal{H}$. $\mathcal{G}$ is called the Gauss-Manin system (cf. [Pham, 1979]). Decompose f as an embedding $i: X \hookrightarrow X \times S$ on the graph of f, followed by the projection $\pi: X \times S \to S$. Then (cf. Example 1.6.2.5)

$$\mathcal{G} = \mathrm{R}^{n+1}\pi_*(\Omega^{\cdot}_{X\times S/S} \underset{\mathcal{O}_{X\times S}}{\otimes} \mathcal{B}_{X\times S|i(X)}).$$

Here

$$\mathcal{B}_{X\times S|i(X)} = i_+\mathcal{O}_X = \mathcal{O}_{X\times S}[(t-f)^{-1}]/\mathcal{O}_{X\times S}$$

and t is a coordinate on S. The $\mathcal{D}_{X\times S}$-module $\mathcal{B}_{X\times S|i(X)}$ is naturally filtered by pole order. Giving $\Omega^{\cdot}{}_{X\times S/S}$ the stupid filtration yields a filtration on $\mathcal{G}$. (Cf. [Brylinski, 1982a]). Now $\mathcal{G}$ contains the canonical lattice $\mathcal{H}_X$ (cf. [Pham, 1983] and also [Barlet and Kashiwara, 1986]) and $\mathcal{H}_{X,0}/t\mathcal{H}_{X,0} \cong H$. The induced filtration on H gives rise to the mixed Hodge structure of [Steenbrink, 1976] (cf. [Scherk and Steenbrink, 1985] and [Saito, 1982]).

Remark. Consider again the particular case that f is a projective morphism. Then $i_s: X_s \hookrightarrow \mathbf{P}^{n+1}$ is a closed submanifold and

$$\mathcal{G} = \mathcal{H}^{n+1}(\mathbf{P}^{n+1}, \Omega^{\cdot}_{\mathbf{P}^{n+1}/X_s} \otimes \mathcal{B}_{\mathbf{P}^{n+1}|X_s}).$$

Since $\Omega^{\cdot}_{\mathbf{P}^{n+1}/X_s} \otimes \mathcal{B}_{\mathbf{P}^{n+1}|X_s}$ is quasi-isomorphic to the shifted complex $i_{s*}\Omega^{\cdot}_{X_s}[-1]$ (cf. [Pham, 1979], Ch. I §14.2.2), this implies

$$\mathcal{G}_s = \mathcal{H}^n(X_s, \Omega^{\cdot}_{X_s}) = H^n(X_s, \mathbb{C}).$$

Moreover the induced filtration corresponds to the Hodge filtration on $H^n(X_s, \mathbb{C})$. Thus the filtration on $\mathcal{G}$ is an extension of the Hodge filtration on $\mathcal{H}$. (Cf. [Brylinski, 1982a] and Example 1.11.5.)

2.1.3 Cohomology of singular spaces

2.1.3.1 Let Y be a complex analytic variety of dimension m. Y admits a stratification into disjoint connected nonsingular analytic subvarieties $\{Y_\alpha\}$. $\Sigma := \bigcup_{\dim Y_\alpha < m} Y_\alpha$ contains the singular points of Y. Goresky and MacPherson introduced the notion of intersection cochain complex $IC^{\cdot}(Y)$ on Y. This calculates the intersection cohomology $IH^{\cdot}(Y)$. The intersection cohomology for a singular variety Y satisfies many important properties such as Poincaré duality and hard Lefschetz theorem. We refrain from recalling these, but instead we refer the reader to the nice paper [MacPherson, 1984] (and also [Brylinski, 1982b]).

The notion of intersection cochain can be sheafified and gives rise to a complex of sheaves $\mathcal{IC}^{\cdot}_Y$. Using sheaf theoretic constructions Deligne gave another construction of $\mathcal{IC}^{\cdot}_Y$ (cf. [Beilinson, Bernstein, Deligne, 1982]). The intersection complex $\mathcal{IC}^{\cdot}_Y$ may be characterized by the following properties:

(i) $\mathcal{IC}^i_Y = 0 \quad$ if $i < 0$;
$\mathcal{IC}^{\cdot}_Y$ is a bounded complex with constructible cohomology i.e., for some stratification $\{Y_\alpha\}$ of Y $H^i(\mathcal{IC}^{\cdot}_Y)|Y_\alpha$ is a local system, for all α and i;

(ii) codim supp$(H^i(\mathcal{IC}^{\cdot}_Y)) > i$, for all $i \in \mathbf{Z}$;

(iii) the Verdier dual $(\mathcal{IC}^{\cdot}_Y)^*$ satisfies (i) and (ii);

(iv) for some stratification $\{Y_\alpha\}$ of Y one has: $\mathcal{IC}^{\cdot}_Y|_{Y-\Sigma} \cong \mathbb{C}_{Y-\Sigma}$.

The construction of $\mathcal{IC}^{\cdot}_Y$ generalizes as follows. For any local system V on $Y - \Sigma$ one defines a perverse sheaf $\mathcal{IC}^{\cdot}_Y(V)$. It is characterized as above but (iv) has to be replaced by

(iv') $\mathcal{IC}^{\cdot}_Y(V)|_{Y-\Sigma} \cong V$.

2.1.3.2 The complexes $\mathcal{IC}^{\cdot}_Y(V)$ are examples of perverse sheaves (compare §1.11). For many reasons the category of perverse sheaves on Y is important. We refer the reader to [MacPherson, 1984] for a nice account of this. We single out two major themes.

Let us assume that Y is a closed subvariety of a complex manifold X. Let $i: Y \hookrightarrow X$ denote the inclusion. Put $d = \operatorname{codim}(Y, X)$. The complexes $i_*(\mathcal{IC}^{\cdot}_Y(V))[-d]$ are all perverse sheaves on X. Usually one drops the i_* and considers $\mathcal{IC}^{\cdot}_Y(V)[-d]$ as a perverse sheaf living on X. By the Riemann-Hilbert correspondence the category of perverse sheaves on X, Perv(X), is equivalent to the category of regular holonomic $\mathcal{D}_X$-modules. The regular holonomic $\mathcal{D}_X$-modules $\mathcal{L}(Y, X, V)$, introduced in §1.11, correspond to $\mathcal{IC}^{\cdot}_Y(V)[-d]$.

2.1.3.3 Theorem. *Let Y be a projective irreducible variety. Assume V is the underlying local sytem of a polarized variation of Hodge structure on $Y - \Sigma$. Then the intersection cohomology groups $IH^i(Y, V)$ carry a pure Hodge structure.*

This was conjectured in [Cheeger, Goresky, MacPherson, 1982] and has recently been proved in a pioneering paper by M. Saito [Saito, 1986c]. (Cf. also [Saito, 1985].) In fact he derives a much more general result (cf. [Saito, 1986c], Thm. 5.3.1). In order to establish this he works in the category of filtered $\mathcal{D}_X$-modules and introduces the notion of Hodge modules. He studies the functorial behaviour of the Hodge modules (cf. also [Saito, 1986b]).

2.1.3.4 Another consequence of his work is a proof of the

Decomposition theorem. *Let $f: Z \to Y$ be a projective morphism of complex varieties. Then $f_*\mathcal{IC}^{\cdot}_Z$ is a direct sum of intersection cohomology complexes of the form $\mathcal{IC}^{\cdot}_Y(V)$.*

(Cf. also [Saito, 1983b].) Until then the only known proof used characteristic p methods (cf. [Beilinson, Bernstein, Deligne, 1982], Thm 6.2.5). The decomposition theorem has important consequences; for example see [Goresky and MacPherson, 1981], [MacPherson, 1984] and [Springer, 1982].

2.1.3.5 L^2-cohomology. Finally we would like to mention the following. Let Y be a compact analytic variety. Let V be as above. The question is wether the intersection cohomology groups and the L^2-cohomology groups associated with V coincide (cf. [Zucker, 1982]). In the one-dimensional case this was shown by Zucker [Zucker, 1979]. In case Y is a non-singular Kähler manifold and Σ a divisor with normal crossings it has been confirmed by Kashiwara and Kawai (cf. [Kashiwara and Kawai, 1986]) and independently by Cattani, Kaplan and Schmid [Cattani, Kaplan, Schmid, 1987]. Recently there has appeared a proof by Looijenga [Looijenga, 1987] in the context of locally symmetric varieties.

2.2 Summary of results

We focus our attention on the category of regular holonomic modules. In the preceding examples it has become clear that in general this category is very complicated. One would like to get a better understanding of its structure. Below we summarize results in this direction. First of all we mention those where one works in the category of perverse sheaves. Next we come to the cases where the study is done in the framework of $\mathcal{D}$-modules.

2.2.1 Classifying perverse sheaves

2.2.1.1 Deligne [letter to MacPherson, 1981] gives a combinatorial description of the category $\mathrm{Perv}^{\{0\}}(\mathbb{C})$ of perverse sheaves on $\mathbb{C}$ with respect to the stratification $\{0\}, \mathbb{C} - \{0\}$. It uses a characterization of constructible sheaves given in [Deligne, 1973a,b].

The category $\mathrm{Perv}^{\{0\}}(\mathbb{C})$ *is equivalent to the category of pairs* $E \underset{var}{\overset{can}{\rightleftarrows}} F$ *of finite dimensional complex vector spaces* E, F *and linear maps* can, var, *satisfying* $1 + var \circ can$ *is an automorphism of* E.

If $\mathcal{F}^{\cdot}$ is a perverse sheaf on $\mathbb{C}$ (w.r.t. the stratification $\{0\}, \mathbb{C}-\{0\}$), then $E \cong \mathcal{F}^{\cdot}_1$, the stalk above 1. $(E, 1+var\circ can)$ represents the local system $\mathcal{F}^{\cdot}|_{\mathbb{C}-\{0\}}$. The complex "$E \xrightarrow{u} F$" ($E$ in degree 0) is quasi-isomorphic to $\mathcal{F}^{\cdot}|_{\{0\}}$. In fact $E = \Psi(\mathcal{F}^{\cdot})$, $F = \Phi(\mathcal{F}^{\cdot})$, where Ψ (resp. Φ) denote the nearby cycle functor (resp. the vanishing cycle functor) defined in [Deligne, 1973a]. He generalizes this to the normal crossings case in $\mathbb{C}^n$ (see §1.2.1.2).

2.2.1.2 In [Galligo, Granger, Maisonobe, 1985a] a classification of perverse sheaves on $\mathbb{C}^n$ with regular singularities along normal crossings is given. The result is analogous to that of Deligne, but their method is quite different. Let $T \subset \mathbb{C}^n$ be the divisor with normal crossings given by $x_1 \dots x_n = 0$. Denote $\mathrm{Perv}^T(\mathbb{C}^n)$ the category of perverse sheaves on

$\mathbb{C}^n$ whose singular loci are contained in T. Introduce the category $\mathcal{C}_n$ of 2^n-tuples $\{F_I : I \subset \{1,\dots,n\}\}$ of finite dimensional complex vector spaces related by a set of linear maps

$$u_i\colon F_I \to F_{I\cup\{i\}} \qquad v_i\colon F_{I\cup\{i\}} \to F_I$$

satisfying

$$u_iu_j = u_ju_i, \quad v_iv_j = v_jv_i, \quad u_iv_j = v_ju_i;$$
$$1 + v_iu_i \quad \text{is an isomorphism.}$$

Theorem. *There exists an equivalence of categories*

$$\mathrm{Perv}^T(\mathbb{C}^n) \to \mathcal{C}_n.$$

Their method to show this consists in using suitable real scissions of $\mathbb{C}^n$—or rather of T—and studying the associated distinguished triangles.

An appeal to the Riemann-Hilbert correspondence yields an equivalence between the category of holonomic $\mathcal{D}_{\mathbb{C}^n}$-modules with regular singularities along T and the category $\mathcal{C}_n$. In a second paper [Galligo, Granger, Maisonobe, 1985b] the authors apply their results to obtain some corollaries on holonomic $\mathcal{D}_{\mathbb{C}^n}$-modules.

2.2.1.3 In [Granger and Maisonobe, 1984] one studies the case where the singular locus is a cusp. In [Maisonobe, 1985] this is generalized to the case of a plane curve Δ. They use a suitable scission of the real surface Δ. Their method is very laborious and it requires a good understanding of the topology of the situation. The method is similar to the one for the normal crossings case.

Theorem. *Let $X \subset \mathbb{C}^2$ be an open disc around* 0. *Let $\Delta \subset X$ be a plane curve. There exists an equivalence of categories*

$$\mathrm{Perv}^\Delta(X) \to \mathcal{C}(\Delta).$$

The category $\mathcal{C}(\Delta)$ consists of $(\delta+2)$-tuples of finite dimensional $\mathbb{C}$ vector spaces related by linear maps

$$E \underset{v_k}{\overset{u_k}{\rightleftarrows}} F_k \underset{v'_k}{\overset{u'_k}{\rightleftarrows}} G, \qquad k \in \{1,\dots,\delta\},$$

satisfying a set of conditions (cf. [loc. cit.], Ch. IV, §1). The number δ is given by topological means (cf. [loc. cit.], Ch. I, Prop. 2.2).

2.2.1.4 In [MacPherson and Vilonen, 1986] an elementary construction of perverse sheaves is given. Elementary in the sense that the construction uses only topology and linear algebra. Furthermore it is clear from their construction that the category of perverse sheaves is abelian.

Denote by Perv(X) the category of perverse sheaves with respect to a given stratification of the variety X. Their construction of Perv(X) is by induction on the strata and starts with $X - \Sigma$. The inductive step constructs Perv(X) from Perv($X - S$) for a closed stratum $S \subset X$. Perv(X) is then obtained as a category consisting of objects $A \in$ Perv($X - S$) together with a commutative diagram

$$\begin{tikzcd} F(A) \arrow[rr, "T_A"] \arrow[dr] & & G(A) \\ & B \arrow[ur] & \end{tikzcd}$$

where F and G are certain functors on Perv($X - S$) and T is a natural transformation.

In a second paper [MacPherson and Vilonen, 1985] they use their result to obtain a classification of the perverse sheaves on $\mathbb{C}^2$ with respect to the stratification $\{0\}$, $\{y^n = x^m\} - \{0\}$, $\mathbb{C}^2 - \{y^n = x^m\}$. Their description is analogous to the one of [Maisonobe, 1985] (see §2.2.1.3).

2.2.1.5 Narvaez Macarro [Narvaez Macarro, 1984] considers the following. Let $f: (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an irreducible germ with an isolated singularity. Let $f: X \to D$ be a good representative. Suppose $1 \in D$ and fix $x_0 \in X_1$. Let $\mathcal{L}$ be a local system on $U = X - X_0$. Note that $\Psi_f(\mathcal{L})$ is a perverse sheaf on X_0. Since X_0 is homeomorphic to $\mathbb{C}$, $\Psi_f(\mathcal{L})$ is determined by a pair of vector spaces (cf. 2.2.1.1). Narvaez Macarro gives an explicit description of this pair.

Theorem. *Put $L = \pi_1(X_1, x_0)$ and $E = \mathcal{L}_{x_0}$. Denote $I(L)$ the augmentation ideal of $\mathbb{C}[L]$, then $\Psi_f(\mathcal{L})$ is given by*

$$E \underset{v}{\overset{c}{\rightleftarrows}} \mathrm{Hom}_{\mathbb{C}[L]}(I(L), E).$$

Here $1_E + vc$ gives the monodromy.

Next he applies the extension theorem of Verdier (see 2.2.1.6) to obtain a combinatorial description of $\mathrm{Perv}^{X_0}(X)$, the category of perverse sheaves on X with respect to the stratification $\{0\}$, $X_0 - \{0\}$, U.

Theorem. *$\mathrm{Perv}^{X_0}(X)$ is equivalent to the category consisting of objects*

$$\left(E, \begin{array}{ccc} E & \rightleftarrows & \phi_1 \\ \upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright \\ \mathrm{Hom}_{\mathbb{C}[L]}(I(L), E) & \rightleftarrows & \phi_2 \end{array} \right).$$

Here E is a $\mathbb{C}[\pi_1(U, x_0)]$-module and ϕ_1, ϕ_2 are finite dimensional complex vector spaces. The arrows represent linear maps satisfying some relations (cf. [loc. cit.], Thm 2).

2.2.1.6 Although it is not really a classification of perverse sheaves we like to remark the following. Let $f\colon X \to \mathbb{C}$ be a holomorphic map. Put $X_0 = f^{-1}(0)$, $U = X - X_0$.

Theorem. (Extension theorem) *The functor*

$$\mathcal{F} \mapsto (\mathcal{F}|_U, \Psi_f(\mathcal{F}) \underset{var}{\overset{can}{\rightleftarrows}} \Phi_f(\mathcal{F}), \alpha)$$

defines an equivalence between $\mathrm{Perv}(X)$ *and the category consisting of objects* $(\mathcal{G}, \mathcal{G}_1 \underset{v}{\overset{c}{\rightleftarrows}} \mathcal{G}_2, \alpha)$. *Here* $\mathcal{G} \in \mathrm{Perv}(U)$, $\mathcal{G}_1, \mathcal{G}_2 \in \mathrm{Perv}(X_0)$ *and* $\alpha\colon \Psi_f(\mathcal{G}) \xrightarrow{\simeq} \mathcal{G}_1$ *is an isomorphism such that* $1_{\mathcal{G}} + \alpha^{-1} v c \alpha$ *gives the monodromy of* $\mathcal{G}$.

This is a result of Verdier [Verdier, 1985a] (cf. also [Verdier, 1985b]) on extensions of perverse sheaves over a hypersurface. In Chapter IV we will give a proof of an analogous version in the language of D-modules.

If we take the special case $X = \mathbb{C}$, $X_0 = \{0\}$, we get back the classification of perverse sheaves on $\mathbb{C}$ as mentioned in §2.2.1.1.

2.2.2 Classifying $\mathcal{D}$-modules

2.2.2.1 Let x be a coordinate on $\mathbb{C}$. Boutet de Monvel gives a description of holonomic $\mathcal{D}_{\mathbb{C}}$-modules which are regular at the origin (cf. [Boutet de Monvel, 1983b]).

Theorem. *Let* $\mathcal{M}$ *be a holonomic* $\mathcal{D}_{\mathbb{C}}$*-module, regular at* $\{0\}$. *Then* $\mathcal{M}$ *is isomorphic to a direct sum of indecomposable modules. The indecomposable modules are isomorphic to one of*

$$\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}(x\partial - \alpha)^n, (\alpha \notin \mathbb{Z}); \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}(\dots x\partial x\partial); \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}(\dots x\partial x).$$

For a proof see [Boutet de Monvel, 1983b] and also [Briançon and Maisonobe, 1984], Cor. 19.

Boutet de Monvel proceeds with a classification of holonomic $\mathcal{D}_{\mathbb{C}}$-modules, regular at $\{0\}$. His treatment is not correct but can be easily adapted to yield the following. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{\mathbb{C}}$-module, regular at $\{0\}$. Put $E := \big(\mathcal{H}om_{\mathcal{D}}(\mathcal{O}, \mathcal{M})\big)_1$. Denote by γ the monodromy. Next put (compare Ch. II)

$$V_\alpha := \bigcup_{n \in \mathbb{N}} \mathrm{Ker}\big((\partial x - \alpha)^n, \mathcal{M}_0\big); \qquad V := \bigoplus_{0 \le \mathrm{Re}\,\alpha < 1} V_\alpha;$$

$$W_\alpha := \bigcup_{n \in \mathbb{N}} \mathrm{Ker}\big((x\partial - \alpha)^n, \mathcal{M}_0\big); \qquad W := \bigoplus_{0 \le \mathrm{Re}\,\alpha < 1} W_\alpha.$$

These are finite dimensional vector spaces over $\mathbb{C}$. Multiplication by x (resp. ∂) induces a linear map $u\colon V \to W$ (resp. $v\colon W \to V$). Then $\exp(2\pi i u v)$ agrees with γ. W_α identifies with

$$\bigcup_{n \in \mathbb{N}} \mathrm{Ker}\big((\gamma - e^{2\pi i \alpha} 1_E)^n, E\big)$$

and W identifies with E.

Theorem. *The holonomic $\mathcal{D}_{\mathbb{C}}$-modules, regular at $\{0\}$, are classified by pairs $E \underset{v}{\overset{u}{\rightleftarrows}} V$ such that for every eigenvalue λ of uv, $0 \leq \operatorname{Re}\lambda < 1$.*

A similar discussion applies to holonomic $\mathcal{D}_{\mathbb{C}}$-modules with regular singularities at the points $\{a_1, \ldots, a_n\}$. These are classified by $(n+1)$-tuples $(E, V_1, \ldots, V_n)$ and linear maps

$$E \underset{u_i}{\overset{v_i}{\rightleftarrows}} V_i, \qquad i \in \{1, \ldots, n\}.$$

Furthermore, as Boutet de Monvel noted, essentially the same applies when X is a Riemann surface.

2.2.2.2 Two more cases of a classification of regular holonomic $\mathcal{D}$-modules are described in [Gelfand and Khoroskhin, 1985]. Let $X = \mathbb{C}^n$ and $\sigma = \{\Sigma_1, \ldots, \Sigma_k\}$ a stratification of X. Denote $\mathcal{M}od(\mathcal{D}_X, \sigma)_{\mathrm{hr}}$ the category of regular holonomic $\mathcal{D}_X$-modules $\mathcal{M}$ satisfying

$$SS(\mathcal{M}) \subset \bigcup_{i=1}^{k} T^*_{\Sigma_i} X.$$

Let $n = 2$. Let (x, y) be coordinates on X. Put $X_0 = \{xy(x+y) = 0\}$ and let σ be the obvious stratification. Let $\mathcal{A}$ denote the category of objects (U, V_1, V_2, V_3, W) of finite dimensional complex vector spaces related by linear maps $(i \in \{1, 2, 3\})$

$$a_i : U \to V_i, \quad b_i : V_i \to U, \quad c_i : V_i \to W, \quad d_i : W \to V_i,$$

satisfying

(i) the eigenvalues of $b_i a_i$ (resp. $c_i d_i$) are contained in $\{\, z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z < 1 \,\}$;
(ii) $\sum c_i a_i = \sum b_i d_i = 0$;
(iii) $a_i b_j + d_i c_j = 0$, for all $i \neq j$.

Theorem. *The category $\mathcal{M}od(\mathcal{D}_X, \sigma)_{\mathrm{hr}}$ is equivalent to $\mathcal{A}$.*

A similar description is obtained in [Maisonobe, 1985] (cf. §2.2.1.3).

Let $n \geq 4$ and let $X_0 = \{\sum x_i^2 = 0\}$, $\sigma = \big\{\{0\}, X_0 - \{0\}, X - X_0\big\}$. Denote $\mathcal{B}_n$ the category of objects

$$U \underset{b}{\overset{a}{\rightleftarrows}} V \underset{d}{\overset{c}{\rightleftarrows}} W$$

of finite dimensional vector spaces and linear maps satisfying

(i) the eigenvalues of ba are contained in $\{\, z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z < 1 \,\}$;
(ii) $ab - cd = 0$, if n is even;
$ab - cd = \frac{1}{2} 1_V$, if n is odd.

Theorem. *The category $\mathcal{M}od(\mathcal{D}_X, \sigma)_{\mathrm{hr}}$ is equivalent to $\mathcal{B}_n$.*

In each case the authors give an explicit description of the $\mathcal{D}_X$-module belonging to a given element in $\mathcal{A}$ (resp. $\mathcal{B}_n$).

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Chapter II

CLASSIFICATION OF $\mathcal{D}$-MODULES WITH REGULAR SINGULARITIES ALONG NORMAL CROSSINGS

by

M.G.M. van Doorn

0 Introduction

0.1 To classify regular holonomic $\mathcal{D}_1$-modules Boutet de Monvel [2] uses pairs of finite dimensional $\mathbb{C}$-vector spaces related by certain $\mathbb{C}$-linear maps. Galligo, Granger and Maisonobe [4] obtain, using the Riemann-Hilbert correspondence, a classification of holonomic $\mathcal{D}_n$-modules with regular singularities along $x_1 \dots x_n$ by means of 2^n-tuples of $\mathbb{C}$-vector spaces provided with a set of linear maps. We mention that also Deligne (not published) gets a classification of regular holonomic $\mathcal{D}_1$-modules.

The aim of this paper is to get such a classification in a direct way. The idea is roughly as follows. Denote by $\widetilde{\mathcal{C}}_1$ the category whose objects are diagrams $E \underset{v}{\overset{u}{\rightleftarrows}} F$ of finite dimensional $\mathbb{C}$-vector spaces such that $\{\lambda \mid \lambda \text{ eigenvalue of } vu\} \subset \{\alpha \in \mathbb{C} \mid 0 \leq \operatorname{Re}\alpha < 1\}$. We construct $\mathcal{D}_1$-modules $\mathcal{F}'$ ("Nilsson class functions"), $\mathcal{F}''$ ("micro Nilsson class functions") and $\mathcal{D}_1$-linear maps $\mathcal{U}: \mathcal{F}' \to \mathcal{F}''$ ("canonical map"), $\mathcal{V}: \mathcal{F}'' \to \mathcal{F}'$ ("variation"). For $M \in \mathrm{Mod}_l(\mathcal{D}_1)_{\mathrm{hr}}$ i.e., M is a regular holonomic left $\mathcal{D}_1$-module, we consider the solutions of M with values in $\mathcal{F}'$ (resp. $\mathcal{F}''$) i.e., $\mathrm{Hom}_{\mathcal{D}_1}(M, \mathcal{F}')$ (resp. $\mathrm{Hom}_{\mathcal{D}_1}(M, \mathcal{F}'')$). In this way we get an object in $\widetilde{\mathcal{C}}_1$ i.e., a functor $S: \mathrm{Mod}_l(\mathcal{D}_1)_{\mathrm{hr}} \to \widetilde{\mathcal{C}}_1$. In order to prove that S defines an equivalence of categories we exhibit an inverse functor T of S. As a matter of fact $T(E \rightleftarrows F) = \mathrm{Hom}(E \rightleftarrows F, \mathcal{F}' \rightleftarrows \mathcal{F}'')$. The proof that S and T are inverse to each other reduces to a study of what happens to simple objects of both categories.

The generalization to several variables is more or less straightforward, but the proofs get more involved. In proving statements we use induction on n to step down to the case $n = 1$ (or $n = 0$ if you wish). This causes some technical problems (cf. Lemma 4). At the end the proof of the equivalence (Proposition 3) becomes a formal exercise.

Notations Let $n \in \mathbb{N}$. Write $\partial_i = \frac{\partial}{\partial x_i}$, $i \in \{1, \dots, n\}$. Denote $\mathcal{O} = \mathcal{O}_n = \mathbb{C}[[x_1, \dots, x_n]]$ (resp. $\mathbb{C}\{x_1, \dots, x_n\}$); $\mathcal{D}_n = \mathcal{O}_n[\partial_1, \dots, \partial_n]$. Denote $\mathcal{O}_{(n)} = \mathbb{C}[[x_n]]$ (resp. $\mathbb{C}\{x_n\}$); $\mathcal{D}_{(n)} = \mathcal{O}_{(n)}[\partial_n]$. Let $\mathcal{D}$ be $\mathcal{D}_n$ or $\mathcal{D}_{(n)}$. Denote by $\mathrm{Mod}_l(\mathcal{D})$ the category of left $\mathcal{D}$-modules. If $P \in \mathcal{D}$ the left $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}P$ is denoted by $\mathcal{D}/(P)$. If $M \in \mathrm{Mod}_l(\mathcal{D})$ and $P \in \mathcal{D}$, left multiplication with P on M is denoted by $M \xrightarrow{P} M$.

Finally we put $J := \{\alpha \in \mathbb{C} \mid 0 \leq \operatorname{Re}\alpha < 1\}$.

0.2 Throughout the paper we assume that the reader has some familiarity with the language of $\mathcal{D}$-modules. He may consult e.g., [1], [7].

Let $M, N \in \mathrm{Mod}_l(\mathcal{D}_n)$. Then the tensor product $M \otimes_{\mathcal{O}} N$ has in a natural way a left $\mathcal{D}_n$-module structure, namely given by

$$\partial_i = \partial_i(m) \otimes n + m \otimes \partial_i(n), \qquad \text{all } i.$$

Let $M \in \mathrm{Mod}_l(\mathcal{D}_{n-1})$, then $\mathcal{O} \otimes_{\mathcal{O}_{n-1}} M$ has a left $\mathcal{D}_n$-module structure given by (cf. [7], Ch. 2, 12.2)

$$\begin{aligned} &\partial_i(a \otimes m) = \partial_i(a) \otimes m + a \otimes \partial_i(m), \quad \text{all } i \in \{1, \dots, n-1\} \\ &\partial_n(a \otimes m) = \partial_n(a) \otimes m. \end{aligned}$$

In a similar way $\mathcal{O} \otimes_{\mathcal{O}_{(n)}} N$ has a left $\mathcal{D}_n$-structure if $N \in \mathrm{Mod}_l(\mathcal{D}_{(n)})$. If $Q_i \in \mathcal{D}_{(i)}$, the following is easily verified

$$\Big(\mathcal{O} \underset{\mathcal{O}_{n-1}}{\otimes} \mathcal{D}_{n-1}/(Q_1, \dots, Q_{n-1})\Big) \underset{\mathcal{O}}{\otimes} \Big(\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathcal{D}_{(n)}/Q_{(n)}\Big) \cong \mathcal{D}/(Q_1, \dots, Q_n).$$

1 The operation $\mathcal{C}$

1.1 In order to state the results in a neat way we introduce some general notions. Let $\mathcal{A}$ be a category. Denote $\mathcal{C}(\mathcal{A})$ the category whose objects are quadruples (E, F, u, v), where E, F are objects of $\mathcal{A}$, $u \in \mathrm{Hom}_{\mathcal{A}}(E, F)$ and $v \in \mathrm{Hom}_{\mathcal{A}}(F, E)$. If (E, F, u, v) and (E', F', u', v') belong to $\mathcal{C}(\mathcal{A})$, then

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}\big((E, F, u, v), (E', F', u', v')\big) \\ &\qquad = \{\, (f, g) \in \mathrm{Hom}_{\mathcal{A}}(E, E') \\ &\qquad\qquad \times \mathrm{Hom}_{\mathcal{A}}(F, F') \mid u'f = gu, fv = v'g \,\}. \end{aligned}$$

Hence $\mathcal{C}(\mathcal{A})$ is the category of diagrams in $\mathcal{A}$ over the scheme "$\cdot \rightleftarrows \cdot$". Cf. Grothendieck [5] and Mitchell [6], Ch. II §1. $\mathcal{C}(\mathcal{A})$ may be seen as a functor category and as such it inherits the properties of $\mathcal{A}$. In particular $\mathcal{C}(\mathcal{A})$ is an abelian category if $\mathcal{A}$ is abelian. We have two evaluation functors e_0 and e_1 from $\mathcal{C}(\mathcal{A})$ to $\mathcal{A}$. If $X = (E, F, u, v) \in \mathcal{C}(\mathcal{A})$ then $e_0(X) = E$, $e_1(X) = F$. If $\mathcal{A}$ is an abelian category these functors are exact and collectively faithful. Hence in particular: $X' \to X \to X''$ is exact in $\mathcal{C}(\mathcal{A})$ if and only if $e_i(X') \to e_i(X) \to e_i(X'')$ is exact in $\mathcal{A}$, for all $i \in \{0, 1\}$. Notice that we have natural transformations $u: e_0 \to e_1$, $v: e_1 \to e_0$. If $F: \mathcal{A} \to \mathcal{B}$ is a functor between categories $\mathcal{A}$ and $\mathcal{B}$, there is obviously an induced functor $\mathcal{C}(F): \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$. Clearly if $\mathcal{A}$ and $\mathcal{B}$ are additive and F is an additive functor, then $\mathcal{C}(F)$ is additive. Exactness properties of F are transferred to $\mathcal{C}(F)$. Furthermore, if $G: \mathcal{A} \to \mathcal{B}$ is another functor and $\eta: F \to G$ is a natural transformation (resp. equivalence), there is a natural transformation (resp. equivalence) $\mathcal{C}(\eta): \mathcal{C}(F) \to \mathcal{C}(G)$.

[3]

1.2 Let $\mathcal{A}$ be a category. For all $n \in \mathbf{N}$ we define inductively

$$\mathcal{C}_0(\mathcal{A}) := \mathcal{A}$$
$$\mathcal{C}_{n+1}(\mathcal{A}) := \mathcal{C}\big(\mathcal{C}_n(\mathcal{A})\big).$$

For each $n \in \mathbf{N}$ we have 2^n evaluation functors defined inductively as follows: for all $i_1, \dots, i_{n+1} \in \{0,1\}$

$$e_{i_1 \cdots i_{n+1}} = e_{i_1 \cdots i_n} \circ e_{i_{n+1}}.$$

If $E \in \mathcal{C}_n(\mathcal{A})$ and $i_1, \dots, i_n \in \{0,1\}$ we mostly write $E(i_1 \dots i_n)$ or $E_{i_1 \cdots i_n}$ instead of $e_{i_1 \cdots i_n}(E)$.
For every $j \in \{1, \dots, n\}$ and all $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_n \in \{0,1\}$ we get $\mathcal{A}$-morphisms

$$E(i_1 \dots i_{j-1} 0 i_{j+1} \dots i_n) \to E(i_1 \dots i_{j-1} 1 i_{j+1} \dots i_n)$$
$$E(i_1 \dots i_{j-1} 1 i_{j+1} \dots i_n) \to E(i_1 \dots i_{j-1} 0 i_{j+1} \dots i_n)$$

It is easily seen that the category $\mathcal{C}_n(\mathcal{A})$ can be identified with the category whose objects are 2^n-tuples $\big(E(i_1 \dots i_n);\, i_1, \dots, i_n \in \{0,1\}\big)$ of objects of $\mathcal{A}$, connected by $\mathcal{A}$-morphisms, for all $j \in \{1, \dots, n\}$, all $i_1, \dots, i_n \in \{0,1\}$,

$$u\colon E(-0-) \to E(-1-) \qquad v\colon E(-1-) \to E(-0-),$$

where $E(-r-)$ stands for $E(i_1 \dots i_{j-1} r i_{j+1} \dots i_n)$. The following diagrams have to commute

$$\begin{array}{ccc} E_{00} & \xrightarrow{u} & E_{01} \\ \downarrow u & & u \downarrow \\ E_{10} & \xrightarrow{u} & E_{11} \end{array} \qquad \begin{array}{ccc} E_{00} & \xleftarrow{v} & E_{01} \\ \uparrow v & & v \uparrow \\ E_{10} & \xleftarrow{v} & E_{11} \end{array} \qquad \begin{array}{ccc} E_{00} & \xrightarrow{u} & E_{01} \\ \uparrow v & & v \uparrow \\ E_{10} & \xrightarrow{u} & E_{11} \end{array} \qquad \begin{array}{ccc} E_{00} & \xleftarrow{v} & E_{01} \\ \downarrow u & & u \downarrow \\ E_{10} & \xleftarrow{v} & E_{11} \end{array}$$

where for simplicity we have written E_{rs} instead of

$$E(i_1 \dots i_{j-1} r i_{j+1} \dots i_{k-1} s i_{k+1} \dots i_n), \quad \text{all } r, s \in \{0,1\}.$$

Remark Let A be a ring and let $\mathrm{Mod}_l(A)$ be the category of left A-modules. We write $\mathcal{C}_n(A)$ instead of $\mathcal{C}_n\big(\mathrm{Mod}_l(A)\big)$. Furthermore we put $\mathcal{C}_n = \mathcal{C}_n(\mathbf{C})$.

2 Definition and properties of $\mathcal{F}_n$

2.1 Our next goal is to construct a particular object $\mathcal{F}_n$ of $\mathcal{C}_n(\mathcal{D}_n)$. Let therefore $n \in \mathbf{N}$, $n \neq 0$. For $\alpha \in J$, $i \in \mathbf{N} - \{0\}$ define

$$\mathcal{F}'_{(n),\alpha,i} := \mathcal{D}_{(n)}/\big((\partial_n x_n - \alpha)^i\big) \qquad \mathcal{F}''_{(n),\alpha,i} := \mathcal{D}_{(n)}/\big((x_n\partial_n - \alpha)^i\big).$$

For each $\alpha \in J$, the $\mathcal{D}_{(n)}$-linear maps

$$\mathcal{F}'_{(n),\alpha,i} \to \mathcal{F}'_{(n),\alpha,i+1}, \qquad \text{induced by } P \mapsto P(\partial_n x_n - \alpha)$$

and

$$\mathcal{F}''_{(n),\alpha,i} \to \mathcal{F}''_{(n),\alpha,i+1}, \qquad \text{induced by } P \mapsto P(x_n\partial_n - \alpha)$$

yield inductive systems $(\mathcal{F}'_{(n),\alpha,i})_i$ and $(\mathcal{F}''_{(n),\alpha,i})_i$. Define

$$\mathcal{F}'_{(n)} := \bigoplus_{\alpha\in J} \varinjlim_i \mathcal{F}'_{(n),\alpha,i} \qquad \mathcal{F}''_{(n)} := \bigoplus_{\alpha\in J} \varinjlim_i \mathcal{F}''_{(n),\alpha,i}.$$

Furthermore, the $\mathcal{D}_{(n)}$-linear maps

$$\begin{aligned} \mathcal{F}'_{(n),\alpha,i} &\to \mathcal{F}''_{(n),\alpha,i}, \qquad \text{induced by } P \mapsto P\partial_n, \\ \mathcal{F}''_{(n),\alpha,i} &\to \mathcal{F}'_{(n),\alpha,i}, \qquad \text{induced by } P \mapsto Px_n, \end{aligned}$$

give rise to $\mathcal{D}_{(n)}$-linear maps

$$\mathcal{U}_{(n)}: \mathcal{F}'_{(n)} \to \mathcal{F}''_{(n)} \qquad \mathcal{V}_{(n)}: \mathcal{F}''_{(n)} \to \mathcal{F}'_{(n)}.$$

Hence we have constructed an object

$$\mathcal{F}_{(n)} := \big(\mathcal{F}'_{(n)}, \mathcal{F}''_{(n)}, \mathcal{U}_{(n)}, \mathcal{V}_{(n)}\big) \in \mathcal{C}_1(\mathcal{D}_{(n)}).$$

By extending coefficients we get $\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathcal{F}_{(n)} \in \mathcal{C}_1(\mathcal{D}_n)$.

Remark Instead of the clumsy notation $\mathcal{C}_1(\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \cdot)(\mathcal{F}_{(n)})$ we prefer to write $\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathcal{F}_{(n)}$.

2.2 The preceding constructions lead immediately to

Lemma 1. *There exist short exact sequences of $\mathcal{D}_{(n)}$-modules*

$$\mathcal{O}_{(n)} = \mathcal{D}_{(n)}/(\partial_n) \hookrightarrow \mathcal{F}'_{(n)} \overset{\mathcal{U}_{(n)}}{\twoheadrightarrow} \mathcal{F}''_{(n)}$$

$$\mathcal{D}_{(n)}/(x_n) \hookrightarrow \mathcal{F}''_{(n)} \overset{\mathcal{V}_{(n)}}{\twoheadrightarrow} \mathcal{F}'_{(n)}$$

$$\mathcal{D}_{(n)}/(\partial_n x_n - \alpha) \hookrightarrow \mathcal{F}'_{(n)} \overset{\mathcal{V}_{(n)}\mathcal{U}_{(n)} - \alpha 1}{\twoheadrightarrow} \mathcal{F}'_{(n)}, \quad \alpha \in J - \{0\}.$$

Proof. Let $\alpha \in J - \{0\}$. The $\mathcal{D}_{(n)}$-linear map $\mathcal{D}_{(n)}/(\partial_n x_n - \alpha) \to \mathcal{D}_{(n)}/(x_n\partial_n - \alpha)$, induced by $P \mapsto P\partial_n$ is an isomorphism (left to the reader). We have the commutative diagram with exact rows

$$\begin{array}{ccccc}
\mathcal{F}'_{(n),\alpha,i} & \xhookrightarrow{1\mapsto \partial_n x_n - \alpha} & \mathcal{F}'_{(n),\alpha,i+1} & \xrightarrow{1\mapsto 1} & \mathcal{D}_{(n)}/(\partial_n x_n - \alpha) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F}''_{(n),\alpha,i} & \xhookrightarrow{1\mapsto x_n\partial_n - \alpha} & \mathcal{F}''_{(n),\alpha,i+1} & \xrightarrow{1\mapsto 1} & \mathcal{D}_{(n)}/(x_n\partial_n - \alpha)
\end{array}$$

where the vertical maps are induced by $P \mapsto P\partial_n$. Hence, by induction on i, it follows that $\mathcal{F}'_{(n),\alpha,i} \to \mathcal{F}''_{(n),\alpha,i}$, $1 \mapsto \partial_n$ is an isomorphism for all $i \in \mathbb{N} - \{0\}$.

It is easily verified that we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathcal{D}_{(n)}/(\partial_n) & \xhookrightarrow{1\mapsto x_n(\partial_n x_n)^{i-1}} & \mathcal{F}'_{(n),0,i} & \xrightarrow{1\mapsto \partial_n} & \mathcal{F}''_{(n),0,i} & \xrightarrow{1\mapsto 1} & \mathcal{D}_{(n)}/(\partial_n) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1 \mapsto \partial_n x_n} & & \downarrow{\scriptstyle 1 \mapsto x_n\partial_n} & & \downarrow{\scriptstyle 0} \\
\mathcal{D}_{(n)}/(\partial_n) & \xhookrightarrow{1\mapsto x_n(\partial_n x_n)^{i}} & \mathcal{F}'_{(n),0,i+1} & \xrightarrow{1\mapsto \partial_n} & \mathcal{F}''_{(n),0,i+1} & \xrightarrow{1\mapsto 1} & \mathcal{D}_{(n)}/(\partial_n)
\end{array}$$

Taking the direct limit and summing over $\alpha \in J$ we obtain the exact sequence of $\mathcal{D}_{(n)}$-modules

$$\mathcal{D}_{(n)}/(\partial_n) \hookrightarrow \mathcal{F}'_{(n)} \xrightarrow{\mathcal{U}_{(n)}} \mathcal{F}''_{(n)}.$$

The other two sequences are obtained in a similar way.

2.3 Consider the bifunctor $\otimes_{\mathcal{O}}: \mathrm{Mod}_l(\mathcal{D}_n) \times \mathrm{Mod}_l(\mathcal{D}_n) \to \mathrm{Mod}_l(\mathcal{D}_n)$, $(M, N) \mapsto M \otimes_{\mathcal{O}} N$. It induces a bifunctor from $\mathcal{C}_{n-1}(\mathcal{D}_n) \times \mathcal{C}_1(\mathcal{D}_n)$ to $\mathcal{C}_n(\mathcal{D}_n)$, also denoted by $\otimes_{\mathcal{O}}$. Keeping this in mind we define inductively on $n \in \mathbb{N}$

$$\mathcal{F}_0 := \mathbb{C}$$
$$\mathcal{F}_n := \Big(\mathcal{O} \underset{\mathcal{O}_{n-1}}{\otimes} \mathcal{F}_{n-1}\Big) \underset{\mathcal{O}}{\otimes} \Big(\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathcal{F}_{(n)}\Big) \in \mathcal{C}_n(\mathcal{D}_n).$$

Hence

$$\mathcal{F}_n(i_1 \dots i_n) = (\mathcal{O} \underset{\mathcal{O}_{(1)}}{\otimes} \mathcal{F}_{(1)}(i_1)) \underset{\mathcal{O}}{\otimes} \cdots \underset{\mathcal{O}}{\otimes} (\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathcal{F}_{(n)}(i_n))$$

for all $i_1, \dots, i_n \in \{0, 1\}$. The $\mathcal{D}_n$-linear maps are identified as

$$\mathcal{F}_n(i_1 \dots i_{j-1} 0 i_{j+1} \dots i_n) \underset{1\otimes\mathcal{V}_{(j)}\otimes 1}{\overset{1\otimes\mathcal{U}_{(j)}\otimes 1}{\rightleftarrows}} \mathcal{F}_n(i_1 \dots i_{j-1} 1 i_{j+1} \dots i_n).$$

We are ready now to define the functor S_n. Therefore consider the bifunctor $H_n\colon \mathrm{Mod}_l(\mathcal{D}_n) \times \mathrm{Mod}_l(\mathcal{D}_n) \to \mathcal{C}_0$, $(M,N) \mapsto \mathrm{Hom}_{\mathcal{D}_n}(M,N)$. It induces a bifunctor $\mathcal{C}_n(H_n)\colon \mathrm{Mod}_l(\mathcal{D}_n) \times \mathcal{C}_n(\mathcal{D}_n) \to \mathcal{C}_n$. So there arises a contravariant functor

$$S_n\colon \mathrm{Mod}_l(\mathcal{D}_n) \to \mathcal{C}_n \qquad S_n(M) := \mathcal{C}_n(H_n)(M, \mathcal{F}_n).$$

Notice that S_n is characterized by

$$S_n(M)(i_1 \dots i_n) = \mathrm{Hom}_{\mathcal{D}_n}\big(M, \mathcal{F}_n(i_1 \dots i_n)\big)$$

for all $i_1, \dots, i_n \in \{0,1\}$.

3 Study of the functor S_n

3.1 We restrict our attention to the category $\mathrm{Mod}_l(\mathcal{D}_n)_{\mathrm{hr}}^{x_1 \cdots x_n}$ the full subcategory of $\mathrm{Mod}_l(\mathcal{D}_n)$ consisting of holonomic $\mathcal{D}_n$-modules with regular singularities along $x_1 \dots x_n$. For a definition we refer to van den Essen [3], Ch. I, Def. 1.16. He gives also a description of the simple objects in $\mathrm{Mod}_l(\mathcal{D}_n)_{\mathrm{hr}}^{x_1 \cdots x_n}$ (loc. cit. Ch. I, Th. 2.7). They are of the form $\mathcal{D}_n/(q_1, \dots, q_n)$ with $q_i \in \{x_i, \partial_i\} \cup \{\, \partial_i x_i - \alpha_i \mid \alpha_i \in \mathbb{C},\ 0 < \mathrm{Re}\, \alpha_i < 1 \,\}$ for all $i \in \{1, \dots, n\}$. It is suitable for us to write this as

$$\Big(\mathcal{O} \underset{\mathcal{O}_{n-1}}{\otimes} N\Big) \underset{\mathcal{O}}{\otimes} \Big(\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathcal{D}_{(n)}/(q_n)\Big)$$

where $N = \mathcal{D}_{n-1}/(q_1, \dots, q_{n-1})$ is a simple object from the category $\mathrm{Mod}_l(\mathcal{D}_{n-1})_{\mathrm{hr}}^{x_1 \cdots x_{n-1}}$. To simplify notations we introduce:

For $\alpha \in J \cup \{1\}$ define $q_n(\alpha) \in \mathcal{D}_{(n)}$ as

$$q_n(0) := \partial_n; \quad q_n(1) := x_n; \quad q_n(\alpha) := \partial_n x_n - \alpha, \quad \alpha \in J - \{0\}.$$

For $N \in \mathrm{Mod}_l(\mathcal{D}_{n-1})$ define

$$P_\alpha(N) := \Big(\mathcal{O} \underset{\mathcal{O}_{n-1}}{\otimes} N\Big) \underset{\mathcal{O}}{\otimes} \Big(\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathcal{D}_{(n)}/\big(q_n(\alpha)\big)\Big).$$

For $M \in \mathrm{Mod}_l(\mathcal{D}_n)$ define

$$Q_\alpha(M) := \mathrm{Ker}(M \xrightarrow{q_n(\alpha)} M).$$

So for each $\alpha \in J \cup \{1\}$ we have a pair of functors (P_α, Q_α)

$$P_\alpha\colon \mathrm{Mod}_l(\mathcal{D}_{n-1}) \to \mathrm{Mod}_l(\mathcal{D}_n) \quad Q_\alpha\colon \mathrm{Mod}_l(\mathcal{D}_n) \to \mathrm{Mod}_l(\mathcal{D}_{n-1}).$$

Obviously:

- Q_α is left exact;
- P_α is exact because $\mathcal{O} \otimes_{\mathcal{O}_{(n)}} \mathcal{D}_{(n)}/\big(q_n(\alpha)\big)$ is a flat $\mathcal{O}_{n-1}$-module;
- P_α is a left adjoint of Q_α.

3.2 By a direct calculation, using the definitions of $\mathcal{F}'_n$ and $\mathcal{F}''_n$ one establishes the following. For convenience we write $\otimes$ instead of $\underset{\mathcal{O}_{(n)}}{\otimes}$.

Lemma 2. *There exist short exact sequences of $\mathcal{D}_{n-1}$-modules*

$$\mathcal{O}\otimes\mathcal{F}''_{(n)} \overset{\partial_n}{\hookrightarrow\!\!\!\rightarrow} \mathcal{O}\otimes\mathcal{F}''_{(n)} \qquad \mathcal{O}_{n-1}\hookrightarrow \mathcal{O}\otimes\mathcal{F}'_{(n)} \xrightarrow{\partial_n} \mathcal{O}\otimes\mathcal{F}'_{(n)}$$

$$\mathcal{O}\otimes\mathcal{F}'_{(n)} \overset{x_n}{\hookrightarrow\!\!\!\rightarrow} \mathcal{O}\otimes\mathcal{F}'_{(n)} \qquad \mathcal{O}_{n-1}\hookrightarrow \mathcal{O}\otimes\mathcal{F}''_{(n)} \xrightarrow{x_n} \mathcal{O}\otimes\mathcal{F}''_{(n)}$$

$$\mathcal{O}_{n-1}\hookrightarrow \mathcal{O}\otimes\mathcal{F}'_{(n)} \xrightarrow{q_n(\alpha)} \mathcal{O}\otimes\mathcal{F}'_{(n)} \qquad \mathcal{O}_{n-1}\hookrightarrow \mathcal{O}\otimes\mathcal{F}''_{(n)} \xrightarrow{q_n(\alpha)} \mathcal{O}\otimes\mathcal{F}''_{(n)}$$

for all $\alpha\in J-\{0\}$.

Proof. During the proof we write $\otimes$ instead of $\otimes_{\mathcal{O}_{(n)}}$. Let $\alpha\in J$. It is straightforward to verify that $\mathcal{O}_{n-1}\cong Q_\alpha(\mathcal{O}\otimes\mathcal{F}'_{(n),\alpha,i})$. One may use e.g.,

$$\mathcal{O}\otimes\mathcal{F}'_{(n),\alpha,i}=\mathcal{O}[x_n^{-1}]x_n^{\alpha-1}(\log x_n)^{i-1}+\cdots+\mathcal{O}[x_n^{-1}]x_n^{\alpha-1}$$

or the lemma on page 39 in [7]. Furthermore

$$\mathcal{O}_{n-1}\cong \operatorname{Coker}\Big(\mathcal{O}\otimes\mathcal{D}_{(n)}/(\partial_n x_n-\alpha)\xrightarrow{(\partial_n x_n-\alpha)}\mathcal{O}\otimes\mathcal{D}_{(n)}/(\partial_n x_n-\alpha)\Big).$$

Consider the short exact sequences of $\mathcal{D}$-modules

$$\mathcal{O}\otimes\mathcal{F}'_{(n),\alpha,i}\hookrightarrow\mathcal{O}\otimes\mathcal{F}'_{(n),\alpha,i+1}\twoheadrightarrow\mathcal{O}\otimes\mathcal{D}_{(n)}/(\partial_n x_n-\alpha).$$

Writing ϕ_j for the map: left multiplication with $\partial_n x_n-\alpha$ on $\mathcal{O}\otimes\mathcal{F}'_{(n),\alpha,j}$ for all $j\in\mathbb{N}-\{0\}$, we obtain a long exact sequence

$$\begin{aligned}\mathcal{O}_{n-1}=\operatorname{Ker}\phi_i\hookrightarrow\operatorname{Ker}\phi_{i+1}=\mathcal{O}_{n-1}\longrightarrow\operatorname{Ker}\phi_0=\mathcal{O}_{n-1}\\ \xrightarrow{\delta}\operatorname{Coker}\phi_i\xrightarrow{\epsilon}\operatorname{Coker}\phi_{i+1}\twoheadrightarrow\operatorname{Coker}\phi_0=\mathcal{O}_{n-1}\end{aligned}$$

where every map is $\mathcal{D}_{n-1}$-linear. By induction on i we have that $\operatorname{Coker}\phi_i=\mathcal{O}_{n-1}$. Now $\mathcal{O}_{n-1}$ is a simple $\mathcal{D}_{n-1}$-module, hence δ is an isomorphism. Moreover $\epsilon=0$ and $\operatorname{Coker}\phi_{i+1}=\mathcal{O}_{n-1}$. So we have, for all $i\in\mathbb{N}-\{0\}$, a commutative diagram with exact rows

$$\begin{tikzcd}
\mathcal{O}_{n-1} \arrow[r, hook] \arrow[d, "1"] & \mathcal{O}\otimes\mathcal{F}'_{(n),\alpha,i} \arrow[r, "(\partial_n x_n-\alpha)"] \arrow[d] & \mathcal{O}\otimes\mathcal{F}'_{(n),\alpha,i} \arrow[r, two heads] \arrow[d] & \mathcal{O}_{n-1} \arrow[d, "0"] \\
\mathcal{O}_{n-1} \arrow[r, hook] & \mathcal{O}\otimes\mathcal{F}'_{(n),\alpha,i+1} \arrow[r, "(\partial_n x_n-\alpha)"] & \mathcal{O}\otimes\mathcal{F}'_{(n),\alpha,i+1} \arrow[r, two heads] & \mathcal{O}_{n-1}
\end{tikzcd}$$

Another calculation learns that left multiplication with $\partial_n x_n-\alpha$ on $\mathcal{O}\otimes\mathcal{F}'_{(n),\beta,i}$ is a bijection for all $i\in\mathbb{N}-\{0\}$, all $\beta\in J$, $\beta\neq\alpha$ (use induction on i). After taking the direct limit and summing over $\beta \in J$ we arrive at the short exact sequence

$$\mathcal{O}_{n-1} \hookrightarrow \mathcal{O} \otimes \mathcal{F}'_{(n)} \xrightarrow{(\partial_n x_n - \alpha)} \mathcal{O} \otimes \mathcal{F}'_{(n)}.$$

Using that left multiplication with $\partial_n x_n - \alpha$ on $\mathcal{O}$ is bijective and the commutativity of the next diagram with exact rows (cf. Lemma 1)

$$\begin{array}{ccccc}
\mathcal{O} & \hookrightarrow & \mathcal{O} \otimes \mathcal{F}'_{(n)} & \twoheadrightarrow & \mathcal{O} \otimes \mathcal{F}''_{(n)} \\
\downarrow{\scriptstyle (\partial_n x_n - \alpha)} & & \downarrow{\scriptstyle (\partial_n x_n - \alpha)} & & \downarrow{\scriptstyle (\partial_n x_n - \alpha)} \\
\mathcal{O} & \hookrightarrow & \mathcal{O} \otimes \mathcal{F}'_{(n)} & \twoheadrightarrow & \mathcal{O} \otimes \mathcal{F}''_{(n)}
\end{array}$$

one establishes the exactness of

$$\mathcal{O}_{n-1} \hookrightarrow \mathcal{O} \otimes \mathcal{F}''_{(n)} \xrightarrow{(\partial_n x_n - \alpha)} \mathcal{O} \otimes \mathcal{F}''_{(n)}.$$

It is immediately verified that left multiplication with x_n on $\mathcal{O} \otimes \mathcal{F}'_{(n)}$ is bijective. Furthermore left multiplication with x_n on $\mathcal{O} \otimes \mathcal{D}_{(n)}/(x_n)$ is surjective and has Ker $\cong \mathcal{O}_{n-1}$. Consider the second sequence in Lemma 1, argue as above and obtain the exactness of

$$\mathcal{O}_{n-1} \hookrightarrow \mathcal{O} \otimes \mathcal{F}''_{(n)} \xrightarrow{x_n} \mathcal{O} \otimes \mathcal{F}''_{(n)}.$$

Combining results on left multiplication with $\partial_n x_n$ and left multiplication with x_n yields the exactness of the upper sequences in the lemma.

3.3 At this point we introduce a category $\widetilde{\mathcal{C}}$ as follows.

$\widetilde{\mathcal{C}}_0$ is the category of finite dimensional $\mathbb{C}$-vector spaces;
$\widetilde{\mathcal{C}}_{n+1}$ is the full subcategory of $\mathcal{C}_{n+1}$ consisting of all the objects $(E, F, u, v) \in \mathcal{C}_{n+1}$ that satisfy:
(i) $E, F \in \widetilde{\mathcal{C}}_n$;
(ii) $\{\lambda \mid \lambda \text{ eigenvalue of } e_{i_1 \cdots i_n}(vu)\} \subset J$ for all $i_1, \ldots, i_n \in \{0, 1\}$.

Notice that $\widetilde{\mathcal{C}}_n$ is a thick abelian subcategory of $\mathcal{C}_n$. For each $\alpha \in J \cup \{1\}$ we introduce a functor $L_\alpha: \mathcal{C}_{n-1} \to \mathcal{C}_n$ by putting for all $E \in \mathcal{C}_{n-1}$:

$$\begin{aligned}
L_0(E) &:= (E, 0, 0, 0) \\
L_1(E) &:= (0, E, 0, 0) \\
L_\alpha(E) &:= (E, E, 1, \alpha 1) \qquad \text{for all } \alpha \in J - \{0\}.
\end{aligned}$$

These are all exact functors. Clearly for each $\alpha \in J \cup \{1\}$ L_α restricts to a functor from $\widetilde{\mathcal{C}}_{n-1}$ to $\widetilde{\mathcal{C}}_n$, also denoted by L_α.

3.4 Putting $n = 1$ in Lemma 2 we may reformulate it as

$$S_1\Big(\mathcal{D}_1/(q_1(\alpha))\Big) = L_\alpha(\mathbb{C}) \in \widetilde{\mathcal{C}}_1$$
$$\mathrm{Ext}^1_{\mathcal{D}_1}\Big(\mathcal{D}_1/(q_1(\alpha)), \mathcal{F}_1(i)\Big) = 0 \quad \text{all } \alpha \in J \cup \{1\}, \text{ all } i \in \{0,1\}.$$

However elements of $\mathrm{Mod}_l(\mathcal{D}_1)_{\mathrm{hr}}$ have finite length. Hence S_1 induces a contravariant exact functor—denoted S_1—from $\mathrm{Mod}_l(\mathcal{D}_1)_{\mathrm{hr}}$ to $\widetilde{\mathcal{C}}_1$. This result generalizes to

Proposition 1. *S_n induces a contravariant exact functor*

$$S_n : \mathrm{Mod}_l(\mathcal{D}_n)^{x_1 \cdots x_n}_{\mathrm{hr}} \to \widetilde{\mathcal{C}}_n.$$

Proof. By induction on n. We need only to consider a simple module $M \in \mathrm{Mod}_l(\mathcal{D}_n)^{x_1 \cdots x_n}_{\mathrm{hr}}$. Hence let $\alpha \in J \cup \{1\}$, $N \in \mathrm{Mod}_l(\mathcal{D}_{n-1})^{x_1 \cdots x_{n-1}}_{\mathrm{hr}}$ such that $M = P_\alpha N$. Let $i_1, \ldots, i_n \in \{0,1\}$. Write $q = q_n(\alpha)$, $P = P_\alpha$, $Q = Q_\alpha$, $L = L_\alpha$. Lemma 2 says that left multiplication with q is surjective on $\mathcal{O} \otimes_{\mathcal{O}_{(n)}} \mathcal{F}_{(n)}(i_n)$. Furthermore $\mathcal{O} \otimes_{\mathcal{O}_{(n)}} \mathcal{F}_{(n)}(i_n)$ is a flat $\mathcal{O}_{n-1}$-module and $q \in \mathcal{D}_{(n)}$, hence

$$Q(\mathcal{F}_n(i_1 \ldots i_n)) = \mathcal{F}_{n-1}(i_1 \ldots i_{n-1}) \underset{\mathcal{O}_{n-1}}{\otimes} Q\Big(\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathcal{F}_{(n)}(i_n)\Big).$$

Again using Lemma 2 we get $\mathcal{C}_1(Q)(\mathcal{O} \otimes_{\mathcal{O}_{(n)}} \mathcal{F}_{(n)}) = L(\mathcal{O}_{n-1})$. It follows that

$$\begin{aligned}
S_n(PN) &= \mathcal{C}_n(H_n)(PN, \mathcal{F}_n) = \mathcal{C}_n(H_{n-1})\big(N, \mathcal{C}_n(Q)(\mathcal{F}_n)\big) \\
&= \mathcal{C}_n(H_{n-1})\Big(N, \mathcal{F}_{n-1} \underset{\mathcal{O}_{n-1}}{\otimes} \mathcal{C}_1(Q)\Big(\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathcal{F}_{(n)}\Big)\Big) \\
&= \mathcal{C}_n(H_{n-1})\Big(N, \mathcal{F}_{n-1} \underset{\mathbb{C}}{\otimes} L(\mathbb{C})\Big) \\
&= \big(\mathcal{C}_{n-1}(H_{n-1})(N, \mathcal{F}_{n-1})\big) \underset{\mathbb{C}}{\otimes} L(\mathbb{C}) = LS_{n-1}N
\end{aligned}$$

The exactness of S_n follows, by induction, from the next general result.

Lemma 3. *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories with enough injectives. Let $G: \mathcal{B} \to \mathcal{A}$ be a left adjoint of $F: \mathcal{A} \to \mathcal{B}$ and assume that G is exact. Furthermore, let $A \in \mathcal{A}$ be such that $\mathrm{R}^1F(A) = 0$. Then one has that $\mathrm{Ext}^1_{\mathcal{A}}\big(G(B), A\big) \cong \mathrm{Ext}^1_{\mathcal{B}}\big(B, F(A)\big)$, for all $B \in \mathcal{B}$.*

Remark $\mathrm{R}^1Q(\mathcal{F}_n(i_1 \ldots i_n)) = 0$ because left multiplication with q is surjective.

PROOF. Notice that for an injective object $I \in \mathcal{A}$, $F(I)$ is injective in $\mathcal{B}$ because one has $\mathrm{Hom}_{\mathcal{B}}(\cdot, F(I)) \cong \mathrm{Hom}_{\mathcal{A}}(G(\cdot), I)$ and this last functor is exact. Consider a short exact sequence $A \hookrightarrow I \twoheadrightarrow R$ in $\mathcal{A}$ with I injective object in $\mathcal{A}$. Since $\mathrm{R}^1F(A) = 0$ we get an exact sequence in $\mathcal{B}$

$$F(A) \hookrightarrow F(I) \twoheadrightarrow F(R).$$

(Obvious F is left exact.) Let $B \in \mathcal{B}$. There results a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{A}}(G(B), A) & \hookrightarrow & \mathrm{Hom}_{\mathcal{A}}(G(B), I) & & & & \\
\| & & \| & & & & \\
\mathrm{Hom}_{\mathcal{B}}(B, F(A)) & \hookrightarrow & \mathrm{Hom}_{\mathcal{B}}(B, F(I)) & & & & \\
& & & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(G(B), R) & \twoheadrightarrow & \mathrm{Ext}^1_{\mathcal{A}}(G(B), A) \\
& & & & \| & & \\
& & & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(B, F(R)) & \twoheadrightarrow & \mathrm{Ext}^1_{\mathcal{B}}(B, F(A))
\end{array}$$

Hence the lemma follows.

4 The inverse functor

4.1 In order to prove that S_n defines an equivalence of categories we come up with an inverse functor. First some generalities. Let $\mathcal{A}$ be an additive category and let R be a ring. A left R-object in $\mathcal{A}$ is an object $A \in \mathcal{A}$ together with a homomorphism of rings $\rho: R \to \mathrm{Hom}_{\mathcal{A}}(A, A)$. (Cf. Mitchell [6], Ch. II, §13.) For instance the objects of $\mathcal{C}_n(R)$ are R-objects. Further if $A \in \mathcal{A}$ is any left R-object, then the abelian group $\mathrm{Hom}_{\mathcal{A}}(B, A)$ gets in a canonical way a left R-module structure. If $\alpha \in \mathrm{Hom}_{\mathcal{A}}(B, B')$ then $\mathrm{Hom}_{\mathcal{A}}(\alpha, A)$ is a morphism of left R-modules. In particular we have a left exact contravariant functor

$$T_n: \mathcal{C}_n \to \mathrm{Mod}_l(\mathcal{D}_n) \qquad E \mapsto \mathrm{Hom}_{\mathcal{C}_n}(E, \mathcal{F}_n).$$

4.2 In order to study this functor T_n we first consider the operation $\mathcal{C}$. We recall that for any additive category $\mathcal{A}$ we defined $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(E, F)$ for all $E, F \in \mathcal{A}$ in such a way that the following sequence of abelian groups is exact

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(E, F) \hookrightarrow \mathrm{Hom}_{\mathcal{A}}(E_0, F_0) \times \mathrm{Hom}_{\mathcal{A}}(E_1, F_1) \\
\longrightarrow \mathrm{Hom}_{\mathcal{A}}(E_0, F_1) \times \mathrm{Hom}_{\mathcal{A}}(E_1, F_0) \\
(f, g) \mapsto (u_F f - g u_E, f v_E - v_F g)
\end{aligned}$$

This observation enables us to prove the following.

Lemma 4. *Let A be a $\mathbb{C}$-algebra, B an A-algebra. Let $\mathcal{A}$ be an abelian subcategory of $\mathcal{C}_0$. Suppose $G\colon \mathrm{Mod}_l(B) \to \mathrm{Mod}_l(A)$ is a left exact functor. Let $\theta_0\colon G(\mathrm{Hom}_{\mathbb{C}}(\cdot, \cdot)) \to \mathrm{Hom}_{\mathbb{C}}(\cdot, G(\cdot))$ be a natural transformation (resp. equivalence) of bifunctors from $\mathcal{A} \times \mathrm{Mod}_l(B)$ to $\mathrm{Mod}_l(A)$. Then there is a natural transformation (resp. equivalence)*

$$\theta_n\colon G(\mathrm{Hom}_{\mathcal{C}_n}(\cdot, \cdot)) \to \mathrm{Hom}_{\mathcal{C}_n}(\cdot, \mathcal{C}_n(G)(\cdot))$$

of bifunctors from $\mathcal{C}_n(\mathcal{A}) \times \mathcal{C}_n(B)$ to $\mathrm{Mod}_l(A)$.

4.3 Finally let us define for each $\alpha \in J \cup \{1\}$ a functor $K_\alpha\colon \mathcal{C}_n \to \mathcal{C}_{n-1}$ as follows:

$$\begin{aligned} K_0(E, F, u, v) &:= \operatorname{Ker} u \\ K_1(E, F, u, v) &:= \operatorname{Ker} v \\ K_\alpha(E, F, u, v) &:= \operatorname{Ker}(vu - \alpha 1), \qquad \text{all } \alpha \in J - \{0\}. \end{aligned}$$

Clearly K_α is left exact for all $\alpha \in J \cup \{1\}$. Furthermore, as one easily verifies, L_α is a left adjoint of K_α.

4.4 Before we return to the functor T_n we need a description of the simple objects of $\widetilde{\mathcal{C}}_n$. We leave it to the reader to verify

Lemma 5. *(i) Every $F \in \widetilde{\mathcal{C}}_n$, $F \neq 0$, has a subobject of the form $L_\alpha E$, for some $\alpha \in J \cup \{1\}$ and some simple object $E \in \widetilde{\mathcal{C}}_{n-1}$.*
(ii) The simple objects in $\widetilde{\mathcal{C}}_n$ are those of the form $L_\alpha E$ for some $\alpha \in J \cup \{1\}$ and some simple object $E \in \widetilde{\mathcal{C}}_{n-1}$.
(iii) Every object in $\widetilde{\mathcal{C}}_n$ has a finite length.

4.5 Now we are ready to prove

Proposition 2. *T_n restricts to a contravariant exact functor $\widetilde{\mathcal{C}}_n \to \mathrm{Mod}_l(\mathcal{D}_n)_{\mathrm{hr}}^{x_1 \cdots x_n}$, that takes simple objects to simple objects (and is still denoted T_n).*

Proof. By induction on n. We may assume $F \in \widetilde{\mathcal{C}}_n$ to be simple. Let us say $F = L_\alpha E$, $\alpha \in J \cup \{1\}$, $E \in \widetilde{\mathcal{C}}_{n-1}$ simple. Write $L = L_\alpha$, $K = K_\alpha$, $P = P_\alpha$. For each $i \in \{0, 1\}$ $\mathcal{O} \otimes_{\mathcal{O}_{(n)}} \mathcal{F}_{(n)}(i)$ is a flat $\mathcal{O}_{n-1}$-module. Hence in virtue of Lemma 1 we get

$$K(\mathcal{F}_n) = \mathcal{F}_{n-1} \underset{\mathcal{O}_{n-1}}{\otimes} \Big(\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathcal{D}_{(n)}/(q_n(\alpha))\Big) = \mathcal{C}_{n-1}(P)(\mathcal{F}_{n-1}).$$

Apply Lemma 4 to the well-known equivalence $\mathrm{Hom}_{\mathbb{C}}(F, M) \otimes_{\mathcal{O}} N \cong \mathrm{Hom}_{\mathbb{C}}(F, M \otimes_{\mathcal{O}} N)$, where F is a finite dimensional $\mathbb{C}$-vector space, $M \in \mathrm{Mod}(\mathcal{O})$, N a flat $\mathcal{O}$-module, gives

$$\begin{aligned} T_n(LE) &\cong \mathrm{Hom}_{\mathcal{C}_{n-1}}(E, K(\mathcal{F}_n)) \cong \mathrm{Hom}_{\mathcal{C}_{n-1}}(E, \mathcal{C}_{n-1}(P)(\mathcal{F}_{n-1})) \\ &\cong PT_{n-1}E \end{aligned}$$

In fact these isomorphisms are $\mathcal{D}_n$-linear. To exhibit the exactness of T_n we use

Lemma 6. *Let* $R\colon \mathrm{Mod}_l(\mathcal{D}_n) \to \mathcal{C}_0$ *be an exact functor. Then, for all* $E \in \tilde{\mathcal{C}}_n$, $\mathrm{Ext}^1_{\mathcal{C}_n}\big(E, \mathcal{C}_n(R)(\mathcal{F}_n)\big) = 0$.

PROOF. According to Lemma 1 $\mathcal{U}_{(n)}$, $\mathcal{V}_{(n)}$ and $\mathcal{V}_{(n)}\mathcal{U}_{(n)} - \alpha 1$ are surjective, hence $\mathrm{R}^1 K(\mathcal{F}_n) = 0$. Since R is exact it commutes with K and $\mathrm{R}^1 K\big(\mathcal{C}_n(R)(\mathcal{F}_n)\big) = 0$. Hence according to Lemma 3 it follows that

$$\mathrm{Ext}^1_{\mathcal{C}_n}\big(LE, \mathcal{C}_n(R)(\mathcal{F}_n)\big) \cong \mathrm{Ext}^1_{\mathcal{C}_n}\big(E, \mathcal{C}_{n-1}(RP)(\mathcal{F}_{n-1})\big) = 0.$$

Remark According to Mitchell [6], Ch. VI, Corollary 4.2 (with $R = \mathbb{C}$), $\mathcal{C}_n$ is equivalent to a category of right modules over a certain ring of endomorphisms. (Recall, cf. §1, that $\mathcal{C}_n$ is a functor category of the kind mentioned in this Corollary.) Hence $\mathcal{C}_n$ has enough injectives.

5 The equivalence of categories

In the preceding pages we have shown the existence of two contravariant exact functors

$$S_n\colon \mathrm{Mod}_l(\mathcal{D}_n)^{x_1 \quad x_n}_{\mathrm{hr}} \to \tilde{\mathcal{C}}_n \qquad T_n\colon \tilde{\mathcal{C}}_n \to \mathrm{Mod}_l(\mathcal{D}_n)^{x_1 \quad x_n}_{\mathrm{hr}}$$

By some formal considerations it follows now that S_n defines an equivalence of categories with inverse T_n.

Proposition 3. S_n *and* T_n *are inverse to each other.*

PROOF. First we mention the natural equivalence of $\mathbb{C}$-vector spaces $\mathrm{Hom}_{\mathbb{C}}\big(E, \mathrm{Hom}_{\mathcal{D}_n}(M,N)\big) \cong \mathrm{Hom}_{\mathcal{D}_n}\big(M, \mathrm{Hom}_{\mathbb{C}}(E,N)\big)$, where $E \in \mathcal{C}_0$, M, $N \in \mathrm{Mod}_l(\mathcal{D}_n)$. By Lemma 4 there results a natural equivalence

$$\mathrm{Hom}_{\mathcal{C}_n}\big(E, \mathcal{C}_n(H_n)(M,F)\big) \cong \mathrm{Hom}_{\mathcal{D}_n}\big(M, \mathrm{Hom}_{\mathcal{C}_n}(E,F)\big)$$

where $E \in \mathcal{C}_n$, $M \in \mathrm{Mod}_l(\mathcal{D}_n)$, $F \in \mathcal{C}_n(\mathcal{D}_n)$. So in particular we get a natural equivalence

$$\mathrm{Hom}_{\mathcal{C}_n}\big(E, S_n(M)\big) \cong \mathrm{Hom}_{\mathcal{D}_n}\big(M, T_n(E)\big)$$

where $E \in \tilde{\mathcal{C}}_n$, $M \in \mathrm{Mod}_l(\mathcal{D}_n)^{x_1 \quad x_n}_{\mathrm{hr}}$. Or, working in the dual category $\mathcal{C}_n^{\circ}$,

$$\mathrm{Hom}_{\mathcal{C}_n^{\circ}}\big(S_n^{\circ}(M), E\big) \cong \mathrm{Hom}_{\mathcal{D}_n}\big(M, T_n^{\circ}(E)\big).$$

Hence S_n° is a left adjoint of T_n°. This gives rise to natural transformations $\psi\colon 1 \to T_n^{\circ}S_n^{\circ} = T_nS_n$, $\phi\colon S_n^{\circ}T_n^{\circ} \to 1$ and dual $\phi^{\circ}\colon 1 \to S_nT_n$. Both S_n° and T_n° are exact and take simple objects to simple objects. Hence in particular both functors are faithful. Hence $\psi(M)$ and $\phi^{\circ}(E)$ are monomorphisms if $M \in \mathrm{Mod}_l(\mathcal{D}_n)^{x_1 \quad x_n}_{\mathrm{hr}}$, $E \in \tilde{\mathcal{C}}_n$. Hence both are isomorphisms in case the object is simple. So, by induction on the length, ψ and ϕ° are equivalences.

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Chapter III

$\mathcal{D}$-MODULES WITH SUPPORT ON A CURVE

by

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0 Introduction

0.1 In [10] (see also [2], [4]) Kashiwara proves the following

Theorem. *Let X be a complex manifold and Y a closed submanifold. Let $i: Y \hookrightarrow X$ be the inclusion. The direct image functor i_+ establishes an equivalence between the category of $\mathcal{D}_Y$-modules and the category of $\mathcal{D}_X$-modules with support contained in Y.*

What happens in case Y is no longer smooth, but only a closed subvariety? Following [9], [5], [14], [13] one defines the ring $\mathcal{D}_Y$ of differential operators on Y. In case Y is non-singular this definition coincides with the usual one i.e., the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_Y)$ generated by $\mathcal{O}_Y$ and $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_Y)$.

Bloom [5], [6], Vigué [14], Bernstein a.o. [1], and recently Smith and Stafford [13] in the algebraic case, investigated these kind of rings and showed that in general they fail to have some nice properties such as being left or right noetherian. However, as already Bloom and Vigué noticed, in case Y is a curve the situation is more pleasant. Investigations have culminated in a nice

Theorem. (Smith and Stafford [13], Thm B.) *Let X be an affine curve and $\pi: \tilde{X} \to X$ the normalization. Assume π is injective. Then $\mathcal{D}(X)$ is Morita equivalent to $\mathcal{D}(\tilde{X})$.*

It goes without saying that $\tilde{X}$ is non-singular, hence $\mathcal{D}(\tilde{X})$ is well-known. (See e.g., [3].) Using this we are able to modify Kashiwara's theorem as follows.

Theorem. *Let $(X, 0)$ be an irreducible germ of a curve in $(\mathbb{C}^n, 0)$. Then the category of $\mathcal{D}_{X,0}$-modules is equivalent to the category of $\mathcal{D}_{n,0}$-modules with support contained in X.*

0.2 In this paper we take a ringtheoretic point of view. $\mathcal{O}_n$ is the formal (resp. convergent) power series ring in n indeterminates over k, an algebraically closed field of characteristic zero (resp. $\mathbb{C}$). $\wp \subset \mathcal{O}_n$ is a prime ideal of height $n-1$. Our aim is to prove that the category of $\mathcal{D}_n$-modules with support contained in $V(\wp)$ is equivalent to the category of $\mathcal{D}(\mathcal{O}_n/\wp)$-modules and thus to the category of $\mathcal{D}_1$-modules.

In §1 we collect some facts concerning differential operators over a commutative k-algebra. In §2 we introduce the functors which are going to establish the required equivalence. We derive a necessary and sufficient condition for the equivalence to hold. In §3 we investigate "distribution" modules i.e., $\mathcal{D}(\mathcal{O}_n/\wp)$-modules with support at the origin. We exhibit the equivalence for these modules. In §4 we use Kashiwara's theorem for the regular case and the result of §3 to obtain that the afore mentioned condition in §2 is fulfilled. §5 contains an application. We show that for irreducible $f \in \mathcal{O}_2 = \mathcal{O}$ the left $\mathcal{D}$-module $\mathcal{O}_f/\mathcal{O}$ is simple. (Cf. [8], [15].)

We should like to thank Prof. S.P. Smith for the many valuable discussions during his short stay in Nijmegen. Much of the formalism and facts of differential operators as in §1 we learned from him.

1 Generalities on differential operators

1.1 Let A be a commutative k-algebra. Throughout this paper k will denote an algebraically closed field of characteristic zero. Let M and N be A-modules. One defines $\mathcal{D}_A^n(M,N)$, the space of k-linear differential operators from M to N of order $\le n$, inductively by $\mathcal{D}_A^{-1}(M,N) := 0$ and for $n \ge 0$

$$\mathcal{D}_A^n(M,N) := \{\theta \in \mathrm{Hom}_k(M,N) \mid [\theta, a] \in \mathcal{D}_A^{n-1}(M,N), \quad \text{all } a \in A\}.$$

Put

$$\mathcal{D}_A(M,N) := \bigcup_{n=0}^{\infty} \mathcal{D}_A^n(M,N).$$

$\mathcal{D}_A(M) := \mathcal{D}_A(M,M)$ is a k-subalgebra of $\mathrm{End}_k(M)$. $\mathcal{D}_A(M,N)$ is a $\mathcal{D}_A(N)$-$\mathcal{D}_A(M)$ bimodule. The module action is given by composition of maps. We refer the reader to the paper of Smith and Stafford [13], §1, where a nice survey of results on differential operators is given. The reader may also consult [9] or [11].

1.2 We would like to add the following observation:

Let M be an A-module of finite presentation. Then $\mathcal{D}_A(M,N) \cong \mathrm{Hom}_A(M, \mathcal{D}_A(A,N))$ as A-modules, where the A-module structure on $\mathcal{D}_A(A,N)$ is the one coming from the right $\mathcal{D}_A(A)$-structure.

The short proof runs as follows:

$$\mathcal{D}_A^n(M,N) \cong \mathrm{Hom}_A(P_A^n \underset{A}{\otimes} M, N) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(P_A^n, N)),$$

where $\mathrm{Hom}_A(P_A^n, N)$ is formed by viewing P_A^n as an A-module through the left action of A (see note below). $\mathrm{Hom}_A(P_A^n, N)$ is considered as an A-module through the right action of A on P_A^n. As M is finitely presented we may apply [7], §1, Prop. 8a and conclude

$$\varinjlim \mathcal{D}_A^n(M,N) \cong \mathrm{Hom}_A(M, \varinjlim \mathrm{Hom}_A(P_A^n, N)).$$

Note Let $\mu: A \otimes_k A \to A$ be the multiplication map $a \otimes b \mapsto ab$. Put $J_A = \operatorname{Ker}\mu$, $P_A^n = A \otimes_k A/J_A^{n+1}$. P_A^n has two structures of an A-module. Namely multiplication on the left, giving the "left" structure and multiplication on the right, giving the "right" structure. (See e.g., [9].) Observe that in case the P_A^n are projective A-modules of finite type, then

$$\mathcal{D}_A(A,N) = N \underset{A}{\otimes} \mathcal{D}_A(A)$$

and $\mathcal{D}_A(A)$ is a flat A-module. This occurs for instance when A is a regular k-algebra of finite type or when $A = \mathcal{O}_n$.

1.3 Due to the absence of an appropriate reference we mention the following: (See also [13], end of §4.)

Let $I \subset A$ be an ideal, then (by induction on the order of an operator)

$$\mathcal{D}_A(A, A/I) \subset \bigcup_{n=1}^{\infty} \{\, \theta \in \operatorname{Hom}_k(A, A/I) \mid \theta(I^n) = 0 \,\}.$$

Hence

$$\mathcal{D}_A(A, A/I) \cong \varinjlim \operatorname{Hom}_A\bigl(A/I^n, \mathcal{D}_A(A, A/I)\bigr)$$

i.e., $\operatorname{supp}\bigl(\mathcal{D}_A(A, A/I)\bigr) \subset V(I)$. In particular, if $\mathcal{M} \subset A$ is a maximal ideal such that $k \cong A/\mathcal{M}$, then as one easily verifies by induction on n:

$$\begin{aligned}\mathcal{D}_A(A, k) &\cong \bigcup_{n=1}^{\infty} \{\, \theta \in \operatorname{Hom}_k(A, k) \mid \theta(\mathcal{M}^n) = 0 \,\} \\ &\cong \varinjlim \operatorname{Hom}_k(A/\mathcal{M}^n, k).\end{aligned}$$

Hence in case A is a noetherian, local k-algebra with maximal ideal $\mathcal{M}$ such that $A/\mathcal{M} \cong k$ then, according to [7], exercise 32 of §1, $\mathcal{D}_A(A, k)$ is a dualizing module for A. So in particular $\mathcal{D}_A(A, k)$ is the injective hull of the A-module k. (This fact was kindly pointed out to us by S.P. Smith.)

Note that we are considering $\mathcal{D}_A(A, A/I)$ as an A-module through it's right $\mathcal{D}(A)$-structure.

1.4 To finish this section we fix the setting for the rest of the paper. Let $n \in \mathbb{N}$, $n \neq 0$. $\mathcal{O} := \mathcal{O}_{n+1}$ denotes the formal (resp. convergent) power series ring in the indeterminates $x, x_1, \ldots, x_n$ over k (resp. $\mathbb{C}$). $\mathcal{O}_1$ denotes the formal (resp. convergent) power series ring in the indeterminate t over k (resp. $\mathbb{C}$). $\mathcal{O}_0$ denotes the formal (resp. convergent) power series ring in the indeterminate x over k (resp. $\mathbb{C}$).

$$\mathcal{D} := \mathcal{D}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) = \mathcal{O}[\partial, \partial_1, \ldots, \partial_n] \qquad \mathcal{D}_1 := \mathcal{D}_{\mathcal{O}_1}(\mathcal{O}_1, \mathcal{O}_1) = \mathcal{O}_1[\partial_t]$$

$\mathcal{M} := (x, x_1, \ldots, x_n)$ denotes the maximal ideal in $\mathcal{O}$.

Let $\wp \subset \mathcal{O}$ be a prime ideal of height n such that $x \notin \wp$. $A := \mathcal{O}/\wp$ is a local ring of dimension 1 with maximal ideal $\overline{\mathcal{M}} = \mathcal{M}/\wp$. The normalization of A i.e., the integral closure of A in its field of fractions, is $\mathcal{O}_1$. In the sequel we will identify $\mathcal{O}/\mathcal{M} = k = A/\overline{\mathcal{M}}$, $k = \mathcal{O}_1/t\mathcal{O}_1$. We fix once and for all some canonical maps

$$\pi: \mathcal{O} \twoheadrightarrow A \quad \tau: \mathcal{O} \twoheadrightarrow k \quad \overline{\tau}: A \twoheadrightarrow k$$

with $\overline{\tau}\pi = \tau$.
We have

$$\mathcal{D}(A) := \mathcal{D}_A(A, A) \cong I(\wp\mathcal{D})/\wp\mathcal{D} \cong \mathrm{End}_{\mathcal{D}}(\mathcal{D}/\wp\mathcal{D})$$

where

$$\begin{aligned} I(\wp\mathcal{D}) &= \text{the idealizer of } \wp\mathcal{D} \text{ in } \mathcal{D} \\ &:= \{ D \in \mathcal{D} \mid D(\wp) \subset \wp \}. \end{aligned}$$

See [13], 1.6 or [5], [11], [14]. The identification arises as follows. If $D \in \mathcal{D}$ such that $D(\wp) \subset \wp$, then $\pi D(\wp) = 0$. Hence it induces a k-linear map $\widetilde{D}: A \to A$ such that $\pi D = \widetilde{D}\pi$. In fact $\widetilde{D} \in \mathcal{D}(A)$. Note that 1.2 implies that, at least as left $\mathcal{D}(A)$-modules,

$$\mathcal{D}(A) = \mathcal{D}_{\mathcal{O}}(A, A) \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\wp, \mathcal{D}_{\mathcal{O}}(\mathcal{O}, A)) = \mathrm{End}_{\mathcal{D}}(\mathcal{D}/\wp\mathcal{D}).$$

1.5 Morita equivalence. The $\mathcal{D}(A)$-$\mathcal{D}_1$ bimodule $P := \mathcal{D}_A(\mathcal{O}_1, A)$ is isomorphic to a right ideal in $\mathcal{D}_1$. Hence P is projective and a generator because $\mathrm{gl.dim}\,\mathcal{D}_1 = 1$. The rings $\mathcal{D}(A)$ and $\mathcal{D}_1$ are Morita equivalent if the natural map

$$P \underset{\mathcal{D}_1}{\otimes} \mathcal{O}_1 \to A, \qquad p \otimes f \mapsto p(f)$$

is surjective. (See [13], Prop. 3.3.) As P is a left $\mathcal{D}(A)$-module we only need verify that 1 is in the image. That is the case; arguing as in [13] we get $\mathrm{Ann}_A(\mathcal{O}_1/A) \supset t^N\mathcal{O}_1$, for some $N \in \mathbf{N}$. Put $p = \prod_{j=1}^{N-1}(t\partial - j)$, then $p(t^j) = 0$ for all $j \in \{1, \ldots, N-1\}$, $p(t^N) = (N-1)!t^N$ and $p(1) = (-1)(-2)\ldots(-N+1)$, thus $p \in P$. So $\mathcal{D}(A)$ and $\mathcal{D}_1$ are Morita equivalent.

The functor

$$N \mapsto N \underset{\mathcal{D}(A)}{\otimes} P$$

from Mod-$\mathcal{D}(A)$, the category of right $\mathcal{D}(A)$-modules, to Mod-$\mathcal{D}_1$, the category of right $\mathcal{D}_1$-modules, is an equivalence of categories. The inverse functor is $M \mapsto \mathrm{Hom}_{\mathcal{D}_1}(P, M)$.

Similarly

$$N \mapsto \mathrm{Hom}_{\mathcal{D}(A)}(P, N) = P^* \underset{\mathcal{D}(A)}{\otimes} N$$

gives an equivalence between $\mathcal{D}(A)$-Mod, the category of left $\mathcal{D}(A)$-modules and $\mathcal{D}_1$-Mod, the category of left $\mathcal{D}_1$-modules. One has

$$P^* := \mathrm{Hom}_{\mathcal{D}(A)}(P, \mathcal{D}(A)) \cong \mathcal{D}_A(A, \mathcal{O}_1).$$

The reader is referred to [13], §2, 3 for the details.

Remark Let $(X,0) \subset (\mathbb{C}^n,0)$ be a germ of a curve. Let $\pi: \tilde{X} \to X$ be the normalization. According to [13], Thm 3.13: If $\#\pi^{-1}(0) = 1$, then $\mathcal{D}_{\tilde{X},\pi^{-1}(0)}$ is Morita equivalent to $\mathcal{D}_{X,0}$. Now $\#\pi^{-1}(0) =$ number of the irreducible components of the germ $(X,0)$. Hence in case $(X,0)$ is an irreducible germ of a curve $\mathcal{D}_{X,0}$ is Morita equivalent to $\mathcal{D}_{\mathbb{C},0}$. (Cf. [14], II.)

2 The main theorem

2.1 As we have mentioned in the introduction we want to compare $\mathcal{D}$-modules with support contained in $V(\wp)$ and $\mathcal{D}(A)$-modules. Let $\mathrm{Mod}_\wp$-$\mathcal{D}$ denote the category of right $\mathcal{D}$-modules M such that, considered as $\mathcal{O}$-module $\mathrm{supp}(M) \subset V(\wp)$. It is a full abelian subcategory of Mod-$\mathcal{D}$, which is closed under extensions. In case A is regular i.e., $V(\wp)$ is non-singular, the A-module $A \otimes_{\mathcal{O}} \mathcal{D}$ can be given the structure of a left $\mathcal{D}(A)$-module. (See e.g., [2], [3].) This enables one to define inverse images of $\mathcal{D}$-modules. Now $A \otimes_{\mathcal{O}} \mathcal{D} = \mathcal{D}_{\mathcal{O}}(\mathcal{O}, A)$ and it is not difficult to show that the above mentioned left $\mathcal{D}(A)$-module structure on $A \otimes_{\mathcal{O}} \mathcal{D}$, in case A is regular, coincides with the usual left $\mathcal{D}(A)$-module structure on $\mathcal{D}_{\mathcal{O}}(\mathcal{O}, A)$.

This motivates the following

Definition. $$B := \mathcal{D}_{\mathcal{O}}(\mathcal{O}, A).$$

B is a $\mathcal{D}(A)$-$\mathcal{D}$ bimodule and as we already saw, $\mathrm{supp}_{\mathcal{O}}(B) \subset V(\wp)$, where B is considered as an $\mathcal{O}$-module via the action of $\mathcal{D}$. Moreover the natural inclusion

$$\varinjlim_n \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\wp^n, N \underset{\mathcal{D}(A)}{\otimes} B) \hookrightarrow N \underset{\mathcal{D}(A)}{\otimes} B$$

is an isomorphism for all $N \in$ Mod-$\mathcal{D}(A)$. So $N \otimes_{\mathcal{D}(A)} B$ is a right $\mathcal{D}$-module with $\mathrm{supp}(N \otimes_{\mathcal{D}(A)} B) \subset V(\wp)$. This justifies the following

Definition.

$$i_+ : \text{Mod-}\mathcal{D}(A) \to \mathrm{Mod}_\wp\text{-}\mathcal{D}, \quad N \mapsto N \underset{\mathcal{D}(A)}{\otimes} B;$$
$$i^+ : \text{Mod-}\mathcal{D} \to \text{Mod-}\mathcal{D}(A), \quad M \mapsto \mathrm{Hom}_{\mathcal{D}}(B, M).$$

We make the following observations:
- i_+ is a left adjoint of i^+.
- i^+ is left exact; i_+ is right exact.
- If $M \in \mathrm{Mod}_\wp$-$\mathcal{D}$, $M \neq 0$, then $i^+(M) \neq 0$. This is obvious since $i^+M = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\wp, M)$.

2.2 Theorem. *i_+ defines an equivalence between the category of right $\mathcal{D}(A)$-modules and the category of right $\mathcal{D}$-modules with support contained in $V(\wp)$.*

2.2.1 In the remainder of the paper we shall be mainly concerned with the proof of this theorem. In fact the theorem follows directly from Proposition 2 and the Corollary to Proposition 5. As a first step we have

Proposition 1. *i_+ is an exact, faithful functor.*

PROOF. We want to prove that i_+ is an exact functor. Or what amounts to the same B is a flat $\mathcal{D}(A)$-module. Now as we already saw $B = \varinjlim B_n$, where

$$B_n := \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\wp^n, B).$$

By induction on n we show that each B_n is a projective left $\mathcal{D}(A)$-module. This certainly implies the flatness of B. For $n = 1$ we have

$$B_1 = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\wp, B) = \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/\wp\mathcal{D}, \mathcal{D}/\wp\mathcal{D}) = \mathcal{D}(A),$$

hence B_1 is projective.

For each $n \in \mathrm{N}$ we have an exact sequence of $\mathcal{O}$-modules

$$\wp^n/\wp^{n+1} \hookrightarrow \mathcal{O}/\wp^{n+1} \twoheadrightarrow \mathcal{O}/\wp^n,$$

which gives rise to an exact sequence of left $\mathcal{D}(A)$-modules

$$B_n \hookrightarrow B_{n+1} \longrightarrow \mathrm{Hom}_{\mathcal{O}}(\wp^n/\wp^{n+1}, B).$$

Now $\wp^n/\wp^{n+1}$ is an $\mathcal{O}/\wp$-module of finite type. Hence we have a surjection

$$\bigoplus_{i=1}^{r} \mathcal{O}/\wp \twoheadrightarrow \wp^n/\wp^{n+1}$$

and an injection

$$\mathrm{Hom}_{\mathcal{O}}(\wp^n/\wp^{n+1}, B) \hookrightarrow \bigoplus_{i=1}^{r} \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\wp, B) = \bigoplus_{i=1}^{r} \mathcal{D}(A).$$

So B_{n+1}/B_n may be identified with a submodule of $\bigoplus_{i=1}^{r} \mathcal{D}(A)$. Since $\mathcal{D}(A)$ is Morita equivalent to $\mathcal{D}_1$, $\mathrm{gl.dim}\,\mathcal{D}(A) = 1$. This implies that every submodule of a projective $\mathcal{D}(A)$-module is itself projective. So B_{n+1}/B_n is a projective left $\mathcal{D}(A)$-module and we have a split exact sequence

$$B_n \hookrightarrow B_{n+1} \twoheadrightarrow B_{n+1}/B_n.$$

By induction on n, B_{n+1} is a projective left $\mathcal{D}(A)$-module. We can say even more, namely

$$B \cong B_1 \oplus B_2/B_1 \oplus B_3/B_2 \oplus \cdots .$$

So $B_1 \cong \mathcal{D}(A)$ is a direct sum factor of B and this implies

$$N \underset{\mathcal{D}(A)}{\otimes} B = 0 \qquad \text{iff} \qquad N = 0.$$

Hence i_+ is faithful.

2.2.2 Our aim is to show that i_+ defines an equivalence of categories. Now Mod_p-$\mathcal{D}$ is closed under extensions in Mod-$\mathcal{D}$, hence we should have

$$\mathrm{Ext}^1_{\mathcal{D}}(B, i_+N) \cong \mathrm{Ext}^1_{\mathcal{D}(A)}(\mathcal{D}(A), N) = 0, \quad \text{for all } N \in \text{Mod-}\mathcal{D}(A).$$

We claim that this is also sufficient. Let us first mention the existence of natural transformations $\eta\colon 1 \to i^+i_+$, $\varepsilon\colon i_+i^+ \to 1$, arising from the adjointness of i_+ and i^+.

Proposition 2. *Assume* $\mathrm{Ext}^1_{\mathcal{D}}(B, i_+N) = 0$, *for all* $N \in$ Mod-$\mathcal{D}(A)$. *Then* i_+ *is an equivalence of categories.*

Proof. Let N be a right $\mathcal{D}(A)$-module. Since $\mathrm{gl.dim}\,\mathcal{D}(A) = 1$, N has a projective resolution of length 1

$$P_1 \hookrightarrow P_0 \twoheadrightarrow N.$$

Applying i^+i_+ we get a commutative diagram with exact rows

$$\begin{array}{ccccccc}
i^+i_+P_1 & \hookrightarrow & i^+i_+P_0 & \longrightarrow & i^+i_+N & \longrightarrow & \mathrm{Ext}^1_{\mathcal{D}}(B, i_+P_1) = 0 \\
\uparrow{\scriptstyle \eta(P_1)} & & \uparrow{\scriptstyle \eta(P_0)} & & \uparrow{\scriptstyle \eta(N)} & & \\
P_1 & \hookrightarrow & P_0 & \twoheadrightarrow & N & & .
\end{array}$$

Now $\eta(P_1)$ and $\eta(P_0)$ are isomorphisms, [7], §6, Prop. 7. Hence $\eta(N)$ is an isomorphism. Hence η is an equivalence.

Furthermore we have for any $M \in \mathrm{Mod}{-}\mathcal{D}$ a composition of maps

$$i^+M \xrightarrow{\eta(i^+M)} i^+i_+i^+M \xrightarrow{i^+\varepsilon(M)} i^+M$$

$$i^+\varepsilon(M) \circ \eta(i^+M) = 1.$$

Since $\eta(i^+M)$ is bijective, $i^+\varepsilon(M)$ is bijective. Hence $i^+(\mathrm{Ker}\,\varepsilon(M)) = 0$, implying $\mathrm{Ker}\,\varepsilon(M) = 0$ because $\mathrm{Ker}\,\varepsilon(M)$ is a submodule of i_+i^+M, hence $\mathrm{Ker}\,\varepsilon(M) \in \mathrm{Mod}_p$-$\mathcal{D}$. Consider

$$i_+i^+M \xrightarrow{\varepsilon(M)} M \twoheadrightarrow \mathrm{Coker}\,\varepsilon(M).$$

Applying i^+ yields an exact sequence

$$i^+i_+i^+M \xhookrightarrow{i^+\varepsilon(M)} i^+M \longrightarrow i^+\,\mathrm{Coker}\,\varepsilon(M) \longrightarrow \mathrm{Ext}^1_{\mathcal{D}}(B, i_+i^+M) = 0.$$

Hence $i^+\,\mathrm{Coker}\,\varepsilon(M) = 0$ because $i^+\varepsilon(M)$ is surjective. Now if $M \in \mathrm{Mod}_p$-$\mathcal{D}$, then $\mathrm{Coker}\,\varepsilon(M) \in \mathrm{Mod}_p$-$\mathcal{D}$. It follows that $\mathrm{Coker}\,\varepsilon(M) = 0$. This proves that $\varepsilon(M)$ is an isomorphism for all $M \in \mathrm{Mod}_p$-$\mathcal{D}$.

2.3 So we see that a necessary and sufficient condition for i_+ to be an equivalence is the vanishing of the $\mathrm{Ext}^1_{\mathcal{D}}(B, i_+N)$. Later we will see that it suffices to show $\mathrm{Ext}^k_{\mathcal{D}}(B,B) = 0$ for all $k \in \{1,\dots,n\}$. Even $k \in \{1,2\}$ suffices.

3 The module $\mathcal{D}_A(A,k)$

3.1 Before we proceed, we focus our attention on a $\mathcal{D}(A)$-module with support $\{\overline{\mathcal{M}}\}$ namely $\mathcal{D}_A(A,k)$. We already mentioned in the introduction (1.2) that, for A-modules of finite presentation, one has an isomorphism of A-modules $\mathcal{D}_A(M,N) \cong \mathrm{Hom}_A\big(M, \mathcal{D}_A(A,N)\big)$. It follows immediately that

$$i^+\big(\mathcal{D}_{\mathcal{O}}(\mathcal{O},k)\big) \cong \mathrm{Hom}_{\mathcal{O}}\big(A, \mathcal{D}_{\mathcal{O}}(\mathcal{O},k)\big) \cong \mathcal{D}_{\mathcal{O}}(A,k) = \mathcal{D}_A(A,k).$$

This means that we have a bijective map

$$\phi: \mathcal{D}_A(A,k) \to i^+\big(\mathcal{D}_{\mathcal{O}}(\mathcal{O},k)\big),$$

which is A-linear. It is straightforward to check that for every $D \in \mathcal{D}_A(A,k)$, $\phi(D) \in i^+\big(\mathcal{D}_{\mathcal{O}}(\mathcal{O},k)\big) = \mathrm{Hom}_{\mathcal{D}}\big(B, \mathcal{D}_{\mathcal{O}}(\mathcal{O},k)\big)$ is the $\mathcal{D}$-linear map

$$E \mapsto DE, \qquad \text{for all } E \in B = \mathcal{D}_{\mathcal{O}}(\mathcal{O},A).$$

Hence ϕ is a right $\mathcal{D}(A)$-linear isomorphism.

3.2 Let $I \subset \mathcal{O}$ be an ideal containing $\wp$ and let $\overline{I} = I/\wp$ the corresponding ideal in A. Then

$$i_+\big(\mathcal{D}(A)/\overline{I}\mathcal{D}(A)\big) = A/\overline{I} \otimes_A \mathcal{D}_{\mathcal{O}}(\mathcal{O},A) = \mathcal{D}_{\mathcal{O}}(\mathcal{O},\mathcal{O}/I).$$

Applied to $I = \mathcal{M}$ this gives

$$i_+\big(\mathcal{D}(A)/\overline{\mathcal{M}}\mathcal{D}(A)\big) = \mathcal{D}_{\mathcal{O}}(\mathcal{O},k).$$

The faithfulness of i_+ and the fact that $\mathcal{D}_{\mathcal{O}}(\mathcal{O},k)$ is a simple right $\mathcal{D}$-module imply that also $\mathcal{D}(A)/\overline{\mathcal{M}}\mathcal{D}(A)$ is a simple right $\mathcal{D}(A)$-module. Hence the natural map

$$\mathcal{D}(A)/\overline{\mathcal{M}}\mathcal{D}(A) \to \mathcal{D}_A(A,k)$$

is injective. The surjectivity is established by the following

Lemma 1. *$\mathcal{D}_A(A,k)$ is a simple right $\mathcal{D}(A)$-module.*

PROOF. Consider the canonical map

$$(\mathcal{D}_1/t\mathcal{D}_1)\underset{\mathcal{D}_1}{\otimes} P^* = \mathcal{D}_{\mathcal{O}_1}(\mathcal{O}_1,k)\underset{\mathcal{D}_1}{\otimes}\mathcal{D}_A(A,\mathcal{O}_1)\to\mathcal{D}_A(A,k).$$

Clearly this map is non zero and hence injective because $\mathcal{D}_1/t\mathcal{D}_1$ is a simple right $\mathcal{D}_1$-module. It remains to show the surjectivity. According to 1.5 $P\otimes_{\mathcal{D}_1}P^*\cong\mathcal{D}(A)$. Hence $1=\sum_{\alpha\in I}p_\alpha q_\alpha$ (for some finite set I), with $p_\alpha\in P=\mathcal{D}_A(\mathcal{O}_1,A)$, $q_\alpha\in P^*=\mathcal{D}_A(A,\mathcal{O}_1)$, for all $\alpha\in I$. Now let $\theta\in\mathcal{D}_A(A,k)$. Then $\theta p_\alpha\in\mathcal{D}_{\mathcal{O}_1}(\mathcal{O}_1,k)$, for all $\alpha\in I$ and

$$\sum_{\alpha\in I}(\theta p_\alpha)\otimes q_\alpha\mapsto\theta\Big(\sum_{\alpha\in I}p_\alpha q_\alpha\Big)=\theta.$$

Corollary. $i^+i_+\big(\mathcal{D}_A(A,k)\big)\cong\mathcal{D}_A(A,k)$.

3.3 Remark In case of a right ideal $I\subset\mathcal{D}(A)$, $I=(A_1,\ldots,A_m)\mathcal{D}(A)$ one finds a right ideal $J\subset\mathcal{D}$ such that

$$i_+\big(\mathcal{D}(A)/I\big)=\mathcal{D}/J.$$

One may argue as follows. Choose a finite presentation of $\mathcal{D}(A)/I$ i.e., an exact sequence

$$\mathcal{D}(A)^m\xrightarrow{\alpha}\mathcal{D}(A)\twoheadrightarrow\mathcal{D}(A)/I.$$

Apply i_+ and recall that $B=\mathcal{D}/\wp\mathcal{D}$; the map $i_+(\alpha)$ lifts to a map $\overline{\alpha}:\mathcal{D}^m\to\mathcal{D}$ to give a commutative diagram with exact rows

$$\begin{tikzcd}
B^m \arrow[r, "i_+(\alpha)"] & B \arrow[r, two heads] & i_+\big(\mathcal{D}(A)/I\big) \\
\mathcal{D}^m \arrow[u] \arrow[r, "\overline{\alpha}"] & \mathcal{D} \arrow[u] \arrow[r, two heads] & \mathcal{D}/\operatorname{Im}\overline{\alpha} \arrow[u, "\beta"']
\end{tikzcd}$$

and $\operatorname{Ker}\beta=\wp\mathcal{D}/(\operatorname{Im}\overline{\alpha}\cap\wp\mathcal{D})=(\wp\mathcal{D}+\operatorname{Im}\overline{\alpha})/\operatorname{Im}\overline{\alpha}$. Furthermore $\operatorname{Im}\overline{\alpha}=(D_1,\ldots,D_m)\mathcal{D}$ with $\pi D_j=A_j\pi$ for all $j\in\{1,\ldots,m\}$. So $D_j(\wp)\subset\wp$ and D_j induces $A_j\in\mathcal{D}(A)$. One concludes that

$$i_+\big(\mathcal{D}(A)/I\big)=\mathcal{D}/J,$$

where $J=\wp\mathcal{D}+\operatorname{Im}\overline{\alpha}=\wp\mathcal{D}+(D_1,\ldots,D_m)\mathcal{D}$.

4 The modules $\mathrm{Ext}^i_{\mathcal{O}}(A,B)$

4.1 Let us now attack the problem of proving $\mathrm{Ext}^1_{\mathcal{D}}(B, i_+N) = 0$ for all right $\mathcal{D}(A)$-modules N. As noted before such a module N has a projective resolution of length 1

$$P_1 \hookrightarrow P_0 \twoheadrightarrow N.$$

Applying i^+i_+ one finds a commutative diagram with exact rows

$$\begin{tikzcd} i^+i_+P_1 \arrow[r, hook] & i^+i_+P_0 \arrow[r, "\alpha"] & i^+i_+N \\ P_1 \arrow[r, hook] \arrow[u, "\eta(P_1)"] & P_0 \arrow[r, two heads] \arrow[u, "\eta(P_0)"] & N \arrow[u, "\eta(N)"] \end{tikzcd} \tag{1}$$

and long exact sequences

$$\begin{aligned}(2)\quad \mathrm{Ext}^k_{\mathcal{D}}(B, i_+P_0) \to \mathrm{Ext}^k_{D}(B, i_+N) \to \\ \mathrm{Ext}^{k+1}_{\mathcal{D}}(B, i_+P_1) \to \mathrm{Ext}^{k+1}_{\mathcal{D}}(B, i_+P_0).\end{aligned}$$

For a projective right $\mathcal{D}(A)$-module P the obvious map

$$P \underset{\mathcal{D}(A)}{\otimes} \mathrm{Ext}^k_{\mathcal{D}}(B,B) \to \mathrm{Ext}^k_{\mathcal{D}}(B, P \underset{\mathcal{D}(A)}{\otimes} B)$$

is an isomorphism. ([7], §6, Prop. 7.) It follows that $\eta(N)$ is injective, establishing again that i_+ is faithful. Furthermore

$$\begin{aligned}(3)\quad \mathrm{Coker}\,\eta(N) &\cong \mathrm{Coker}\,\alpha \\ &\cong \mathrm{Ker}\big(\mathrm{Ext}^1_{\mathcal{D}}(B, i_+P_1) \to \mathrm{Ext}^1_{\mathcal{D}}(B, i_+P_0)\big) \\ &\cong \mathrm{Ker}\big(P_1 \underset{\mathcal{D}(A)}{\otimes} \mathrm{Ext}^1_{\mathcal{D}}(B,B) \to P_0 \underset{\mathcal{D}(A)}{\otimes} \mathrm{Ext}^1_{\mathcal{D}}(B,B)\big) \\ &\cong \mathrm{Tor}_1^{\mathcal{D}(A)}\big(N, \mathrm{Ext}^1_{\mathcal{D}}(B,B)\big).\end{aligned}$$

Observe that $\mathrm{Ext}^k_{\mathcal{D}}(B,B)$, for all $k \in \{1,\dots,n\}$, has a left and a right $\mathcal{D}(A)$-module structure. Since we are only interested here in the left one, we prefer to write $\mathrm{Ext}^k_{\mathcal{O}}(A,B)$ instead of $\mathrm{Ext}^k_{\mathcal{D}}(B,B)$. (Note that $\mathrm{Ext}^k_{\mathcal{O}}(A,B) \cong \mathrm{Ext}^k_{\mathcal{D}}(B,B)$; this uses $A \otimes_{\mathcal{O}} \mathcal{D} = \mathcal{D}_{\mathcal{O}}(\mathcal{O}, A) = B$ and the fact that $\mathcal{D}$ is a flat $\mathcal{O}$-module. [7], §6, Prop. 8.)

For notational convenience we introduce the left $\mathcal{D}(A)$-modules

$$C^k := \mathrm{Ext}^k_{\mathcal{O}}(A,B), \qquad \text{for all } k \in \{1,\dots,n\}.$$

The previous observations (2) and (3) can be reformulated as

$$\mathrm{Coker}\,\eta(N) = \mathrm{Tor}_1^{\mathcal{D}(A)}(N, C^1);$$

(4)

$$P_0 \underset{\mathcal{D}(A)}{\otimes} C^k \to \mathrm{Ext}^k_{\mathcal{D}}(B, i_+N) \to P_1 \underset{\mathcal{D}(A)}{\otimes} C^{k+1} \to P_0 \underset{\mathcal{D}(A)}{\otimes} C^{k+1},$$

are exact sequences.

4.2 Our aim is to prove $C^k = 0$ for all $k \in \{1, \dots, n\}$ and thereby establishing $\mathrm{Ext}^k_{\mathcal{D}}(B, i_+ N) = 0$ for all $k \in \{1, \dots, n\}$. The first result to this end is the following proposition, whose proof is postponed till the end of this section. We are still considering $C^k = \mathrm{Ext}^k_{\mathcal{O}}(A, B)$ as a left $\mathcal{D}(A)$-module. In particular C^k inherits the structure of an A-module. We discard the other A-module structure. Note that in forming $\mathrm{Ext}^k_{\mathcal{O}}(A, B)$, B is viewed as an $\mathcal{O}$-module through its right $\mathcal{D}$-module structure.

Proposition 3. $\mathrm{supp}_A(C^l) \subset \{\overline{\mathcal{M}}\}$, *for all* $l \in \{1, \dots, n\}$.

The proposition emphasizes that the left $\mathcal{D}(A)$-modules C^l are supported on the singular point $\{\overline{\mathcal{M}}\}$. Before we proceed we need a technical

Lemma 2. *Let* $p \in P^* = \mathcal{D}_A(A, \mathcal{O}_1)$, $m \in \mathbf{N}$ *and* $t^m \in \mathcal{D}(A)$. *There exist* $q \in P^*$, $N \in \mathbf{N}$ *such that* $t^N p = q t^m$.

PROOF. By induction on the order of p.
(i) $p \in \mathcal{D}^0_A(A, \mathcal{O}_1) = \mathrm{Hom}_A(A, \mathcal{O}_1)$. Take $q = p$, $N = m$.
(ii) $p \in \mathcal{D}^{d+1}_A(A, \mathcal{O}_1)$. Then $[p, t^m] \in \mathcal{D}^d_A(A, \mathcal{O}_1)$. Hence there exist $q \in P^*$, $N \in \mathbf{N}$ such that $t^N[p, t^m] = q t^m$. It follows that $t^N p t^m - t^{N+m} p = q t^m$ and thus $t^{N+m} p = (t^N p - q) t^m$.

Proposition 4. *Let* $l \in \{1, \dots, n\}$. *Assume*

$$\mathrm{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A, k), C^l) = 0.$$

Then $C^l = 0$.

PROOF. Let $l \in \{1, \dots, n\}$ and put $C = C^l$. Assume to the contrary that $C \neq 0$. It implies $P^* \otimes_{\mathcal{D}(A)} C \neq 0$. Hence there exist $p \in P^*$, $c \in C$, such that $p \otimes c \neq 0$. According to Proposition 3 some power M of $\overline{\mathcal{M}}$ annihilates c. Choose $m \in \mathbf{N}$ big enough such that $t^m \in \overline{\mathcal{M}}^M \subset \mathcal{D}(A)$. (This is possible because $\mathrm{Ann}_A(\mathcal{O}_1/A) \neq 0$.) By Lemma 2 we can find $q \in P^*$, $N \in \mathbf{N}$ such that $t^N p = q t^m$. It follows that $t^N(p \otimes c) = q t^m \otimes c = q \otimes t^m c = 0$. We arrive at the conclusion that $P^* \otimes_{\mathcal{D}(A)} C$ contains a non-zero element which is annihilated by t. A contradiction because

$$\begin{aligned}\mathrm{Ker}(t\cdot, P^* \underset{\mathcal{D}(A)}{\otimes} C) &= \mathrm{Tor}_1^{\mathcal{D}_1}(\mathcal{D}_1/t\mathcal{D}_1, P^* \underset{\mathcal{D}(A)}{\otimes} C) \\ &= \mathrm{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A, k), C)\end{aligned}$$

which by assumption vanishes.

So we are reduced to prove that all these Tor_1's vanish. This is the content of

Proposition 5. *For all $l \in \{1, \dots, n\}$*

$$\mathrm{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A,k), C^l) = 0.$$

Proof. By induction on l.
(i) $l = 1$:

$$\mathrm{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A,k), C^1) = \mathrm{Coker}\,\eta(\mathcal{D}_A(A,k)) = 0$$

by the Corollary at the end of §3.
(ii) Assume the proposition has been proven for $1, \dots, l$. By the previous Proposition $C^l = 0$. Applying the long exact sequence (4) with $k = l$ we get

$$\begin{aligned}\mathrm{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A,k), C^{l+1}) &\cong \mathrm{Ext}^l_{\mathcal{D}}(B, i_+\mathcal{D}_A(A,k)) \\ &\cong \mathrm{Ext}^l_{\mathcal{O}}(A, \mathcal{D}_{\mathcal{O}}(\mathcal{O},k)) = 0\end{aligned}$$

because $\mathcal{D}_{\mathcal{O}}(\mathcal{O},k)$ is an injective $\mathcal{O}$-module. (Cf. 1.3.)

Corollary. $\mathrm{Ext}^l_{\mathcal{D}}(B, i_+N) = 0$, *for all $l \in \{1, \dots, n\}$, $N \in$ Mod-$\mathcal{D}(A)$.*

4.3 Proof of Proposition 3.

Let $l \in \{1, \dots, n\}$. We have to show that $\mathrm{supp}_A(C^l) \subset \{\overline{\mathcal{M}}\}$. Now A is a local ring with only two prime ideals (0) and $\overline{\mathcal{M}}$. We need only show that $(0) \notin \mathrm{supp}(C^l)$. Now

$$A_{(0)} \underset{A}{\otimes} C^l = \mathcal{O}_\wp \underset{\mathcal{O}}{\otimes} C^l = \mathrm{Ext}^l_{\mathcal{O}}(A, \mathcal{O}_\wp \underset{\mathcal{O}}{\otimes} B).$$

Furthermore

$$\mathcal{O}_\wp \underset{\mathcal{O}}{\otimes} B = \mathcal{O}_\wp \underset{\mathcal{O}}{\otimes} \mathcal{D}_{\mathcal{O}}(\mathcal{O}, A) = \mathcal{D}_{\mathcal{O}_\wp}(\mathcal{O}_\wp, \mathcal{O}_\wp/\wp\mathcal{O}_\wp),$$

so it is a right $\mathcal{D}(\mathcal{O}_\wp)$-module and we are done if we can show that

$$\mathrm{Ext}^l_{\mathcal{O}_\wp}(\mathcal{O}_\wp/\wp\mathcal{O}_\wp, \mathcal{D}_{\mathcal{O}_\wp}(\mathcal{O}_\wp, \mathcal{O}_\wp/\wp\mathcal{O}_\wp)) = 0.$$

In fact we will prove that for any right $\mathcal{D}_\wp = \mathcal{O}_\wp[\partial, \partial_1, \dots, \partial_n]$-module M with $\mathrm{supp}(M) \subset V(\wp\mathcal{O}_\wp)$

$$\mathrm{Ext}^l_{\mathcal{O}_\wp}(\mathcal{O}_\wp/\wp\mathcal{O}_\wp, M) = 0, \qquad \text{for all } l \in \{1, \dots, n\}.$$

By a suitable change of coordinates we can manoeuvre ourselves into the following situation (Normalization theorem. See [3], Ch. 3, 3.22):

(i) $x \notin \wp$ (this we already assumed from start);

(ii) $\wp$ contains an element f_1 such that $f_1 \in \mathcal{O}_0[x_1]$ is an irreducible Weierstrass polynomial in x_1;

(iii) Let $\Delta \in \mathcal{O}_0$ be the discriminant of f_1. $\wp$ contains elements f_2, $\ldots, f_n$, such that for any $i \in \{2,\ldots,n\}$ $f_i = \Delta x_i - T_i$, for certain $T_i \in \mathcal{O}_0[x_1]$;

(iv) $\wp\mathcal{O}_\Delta = (f_1,\ldots,f_n)$;

(v) Notice that $\Delta \notin \wp$; hence $\wp\mathcal{O}_\wp = (f_1,\ldots,f_n)$ and $(f_1,\ldots,f_n)$ is a regular sequence in $\wp\mathcal{O}_\wp$;

(vi) $\partial_i(f_i) \in \mathcal{O}_\wp - \wp\mathcal{O}_\wp$ i.e., $\partial_i(f_i)$ is a unit in $\mathcal{O}_\wp$ for all $i \in \{1,\ldots,n\}$. $\partial_i(f_j) = 0$, if $i,j \in \{1,\ldots,n\}$, $i > j$.

By induction on d one proves

Sublemma. *Let M be a right $\mathcal{O}_\wp[\partial_1,\ldots,\partial_d]$-module with* $\mathrm{supp}(M) \subset V(f_1,\ldots,f_d)$. *Then $(f_1,\ldots,f_d)$ is an M-coregular sequence.*

(The reader is referred to [7], §9, No. 6 for the definition of coregular sequence.)

Proof. By induction on d $(f_1,\ldots,f_{d-1})$ is an M-coregular sequence. We need to verify that right multiplication by f_d is surjective on

$$M' := \mathrm{Ker}(f_1, M) \cap \ldots \cap \mathrm{Ker}(f_{d-1}, M).$$

Put $f := f_d$, $\delta := \partial_d$. The right $\mathcal{O}_\wp[\delta]$-module M' has $\mathrm{supp}(M') \subset V(f)$. Let $m \in M'$. Some power N of f annihilates m i.e., $mf^N = 0$. Hence $0 = mf^N\delta = \big(m\delta f - mN\delta(f)\big)f^{N-1}$. By induction on N we may assume that $m\delta f - mN\delta(f) = m_0 f$, for some $m_0 \in M'$. Hence $m = (m\delta - m_0)\big(N\delta(f)\big)^{-1} f$ because $\delta(f)$ is a unit in $\mathcal{O}_\wp$.

It follows that $(f_1,\ldots,f_n)$ is M-coregular for any right $\mathcal{D}_\wp$-module M with $\mathrm{supp}(M) \subset V(\wp\mathcal{O}_\wp)$. Hence for any such module

$$\mathrm{Ext}^l_{\mathcal{O}_\wp}(\mathcal{O}_\wp/\wp\mathcal{O}_\wp, M) = H_{n-l}\big((f_1,\ldots,f_n), M\big) = 0,$$

for all $l \in \{1,\ldots,n\}$ according to [7], §9, No. 7.

Remark No doubt the reader familiar with the theory of $\mathcal{D}$-modules will have recognized this proof as one for a special case of Kashiwara's theorem. (Cf. e.g., [4].)

4.4 Remark Let Y be an affine non-singular variety over k, an algebraically closed field of characteristic zero. Let $i: X \hookrightarrow Y$ be a closed subvariety of $\dim X = 1$. Assume that the normalization map $\pi: \widetilde{X} \to X$ is injective. Then $\mathrm{Mod}_X\text{-}\mathcal{D}(Y)$, the category of right $\mathcal{D}(Y)$-modules with support contained in X, is equivalent to $\mathrm{Mod}\text{-}\mathcal{D}(X)$.

Of course the "same" proof as above applies. Put

$$B := \mathcal{D}_{\mathcal{O}(Y)}(\mathcal{O}(Y), \mathcal{O}(X))$$

as in §2. Note that in case X is non-singular, B corresponds to the sheaf $\mathcal{D}_{X\to Y}$. Put

$$i_+ := \cdot \underset{\mathcal{D}(X)}{\otimes} B,$$
$$i^+ := \mathrm{Hom}_{\mathcal{D}(Y)}(B, \cdot),$$

a pair of adjoint functors. $\mathcal{D}(X)$ is Morita equivalent to $\mathcal{D}(\widetilde{X})$ (cf. [10], Thm B), which achieves that i_+ is faithful and exact.

By Kashiwara's theorem (cf. [2], [4])

$$\mathrm{Ext}^l_{\mathcal{D}(Y)}(B, B) = \mathrm{Ext}^l_{\mathcal{O}(Y)}(\mathcal{O}(X), B) =: C^l,$$

to be viewed as a left $\mathcal{D}(X)$-module, is for $l \neq 0$ supported at the singular points of X. Write $A := \mathcal{O}(X)$, $\widetilde{A} := \mathcal{O}(\widetilde{X})$. Let $x \in X$ be a singular point corresponding to a maximal ideal $\mathcal{M}$ in A. As π is injective, let $\widetilde{\mathcal{M}}$ be the unique maximal ideal of $\widetilde{A}$, which is above $\mathcal{M}$. Identify $k = A/\mathcal{M} = \widetilde{A}/\widetilde{\mathcal{M}}$. The $\mathcal{D}(X)$-module $\mathcal{D}_A(A, k)$ has support $\{\mathcal{M}\}$. As in Lemma 1 one obtains

$$\mathcal{D}_A(A, k) = \mathrm{Hom}_{\mathcal{D}(\widetilde{A})}(P, \mathcal{D}_{\widetilde{A}}(\widetilde{A}, k)),$$

where $P = \mathcal{D}(\widetilde{X}, X)$ is the bimodule establishing the Morita equivalence. One derives that

$$\begin{aligned} 0 = \mathrm{Coker}\,\eta(\mathcal{D}_A(A, k)) &= \mathrm{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A, k), C^1) \\ &= \mathrm{Tor}_1^{\mathcal{D}(\widetilde{A})}(\mathcal{D}_{\widetilde{A}}(\widetilde{A}, k), P^* \underset{\mathcal{D}(A)}{\otimes} C^1). \end{aligned}$$

Now $R := \widetilde{A}_{\widetilde{\mathcal{M}}}$ is a regular local ring, whose maximal ideal $\widetilde{\mathcal{M}}A_{\widetilde{\mathcal{M}}}$ is a principal ideal; say $t\widetilde{A}_{\widetilde{\mathcal{M}}} = \widetilde{\mathcal{M}}A_{\widetilde{\mathcal{M}}}$. Then

$$0 = \mathrm{Tor}_1^{\mathcal{D}(R)}(\mathcal{D}(R)/t\mathcal{D}(R), P^*_{\mathcal{M}} \otimes C^1_{\mathcal{M}}),$$

which means that $C^1_{\mathcal{M}} = (C^1)_x$, the stalk at x, has no t-torsion. But then $C^1_{\mathcal{M}} = 0$ because $\mathrm{supp}(C^1_{\mathcal{M}}) \subset \{\mathcal{M}A_{\mathcal{M}}\}$. By induction on l one derives that $C^l_{\mathcal{M}} = 0$.

4.5 Let $f \in \mathcal{O}_2$ be irreducible and let M be a right $\mathcal{D}_2$-module with $\mathrm{supp}(M) \subset V(f)$ i.e., $M_f = 0$. Then the Corollary to Proposition 5 implies that $\mathrm{Ext}^1_{\mathcal{D}_2}(B, M) = 0$. Hence the right multiplication by f on M is surjective.

5 An application

5.1 The application we have in mind is to show the following proposition. We will not dwell on its meaning but refer the reader to [8] or [15].

Proposition 6. *Let $f \in \mathcal{O}_2 =: \mathcal{O}$ be irreducible. Then $\mathcal{O}_f/\mathcal{O}$ is a simple left $\mathcal{D}$-module.*

PROOF. There exists a k-linear involution on $\mathcal{D}$—transposition of differential operators—determined by

(i) $a^t = a$, for all $a \in \mathcal{O}$;
(ii) $\partial_i^t = -\partial_i$, for all i;
(iii) $(PQ)^t = Q^t P^t$, for all $P, Q \in \mathcal{D}$.

Clearly this involution turns every left $\mathcal{D}$-module into a right one, denoted by M^t, and vice versa. This involution induces a k-algebra anti-isomorphism

$$I(\mathcal{D}f)/\mathcal{D}f \cong I(f\mathcal{D})/f\mathcal{D} = \mathcal{D}(A).$$

Furthermore there exists a k-algebra isomorphism

$$\phi: I(f\mathcal{D})/f\mathcal{D} \cong I(\mathcal{D}f)/\mathcal{D}f,$$

induced by the map: for all $D \in I(f\mathcal{D})$, $D \mapsto D'$, where $D' \in I(\mathcal{D}f)$ is the unique element such that $Df = fD'$.

Composing both maps gives a k-linear involution on $\mathcal{D}(A)$, which turns A into a right $\mathcal{D}(A)$-module, provisionally denoted by A^t. It is straightforward to check that

$$i^+((\mathcal{O}_f/\mathcal{O})^t) \cong A^t.$$

Since A is a simple left $\mathcal{D}(A)$-module (as $\mathcal{D}(A)$ is Morita equivalent to $\mathcal{D}_1$), it follows that A^t is a simple right $\mathcal{D}(A)$-module. Hence $(\mathcal{O}_f/\mathcal{O})^t$ is a simple right $\mathcal{D}$-module, which implies that $\mathcal{O}_f/\mathcal{O}$ is a simple left $\mathcal{D}$-module.

Remark The above proposition has been obtained independently by S.P. Smith [12] by a quite different method.

Appendix

For later use we derive a slight reformulation of our main theorem. Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be an irreducible germ of a curve; denote by $i:(X,0) \hookrightarrow (\mathbb{C}^{n+1},0)$ the inclusion. Let $\pi:(\widetilde{X},0) \to (X,0)$ be the normalization. Note that $(\widetilde{X},0) \cong (\mathbb{C},0)$ (cf. §1.5 Remark). Finally put $\tilde{\pi} := i \circ \pi$.

Theorem. *The direct image functor $\tilde{\pi}_+$ defines an equivalence between the category of right $\mathcal{D}_{\widetilde{X},0}$-modules and the category of right $\mathcal{D}_{\mathbb{C}^{n+1},0}$-modules with support contained in $(X,0)$.*

Proof. Put $\mathcal{O} = \mathcal{O}_{\mathbb{C}^{n+1},0}$, $A = \mathcal{O}_{X,0}$, $\widetilde{A} = \mathcal{O}_{\widetilde{X},0}$. We will write $\mathcal{D} = \mathcal{D}_{\mathbb{C}^{n+1},0}$, $\mathcal{D}(A) = \mathcal{D}_{X,0}$, $\mathcal{D}(\widetilde{A}) = \mathcal{D}_{\widetilde{X},0}$. Recall that the direct image functor $\tilde{\pi}_+$ is defined by

$$N \mapsto N \underset{\mathcal{D}(\widetilde{A})}{\otimes} \mathcal{D}_{\widetilde{X}\to\mathbb{C}^{n+1},0}$$

for all right $\mathcal{D}(\widetilde{A})$-modules N, where

$$\mathcal{D}_{\widetilde{X}\to\mathbb{C}^{n+1},0} = \widetilde{A} \underset{\mathcal{O}}{\otimes} \mathcal{D} = \mathcal{D}_{\mathcal{O}}(\mathcal{O},\widetilde{A}).$$

The bimodule $\mathcal{D}_{\mathcal{O}}(\mathcal{O},\widetilde{A})$ viewed as a right $\mathcal{D}$-module is supported at X. If for a moment we forget the left $\mathcal{D}(A)$-structure on B, we obtain isomorphisms of A-modules

$$\begin{aligned} i^+(\mathcal{D}_{\mathcal{O}}(\mathcal{O},\widetilde{A})) &= \mathrm{Hom}_{\mathcal{D}}(B,\mathcal{D}_{\mathcal{O}}(\mathcal{O},\widetilde{A})) \\ &\cong \mathrm{Hom}_{\mathcal{O}}(A,\mathcal{D}_{\mathcal{O}}(\mathcal{O},\widetilde{A})) \\ &\cong \mathcal{D}_{\mathcal{O}}(A,\widetilde{A}) \qquad \text{by } 1.2 \\ &= \mathcal{D}_A(A,\widetilde{A}) \cong P^*. \end{aligned}$$

Obviously the isomorphisms are linear for the left $\mathcal{D}(\widetilde{A})$-structure. Further it is straightforward to check that this yields an isomorphism $i^+(\mathcal{D}_{\mathcal{O}}(\mathcal{O},\widetilde{A})) \cong P^*$ of right $\mathcal{D}(A)$-modules. In virtue of the main theorem 2.2 this yields an isomorphism of $\mathcal{D}(\widetilde{A})$-$\mathcal{D}$ bimodules

$$\mathcal{D}_{\mathcal{O}}(\mathcal{O},\widetilde{A}) \cong P^* \underset{\mathcal{D}(A)}{\otimes} B.$$

Now we are done since

$$\tilde{\pi}_+(N) \cong (N \underset{\mathcal{D}(\widetilde{A})}{\otimes} P^*) \underset{\mathcal{D}(A)}{\otimes} B = i_+(N \underset{\mathcal{D}(\widetilde{A})}{\otimes} P^*)$$

i.e., $\tilde{\pi}_+ = \pi_+ \circ i_+$ where $\pi_+(\,\cdot\,) := \,\cdot\, \underset{\mathcal{D}(\widetilde{A})}{\otimes} P^*$ establishes the Morita equivalence (cf. 1.5) and i_+ is the equivalence of the main theorem 2.2.

Remark Since $\tilde{\pi}:(\widetilde{X},0) \to (\mathbb{C}^{n+1},0)$ is a holomorphic map of smooth germs the ordinary theory of $\mathcal{D}$-modules applies to it.

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Chapter IV

VANISHING CYCLES AND $\mathcal{D}$-MODULES

by

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0 Introduction

This paper arose while we where working on the problem of classifying regular holonomic $\mathcal{D}_X$-modules (X a complex manifold) with a prescribed singular support i.e., the projection under $\pi: T^*X \to X$ of the characteristic variety. In [vD] we treated the normal crossings case. We believe that such classifications should be done by "$\mathcal{D}$-module theoretic" methods. In other words one should not start to translate the problem into one on classifying perverse sheaves.

The theorem on extensions of perverse sheaves of Verdier [V] has of course an analogue in the framework of $\mathcal{D}$-modules. This analogy arises by means of the Riemann-Hilbert correspondence (Cf. [Me] or [KK2]). The main goal of this paper is to give a "$\mathcal{D}$-module theoretic" proof of the analogue of the extension theorem (cf. Thm 3.2). Meanwhile we establish some results (Prop. 1.5.1 and Prop. 2.4.3) which might be important on their own.

As a preliminary task one is forced to find analogous versions of the nearby cycle functor Ψ_f and the vanishing cycle functor Φ_f of Deligne [D], (where $f: X \to \mathbb{C}$ is a non-constant holomorphic function) and the natural morphisms $can: \Psi_f \to \Phi_f$, $var: \Phi_f \to \Psi_f$. In [Ma] Malgrange considered the structure sheaf $\mathcal{O}_X$ and defined $\mathcal{D}_X$-modules corresponding with $\Psi_f \mathbb{C}_X$ and $\Phi_f \mathbb{C}_X$. In [K1] Kashiwara treats the general case; he defines functors ϕ and ψ such that for every regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, $\phi\mathcal{M}$ resp. $\psi\mathcal{M}$ agree with $\Phi_f \mathcal{F}$ resp. $\Psi_f \mathcal{F}$, where $\mathcal{F} = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Furthermore there are natural morphisms $c(\mathcal{M}): \phi\mathcal{M} \to \psi\mathcal{M}$, $v(\mathcal{M}): \psi\mathcal{M} \to \phi\mathcal{M}$ corresponding with *can*, *var*. The main result (Theorem 3.2) is then as follows (cf. [V], Cor.1).

Theorem. *The functor*

$$F: \mathcal{M} \mapsto \left(\mathcal{M}[f^{-1}], \phi\mathcal{M} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathcal{M}, \psi(\pi)\right)$$

defines an equivalence between the category of regular holonomic $\mathcal{D}_X$-modules and the category of triples $(\mathcal{N}, \mathcal{N}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathcal{N}_2, \alpha)$. Here $\pi: \mathcal{M} \to \mathcal{M}[f^{-1}]$ denotes the canonical map and $\mathcal{N}$, $\mathcal{N}_1$, $\mathcal{N}_2$ are regular holonomic $\mathcal{D}_X$-modules such that:

$\mathcal{N} \cong \mathcal{N}[f^{-1}]$;

$\mathcal{N}_1$, $\mathcal{N}_2$ are supported by $X_0 = f^{-1}(0)$;

U, V are $\mathcal{D}_X$-morphisms;

$\alpha: \mathcal{N}_2 \xrightarrow{\cong} \psi\mathcal{N}$ is an isomorphism satisfying $\alpha UV = c(\mathcal{N})v(\mathcal{N})\alpha$.

In §1 we introduce Kashiwara's filtration and state the main properties (Thm. 1.4). Moreover we put forward a nice description of this filtration (Prop. 1.5.1). This enables a rather easy proof of the Artin-Rees property (cf. 1.6.3) .

In §2 following Kashiwara (cf. [K1] and also [Ma]) we introduce functors ϕ, ψ on $\mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ We list some properties. Using material from §1 we deduce the existence of a distinguished triangle in $\mathrm{D}_{\mathrm{hr}}(\mathcal{D}_X)$

$$\begin{tikzcd} & R\Gamma_{[X_0]}\mathcal{M}[1] \arrow[dl, "+"'] & \\ \phi\mathcal{M} \arrow[rr] & & \psi\mathcal{M} \arrow[ul] \end{tikzcd} .$$

We obtain some corollaries to be used later. We make some comment on relations with Deligne's functors and add a remark concerning why one should restrict attention to regular holonomic $\mathcal{D}_X$-modules.

In §3 we formulate the main theorem 3.2. This section is rather technical. By then it is obvious that the functor F is exact and faithful. However the difficulty is to show that F is essentially surjective. To solve this problem we introduce an inverse functor G, which does the reconstruction for us. The details are in 3.2.3.

For a moment we return to the classification problem mentioned at the beginning. Suppose one wants to classify holonomic $\mathcal{D}_X$-modules with regular singularities along X_0. The main theorem reduces this to a problem of classifying pairs $\mathcal{N}_1 \rightleftarrows \mathcal{N}_2$ of regular holonomic $\mathcal{D}_X$-modules with support contained in X_0. In a subsequent paper we will return to this question.

N.B. If we write "module" we always mean "left module".
$\mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ denotes the category of regular holonomic $\mathcal{D}_X$-modules.

[3]

1 The canonical good filtration

1.1 Definition

Let Y be a complex manifold and let Z be a closed submanifold of Y. Let $\mathcal{I}$ be the defining ideal of $Z \subset Y$ i.e., the sections of $\mathcal{O}_Y$ vanishing on Z. Following Kashiwara [K1], (see also [S]), we define a descending filtration $F^{\cdot}\mathcal{D}_Y$ on $\mathcal{D}_Y$ by

$$F^k\mathcal{D}_Y := \{\, P \in \mathcal{D}_Y \mid P\mathcal{I}^j \subset \mathcal{I}^{j+k}, \ \text{all } j \in \mathbf{N}\,\}, \qquad \text{for all } k \in \mathbf{Z}.$$

In local coordinates $(y_1, \ldots, y_m, z_1, \ldots, z_n)$ on Y such that Z is given by $y_1 = 0, \ldots, y_m = 0$, one has

$$z_i, \frac{\partial}{\partial z_i} \in F^0\mathcal{D}_Y, \qquad y_j \in F^1\mathcal{D}_Y, \qquad \frac{\partial}{\partial y_j} \in F^{-1}\mathcal{D}_Y;$$

$F^0\mathcal{D}_Y$ is the subring of $\mathcal{D}_Y$ generated over $\mathcal{O}_Y$ by

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, y_1\frac{\partial}{\partial y_1}, \ldots, y_m\frac{\partial}{\partial y_m}.$$

$F^0\mathcal{D}_Y$ is a noetherian sheaf of rings. (Cf. [S], Ch.III §1.4 and appendix C.5; cf. also [KK2], Ch.I §1.1.) The $F^k\mathcal{D}_Y$ are coherent modules over $F^0\mathcal{D}_Y$. $F^0\mathcal{D}_Y/F^1\mathcal{D}_Y$ is a coherent sheaf of rings.

1.2 Definition of good filtration

Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module. A descending filtration $F^{\cdot}\mathcal{M}$ on $\mathcal{M}$ is called a good filtration if

(1) $F^l\mathcal{D}_Y F^k\mathcal{M} \subset F^{k+l}\mathcal{M}$, for all $k, l \in \mathbf{Z}$.
(2) $\mathcal{M} = \bigcup_{k\in\mathbf{Z}} F^k\mathcal{M}$.
(3) $F^k\mathcal{M}$ is a coherent $F^0\mathcal{D}_Y$-module, for all $k \in \mathbf{Z}$.
(4) Locally one has:
$F^l\mathcal{D}_Y F^k\mathcal{M} = F^{k+l}\mathcal{M}$ if $(l \geq 0, k \gg 0)$ or $(l \leq 0, k \ll 0)$.

If this is the case then for any $k \in \mathbf{Z}$, $\mathrm{gr}^k\mathcal{M} := F^k\mathcal{M}/F^{k+1}\mathcal{M}$ is a coherent $F^0\mathcal{D}_Y/F^1\mathcal{D}_Y$-module. Notice that a coherent $\mathcal{D}_Y$-module has locally a good filtration. Moreover if $\mathcal{M}$ is a regular holonomic $\mathcal{D}_Y$-module, such a filtration exists globally.

1.3 Definition of canonical good filtration
From now on we assume $Y = X \times \mathbb{C}$, X a complex manifold. Let t be a coordinate on $\mathbb{C}$. The ideal $\mathcal{I} = t\mathcal{O}_Y$ defines the closed submanifold $X \times \{0\}$, which we identify with X. Let θ denote the vectorfield $t\partial$ on Y, where $\partial = \frac{\partial}{\partial t}$. Clearly we have

$$F^k\mathcal{D}_Y = \begin{cases} F^0\mathcal{D}_Y t^k, & \text{if } \ k \in \mathbf{N} \\ \sum_{j=0}^{-k} F^0\mathcal{D}_Y \partial^j, & \text{if } -k \in \mathbf{N}. \end{cases}$$

The coherent sheaf of rings $F^0\mathcal{D}_Y/F^1\mathcal{D}_Y$ may be identified with $\mathcal{D}_X[\theta]$.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module with a good filtration $F^{\cdot}\mathcal{M}$. The filtration is called a canonical good filtration if

(5) there exists a non-zero polynomial $b \in \mathbb{C}[\Theta]$ such that
  (i) $b(\theta - k)F^k\mathcal{M} \subset F^{k+1}\mathcal{M}$, for all $k \in \mathbf{Z}$,
  (ii) $b^{-1}(0) \subset \{ z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z < 1\}$.

1.4 Theorem. (Cf. [K1], Thm 1 and [L], Thm 3.1) *Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module. Then:*

(i) *$\mathcal{M}$ admits at most one canonical good filtration.*
(ii) *If $\mathcal{M}$ is holonomic, then $\mathcal{M}$ carries locally a canonical good filtration.*
(iii) *If $\mathcal{M}$ is regular holonomic, then $\mathcal{M}$ has a canonical good filtration.*
(iv) *If $\mathcal{M}$ is regular holonomic, then for all $k \in \mathbf{Z}$, $\mathrm{gr}^k\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, where $F^{\cdot}\mathcal{M}$ denotes the canonical good filtration. [Notice that $\mathcal{D}_X \subset \mathcal{D}_X[t\partial] = F^0\mathcal{D}_Y/F^1\mathcal{D}_Y$ and in general $\mathrm{gr}^k\mathcal{M}$ is only coherent over $\mathcal{D}_X[t\partial]$.] In that case $\mathrm{gr}^k\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module.*

1.5 Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module equipped with a canonical filtration $F^{\cdot}\mathcal{M}$. We want to give a more explicit description of this filtration. Therefore we introduce the following notation: for a linear subspace $\mathcal{L} \subset \mathcal{M}$ and any $k \in \mathbf{N}$ put

$$t^{-k}\mathcal{L} := \{ m \in \mathcal{M} \mid t^k m \in \mathcal{L} \}.$$

Consider the descending chain of subspaces of $\mathcal{M}$

$$\cdots \subset t^{-2}F^0\mathcal{M} \subset t^{-1}F^0\mathcal{M} \subset F^0\mathcal{M} \subset tF^0\mathcal{M} \subset t^2F^0\mathcal{M} \subset \cdots.$$

We claim that this is just the canonical good filtration.

1.5.1 Proposition. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module carrying a canonical good filtration $F^{\cdot}\mathcal{M}$. Then for all $k \in \mathbf{Z}$*

$$F^k\mathcal{M} = t^k F^0\mathcal{M}.$$

PROOF. Let $b \in \mathbb{C}[\Theta]$ be a non-zero polynomial satisfying condition (5) of 1.3. Observe that for $k \in \mathbf{N}^*$,

$$F^{-k}\mathcal{M} \subset t^{-k}F^0\mathcal{M}, \qquad t^kF^0\mathcal{M} \subset F^k\mathcal{M}.$$

Let us prove the other inclusions.
(1) Let $k \in \mathbf{N}^*$. Suppose $m \in t^{-k}F^0\mathcal{M}$ i.e., $t^km \in F^0\mathcal{M}$. Hence $\partial^kt^km \in F^{-k}\mathcal{M}$; thus $(\theta+k)\dots(\theta+1)m \in F^{-k}\mathcal{M}$. On the other hand there exists $N \in \mathbf{N}$, $N \geq k+1$, such that $m \in F^{-N}\mathcal{M}$; thus $b(\theta+k+1)\dots b(\theta+N)m \in F^{-k}\mathcal{M}$. Because of condition (ii) of 1.3(5) $(\theta+k)\dots(\theta+1)$ and $b(\theta+k+1)\dots b(\theta+N)$ are relatively prime. This yields $m \in F^{-k}\mathcal{M}$.
(2) Locally there exists $j_0 \in \mathbf{N}$ such that $F^j\mathcal{D}_YF^{j_0}\mathcal{M} = F^{j+j_0}\mathcal{M}$, for all $j \geq 0$. It follows that $F^{j_0+j}\mathcal{M} = t^jF^{j_0}\mathcal{M}$, for all $j \geq 0$. If $j_0 = 0$ we are done, so assume $j_0 > 0$. We will derive that $F^{j_0}\mathcal{M} = tF^{j_0-1}\mathcal{M}$. Let $m \in F^{j_0}\mathcal{M}$. Then $b(\theta - j_0)m \in F^{j_0+1}\mathcal{M}$. Hence there exists $m_0 \in F^{j_0}\mathcal{M} \subset F^{j_0-1}\mathcal{M}$ such that $b(\theta-j_0)m = tm_0$. Writing $b(\theta-j_0) = \theta\tilde{b}(\theta) + b(-j_0)$, with $\tilde{b} \in \mathbb{C}[\Theta]$, we have $b(-j_0)m = t(m_0 - \partial\tilde{b}(\theta)m)$. Note that $b(-j_0) \in \mathbb{C}^*$, $m_0 - \partial\tilde{b}(\theta)m \in F^{j_0-1}\mathcal{M}$ and thus $m \in tF^{j_0-1}\mathcal{M}$. This yields $F^{j_0}\mathcal{M} \subset tF^{j_0-1}\mathcal{M}$. The other inclusion is obvious. It follows that already $F^{j_0-1+j}\mathcal{M} = t^jF^{j_0-1}\mathcal{M}$, for all $j \geq 0$. By descending induction we arrive at $F^j\mathcal{M} = t^jF^0\mathcal{M}$, for all $j \geq 0$.

1.6 Using this description of the canonical good filtration we will derive that a morphism $\varphi: \mathcal{M}_1 \to \mathcal{M}_2$ of coherent $\mathcal{D}_Y$-modules, carrying a canonical good filtration, is a strict morphism between the filtered modules. By this we mean that

$$\operatorname{Im}\varphi \cap F^k\mathcal{M}_2 = \varphi(F^k\mathcal{M}_1), \qquad \text{for all } k \in \mathbf{Z}.$$

This will be done by proving an Artin-Rees lemma for canonical good filtrations. As a preliminary step we have:

1.6.1 The canonical filtration on $\mathcal{H}^0_{[X]}\mathcal{M}$

Let us first recall the following. Let $Z \subset Y$ be a subvariety defined by an ideal $\mathcal{I}$. For a $\mathcal{D}_Y$-module $\mathcal{M}$ one defines

$$\Gamma_{[Z]}\mathcal{M} = \varinjlim \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}^n, \mathcal{M}).$$

This is a $\mathcal{D}_Y$-module with support contained in Z. (Cf. [K2], §1 or [Me].) Let $\mathcal{H}^k_{[Z]}$ denote the k-th derived functor of $\Gamma_{[Z]}$. If $\mathcal{M}$ is coherent it is not necessarily the case that $\mathcal{H}^k_{[Z]}\mathcal{M}$ is coherent. However Kashiwara proved:

- If $\mathcal{M}$ is holonomic, then $\mathcal{H}^k_{[Z]}\mathcal{M}$ is holonomic. ([K2], Thm 1.4).

- If $\mathcal{M}$ is regular holonomic, then also $\mathcal{H}^k_{[Z]}\mathcal{M}$. ([KK2], Thm 5.4.1).

The ideal $t\mathcal{O}_Y$ defines the closed submanifold $X \times \{0\} = X$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module and assume that $\mathcal{H}^0_{[X]}\mathcal{M}$ is also coherent. We make the following observations:

(1) $$\mathcal{H}^0_{[X]}\mathcal{M} = \{\, m \in \mathcal{M} \mid \text{there exists } N \in \mathbf{N} \text{ such that } t^N m = 0 \,\} = \bigcup_{k\in\mathbf{N}} \operatorname{Ker}(t^k, \mathcal{M}).$$

(2) $\mathcal{H}^0_{[X]}\mathcal{M}$ carries a descending filtration given by

$$F^k\,\mathcal{H}^0_{[X]}\mathcal{M} := \begin{cases} \operatorname{Ker}(t^{-k}, \mathcal{M}) & \text{if } -k \in \mathbf{N} \\ 0 & \text{if } \;\; k \in \mathbf{N}. \end{cases}$$

(3) This is a good filtration, because:
  (i) Conditions (1) and (2) of 1.2 are trivially satisfied.
  (ii) 1.2(4) is true, because $\operatorname{Ker}(t^l, \mathcal{M}) \cong \bigoplus_{j=0}^{l-1} \partial^j \operatorname{Ker}(t, \mathcal{M})$.
  (iii) $\operatorname{Ker}(t, \mathcal{M})$ is a coherent $\mathcal{D}_X$-module ([K2], Prop. 4.2), and thus a coherent $F^0\mathcal{D}_Y$-module. This implies 1.2(3).

(4) Furthermore $(t\partial - k)F^k\,\mathcal{H}^0_{[X]}\mathcal{M} \subset F^{k+1}\,\mathcal{H}^0_{[X]}\mathcal{M}$, for all $k \in \mathbf{Z}$ as one easily verifies. Thus the filtration given by (2) on $\mathcal{H}^0_{[X]}\mathcal{M}$ is the canonical good filtration.

1.6.2 The induced filtration on a quotient

Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module and let $\mathcal{N} \subset \mathcal{M}$ be a coherent $\mathcal{D}_Y$-submodule. Suppose $\mathcal{M}$ is equipped with a filtration $F^{\cdot}\mathcal{M}$ which is canonical good. There are induced filtrations on $\mathcal{N}$ and $\mathcal{M}/\mathcal{N}$ defined by

$$F^k\mathcal{N} := \mathcal{N} \cap F^k\mathcal{M}, \qquad \text{for all } k \in \mathbf{Z},$$

$$F^k(\mathcal{M}/\mathcal{N}) := F^k\mathcal{M}/F^k\mathcal{N}, \qquad \text{for all } k \in \mathbf{Z}.$$

Proposition. *The induced filtration $F^{\cdot}(\mathcal{M}/\mathcal{N})$ is canonical good.*

Proof. Clearly the induced filtration satisfies properties (1), (2), (4) and (5) of the definition of canonical good filtration. Condition (3) is fulfilled, because locally $F^k(\mathcal{M}/\mathcal{N})$ is a $F^0\mathcal{D}_Y$-module of finite type and it is a $F^0\mathcal{D}_Y$-submodule of the coherent $\mathcal{D}_Y$-module $\mathcal{M}/\mathcal{N}$. By [S], Prop. 1.4.2 and the last lines of 1.1 $F^k(\mathcal{M}/\mathcal{N})$ is a coherent $F^0\mathcal{D}_Y$-module, for all $k \in \mathbf{Z}$.

Note. By the last line $F^k\mathcal{N}$ is a coherent $F^0\mathcal{D}_Y$-module, all $k \in \mathbf{Z}$.

1.6.3 The induced filtration on a holonomic submodule

Proposition. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module and let $\mathcal{N} \subset \mathcal{M}$ be a coherent submodule. Assume $\mathcal{M}$ carries a canonical good filtration $F^{\cdot}\mathcal{M}$. Then:*

(i) $\mathrm{Ker}(t, \mathcal{M}) \cap F^0\mathcal{M} = 0$;
(ii) the induced filtration on $\mathcal{N}$ is canonical good.

PROOF. We begin the proof of (ii) and meanwhile obtain (i) as a special case. The induced filtration satisfies conditions (1), (2), (3) and (5). The problem is to show that $F^{\cdot}\mathcal{N}$ satisfies condition (4) i.e., locally

$$F^l\mathcal{D}_Y F^k\mathcal{N} = F^{k+l}\mathcal{N} \quad \text{if} \quad (l \geq 0, k \gg 0) \text{ or } (l \leq 0, k \ll 0).$$

We proof this in two steps.
1: $F^l\mathcal{D}_Y F^k\mathcal{N} = F^{k+l}\mathcal{N}$ for all $l \leq 0$, $k \leq -1$.
It is enough to prove

$$F^{-k-1}\mathcal{N} = F^{-1}\mathcal{D}_Y F^{-k}\mathcal{N}, \qquad \text{for all } k \geq 1.$$

Therefore let $k \in \mathbf{N}^*$ and $n \in F^{-k-1}\mathcal{N}$. Let $b \in \mathbb{C}[\Theta]$ be a non-zero polynomial belonging to the canonical good filtration $F^{\cdot}\mathcal{M}$ (cf. 1.3(5)). Then $b(\theta + k + 1)n \in \mathcal{N} \cap F^{-k}\mathcal{M} = F^{-k}\mathcal{N}$. Write $b(\theta + k + 1) = \partial t\tilde{b}(\theta + 1) + b(k)$, $\tilde{b} \in \mathbb{C}[\Theta]$ and $b(k) \in \mathbb{C}^*$. Further $t\tilde{b}(\theta + 1)n \in F^{-k}\mathcal{N}$, yielding that $n \in F^{-k}\mathcal{N} + \partial F^{-k}\mathcal{N} = F^{-1}\mathcal{D}_Y F^{-k}\mathcal{N}$.
2: The problem seems to be in the tail of the filtration. We shall derive that

$$F^l\mathcal{D}_Y F^k\mathcal{N} = F^{k+l}\mathcal{N}, \qquad \text{for all } k, l \in \mathbf{N}.$$

It suffices to show that $F^{k+1}\mathcal{N} \subset tF^k\mathcal{N}$, for all $k \in \mathbf{N}$.
Let us first treat the special case
2a: $\mathcal{N} = \mathcal{H}^0_{[X]}\mathcal{M}$.
Let $k \in \mathbf{N}$ and $n \in F^{k+1}\mathcal{N} = \mathcal{N} \cap F^{k+1}\mathcal{M}$. By proposition 1.5.1. there exists $m \in F^k\mathcal{M}$ such that $n = tm$. Because $\mathcal{N} = \mathcal{H}^0_{[X]}\mathcal{M}$ there exists $N \in \mathbf{N}$ such that $t^N n = 0$. It follows also $t^{N+1}m = 0$, hence $m \in \mathcal{H}^0_{[X]}\mathcal{M} = \mathcal{N}$. So $n = tm$ with $m \in F^k\mathcal{N}$. This yields $F^{k+1}\mathcal{N} \subset tF^k\mathcal{N}$.
Hence in the particular case $\mathcal{N} = \mathcal{H}^0_{[X]}\mathcal{M}$ we have established that the induced filtration is canonical good and by unicity (Th. 1.4 (i)) equals the filtration given in 1.6.1. In particular $\mathrm{Ker}(t, \mathcal{M}) \cap F^0\mathcal{M} \subset \mathcal{N} \cap F^0\mathcal{M} = 0$, so this yields part (i).
2b: The general case.
Let $k \in \mathbf{N}$ and let $n \in F^{k+1}\mathcal{N}$. There exists $m \in F^k\mathcal{M}$ such that $n = tm$ (Prop. 1.5.1). Denote with $\overline{m}$ the image of m in $\mathcal{M}/\mathcal{N}$. Then $t\overline{m} = 0$ in $\mathcal{M}/\mathcal{N}$. Hence $\overline{m} \in \mathrm{Ker}(t, \mathcal{M}/\mathcal{N}) \cap F^k(\mathcal{M}/\mathcal{N}) \subset$

$\operatorname{Ker}(t, \mathcal{M}/\mathcal{N}) \cap F^0(\mathcal{M}/\mathcal{N})$. By proposition 1.6.2 the induced filtration on $\mathcal{M}/\mathcal{N}$ is canonical good, hence by (a) above $\overline{m} = 0$. It follows that $m \in \mathcal{N} \cap F^k\mathcal{M} = F^k\mathcal{N}$ and $n = tm \in tF^k\mathcal{N}$. This yields $F^{k+1}\mathcal{N} \subset tF^k\mathcal{N}$ for all $k \in \mathbf{N}$.

1.6.4 Corollary. *Let $\varphi: \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism of coherent $\mathcal{D}_Y$-modules. Assume $\mathcal{M}_1$ and $\mathcal{M}_2$ carry canonical good filtrations $F^{\cdot}\mathcal{M}_1$, $F^{\cdot}\mathcal{M}_2$. Then φ is a strict morphism of filtered modules.*

Proof. Put $\mathcal{N} = \operatorname{Im}\varphi \subset \mathcal{M}_2$. Then $F^{\cdot}\mathcal{M}_1$ induces a canonical good filtration on$\mathcal{N}$ (Prop. 1.6.2). Also $F^{\cdot}\mathcal{M}_2$ induces a canonical good filtration on $\mathcal{N}$ (Prop. 1.6.3). But there can be only one canonical good filtration on $\mathcal{N}$, hence $\varphi(F^k\mathcal{M}_1) = \mathcal{N} \cap F^k\mathcal{M}_2$, for all $k \in \mathbf{Z}$.

1.6.5 Corollary. *Let $\mathcal{M}_1 \hookrightarrow \mathcal{M}_2 \twoheadrightarrow \mathcal{M}_3$ be a short exact sequence of coherent $\mathcal{D}_Y$-modules with a canonical good filtration. Then for all $k \in \mathbf{Z}$ we have exact sequences:*

(i) $F^k\mathcal{M}_1 \hookrightarrow F^k\mathcal{M}_2 \twoheadrightarrow F^k\mathcal{M}_3$ *of $F^0\mathcal{D}_Y$-modules;*
(ii) $\operatorname{gr}^k\mathcal{M}_1 \hookrightarrow \operatorname{gr}^k\mathcal{M}_2 \twoheadrightarrow \operatorname{gr}^k\mathcal{M}_3$ *of $\mathcal{D}_X[\theta]$-modules.*

1.6.6 Remark
Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module admitting a canonical good filtration $F^{\cdot}\mathcal{M}$. The multiplication with t induces, for all $k \in \mathbf{N}$, a bijection $\operatorname{gr}^k\mathcal{M} \xrightarrow{\cong} \operatorname{gr}^{k+1}\mathcal{M}$. This follows from 1.6.3(i) and 1.5.1.

2 Vanishing cycles and nearby cycles

Let X, Y be as in 1.3. Let $f: X \to \mathbf{C}$ be a non-constant holomorphic function on X.
Let

$$i: X \to Y = X \times \mathbf{C}, \qquad x \mapsto (x, f(x))$$

be the embedding on the graph of f.
Finally put $X_0 := f^{-1}(0)$.

2.1 Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then

$$i_*\mathcal{M} = (\mathcal{D}_Y/\mathcal{D}_Y(t-f)) \underset{\mathcal{D}_X}{\otimes} \mathcal{M}$$

(where we have identified X with the graph of f) is a coherent $\mathcal{D}_Y$-module supported on the graph of f. If $\mathcal{M}$ is holonomic, then $i_*\mathcal{M}$ is holonomic (cf. [KK2], Lemma 5.1.9.). If $\mathcal{M}$ has regular singularities, then $i_*\mathcal{M}$ has regular singularities (ibid.). In fact i_* is an exact functor and establishes an equivalence between the category of coherent $\mathcal{D}_X$-modules and the category of coherent $\mathcal{D}_Y$-modules with support contained in the graph of f (cf. [K2], Prop. 4.2.). The inverse functor of i_* is given by

$$\operatorname{Ker}(t-f, \cdot) = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/(t-f)\mathcal{O}_Y, \cdot).$$

In fact one has the identification

$$i_*\mathcal{M} \cong \mathbb{C}[\partial] \otimes \mathcal{M},$$

where the $\mathcal{D}_Y$-structure on the right-hand side is determined by (in local coordinates $x_1, \ldots, x_d, t$ on Y):
for all $m \in \mathcal{M}$, $i \in \mathbf{N}$:

$$\begin{aligned} t(\partial^i \otimes m) &= -i\partial^{i-1} \otimes m + \partial^i \otimes fm \\ \partial(\partial^i \otimes m) &= \partial^{i+1} \otimes m \\ \partial_\alpha(\partial^i \otimes m) &= \partial^i \otimes \partial_\alpha m - \partial^{i+1} \otimes \partial_\alpha(f)m, \end{aligned}$$
$$\text{for all } \alpha \in \{1, \ldots, d\},\ (\partial_\alpha = \tfrac{\partial}{\partial x_\alpha}).$$

2.2 Definitions

2.2.1 The category of coherent $\mathcal{D}_X$-modules $\mathcal{M}$ satisfying the requirement that $i_*\mathcal{M}$ admits a canonical good filtration is by 1.6.2, 1.6.3 and 1.6.4 an abelian category. Let us denote this category by $\mathcal{R}$. Following Kashiwara [K1] (see also Malgrange [Ma] and M. Saito [Sa]) we define for any $\mathcal{M} \in \mathcal{R}$

$$\begin{aligned} \psi\mathcal{M} &:= F^0(i_*\mathcal{M})/F^1(i_*\mathcal{M}) \\ \phi\mathcal{M} &:= F^{-1}(i_*\mathcal{M})/F^0(i_*\mathcal{M}), \end{aligned}$$

where $F^{\cdot}(i_*\mathcal{M})$ denotes the canonical good filtration on $i_*\mathcal{M}$.

2.2.2 Left multiplication with t resp. ∂ induces maps

$$c(\mathcal{M}) \colon \phi\mathcal{M} \to \psi\mathcal{M},$$

$$v(\mathcal{M}) \colon \psi\mathcal{M} \to \phi\mathcal{M}.$$

2.2.3 If we make the identification $X = X \times \{0\}$ (see 1.3), then $\psi\mathcal{M}$ and $\phi\mathcal{M}$ have the structure of a module over $F^0\mathcal{D}_Y/F^1\mathcal{D}_Y = \mathcal{D}_X[t\partial]$. Moreover $\psi\mathcal{M}$ and $\phi\mathcal{M}$ are coherent $\mathcal{D}_X[t\partial]$-modules. The mappings $c(\mathcal{M})$ and $v(\mathcal{M})$ are $\mathcal{D}_X$-linear and the action of $t\partial$ on $\phi\mathcal{M}$ (resp. $\psi\mathcal{M}$) is given by $v(\mathcal{M}) \circ c(\mathcal{M}) - 1_{\phi\mathcal{M}}$ (resp. $c(\mathcal{M}) \circ v(\mathcal{M})$). The $\mathcal{D}_X$-modules $\psi\mathcal{M}$ and $\phi\mathcal{M}$ have their support contained in

$$i(X) \cap (X \times \{0\}) = \mathrm{graph}(f) \cap (X \times \{0\}) = X_0 \times \{0\}.$$

2.3 Restriction to regular holonomic modules
By Thm 1.4(iii) the category $\mathcal{R}$ contains the category of the regular holonomic modules $\mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$. According to theorem 1.4(iv) $\psi\mathcal{M}$ and $\phi\mathcal{M}$ are regular holonomic $\mathcal{D}_X$-modules if $\mathcal{M}$ is regular holonomic. The restrictions of ψ and ϕ to $\mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ are still denoted ψ and ϕ. We view them as functors from $\mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ to itself. There exist natural transformations $c\colon\phi \to \psi$, $v\colon\psi \to \phi$. These satisfy the condition that for any $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ there exists a non-zero polynomial $b \in \mathbb{C}[\Theta]$ such that:

(i) the $\psi\mathcal{M}$-endomorphism $c(\mathcal{M})v(\mathcal{M})$ satisfies $b(c(\mathcal{M})v(\mathcal{M})) = 0$,
(ii) $b^{-1}(0) \subset \{\, z \in \mathbb{C} \mid 0 \le \operatorname{Re} z < 1 \,\}$.

2.4 A distinguished triangle
Our goal in this subsection is to show the existence of a distinguished triangle in $\mathrm{D}_{\mathrm{hr}}(\mathcal{D}_X)$, the derived category of bounded complexes of $\mathcal{D}_X$-modules with regular holonomic cohomology. For any $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ there exists a distinguished triangle

$$\begin{tikzcd} & \mathrm{R}\Gamma_{[X_0]}\mathcal{M}[1] \arrow[dl, "+"'] & \\ \phi\mathcal{M} \arrow[rr] & & \psi\mathcal{M} \arrow[ul] \end{tikzcd}.$$

Or in more down to earth terms, there exists an exact sequence of regular holonomic $\mathcal{D}_X$-modules

$$\mathcal{H}^0_{[X_0]}\mathcal{M} \hookrightarrow \phi\mathcal{M} \xrightarrow{c} \psi\mathcal{M} \twoheadrightarrow \mathcal{H}^1_{[X_0]}\mathcal{M}.$$

We start with a lemma; its proof is a bit technical.

2.4.1 Lemma. *Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. There exists a natural isomorphism of $\mathcal{D}_X$-modules*

$$\mathcal{H}^0_{[X_0]}\mathcal{M} \xrightarrow{\cong} \operatorname{Ker}(t, i_*\mathcal{M}) = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, i_*\mathcal{M}).$$

Proof. Recall that (cf. 1.6.1, 2.1)

$$\mathcal{H}^0_{[X_0]}\mathcal{M} = \bigcup_{n\in\mathbf{N}} \operatorname{Ker}(f^n, \mathcal{M}) \qquad i_*\mathcal{M} = \mathbb{C}[\partial] \otimes \mathcal{M}.$$

Let $p = \sum_{j=0}^n \partial^j \otimes m_j \in i_*\mathcal{M}$.
Then $tp = \sum_{j=0}^{n-1} \partial^j \otimes (-(j+1)m_{j+1} + fm_j) + \partial^n \otimes fm_n$. Hence

$$\begin{aligned} p \in \operatorname{Ker}(t, i_*\mathcal{M}) \quad &\text{iff} \quad fm_n = 0, nm_n = fm_{n-1}, \dots, m_1 = fm_0 \\ &\text{iff} \quad f^{n+1}m_0 = 0, \\ &\qquad\; j!m_j = f^j m_0 \qquad \text{for all } j \in \{\,1, \dots, n\,\}. \end{aligned}$$

This clearly implies that the injective maps

$$\varepsilon_n(\mathcal{M})\colon \mathrm{Ker}(f^n,\mathcal{M}) \hookrightarrow \mathrm{Ker}(t,i_*\mathcal{M}),$$
$$m \mapsto \sum_{j=0}^{n} \frac{1}{j!}\partial^j \otimes f^j m, \quad \text{for all } m,$$

induce a bijective $\mathcal{O}_X$-linear map

$$\varepsilon(\mathcal{M})\colon \mathcal{H}^0_{[X_0]}\mathcal{M} \to \mathrm{Ker}(t,i_*\mathcal{M}).$$

Clearly ε is functorial in $\mathcal{M}$, so it remains to check that $\varepsilon(\mathcal{M})$ is $\mathcal{D}_X$-linear. Therefore let $\xi \in \mathcal{D}er(\mathcal{O}_X)$, $m \in \mathrm{Ker}(f^n,\mathcal{M})$. Then $\xi m \in \mathrm{Ker}(f^{n+1},\mathcal{M})$. Using the description of the $\mathcal{D}_Y$-structure on $\mathbb{C}[\partial]\otimes\mathcal{M}$ in 2.1 one obtains

$$\begin{aligned}
\xi\varepsilon(\mathcal{M})(m) &= \xi\sum_{j=0}^{n}\frac{1}{j!}\partial^j\otimes f^j m \\
&= \sum_{j=0}^{n}\frac{1}{j!}\partial^j\otimes \xi f^j m - \sum_{j=0}^{n}\frac{1}{j!}\partial^{j+1}\otimes \xi(f)f^j m \\
&= \sum_{j=0}^{n}\frac{1}{j!}\partial^j\otimes (f^j\xi + j\xi(f)f^{j-1})m \\
&\qquad\qquad - \sum_{j=1}^{n+1}\frac{1}{(j-1)!}\partial^j\otimes \xi(f)f^{j-1}m \\
&= \sum_{j=0}^{n}\frac{1}{j!}\partial^j\otimes f^j\xi m = \varepsilon(\mathcal{M})(\xi m).
\end{aligned}$$

2.4.2 Corollary. *Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. Then*

$$\mathrm{R}\Gamma_{[X_0]}\mathcal{M} \xrightarrow{\simeq} \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, i_*\mathcal{M}).$$

PROOF. The result follows once we have checked that

$$\mathcal{M} \text{ injective } \mathcal{D}_X\text{-module} \Rightarrow i_*\mathcal{M} \text{ is acyclic for } \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y,-).$$

But this is clear, because an injective $\mathcal{D}_X$-module $\mathcal{M}$ is injective when considered as an $\mathcal{O}_X$-module. Hence $\mathcal{M}$ is divisible by f i.e., multiplication by f on $\mathcal{M}$ is surjective. This implies that the multiplication with t on $i_*\mathcal{M}$ is surjective i.e., $\mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{O}_Y t, i_*\mathcal{M}) = 0$, for all $i > 0$. The argument is as follows: Let $m \in \mathcal{M}$. There exists $n \in \mathcal{M}$ such that $fn = m$. Proceed by induction on j, using $t(\partial^j\otimes n) = -j\partial^{j-1}\otimes n + \partial^j\otimes m$.

2.4.3 Proposition. *Let $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$. In $\mathrm{D}_{\mathrm{hr}}(\mathcal{D}_X)$ we have a distinguished triangle, functorial in $\mathcal{M}$,*

$$\begin{tikzcd} & \mathrm{R}\Gamma_{[X_0]}\mathcal{M}[1] \arrow[dl, "+"'] & \\ \phi\mathcal{M} \arrow[rr] & & \psi\mathcal{M}. \arrow[ul] \end{tikzcd}$$

Proof. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module. Consider the short exact sequence of $F^0\mathcal{D}_Y$-modules

$$F^0 i_*\mathcal{M} \hookrightarrow i_*\mathcal{M} \twoheadrightarrow i_*\mathcal{M}/F^0 i_*\mathcal{M},$$

where $F^{\cdot} i_*\mathcal{M}$ denotes the canonical good filtration on $i_*\mathcal{M}$. The functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, -)$ applied to this sequence yields a distinguished triangle

$$\begin{tikzcd} & \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, i_*\mathcal{M}/F^0 i_*\mathcal{M}) \arrow[dl, "+"'] & \\ \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, F^0 i_*\mathcal{M}) \arrow[rr] & & \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, i_*\mathcal{M}). \arrow[ul] \end{tikzcd}$$

By 2.4.2 we have

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, i_*\mathcal{M}) = \mathrm{R}\Gamma_{[X_0]}\mathcal{M}.$$

By Proposition 1.6.3(i) and Proposition 1.5.1 it follows

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, F^0 i_*\mathcal{M}) = F^0 i_*\mathcal{M}/tF^0 i_*\mathcal{M}[-1] = \psi\mathcal{M}[-1].$$

By Proposition 1.5.1

$$\mathrm{Ker}(t, i_*\mathcal{M}/F^0 i_*\mathcal{M}) = t^{-1}F^0 i_*\mathcal{M}/F^0 i_*\mathcal{M} = \phi\mathcal{M}.$$

Let us investigate $\mathrm{Coker}(t, i_*\mathcal{M}/F^0 i_*\mathcal{M})$. Therefore let $m \in i_*\mathcal{M}$. For some $N \in \mathbf{N}^*$, $m \in F^{-N} i_*\mathcal{M}$. It follows that $a(\theta)m \in F^0 i_*\mathcal{M}$, where $a(\theta) = b(\theta+1)\ldots b(\theta+N)$ and $b \in \mathbb{C}[\Theta]$ is a non-zero polynomial satisfying (5) of subsection 1.3. Then $a(\theta) = t\partial\tilde{a}(\theta) + a(0)$ with $\tilde{a} \in \mathbb{C}[\Theta]$ and $a(0) \in \mathbb{C}^*$. Thus $a(0)m + t\partial\tilde{a}(\theta)m \in F^0 i_*\mathcal{M}$. We conclude that $\mathrm{Coker}(t, i_*\mathcal{M}/F^0 i_*\mathcal{M}) = 0$.

Finally collecting things we end up with a distinguished triangle

$$\begin{tikzcd} & \phi\mathcal{M} \arrow[dl, "+"'] & \\ \psi\mathcal{M}[-1] \arrow[rr] & & \mathrm{R}\Gamma_{[X_0]}\mathcal{M}. \arrow[ul] \end{tikzcd}$$

2.4.4 Some easy consequences

Let $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$. There are two natural distinguished triangles in $\mathrm{D}_{\mathrm{hr}}(\mathcal{D}_X)$

$$\begin{tikzcd} & \mathrm{R}\Gamma_{[X_0]}\mathcal{M}[1] \arrow[dl, "+"'] & \\ \phi\mathcal{M} \arrow[rr] & & \psi\mathcal{M} \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{R}\Gamma_{[X_0]}\mathcal{M}[1] \arrow[dl, "+"'] & \\ \mathcal{M} \arrow[rr] & & \mathcal{M}[f^{-1}]. \arrow[ul] \end{tikzcd}$$

It follows immediately from these triangles.

2.4.4.1 Proposition. *For every $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ the following are equivalent:*

(i) $\mathcal{M} \xrightarrow{\cong} \mathcal{M}[f^{-1}]$;
(ii) $\mathrm{R}\Gamma_{[X_0]}\mathcal{M} = 0$;
(iii) $c(\mathcal{M}): \phi\mathcal{M} \to \psi\mathcal{M}$ *is an isomorphism.*

2.4.4.2 Proposition. *For every $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ the following are equivalent:*

(i) $\mathrm{supp}(\mathcal{M}) \subset X_0$;
(ii) $\mathcal{H}^0_{[X_0]}\mathcal{M} = \mathcal{M}$;
(iii) $\mathcal{M}[f^{-1}] = 0$;
(iv) $\phi\mathcal{M} = \mathcal{M}$.
Furthermore any of these conditions implies $\psi\mathcal{M} = 0$.

PROOF. The equivalence of (i), (ii) and (iii) is well-known. (iv) $\Rightarrow$ (i) is clear. Now (ii) implies $i_*\mathcal{M} = \mathcal{H}^0_{[X\times\{0\}]}i_*\mathcal{M}$, hence $\psi\mathcal{M} = 0$ (cf. 1.6.1.) and thus $\phi\mathcal{M} = \mathrm{R}\Gamma_{[X_0]}\mathcal{M} = \mathcal{M}$.

Remark. With a little more effort one can show that $\psi\mathcal{M} = 0$ implies $\phi\mathcal{M} = \mathcal{M}$.

2.4.4.3 Corollary. *Let $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ and let $\pi: \mathcal{M} \to \mathcal{M}[f^{-1}]$ be the canonical map. Then:*

(i) $\psi(\pi): \psi\mathcal{M} \to \psi(\mathcal{M}[f^{-1}])$ *is an isomorphism;*
(ii) *there exists an exact sequence*

$$\mathcal{H}^0_{[X_0]}\mathcal{M} \hookrightarrow \phi\mathcal{M} \xrightarrow{\phi(\pi)} \phi(\mathcal{M}[f^{-1}]) \twoheadrightarrow \mathcal{H}^1_{[X_0]}\mathcal{M};$$

(iii) $c(\mathcal{M}[f^{-1}]) \circ \phi(\pi) = \psi(\pi) \circ c(\mathcal{M})$.

2.4.5 An alternative proof of 2.4.3
The reader who is not happy with the given proof of 2.4.3 is offered a different approach. We give a derivation, in the category $\mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ of an equivalent formulation of 2.4.3. We avoid the use of Corollary 2.4.2. First we need some preliminary results.

2.4.5.1 Sublemma. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module with support contained in X_0 (thus $\mathcal{M} = \mathcal{H}^0_{[X_0]}\mathcal{M}$). Then $\phi\mathcal{M} = \mathcal{M}$ and $\psi\mathcal{M} = 0$.*

PROOF. Note that $i_*\mathcal{M}$ is coherent $\mathcal{D}_Y$-module supported on $X = X \times \{0\}$ i.e., $i_*\mathcal{M} = \mathcal{H}^0_{[X]}i_*\mathcal{M}$. Thus the canonical good filtration $F^{\cdot}i_*\mathcal{M}$ satisfies (cf. 1.6.1) $F^{-1}i_*\mathcal{M} = \mathrm{Ker}(t, i_*\mathcal{M})$ and $F^k i_*\mathcal{M} = 0$, for all $k \in \mathbb{N}$. Thus $\psi\mathcal{M} = 0$ and by Lemma 2.4.1 $\phi\mathcal{M} = \mathcal{M}$.

2.4.5.2 Sublemma. *Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. Then the map "multiplication by t"*

$$i_*(\mathcal{M}[f^{-1}]) \xrightarrow{t\cdot} i_*(\mathcal{M}[f^{-1}])$$

is bijective.

PROOF. The injectivity follows using Lemma 2.4.1. The surjectivity follows (as in the proof of Corollary 2.4.2) by induction on j, using $t(\partial^j \otimes f^{-1}m) = -j\partial^{j-1} \otimes f^{-1}m + \partial^j \otimes m$ for all $m \in \mathcal{M}[f^{-1}]$.

2.4.5.3 Corollary. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module such that $i_*\mathcal{M}$ admits a canonical good filtration. Assume that the canonical map $\pi: \mathcal{M} \to \mathcal{M}[f^{-1}]$ is an isomorphism. Then $c(\mathcal{M}): \phi\mathcal{M} \to \psi\mathcal{M}$ is an isomorphism.*

PROOF. Consider the commutative diagram with exact rows

$$\begin{array}{ccccc} F^0 i_*\mathcal{M} & \hookrightarrow & F^{-1}i_*\mathcal{M} & \twoheadrightarrow & \phi\mathcal{M} \\ \cong\big\downarrow t\cdot & & \big\downarrow t\cdot & & \big\downarrow c \\ F^1 i_*\mathcal{M} & \hookrightarrow & F^0 i_*\mathcal{M} & \twoheadrightarrow & \psi\mathcal{M} \end{array}$$

The left arrow is bijective by Prop. 1.5.1. and Prop. 1.6.3(i). By Sublemma 2.4.5.2 the middle arrow is injective. This Lemma together with the fact that $F^{-1}i_*\mathcal{M} = t^{-1}F^0 i_*\mathcal{M}$ (cf. Prop. 1.5.1) yield the surjectivity of the middle arrow. Hence c is bijective.

2.4.5.4 Proposition. *Let $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$. There exists a natural exact sequence of regular holonomic $\mathcal{D}_X$-modules*

$$\mathcal{H}^0_{[X_0]}\mathcal{M} \hookrightarrow \phi\mathcal{M} \xrightarrow{c} \psi\mathcal{M} \twoheadrightarrow \mathcal{H}^1_{[X_0]}\mathcal{M}.$$

PROOF. Consider the short exact sequences of $\mathcal{D}_X$-modules

$$\begin{array}{ccccc} \mathcal{H}^0_{[X_0]}\mathcal{M} & \hookrightarrow & \mathcal{M} & \twoheadrightarrow & \widetilde{\mathcal{M}} \\ \widetilde{\mathcal{M}} & \hookrightarrow & \mathcal{M}[f^{-1}] & \twoheadrightarrow & \mathcal{H}^1_{[X_0]}\mathcal{M} \end{array}$$

where $\widetilde{\mathcal{M}} = \mathrm{Im}(\mathcal{M} \to \mathcal{M}[f^{-1}])$. In view of Sublemma 2.4.5.1 these give rise to two commutative diagrams with exact rows

$$\begin{tikzcd}
\mathcal{H}^0_{[X_0]}\mathcal{M} \arrow[r, hook] \arrow[d] & \phi\mathcal{M} \arrow[r, two heads] \arrow[d, "c"] & \phi\widetilde{\mathcal{M}} \arrow[d, "c"] \\
0 \arrow[r, hook] & \psi\mathcal{M} \arrow[r, two heads] & \psi\widetilde{\mathcal{M}}
\end{tikzcd}$$

$$\begin{tikzcd}
\phi\widetilde{\mathcal{M}} \arrow[r, hook] \arrow[d, "c"] & \phi(\mathcal{M}[f^{-1}]) \arrow[r, two heads] \arrow[d, "c", "\cong"'] & \mathcal{H}^1_{[X_0]}\mathcal{M} \arrow[d] \\
\psi\widetilde{\mathcal{M}} \arrow[r, hook] & \psi(\mathcal{M}[f^{-1}]) \arrow[r, two heads] & 0
\end{tikzcd}$$

By Corollary 2.4.5.3 $c(\mathcal{M}[f^{-1}])$ is bijective. Hence the second diagram gives $\mathrm{Ker}\, c(\widetilde{\mathcal{M}}) = 0$, $\mathrm{Coker}\, c(\widetilde{\mathcal{M}}) = \mathcal{H}^1_{[X_0]}\mathcal{M}$. Now using the first diagram it follows that

$$\mathrm{Ker}\, c(\mathcal{M}) = \mathcal{H}^0_{[X_0]}\mathcal{M}, \qquad \mathrm{Coker}\, c(\mathcal{M}) = \mathcal{H}^1_{[X_0]}\mathcal{M}.$$

2.5 Relation with Deligne's functors

2.5.1 It is well-known that the contravariant "solution" functor

$$S: \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}} \to \mathrm{Perv}(X)$$

defined by

$$S(\mathcal{M}) = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X),$$

establishes an (anti-)equivalence of categories. Here $\mathrm{Perv}(X)$ denotes the category of perverse sheaves on X. This is known as the Riemann-Hilbert correspondence (cf. [Me] or [KK2]). Via this equivalence the functors ϕ resp. ψ correspond to the vanishing cycle functor Φ_f resp. the nearby cycle functor Ψ_f as introduced by Deligne [D]. More precise for $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ there are natural isomorphisms (cf. [K1], Thm 2)

$$S(\phi\mathcal{M})|_{X_0} \cong \Phi_f(S\mathcal{M})[-1],$$

$$S(\psi\mathcal{M})|_{X_0} \cong \Psi_f(S\mathcal{M})[-1].$$

c agrees with the canonical map $can: \Psi_f \to \Phi_f$ and $v\frac{\exp(2\pi i\theta)-1}{\theta}$ agrees with the variation map $var: \Phi_f \to \Psi_f$, where $\theta = cv$. The monodromy on Ψ_f is given by $S(\exp 2\pi i\theta)$ (loc. cit.).

Furthermore

$$S(\mathrm{R}\Gamma_{[X_0]}\mathcal{M}) \cong S(\mathcal{M})|_{X_0}$$

for every regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$. (Cf. [Me], Prop. 1.2.1). So our distinguished triangle corresponds to the fundamental distinguished triangle in Perv(X)

$$\begin{tikzcd} & \Phi_f(S\mathcal{M}) \arrow[dl, "+1"'] & \\ S(\mathcal{M})|_{X_0} \arrow[rr] & & \Psi_f(S\mathcal{M}). \arrow[ul] \end{tikzcd}$$

2.5.2 We have seen that we might define functors ϕ and ψ on a somewhat bigger category $\mathcal{R}$ (cf. 2.2.1), the abelian category of coherent $\mathcal{D}_X$-modules $\mathcal{M}$ such that $i_*\mathcal{M}$ carries a canonical good filtration. To assure that $\phi\mathcal{M}$ and $\psi\mathcal{M}$ are again regular holonomic $\mathcal{D}_X$-modules we restricted those functors to the category $\mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ (cf. Thm. 1.4(iv)). We shall indicate that in some sense this is necessary too.

Note that, for $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{h}}$, $S(\mathcal{M}) \in \mathrm{D}_c(X)$, the derived category of bounded complexes of $\mathbb{C}_X$-modules with constructible cohomology. On this level S is not an equivalence. The functors Φ_f and Ψ_f are defined on $\mathrm{D}_c(X)$ and take their values in $\mathrm{D}_c(X_0)$. In $\mathrm{D}_c(X_0)$ there exists, for all $\mathcal{F} \in \mathrm{D}_c(X)$, a distinguished triangle

$$\begin{tikzcd} & \Phi_f\mathcal{F} \arrow[dl, "+1"'] & \\ \mathcal{F}|_{X_0} \arrow[rr] & & \Psi_f\mathcal{F}. \arrow[ul] \end{tikzcd}$$

- **Assume** that for any $\mathcal{M} \in \mathcal{R}_{\mathrm{h}} := \mathcal{R} \cap \mathcal{M}od(\mathcal{D}_X)_{\mathrm{h}}$:
  - there exist natural isomorphisms

$$S(\phi\mathcal{M})|_{X_0} \cong \Phi_f(S\mathcal{M})[-1], \qquad S(\psi\mathcal{M})|_{X_0} \cong \Psi_f(S\mathcal{M})[-1].$$

Note that $\phi\mathcal{M}$ and $\psi\mathcal{M}$ are holonomic $\mathcal{D}_X$-modules.

Under these assumptions 2.4.3 still holds i.e., for any $\mathcal{M} \in \mathcal{R}_{\mathrm{h}}$ we have a distinguished triangle in $\mathcal{M}od(\mathcal{D}_X)_{\mathrm{h}}$

$$\begin{tikzcd} & \mathrm{R}\Gamma_{[X_0]}\mathcal{M}[1] \arrow[dl, "+1"'] & \\ \phi\mathcal{M} \arrow[rr] & & \psi\mathcal{M}. \arrow[ul] \end{tikzcd}$$

This triangle corresponds via S to the triangle above (take $\mathcal{F} = S(\mathcal{M})$). This yields $S(\mathrm{R}\Gamma_{[X_0]}\mathcal{M}) = S(\mathcal{M})|_{X_0}$ i.e., $\mathcal{M}$ is regular along X_0. So in order that (•) holds, we have to limit ourselves to regular holonomic $\mathcal{D}_X$-modules.

3 The main theorem

Let X, Y, f, X_0 be as in §2. In this section we prove that the mapping, for all $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\text{hr}}$:

$$\mathcal{M} \mapsto (\mathcal{M}[f^{-1}], \phi\mathcal{M} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathcal{M}, \psi(\pi))$$

(with $\pi: \mathcal{M} \to \mathcal{M}[f^{-1}]$ the canonical map), defines an equivalence of categories. By the Riemann-Hilbert correspondence (cf. 2.5.1) this corresponds to Verdier's extension theorem of perverse sheaves (cf. [V]). Of course this offers a way to prove the above claim, but we prefer to give a derivation using only the language of $\mathcal{D}$-modules (without an appeal to the Riemann-Hilbert correspondence).

3.1 Definitions and notations

First of all we introduce some notations in order to be able to formulate the theorem correctly. Let $\mathcal{M}od(\mathcal{D}_X)_{X_0,\text{hr}}$ denote the category of regular holonomic $\mathcal{D}_X$-modules with support contained in X_0.

3.1.1 Let $\mathcal{C}(\mathcal{D}_X)_{X_0,\text{hr}}$ denote the category determined as follows:

- Objects: quadruples $(\mathcal{M}_1, \mathcal{M}_2, U, V)$ where
  $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{M}od(\mathcal{D}_X)_{X_0,\text{hr}}$,
  $U \in \text{Hom}_{\mathcal{D}_X}(\mathcal{M}_1, \mathcal{M}_2)$, $V \in \text{Hom}_{\mathcal{D}_X}(\mathcal{M}_2, \mathcal{M}_1)$.
- Morphisms: $(\alpha_1, \alpha_2) \in \text{Hom}((\mathcal{M}_1, \mathcal{M}_2, U, V), (\mathcal{M}'_1, \mathcal{M}'_2, U', V'))$
  iff $\alpha_1 \in \text{Hom}_{\mathcal{D}_X}(\mathcal{M}_1, \mathcal{M}'_1)$, $\alpha_2 \in \text{Hom}_{\mathcal{D}_X}(\mathcal{M}_2, \mathcal{M}'_2)$ such that
  $U'\alpha_1 = \alpha_2 U$, $\alpha_1 V = V'\alpha_2$.

$\mathcal{C}(\mathcal{D}_X)_{X_0,\text{hr}}$ is an abelian category. For details we refer to [vD, §1]. Note that for all $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\text{hr}}$ we have $(\phi\mathcal{M}, \psi\mathcal{M}, c(\mathcal{M}), v(\mathcal{M})) \in \mathcal{C}(\mathcal{D}_X)$, where for convenience we dropped "X_0, hr". If $\alpha: \mathcal{M} \to \mathcal{M}'$ is a morphism, then $(\phi(\alpha), \psi(\alpha))$ is a morphism in $\mathcal{C}(\mathcal{D}_X)$.

Notation. In the sequel we use the notation $\mathcal{M}_1 \underset{V}{\overset{U}{\rightleftarrows}} M_2$ to denote the object $(\mathcal{M}_1, \mathcal{M}_2, U, V) \in \mathcal{C}(\mathcal{D}_X)$.

3.1.2 Let $\text{Rc}(X, X_0)$ denote the category determined as follows:

- Objects: triples $(\mathcal{M}, \mathcal{M}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathcal{M}_2, \alpha)$ where
  $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\text{hr}}$ such that $\mathcal{M} \xrightarrow{\cong} \mathcal{M}[f^{-1}]$ (canonical map);
  $\mathcal{M}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathcal{M}_2 \in \mathcal{C}(\mathcal{D}_X)$;
  $\alpha: \mathcal{M}_2 \xrightarrow{\cong} \psi\mathcal{M}$ is a $\mathcal{D}_X$-linear isomorphism such that
  $\alpha UV = c(\mathcal{M})v(\mathcal{M})\alpha$.
- Morphisms:
  $(\beta, \beta_1, \beta_2) \in \text{Hom}((\mathcal{M}, \mathcal{M}_1 \rightleftarrows \mathcal{M}_2, \alpha), (\mathcal{M}', \mathcal{M}'_1 \rightleftarrows \mathcal{M}'_2, \alpha'))$
  iff $\beta: \mathcal{M} \to \mathcal{M}'$ is $\mathcal{D}_X$-linear and
  $(\beta_1, \beta_2) \in \text{Hom}_{\mathcal{C}}(\mathcal{M}_1 \rightleftarrows \mathcal{M}_2, \mathcal{M}'_1 \rightleftarrows \mathcal{M}'_2)$ such that
  $\alpha'\beta_2 = \psi(\beta)\alpha$.

$\mathrm{Rc}(X, X_0)$ is an abelian category. Note that for all $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ $(\mathcal{M}, \mathcal{M}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathcal{M}_2, \alpha) \in \mathrm{Rc}(X, X_0)$, $c(\mathcal{M}): \phi\mathcal{M} \to \psi\mathcal{M}$ is an isomorphism (by 2.4.4.1(iii)). Note furthermore that we have a morphism $(c(\mathcal{M})^{-1}\alpha U, \alpha) \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}_1 \rightleftarrows \mathcal{M}_2, \phi\mathcal{M} \rightleftarrows \psi\mathcal{M})$.

3.1.2.1 Remark
Let $(\mathcal{M}, \mathcal{M}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathcal{M}_2, \alpha) \in \mathrm{Rc}(X, X_0)$. $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) can be given the structure of a $\mathcal{D}_X[t\partial]$-module by defining the action of $t\partial$ as the $\mathcal{D}_X$-endomorphism $VU - 1$ (resp. UV) (1 denotes the identity map on $\mathcal{M}_1$). Let $b \in \mathbb{C}[\Theta]$ be a non-zero polynomial belonging to the canonical good filtration $F^{\cdot} i_* \mathcal{M}$ (cf. (5) of 2.1). This implies:

$$b(t\partial)\psi\mathcal{M} = 0,$$
$$b^{-1}(0) \subset \{ z \in \mathbb{C} \mid 0 \le \mathrm{Re}\, z < 1 \}.$$

Hence $b(t\partial)\mathcal{M}_2 = 0$. Further $Ub(t\partial + 1)\mathcal{M}_1 = b(t\partial)U\mathcal{M}_1 = 0$ i.e., $\partial t b(\partial t)\mathcal{M}_1 = 0$. So we see that $\mathcal{M}_1 \rightleftarrows \mathcal{M}_2$ satisfies the additional requirement (compare with 2.3): there exists a non-zero polynomial $a \in \mathbb{C}[\Theta]$ with:

$$a(t\partial)\mathcal{M}_2 = 0,$$
$$a(t\partial + 1)\mathcal{M}_1 = 0,$$
$$a^{-1}(0) \subset \{ z \in \mathbb{C} \mid 0 \le \mathrm{Re}\, z < 1 \}.$$

3.1.2.2 Remark
Let $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ and denote $\pi: \mathcal{M} \to \mathcal{M}[f^{-1}]$ the canonical map. By Corollary 2.4.4.3 we have that $\psi(\pi)$ is an isomorphism and that $\psi(\pi)c(\mathcal{M})v(\mathcal{M}) = v(\mathcal{M}[f^{-1}])c(\mathcal{M}[f^{-1}])\psi(\pi)$. From this it follows that $(\mathcal{M}[f^{-1}], \phi\mathcal{M} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathcal{M}, \psi(\pi)) \in \mathrm{Rc}(X, X_0)$.
Furthermore if $\alpha: \mathcal{M} \to \mathcal{M}'$ is a morphism, then $(\alpha[f^{-1}], \phi(\alpha), \psi(\alpha))$ is a morphism in $\mathrm{Rc}(X, X_0)$.

3.2 Theorem. *The functor*

$$F: \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}} \to \mathrm{Rc}(X, X_0)$$

defined by, for all $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$

$$\mathcal{M} \mapsto (\mathcal{M}[f^{-1}], \phi\mathcal{M} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathcal{M}, \psi(\pi))$$

establishes an equivalence of categories.

The rest of this subsection is devoted to the proof of the theorem. As we mentioned already this theorem is an analogue of a theorem on extensions of perverse sheaves due to Verdier [V]. Before we begin with the proof we derive two lemmas. These throw some light on how to reconstruct $\mathcal{M}$ from the data $F(\mathcal{M})$. Needless to say that they will be used in the derivation of 3.2.

The difficulty of the derivation lies in the reconstruction. Given an object $(\mathcal{N}, \mathcal{N}_1 \rightleftarrows \mathcal{N}_2, \alpha) \in \mathrm{Rc}(X, X_0)$, find a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ such that $F(\mathcal{M})$ is isomorphic to the given element in $\mathrm{Rc}(X, X_0)$. The idea is to recapture the various levels of the filtration from the given data and thereby reconstructing $\mathcal{M}$. By considerations in §1 $F^0 i_* \mathcal{M}$, the zeroth-level, must equal $F^0 i_* \mathcal{N}$. The first lemma we derive, Lemma 3.2.1 tells us how the (-1)-level can be regained. Successive applications of the second Lemma 3.2.2 take care of the $(-k)$-levels for all $k \geq 2$.

During this process a lot of things need to be checked. This we plan to do in subsection 3.2.3. Finally in subsection 3.2.4 we finish the proof of theorem 3.2.

3.2.1 Lemma. *Let $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$. Denote $F^{\cdot} i_* \mathcal{M}$ the canonical good filtration on $i_* \mathcal{M}$. Denote $\pi: \mathcal{M} \to \mathcal{M}[f^{-1}]$ the canonical map. Consider the commutative diagrams*

$$\begin{tikzcd} F^0 i_* \mathcal{M} \arrow[r, two heads] \arrow[d, "\cong"', "i_*\pi"] & \psi\mathcal{M} \arrow[d, "\cong"', "\psi(\pi)"] \\ F^0 i_* \mathcal{M}[f^{-1}] \arrow[r, two heads] & \psi\mathcal{M}[f^{-1}], \end{tikzcd} \qquad \begin{tikzcd} F^{-1} i_* \mathcal{M} \arrow[r, two heads] \arrow[d, "i_*\pi"] & \phi\mathcal{M} \arrow[d, "\phi(\pi)"] \\ F^{-1} i_* \mathcal{M}[f^{-1}] \arrow[r, two heads] & \phi\mathcal{M}[f^{-1}] \end{tikzcd}$$

where the horizontal arrows are the obvious projections.
Then these are pull-back diagrams of $F^0\mathcal{D}_Y$-modules.

PROOF. By Cor. 2.4.4.3(i) $\psi(\pi)$ is an isomorphism. It follows from the Corollaries 1.6.4 and 1.6.1(2) that $i_*\pi: F^0 i_* \mathcal{M} \to F^0 i_* \mathcal{M}[f^{-1}]$ is an isomorphism. This settles the diagram on the left. The assertion about the diagram on the right is easily verified by chasing in the following commutative diagram with exact rows and exact columns. We leave it to the reader.

$$\begin{tikzcd} & F^{-1} i_* \mathcal{H}^0_{[X_0]}\mathcal{M} \arrow[r, equal] \arrow[d] & \phi(\mathcal{H}^0_{[X_0]}\mathcal{M}) \arrow[d] \\ F^0 i_* \mathcal{M} \arrow[r, hook] \arrow[d, "\cong"', "i_*\pi"] & F^{-1} i_* \mathcal{M} \arrow[r, two heads] \arrow[d, "i_*\pi"] & \phi\mathcal{M} \arrow[d, "\phi(\pi)"] \\ F^0 i_* \mathcal{M}[f^{-1}] \arrow[r, hook] & F^{-1} i_* \mathcal{M}[f^{-1}] \arrow[r, two heads] & \phi(\mathcal{M}[f^{-1}]) \end{tikzcd}$$

3.2.2 Lemma. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module carrying a canonical good filtration $F^{\cdot}\mathcal{M}$. Then, for any $k \in \mathbf{N}$, $k \neq 0$, we have a push-out diagram of $\mathrm{pr}^{-1}\mathcal{D}_X$-modules ($\mathrm{pr}: X \times \mathbf{C} \to X$ denotes the projection)*

$$\begin{tikzcd} F^{-k+1}\mathcal{M} \arrow[r, hook] \arrow[d, "\partial"] & F^{-k}\mathcal{M} \arrow[d, "\partial"] \\ F^{-k}\mathcal{M} \arrow[r, hook] & F^{-k-1}\mathcal{M}. \end{tikzcd}$$

Proof. Let $k \in \mathbb{N}$, $k \neq 0$. We must show that the following sequence is exact (as $\mathrm{pr}^{-1}\mathcal{D}_X$-modules)

$$\begin{array}{ccccc} F^{-k+1}\mathcal{M} & \xrightarrow{r} & F^{-k}\mathcal{M} \oplus F^{-k}\mathcal{M} & \xrightarrow{s} & F^{-k-1}\mathcal{M} \longrightarrow 0. \\ & m \mapsto (-\partial m, m) & & (m, m') \mapsto m + \partial m' & \end{array}$$

Clearly $sr = 0$. Now let $m, m' \in F^{-k}\mathcal{M}$ be such that $s(m, m') = m + \partial m' = 0$. Hence $t\partial m' = -tm \in F^{-k+1}\mathcal{M}$. But $b(t\partial + k)m' \in F^{-k+1}\mathcal{M}$, where $b \in \mathbb{C}[\Theta]$ is a non-zero polynomial as in (5) of 1.3. As $k \neq 0$, $t\partial$ and $b(t\partial + k)$ are relatively prime, this implies that $m' \in F^{-k+1}\mathcal{M}$ and thus $(m, m') = r(m')$. This establishes the exactness in the middle.

Finally let us show that s is surjective. Let $n \in F^{-k-1}\mathcal{M}$. Then $b(t\partial + k + 1)n \in F^{-k}\mathcal{M}$. As $b(t\partial + k + 1) = \partial t\tilde{b}(t\partial + 1) + b(k)$, with $b(k) \in \mathbb{C}^*$ (because $k \neq 0$) and some $\tilde{b} \in \mathbb{C}[\Theta]$, it follows that $n \in F^{-k}\mathcal{M} + \partial F^{-k}\mathcal{M} = \mathrm{Im}\, s$.

Remark. We have seen in 1.1 that $\mathrm{pr}^{-1}\mathcal{D}_X$ is a subring of $F^0\mathcal{D}_Y$. Elements of $\mathrm{pr}^{-1}\mathcal{D}_X$ commute with ∂. In fact $\mathrm{pr}^*\mathcal{D}_X[t\partial] = F^0\mathcal{D}_Y$.

Remark. The lemma implies that $\partial\colon \mathrm{gr}^{-k}\mathcal{M} \to \mathrm{gr}^{-k-1}\mathcal{M}$ is bijective for $k \geq 1$. Compare this with 1.6.6.

3.2.3 The reconstruction procedure

The reconstruction is rather technical. We begin with defining an abelian category $\mathcal{A}$ and an additive subcategory $\mathcal{A}^*$. The category $\mathcal{A}^*$ serves as an intermediate in the construction of a $\mathcal{D}_X$-module $\mathcal{M}$ from the given data $N = (\mathcal{M}, \mathcal{N}_1 \rightleftarrows \mathcal{N}_2, \alpha) \in \mathrm{Rc}(X, X_0)$, such that $F(\mathcal{M}) = N$. We define a functor $P\colon \mathcal{A}^* \to \mathcal{A}^*$ that takes us from the $(-k)$-level to the $(-k-1)$-level of the filtration to exist on $i_*\mathcal{M}$. Repeated applications of P yield an inductive system. Taking the direct limit gives a functor $P^\infty\colon \mathcal{A}^* \to \mathcal{M}od(\mathcal{D}_Y)$ which regains $i_*\mathcal{M}$ from the (-1)-level of the filtration.

Finally in 3.2.3.5 we define a functor $Q\colon \mathrm{Rc}(X, X_0) \to \mathcal{A}^*$ that extracts the (-1)-level from the given element $N \in \mathrm{Rc}(X, X_0)$.

In 3.2.3.6 we consider the composition $P^\infty Q$ and introduce an inverse functor for F, namely $G\colon \mathrm{Rc}(X, X_0) \to \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$.

3.2.3.1 Denote $\mathcal{A}$ the category defined as follows:

- Objects: (G, H, ι, δ) where G and H are $\mathrm{pr}^{-1}\mathcal{D}_X$-modules and $\iota, \delta\colon G \to H$ are $\mathrm{pr}^{-1}\mathcal{D}_X$-morphisms.
- Morphisms: $(\alpha, \beta) \in \mathrm{Hom}_{\mathcal{A}}((G, H, \iota, \delta), (G', H', \iota', \delta'))$ iff $\alpha\colon G \to G'$, $\beta\colon H \to H'$ are $\mathrm{pr}^{-1}\mathcal{D}_X$-morphisms satisfying $\beta\iota = \iota'\alpha$, $\beta\delta = \delta'\alpha$.

$\mathcal{A}$ is an abelian category (because it is a functor category. Cf. [vD], §1). The kernel and the cokernel of a morphism in $\mathcal{A}$ are evident.

Define a map $P: \mathcal{A} \to \mathcal{A}$ as follows: for every $(G, H, \iota, \delta) \in \mathcal{A}$, let $P(G, H, \iota, \delta) = (H, I, \iota_1, \delta_1) \in \mathcal{A}$ be given by the push-out diagram of $\mathrm{pr}^{-1}\mathcal{D}_X$-modules

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[d, "\delta"] & H \arrow[d, "\delta_1"] \\ H \arrow[r, "\iota_1"] & I \end{tikzcd}$$

Because of the universal property of push-out, P is functorial. This yields also that P is rightexact.

3.2.3.2 Denote $\mathcal{A}^*$ the subcategory of $\mathcal{A}$ given as follows:

- Objects: $(G, H, \iota, \delta) \in \mathcal{A}$ that satisfy the additional requirements:
  (i) the $\mathrm{pr}^{-1}\mathcal{D}_X$-structure on G (resp. H) comes from a $F^0\mathcal{D}_Y$-structure on G (resp. H);
  (ii) ι is a $F^0\mathcal{D}_Y$-linear injection;
  (iii) for all $h \in \mathcal{O}_Y$: $\delta h - h\delta = \iota\partial(h)$;
  (iv) $tH \subset \operatorname{Im}\iota$ i.e., $t: H \to H$ factors through ι; [Abusing language we denote the factorisation by $t: H \to G$ (thus $\iota t = t$). There will be no ambiguity for ι is injective.]
  (v) the action of $t\partial \in F^0\mathcal{D}_Y$ on G (resp. H) is given by the $\mathrm{pr}^{-1}\mathcal{D}_X$-endomorphism $t\delta$ (resp. $\delta t - 1$).
- Morphisms: (α, β) as above but α and β are now supposed to be $F^0\mathcal{D}_Y$-linear.

Certainly $\mathcal{A}^*$ is an additive subcategory of $\mathcal{A}$; it is not abelian, because of the injectivity condition in (ii). But it perfectly makes sense to talk about exact sequences in $\mathcal{A}^*$, namely those sequences which are exact when considered in $\mathcal{A}$.

3.2.3.3 Lemma. *P restricts to an exact functor from $\mathcal{A}^*$ to $\mathcal{A}^*$ which we still denote by P; let $(G, H, \iota, \delta) \in \mathcal{A}^*$ and let us put $(H, I, \iota_1, \delta_1) = P(G, H, \iota, \delta)$. Then there exists a unique structure of a $F^0\mathcal{D}_Y$-module on I that satisfies (i) to (v) above.*

Proof. Let us first see that $P(\mathcal{A}^*) \subset \mathcal{A}^*$. Let $(G, H, \iota, \delta) \in \mathcal{A}^*$. Then $P(G, H, \iota, \delta)$ is given by the push-out of $\mathrm{pr}^{-1}\mathcal{D}_X$-modules

$$\begin{tikzcd} G \arrow[r, hook, "\iota"] \arrow[d, "\delta"] & H \arrow[d, "\delta_1"] \\ H \arrow[r, "\iota_1"] & I \end{tikzcd}$$

I is a $\mathrm{pr}^{-1}\mathcal{D}_X$-module. We extend this structure to one over $F^0\mathcal{D}_Y$ as follows. Let $h \in \mathcal{O}_Y$. Consider the $\mathbb{C}$-linear mappings

$$\delta_1 h - \iota_1\partial(h): H \to I, \qquad \iota_1 h: H \to I.$$

These satisfy

$$\begin{aligned}(\delta_1 h - \iota_1\partial(h))\iota &= \delta_1\iota h - \iota_1\iota\partial(h) \qquad (\iota \text{ is } F^0\mathcal{D}_Y\text{-linear})\\ &= \iota_1(\delta h - \iota\partial(h))\\ &= \iota_1 h\delta \qquad (\text{by (iii) of 3.2.3.2}).\end{aligned}$$

Thus by the universal property of push-out there exists a unique $\mathbb{C}$-linear map, denoted $h: I \to I$, satisfying $h\delta_1 = \delta_1 h - \iota_1\partial(h)$ and $h\iota_1 = \iota_1 h$. This defines the structure of a $\mathrm{pr}^*\mathcal{D}_X$-module on I, extending the $\mathrm{pr}^{-1}\mathcal{D}_X$-structure and satisfying (iii) of 3.2.3.2.

Further ι_1 becomes $\mathrm{pr}^*\mathcal{D}_X$-linear. Observe that ι_1 is injective. Note that $\delta_1 t - \iota_1 = \iota_1(\delta t - 1)$, so $t: I \to I$ factors through $\iota_1: H \to I$. This yields 3.2.3.2(iv).

Denote $t: I \to H$ the factorisation. One has $t\delta_1 = \delta t - 1$ i.e., the action of $t\partial \in F^0\mathcal{D}_Y$ on H is given by $t\delta_1$. Define the action of $t\partial \in F^0\mathcal{D}_Y$ on I to be $\delta_1 t - 1$. This gives I the desired structure of a $F^0\mathcal{D}_Y$-module i.e., 3.2.3.2(i) and establishes 3.2.3.2(iv).

Note further that $(t\delta)\iota_1 = (\delta_1 t - 1)\iota_1 = \delta_1 t - \iota_1 = \iota_1(\delta t - 1) = \iota_1(t\partial)$, hence ι_1 is $F^0\mathcal{D}_Y$-linear. This establishes 3.2.3.2(ii) and all together $P(\mathcal{A}^*) \subset \mathcal{A}^*$.

The next thing we must check is the functoriality of $P|_{\mathcal{A}^*}$. So let $(\alpha,\beta): (G, H, \iota, \delta) \to (G', H', \iota', \delta')$ be a morphism in $\mathcal{A}^*$. By the universal property of push-out there exists a unique $\mathrm{pr}^{-1}\mathcal{D}_X$-linear map $\gamma: I \to I'$ satisfying $\gamma\delta_1 = \delta_1'\beta$, $\gamma\iota_1 = \iota_1'\beta$, where I is as above and $(H', I', \iota_1', \delta_1') = P(G', H', \iota', \delta')$. Thus $P(\alpha,\beta) = (\beta,\gamma)$. We must verify that γ is $F^0\mathcal{D}_Y$-linear. Therefore let $h \in \mathcal{O}_Y$ and consider the $\mathbb{C}$-linear mapping $\gamma h - h\gamma: I \to I'$. It satisfies:

$$\begin{aligned}(\gamma h - h\gamma)\delta_1 &= \gamma h\delta_1 - h\gamma\delta_1\\ &= \gamma(\delta_1 h - \iota_1\partial(h)) - h\delta_1'\beta\\ &= \delta_1'\beta h - \iota_1'\beta\partial(h) - h\delta_1'\beta\\ &= (\delta_1' h - \iota_1'\partial(h) - h\delta_1')\beta = 0\end{aligned}$$

and

$$(\gamma h - h\gamma)\iota_1 = \gamma\iota_1 h - h\iota_1'\beta = \iota_1'(\beta h - h\beta) = 0.$$

It follows, by the universal property of push-out, that $\gamma h - h\gamma = 0$. Consequently γ is $\mathrm{pr}^*\mathcal{D}_X$-linear. Especially we have $\iota_1'\beta t = \gamma\iota_1 t = \iota_1' t\gamma$, yielding $\beta t = t\gamma$ as ι_1' is injective. This implies

$$\gamma(t\partial) = \gamma(\delta_1 t - 1) = \gamma\delta_1 t - \gamma = \delta_1'\beta t - \gamma = (\delta_1' t - 1)\gamma = (t\partial)\gamma$$

i.e., γ is $F^0\mathcal{D}_Y$-linear.

Finally the fact that ι is injective implies that P is exact.

3.2.3.4 From $\mathcal{A}^*$ to $\mathcal{M}od(\mathcal{D}_Y)$

Let $A \in \mathcal{A}^*$. Lemma 3.2.2 suggests that we should take a series of push-outs, yielding an inductive system $\{((P^k A)_1, \iota_k) \mid k \in \mathrm{N}\}$ of $F^0\mathcal{D}_Y$-modules. Here for any $j \in \{1,2,3,4\}$ $(\cdot)_j$ denotes projecting on the j-th factor; for all $k \in \mathrm{N}$, $\iota_k := (P^k A)_3$ is an injective $F^0\mathcal{D}_Y$-morphism from $(P^k A)_1$ into $(P^{k+1}A)_1$.

Put $P^\infty A := \text{inj.}\lim (P^k A)_1 \in \mathcal{M}od(F^0\mathcal{D}_Y)$.

We give $P^\infty A$ a $\mathcal{D}_Y$-structure as follows. For any $k \in \mathrm{N}$ define $\delta_k := (P^k A)_4$ a $\mathrm{pr}^{-1}\mathcal{D}_X$-linear map from $(P^k A)_1$ to $(P^{k+1}A)_1$. These satisfy $\delta_{k+1}\iota_k = \iota_{k+1}\delta_k$, for all $k \in \mathrm{N}$, yielding a $\mathrm{pr}^{-1}\mathcal{D}_X$-endomorphism δ of $P^\infty A$. For any $h \in \mathcal{O}_Y$ we have

$$\delta h = \text{inj.}\lim(\delta_k h) = \text{inj.}\lim(h\delta_k + \iota_k\partial(h)) = h\delta + \partial(h).$$

Consequently $P^\infty A$ becomes a $\mathcal{D}_Y$-module by defining the action of $\partial \in \mathcal{D}_Y$ as the endomorphism δ. Obviously $P^\infty A$ is functorial in A i.e., we get an exact functor

$$P^\infty : \mathcal{A}^* \to \mathcal{M}od(\mathcal{D}_Y).$$

Note that $(P^k A)_1$ may be regarded as a $F^0\mathcal{D}_Y$-submodule of $P^\infty A$. These induce a filtration on $P^\infty A$.

3.2.3.5 From $\mathrm{Rc}(X, X_0)$ to $\mathcal{A}^*$

The final step (that is the first step of the reconstruction) is to define a functor $Q: \mathrm{Rc}(X, X_0) \to \mathcal{A}^*$. Its definition is suggested by Lemma 3.2.1.

Let $(\mathcal{N}, \mathcal{N}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathcal{N}_2, \alpha) \in \mathrm{Rc}(X, X_0)$. One has:

(i) $\mathcal{N} = \mathcal{N}[f^{-1}]$, thus $c(\mathcal{N})$ is an isomorphism (cf. Prop. 2.4.4.1.);

(ii) $\mathcal{N}_1$, $\mathcal{N}_2$, $\phi\mathcal{N}$, $\psi\mathcal{N}$ have the structure of a module over $\mathrm{gr}^0\mathcal{D}_Y = \mathcal{D}_X[t\partial]$ (see 2.2.3) and thus a $F^0\mathcal{D}_Y$-structure;

(iii) α and $c(\mathcal{N})^{-1}\alpha U$ are $F^0\mathcal{D}_Y$-linear;

(iv) U, V, $c(\mathcal{N})$, $v(\mathcal{N})$ commute with $\mathrm{pr}^{-1}\mathcal{D}_X \subset F^0\mathcal{D}_Y$.

Denote $F^\cdot i_*\mathcal{N}$ the canonical good filtration on $i_*\mathcal{N}$ and define $G := F^0 i_*\mathcal{N}$. Consider the diagram of $F^0\mathcal{D}_Y$-modules

$$\begin{tikzcd} & \mathcal{N}_1 \arrow[d, "c(\mathcal{N})^{-1}\alpha U"] \\ F^{-1}i_*\mathcal{N} \arrow[r, two heads] & \phi\mathcal{N}. \end{tikzcd}$$

Define H to be the pull-back (as $F^0\mathcal{D}_Y$-modules). This yields a commutative diagram with exact rows

$$\begin{tikzcd} G \arrow[r, hook, "\iota"] \arrow[d, equal] & H \arrow[r, two heads] \arrow[d] & \mathcal{N}_1 \arrow[d, "c(\mathcal{N})^{-1}\alpha U"] \\ F^0 i_*\mathcal{N} \arrow[r, hook] & F^{-1}i_*\mathcal{N} \arrow[r, two heads] & \phi\mathcal{N} \end{tikzcd}$$

Evidently we have a pull-back diagram

$$\begin{tikzcd} G \arrow[r, two heads] \arrow[d, equal] & \mathcal{N}_2 \arrow[d, "\alpha", "\cong"'] \\ F^0 i_*\mathcal{N} \arrow[r, two heads] & \psi\mathcal{N}. \end{tikzcd}$$

By the universal property of pull-backs $V: \mathcal{N}_2 \to \mathcal{N}_1$ induces an unique $\mathrm{pr}^{-1}\mathcal{D}_X$-linear map $\delta: G \to H$.

We verify that $(G, H, \iota, \delta) \in \mathcal{A}^{\bullet}$. Clearly 3.2.3.2(i), (ii) are true. Also 3.2.3.2(iii) is easily checked. Of course $U: \mathcal{N}_1 \to \mathcal{N}_2$ agrees with $t: H \to G$, which establishes 3.2.3.2(iv). Finally the action of $t\partial$ on G (resp. H) is given by $t\delta$ (resp. $\delta t - 1$), which takes care of 3.2.3.2(v).

Clearly this construction is functorial and therefore yields a functor

$$Q: \mathrm{Rc}(X, X_0) \to \mathcal{A}^{\bullet}.$$

Note furthermore that Q is exact.

3.2.3.6 The inverse functor G

In this subsection we investigate the effect of the functor $P^{\infty}Q$ on objects in the image of F, the one in theorem 3.2. Does the reconstruction work well?

Let $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ and denote $F^{\cdot} i_*\mathcal{M}$ the canonical good filtration on $i_*\mathcal{M}$. For all $k \in \mathbf{N}$ denote by $\iota_k: F^{-k} i_*\mathcal{M} \to F^{-k-1} i_*\mathcal{M}$ the inclusion. By Lemma 3.2.1 and the definition of Q there exists a natural isomorphism

$$Q(F\mathcal{M}) \cong (F^0 i_*\mathcal{M}, F^{-1} i_*\mathcal{M}, \iota_0, \partial).$$

Applying P^k to both sides and using Lemma 3.2.2 yields a natural isomorphism

$$P^k Q(F\mathcal{M}) \cong (F^{-k} i_*\mathcal{M}, F^{-k-1} i_*\mathcal{M}, \iota_k, \partial), \qquad \text{for all } k \in \mathbf{N}.$$

Hence there exists a natural isomorphism

$$P^{\infty} Q(F\mathcal{M}) \cong i_*\mathcal{M}.$$

Therefore the next definition doesn't come as a surprise. Define

$$G: \mathrm{Rc}(X, X_0) \to \mathcal{M}od(\mathcal{D}_X)$$

by putting for all $M \in \mathrm{Rc}(X, X_0)$,

$$G(M) := \mathrm{Ker}(t - f, P^{\infty} Q(M)).$$

The foregoing can then be restated as: there exists a natural isomorphism $GF(\mathcal{M}) \cong \mathcal{M}$, for all $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$.

3.2.4 Proof of theorem 3.2

It remains to verify that for all $M \in \mathrm{Rc}(X, X_0)$:

(i) $G(M) \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$;
(ii) $FG(M) \cong M$, functorial in M.

(i) Let $M = (\mathcal{N}, \mathcal{N}_1 \overset{U}{\underset{V}{\rightleftarrows}} \mathcal{N}_2, \alpha) \in \mathrm{Rc}(X, X_0)$. In $\mathcal{C}(\mathcal{D}_X)$ we have an exact sequence

$$\begin{array}{ccccccc} \operatorname{Ker} U & & \mathcal{N}_1 & & \phi\mathcal{N} & & \operatorname{Coker} U \\ \uparrow\downarrow & \hookrightarrow & {\scriptstyle V}\upharpoonleft\downharpoonright{\scriptstyle U} & \xrightarrow{\binom{c^{-1}\alpha U}{\alpha}} & {\scriptstyle v}\upharpoonleft\downharpoonright{\scriptstyle c} & \twoheadrightarrow & \upharpoonleft\downharpoonright \\ 0 & & \mathcal{N}_2 & & \psi\mathcal{N} & & 0 \end{array}$$

This yields an exact sequence in $\mathrm{Rc}(X, X_0)$

$$(*) \qquad F(\operatorname{Ker} U) \hookrightarrow M \longrightarrow F(\mathcal{N}) \twoheadrightarrow F(\operatorname{Coker} U).$$

Here

$$F(\operatorname{Ker} U) = (0, \operatorname{Ker} U \rightleftarrows 0, 1),$$

$$F(\operatorname{Coker} U) = (0, \operatorname{Coker} U \rightleftarrows 0, 1),$$

because $\operatorname{Ker} U$ and $\operatorname{Coker} U$ are regular holonomic $\mathcal{D}_X$-modules with support contained in X_0 (cf. Prop. 2.4.4.2);

$$F(\mathcal{N}) = (\mathcal{N}, \phi\mathcal{N} \overset{c}{\underset{v}{\rightleftarrows}} \psi\mathcal{N}, 1)$$

because $\mathcal{N} = \mathcal{N}[f^{-1}]$. By 3.2.3.6 we obtain an exact sequence (for P^∞ and Q are exact) of $\mathcal{D}_Y$-modules

$$i_* \operatorname{Ker} U \hookrightarrow P^\infty Q(M) \longrightarrow i_*\mathcal{N} \twoheadrightarrow i_* \operatorname{Coker} U.$$

It follows that $P^\infty Q(M)$ is supported on $i(X)$, so applying the functor $\operatorname{Ker}(t - f, \cdot)$ yields an exact sequence of $\mathcal{D}_X$-modules

$$\operatorname{Ker} U \hookrightarrow G(M) \longrightarrow \mathcal{N} \twoheadrightarrow \operatorname{Coker} U.$$

So finally we arrive at the conclusion that $G(M)$ is a regular holonomic $\mathcal{D}_X$-module and $G(M)[f^{-1}] \xrightarrow{\sim} \mathcal{N}$.

(ii) Let $M \in \mathrm{Rc}(X, X_0)$ be as above; then $i_*G(M) = P^\infty Q(M)$. Let $k \in \mathbb{N}$. Applying the exact functor $P^k Q$ to the exact sequence $(*)$ yields (by 3.2.3.6) an exact sequence of $F^0\mathcal{D}_Y$-modules

$$F^{-k} i_* \operatorname{Ker} U \hookrightarrow (P^k Q(M))_1 \longrightarrow F^{-k} i_*\mathcal{N} \twoheadrightarrow F^{-k} i_* \operatorname{Coker} U.$$

It follows that $(P^kQ(M))_1$ is a coherent $F^0\mathcal{D}_Y$-module, for every $k \in \mathbf{N}$. By construction $i_*G(M) = P^\infty Q(M)$ carries a filtration $F^{\cdot}i_*G(M)$, where for $k \in \mathbf{Z}$

$$F^k i_*G(M) := \begin{cases} \mathrm{Im}((P^{-k}Q(M))_1 \hookrightarrow i_*G(M)), & \text{if } -k \in \mathbf{N} \\ t^k F^0 i_*G(M), & \text{if } \quad k \in \mathbf{N}^*. \end{cases}$$

So we have established that this filtration satisfies 1.2 (3). By construction of P^∞ it satisfies 1.2, (2) and (4). The definition of the $\mathcal{D}_Y$-structure on $P^\infty Q(M)$ implies that the filtration fulfils 1.2 (1). Hence it is a good filtration. By 3.1.2.1 there exists a non-zero polynomial $b \in \mathbb{C}[\Theta]$ with: $b^{-1}(0) \subset \{ z \in \mathbb{C} \mid 0 \leq \mathrm{Re}\, z < 1 \}$, $b(t\partial)\mathcal{N}_2 = 0$ and $b(t\partial + 1)\mathcal{N}_1 = 0$. Clearly this implies 1.3 (5) i.e., it is the canonical good filtration on $i_*G(M)$. Consequently $FG(M) \cong M$.

We leave it to the reader to verify that the isomorphism is functorial in M.

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CLASSIFICATIE VAN REGULIERE HOLONOME $\mathcal{D}$-MODULEN

SAMENVATTING

Dit proefschrift vormt een neerslag van onderzoek aan het classificatieprobleem van reguliere holonome $\mathcal{D}$-modulen. Het proefschrift bestaat uit vier hoofdstukken. Te weten een inleiding gevolgd door drie hoofdstukken die afzonderlijk gepubliceerd zijn en/of zullen worden.

De strategie die bij het onderzoek gevolgd werd komt het best tot uiting in hoofdstuk vier. Het betreft de reductie van het classificatieprobleem tot een probleem in lagere dimensie. We bewijzen hier een stelling analoog aan Verdier's uitbreidingsstelling voor perverse schoven. In het bijzonder wordt de classificatie van reguliere holonome $\mathcal{D}$-modulen met als singulier support een irreducibele vlakke kromme herleid tot een probleem van classificeren van paren van $\mathcal{D}$-modulen met support op die kromme.

Hoofdstuk drie betreft gezamelijk werk van de auteur met Dr. A.R.P. van den Essen. Hierin wordt aangetoond dat de twee categorieën "$\mathcal{D}$-modulen met support op een irreducibele kromme" en "$\mathcal{D}_1$-modulen met reguliere singulariteiten te 0", equivalent zijn. Aldus vindt een herleiding plaats van bovenstaand classificatieprobleem tot een eenvoudigere situatie met normale kruisingen.

De classificatie in het laatste geval gebeurt onder meer in het tweede hoofdstuk van het proefschrift. Dit deel betreft de eerste kennismaking van de auteur met het classificatieprobleem van reguliere holonome $\mathcal{D}$-modulen. We behandelen hier het geval van normale kruisingen.

In hoofdstuk één wordt allereerst een overzicht van de theorie van $\mathcal{D}$-modulen gegeven. Daarna volgt een drietal toepassingen van reguliere holonome $\mathcal{D}$-modulen. Dit ter motivatie voor het onderzoek van de categorie van reguliere holonome $\mathcal{D}$-modulen. Tenslotte geven we een overzicht van de tot nu toe bekende resultaten betreffende de classificatie.

CURRICULUM VITAE

De auteur van dit proefschrift werd op 14 oktober 1953 te Vinkel (gem. Nuland) geboren. Van september 1966 tot juli 1970 bezocht hij de R.K. Streekschool voor ULO te Rosmalen. Hier behaalde hij in juni 1970 het diploma MULO-B. Vervolgens bezocht hij van september 1970 tot juli 1975 de HTS te 's-Hertogenbosch. Hier studeerde hij elektrotechniek en behaalde in juni 1975 het diploma HTS-elektrotechniek. In september 1975 werd hij onder de wapenen geroepen. In september 1976 kreeg de auteur vervroegd verlof om wiskunde te gaan studeren. Van september 1976 tot juli 1981 studeerde hij aan de Katholieke Universiteit Nijmegen. In september 1978 behaalde hij het MO-A diploma een jaar later gevolgd door het MO-B diploma en het kandidaats wiskunde (cum laude). In juni 1981 legde hij het doctoraalexamen wiskunde af (cum laude). In juli 1982 trad de auteur als wetenschappelijk medewerker in dienst bij het Mathematisch Instituut van de Katholieke Universiteit Nijmegen. In 1983 begon de confrontatie met de $\mathcal{D}$-modulen onder leiding van Prof. Dr. A.H.M. Levelt.

STELLINGEN

behorende bij het proefschrift

"Classification of regular holonomic $\mathcal{D}$-modules"

van

M.G.M. van Doorn

1. De definitie van het begrip "duale connectie" in hoofdstuk 3, §2.2 in het erg mooie boek *"Algebraic $\mathcal{D}$-Modules"* van A. Borel *et al.* is niet correct.

   Cf. A. Borel *et al.*, *Algebraic $\mathcal{D}$-Modules*. Perspectives in Mathematics, 2. Academic Press (1987).

2. Met het computeralgebrasysteem MACSYMA kan men soms fouten in integraaltafels vinden. Soms lukt ook het omgekeerde, zoals blijkt bij berekening door MACSYMA van $\int_0^{4\pi}(2+\sin x)^{-1}\,dx$.

   Cf. [1] R. Pavelle en P.S. Wang, MACSYMA from F to G, *J. Symbolic Computation* **1** (1985), p.81.

   [2] *Standard Mathematical Tables*. Twentieth edition. The Chemical Rubber Co. (1972). Voorbeeld op pagina 394.

3. Het bewijs van Habicht's stelling dat R. Loos geeft is niet vlekkeloos. In tegenstelling tot zijn bewering is bovendien een rechtstreeks bewijs van de subresultantenstelling eenvoudiger dan het zijne dat gebaseerd is op Habicht's stelling.

   Cf. R. Loos, Generalized Polynomial Remainder Sequences, in *Computer Algebra, Symbolic and Algebraic Computation*. Second edition. Springer-Verlag (1983), 115–137.

4. Het bewijs van Proposition 14.1 in F. Pham, *Singularités des Systèmes Différentiels de Gauss-Manin*. Progress in Mathematics, 2. Birkhäuser (1979), p.136 deugt niet.

5. Het is jammer dat zo weinig wiskundigen kennis nemen van moderne verworvenheden in de wiskunde.

   "How sad that modern science has come to this pass. Not only are we misunderstood by the world at large, but so few of us actually understand the greatest achievements in our own fields."

   Cf. S. Bloch, in *Bull. Amer. Math. Soc. (N.S.)* **4** (1981), 235–239.

6. Het schort veel beginnende studenten aan elementaire vaardigheid in logisch redeneren. Ook het handschrift is vaak niet om over naar huis te schrijven.

7. Een recensent dient zich bij de bespreking van een artikel niet te beperken tot een citaat uit de inleiding.

8. Het is vermetel om te menen dat een theorie die de vier fundamentele natuurkrachten in zich verenigt het einde van de fysica zou betekenen.
   Cf. J. de Kam, 'Het einde van de fysica?', *Intermediair* **19** (1987).

9. Het verband dat gelegd wordt tussen Oosterse mystiek en fysica is slechts schijnbaar en moet niet serieus genomen worden.

10. Het is een vergissing aan te nemen dat goede preventie de gezondheidszorg goedkoper maakt.

11. De Nederlandse agrariër dreigt aan zijn eigen vlijt ten onder te gaan.

12. In het tijdperk van de glasvezelkabel worden de PC's in het Mathematisch Instituut d.m.v. staalkabel met elkaar verbonden.