MASTERARBEIT

TITEL:

"An alternative approach to the resolution of singularities of toric varieties."

Verfasserin

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Introduction

The problem of *Resolution of Singularities* consists of interpreting an algebraic variety with singular points as the projection of a smooth algebraic variety, and finding out how the projection map can be constructed. Projection means here a specific kind of morphism, namely a birational proper map, which defines an isomorphism outside the singular points of the variety.

Mathematicians have tried several techniques to address the problem. Examples of these techniques are normalization and blowups. A normalization is a transformation of the variety which, in terms of its coordinate ring[1], is given by its integral closure. Geometrically, it reduces all singularities to codimension ≥ 2. In the case of curves this gives a smooth variety (and thus a resolution), for surfaces it gives a variety with at most isolated singularities, and for higher dimension a variety whose singular locus is relatively small. On the other hand, blowups are elementary transformations given by morphisms. Each blowup simplifies[2] the singularities of the variety in some level, by replacing a subvariety which we will call the center by a hypersurface. This transformation is just the projection we already mentioned. Each technique uses one or several properties of the variety, and defines an invariant to measure the improvement of the singular locus along the resolution process. This invariant can be computed using, for instance, the coordinate ring or the equations of the variety.

Nowadays, the case of varieties over fields of characteristic zero is solved. Hironaka proved (see [28]) that for this kind of varieties, a suitably defined finite sequence of blowups with centers inside of the singular locus gives a resolution of the singularities. However, the proof given by Hironaka is quite complicated, and many authors have worked on simplifications of the proof, see [41], [42], [3], [15], [16], [14]. For varieties over fields of positive characteristic, some results are also known for low dimensions (see [1], [8], [9], [34]), but the general case is still an open problem.

The present work is focused on a particular class of varieties, namely toric varieties, which we will also call here *monomial varieties*.[3] The coordinate ring of a monomial variety[4] is a K-algebra generated by a finite set of monomials. Equivalently, they are defined as the zero set of a finite number of binomial

[1] We will look at varieties locally, so that we can reduce to the affine case. To do this, one can consider each affine chart of an abstract variety separately as an affine variety. Then we have one coordinate ring for each affine cahrt.

[2] Provided that smooth centers are chosen inside the singular locus

[3] The purpose of this name is to remark the property we are more interested in.

[4] Locally, the coordinate ring of an affine chart of an abstract monomial variety.

equations. Therefore, a set of vectors can be associated to each monomial variety, see [11]. These vectors correspond to the exponents of the generators of the K-algebra. This collection of vectors is a set of combinatorial data about the variety, which we want to use for the resolution of its singularities. In fact, it is a system of generators of a subsemigroup of $\mathbb{Z}^n$, where n is the dimension of the variety. We will define certain transformations of this semigroup that can be translated into morphisms of the algebraic varieties represented by them. Our interest in this property relies on the fact that certain blowups can be codified this way.

One of the objectives here is to describe blowups using a system of generators of the semigroup of the variety, and find with this description an algorithm for resolution of singularities of monomial varieties. By an algorithm we mean a set of rules to find well defined transformations on a given variety, and which guarantee resolution after a finite number of transformations. A variety will be resolved when all affine charts are smooth.

To find a resolution of singularities of a monomial variety, one could use the toric resolution (see [18], [32], [11]) which associates a fan to each variety, and then divides the cones in this fan, the so called star subdivision, in order to achieve certain properties associated to smooth varieties. Still, this machinary is relatively complicated, and this decomposition is not unique. Also the computations are highly complex. One could also use the classical way of performing sequences of blowups using the binomial ideal defining the variety (see [40]). The approach we purpose here does not use the equations or the fans of the varieties. Our goal is to construct a method in which all computations can be done through the semigroup associated to the variety: we wish to prescribe for each semigroup a center of a blowup based on a minimal system of generators of this semigroup, such that blowing up along this center gives a resolution of singularities after finitely many iterations. Once the center is clear, blowups can be performed by transformations of this system of generators. Some work using this combinatorial data can be found in [4], [5], and in [20] it is also used for the nash blowup.

In the present work, we will not be able to carry out this program completely. Nevertheless, we present all the ingredients in detail, we provide some results obtained from the study of this problem, as well as the difficulties found.

A simple example that illustrates this idea for plane affine curves is the resolution of the singularities of the cusp, of equation $C : \left\{x^3 - y^2 = 0\right\}$. We start with the coordinate ring of this curve,

$$K[x,y]/(x^3 - y^2) \cong K[t^2, t^3].$$

The semigroup associated to this curve is generated by the set $\{2,3\} \subset \mathbb{Z}$. We call this semigroup $\Gamma = 2\mathbb{N} + 3\mathbb{N} \subset \mathbb{Z}$, and we denote $K[t^2, t^3] = K[\Gamma]$. The only possible center inside of the singular locus of C is the origin $\{0\} = V(x, y)$. The blowup of C with this center has two affine charts, one for x and one for y. Alternatively, we can construct two new semigroups, Γ_1 and Γ_2 from Γ. Each of these new semigroups characterizes one affine chart of the blowup. The semigroup corresponding to the affine chart for x is obtained by subtracting the first element of the system of generators of the semigroup to the second one. The construction of Γ_2 is symmetric.

What is being performed here is what we could call a symmetric division algorithm: when we subtract one integer a to an integer b several times, before obtaining an integer of the opposite sign of b we obtain the rest r of the euclidean division b/a. Here we subtract in parallel calculations $b-a$ and $a-b$. Mantaining this symmetry for each step we obtain a ramification represented in the following tree

$$\begin{tikzcd}[column sep=small]
 & & & \Gamma \arrow[dll] \arrow[d] \arrow[dr] \arrow[drr] & & & \\
 & \Gamma_1 \arrow[dl] \arrow[d] \arrow[dr] & & \Gamma_2 & \cdots & \Gamma_n \arrow[dl] \arrow[d] \arrow[dr] & \\
\Gamma_{1,1} & \cdots & \Gamma_{1,n} & \cdots & \Gamma_{n,1} & \cdots & \Gamma_{n,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Each branch contains the semigroups associated to the intermediate charts for each blowup of a sequence, leading to one of the final affine charts of the sequence. Here n is the number of monomials generating $K[\Gamma]$.

For the last example we obtain

$$\begin{tikzcd}
 & \{2,3\} \arrow[dl] \arrow[dr] & \\
\{2,1\} & & \{-1,3\}
\end{tikzcd}$$

The corresponding coordinate rings are

$$\begin{tikzcd}
 & K[t^2,t^3] \arrow[dl] \arrow[dr] & \\
K[t^2,t] \cong K[t] & & K[\frac{1}{t},t^3] \cong K[\frac{1}{t},t]
\end{tikzcd}$$

The affine charts of this blowup correspond to the affine line and to the hyperbola respectively, which implies that the variety is resolved.

This process is quite easy for curves, and success is guaranteed after a finite number of steps by the Euclidean division algorithm in $\mathbb{Z}$ for each chart, as Bézout identity asserts: "Let a, b be two integers with $\gcd(a, b) = c$. Then there exist integers p, q such that $ap + bq = c$." The center is unique, and it is easy to determine the improvement in this case, as it will be shown in section 3.5. Unfortunately, for monomial varieties of higher dimension, there are many things to take into account. Already for monomial surfaces, there is more than one possible choice of the center. Moreover, it is not clear how to measure the improvement of the singularities by the semigroup, and there is not an analogue theorem to the division algorithm for pairs of integers.

We want to develop a strategy for resolution on monomial varieties using this. For this reason, we study here the way in which the information given by the sets of integers above mentioned can be interpreted. We will try to decide which of this information is relevant for the process of resolution. Of special interest is to find a description of the condition of smoothness in terms of this set of integers. We will see that the relative position of the vectors in space determines whether the variety is smooth or not.

The material presented is organized in the following way: The first chapter is dedicated to the basics of algebraic geometry, necessary for resolution of singularities. Basic knowledge about commutative algebra, concerning ring theory, as well as Zariski topology is assumed. In the second chapter, the concept of *toric variety* is introduced, together with the basics of the theory developed on this field. This includes some notions on convex geometry, which is used for the study of toric varieties. In the third chapter, the ideas that we want to apply for the resolution of singularities in this particular approach are presented, together with conclusions and results. Finally, the fourth chapter contains some analysis of particular aspects of monomial varieties, which have appeared during the study of their properties.

Note: All the images that appear in this thesis have been created using *surfex* for surfaces in 3 dimensions, and *pstricks* for cones, lattices, affine spaces and varieties of dimension 2.

Chapter 1

Introduction to monomial varieties

In this first section, we review some general theory of algebraic geometry. For basic definitions we refer to the apendix. We intend to build the necessary tools to introduce the basic ideas about resolution of singularities. Morphisms and regular functions are presented. Then we will explain a method of resolution: the blowup. To finish this section, we will introduce the kind of varieties we want to focus on, which we will call monomial.
Basic knowledge on commutative algebra is assumed. For a complete vision of the bases of algebraic geometry, see [22], [13], [39], [37], [38] and [21]. For results on commutative algebra see [35].

1.1 Algebraic varieties

Let K be an algebraic closed field of characteristic zero. We understand by an *n-dimensional affine algebraic variety* $X \subset \mathbb{A}^n_K$ over K the zero locus of a prime ideal $I = I(X) \subset K[x_1, \ldots, x_n]$. We denote by $K[X]$ the coordinate ring of the affine algebraic variety X.

A *projective algebraic variety* $Y \subset \mathbb{P}^n_K$ over K is the zero locus of a collection of homogeneous polynomials in $K[x_1, \ldots, x_{n+1}]$.

More generally, a *quasi-projective algebraic variety* $Z \subset \mathbb{P}^n_K$ is an open subset of a projective variety. Both affine and projective algebraic varieties are quasi-projective varieties.

In what follows, we will work with algebraic varieties over a fixed field K as above. For more background on algebraic varieties and properties see the Appendix.

1.2 Regular functions and morphisms

1.2.1 Regular functions

Let X be an n-dimensional affine algebraic variety over K, and let $p \in X$ a point. A *regular function at* p is a map $f : X \to K$ for which there exists an open neighborhood $U \subset X$ of p such that $f = \frac{g}{h}$ on U for some polynomials $g, h \in K[x_1, \ldots, x_n]$, with $h \neq 0$ on U. A *regular function on* X is a function $f : X \to K$ such that f is regular at every point of X. The ring of all regular functions on X is denoted by $\mathcal{O}_X(X)$.

Remark 1. If $K[X]$ is the affine coordinate ring of the affine variety X, then $\mathcal{O}_X(X) \cong K[X]$ (see [22, Theorem 3.2]).

Now let $X \subset \mathbb{P}^n$ be a quasi-projective variety over K of dimension n. A function $f : X \to K$ is *regular* at a point $p \in X$ if there exists an open neighborhood $U \subset X$ of p such that $f = \frac{g}{h}$ on U for some homogeneous polynomials $g, h \in K[x_0, \ldots, x_n]$ of the same degree, $h \neq 0$ on U. The function f is regular in X if it is regular at every point of X. We denote $\mathcal{O}_X$ the set of regular functions on X, while $\mathcal{O}_X(U)$ denotes the ring of regular functions defined on the open subset $U \subset X$. Since X can be covered by $n+1$ affine charts, $\mathcal{O}_X$ can be constructed from the rings of regular functions corresponding to these affine charts, with a sheaf structure.

1.2.2 Morphisms

Let $X, Y \subset \mathbb{P}^n$ be two affine or quasi-projective algebraic varieties. A *morphism* of quasi-projective varieties

$$F : X \longrightarrow Y$$

is a continuous map in the Zariski topology such that the composition with any regular function defined in an open set $U \subset Y$, is a regular function defined in the open set $F^{-1}(U) \subset X$. An *isomorphism* is a morphism which admits an inverse that is also a morphism. We say two varieties X and Y are *isomorphic* if there exists an isomorphism $\varphi : X \longrightarrow Y$ between them. An isomorphism φ between X and Y defines an isomorphism of K-algebras between their rings of regular functions, maping regular functions on Y into regular functions on X:

$$\begin{aligned} \varphi^* K[Y] &\longrightarrow K[X] \\ f &\mapsto f \circ \varphi. \end{aligned}$$

Remark 2. Smoothness is preserved by isomorphisms. This is due to the fact that regularity is preserved under ring isomorphisms.

A morphism $F : X \to Y$ is *proper* if it is universally closed, that is, if for any affine variety Z and any morphisms $f : X \to Y$ and $g : Z \to Y$, the projection onto Z of the fiber product

$$X \times_Y Z = \{(x, z) \in X \times Z : f(x) = g(z) \in Y\},$$

is a closed map.

1.2.3 Abstract varieties

An *abstract variety* X is the result of gluing together a collection of open subsets of affine varieties (see [39]). The main example of this process is the construction of the projective space $\mathbb{P}^n$ as it is usually given in the literature.

Consider a finite collection of affine varieties $\{V_\alpha\}_{\alpha\in\Lambda}$, having for each pair $\alpha, \beta \in \Lambda$, Zariski open sets $V_{\beta\alpha} \subset V_\alpha$, $V_{\alpha\beta} \subset V_\beta$ and isomorphisms of affine open subsets $g_{\beta\alpha} : V_\alpha \to V_\beta$ such that:

1. $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$
2. For any γ, $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$ on $V_{\beta\alpha} \cap V_{\gamma\alpha}$, and $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta} \subseteq V_\beta$.

This allows us to consider an equivalence relation between points of different affine varieties in the collection, for example $x \in V_\alpha$, $y \in V_\beta$:

$$x \sim y \Leftrightarrow g_{\beta\alpha}(x) = y.$$

Consider the quotient space of the union of the V_α by this equivalence relation and call it X. Notice that X can be seen locally as an affine variety, via the homomorphism which takes each element to its equivalence class in X,

$$\phi_\alpha : V_\alpha \longrightarrow U_\alpha \subset X,$$

where U_α is the set of all equivalence classes in X which have a representative in V_α. The resulting variety of this gluing is an abstract variety.

We will refer to abstract varieties simply as varieties.

Examples 3. Projective and quasi-projective varieties are abstract varieties.

The same constructions and properties which are studied in affine varieties (e.g. normalization, smoothness) can be studied locally for abstract varieties. To do this, one must consider each affine chart of the abstract variety separately, and consider it locally as an affine variety. The following results are an example.

An abstract variety X is *normal* if for every point $p \in X$, the local ring of X at p is normal. The same definition of the Zariski tangent space at a point $p \in X$ given for affine varieties is valid for general varieties. With this definition, the concept of smoothness at a point is still the same as it was for affine varieties. Again, X is *smooth* if it is smooth at every point $p \in X$.

Proposition 4. *Let X be an irreducible variety given by a cover of affine open sets U_α. Then X is a normal variety if and only if every U_α is normal.*

Proof. See [11, Proposition 3.0.12]. □

1.2.4 Rational maps

A *rational map*

$$\phi : X \dashrightarrow Y$$

is an equivalence class of pairs (U, ϕ_U), with $U \subset X$ a non empty open subset and $\phi_U : U \longrightarrow Y$ a morphism, and where $(U, \phi_U) \sim (V, \phi_V)$ if ϕ_U and ϕ_V agree in the intersection $U \cap V$. A *birational map*

$$r : X \dashrightarrow Y$$

is a rational map which admits an inverse that is rational. That is, if there exist non empty open subsets $U \subset X$, $V \subset Y$ such that r is an isomorphism between U and V. Two affine varieties X, X' with respective coordinate rings B, B' are called *birationally equivalent* if there exists a birational map between them. This is equivalent to saying that the fields of fractions $\mathrm{Q}(K[X])$, $\mathrm{Q}(K[X'])$ are isomorphic (see [22, Corollary 4.5]).

Now we have the necessary tools to talk about resolution of singularities. However, in the next section we will give more details including blowups.
Let X be an algebraic variety. A *weak resolution of singularities* of X is a proper birational morphism

$$f : X' \to X$$

such that X' is smooth. If $\mathrm{Sing}(X)$ is the singular locus of X, it is sometimes required that in addition $f : X' \setminus f^{-1}(\mathrm{Sing}(X)) \to X \setminus \mathrm{Sing}(X)$ is an isomorphism. To be a *strong resolution of singularities* X' must satisfy some more restrictive conditions. For instance, it is also required that f is a composition of blowups in closed centers which are transversal to the exceptional locus, and that f does not depend on the embedding of X in the ambient space.

1.3 Resolution of singularities and blowups.

In this section the concept of blowup and its importance in the resolution of singularities is explained. It will be used to resolve the singularities of monomial varieties in next sections.

The *blowup* Y *of* $\mathbb{A}^n$ *at the origin* is defined as the subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ given by the pairs (x, l), where $l \in \mathbb{P}^{n-1}$ is the line through the origin containing the point $x \in \mathbb{A}^n$, together with its projection φ to $\mathbb{A}^n$ as in the diagram:

$$\begin{tikzcd}
Y = \{(x,l) \in \mathbb{A}^n \times \mathbb{P}^{n-1}\} \arrow[r, hook] \arrow[dr, "\varphi"] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d, "\pi_1"] \\
 & \mathbb{A}^n
\end{tikzcd}$$

The variety Y can also be seen as the closure of a graph. It is the closure of the graph of the rational map which associates to each point $x = (x_1, \dots x_n) \in \mathbb{A}^n$ the line $l = [x_1 : \dots : x_n] \in \mathbb{P}^{n-1}$, together with the projection onto the affine space.

The *blowup of* $\mathbb{A}^n$ *at a point* $p \in \mathbb{A}^n$ is the result of finding a coordinate change that sends p to the origin, and performing a blowup of $\mathbb{A}^n$ at the origin after applying this coordinate change.

Let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety, and let p be a point in X. The *blowup* of X at p is the Zariski closure X' of the preimage $\varphi^{-1}(X \setminus \{p\})$, where φ is the blowup of $\mathbb{A}^n$ at p, together with the projection of X' in the affine factor

$$\varphi' : X' \longrightarrow X \subseteq \mathbb{A}^n. \tag{1.1}$$

Let the prime ideal $I \subseteq K[X]$ be the ideal generated by $f_1, \ldots, f_r \in K[X]$. The *blowup of* X *along* I, or along the subvariety $V(I) \in X$ is the closure of the graph of the rational map defined outside of $V(I)$

$$\begin{aligned} X &\overset{F}{\dashrightarrow} \mathbb{P}^{r-1} \\ x &\longmapsto [f_1(x) : \ldots : f_r(x)], \end{aligned}$$

that is $X' = \overline{\mathrm{Graph}(F)} = \overline{\{(x, F(x)) \in \mathbb{A}^n \times \mathbb{P}^{r-1}, x \notin V(I)\}}$, together with the projection onto $\mathbb{A}^n$:

$$\varphi' : X' \dashrightarrow X \subseteq \mathbb{A}^n. \tag{1.2}$$

The ideal I is the *center* of the blowup. It is clear that blowing X up at a point is just blowing up along the maximal ideal corresponding to this point, and thus a particular case of this definition of blowup. The blowup of X along the ideal I does not depend on the choice of the generators of I, although they appear in the definition (see [39, sec. 7.4]).

Remark 5.

- The center Z of a blowup is the set of points of X in which φ' is not an isomorphism.
- A blowup is completely determined by its center.

The map φ' is a birational map, and it induces an isomorphism between $\mathrm{Graph}(F)$ and $X \setminus V(I) \subseteq \mathbb{A}^n$. The preimage of the center $Z = V(I)$ under the projection, $Z' = (\varphi')^{-1}(Z)$ is a hypersurface, called the *exceptional divisor* of the blowup. The preimage of X, $X^* = (\varphi')^{-1}(X)$ is called the *total transform*, and contains the whole inverse image of X under the projection onto the affine space. In particular, it contains X' and the exceptional divisor Z' (see [25]).
Geometrically, the irreducible components of X' are the components of the total transform which are not equal to the exceptional divisor. The set X' is called the *strict transform*. This name is also used for the ideal defining X'. Notice that $X' \subseteq \mathbb{A}^n \times \mathbb{P}^{r-1}$. That is, X' is a quasi-projective variety. It can be covered by r affine charts, in the same way that $\mathbb{P}^{r-1}$ can be covered by r open subsets of $\mathbb{A}^{r-1}$. Each chart will be the restriction to X' of the cartesian product of $\mathbb{A}^n$ and an open subset of $\mathbb{A}^{r-1}$ which is an affine chart of $\mathbb{P}^{r-1}$.

Example 6. Let us go back to Example 8.2, in order to show a possible resolution of singularities of the Whitney Umbrella. Using the expression of its coordinate ring in (1.2), it is possible to compute the blowup along the ideal $I = (x, y)$. That is, along the z-axis, $Z = V(I) \subset W$. Let $\mathbb{P}^1$ be the projective line with homogeneous coordinates given by s, t. Define the rational map

$$\begin{aligned} W &\overset{F}{\dashrightarrow} \mathbb{P}^1 \\ (x, y, x) &\longmapsto [s : t] = [x : y], \end{aligned}$$

so that the blowup of W with center Z is given by

$$W' = \overline{\mathrm{Graph}(F)} = \{((x, y, z), [s : t]) | x^2 - y^2 z = 0, xt - sy = 0\}.$$

together with the projection

$$\varphi' : W' \dashrightarrow W \subseteq \mathbb{A}^3.$$

It can be covered by two affine charts, each of which is an open subset of $\mathbb{A}^3 \times \mathbb{A}^1 = \mathbb{A}^4$. Call them W'_s, W'_t, for $s \neq 0$, $t \neq 0$ respectively. The affine expression of W' for $x \neq 0$ is

$$W'_s = \left\{ \left(x, y, z, u = \frac{t}{s} \right) \in \mathbb{A}^4 | y = ux, x^2 - y^2 z = 0 \right\},$$

and it can be projected onto $\mathbb{A}^3$ by

$$\begin{aligned} \varphi' : W'_s &\longrightarrow \mathbb{A}^3 \\ (x, y, z, u) &\longmapsto (x, z, u). \end{aligned}$$

This projection gives new coordinates x, z, $u = \frac{t}{s} = \frac{y}{x}$ for the coordinate ring of the blowup W' in this chart:

$$\begin{aligned} K[W'_s] = K[x, z, \frac{y}{x}]/(\quad x^2 - y^2 z) \cong K[x, z, u]/(x^2 - u^2 x^2 z) \cong K[x, \frac{1}{x^2}, u] \cong \\ \cong K[x, \frac{1}{x}, u]. \end{aligned}$$

An analogous computation shows that $W'_t = \{(x, y, z, v) \in \mathbb{A}^4 | x = vy, x^2 - y^2 z = 0\}$ is the affine chart for $y \neq 0$, and its coordinate ring in this chart is

$$K[W'_t] = K[y, z, \frac{x}{y}]/(x^2 - y^2 z) \cong K[y, z, v]/(v^2 - z) \cong K[x, v^2, v] \cong K[x, v].$$

The coordinate ring in the chart s can also be expressed as

$$K[W'_s] \cong K[x, y, z]/(yz - 1).$$

The Jacobian criterion shows that it is a smooth surface. Moreover, it is the cartesian product of an affine line and a hyperbola. The second affine chart is an affine plane $\mathbb{A}^2$.

Hironaka proved that given a quasi-projective variety $X \subset \mathbb{P}^n$ over a field K of characteristic zero, there exists a sequence of blowups with smooth centers inside the singular locus that gives a resolution of singularities of X. Villamayor found a method to find a right choice of the centers of this sequence of blowups. However, it is not easy to determine the ideals corresponding to these centers. This resolution is not unique. In particular, given an affine variety $X \subseteq \mathbb{A}^n$ over a field of characteristic zero, there exist polynomials

$$f_1, \ldots, f_d \in K[x_1, \ldots, x_n]$$

such that the blowup of X with center the ideal $I = (f_1, \ldots, f_d)$ is a resolution of singularities of X.

1.4 Monomial varieties and monomial algebras

Let X be an affine algebraic variety in $\mathbb{A}^n_K$. We denote by t^{α_i}, where $t = (t_1, \ldots, t_m)$ and $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,m}) \in \mathbb{Z}^m$, the monomial $t^{\alpha_i} = \prod_{j=1}^m t_j^{\alpha_{i,j}}$. A map

$$\begin{aligned} \varphi : U &\longrightarrow \mathbb{A}^n_K \\ t = (t_1, \ldots, t_m) &\longmapsto (t^{\alpha_1}, \ldots, t^{\alpha_n}), \ \alpha_i \in \mathbb{Z}^m, \end{aligned}$$

defined over some subset $U \subseteq \mathbb{A}^m_K$, is a *monomial parametrization* of X if

$$X = \overline{\mathrm{Im}(\varphi)}, \tag{1.3}$$

where $\overline{\mathrm{Im}(\varphi)}$ denotes the closure in the Zariski topology.

A variety which can be expressed in the form (1.3) is called a *monomial variety*. The coordinate ring of a monomial affine algebraic variety X is given by

$$B = K[x_1, \ldots, x_n]/I, \tag{1.4}$$

where I is the ideal given by the relations between the components $t^{\alpha_1}, \ldots, t^{\alpha_n}$ of φ.

Lemma 7. *Let $X = \overline{\mathrm{Im}(\varphi)}$ be an affine variety given by a monomial parametrization as above. Then $K[X] = K[x_1, \ldots, x_n]/I \cong K[t^{\alpha_1}, \ldots, t^{\alpha_n}]$.*

Proof. Consider the homomorphism:

$$\begin{aligned} h : K[x_1, \ldots, x_n] &\longrightarrow K[t^{\alpha_1}, \ldots, t^{\alpha_n}] \\ x_i &\longmapsto t^{\alpha_i} \end{aligned}$$

whose kernel is given exactly by the relations between the images of the x_i, induced by the relations of the t^{α_i}. $\square$

A K-algebra that can be generated by a finite set of monomials is called a *monomial algebra.*
Let $\varphi = (t^{\alpha_1}, \ldots, t^{\alpha_n})$ be a monomial parametrization of a variety X as above, for $t = (t_1, \ldots, t_m)$. We will denote by

$$\mathcal{B}_\Gamma = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{Z}^m \tag{1.5}$$

the set of exponents of the monomial components of φ. By the lemma we know that those monomials are a set of generators for $K[X]$. By taking positive linear combinations of the elements in $\mathcal{B}_\Gamma$ we get a commutative semigroup Γ with identity. Recall that a semigroup is a set with an associative binary operation (more information about semigroups is given in section 2). Conversely, any semigroup defines a K-algebra: for any monomial $t^\alpha \in K[X]$, the exponent α belongs to the commutative semigroup

$$\Gamma = \left\{ \sum_{\alpha \in \Gamma} a_\alpha \alpha : a_\alpha \in \mathbb{N} \right\}.$$

We denote by $K[\Gamma]$ the K-algebra defined by a semigroup Γ as

$$K[t^\alpha : \alpha \in \Gamma]. \tag{1.6}$$

Observe that if a semigroup Γ has a finite set of generators $\mathcal{B}_\Gamma = \{\alpha_1, \ldots \alpha_n\}$, then the K-algebra is also finitely generated as

$$K[\Gamma] = K[t^{\alpha_1}, \ldots, t^{\alpha_n}]. \tag{1.7}$$

Examples 8. 1. The cusp is given by the equation:

$$C : \left\{ x^3 - y^2 = 0 \right\}. \tag{1.8}$$

A parametrization for C is:

$$\begin{aligned} \gamma_1 : \mathbb{A}^1_K &\longrightarrow \mathbb{A}^2_K \\ t &\longmapsto (t^2, t^3) \end{aligned}$$

Using the defining equation of C and the monomials of the parametrization, we can describe its coordinate ring:

$$K[x, y]/(x^3 - y^2) \cong K[t^2, t^3]. \tag{1.9}$$

Figure 1.1 shows this curve. The cusp has a singular point at the origin. By Remark 70 we see that it is not normal.

2. The surface $\left\{ W : x^2 - y^2 z = 0 \right\}$ is called Whitney Umbrella. The real part of W is shown in figure 1.2. It can be parametrized by

$$\begin{aligned} \gamma_2 : \mathbb{A}^2_K &\longrightarrow \mathbb{A}^3_K \\ (t, s) &\longmapsto (ts, t, s^2). \end{aligned}$$

Its coordinate ring is given by:

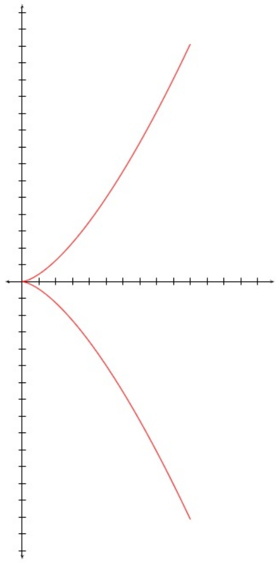

Figure 1.1: Cusp.

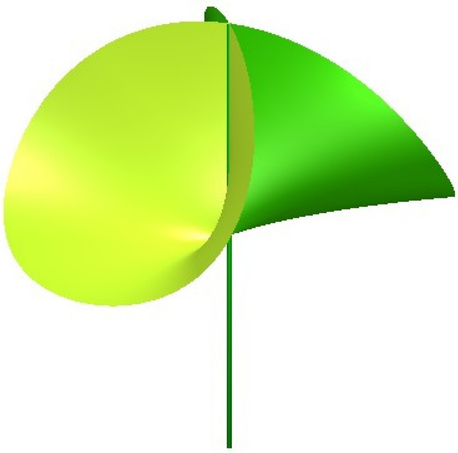

Figure 1.2: Whitney Umbrella: $x^2 - y^2 z = 0$.

$$K[x, y, z]/(x^2 - y^2 z) \cong K[ts, t, s^2]. \tag{1.10}$$

Notice that W is singular along the z-axis. This surface will appear in future computations.

In the last two examples, the corresponding semigroups defining C and W are subsets of $\mathbb{N}$ and $\mathbb{N}^2$ respectively. The following example uses a parametrization which is rational instead of polynomial, which implies that the semigroup is a subset of $\mathbb{Z}^2$.

3. Consider the parametrization

$$\begin{aligned} \gamma_3 : \mathbb{A}^2_K &\longrightarrow \mathbb{A}^3_K \\ (t, s) &\longmapsto (t, ts^{-2}, s). \end{aligned}$$

The surface $Y = \overline{\mathrm{Im}(\gamma_3)}$ is given by the equation $\{Y : x - yz^2 = 0\}$, and it is everywhere smooth. Its coordinate ring is

$$K[x,y,z]/(x-yz^2) \cong K[t, ts^{-2}, s].$$

This surface is shown in figure 1.3.

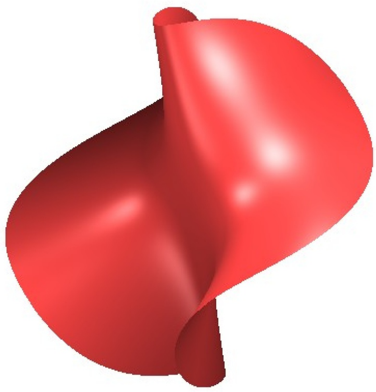

Figure 1.3: Surface $Y : x - yz^2$.

In what follows, we will see that for a given affine variety X, there is not a unique parametrization φ such that $X = \overline{\mathrm{Im}(\varphi)}$. It is necessary to guarantee that, in this situation, for any two different parametrizations of X the corresponding semigroups can be identified in a clear way.

Example 9. Consider the cone with equation $\left\{V : xy - z^2 = 0\right\}$ (see Example 71.2 in the Appendix). Three different parametrizations for the cone are:

$$\varphi_1(t,s) = (t^2, s^2, ts)$$

$$\varphi_2(t,s) = (ts, ts^{-1}, t)$$

$$\varphi_3(t,s) = (ts, t^3 s, t^2 s)$$

They respectively define the following semigroups:

$$\Gamma_1 = \langle (2,0), (0,2), (1,1) \rangle_{\mathbb{N}}$$

$$\Gamma_2 = \langle (1,1), (1,-1), (1,0) \rangle_{\mathbb{N}}$$

$$\Gamma_3 = \langle (1,1), (3,1), (2,1) \rangle_{\mathbb{N}}$$

These sets of generators of Γ_1, Γ_2 and Γ_3 are shown in figure 1.4.

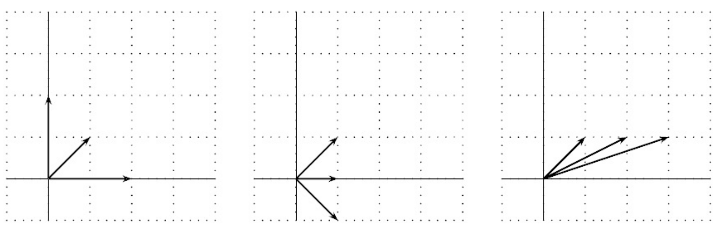

Figure 1.4: Generating sets of Γ_1, Γ_2 and Γ_3.

However, different parametrizations for the same variety are equivalent in the following sense: the monomial K-algebras generated by both parametrizations are isomorphic or, equivalently, the corresponding semigroups are isomorphic, as we will show. Let φ_1 and φ_2 be parametrizations for X satisfying (1.3), and let Γ_1 and Γ_2 be the semigroups that they define. Let $\mathcal{B}_{\Gamma_1} = \{\alpha_1, \ldots, \alpha_{r_1}\}$ and $\mathcal{B}_{\Gamma_2} = \{\beta_1, \ldots, \beta_{r_2}\}$ be respective sets generating the semigroups. We know that $K[X] \cong K[\Gamma_1]$ and $K[X] \cong K[\Gamma_2]$, and therefore there exists a K-algebra isomorphism

$$\phi : K[t^{\alpha_1}, \ldots, t^{\alpha_{t_1}}] \longrightarrow K[t^{\beta_1}, \ldots, t^{\beta_{r_2}}].$$

This algebra isomorphism induces a semigroup isomorphism between Γ_1 and Γ_2. This implies, in particular, that $\dim(\Gamma_1)$ $=\dim(\Gamma_2) = d$, and there exist generating sets of Γ_1 and Γ_2 with the same number of elements. In fact, all minimal systems of generators of Γ_1, respectively Γ_2, have d elements, where d is the *dimension of immersion* of X. This is a consequence of the way in which d is defined: it is the minimal integer such that X can be embedded in $\mathbb{A}^d$.

Example 10. We will show how the isomorphism is built for the semigroups of Example 9. Consider the semigroups Γ_1 and Γ_2 and let ϕ_1 and ϕ_2 be two K-algebra isomorphisms

$$K[t^2, s^2, ts] \xrightarrow{\phi_1} K[x, y, z]/(xy - z^2) \xleftarrow{\phi_2} K[ts, ts^{-1}, t].$$

Both homomorphisms are defined by the images of the generators of the algebras. Set $\phi_1(t^2) = \overline{x} \in K[x, y, z]/(xy - z^2)$ and $\phi_1(ts) = \overline{z}$. This implies $\phi_1(s^2)x = z^2$, so $\phi_1(s^2) = \overline{y}$. Similarly, set $\phi_2(ts) = \overline{x}$ and $\phi_2(t) = \overline{z}$. The induced isomorphism between Γ_1 and Γ_2 is determined by

$$\begin{aligned} \phi : \Gamma_1 &\longrightarrow \Gamma_2 \\ (2, 0) &\longmapsto (1, 1) \\ (1, 1) &\longmapsto (1, 0), \end{aligned}$$

Now $\phi(0, 2)$ must be equal to $(1, -1)$, and it is easy to check:

$$\phi(0, 2) = \phi((2(1, 1) - (2, 0)) = 2(1, 0) - (1, 0) = (1, -1).$$

Notice that this will always happen because the generators of both K-algebras, $K[\Gamma_1]$ and $K[\Gamma_2]$, must satisfy the equation of V, so the isomorphism will be well defined.

Chapter 2

Toric geometry

In this chapter, we give an introduction to toric varieties, focusing on affine toric varieties, and relating this theory with our initial problem. Further information about this topic can be found in [11], [18] or [32].

2.1 Semigroups, cones and lattices

Some concepts about convex geometry will be introduced here. As we will see, they make it possible the use of combinatoric tools for the kind of varieties that we will be considering.

Along these lines, we will consider subsemigroups $\Gamma \subset (\mathbb{Z}^m, +)$.

2.1.1 Semigroup and lattice.

A subset $\mathcal{B}_\Gamma \subset \Gamma$ is said to *generate* the semigroup Γ, or to be a set of generators of Γ, if any element in the semigroup is a finite linear combination of elements $\alpha \in \mathcal{B}_\Gamma$ with coefficients in $\mathbb{N}$. In this case we will write $\Gamma = \langle \mathcal{B}_\Gamma \rangle$.

A *lattice* M is a free abelian group of finite rank m. That is, a group isomorphic to $\mathbb{Z}^m$. The *dimension of a lattice* M is the dimension of the $\mathbb{R}$-vector space $\mathbb{R} \otimes_{\mathbb{Z}} M$.

Remark 11. A lattice is a semigroup.

Let Γ be a finitely generated semigroup. Then the set $\left\{\sum_{\alpha \in \Gamma} a_\alpha \alpha : a_\alpha \in \mathbb{Z}\right\}$ is a lattice, and we will denote it by $\mathbb{Z}\Gamma$. The *dimension of a semigroup* Γ is the dimension of the lattice $\mathbb{Z}\Gamma$.

Examples 12. 1. Let Γ_1 be the semigroup generated by $\mathcal{B}_1 = \{(1,1)\}$. Figure 2.1 shows the semigroup and the lattice generated by $\mathcal{B}_1$. Here $\dim(\Gamma_1) = 1$.

2. The semigroup Γ_2 generated by the set $\mathcal{B}_2 = \{(1,1),(2,0),(0,2)\}$ is shown in figure 2.2, together with the lattice $\mathbb{Z}\Gamma_2$.

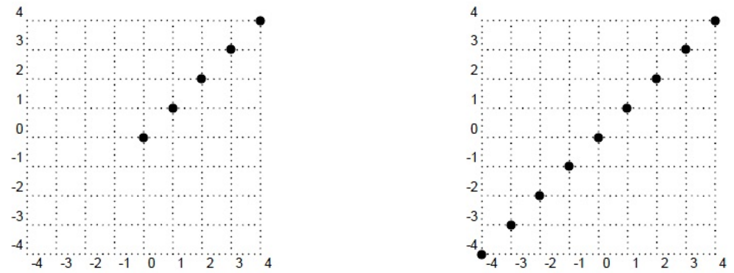

Figure 2.1: Semigroup and lattice of $\{(1,1)\}$.

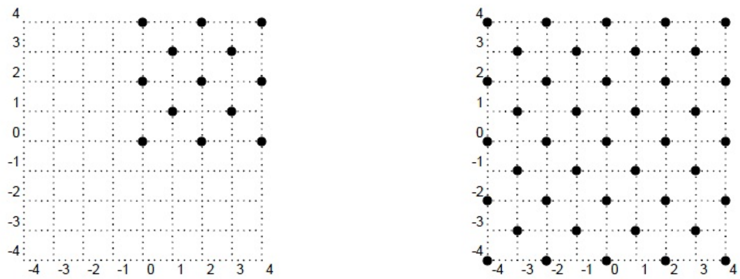

Figure 2.2: Semigroup and lattice of $\{(1,1),(2,0),(0,2)\}$

Notice that the set $\mathcal{B}_2' = \{(2,0),(1,1)\}$ is a basis of this lattice although it doesn't generate Γ_2 as a semigroup. We have that $\dim(\Gamma_2) = 2$.

A semigroup $\Gamma \subseteq M$ is said to be *saturated in M* if for any element $m \in M$ and any constant $k \in \mathbb{Z}_{>0}$, the fact that $km \in \Gamma$ means that m is in Γ. An element $m \in \Gamma$ is *primitive* if it can not be written as $m = n + r$ for some non zero elements $n, r \in \Gamma$.

A semigroup Γ is said to be *pointed* if $\Gamma \cap (-\Gamma) = \{0\}$.

Examples 13. Figure 2.3 shows two examples of semigroups of dimension 2. The semigroup Γ_1 generated by $\{(1,2),(2,1)\}$ is pointed. For the semigroup Γ_2, generated by $\{(1,0),(-1,0),(0,1)\}$, we have that $\Gamma \cap -\Gamma$ is equal to all the integer points in the x-axis, that is, a lattice isomorphic to $\mathbb{Z}$. This semigroup is not pointed.

2.1.2 The semigroup algebra

Semigroups will be important for our purpose because, as it was said in section 1.4, monomial algebras can be described via the semigroup formed by the exponents of the monomials in the algebra.
Let Γ be a finitely generated subsemigroup of $\mathbb{Z}^m$, and let R be a commutative ring.
The *semigroup algebra* $R[\Gamma]$ of Γ with coefficients in R is the subring of

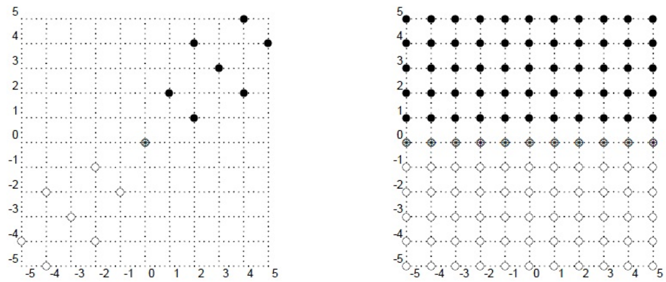

Figure 2.3: Semigroups Γ_1 and $-\Gamma_1$ share the point $(0,0)$, while semigroups Γ_2 and $-\Gamma_2$ have infinitely many points in common.

$R(t_1,\ldots,t_m)$, with $t_i, i=1,\ldots,m$ variables, whose elements have the form:

$$\sum_{\alpha\in\Gamma} a_\alpha t^\alpha,\ a_\alpha \in R,$$

where $t=(t_1,\ldots,t_m)$, $t^\alpha = t_1^{\alpha_1}\cdot\ldots\cdot t_m^{\alpha_n}$ and only a finite number of the a_α are different from zero. This ring is equipped with multiplication

$$\left(\sum_{\alpha\in\Gamma} a_\alpha t^\alpha\right)\left(\sum_{\beta\in\Gamma} b_\beta t^\beta\right) = \sum_{\alpha,\beta\in\Gamma:\lambda=\alpha+\beta} a_\alpha b_\beta t^\lambda,\ a_\alpha, b_\beta \in \mathbb{Z},$$

induced by the binary operation of the semigroup.

Now, assume that $R=K$ an algebraically closed field of characteristic zero. Then an affine variety of dimension $d=\dim(\Gamma)$ in $\mathbb{A}^n_K$ is naturally defined by Γ: Consider the surjective map:

$$\rho^{(+)} : K[x_1,\ldots,x_n] \longrightarrow K[\Gamma] \tag{2.1}$$

$$x_i \longmapsto t^{\alpha_i} \tag{2.2}$$

which induces an inclusion of varieties:

$$\nu^{(+)} : X^\Gamma \hookrightarrow \mathbb{A}^n_K.$$

We will see that X^Γ is truly an affine variety whose coordinate ring $K[\Gamma]$ is generated by monomials in $t=(t_1,\ldots,t_m)$ with exponents in the semigroup Γ. The equations of X^Γ are given by the kernel of $\rho^{(+)}$, and they form a prime ideal (see [4] and section 2.2 of these notes).

Lemma 14. *The field of fractions of $K[\Gamma]$ is the semigroup algebra $K[\mathbb{Z}\Gamma]$.*

Proof. It is easy to see that $Q(K[\Gamma])$ is the K-algebra generated by all elements in Γ and their additive inverses. In other words, it is the semigroup algebra generated by the set of all possible linear combinations of elements from Γ with either positive or negative coefficients. This set corresponds exactly to the definition of the lattice M, and therefore:

$$Q(K[\Gamma]) = K[t^{\pm\alpha} : \alpha\in\Gamma] = K[t^\alpha : \alpha \in \mathbb{Z}\Gamma] = K[M]. \tag{2.3}$$

□

Proposition 15. *Let Γ be a finitely generated semigroup, let $\mathbb{Z}\Gamma$ be the lattice generated by integral linear combinations of Γ. Then the following are equivalent:*
1. The ring $K[\Gamma]$ is integrally closed, and therefore X^{Γ} is a normal variety.
2. The semigroup Γ is saturated in M.

Proof. Suppose first that $K[\Gamma]$ is integrally closed. Consider any element of the form $a\alpha \in \Gamma$ for some $\alpha \in M$ and $a \in \mathbb{N}$. By Lemma 14 we know $t^{\alpha} \in Q(K[\Gamma])$, and one can find a monic polynomial $P(x) = x^{a} - t^{a\alpha}$, with $P(t^{\alpha}) = 0$. This implies that t^{α} is integral over $K[\Gamma]$ and, since this ring is integrally closed, it also means that $t^{\alpha} \in K[\Gamma]$. Therefore $a\alpha \in \Gamma \Rightarrow \alpha \in \Gamma$ and Γ is a saturated semigroup.

Suppose now that Γ is saturated. Let us choose an element $t^{\alpha} \in Q(K[\Gamma])$, with t^{α} integral over $K[\Gamma]$. By Lemma 14 we know that $\alpha \in M$. There exists a monic polynomial

$$P(x) = x^{n} + a_1 x^{n-1} + \ldots + a_n \in K[\Gamma][x],$$

with $a_i \in K[\Gamma]$, $i = 1, \ldots, n$, such that $P(t^{\alpha}) = 0$ for all $t \in \mathbb{R}^{m}$, where m is the dimension of Γ. The a_i are of the form

$$a_i = \sum_{\gamma \in \Gamma} a_i^{\gamma} t^{\gamma}, \;\; a_i^{\gamma} \in K.$$

Now $P(t^{\alpha})$ is a polynomial in t that must be zero por all $t \in \mathbb{R}^{m}$, and therefore all of the homogeneous components must be zero for all t. One homogeneous component is of the form

$$P_{\alpha n}(t) = t^{\alpha n} + \tilde{a_1} t^{v_1} t^{\alpha(n-1)} + \ldots + \tilde{a_n} t^{v_n},$$

with $\tilde{a_i} \in K$ and $v_i \in \Gamma$. The monomials in $P_{\alpha n}$ with $\tilde{a_i} \neq 0$ must satisfy $t^{v_i} t^{\alpha(n-i)} = t^{\alpha n}$ or equivalently $v_i = \alpha i$. Note that there must exist some monomial with $\tilde{a_i} \neq 0$. We can conclude that there exists some $v_i \in \Gamma$ with $v_i = \alpha i$, and since Γ is saturated, $\alpha \in \Gamma$ and $t^{\alpha} \in K[\Gamma]$. As this happens for any element from $Q(K[\Gamma])$ which is integral over $K[\Gamma]$, $K[\Gamma]$ is integrally closed.

□

2.1.3 Convex cones and dual cones

Another important object for the study of toric varieties is the convex cone of a semigroup.
Let M be a lattice of dimension m: $M = \mathbb{Z}^{m}$. A *rational polyhedral cone* σ is a subset of $M \otimes_{\mathbb{Z}} \mathbb{R}$, generated as $\mathbb{R}_{\geq 0}$-linear combinations of a subset of elements in M. It is *convex* if for any $x, y \in \sigma$, we have that $\lambda x + (1-\lambda) y \in \sigma$, with $0 \leq \lambda \leq 1$.
We define the *dimension of a cone* σ as the dimension of the vector space $\sigma \otimes_{\mathbb{Z}} \mathbb{R}$.
A cone is *strongly convex* if it does not contain any line. Figure 2.4 shows strongly convex cones of dimensions 2 and 3.

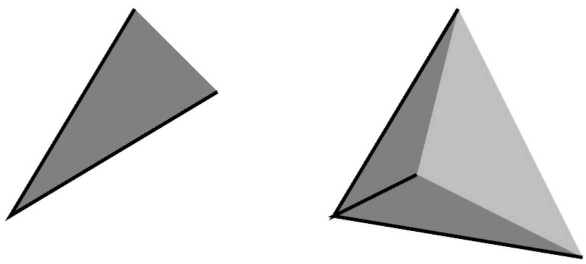

Figure 2.4: Strongly convex cones of dimensions 2 and 3.

A semiplane is a non strongly convex cone of dimension 2, and the cartesian product of a strongly convex cone of dimension 2 with a third axis is a non strongly convex cone of dimension 3. Both are show in figure 2.5.

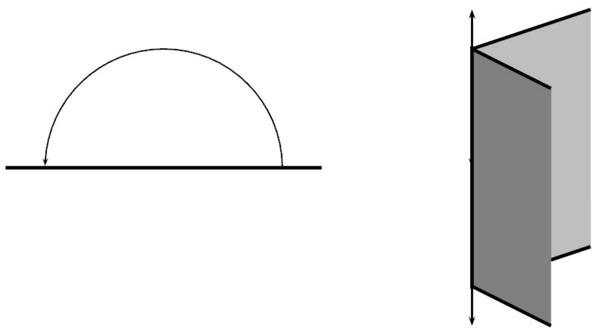

Figure 2.5: Non strongly convex cones of dimensions 2 and 3.

A *face* τ of a cone σ is the intersection of σ with an affine variety $V(l)$ defined by a linear form $l : \mathbb{R}^m \to \mathbb{R}$ with only positive or only negative values in σ. Among the set of faces, one can distinguish the whole cone σ and proper faces, which are defined by hyperplanes l. The *dimension of a face* τ is the dimension of the vector space given by all linear combinations of elements in τ with real coefficients. If σ is a cone of dimension m, the $(m-1)$-dimensional faces of σ are called *facets*.

Given a rational convex polyhedral cone σ, its *dual* cone $\sigma^\vee$ is defined as

$$\sigma^\vee = \{w \in \mathbb{R}^m : \ v \cdot w \geq 0 \ \text{ for all } v \in \sigma \subset \mathbb{R}^m\}. \tag{2.4}$$

Example 16. Figure 2.6 gives some examples of comparison between cones in $\mathbb{R}^2$ and their dual cones. One can observe that the bigger a cone is, the smaller its dual cone will be, and vice versa. For instance, all vectors in the closed upper halfplane, $(m, n) \in \mathbb{R} \times \mathbb{R}^+$ have scalar product greater or equal than zero with the vectors $(0, r) \in \{0\} \times \mathbb{R}^+$. The intersection of the dual cones of of $(0, n), (0, -m) \in \{0\} \times \mathbb{R}^+$ is the x-axis, $(r, 0) \in \smallsetminus \times \{0\}$. The dual cone $\sigma^\vee$ of a cone σ whose facets form an angle $\theta = \pi$ is equal to σ.

In general, if σ is a strongly convex cone in $\mathbb{R}^2$, and if its facets form an angle θ, then the facets of the dual cone form an angle of degree $\pi - \theta$. If the facets of σ form an angle smaller than $\pi/2$, then the facets of $\sigma^\vee$ form an angle greater than $\pi/2$. This can be seen by considering the dual cones to the facets and taking their intersection, as in the proof of the following proposition.

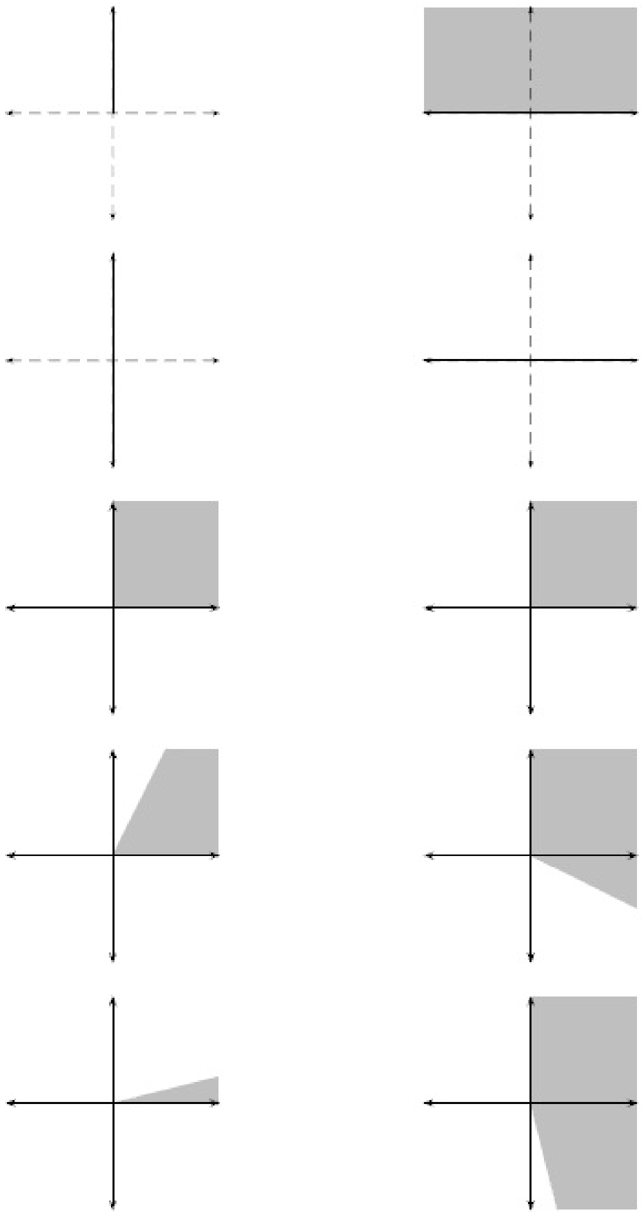

Figure 2.6: Examples of duality between cones in $\mathbb{R}^2$.

Proposition 17. *The dual cone of $\sigma^\vee$ is σ, i.e. $(\sigma^\vee)^\vee = \sigma$.*

Proof. By definition,

$$(\sigma^\vee)^\vee \supseteq \left\{ u \in \mathbb{R}^m : uw \geq 0 \ \ \forall w \in \sigma^\vee \right\}.$$

It is clear that every vector in σ satisfies this, by the way $\sigma^\vee$ is constructed, thus:

$$\sigma \subseteq (\sigma^\vee)^\vee. \tag{2.5}$$

So we are left to prove that the equality holds. For a given vector $v \in \sigma$, consider the set of vectors $H_v = \{w \in \mathbb{R}^m : wv \geq 0\}$, which is a half-space. It is obvious that $H_v = H'_v$ implies $v = \lambda v'$ for some $\lambda \in \mathbb{R}_{>0}$. We can construct $\sigma^\vee$ as

$$\sigma^\vee = \bigcap_{v\in\sigma} H_v.$$

Suppose now that there exists a vector $\tilde{v} \in (\sigma^\vee)^\vee$ but $\tilde{v} \notin \sigma$. Then all vectors in $\sigma^\vee$ would belong to $H_{\tilde{v}}$, but as $\tilde{v} \notin \sigma$, then

$$(\cap_{v\in\sigma} H_v) \cap H_{\tilde{v}} \subsetneq \cap_{v\in\sigma} H_v = \sigma^\vee,$$

and since $H_{\tilde{v}}$ is a half-space, and $\sigma^\vee$ is contained in one (in fact, it is contained in all the H_v for $v \in \sigma$), $H_{\tilde{v}} \cap (\sigma^\vee)^c \neq \emptyset$ and we get a contradiction. $\square$

2.1.4 The cone and the dual cone of a monomial variety

For a given semigroup $\Gamma = \langle \mathcal{B}_\Gamma \rangle$, we define $\sigma_\Gamma^\vee$ as the rational polyhedral cone generated by $\mathcal{B}_\Gamma$

$$\sigma_\Gamma^\vee = \{x = \sum_{\alpha\in\mathcal{B}_\Gamma} a_\alpha \alpha,\ a_\alpha \in \mathbb{R}_{\geq 0}\} = \mathbb{R}_{\geq 0}\Gamma.$$

This is the *dual cone associated to* X^Γ. The dual cone to $\sigma_\Gamma^\vee$, denoted by $\sigma_\Gamma = (\sigma_\Gamma^\vee)^\vee$, is known as the *cone associated to* X^Γ. We will simply write σ and $\sigma^\vee$ instead of σ_Γ and $\sigma_\Gamma^\vee$ for the cone and the dual cone associated to X^Γ when the semigroup Γ is clear.

Example 18. Consider the set of generators $\mathcal{B}_\Gamma = \{(1,1),(1,0),(0,2)\}$ of the semigroup $\Gamma \subset \mathbb{N}^2$. The coordinate ring associated to this semigroup is given by $K[\Gamma] = K[t_1t_2, t_1, t_2^2]$. This way we obtain the variety $X^\Gamma = W : \left\{x^2 - y^2z = 0\right\}$ which corresponds to the Whitney Umbrella. The coordinate ring of W is given by the K-algebra

$$K[\Gamma] = K[t_1t_2, t_1, t_2^2] \cong K[x,y,z]/(x^2 - y^2z).$$

The lattice generated by Γ is $M = \mathbb{Z}^2$. The dual cone associated to W consists on the first quadrant of $\mathbb{R}^2$. As one can see in figure 2.7, Γ is not saturated: the element $(0,1)$ belongs to M, but is not in Γ, whereas $(0,2) = 2\cdot(0,1) \in M$ is in Γ. The element $(1,0)$ is a primitive element of Γ and it is also a primitive element of the lattice M. The element $(0,2)$, which is primitive in Γ, is not primitive in M. Using Proposition 15 below, we can prove that W is not a normal variety, as it was already proved in section 1 when this surface was presented. In the picture, black points correspond to the semigroup, and white points together with black points form the lattice. The dual cone of Whitney Umbrella $\sigma^\vee$ is coloured in grey.

The field of fractions of $K[W]$ is given by

$$\mathrm{Q}(K[\Gamma]) \cong K[t_1, t_1^{-1}, t_2, t_2^{-1}].$$

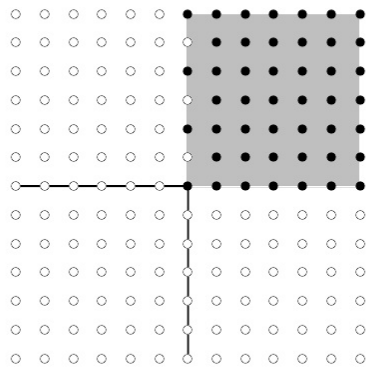

Figure 2.7: Semigroup, lattice and dual cone of Whitney Umbrella.

Proposition 19. *Let Γ be a finitely generated subsemigroup of a lattice $M \subset \mathbb{Z}^m$ and let $\sigma^\vee = \mathbb{R}_{\geq 0}\Gamma$ the rational convex cone generated by Γ. Under these conditions, the saturation of Γ in M is $\sigma^\vee \cap M$.*

Proof. It is easy to see that $\sigma^\vee \cap M$ is saturated in M: for any $m \in M$ and any $k \in \mathbb{N} \setminus \{0\}$, we have that $l = km \in \sigma^\vee \cap M$ implies that $l\frac{1}{k} = m \in \sigma^\vee$, since $\frac{1}{k} \in \mathbb{R}_{\geq 0}$, and therefore $m \in \sigma^\vee \cap M$.
On the other hand, we will see that any element in $\sigma^\vee \cap M$ belongs to the saturation $\widetilde{\Gamma}$ of Γ. Let e be an element in $\sigma^\vee \cap M$. Let $\{e_1, \ldots, e_m\}$ be a basis of the lattice M, and $\{\alpha_1, \ldots \alpha_n\}$ a set generating Γ. Since $e \in M$, there exist $a_i \in \mathbb{Z}$, $i = 1, \ldots, m$, with

$$e = \sum_{i=1}^{m} a_i e_i.$$

At the same time, $e \in \sigma^\vee$, so there exist $a'_j \in \mathbb{R}_{\geq 0}$, $j = 1, \ldots n$ such that

$$e = \sum_{j=1}^{n} a'_j \alpha_j.$$

Notice that, since $\Gamma \subseteq M$, each α_j can be written in the form $\alpha_j = \sum_{i=1}^{m} b_i^j e_i$, for some $b_i^j \in \mathbb{N}$, $j = 1, \ldots, n$, $i = 1, \ldots, m$. This leads to the expression

$$e = \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} (\sum_{j=1}^{n} a'_j b_i^j) e_i.$$

This means that for every $i = 1, \ldots, m$, $a_i = \sum_{j=1}^{n} a'_j b_i^j \in \mathbb{N}$. The b_i^j are positive integers, and the a'_j are positive rational numbers which we will denote by $\frac{p_j}{q_j}$. Thus every term of the sum must be in $\mathbb{Q}_{\geq 0}$. By multiplying the element e by $q = q_1 \cdot \ldots \cdot q_n$, we obtain that qe is a linear combination of the α_i with coefficients in $\mathbb{N}$. We conclude that qe is in Γ, and therefore e is in $\widetilde{\Gamma}$.

$\square$

The next result follows directly from Propositions 15 and 19:

Corollary 20. *Under the assumptions of Proposition 19, the integral closure of $K[\Gamma]$ in its field of fractions $K[M]$, is $K[\sigma^\vee \cap M]$. This means that $Y = \mathrm{Spec}(K[\sigma^\vee \cap M])$ is a normal variety.*

Example 21. The rational cone σ associated to Whitney Umbrella is the dual cone of $\sigma^\vee$, that is $\sigma = \{w \in \mathbb{R}^2 : \ v \cdot w \geq 0 \ \text{ for all } v \in \sigma^\vee \subset \mathbb{R}^2\}$, which is represented in figure 2.8. In this case, $\sigma = \sigma^\vee$.

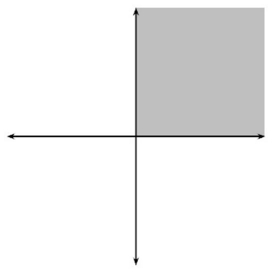

Figure 2.8: The cone of Whitney Umbrella.

2.1.5 Faces

The concept of face of a rational polyhedral convex cone was already introduced in section 2.1.3. In this section we describe them in detail, in the context of cones, dual cones and semigroups associated to a monomial variety.

Proposition 22. *Let σ be a rational polyhedral convex cone, and $\sigma^\vee$ its dual cone. Any face of $\sigma^\vee$ is of the form $\sigma^\vee \cap \tau^\perp$ for a unique face τ of σ, where $\tau^\perp = \{w \in M : \ v \cdot w = 0 \ \text{ for all } v \in \tau\}$.*

Proof. See [11, Proposition 1.2.10]. □

The faces of the dual cones, named $\tilde{\tau}_i$ correspond to $\tau_i^\perp \cap \sigma^\vee$, that is, the face of the dual cone $\sigma^\vee$ associated to the face τ_i of σ.

Examples 23. Figure 2.9 shows the duality of cones and their faces for some cones in $\mathbb{R}^2$.

A subsemigroup F of a semigroup Γ is a *face of* Γ if for any $x, y \in \Gamma$ with $x + y \in F$ we have $x, y \in F$.

Proposition 24. *Any face of Γ is of the form $\Gamma \cap \tau^\perp$ for some face τ of σ_Γ.*

Proof. See [11, Proposition 1.2.10]. □

Example 25. Going back to the Whitney Umbrella, the cone associated to this surface has three proper faces, given, respectively, by the intersection of σ with

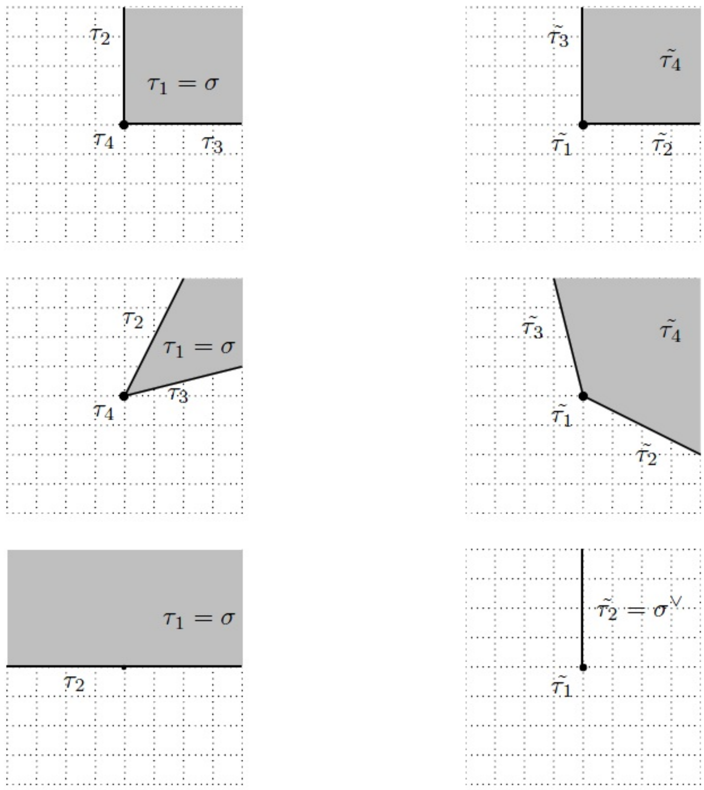

Figure 2.9: Duality between faces of cones in $\mathbb{R}^2$.

the horizontal axis, the vertical axis, and with some line with negative slope. The first two faces have dimension 1, and the last one is the origin. Therefore, this cone is strongly convex.
The dual face to the origin is the whole $\sigma^\vee$. The dual face for σ is the origin. For each of the 1-dimensional faces, the corresponding dual face is the perpendicular positive semi-axis.
The corresponding faces of Γ are the semigroup $F_1 = \{(0,0)\}$, the semigroup $\mathbb{N}^2$, and the semigroups of dimension 1 $F_2 = \langle(1,0)\rangle_\mathbb{N}$ and $F_3\langle(0,2)\rangle_\mathbb{N}$. They are represented in figure 2.10.

Remark 26. Each face F of the semigroup Γ is a subsemigroup with finite index of Γ. Each lattice $\mathbb{Z}F$ generated by one of these faces is a sublattice with finite index (a subgroup of finite index) of the lattice generated by the corresponding face τ of the cone σ by $M \cap \tau^\perp$:

$$\mathbb{Z}(\Gamma \cap \tau^\perp) \subset \mathbb{Z}(M \cap \tau^\perp).$$

We will denote this sublattice by $M(\tau, \Gamma)$ to use it later. The lattice generated by $M \cap \tau^\perp$ will be denoted as $M(\tau)$.

In the previous example, the lattices $\mathbb{Z}F_1$, $\mathbb{Z}F_2$ and $\mathbb{Z}F_3$ are subgroups of $\mathbb{Z}^2$ of indices 1, 2 and 1 respectively.

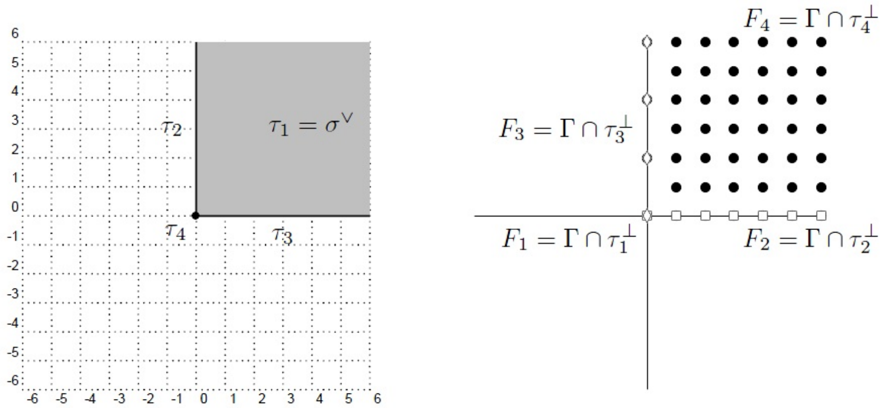

Figure 2.10: Faces of the dual cone and of the semigroup of Whitney Umbrella.

Remark 27. Since the faces of a semigroup are again semigroups, there is a monomial variety associated to each face F of a semigroup Γ, whose associated convex cone is the unique face τ of the convex cone σ_Γ satisfying $F = \Gamma \cap \tau^\perp$. For monomial varieties constructed in this way, we have that X^τ is an open subset of X^σ, where $K[X^\tau]$ is the localization of $K[X^\sigma]$ along the closed subvariety given by $X^{\tau^\perp \cap \mathbb{Z}\Gamma}$ (see [11, Proposition 1.3.16]). In particular, the intersection of two monomial varieties $X^\sigma, X^{\sigma'}$, where σ, σ' are rational polyhedral convex cones sharing a face τ, is an affine open subset X^τ.

As we will show in section 2.4, the faces of a cone σ_Γ are related to the orbits under the action of an algebraic group (an algebraic torus) over the variety X^Γ. In section 2.5 we will see that these orbits help to determine the singular locus of X^Γ.

2.2 Affine toric varieties

2.2.1 Torus

Let K be a field. The set $K^* = K \setminus \{0\}$ is a multiplicative group with the structure of an algebraic group over K given by $\mathrm{Spec}(K[t^{\pm 1}])$, where t is a variable.
A d-dimensional *torus* over K is an algebraic group isomorphic to $(K^*)^d$.

A lattice M of rank m gives, as a semigroup, the ring $K[M]$ of an algebraic torus over K of dimension m, which we will call

$$T^M = \mathrm{Spec}(K[M])$$

(that is, $X^{\mathbb{Z}}$). To see that T^M is isomorphic to $(K^*)^m$, consider the map:

$$\nu : T^M \longrightarrow (K^*)^m$$

induced by

$$\begin{aligned} \rho : K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}] &\longrightarrow K[M] \\ x_i &\longmapsto t^{\gamma_i}, \end{aligned}$$

where $\{\gamma_1, \ldots, \gamma_m\}$ is a basis of M.
This last homomorphism is also determined by the group homomorphism:

$$\begin{aligned} \tilde{\rho} : \mathbb{Z}^m &\longrightarrow M \\ e_i &\longmapsto \gamma_i \end{aligned}$$

in the sense that

$$\rho\left(x_1^{a_1} \cdot \ldots \cdot x_m^{a_m}\right) = t^{\tilde{\rho}(a_1 e_1 + \ldots + a_m e_m)}.$$

It is easy to see that $\tilde{\rho}$ is an isomorphism:
It is injective because $\tilde{\rho}(a) = \tilde{\rho}(b)$, $a, b \in \mathbb{Z}^m$, $a \neq b$ would mean $(a_1 - b_1)\gamma_1 + \ldots + (a_m - b_m)\gamma_m = 0$, which is only possible if $a_i - b_i = 0$ for all $i = 1, \ldots, m$ since the γ_i form a basis of M. It is surjective because any element v in M can be written as a linear combination of the canonical basis of $\mathbb{Z}^m$ by just using the i-th component of v as i-th coordinate.
This makes ρ an isomorphism by the way it is defined from $\tilde{\rho}$.

We now consider the surjective map $\tilde{\rho}_{(n)} : \mathbb{Z}^n \to M$, where $n \geq m$ which extends $\tilde{\rho}$ by first projecting into the first m components and then applying $\tilde{\rho}$. The kernel of $\tilde{\rho}_{(n)}$ is isomorphic to $\mathbb{Z}^{n-m}$.
One can consider now the restriction of $\tilde{\rho}_{(n)}$ to m-tuples of natural numbers is the same map we defined some lines above as $\tilde{\rho}^{(+)}$:

$$\begin{aligned} \tilde{\rho}^{(+)} : \mathbb{N}^n &\longrightarrow \Gamma \\ e_i &\longmapsto \tilde{\rho}_{(n)}(e_i) \end{aligned}$$

If we want this map to be the restriction of $\tilde{\rho}_{(n)}$, i.e.

$$\tilde{\rho}_{(n)}(v) = \tilde{\rho}^{(+)}(v) \forall v \in \mathbb{N}^m, \tag{2.6}$$

it is necessary to choose the basis of M and the basis of Γ in such a way that the second one is a subset of the first one. This map induces the construction of $K[\Gamma]$ in (2.1).

2.2.2 Affine toric varieties

An *algebraic action* is the action of an algebraic group G (for example, $(K^*)^n$) on a variety X by a morphism

$$\begin{aligned} \varphi : G \times X &\longrightarrow X \\ (g, x) &\longmapsto g \cdot x. \end{aligned}$$

An *affine toric variety* is an affine variety X which contains a torus as a Zariski open subset, and on which the action of the torus on itself

$$T \times T \to T \tag{2.7}$$

can be extended as an algebraic action

$$T \times X \to X. \tag{2.8}$$

A discussion about this action is given in section 2.4.

Proposition 28. *For any semigroup Γ, the torus T^M is embedded in X^Γ.*

Proof. Consider bases $\{\gamma_1, \dots, \gamma_m\} \subset \mathbb{N}^m$ and $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{N}^m$ of M and Γ respectively. That is, parametrizations $t \longmapsto (t^{\gamma_1}, \dots, t^{\gamma_m})$ of T^M and $t \longmapsto (t^{\alpha_1}, \dots, t^{\alpha_n})$ of X^Γ. Each α_i can be written as

$$\alpha_i = \sum_{j=1}^{m} a_j^i \gamma_j. \tag{2.9}$$

Each point of the torus can be mapped into X^Γ by

$$\begin{aligned} T^M &\overset{i}{\hookrightarrow} X^\Gamma \\ (x_1, \dots, x_m) &\longmapsto (\prod_{j=1}^{m} x_j^{a_j^1}, \dots, \prod_{j=1}^{m} x_j^{a_j^n}). \end{aligned}$$

To see that i is inyective, notice that each point of the torus can be written as $(x_1, \dots, x_m) = (t^{\gamma_1}, \dots, t^{\gamma m}) \in T^M$ for some $t \in K$. Hence

$$i(x_1, \dots, x_m) = (\prod_{j=1}^{m} (t^{\gamma_j})^{a_j^1}, \dots, \prod_{j=1}^{m} (t^{\gamma_j})^{a_j^n}) = (t^{\sum_{j=1}^m \gamma_j a_j^1}, \dots, t^{\sum_{j=1}^m \gamma_j a_j^n}).$$

For any two different points

$$\begin{aligned} p_1 = i(q_1) &= (t^{\sum_{j=1}^m \gamma_j a_j^1}, \dots, t^{\sum_{j=1}^m \gamma_j a_j^n}), \\ p_2 = i(q_2) &= (\tilde{t}^{\sum_{j=1}^m \gamma_j a_j^1}, \dots, \tilde{t}^{\sum_{j=1}^m \gamma_j a_j^n}) \end{aligned}$$

in X^Γ, $p_1 = p_2$ implies $t^{\sum_{j=1}^m \gamma_j a_j^i} = \tilde{t}^{\sum_{j=1}^m \gamma_j a_j^i}$ for every $i = 1, \dots, n$. As this happens for any different parametrization of T^M and X^Γ, or equivalently, for any choice of basis of M and Γ and the corresponding a_j^i satisfying (2.9), then

$$p_1 = p_2 \implies t = \tilde{t} \implies q_1 = q_2.$$

□

Given a semigroup Γ, the kernel of $\rho^{(+)} : K[x_1, \dots, x_n] \to k[\Gamma]$ as defined in (2.1) is an ideal $I \subset K[x_1, \dots x_n]$ defined by binomials, and it is called the *toric ideal* associated to $\rho^{(+)}$. The *affine toric variety* $X^\Gamma :=$ Spec$(K[\Gamma])$ is the subvariety of the affine space $\mathbb{A}^n$ whose defining ideal is the toric ideal I.

2.3 Abstract toric varieties

We can define now a toric variety in general:
A *toric variety* is an algebraic variety X containing a torus as a Zariski open subset, and such that the action of the torus on itself extends to an action on X.

The structure of fans presented in this section contains the combinatorial data which will be necessary for the result of blowing up one of the varieties we will consider in this text.

A *fan* is a finite set Σ of strongly convex polyhedral rational cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$ for a lattice N, such that for each pair $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\tau = \sigma_1 \cap \sigma_2$ is a face of both cones, and $\tau \in \Sigma$.

Let N be a lattice of rank n, and let Σ be a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$. We define the toric variety X_Σ^Γ by a triple (N, Σ, Γ) where $\Gamma = \{\Gamma_\sigma \subset \sigma^\vee \cap M\}_{\sigma \in \Sigma}$ is a family of finitely generated semigroups of a lattice $M = \mathrm{Hom}(N, \mathbb{Z})$ such that:

1. $\mathbb{Z}\Gamma_\sigma = M$ and $\mathbb{R}_{\geq 0}\Gamma_\sigma = \sigma^\vee$ for any $\sigma \in \Sigma$,

2. $\tau = \sigma + M\,(\tau, \sigma)$ for each $\sigma \in \Sigma$ and any face τ of σ.

This toric variety is given by the union of the affine varieties X^σ, $\sigma \in \Sigma$ where, for any pair $\sigma, \sigma' \in \Sigma$ we glue up X^σ and $X^{\sigma'}$ along their common open affine set $X^{\sigma \cap \sigma'}$.

2.4 Action of the torus and orbits

By definition, given a toric variety X, it is required that the action of the torus $T \subset X$ on itself is extended to an algebraic action over X. We will try to clarify in this subsection what this means, and some properties of this action will be explained.

Let us describe now the following relation between the torus T^M and the lattice M:
A *character* of a torus T^M is a group homomorphism $\chi : T^M \longrightarrow K^*$. The characters of a torus form an abelian group M whose rank (n) is equal to the dimension of the torus (see [11, section 1.1]). An element $m = (m_1, \ldots, m_n) \in M$ induces the character $\chi^m(t_1, \ldots, t_n) = t_1^{m_1} \cdot \ldots \cdot t_n^{m_n}$.
The dual lattice N of M can be identified with $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. For a given semigroup Γ with dual cone $\sigma^\vee$, we have that $\sigma = (\sigma^\vee)^\vee \in N \otimes_{\mathbb{Z}} \mathbb{R}$ and $\sigma \in M \otimes_{\mathbb{Z}} \mathbb{R}$.
Notice that for $M \cong \mathbb{Z}^n$, we have that $N \cong \mathbb{Z}^n$. Here, M is always considered as $\mathbb{Z}^n$, and that is the reason why we usually identify here M and N.

A different description of the points of a toric variety helps to understand the action of the embedded torus on it. Let X^Σ be a toric variety with $\Sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$ a fan for the lattice N, and let Γ be a family of semigroups. Pick a cone $\sigma \in \Sigma$

and consider the affine toric variety X^σ. First of all, notice that the ring of regular functions in X^σ is $K[\Gamma^\sigma]$, that is, any regular function on X^σ is of the form

$$\begin{aligned} \chi^\gamma : X^\sigma &\longrightarrow K^* \\ p = (p_1, \ldots, p_n) &\longmapsto p_1^{\gamma_1} \cdot \ldots \cdot p_n^{\gamma_n}, \end{aligned}$$

for some $\gamma \in \Gamma^\sigma$. The semigroup homomorphisms χ^γ, with $\gamma \in M$, are the characters of the torus T^M.

The points of X^σ are in bijective correspondence with the semigroup homomorphisms $\phi : \Gamma^\sigma \to K$, by associating each point $p \in X^\sigma$ to the semigroup homomorphism

$$\begin{aligned} \phi_p : \Gamma^\sigma &\longrightarrow K^* \\ \gamma &\longmapsto \chi^\gamma(p). \end{aligned}$$

(See [11, Proposition 1.3.1], [18, p.19].)
As a particular case of toric variety, the points of the torus $y \in X^M$ can also be identified with the group homomorphisms $\phi_y : M \to K^*$.
Using both correspondences, the action of the torus on X^σ can be described as follows:

$$\begin{aligned} X^M \times X^\sigma &\longrightarrow X^\sigma \\ (y, p) &\longmapsto y \cdot p : \Gamma^\sigma \to K, \end{aligned}$$

where $y{\cdot}p$ is a semigroup homomorphism defined on Γ^σ, and thus a point of X^σ.

As for any group action, it is possible to distinguish the orbits by the action of the torus. The following theorem yields:

Theorem 29. *Let $N \cong \mathrm{Hom}(M, K)$ be a lattice, let Σ be a fan and let X^Σ be the normal toric variety defined by Σ. Then:*

1. *There is a bijection between the cones in Σ and the orbits by the action of the torus X^N on X^Σ.:*

$$\sigma \leftrightarrow O(\sigma) \cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, K^*).$$

2. *For each $\sigma \in \Sigma$, dim$(O(\sigma))$ =dim$(N)-$dim(σ).*

3. *The affine open subset X_σ is the union of the orbits of all faces of σ.*

4. *The cone τ is a face of σ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$ and $\overline{O(\tau)}$ is the union of the orbits of all cones such that τ is a face of them.*

Proof. See [11, Theorem 3.2.6]. □

Notice that each face of a cone is associated to a unique orbit and vice versa.

Remark 30. The closure of an orbit by this action, $\overline{O(\sigma)}$ has the structure of a toric variety (see [11, p. 121]).

Example 31. Let M be the lattice $M = \langle (2,0),(1,1)\rangle_{\mathbb{Z}}$, and consider the cone $\sigma = \langle (1,0),(0,1)\rangle_{\mathbb{R}_{\geq 0}}$. Let Γ be the semigroup given by the intersection $\sigma^\vee \cap M$, that is $\Gamma = \langle (2,0),(1,1),(0,2)\rangle_{\mathbb{N}}$.
The face $\{(0,0)\}$ corresponds, by Theorem 29.2, to the whole X^Γ, which is the cone presented in Example 5.2. The orbit consisting on the whole cone σ corresponds to the origin, as one can check using the same argument. As for the faces, denote them by $\tau_1 = \langle (1,0)\rangle_{\mathbb{R}_{\geq 0}}$, $\tau_2 = \langle (0,1)\rangle_{\mathbb{R}_{\geq 0}}$. Using the first point of the theorem, we have that $O(\tau_1) \cong \mathrm{Hom}_{\mathbb{Z}}(\tau^\perp \cap M, K^*)$, where $\tau^\perp = \langle (0,2)\rangle_{\mathbb{R}_{\geq 0}}$. This is the subvariety given by the generator $t^{(0,2)}$, and therefore the z-axis.

The following proposition allows to decide whether a variety has a fixed point by the action of the torus or not. It will be necessary for a theorem in next section.

Proposition 32. *Let X^Σ an affine toric variety. Then there exists a fixed point under the action of the torus if and only if Γ is pointed.*

Proof. See [11, Proposition 1.3.2]. □

2.5 Toric varieties and smoothness

This subsection is dedicated to one of the properties of toric varieties that can sometimes be checked without looking at the equations: smoothness. The information of the variety contained in the rational cone associated to it, allows sometimes to decide whether it is smooth or not by just testing some properties of this cone. We discuss here the property of smoothness for toric varieties and present a result for normal toric varieties. Nevertheless, it can be useful for the combinatorial resolution process we are working in. For more details see [11].

A strongly convex rational polyhedral cone $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ is said to be a *smooth* or a *regular cone* if there exists a set of generators in N of σ that form a $\mathbb{Z}$-basis of N.
A cone $\sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$ is called *simplicial* if its minimal[1] generators are linearly independent over $\mathbb{R}^m$, where $m = \mathrm{rank}(N)$.

Notice that being smooth is a stronger condition than being simplicial:
smooth $\Rightarrow$ simplicial: Linear independence is required to form a basis.
simplicial $\not\Rightarrow$ smooth: Recall that the definition of smoothness implies that the generators of the cone can be extended to a $\mathbb{Z}$-basis, not to an $\mathbb{R}$-basis. Consider the following example: $M = \langle (2,0),(1,1)\rangle_{\mathbb{Z}}$, $\sigma = \langle (1,0),(0,1)\rangle_{\mathbb{R}_{\geq 0}}$. Let Γ be the semigroup given by the intersection $\sigma^\vee \cap M$, that is

$$\Gamma = \langle (2,0),(1,1),(0,2)\rangle_{\mathbb{N}}.$$

The minimal generators of σ are linearly independent, and since they are two, they form an $\mathbb{R}$-basis of $\mathbb{R}^2$, but they cannot be extended to a $\mathbb{Z}$-basis of N,

[1] Of minimal length in N

since they already generate points that are outside of N.

Remark 33. All strongly convex cones of dimension 2 are simplicial. On the other hand, in dimension 3 it is already possible to find strongly convex cones which are not simplicial: figure 2.11 shows a simplicial cone of dimension 3, while the cone shown in figure 2.12 is not simplicial although it is strongly convex.

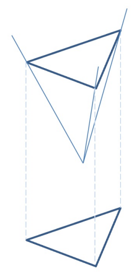

Figure 2.11: Triangular cone.

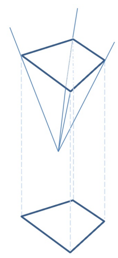

Figure 2.12: Quadrangular cone.

Theorem 34. *Let $\sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$ be a strongly convex rational polyhedral cone. The (normal) affine toric variety $X^{\sigma^\vee \cap M}$ is smooth if and only if σ is a smooth cone. Furthermore, all smooth affine toric varieties are of this form.*
(See [11, Theorem 1.3.12][32, Theorem 4].)

For the proof of Theorem 34, we need a lemma concerning the Hilbert basis of a semigroup. We call the set $\mathcal{H} = \{\alpha \in \sigma^\vee \cap M | \alpha \text{ is primitive } \}$ the *Hilbert basis of* $\sigma^\vee \cap M$. Notice that this set contains the ray generators of the edges of $\sigma^\vee$, that is, the sets of the form $\mathbb{R}_{\geq 0}\tau$ where τ is a face of $\sigma^\vee$ of dimension 1. The Hilbert basis is finite and generates $\sigma^\vee \cap M$. Furthermore, it is the minimal generating set of the semigroup (with respect to inclusion).

Lemma 35. *Let $\sigma \subset N$ be a strongly convex rational polyhedral cone of maximal dimension. Then the Zariski tangent space of $X^{\sigma^\vee \cap M}$ at its fixed point has dimension equal to $|\mathcal{H}|$, where $\mathcal{H}$ is the Hilbert basis of $\sigma^\vee \cap M$.*

Proof. See [11, Lemma 1.3.10].

□

Proof of Theorem 34: Suppose first that we have a smooth cone σ of dimension $r \leq m$ in $\mathbb{R}^m$. By definition of smoothness for a cone, there is a set of generators of σ that can be extended to a basis of the lattice N. W.l.o.g we can suppose that these generators are $e_1, \ldots, e_r$. Its dual cone will be $\sigma^\vee = \langle e_1, \ldots, e_r, \pm e_{r+1}, \ldots, \pm e_m \rangle \subset \mathbb{R}^n$. One can see that the affine variety given by σ is

$$X^{\sigma^\vee \cap M} = \mathrm{Spec}(K[\sigma^\vee \cap M]) = \mathrm{Spec}(K[x_1, \ldots, x_r, x_{r+1}^{\pm 1}, \ldots x_m^{\pm 1}]),$$

or in other words

$$X^{\sigma^\vee \cap M} \cong K^r \times (K^*)^{m-r},$$

which is clearly smooth.

Now suppose that we have a smooth affine toric variety of dimension m. Since it is smooth, it will necessarily be normal and therefore can be written as $X^{\sigma^\vee \cap M}$ for a certain cone σ. The dimension of the lattice M is also m. We separately consider the cases $\mathrm{rank}(\sigma) = m$ and $\mathrm{rank}(\sigma) < m$.

1. Suppose $\mathrm{rank}(\sigma) = m$. As σ is strongly convex by hypothesis, the semigroup $\sigma^\vee \cap M$ is necessarily pointed, and by Proposition 32 it has a unique fixed point p under the action of the torus.

Since $X^{\sigma^\vee \cap M}$ is smooth, it is in particular smooth at p and therefore

$$\dim(T_p X^{\sigma^\vee \cap M}) = \dim(X^{\sigma^\vee \cap M}) = m.$$

By Lemma 35 we know that $|\mathcal{H}| = \dim(T_p X^{\sigma^\vee \cap M}) = m$. Notice that $\mathcal{H}$ contains the ray generators of the edges of $\sigma^\vee$, and therefore the number of these edges is less than or equal to m. Notice also that σ must have at least as many edges as its dimension, namely m, so it has exactly m edges.
We already said that $\mathcal{H}$ contains the ray generators of the edges of σ, and now that we know the dimensions of both sets coincide, we can assert those ray generators are all the elements $\mathcal{H}$ contains. By definition, those rays generate the semigroup $\sigma^\vee \cap M$, and $M = \mathbb{Z}(\sigma^\vee \cap M)$. This is the same as saying that the edges of $\sigma^\vee$ generate M, and this is the simplest example of a smooth cone.
The last thing to use is that duality preserves smoothness. This is due to the fact that the edges of the dual cone are perpendicular to the edges of the original cone. Because of this we have that $(\sigma^\vee)^\vee = \sigma$ is smooth.

2. Suppose now $0 < \mathrm{rank}(\sigma) = r < m$. We consider the smallest sublattice $N_1 \subseteq N$ such that it is saturated and contains the generators of σ. Since N_1 is saturated, then N/N_1 is torsion-free, and there exists a sublattice $N_2 \subseteq N$ with $N = N_1 \oplus N_2$. We will call M_1 and M_2 their respective dual lattices. One can consider now two different varieties for the cone σ using the different lattices M, M_1, namely $X = \mathrm{Spec}(K[\sigma^\vee \cap M])$ and $X_1 = \mathrm{Spec}(K[\sigma^\vee \cap M_1])$. As $N_2 \cap \sigma = \{0\}$, we have that $M_2 \cap \sigma^\vee = M_2$, and from this comes

$$M \cap \sigma^\vee = (M_1 \cap \sigma^\vee) \oplus M_2.$$

This implies

$$K[M \cap \sigma^\vee] = \{ \sum_{\alpha \in M \cap \sigma^\vee} a_\alpha t^\alpha \} = \{ \sum_{\alpha \in M_1 \cap \sigma^\vee, \beta \in M_2} a_\alpha t^\alpha t^\beta \} = K[M_1 \cap \sigma^\vee][M_2],$$

and induces an isomorphism of K-algebras

$$K[M \cap \sigma^\vee] \cong K[M_1 \cap \sigma^\vee] \otimes_K K[M_2]$$

(see [2]) and hence an isomorphism of varieties

$$X = \mathrm{Spec}(K[M \cap \sigma^\vee]) \cong \mathrm{Spec}(K[M_1 \cap \sigma^\vee]) \times T^{M_2} \cong X_1 \times (K^*)^{m-r}.$$

Claim 36. $X_1 \times (K^*)^{m-r}$ smooth implies X_1 is smooth.

Proof of claim: Notice that

$$(K^*)^{m-r} = \mathrm{Spec}(K[x_{r+1}^{\pm 1}, \ldots, x_m^{\pm 1}])$$

and

$$X_1 \times (K^*)^{m-r} = \mathrm{Spec}(K[X_1][x_{r+1}^{\pm 1}, \ldots, x_m^{\pm 1}]).$$

Suppose X_1 is not smooth. Then $K[X_1]$ is not a regular ring. But this would mean $K[X_1][x_{r+1}]$ is not regular, and by induction $K[X_1 \times (K^*)^{m-r}]$ is not regular. So X_1 non smooth implies that $X_1 \times (K^*)^{m-r}$ is not smooth. □

Now we can use the first case because we have $\dim(\sigma) = \dim(N_1)$, which completes the proof. □

The hypothesis of normality makes it possible to study properties of a variety through the cone, instead of the dual cone. Considering a semigroup and the variety associated to it (because of the way the dual cone is constructed) it is not possible to detect whether the semigroup where it comes from is saturated. This information gets lost when passing to the dual cone. To see this, notice that many semigroups can have the same associated dual cone, but when we define a semigroup from a cone σ and a lattice M, namely $\sigma^\vee \cap M$, we make a choice on the generators of this semigroup. In particular, we choose primitive elements as generators.

There is a more general result about smooth cones and the singular locus of a (not necessarily affine) toric variety:

Theorem 37. *Using the relation between cones and orbits by the action of the torus, we have the following result: Let X_Σ be the normal toric variety of the fan Σ. We have:*
1. The singular locus of X_Σ is the union of the varieties corresponding to non smooth cones in

$$\Sigma\text{: } (X_\Sigma)_{\mathrm{sing}} = \cup_{\sigma \textit{ not smooth}} V(\sigma),$$

where $V(\sigma) = \overline{O(\sigma)}$ is the closure of the orbit corresponding to σ. 2. The union of all open sets (affine toric varieties) given by smooth cones gives all smooth points of the

$$X_\Sigma\text{: } X_\Sigma \backslash (X_\Sigma)_{\mathrm{sing}} = \cup_{\sigma \mathrm{smooth}} X_\sigma.$$

Proof. See [11, Proposition 11.1.2]. □

To find a resolution of the singularities of a toric variety X, we will usually blow X up along an affine subspace contained in X with certain properties. This kind of center will always be given by a monomial ideal. The following concept will help simplifying such blowups.

The *Newton Polyhedron* of the monomial ideal $I = (t^{\alpha_1}, \dots, t^{\alpha_r}) \subset K[\Gamma] \subset K[t\frac{1}{t}]$ is the convex hull $\mathcal{N}(I)$ of the Minkowski sum of sets $(\alpha_1, \dots, \alpha_r) + \sigma_\Gamma^\vee$.

Example 38. Let W be the Whitney Umbrella (see Example 8.2). Let I be the ideal corresponding to the origin, that is $I = (t^{(1,1)}, t^{(1,0)}, t^{(0,2)})$. Then the Newton polyhedron is the convex hull of all the points of the form $a + b$, where a is one of the exponents of the generators of I: $\{(1,1), (1,0), (0,2)\}$, and $b \in \sigma^\vee$. This set is shown in figure 2.13.

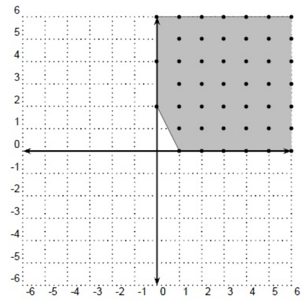

Figure 2.13: Newton Polyhedron of Whitney Umbrella.

The information given by the orbits of a toric variety under the torus action, together with the following result about the Newton polyhedron are used for choosing the center of the blowups.

Proposition 39. *The blowup of the ideal I is covered by the charts corresponding to the points of the lattice which are vertices of the Newton Polyhedron of this ideal.*

Proof. See [11, Prop. 2.1.9].

□

Chapter 3

Resolution of singularities of a monomial variety

Let $\mathcal{B}_\Gamma$ be a set of integer vectors with m components generating the semigroup Γ. Let $B = K[\Gamma]$ be the coordinate ring of the monomial variety X^Γ, defined by the semigroup Γ as in section 1.4. Let $\mathrm{Q}(B)$ be the field of fractions of B. In this section, we try to develop an algorithm to find a resolution of the singularities of X^Γ, using the combinatorial data of $\mathcal{B}_\Gamma$. In other words, what we intend to describe is a method to find, by performing blowups, a smooth algebra B' such that $\mathrm{Q}(B)$ and $\mathrm{Q}(B')$ are isomorphic as K-algebras. But we want to do this using the set of exponents of the generators of B. To make this possible, we define blowups as transformations of Γ.

All semigroups considered here will be subsemigroups of $\mathbb{Z}^m$. It will also be assumed that $\mathcal{B}_\Gamma$ is a minimal set of generators of Γ, and therefore, that we have B given by a minimal set of monomial generators as a K-algebra, unless it is stated otherwise.

3.1 Examples

In this subsection, some examples are presented, which intend to help understanding the kind of varieties considered here. Examples of resolution can be found in later sections.

Example 40. Let $K[X] = K[t^\alpha] \subset K[t]$, for some $\alpha \in \mathbb{N}$. It can be assumed that $\alpha = 1$: otherwise it is enough to consider the variable change $t'_i = t_i^{\alpha_i}$. In this case, X is isomorphic to the affine line $\mathbb{A}^1$. This variety is a smooth curve. Notice that this is the same case as $K[X] \cong K[t_i^\alpha] \subset K[t_1, \ldots, t_m]$ for some $i \in [m] = \{1, \ldots, m\}$ and some $\alpha \in \mathbb{Z}$, which is just an immersion in $\mathbb{A}^m$.

Example 41. Let $m = 1$ and let $\Gamma \subseteq \mathbb{Z}$. Then $K[X] \cong K[t^{\pm 1}] = K[\Gamma]$ is the coordinate ring of $\mathbb{A}^1 \setminus \{0\}$, an algebraic torus of dimension 1. Since this set cannot be expressed as the zero set of an ideal of $K[x]$, it is not an affine algebraic variety embedded in $\mathbb{A}^1$ but only in $\mathbb{A}^2$ (see section 4.2 for a discussion about dimension). This set is a quasi-affine variety, or an open set in the Zariski topology. Nevertheless, X can be seen as an affine variety in

$\mathbb{A}^2$, with

$$K[X] \cong K[t^{\pm 1}] \cong K[x,y]/(xy-1). \tag{3.1}$$

Example 42. Let $m > 1$. Let $K[X] \cong K[t_1^{\alpha_1}, \ldots, t_m^{\alpha_m}]$, with $\alpha_i \in \mathbb{Z}$. This means that Γ is a subsemigroup of $\mathbb{Z}^m$ given by linearly independent generators

$$e_i = (0, \ldots, 0, \alpha_i, 0, \ldots, 0).$$

Suppose, by the argument used in Example 40, that $\alpha_i = 1$ for all $i = 1, \ldots, m$. Here, X is again smooth (see Lemma 69). It is isomorphic to the m-dimensional affine space.

Example 43. Let $\mathcal{B}_\Gamma = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{Z}^m$ be a minimal system generating a semigroup Γ as $\mathbb{N}$-linear combinations, where $n < m$. These generators define a sublattice of $\mathbb{Z}^m$ of dimension $k \leq n$ isomorphic to $\mathbb{Z}^k$, with $k = n$ if and only if they are linearly independent. The projection of $\mathbb{Z}^m$ onto this sublattice is a homomorphism[1] γ between $\mathbb{Z}^m$ and $\mathbb{Z}^k$. The restriction of this homomorphism to Γ is injective. Notice that the image of Γ by this homomorphism is a semigroup, since it is closed under addition by the properties of homomorphisms. Thus we can identify Γ and $\gamma(\Gamma)$, which implies $X^\Gamma \cong X^{\gamma(\Gamma)} \subset \mathbb{A}^k$.

Example 44. Let $\Gamma = \langle (1,0), (0,3), (1,2) \rangle_{\mathbb{N}}$ be the semigroup defining a variety X^Γ. This variety is given by the equation

$$x^3 z^2 - y^3 = 0,$$

and in figure 3.1 a visualization of it in $\mathbb{R}^3$ is shown.

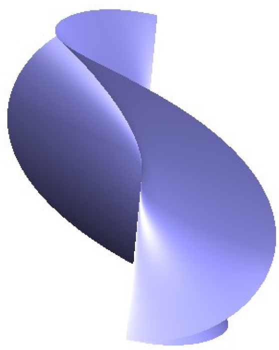

Figure 3.1: Surface in $\mathbb{A}^3$: $x^3z^2 - y^3$.

Examples 40 and 42 relate smoothness of the variety to linear independence of the elements in $\mathcal{B}_\Gamma$. Nevertheless, examples 41 and 44 show varieties given by semigroups, each of them generated by a linearly dependent set of elements. But while Example 41 shows a smooth variety, the one in Example 44 is not smooth.

We can conclude that linear independence is not necessary for smoothness,

[1]This is a group homomorphism, since a lattice is also a group.

although it is enough. The next subsection is dedicated to the analysis of a sharper condition for smoothness based on the relations between the elements in $\mathcal{B}_\Gamma$.

3.2 Smoothness condition

In this section we will analyze smoothness of monomial varieties X^Γ given by a semigroup Γ. This analysis will focus on the appearence of the semigroup.

Consider a semigroup $\Gamma = \langle \alpha_1, \ldots, \alpha_n \rangle_{\mathbb{N}}$, with $\alpha_i \in \mathbb{Z}^m$ for $i = 1, \ldots, n$. We will suppose that the set of generators of the semigroup $\mathcal{B}_\Gamma$ is minimal, and Γ has dimension m. The condition of smoothness of the monomial variety X^Γ is related to the algebraic dependence between the generators $t^{\alpha_1}, \ldots, t^{\alpha_n}$ of its coordinate ring $K[\Gamma]$. We will see that one can find a correspondence between the relations of algebraic dependence of these generators, and the relations of linear dependence between their exponents $\alpha_1, \ldots, \alpha_n$ over $\mathbb{Z}$. That makes it possible to study the different possible equations of monomial varieties by just looking at a generating set of the linear relations between vectors in $\mathbb{Z}^m$.

First of all, we must consider the case in which $\mathcal{B}_\Gamma \subset \{t^\alpha\}_{\alpha \in \Gamma}$ is a set of algebraically independent elements. This means that we have a basis of the semigroup Γ which is also a basis of the lattice $\mathbb{Z}\Gamma$.

Remark 45. Let Γ be a semigroup and let $t^\Gamma := \{t^\alpha\}_{\alpha \in \Gamma}$ a set of algebraically independent generators. The K-algebra generated by t^Γ, $K[\Gamma]$, is a smooth algebra. Then X^Γ is isomorphic to an affine space of dimension $|\Gamma|$ by Lemma 69.

Claim 46. Let Γ be a finitely generated semigroup, with minimal system of generators $\{\alpha_1, \ldots, \alpha_n\}$. The ideal I of algebraic relations between the monomials t^{α_i} is given, as it was explained in section 2.2, by binomials. The binomials constituting a minimal system of generators of I are in one to one correspondence with the minimal generators of the set of linear relations between the α_i.

Proof. A binomial in I is of the form

$$f(x_1, \ldots, x_n) = \prod_{i \in \Delta_+ \subseteq [n]} x_i^{a_i} - \prod_{j \in \Delta_- \subseteq [n]} x_j^{-a_j}, \tag{3.2}$$

with $a_i, a_j \in \mathbb{Z}$, and where $\Delta_+ \uplus \Delta_-$ is a partition of $\{i \in [n] : a_i \neq 0\}$ such that $i \in \Delta_+$ if $a_i > 0$ and $a_i \in \Delta_-$ if $a_i < 0$. The monomials t^{α_i} satisfy

$$f(x_1, \ldots, x_n) = \prod_{i : \in \Delta_+} (t^{\alpha_i})^{a_i} - \prod_{j : \in \Delta_-} (t^{\alpha_j})^{-a_j} = 0. \tag{3.3}$$

This clearly induces a linear relation between the α_i, that is, a linear polynomial in n variables

$$P(z_1,\ldots,z_n)=\sum_{i\in\Delta_+}a_iz_i-\sum_{j\in\Delta_-}-a_jz_j=\sum_{i\in\Delta_+\cup\Delta_-}a_iz_i,\text{ where }a_i,a_j\in\mathbb{Z}, \tag{3.4}$$

which is zero in $(\alpha_1,\ldots,\alpha_n)$. On the other hand, a linear relation as in (3.4) induces a binomial equation as (3.2) by the inverse process.
Choose a polynomial P of the form (3.4). Note that, if there exists a polynomial $Q\in Z[z_1,\ldots,z_n]$ such that $Q|P$, $Q\neq P$, then necessarily $Q\in\mathbb{Z}$. In this case, then the f (as in (3.2)) associated to P satisfies $f=g^Q$ for some binomial g. This g induces the linear relation $\tilde{P}=P/Q$. Therefore, every irreducible P in $\mathbb{Z}[z_i,\ldots,z_n]$ of the form (3.4) is uniquely associated to an irreducible $f\in K[x_1,\ldots,x_n]$ of the form (3.2).

□

If one writes the generators of Γ in matrix form, considering each α_i as a column of a matrix $G_{\mathcal{B}_\Gamma}$, one gets an easy representation of the configuration of the vectors in $\mathcal{B}_\Gamma$, see [33].
By just computing the determinant of this matrix, it is possible to decide whether the vectors are linearly independent or not. This will give us a criterion to decide if there are any linear relations to search for in case $\det(G_{\mathcal{B}_\Gamma})=0$, or the generators of $K[\Gamma]$ are algebraically independent in case $\det(G_{\mathcal{B}_\Gamma})\neq 0$. In the last case we can use the criterion in Remark 45 above to say X^Γ is smooth.

If $\det(G_{\mathcal{B}_\Gamma})=0$, we can compute the kernel of the map given by $G_{\mathcal{B}_\Gamma}$ in order to obtain a set of generators of the space of linear relations between the α_i. This is the same as solving the linear system

$$G_{\mathcal{B}_\Gamma}\cdot\overline{x}=0, \tag{3.5}$$

where $\overline{x}$ is a column vector of n variables $x_1,\ldots,x_n$.
The lattice formed by all integer solutions of this system is the dual lattice to the one generated by $\mathcal{B}_\Gamma$. The matrix whose rows are a minimal generating system of this lattice is known as the *Gale transform* of $\mathcal{B}_\Gamma$ (see [33]), and we will denote it by $\mathcal{G}_{\mathcal{B}_\Gamma}$.

We will consider the case of a hypersurface X^Γ. The matrix $G_{\mathcal{B}_\Gamma}$ associated to X^Γ has size $m\times(m+1)$, and rank m over $\mathbb{Z}$. The space of $\mathbb{N}$-linear relations between the α_i is generated then by a unique vector. The linear relation given by this vector induces the equation of the hypersurface. We will study the different situations which can happen for different linear relations.

Calculating an element that generates these linear relations as $\mathbb{N}$-linear combinations is easy in this situation. Let $G_{\mathcal{B}_\Gamma}=\{a_{i,j}\}$ be the matrix associated to $\mathcal{B}_\Gamma$. Then the vector given by

$$\left(a_i,i=1,\ldots,m\right)=\left((-1)^{i+1}\det(A_i),i=1,\ldots,m+1\right), \tag{3.6}$$

where

$$A_i=\begin{pmatrix}\alpha_{1,1}&\cdots&\alpha_{1,i-1}&\alpha_{1,i+1}&\cdots&\alpha_{1,m+1}\\ \vdots&\ddots&\vdots&\vdots&\ddots&\vdots\\ \alpha_{m,1}&\cdots&\alpha_{m,i-1}&\alpha_{m,i+1}&\cdots&\alpha_{m,m+1}\end{pmatrix},$$

generates the space of linear relations between the vectors in $\mathcal{B}_\Gamma$.

If X is not a hyperfurface, the kernel of the map has more than one generator. It is important to notice that what we need is a set of generators of the kernel of the map in $\mathbb{Z}$.

Example 47. The parametrization

$$t \longmapsto (t^3, t^2 s, ts^2, s^3)$$

gives the matrix

$$G_{\mathcal{B}_\Gamma} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix},$$

of size 2×4 and rank 2. The kernel of this map is generated by $\mathbb{R}$-linear combinations of two vectors. However, to generate it as $\mathbb{Z}$-linear combinations we need three vectors. This difficulty is related to the problem of not complete intersections. Figure 3.2 shows the semigroup generated by the columns of $G_{\mathcal{B}_\Gamma}$. The following relations generate the kernel as a $\mathbb{Z}$-linear combination:

$$\begin{aligned}(3,0)+(1,2) &= 2(2,1),\\ (2,1)+(1,2) &= (3,0)+(0,3),\\ (2,1)+(0,3) &= 2(1,2),\end{aligned}$$

which are also represented in figure 3.2.

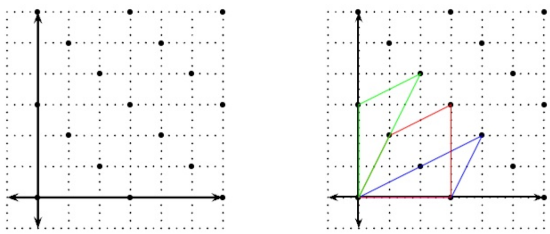

Figure 3.2: Semigroup associated to the cubic $Y : xz = y^2,\ yz = xt,\ yt = z^2$.

The equations induced by these relations are

$$\begin{aligned}xz &= y^2,\\ yz &= xt,\\ yt &= z^2,\end{aligned}$$

which are the equations of the cubic Y, a projective curve in $\mathbb{P}^3$. It is well known that this curve cannot be expressed by less than 3 equations even though it is a curve in a 3-dimensional projective space[2]. The reason for this is that this curve is not a complete intersection.

Computing the equations of a toric variety is a more delicate task when X is not a hypersurface. Sturmfels deals with this problem in [40], using elimination theory and Gröbner bases.

[2]It can also be considered as a variety of dimension 2 in the 4-dimensional affine space.

Consider a relation $(a_1, \ldots, a_n)$ between the elements of Γ as in (3.4).

We will distinguish two cases: in the first one either Δ_+ or Δ_- are empty; in the second both are non-empty.

First case: Suppose Δ_- is empty. This means that all the coefficients are positive. In case Δ_+ was empty, by multiplying the whole equation by -1 we are in the same case. The equation of the variety has the form:

$$\prod_{i \in [n]} (t^{\alpha_i})^{a_i} - 1 = 0, \text{ with } a_i \geq 0. \tag{3.7}$$

Using the Jacobian criterion explained in section 5.1, it is easy to see that X^Γ given by such an equation is smooth at every point.

Second case: Consider in first place $|\Delta_-| = 1$ or $|\Delta_-| = |\Gamma| - 1$. This means that there exists $j \in \{1, \ldots, n\}$ such that

$$\prod_{i \in [n] \setminus j} (t^{\alpha_i})^{a_i} = (t^{\alpha_j})^{a_j}. \tag{3.8}$$

If $a_j = 1$, then t^{α_j} can be generated by a product of some other generators, and α_j is redundant, so it can be removed from Γ. We will omit this case in the future. If $a_j \neq 1$ the Jacobian criterion shows that X^Γ is not smooth at the origin.

In general, using the partitions of the coefficients according to their sign, one will get an equation

$$\prod_{i \in \Delta_+} (t^{\alpha_i})^{a_i} = \prod_{j \in \Delta_-} (t^{\alpha_j})^{|a_j|}, \tag{3.9}$$

which will give a variety X^Γ which is singular at the origin (and possibly somewhere else) whenever Δ_- and Δ_+ are both non empty. This can be checked with the Jacobian criterion.

In order to understand the meaning of the last equation in terms of the vectors α_i, one can look at the linear relation

$$\sum_{i \in \Delta_+} a_i * \alpha_i = \sum_{i \in \Delta_-} |a_j| * \alpha_j, \tag{3.10}$$

induced by the algebraic relation (3.9). The existence of such a relation admits the following geometrical interpretation in $M \otimes_{\mathbb{Z}} \mathbb{R}$: there exists a point $p \in \mathbb{A}^m$ which has two different expressions as linear combination of disjoint subsets of the α_i with positive coefficients, namely

$$p = \sum_{i \in \Delta_+} a_i * \alpha_i = \sum_{i \in \Delta_-} |a_j| * \alpha_j. \tag{3.11}$$

We will explain what this last condition means for some particular values of m. Remark that for vectors of size m we will only consider cones of $\mathbb{R}^m$. If the cone is of smaller dimension, this means that there exists an affine subspace of dimension $m-1$ in $\mathbb{R}^m$ containing all the vectors in the set. By taking their projections over this subspace, they can be seen as vectors in $\mathbb{Z}^{m-1}$.

Proposition 48. *(Criterion for smoothness of hypersurfaces) Let*

$$\mathcal{B}_\Gamma = \{\alpha_1, \dots, \alpha_{m+1}\} \subset \mathbb{Z}^m$$

be a set of generators of a semigroup Γ of dimension m. Assume there exists some $i \in [m]$ such that the cone $\sigma_i = \langle \alpha_1, \dots, \hat{\alpha_i}, \dots, \alpha_{m+1} \rangle_{\mathbb{R}_{\geq 0}}$ is a cone of dimension m. Then the monomial variety X^Γ is smooth if and only if $-\alpha_i \in \sigma_i$.

Proof. The existence of an m-dimensional cone generated by a subset of m elements from $\mathcal{B}_\Gamma$ is ensured by the dimension of the semigroup. Suppose without loss of generality that the cone σ_{m+1} generated by the first m vectors of $\mathcal{B}_\Gamma$ has dimension m, and let $i = m+1$. There exist positive integers a_j, for $j = 1, \dots, m+1$, such that

$$-a_{m+1}\alpha_{m+1} = \sum_{j=1}^{m} a_j \alpha_j. \tag{3.12}$$

That is, $\sum_{j=1}^{m+1} a_j \alpha_j = 0$. Then the variety X^Γ is given by the equation $\prod_{j=1}^{m+1} x_j^{a_j} - 1 = 0$, and is therefore smooth.
Conversely, it is clear that an equation of a smooth surface, as in (3.7) induces a relation as (3.12), and therefore $-\alpha_{m+1} \in \sigma_{m+1}$. □

This is the only possible description of a smooth toric hypersurface in $\mathbb{A}^m$. For any semigroup Γ with $|\mathcal{B}_\Gamma| = m + d > m + 1$ for a minimal generating set $\mathcal{B}_\Gamma$, the kernel of the associated map has dimension $d > 1$ and therefore any base of this kernel over $\mathbb{Z}$ has at least d elements. This means that X^Γ is given by at least d equations, so it is not a hypersurface.

Example 49. Let $m = 2$. By Remark 45 we know that when Γ is generated by two linearly independent vectors in $\mathbb{Z}^2$, X^Γ is a smooth variety. The last proposition tells us that when $|\mathcal{B}_\Gamma| = 3$, the variety will be smooth as long as the cone generated by a linearly independent pair of the elements in $\mathcal{B}_\Gamma$ contains the inverse of the third element. Some examples of configurations of vectors which give singular and smooth varieties are shown in figures 3.3 and 3.4 respectively.

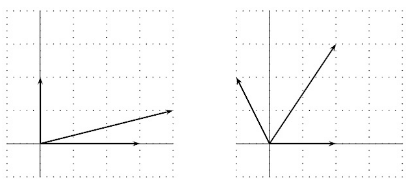

Figure 3.3: Semigroups of a singular algebra.

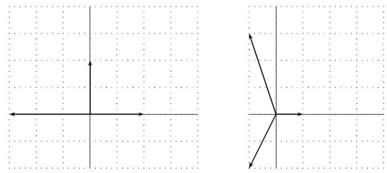

Figure 3.4: Semigroups of smooth algebras.

We would like to use the criterion for more general varieties, with codimension greater than 1. The following results hold for arbitrary monomial varieties.

Proposition 50. *Let $\mathcal{B}_{\Gamma_1}, \ldots, \mathcal{B}_{\Gamma_n}$ be minimal systems of generatos of the semigroups $\Gamma_1, \ldots, \Gamma_n \subset \mathbb{Z}^m$ with $n \leq m$, such that $X_1, \ldots, X_n$, with $X_i = X^{\Gamma_i}$ are smooth monomial hypersurfaces in $\mathbb{A}^m$. Suppose that these hypersurfaces are given by binomial equations $f_1 = 0, \ldots, f_n = 0$ respectively, such that the f_i are algebraically independent elements in $K[x_1, \ldots, x_m]$. Then the intersection $X_1 \cap \ldots \cap X_n = X$ is a smooth monomial variety.*

Proof. It is clear that X is a monomial variety, since it is given as the zero set of a set of binomial equations, $X : f_1 = \ldots = f_n = 0$.
We still need to check that it is smooth. To use the Jacobian criterion, we must compute the determinant of the Jacobian matrix.By by Proposition 48, we may assume that the f_j are given by $\prod_{i=1}^m x_i^{a_i^j} - 1$, with $a_i^j \in \mathbb{N}$. Notice that X has empty intersection with the coordinate axes. Let $F = (f_1, \ldots, f_n)$. The j-th row of the Jacobian matrix DF will be given by the partial derivatives of f_j. The i-th derivative of f_j is $\frac{a_i^j}{x_i} \prod_{k=1}^m x_k^{a_k^j}$. Notice that all elements in a row have a common factor, which is a product of the x_i. Furthermore, all elements in a column also share a common factor: all elements in column i have the factor $\frac{1}{x_i}$. By the properties of determinants, it is clear that for any subset $\{i_1, \ldots, i_n\} \subset \{1, \ldots, m\}$ with $i_1 \neq i_j$ whenever $i \neq j$ we have

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_{i_1}} & \cdots & \frac{\partial f_1}{\partial x_{i_n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_{i_1}} & \cdots & \frac{\partial f_n}{\partial x_{i_n}} \end{pmatrix} = \left(\prod_{j=1}^n \prod_{i=1}^n x_{i_1}^{a_{i_1}^j} \right) \left(\prod_{i=1}^n \frac{1}{x_{i_1}} \right) \det(A(i_1, \ldots, i_n)), \tag{3.13}$$

where $A = \{a_i^j\}_{i,j}$ is the matrix containing for each row j the exponents of the variables $x_i, i = i_1, \ldots, i_n$ in f_j. Since DF is not defined for X when one of the x_i is equal to zero because these points are not part of the variety, we have that the determinant in (3.13) is equal to zero if and only if $\det(A(i_1, \ldots, i_n)) = 0$. But by assumption the $f_i, i = 1, \ldots, n$ are algebraically independent, and by Claim 46 this implies that the vectors $(a_1^i, \ldots, a_n^i)$ for $i = 1, \ldots n$ are linearly independent. Thus, there exists a minor $A(i_1, \ldots, i_n)$ of the matris $A = A(1, \ldots, m)$ whose determinant is different from zero. The determinant of the corresponding minor of the Jacobian matrix is also different from zero for any point of X. Hence X is smooth. $\square$

Proposition 51. *Let $X_1, \ldots, X_n$ be singular monomial hypersurfaces in $\mathbb{A}^m$, with $n \leq m$, given by the binomial equations $f_i, \ldots, f_n$ respectively, such that* *the f_j are algebraically independent. Then the intersection $X = X_1 \cap \ldots \cap X_n$ is a non smooth toric variety. In particular, it is singular at the origin.*

Proof. Again, X is monomial because it is given by n binomial equations. Let $f_j = \prod_{i \in \Delta_+^j} x_i^{a_i^j} - \prod_{i \in \Delta_-^j} x_i^{a_i^j}$, where for every j, $|\Delta_+^j| \leq |\Delta_-^j|$. Recall that since X_j is singular, $|\Delta_+^j| = 1$ implies $a_k^j > 1$ for $k \in \Delta_+^j$, and the same condition holds for Δ_-^j. This implies that every non zero element in DF is a product of powers of some of the x_i. It is clear that $DF(0,\ldots,0)$ is the zero matrix, and therefore $\det(DF(0,\ldots,0)) = 0$. From this follows that X is singular at the origin.

□

The last results can help us to recognize some situations in which the Gale transform defined above shows the smoothness of a monomial variety. We have the following proposition:

Proposition 52. *Let $G_{\mathcal{B}_\Gamma}$ be the matrix of a minimal system of generators of the semigroup Γ as above. Let $\mathcal{G}_{\mathcal{B}_\Gamma}$ be the Gale transform of $\mathcal{B}_\Gamma$. If for each column of $\mathcal{G}$, all the elements in this column share the same sign, then the monomial variety X^Γ is smooth.*

Proof. The matrices $G_{\mathcal{B}_\Gamma}$ and $\mathcal{G}_{\mathcal{B}_\Gamma}$ satisfy $G_{\mathcal{B}_\Gamma} \cdot \mathcal{G}_{\mathcal{B}_\Gamma} = (0)$. Choose a column $\overline{v}^t$ of $\mathcal{G}_{\mathcal{B}_\Gamma}$. Then

$$G_{\mathcal{B}_\Gamma} \cdot \overline{v}^t = (0,\ldots,0)^t,$$

gives a linear relation between the elements of $\mathcal{B}_\Gamma$. By hypothesis, all the coefficients of this relation have the same sign, so it corresponds to an algebraic relation between the generators of $K[\Gamma]$ which gives rise to the equation of a smooth hypersurface.

The columns in $\mathcal{G}_{\mathcal{B}_\Gamma}$ are a minimal system of generators of $\mathrm{Ker}(G_{\mathcal{B}_\Gamma})$, and therefore algebraically independent. This means that X^Γ is defined by the equations associated to these linear relations, and therefore defined as the intersection of the corresponding smooth monomial hypersurfaces. By Proposition 50 it is a smooth monomial variety.

□

Similarly, the following can be proved:

Proposition 53. *Let $G_{\mathcal{B}_\Gamma}$ be the matrix of a minimal system of generators of the semigroup Γ. Let $\mathcal{G}_{\mathcal{B}_\Gamma}$ be the Gale transform of $\mathcal{B}_\Gamma$. If for each column of $\mathcal{G}_{\mathcal{B}_\Gamma}$, there are both positive and negative elements in this column, then the monomial variety X^Γ is singular. In particular it is singular at the origin.*

3.3 Combinatorial process of resolution

Blowups of varieties have already been introduced in section 1.3. In this section we give a construction of blowups for the simpler specific case of monomial varieties X^Γ. For this, we will show how the monomial algebra changes under a blowup with a given center. Remark that we will only choose as centers affine coordinate subspaces.

Let $\Gamma \subset \mathbb{Z}^m$ be a semigroup and let $B = K[\Gamma] \subseteq K[t_1, \ldots, t_n] \subseteq K[t^{\pm 1}]$ be the K-algebra defined by this semigroup, where $t = (t_1, \ldots, t_m)$. Let J be a non-empty subset of $\{1, \ldots, n\}$. For each $i \in J$, define a new K-algebra

$$B_i = K[t^{\alpha_1}, \ldots, t^{\alpha_n}, t^{\alpha_j - \alpha_i} : j \in J, j \neq i]. \tag{3.14}$$

The exponents of the generators of B_i form a semigroup which we will denote by Γ_i, so that we have

$$B_i = K[\Gamma_i]. \tag{3.15}$$

The ring B_i is the coordinate ring corresponding to the i-th chart of the monomial transform of B under the blowup with center the ideal $(x_i, i \in J)$.

This is the combinatorial formulation of the process of performing a blowup of a monomial variety: it consists of constructing a new semigroup for each chart by adding new elements to Γ. Those elements are the result of choosing a subset of a system of generators $\mathcal{B}_\Gamma$ of Γ and subtracting one element from every other element in this subset.

The interest in looking at blowups from this combinatorial point of view lies in the possibility of simplifying the construction of the charts, and computing the blowups in a very fast way through the coordinate ring. It is also possible to understand some properties of the variety from the structure of the semigroup Γ. For instance, we know that if Γ is saturated, then it gives a normal ring $K[\Gamma]$ and therefore a normal variety. It can be easier to look for an invariant to measure the improvement of a variety during the resolution process which can be computed using just the elements of Γ.
As we said, we consider only coordinate subspaces as centers for the blowups. In this case, it could be also interesting to find a criterion for the choice of this center Z by knowing the role that some generator α_i plays in the semigroup, and deciding if the associated coordinate $x_i = t^{\alpha_i}$ should be one of the monomial generating the ideal Z.

Example 54. The variety of equation $W : \left\{x^2 - y^2 z = 0\right\}$ was already resolved in section 1.3, but let us go back to this example to show how a blowup is computed following the notation explained in this subsection. In this case we are going to try two different choices of the center J. Recall that we expressed the coordinate ring of W as $K[W] \cong K[\Gamma]$ for the semigroup $\Gamma = \langle (1,1), (1,0), (0,2) \rangle_{\mathbb{N}}$.

In first place, let J be $\{1,2\} \subset [3]$. That is, we blowup at the z-axis Two charts are required:

- Chart 1: The semigroup of the transform in this chart is

  $$\Gamma_1 = \langle (1,1), (0,-1), (0,2) \rangle_{\mathbb{N}}.$$

  That is, its coordinate ring in the chart corresponding to x is

  $$K[ts, \frac{1}{s}, s^2] \cong K[x, y, y^{-1}].$$

  This corresponds to the cartesian product of the hyperbola and the affine line.

- Chart 2: The semigroup of the transform in this chart is

$$\Gamma_2 = \langle (0,1), (1,0), (0,2) \rangle_{\mathbb{N}}.$$

 The coordinate ring in the chart corresponding to y is now $K[s,t,s^2] \cong K[x,y]$. This affine chart gives the affine plane.

Both charts are smooth, and the resolution process is complete.

Let us try now $J = \{1,2,3\}$. That is, choose the origin as center of the blowup. Now three charts appear:

- Chart 1: The transform in this chart will have coordinate ring

$$K[ts, \frac{1}{s}, \frac{s}{t}] \cong K[t, \frac{1}{s}, \frac{1}{t}, s] \cong K[x^{\pm 1}, y^{\pm 1}].$$

 The variety is the 2-dimensional torus.

- Chart 2: The coordinate ring of the transform in this chart is $K[s,t,\frac{s^2}{t}] \cong K[x,y,z]/(x^2 - yz)$. This is not smooth, but it is normal, so in some sense the singularities have improved.

- Chart 3: In this chart one has $K[\frac{t}{s}, \frac{t}{s^2}, s^2] \cong K[x,y,z]/(x^2 - y^2 z)$, which is the same variety, the Whitney Umbrella. In this case there is no improvement of the singularities.

It is clear that the first choice of the center gives a better result: the second choice reproduces the singularities in one of the charts.

3.4 Choosing the center

Let X be a singular affine algebraic variety. A blowup $\varphi : X' \longrightarrow X$ yields an isomorphism between the complement of the center $Z \subseteq X$ of the blowup and of its preimage by φ. The blowup φ is completely determined by the choice of Z. However, there is no clear criterion for such an election.

Recall that the objective of the resolution process is to find a smooth algebraic variety $\widetilde{X}$ such that it is isomorphic to X everywhere except over the singular locus $\mathrm{Sing}(X)$.

Some observations to take into account when choosing the center of a blowup are the following:

- Blowing up X along the subvariety Z modifies X along Z and not $X \setminus Z$.

- Every singular point in $\mathrm{Sing}(X)$ needs to be modified, and therefore has to be part of the center of some blowup before arriving at a resolution.

- Points outside of X do not need to be modified.

The first approach would be choosing Z as the whole singular locus $\mathrm{Sing}(X)$. However, there is another fact to consider: if Z is smooth, the transform of the ambient space in which X is embedded remains smooth, but if Z is a singular subvariety, the transform of the ambient space can be singular along the preimage of Z under φ.

Choosing smooth centers guarantees that the ambient space does not become singular. If it did, the geometry would become much more complicated. However, choosing a smooth center does not guarantee an improvement of the singularities under blowups. This was clear in Example 54, where choosing the origin as center of the blowup did not improve the singularities of the Whitney Umbrella, while choosing the z-axis led to a resolution.

Nevertheless, some facts have been proved for particular cases, see [23]. Sometimes they will help for this choice. We mention here some of these aspects, although we do not intend to go deeper into them in general.

For curves, choosing the singular locus as center of the blowup works. Since curves only have isolated singularities, all the components of the singular locus are regular. All these points have to be part of the center of a blowup. Choosing them all as center will give, after a finite sequence of blowups, a regular curve, which can be tangent to some exceptional component. After that, a new sequence of blowups, choosing the points in which this tangential intersection takes place, finally makes the curve transversal to the exceptional divisor.

For surfaces, isolated singularities can also be resolved by a sequence of blowups with center the singular point. We will see this in next section. But the singular locus of a surface can also contain a finite number of curves. Those curves can be smooth or not. One could try to resolve first the singularities inside each non-regular curve, as above. In principle, that should not modify the rest of the variety. But it is not clear if the total transforms of these curves have new componens, which can be inside or outside of the singular locus of X. That could modify the singular locus of X, but only by adding new smooth curves. This makes it possible to obtain, by this strategy, a variety whose singular locus is formed by a finite number of points and smooth curves. They can intersect tangentially. More blowups are needed to separate those curves and get at worse normal crossings. Finally, it is proved in [24] that choosing this set of separated or transversal curves and isolated points as center of a new blowup improves the singularities. This ensures that we will have a resolution after a finite number of steps.

3.5 Curves

With the notation for blowups of monomial varieties introduced in section 3.3, we will show how the process of resolution works for monomial curves.

Let B be a monomial algebra generated by a finite set of monomials

$$\{t^{\alpha}\}_{\alpha\in\mathcal{B}_\Gamma\subset\mathbb{Z}},$$

and let $\Gamma \subset \mathbb{Z}$ be the semigroup generated by $\mathcal{B}_\Gamma$. The ring B is the coordinate ring of a monomial curve $C \subset \mathbb{A}^n$, where n is the size of $\mathcal{B}_\Gamma$. There is only one possible center for a blowup, namely $J = [n]$. This choice of center corresponds to the origin, with defining ideal $(x_1, \ldots, x_n) \subset K[x_1, \ldots x_n]$. For each chart of this blowup, we obtain a coordinate ring $B' = K[\{t^{\alpha'}\}_{\alpha' \in \mathcal{B}_{\Gamma'}}]$, where $\mathcal{B}_{\Gamma'}$ is the set of exponents of a minimal system of monomial generators of the ring B'.

For the chart corresponding to α_i, constructed by the subtraction of α_i from the rest of the elements, associate:

$$\begin{aligned} B \longleftrightarrow \{t^\alpha, \alpha \in \Gamma\} &\longrightarrow \{t^{\alpha-\alpha_i}, \alpha \in \Gamma - \{\alpha_i\}\} \cup \{t^{\alpha_i}\} \longleftrightarrow B' \\ t^{\alpha_i} &\longmapsto t^{\alpha_i} \\ t^{\alpha} &\longmapsto t^{\alpha-\alpha_i}, \alpha \neq \alpha_i. \end{aligned}$$

It is easy to see that for any $i = 1, \ldots, n$, we have

$$B' = K[\{t^\alpha\}_{\alpha \in \mathcal{B}_\Gamma} \cup \{t^{(\alpha-\alpha_i)}\}_{\alpha \in \mathcal{B}_\Gamma - \{\alpha_i\}}] = K[\{t^{(\alpha-\alpha_i)}\}_{\alpha \in \mathcal{B}_\Gamma - \{\alpha_i\}} \cup \{t^{\alpha_i}\}].$$

The variety with coordinate ring B' will be smooth, and the algorithm will be finished, if one of the following conditions is satisfied:
1. $|\mathcal{B}_{\Gamma'}| = 1$,
2. $|\mathcal{B}_{\Gamma'}| = 2$, and $\mathcal{B}_{\Gamma'} = \{\alpha, -\alpha\}$, for some $\alpha \in \mathbb{Z}$.

We distinguish three different cases according to the sign of the elements in $\mathcal{B}_\Gamma$:
a) If all the elements in the set $\mathcal{B}_\Gamma$ are negative, it is possible to make a change of variables in the algebra $K[\Gamma]$, namely $t \longmapsto \frac{1}{t}$, which changes the sign of the exponents of the monomial generators. All of them will be positive after that.
b) If there are both positive and negative elements in $\mathcal{B}_\Gamma$, no blowups are needed: it is possible to express $-\gcd(\{\alpha\}_{\alpha \in \mathcal{B}_\Gamma})$ and $+\gcd(\{\alpha\}_{\alpha \in \mathcal{B}_\Gamma})$ as an $\mathbb{N}$-linear combination of the exponents of generators which are already in $\mathcal{B}_\Gamma$. To see this, choose two elements in $\mathcal{B}_\Gamma$, a positive one α and a negative one β. If $\alpha = a \cdot c$ and $\beta = (-b) \cdot c$ for some $a, b \in \mathbb{Z}_{>0}$ with $c = +\gcd(\alpha, -\beta)$. We want to find $a', b' \in \mathbb{N}$ such that

$$a' \cdot \alpha + b' \cdot \beta = -c. \tag{3.16}$$

This is the same as $a' \cdot a \cdot c - b' \cdot b \cdot c = -c$, and as $b' \cdot b - a' \cdot a = 1$. But b and a are coprime, so Bézout identity gives the result we need: such numbers $a', b' \in \mathbb{N}$ exist, the so called Bézout coefficients. The minimal coefficients satisfying (3.16) can be found by the extended Euclidean algorithm. Once we know $-c \in A$, we can express $c = \alpha + (a-1) \cdot (-c)$ (where $a - 1 \geq 0$ because $a > 0$). Imagine now that we have $c = \gcd(\alpha_1, ..., \alpha_n)$. We can arrive at the same result if we begin expressing $c_{1,2} = \gcd(\alpha_1, \alpha_2)$, and then recursively $c_{1,\ldots,n} = \gcd(\alpha_1, ..., \alpha_n) = \gcd(c_{1,\ldots,n-1}, \alpha_n)$.
In this second case, the K-algebra B already satisfies condition 2 above.
c) Finally, if all elements in $\mathcal{B}_\Gamma$ are positive, then we start blowing up. We define

$$\gamma(\mathcal{B}_\Gamma) = \sum_{\alpha \in \mathcal{B}_\Gamma} |\alpha|. \tag{3.17}$$

Lemma 55. *Let $\mathcal{B}_\Gamma$ be a set of positive generators of a semigroup $\Gamma \subset \mathbb{Z}$ with $|\mathcal{B}_\Gamma|$ at least 2, and $\gamma(\mathcal{B}_\Gamma)$ as defined above. Let $\mathcal{B}_{\Gamma'}$ be the resulting set of generators of the ring corresponding to one of the charts after a blowup with center $J = [n]$. Then we have $\gamma(\mathcal{B}_{\Gamma'}) < \gamma(\mathcal{B}_\Gamma)$.*

Proof. First of all, notice that $\gamma(\mathcal{B}_\Gamma) > 0$. For each $\alpha \in \Gamma$, since $\alpha, \alpha_0 > 0$, it follows that $|\alpha - \alpha_0| < |\alpha|$. Then

$$\sum_{\alpha \in \Gamma - \{\alpha_0\}} |\alpha - \alpha_0| < \sum_{\alpha \in \Gamma - \{\alpha_0\}} |\alpha| \ .$$

We have that

$$\gamma(\mathcal{B}_{\Gamma'}) = \sum_{\alpha' \in \Gamma'} |\alpha'| = \sum_{\alpha \in \Gamma - \{\alpha_0\}} |\alpha - \alpha_0| + |\alpha_0| < \sum_{\alpha \in \Gamma - \{\alpha_0\}} |\alpha| + |\alpha_0| =$$
$$= \sum_{\alpha \in \Gamma} |\alpha| = \gamma(\mathcal{B}_\Gamma),$$

and the lemma is proved.

□

Claim 56. After a finite number of blowups, either condition 1 or condition 2 from above are satisfied, and therefore B' is the coordinate ring of a smooth variety. Actually, if c is the positive greatest common divisor of the elements in $\mathcal{B}_\Gamma$ then condition 1 implies $\mathcal{B}_\Gamma = \{c\}$, and condition 2 implies $\mathcal{B}_\Gamma = \{c, -c\}$.

Proof. We are left to prove the case where $\alpha_i > 0$ for any $\alpha_i \in \mathcal{B}_\Gamma$. If $|\mathcal{B}_{\Gamma'}| = 1$, then we are done. Otherwise, since $\gamma(\mathcal{B}_\Gamma)$ strictly decreases whenever $|\mathcal{B}_\Gamma| > 1$, at some point one of the following happens:
1. There is some negative element in $\mathcal{B}_{\Gamma'}$. From that moment on, it is not guaranteed that γ will decrease under blowups. But, as we already discussed, in this situation $c, -c \in \mathcal{B}_\Gamma$ and no further blowup is needed. Thus, we can choose a new system of generators of Γ, namely $\mathcal{B}_{\Gamma'} = \{c, -c\}$.
2. We obtain $\gamma(\mathcal{B}_{\Gamma'}) = c$. Since any element in $\mathcal{B}_{\Gamma'}$ always comes from subtracting multiples of c to each other, this means that $\mathcal{B}_{\Gamma'} = \{c, 0, \ldots, 0\}$ and we can simplify to $\mathcal{B}_{\Gamma'} = \{c\}$.

□

3.6 Surfaces in $\mathbb{A}^3$

This subsection deals with surfaces in $\mathbb{A}^3$ which are given by a binomial equation. Therefore they are monomial varieties as in section 1.4. The objective is to find out the best choice for the center of a blowup for such surfaces to eventually obtain a resolution. Our perspective here is as follows: we wish to determine by combinatorial methods, directly from the system of generators t^α, t^β, t^δ of $K[X]$ the correct choice of the center. Moreover, we wish to define from these generators a numerical invariant which measures the improvement of $K[X]$, and therefore of X, after having applied the blowup along the selected center.

Let X be a surface in $\mathbb{A}^3$ with a monomial parametrization

$$\begin{aligned} f : \mathbb{A}^2 &\longrightarrow \mathbb{A}^3 \\ t &\longmapsto (t^\alpha, t^\beta, t^\delta), \end{aligned}$$

where $t = (t_1, t_2)$ and α, β, δ are vectors in $\mathbb{Z}^2$. This X is the Zariski closure of the image of f. Its coordinate ring $K[X]$ is the monomial algebra $K[t^\alpha, t^\beta, t^\delta]$. The three generators of this algebra will be algebraically dependent over K, and as shown in Lemma 7, the ideal of relations corresponds to the principal binomial ideal $I = I(X) \subset K[x, y, z]$ with $K[X] \cong K[x, y, z]/I$.

A generator of I is a minimal $\mathbb{Z}$-linear relation between α, β and δ. We will distinguish the cases where the coefficients of this relation have the same sign, and those where they don't. Actually, we will distinguish, up to permutation, between two cases: either

$$a\alpha + b\beta + c\delta = 0 \quad \text{or} \quad a\alpha = b\beta + c\delta, \tag{3.18}$$

with a, b, c in $\mathbb{N}$. These correspond to respective generators

$$x^a y^b z^c - 1 \quad \text{or} \quad x^a - y^b z^c \tag{3.19}$$

of I. Without loss of generality we may assume that a, b, and c are all positive. If one of them were zero, then X would be the Cartesian product of a monomial plane curve with a line.

It is easy to see that an equation of the first type, $x^a y^b z^c - 1$, yields a smooth hypersurface X. No resolution of singularities will be necessary here, so we discard this case. The only case left to study is $X : \left\{x^a = y^b z^c\right\}$.

If $a = 1$, X is smooth, as one can see using the Jacobian criterion. Therefore, we will only consider the case $a > 1$.

Our objective now is to resolve the singularities of such a variety X by a sequence of blowups. The centers will be coordinate subspaces of $\mathbb{A}^3$. That this is always possible is ensured by the theory of toric resolutions (see [4]).

Our particular interest is to understand how the monomial algebra

$$K[X] = K[t^\alpha, t^\beta, t^\delta]$$

improves under the blowups described in section 3.3. As it was said in section 3.4, for certain choices of centers the singularities may not improve. Determining the correct center is not an easy task in general. In the general theory of resolution of singularities, it is usually done by using a suitable resolution invariant, which induces a stratification of X (see [25], [27], [6]).

To do so in our context we will nevertheless glimpse at the equation of X in $\mathbb{A}^3$ to determine the correct center, then we will try to express this choice directly through the semigroup in $\mathbb{Z}^2$ generated by α, β and δ, and finally we will try to measure the "distance" of $K[X]$ from being a regular ring. The distance should have decreased when passing to the transform of X under the blowup.

As a matter of fact, the resolution of binomial surfaces follows a case distinction according to the values of a, b, c and $c+b$. This reflects the stratification of X by strata of constant multiplicity, see [37, sec. 1.5]. We will distinguish (up to symmetry) four cases:

(i) $a \leq b+c$ with $a > b$, $a > c$;
(ii) $a > b$, $a \leq c$;
(iii) $a \leq b$, $a \leq c$;
(iv) $a > b+c$.

The centers prescribed by the general theory of resolution are then chosen as follows:

(i) origin: $J = \{1,2,3\}$;
(ii) y-axis: $J = \{1,3\}$;
(iii) union of both the y-axis and the z-axis;
(iv) origin: $J = \{1,2,3\}$.

Considering the (binomial) equation of a non-smooth toric surface in $\mathbb{A}^3$, a is the exponent of the variable which is alone in one of the monomials of the equation, and b, c are defined as the exponents of the other two variables, making them satisfy $b \leq c$. As we specified above the process will be complete when $a \leq 1$. Therefore, it would be reasonable to measure the improvement of the singularities of the variety after a blowup by looking at the decrease of a. However, as we will observe, a sufficient decrease of b or c will lead to case (iv), and in this situation we can make a decrease by blowing up the origin. For this reason, we will use an invariant γ given by $(a, b+c)$, where a, b, c are defined as above: the goal is making $a \leq 1$, but both, the decrease of a and the decrease of $b+c$ mean an improvement in the singularities of X.

Let us now apply the respective blowups to X and see how $K[X]$ transforms. We will look carefully at the order and the degree of the equation of X and of its strict transform. This is the same as looking at the relations between a, b and c, and therefore we have to distinguish these four cases:

(i) Case $a \leq b+c$: This means that the order of the equation $f = x^a - y^b z^c$ at 0is a. The chosen center will be the origin: $J = \{1,2,3\}$ using the notation explained in section 3.3. The three following charts of the blowup have to be considered:

1. First chart (x): We have $\alpha' := \alpha, \beta' := \beta - \alpha, \delta' := \delta - \alpha$. The coordinate ring of the transform X' in the first chart is $K[t^\alpha, t^{\beta-\alpha}, t^{\delta-\alpha}]$. We can determine the equation of the strict transform X' (relating the new generators) based on the previous one:

$$(t^\alpha)^a - (t^\beta)^b (t^\delta)^c = (t^\alpha)^a - (t^{\beta-\alpha})^b (t^\alpha)^b (t^{\delta-\alpha})^c (t^\alpha)^c$$

$$x^a - y^b x^b z^c x^c = 0$$
$$x^a(1 - y^b z^c x^{b+c-a}) = 0$$

where $b+c-a \geq 0$. The factor x^a corresponds to the exceptional divisor. The other irreducible component of X^* is given by the relation

$$b\beta + c\delta + (b+c-a)\alpha = 0.$$

This is the equation of the strict transform X'. The new coefficients are: $a' := b+c-a, b' := b, c' := c$. This equation defines a smooth irreducible variety:

$$X' : \left\{x^{b+c-a} y^b x^c = 1\right\}, \text{ where } b+c-a,\ b,\ c \in \mathbb{Z}_{>0}$$

and no more blowups are needed for this chart.
2. Second chart (y): The exponents of the new generators are $\alpha' := \alpha - \beta, \beta' := \beta, \delta' := \delta - \beta$. Proceeding as in the previous case, one can find the irreducible components of X^+. One of them is y^a, and the other one is given by the relation between the new exponents: $a\alpha = (b+c-a)\beta + c\delta$.

In this new relation, one of the coefficients has changed with respect to the initial one, while the others remain the same: $a' = a, b' \neq b, c' = c$. We check the invariant:

$$\gamma' = (a, b+c-a+c) < \gamma = (a, b+c),$$

whenever $b+c-a+c < b+c$ is satisfied. This will happen if $c < a$. In this case $b' < b$, and $b' + c' < b + c$: b decreases.

3. Third chart (z): The new exponents are $\alpha - \delta, \beta - \delta, \delta$. This situation is symmetric to the one in the second chart: the exceptional divisor is z^a and the new coefficients are $a' = a, b' = b, c' = (b+c-a)$. In this case, $\gamma' < \gamma$ as long as $b < a$. If $b \geq a$ a different center should be chosen.

(ii) When $a > b, a \leq c$, the result in the first chart is the same as in (i): X' is smooth in this chart. However, for the second and the third chart, there is no improvement in the order or degree of the equation. A different choice of the center solves this problem.
The *equimultiple locus* of X at 0 is the set of points of X which have the same multiplicity as the origin. The concept of equimultiple locus often appears in the theory of resolution of singularities because it has important properties (see [25]). Notice that the equation of our variety is contained in the ideal $(x, z)^a$, since both terms of the defining equation, x^a and $y^b z^c$, are in it. This means that X has multiplicity a in the y-axis. The zero locus of the ideal $(x, z)^a$ is the equimultiple locus of X, see [4]. The chosen center will be the y-axis. This means that $J = \{1, 3\}$.

With this choice of J we obtain two charts that cover the blowup, namely:
1. First chart (x): The new exponents are $\alpha' := \alpha, \beta' := \beta, \delta' := \delta - \alpha$. The exceptional divisor of the transform is x^a and the strict transform X' is the smooth component given by $a' = (a-c), b' = b, c' = c$, where $c \geq a$ gives $(-a') \in \mathbb{Z}_{>0}$. That is:

$$X'_x : \left\{x^{(c-a)} y^b z^c - 1 = 0\right\}, \text{ where } c-a, b, c \in \mathbb{N}.$$

2. Second chart (z): We obtain $\alpha' := \alpha - \delta, \beta' := \beta, \delta' := \delta$. The exceptional divisor is z^a and the strict transform is given in this case by the equation where $a' = a, b' = b, c' = (c-a)$:

$$X'_z : \left\{ x^a - y^b z^{c-a} = 0 \right\}, \text{ where } a, b, c-a \in \mathbb{N}.$$

It is clear that here $c' < c$, so $\gamma' < \gamma$.

(iii) If $a \leq b, a \leq c$, there are two singular curves in X. Comparing with (ii), we can choose between one of them as center of the blowup, i.e., $J = \{1,2\}$ or $J = \{1,3\}$. But it is also possible to obtain a smooth surface by blowing up once, choosing $I_Z = I(Z) = (x, yz)(z, y)(x, z)$ as center, which is non-reduced; see [17] for details about this kind of center for monomial varieties. This blowup is equivalent (see [24]) to first blowing up along the ideal $I_Z = (x, yz)$, which is the defining ideal of the union of the y-axis and the z-axis, and then along the unique singular point of the strict transform of X under the first blowup.

The first blowup, with center (x, yz), gives two charts:
1. First chart (x): The transform W'_x of the ambient space is given by

$$K[x, y, z, \frac{yz}{x}] \cong K[x, y, z, w]/(xw - yz),$$

and the strict transform of X is

$$X'_x : \left\{ 1 - x^{b+c-a} y^b z^c = 0 \right\}, \text{ where } b+c-a, b, c \in \mathbb{N},$$

which is smooth.

2. Second chart (yz): The transform W'_{yz} of the ambient space is given by

$$K[x, y, z, \frac{x}{yz}] \cong K[\frac{x}{yz}, y, z] \cong K[x, y, z],$$

The strict transform of X is

$$X'_{yz} : \left\{ x^a - y^{b-a} z^{c-a} = 0 \right\}, \text{ where } a, b-a, c-a \in \mathbb{N}.$$

The center of last blowup is not smooth, and one can see that the ambient space has become singular: the transform of the ambient space in the first chart is singular at the origin $x = y = z = w = 0$. To obtain a smooth ambient space, we perform a second blowup with center this singular point. Note that the second chart is not affected by it, since it does not intersect the center, it is already smootht. On the other hand, the strict transform of X in the first chart does not contain this center either, and therefore is not affected by the blowup wich resolves the singularities of W'_x.

We conclude that in the four affine charts corresponding to the blowup of X'_x, we obtain a smooth variety, and in the second cchart of the first blowup the singularities have improved:

$$\gamma' = (a, b+c-2a) < (a, b+c) = \gamma.$$

(iv) Case $a > b + c$. Note that, since $a, b, c \in \mathbb{N}$, this implies

$$a > b, a > c. \tag{3.20}$$

Choosing $J = \{1, 2, 3\}$, three charts are necessary:
1. First chart (x): $\alpha' := \alpha, \beta' := \beta - \alpha, \delta' := \delta - \alpha$ gives a variety X^* with an exceptional divisor x^{b+c} and X' defined by $a' = (a - b - c), b' = b, c' = c$. Clearly,

$$\gamma' = (a - b - c, b + c) < \gamma = (a, b + c),$$

so a decreases in this chart, and our invariant too.
2. Second chart (y): $\alpha' := \alpha - \beta, \beta' := \beta, \delta' := \delta - \beta$ gives a variety with exceptional divisor y^{b+c} and X' given by the equation with exponents $a' = -a, b' = (a - b - c), c' = -c$. Here

$$2a - b - c > a > c,$$

using $a > b + c$ for both inequalities. Observe that $(-c') < (-a') + b'$. This leads to a variety as in (i) again. In this chart a also decreases.
3. Third chart (z): $\alpha' := \alpha - \delta, \beta' := \beta - \delta, \delta' := \delta$ are the new exponents, and the situation is symmetric to the previous one changing β, b for δ, c: a decreases.

In conclusion, γ decreases under this blowup for every chart.

To assert that the process always leads to a smooth variety for all charts involved, we need to make sure that, at some point, $a \leq 1$. In other words, we must check that every case leads to a relation of the form

$$a\alpha + b\beta + c\delta = 0 \text{ or } \alpha = b\beta + c\delta,\ a, b, c \in \mathbb{N} \tag{3.21}$$

after a finite number of steps. But we have showed that γ decreases under every blowup defined above. This means that either a or $b + c$ decrease. It is also important to notice that whenever $b + c$ increases by some blowup, this blowup makes a decrease, and that a never increases. As a, b and c are finite, this guarantees that after a finite number of blowups, $a \leq 1$ for every affine chart.

To ensure that the blowups are well defined globally, we need to check that after one blowup of X, the center we choose locally for a new blowup of each affine chart X'_i, define a suitable center on X for a blowup after gluing the X'_i.

For (i), notice that the first chart is smooth, and for the second and third chart, b and c decrease respectively and a does not increase. This process must be repeated until b, respectively c, are small enough so that $a > b + c$, and we are in (iv).
The center of the next blowup will be the origin. This blowup X'' of X' will not produce any change in the first chart of X', because the origin does not appear in it. In the discussion above, it was suggested that the center for blowing up

the second chart, as well as the third chart would be the origin, because we are either in (i) or in (iv). The center for the next blowup is therefore well defined by gluing the centers that we choose for each affine chart of X' separately.
For (ii), the first chart is again smooth. For the second chart, the process must be repeated until c has decreased enough so that $c < a$, and then we are back to (i). This means that the center for the next blowup will be the origin or the y-axis, depending on the value of $c - a$. In any case, this center is not contained in the first affine chart of X', so only the second one will be modified. Again, the center is well defined in X' by gluing the centers that one would choose in each affine chart.
For (iii), there is only one chart wich is not smooth after the blowup, and again the choice of a center for a new blowup inside of the singular locus of the affine variety in this chart guarantees that the blwup is well defined globally.
For (iv), a decreases in every chart, so it is necessary to repeat the process as many times as necessary for every chart, until one of the following happens:

- $a \leq 1$ and we are done;
- $a \leq b + c$ and we are back to (i).

To see that in this case we can also define the center of the next blowup by choosing the centers of the affine charts, it is necessary to realize that each affine chart is the complement of a coordinate plane, $x = 0$, $y = 0$ or $z = 0$ respectively, and that we are always choosing the origin or some coordinate axis as center of our blowups. For this reason the part of the center that intersects one affine chart will not intersect the rest of them.

Example 57. The following example illustrates the process for a non-normal surface. Let X be the surface in figure 3.5, of equation $x^3 - y^2z^5 = 0$.

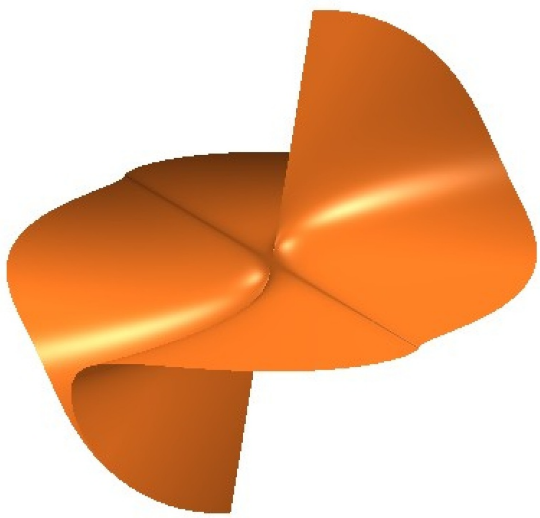

Figure 3.5: Surface $X : \{x^3 - y^2z^5 = 0\}$.

Its coordinate ring is $K[t^2s^5, t^3, s^3]$. With the notation from section 3.3, the set of exponents is $\{(2, 5), (3, 0), (0, 3)\}$. To begin, using the conclusions of the previous discussion, the first center to choose is $J = \{1, 3\}$, since $3 \leq 2 + 5$,

$3 > 2$ and $3 \leq 5$. This blowup gives two charts, corresponding to the sets

$$\{(2,5),(3,0),(-2,-2)\} \text{ and } \{(2,2),(3,0),(0,3)\}$$

respectively. They correspond to the affine varieties with affine coordinate rings

$$K[t^2s^5, t^3, \frac{1}{t^2s^2}] \cong K[x,y,z]/(x^2y^2z^5-1), \text{ and}$$
$$K[t^2s^2, t^3, s^3] \cong K[x,y,z]/(x^3-y^2z^2).$$

Both affine charts can be seen in figure 3.6. The first one is already smooth.

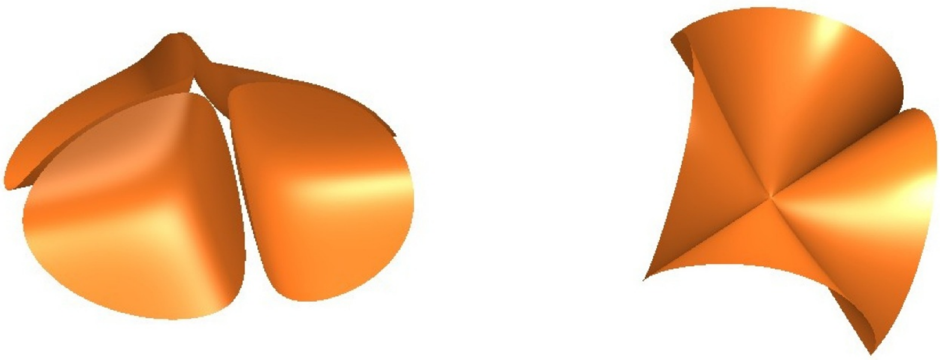

Figure 3.6: Affine charts of the first blowup of X with center $J = \{1,3\}$.

We continue working with the second chart. Now we are in case (i), and therefore $J = \{1,2,3\}$. This blowup gives the following three affine charts:

$$K[t^2s^2, \frac{t}{s^2}, \frac{s}{t^2}] \cong K[x,y,z]/(xy^2z^2-1),$$

$$K[\frac{s^2}{t}, t^3, \frac{s^3}{t^3}] \cong K[x,y,z]/(x^3-yz^2),$$

$$\text{and } K[\frac{t^2}{s}, \frac{t^3}{s^3}, s^3] \cong K[x,y,z]/(x^3-y^2z).$$

Again, the first chart is smooth, and it is equal to the one in figure 3.5. The second and the third chart are isomorphic to each other, so we will perform the next blowup only for the second one. It can be seen in figure 3.7. The center will be $J = \{1,2,3\}$ now. Three new affine charts are necessary:

$$K[\frac{s^2}{t}, \frac{t^4}{s^2}, \frac{s}{t^2}] \cong K[x,y,x]/(yz^2-1),$$

$$K[\frac{s^2}{t^4}, t^3, \frac{s^3}{t^6}] \cong K[x,y,z]/x^3-z^2, \text{ and}$$

$$K[\frac{t^2}{s}, \frac{t^6}{s^3}, \frac{s^3}{t^3}] \cong K[x,y,z].$$

In this case the first and the last chart are smooth. They correspond to the product of a hyperbola with an affine line and to the affine three dimensional

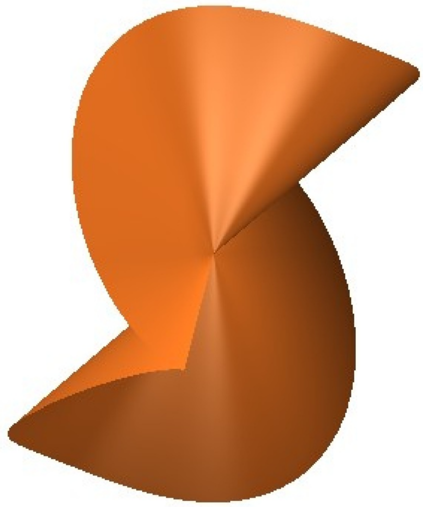

Figure 3.7: Affine chart $K[x, y, z]/(x^3 - yz^2)$.

space. The second one is the cartesian product of a non-smooth curve, namely a cusp, and an affine line. By a process that we will see later, this last surface can be easily blown up obtaining two charts, an affine line and a hyperbola.

Example 58. *Normal surfaces.* Let X be the surface in $\mathbb{A}^3$ given by $x^a - y^b z^c = 0$, with $a, b, c \in \mathbb{N}$. The Jacobian matrix is $J = (ax^{a-1}, -by^{b-1}z^c, -cy^b z^{c-1})$. This matrix will be the zero matrix whenever $x = 0$, $y^{b-1}z^c = 0$ and $z^{c-1}y^b = 0$. Suppose that both $c, b \geq 0$. Then the y-axis and the z-axis are both singular in X. If $c = 1$, then only the y-axis will be singular, but not the z-axis. The only possibility that gives an isolated singularity (of codim 2 in X) is $c = b = 1$. This proves that any normal monomial surface is of the form

$$X : \{x^a - yx = 0\}\,.$$

Now we can use the result from the previous discussion. Recall that if $a = 1$ then X is already smooth. If $a = 2$ then we are in case (i), and choosing the singular point (the origin) as the center for a blowup we will have a resolution of the singularities of X. Otherwise we are in case (iv), and the good choice of the center is still the singular point. In this case, the blowup will make a decrease, and after a finite number of steps with the same choice for the center, we will have $a \leq 2$.

Conclusion: For normal toric surfaces, choosing the respective singular points of the strict transforms as center of blowups leads, in a finite number of steps, to a resolution.

Now let us find a set of conditions between the values of α, β and δ, such that one can decide using these conditions which center should be chosen for the next blowup. We will use the results obtained above for the different relations between the exponents a, b, c of the variables in the equations.

We start with the equation

$$a\alpha = b\beta + c\delta \tag{3.22}$$

as in (3.18). To calculate the values for a, b, c in the case of a hypersurface, consider the following system of linear equations:

$$a\alpha_1 + b\beta_1 + c\delta_1 = 0$$

$$a\alpha_2 + b\beta_2 + c\delta_2 = 0,$$

where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ and $\delta = (\delta_1, \delta_2)$. We obtain the following integer solution for a, b, c:

$$a = \begin{vmatrix} \delta_1 & \beta_1 \\ \delta_2 & \beta_2 \end{vmatrix} = \beta_2\delta_1 - \beta_1\delta_2,$$

$$b = \begin{vmatrix} \alpha_1 & \delta_1 \\ \alpha_2 & \delta_2 \end{vmatrix} = \alpha_1\delta_2 - \alpha_2\delta_1,$$

$$c = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \beta_2\alpha_1 - \beta_1\alpha_2.$$

If we use the last determinants together with the results from the previous analysis it is easy to use the following relations to describe the right center:

Denote

$$\det[\delta, \beta] = \begin{vmatrix} \delta_1 & \beta_1 \\ \delta_2 & \beta_2 \end{vmatrix}, \quad \det[\alpha, \beta + \delta] = \begin{vmatrix} \alpha_1 & \beta_1 + \delta_1 \\ \alpha_2 & \beta_2 + \delta_2 \end{vmatrix},$$

$$\det[\alpha + \beta, \delta] = \begin{vmatrix} \alpha_1 + \beta_1 & \delta_1 \\ \alpha_2 + \beta_2 & \delta_2 \end{vmatrix} \quad \text{and} \quad \det[\alpha - \delta, \beta] = \begin{vmatrix} \alpha_1 - \delta_1 & \beta_1 \\ \alpha_2 - \delta_2 & \beta_2 \end{vmatrix}.$$

The determinant

$$\det[v, u] = \begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix},$$

is equal to the oriented area of the paralellogram determined by the vectors (u_1, u_2) and (v_1, v_2). By oriented we mean that:

$$\det[(1, 0), (0, 1)] = 1 = -\det[(0, 1), (1, 0)].$$

Then
i. If $\det[\delta, \beta] \leq \det[\alpha, \beta + \delta]$ and

$$\det[\alpha + \beta, \delta] < 0, \, \det[\alpha - \delta, \beta] < 0$$

the correct center is $J = \{1, 2, 3\}$.

ii. If $\det[\delta, \beta] \leq \det[\alpha, \beta + \delta]$ and

$$\det[\alpha + \beta, \delta] < 0, \, \det[\alpha - \delta, \beta] \geq 0$$

then $J = \{1, 2\}$.

iii. If $\det[\delta, \beta] \leq \det[\alpha, \beta + \delta]$ and

$$\det[\alpha + \beta, \delta] \geq 0,\ \det[\alpha - \delta, \beta] \geq 0$$

then we will blow X up along both the y-axis and the z-axis simultaneously.

iv. If $\det[\delta, \beta] > \det[\alpha, \beta + \delta]$ the chosen center is again $J = \{1, 2, 3\}$.

3.7 Generalization to higher dimension

In this section we try to adapt for higher dimensional varieties the process developed for curves in section 3.5.

Let $K[\Gamma]$ be the coordinate ring of a singular monomial variety X^{Γ}. Here, the semigroup Γ is contained in $\mathbb{Z}^m$. Let $\mathcal{B}_\Gamma \subset \mathbb{Z}^m$ be a minimal generator system of Γ of cardinality $n \geq m$.

The goal here is to define an algorithm for vectors similar to the Euclidean division. We wish to transform, by performing blowups, the semigroup Γ into the semigroup Γ' of a smooth monomial variety $X^{\Gamma'}$. To do so, we consider the matrix $M_{n\times m} = M(\Gamma)$ with rows $r_1, \ldots, r_n$ the vectors $\alpha_1, \ldots, \alpha_n$ in $\mathcal{B}_\Gamma$, and try to simplify this matrix until it corresponds to the matrix $M(\Gamma')$ of a smooth semigroup.

Note that a diagonal matrix is the canonical example of the matrix of a smooth variety, namely $\mathbb{A}^m$. Therefore, the simplest idea, altogh maybe not optimal, is to try to diagonalize M by row transformations. This will be our first approach. However, the typical transformations used by Gauss to diagonalize a matrix over $\mathbb{Z}$ do not correspond exactly to the kind of transformations that blowups provide for the vectors of $\mathcal{B}_\Gamma$.

The following transformation of rows corresponds to a blowup:
a. Choose the center $J \subseteq [n]$. We associate to this center the rows $\{r_j : j \in J\}$ of the matrix.
b. For each $i \in J$ build a new matrix M'_i whose j-th row r'_j equals r_j in case $j = i$ or $j \notin J$, and $r'_j = r_j - r_i$ otherwise.

With this transformation one obtains $|J|$ new matrices, $M'_i, i \in J$, with the semigroups $\Gamma'_i = \langle \mathcal{B}(M'_i) \rangle_{\mathbb{N}}$. For each of these semigroups, $X^{\Gamma'_i}$ is the i-th affine chart of the transform of X^{Γ} under the blowup with center $(x_j, j \in J)$. The process splits thus into $|J|$ new monomial algebras. Note that this is just the translation to matricial language of what was already explained in section 3.3.

To compare the process with Gauss' algorithm, we will consider the following additional row transformations of a matrix (corresponding to transformations

in Gauss algorithm):
1. Swap rows.
2. Add a new row $r_{n+1} = \sum_{j=1}^{n} c_j r_j$ for some $c_i \in \mathbb{N}$ to the matrix.
2'. Remove a row if it is an $\mathbb{N}$-linear combination of some other rows.
3. Transform M by a blowup with center $J \subseteq [n]$ as described above.

The first and the second transformations are transformations of the matrix M, but not of the semigroup Γ and thus of the algebra $K[\Gamma]$. They correspond to reordering elements of $\mathcal{B}_\Gamma$ and to adding to $\mathcal{B}_\Gamma$ elements of Γ respectively. On the other hand, blowups are transformations of M but also of $K[\Gamma]$. It is clear, by the way in which the transformations are defined, that the size of the matrix can change.

The transformations above are a modification of Gauss' algorithm over $\mathbb{Z}$ with three main differences:
(i) In Gauss' algorithm, one can add to some row a linear combination of others. We can not modify rows in this way, but we can add new rows as the result of these linear combinations.
(ii) The linear combinations considered in transformations 2 and 2' have coefficients in $\mathbb{N}$, while for Gauss' algorithm in $\mathbb{Z}$, integer coefficients are valid.
(iii) After each blowup, the process splits into several subprocesses, one for each matrix (one for each chart).
Gauss' algorithm over a field guarantees that, after finitely many row transformations, we will obtain a diagonal matrix. However, the restrictions we impose give a weaker result. For instance, with Gauss' algorithm, a valid transformation is:

$$\begin{pmatrix} 1 & \cdot & \dots & \cdot \\ -1 & \cdot & \dots & \cdot \\ 0 & \cdot & \dots & \cdot \\ \vdots & & \ddots & \vdots \\ 0 & \cdot & \dots & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \cdot & \dots & \cdot \\ 0 & \cdot & \dots & \cdot \\ 0 & \cdot & \dots & \cdot \\ \vdots & & \ddots & \vdots \\ 0 & \cdot & \dots & \cdot \end{pmatrix},$$

adding the first row to the second. It is not a blowup transformation but it modifies the algebra, since in general

$$K[t^{\alpha_1}, t^{\alpha_2}, t^{\alpha_3}, \dots, t^{\alpha_n}] \ncong K[t^{\alpha_1}, t^{\alpha_2+\alpha_1}, t^{\alpha_3}, \dots, t^{\alpha_n}],$$

so it is not allowed in our version. Nevertheless, for our purpose it would be desirable to obtain matrix of the form:

$$\begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ -c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ 0 & -c_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & \dots & 0 & c_m \\ 0 & \dots & \dots & 0 & -c_m \end{pmatrix} \tag{3.23}$$

for some $c_i \in \mathbb{Z}$ for $i = 1, \dots, m$, or a matrix similar to (3.23), maybe not containing some of the rows with a negative element. The matrix (3.23) corresponds to the matrix of the semigroup generated by

$$\mathcal{B}_\Gamma = \{(c_1, 0, \ldots, 0), (-c_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, c_m), (0, \ldots, 0, -c_m)\},$$

which is isomorphic to

$$\mathbb{Z}^m = \langle (1, 0, \ldots, 0), (-1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1), (0, \ldots, 0, -1) \rangle_{\mathbb{N}}.$$

The monomial algebra corresponding to this matrix is isomorphic to

$$K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}],$$

and it is the coordinate ring of a smooth variety: the m-dimensional torus.

As a step previous to the diagonalization of the matrix, we will try to triangularize it. The strategy for this process is as follows:
Denote by (M_i) for $i > 0$ the minor of size $d \times (m - i + 1)$ containing the elements of the columns c_i to c_m wich are in rows with the first $i-1$ elements zero. Call $M_0 = M$. Start with $i = 1$:
Look at the first column of M_{i-1}.
1. Using the transformations of rows above, transform the matrix in order to obtain for the first column of M_{i-1},

$$c_i' = (a_{1,i}', \ldots, a_{i-1,i}', \gcd(\{a_{j,i}, j = 1, \ldots, d\}), a_{i+1,i}', \ldots, a_{d,i}')^t.$$

Suppose that the size of M is $d \times m$. Consider the first column of the matrix M. One can see the elements in this column as the generators of a semigroup of $\mathbb{Z}$. To complete the first step of the strategy explained above, we will use the proof given for curves in section 3.5, but with some modifications:
Recall that for curves, if $\alpha \in \mathcal{B}_\Gamma$ with α a $\mathbb{N}$-linear combination of the elements in $\mathcal{B}_\Gamma \setminus \{\alpha\}$, we have that $\Gamma = \langle \mathcal{B} \rangle_{\mathbb{N}} = \langle \mathcal{B}_\Gamma \setminus \{\alpha\} \rangle_{\mathbb{N}}$. In this case, the element α is redundant in the generating system, and it can be removed. However, when we are working with M, having that $a_{i,1}$ is a $\mathbb{N}$-linear combination of the $a_{i,j}, j \neq i$ does not imply in general that $\langle r_i, i = 1, \ldots, d \rangle_{\mathbb{N}} = \langle r_i, i = 1, \ldots, \hat{i}, \ldots, d \rangle_{\mathbb{N}}$, so this row cannot be removed. Instead of removing this row, we will exclude it for the computation of the invariant. We redefine the invariant $\gamma(\mathcal{B}_\Gamma)$ used for curves as

$$\gamma_i(M) = \sum |a_{j,i}|,$$

for each row $i \in \{1, \ldots, d\}$, where the sum goes over all the $a_{j,i}$ such that there exists no row r_k in M_{i-1} satisfying $\frac{a_{j,i}}{a_{k,i}} \in \mathbb{Z}_{>0}$. The proof given for curves is not affected by this modification. This proof guarantees that, after finitely many steps (blowups) we will obtain a matrix with the element $c_i = \gcd(a_{j,i}, r_j \in M_{i-1})$ in the i-th column, k-th row for some k with r_k in M_{i-1}. We may also have the element $-c$ in the same column of some row $r_{k'} \in M_{i-1}$ with $k' \neq k$. Using transformation 1 above we can place r_k (and $r_{k'}$) as the first row of the minor M_{i-1}.
2. Using the transformations of rows above, transform the matrix in order to obtain for the first column

$$c_i' = (a_{1,i}', \ldots, a_{i-1,i}', \gcd(\{a_{j,i}, j = 1, \ldots, d\}), 0, \ldots, 0)^t.$$

In a first place we will add r_k to the rows with a negative element in the first column and $r_{k'}$ to those with a positive element in the first column. After

doing this for all rows except r_k and $r_{k'}$, we place at the end of the matrix all of the rows which do not have a zero in this column.
3. Increase i and iterate, considering the minor M_i. After a finite number of steps, we will have a matrix of the form:

$$M_{i-1}=\left(\begin{array}{ccccc|c} c_1 & \cdot & \cdot & \cdots & \cdot & \\ -c_1 & \cdot & \cdot & \cdots & \cdot & \\ 0 & c_2 & \cdot & \cdots & \cdot & \\ 0 & -c_2 & \cdot & \cdots & \cdot & \\ \vdots & & \ddots & & \vdots & P \\ \vdots & & & \ddots & \cdot & \\ 0 & \cdots & \cdots & 0 & c_d & \\ 0 & \cdots & \cdots & 0 & -c_d & \\ \hline & & Q & & & \end{array}\right). \tag{3.24}$$

Note that after repeating these steps for the $i-1$ first columns, transformations for the i-th column only modify the minor M_i. In the following example we transform a matrix with the steps mentioned until now.

Example 59. Consider the matrix M for the semigroup generated by

$$\mathcal{B}_\Gamma = \{(4,2),(6,6),(0,5)\}.$$

Choose $J = \{1,2\}$ as the center of a blowup. The result is given by two charts as represented in the following diagram. After the blowup, steps 2 and 3 above produce the matrices shown in the diagram.

$$\begin{tikzcd}
& \begin{pmatrix} 4 & 2 \\ 6 & 6 \\ 0 & 5 \end{pmatrix} \arrow[dl] \arrow[dr] & \\
\begin{pmatrix} 4 & 2 \\ -2 & 4 \\ 0 & 5 \end{pmatrix} \arrow[d] & & \begin{pmatrix} -2 & -4 \\ 6 & 6 \\ 0 & 5 \end{pmatrix} \arrow[d] \\
\begin{pmatrix} 4 & 2 \\ -2 & 4 \\ 0 & 5 \\ 2 & 6 \end{pmatrix} \arrow[d] & & \begin{pmatrix} -2 & 4 \\ 6 & 6 \\ 0 & 5 \\ 2 & -2 \end{pmatrix} \arrow[d] \\
\left(\begin{array}{cc} -2 & 6 \\ -2 & 4 \\ 0 & 5 \\ \hline 4 & 2 \end{array}\right) & & \left(\begin{array}{cc} 2 & -2 \\ -2 & -4 \\ 0 & 5 \\ \hline 6 & 6 \end{array}\right)
\end{tikzcd}.$$

The next goal is to introduce zeros in the column above the c_i. In a first place, we can use again transformation 2 and add some $\mathbb{N}$-linear combination of rows to a row r_j, and place the result r'_j where r_j was, placing r_j at the end of the matrix.

Example 60. Consider now the following matrix of the semigroup generated by

$$\mathcal{B}_\Gamma = \{(3,2),(0,1),(0,-1),(6,5)\}.$$

It has already zeros as we wanted in the first columns. Now we want to introduce them above the c_i:

$$\left(\begin{array}{cc} 3 & 2 \\ 0 & 1 \\ 0 & -1 \\ \hline 6 & 5 \end{array}\right).$$

$$\downarrow$$

$$\left(\begin{array}{cc} 3 & 2 \\ 0 & 1 \\ 0 & -1 \\ 3 & 0 \\ \hline 6 & 5 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cc} 3 & 0 \\ 0 & 1 \\ 0 & -1 \\ \hline 6 & 5 \\ 3 & 2 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cc} 3 & 0 \\ 0 & 1 \\ 0 & -1 \end{array}\right)$$

When this transformation is not longer helpful for introducing zeros over the c_i, we start using blowups. However, the splitting of the process because of the charts after a blowup, introduces some difficulties in this step. Note that if we choose a center $J = \{j,k\}$, $j < k$, for one of the charts we will subtract the row r_j where the j-th element is nonzero to the row r_k, which has a zero in the j-th position. This changes the zeros we had already under the c_i. Another problem is that, for a given i, c_i does not divide, in general, the elements over it in the matrix, so it is not that easy to introduce zeros over them, even if we had c_i and $-c_i$, when no blowups would be necessary. Let us illustrate it with an example.

Example 61. Let Γ be the semigroup generated by $\{(3,2),(0,4),(6,5)\}$, and M the matrix associated to these generators. We make a blowup with center

$J = \{1, 2\}$ to introduce zeros over c_2:

$$\begin{tikzcd}
 & \begin{pmatrix} 3 & 2 \\ 0 & 4 \\ 6 & 5 \end{pmatrix}. \arrow[dl] \arrow[dr] & \\
\begin{pmatrix} 3 & 2 \\ -3 & 2 \\ 6 & 5 \end{pmatrix} & & \left(\begin{array}{cc} 3 & -2 \\ 0 & 4 \\ \hline 6 & 5 \end{array}\right)
\end{tikzcd}$$

The matrix in the left branch shows the first one of the problems mentioned: the matrix has no longer zeros under c_1. The matrix in the right branch shows the second problem: $c_2 \nmid a_{1,2}$, and for this reason, even though they have opposite sign, it is not possible to introduce zeros over c_2 without blowing up. Note that if we have $c_i = a_{k,i}$ and $-c_i$ en every column i of M, and c_i is a divisor of all the elements $a_{j,i}$ with $j < k$, this is possible without blowups.

Chapter 4

Monomial varieties

In this section some concrete aspects of monomial varieties are discussed. These are concepts that already appeared in previous sections and we did not get deeper into them, but did some assumptions to avoid them.
First, there is a short discussion about parametrizations of toric varieties. Then the concept of dimension is analyzed and applied to monomial varieties. The section also contains a summary with pictures of the different smooth monomial varieties of low dimensions.

4.1 Fully parametrized varieties

The following discussion intends to explain whether the image of a monomial parametrization is equal to the toric variety given by the semigroup of the exponents of the parametrization.

Recall, that in section 1.4, when monomial varieties were introduced, we defined the variety X^Γ for a semigroup Γ as the closure of the image of a monomial parametrization. That parametrization was defined by a set of generators of Γ. Later, we identified this variety with the toric variety whose coordinate ring was the monomial algebra defined by the same set of generators of Γ. It was also explained how to find an isomorphic algebra which was the quotient $K[x_1, \dots, x_n]/(f_1, \dots, f_d)$ of a polynomial ring by the defining ideal of X^Γ. In fact, many monomial algebras can be isomorphic to the same quotient. Equivalently, many monomial parametrizations can define the same monomial variety, as closure of their images. The aim of this subsection is to clarify this situation and show how it is possible to see, for a given variety X^Γ, if it is fully parametrized by a specific monomial parametrization.

Let $\mathcal{B}_\Gamma = \{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{Z}^m$ be a set of integer vectors. Consider the set of $\mathbb{A}^n$ defined by

$$P_\Gamma = \{(x_1, \dots, x_n) : x_i = t^{\alpha_i}, i = 1, \dots, n\}. \tag{4.1}$$

This set is the image of a parametrization $\varphi_\Gamma(t_1, \dots, t_m) = (t^{\alpha_1}, \dots, t^{\alpha_n})$ as the one we proposed in section 1.4. We defined a variety X^Γ as the closure of this image. This is because the set P_Γ is not necessarily a toric variety itself.

Consider the following alternative definition: the variety X^Γ is the variety defined by the kernel of the algebra homomorphism

$$\phi_\Gamma : K[x_1, \dots, x_n] \longrightarrow K[t_1^{\pm 1}, \dots, t^{\pm 1}] \tag{4.2}$$
$$x_i \longmapsto t^{\alpha_i} = t_1^{\alpha_{i,1}} \cdot \ldots \cdot t_m^{\alpha_{i,m}}. \tag{4.3}$$

It is clear that $P_\Gamma \subseteq \mathrm{Ker}(\phi_\Gamma)$, since every point in P_Γ satisfies the relations in $\mathrm{Ker}(\phi_\Gamma)$. On the other hand, it is not clear whether $\mathrm{Ker}(\phi_\Gamma) \subseteq P_\Gamma$. If this happens, we say that X^Γ *is fully parametrized by* φ_Γ.

Example 62. The following example shows that not for any $\mathcal{B}_\Gamma$, the equality $P_\Gamma = \mathrm{Ker}(\phi_\Gamma)$ holds. Consider the set

$$\mathcal{B}_\Gamma = \{(1,2), (-1,0), (0,1)\}$$

generating the semigroup Γ. The set P_Γ is given by the points $(ts^2, \frac{1}{t}, s)$ for $t, s \in K$. The toric variety X^Γ is defined by the equation $xy - z^2 = 0$. The point $(0,0,0) \in \mathbb{A}^n_K$ is contained in X^Γ, but it cannot be written as $(ts^2, \frac{1}{t}, s)$ for any $t, s \in K$. In particular, $\frac{1}{t} \neq 0$ for all $t \in K$.

A criterion for deciding whether a toric variety X^Γ is fully parametrized by φ_Γ is given in [36]. As we are interested in the case of K an algebraically closed field, the following result is enough:

Theorem 63. *Let K be an algebraically closed field, and $\mathcal{B}_\Gamma$ a set of vectors defined as above, generating the semigroup Γ. The toric variety $X^\Gamma = V(\mathrm{Ker}(\phi_\Gamma))$ defined as in (4.2) is fully parametrized by φ_Γ as in section 1.4 if and only if $V(\mathrm{Ker}(\phi_\Gamma), x_i) \subset P_\Gamma$ for $i = 1, \dots, n$.*

Proof. See [36, Corollary 2.5]. □

Example 64. Consider the set $\mathcal{B}_\Gamma = \{(1,1), (1,0), (0,2)\}$. The variety X^Γ is the Whitney Umbrella, $W : \{x^2 - y^2 z = 0\}$. The theorem shows that W is fully parametrized by $\varphi_\Gamma(t,s) = (ts, t, s^2)$:

- $V(x^2 - y^2 z, x) = V(x, y^2 z) = \{(0,0,z) : z \in K\} \cup \{(0,y,0) : y \in K\} \subseteq P_\Gamma$, because any point of the form $(0,0,z)$ is the image of $(0,s) = (0,\sqrt{z})$ under φ_Γ.
- $V(x^2 - y^2 z, y) = V(x^2, y) = \{(0,0,z) : x \in K\} \subseteq P_\Gamma$, because the point $(0,0,z)$ is the image of $(0,\sqrt{z})$ under φ_Γ.
- $V(x^2 - y^2 z, z) = V(x^2, z) = \{(0,y,0) : y \in K\} \subseteq P_\Gamma$, where any point in $V(x^2, z)$ is the image of $(y,0)$ under φ_Γ.

Example 65. Let X^Γ be the toric variety defined by

$$\Gamma = \langle (1,2), (-1,0), (0,1) \rangle_{\mathbb{N}},$$

the semigroup of Example 62. Notice that

$$V(xy - z^2, y) = V(y, z^2) \nsubseteq P_\Gamma = \{(ts^2, \frac{1}{t}, s) : t, s \in K\},$$

because, as it was already shown for this example, $\frac{1}{t} \neq 0$ for any $t \in K$. The theorem states that X^Γ is not fully parametrized by $\varphi_\Gamma(t,s) = (ts^2, \frac{1}{t}, s)$.

4.2 Dimension and dimension of immersion of monomial varieties

Our purpose here is to analyze the concept of dimension of an affine algebraic variety, and to show how to compute it for a monomial variety. We are going to consider two different notions of dimension.

Let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety. In section 5.1, the dimension of X was defined as the Krull dimension of its coordinate ring $K[X]$. Recall that this is the maximal height of a prime ideal $\mathcal{P} \subset K[X]$. We denote the dimension of X by $\dim(X)$. The Krull dimension of $K[X]$ is denoted by $\dim(K[X])$. The following theorem for K-algebras gives us a tool to compute the dimension of a coordinate ring $K[X]$.

Theorem: Let K be a field, and B an integral domain, which is a finitely generated K-algebra.
i) The dimension of the ring B is equal to the transcendence degree of $\mathrm{Q}(B)$ over K.
ii) For any prime ideal $\mathcal{P} \subset B$ we have

$$\dim(B) = \dim(B/\mathcal{P}) + \mathrm{height}(\mathcal{P}).$$

The theorem tells us that given an affine variety $X = V(I)$, we can compute the dimension of X:

$$\dim(K[X]) = \dim(K[x_1, \dots, x_n]/I(X)) = \dim(K[x_1, \dots, x_n]) - \mathrm{height}(I).$$

Suppose that we are given a monomial algebra $K[\Gamma]$. It is not necessary to find the equation of X^Γ. We consider the coordinate ring $B = K[\Gamma]$ and apply the first assertion of the theorem. Note that B it is a finitely generated K-algebra. It is also an integral domain because it is isomorphic to the quotient of a polynomial ring by a prime ideal. Therefore we have

$$\dim(X^\Gamma) = \mathrm{transdeg}(\mathrm{Q}(K[\Gamma])). \tag{4.4}$$

This is the classical algebraic notion of dimension of an affine variety.

Apart from that, we are going to analyze a notion of dimension which does not correspond to the space in which the variety is defined, but to the minimal space in which it can be immersed (globally).

Let $X = \overline{\mathrm{Im}(\varphi)}$ with

$$\varphi : \mathbb{A}^m \to \mathbb{A}^n \tag{4.5}$$
$$t = (t_1, \dots, t_m) \mapsto (t^{\alpha_1} \dots, t^{\alpha_n}), \tag{4.6}$$

be a monomial variety, with coordinate ring $K[X] \cong K[\Gamma]$, the K-algebra defined by the semigroup $\Gamma \in \mathbb{N}$. We define the *dimension of immersion* of the variety X, $\dim_{\mathrm{Imm}}(X)$ as the size of a minimal generating system of Γ in $\mathbb{N}$.

This is equivalent to the dimension of the smallest affine space in which $\mathrm{Im}(\varphi)$ can be immersed as a Zariski closed set.

Considering a parametrization φ as in 4.5 we will see in which cases both concepts of dimension coincide.The following examples show what happens for several values of m and n.

Example 66. Consider a parametrization:

$$\varphi : \mathbb{A}^1 \to \mathbb{A}^n. \tag{4.7}$$

Let X be $\overline{\mathrm{Im}(X)}$. Then $K[X] \cong K[t^{\alpha_1}, \ldots, t^{\alpha_n}]$, where $\alpha_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. From (4.4) it follows that

$$\dim(X) = \mathrm{transdeg}(\mathrm{Q}(K[X])) = \mathrm{transdeg}(K(t)) = 1.$$

Nevertheless, computing $\dim_{\mathrm{Imm}}(X)$ is not that simple. We distinguish three cases:

A) If $n = 1$ then $K[X] \cong K[t^\alpha]$ with $\alpha \in \mathbb{Z}$. This variety is isomorphic to $\mathbb{A}^1$, and therefore smooth. We have

$$\dim_{\mathrm{Imm}}(X) = 1.$$

B) If $n = 2$ and $\varphi(t) = (t^\alpha, t^{-\beta})$ with $\alpha, \beta \in \mathbb{N}$, the image of φ is an open set of $\mathbb{A}^1$: it is the result of removing the origin. This set $\mathbb{A}^1 \setminus \{0\}$ cannot be expressed as the zeros of a finite set of polynomials in $K[x]$. However,

$$K[X] \cong K[t^\alpha, t^{-\beta}] \cong K[x, y]/(x^\beta y^\alpha - 1),$$

so X is an affine algebraic subvariety of $\mathbb{A}^2$. Equivalently, the lattice $\langle \alpha, \beta \rangle_{\mathbb{N}} = \Gamma \otimes_{\mathbb{N}} \mathbb{Z}$ has dimension 1. On the other hand, Γ cannot be generated in $\mathbb{N}$ by a set of cardinality smaller than 2. For this reason $\dim(X) = 2$.

The simplest example of this situation is the semigroup $\Gamma = \langle 1, -1 \rangle_{\mathbb{N}}$. That is, $\varphi(t) = (t, \frac{1}{t})$ is a parametrization of the hyperbola.

C) Suppose now that $n > 1$ and that the t^{α_i} satisfy a certain set of binomial equations of order greater than 1 (that is, $\alpha_i \notin \langle \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \rangle_{\mathbb{N}}$). In this situation the semigroup Γ cannot be generated by a set of size $\dim(\Gamma)$. The variety is not smooth, and the dimension of immersion of X is the cardinality of a minimal set of generators of the semigroup Γ.

For instance, consider the parametrization $\varphi(t) = (t^{p_1}, \ldots, t^{p_k})$ where p_i are the k first prime numbers. The dimension of immersion of $X = \overline{\mathrm{Im}(\varphi)}$ is k.

Example 67. Consider a parametrization

$$\varphi : \mathbb{A}^2 \to \mathbb{A}^n,$$

and $X = \overline{\mathrm{Im}(\varphi)}$. We can see by looking at the coordinate ring $K[X]$ and computing the transcendencce degree of its field of fractions, that $\dim(X) = 2$. In fact, $\dim(X) = 1$ when there exists $i \in \{1, \ldots, n\}$ such that $t^{\alpha_j} = (t^{\alpha_i})^{k_j}$, $k_j \in \mathbb{N}$ for all $i \in \{1, \ldots, i-1, i+1, \ldots, n\}$. We distinguish six cases here:

A) If $n = 2$ and t^{α_1}, t^{α_2} are algebraically independent, then

$$\dim(X) = \dim_{\mathrm{Imm}}(X) = 2.$$

In this case $K[X] \cong K[x, y]$, so X is isomorphic to $\mathbb{A}^2$.

B) If $n = 2$ and t^{α_1}, t^{α_2} are not algebraically independent, then X is a plane curve in $\mathbb{A}^3$, and therefore $\dim(X) = 1$.
Also note that the same conclusion holds for any finite set of generators as long as one cannot find two algebraically independent monomials among them. The criteriafor deciding the dimension of immersion in this case is similar to 2 and 3 of Example 66.

C) Let $n = 3$ and suppose that the α_i satisfy $c\alpha_1 + d\alpha_2 = e\alpha_3$ for some $c, d, e \in \mathbb{N}$, $e > 1$. Then the variety X is not smooth (see section 3.2) and $\dim_{\mathrm{Imm}} = 3$, while $\dim(X) = 2$.

D) Let $n = 3$ and suppose that the α_i satisfy $c\alpha_1 + d\alpha_2 + e\alpha_3 = 0$ for some $c, d, e \in \mathbb{N}$. Then $\dim_{\mathrm{Imm}} = 3$ and $\dim(X) = 2$, but X is not smooth.

Consider, for instance, the semigroup $\Gamma = \langle (1,0), (0,1), (0,-1) \rangle_{\mathbb{N}}$. In this case, X is the cartesian product of $\mathbb{A}^1$ and a hyperbola (as in Example 66.B). It can be seen in figure 4.1, together with its semigroup. The semigroup

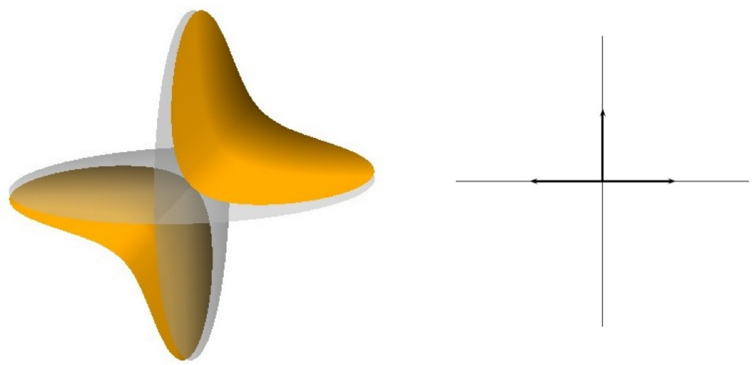

Figure 4.1: yz-1=0

$\Gamma = \langle (1,0), (0,1), (-1,-1) \rangle_{\mathbb{N}}$ is also an example of this situation. The variety associated to this semigroup is shown in figure 4.2.
E) Let $n = 4$, where the components of φ are such that $\alpha_1 = -\alpha_2$, and $\alpha_3 = -\alpha_4$. Even more, suppose α_1 and α_2 are vectors with the same direction and opposite sense, and α_3 and α_4 satisfy the same condition. This case is similar to 66.B. The lattice $\Gamma \otimes_{\mathbb{N}} \mathbb{Z}$ has dimension 2, and therefore $\dim(X) = 2$. Here, $\mathrm{Im}(\varphi)$ is a smooth open subset of $\mathbb{A}^2$ but, as an affine algebraic variety,

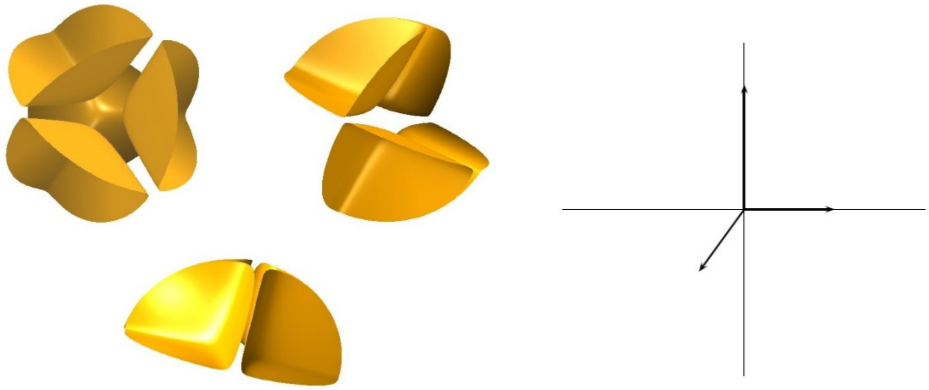

Figure 4.2: Hypersurfaces of the form $x^a y^b z^c - 1 = 0$.

it is immersed in $\mathbb{A}^k$ if and only if $k \geq 4$, and therefore $\dim_{\mathrm{Imm}}(X) = 4$. The affine variety X is the Cartesian product of two hyperbolas.

B.5) In general, if $n > 2$, then $\dim_{\mathrm{Imm}}(X)$ is the cardinality of the smallest set of generators of the semigroup Γ. However, $\dim(X)$=2 if there are two algebraically independent generators among the t^{α_i}.

Example 68. C) Consider the general case

$$\varphi : \mathbb{A}^m \to \mathbb{A}^n,\, m \geq 3.$$

C.1) If the t^{α_i} are algebraically independent, then the variety is the affine space $\mathbb{A}^n$, and its dimension is equal to its dimension of immersion: $\dim(X) = \dim_{\mathrm{Imm}}(X) = n$. Necessarily $n \leq m$.

C.2) If k is the cardinality of a minimal set of generators of the semigroup Γ, where $m \leq k \leq n$, $\dim(X) = m$ and $\dim_{\mathrm{Imm}}(X) = k$.
If $k < m$ is the cardinality of a minimal set of generators of Γ, then

$$\dim_{\mathrm{Imm}}(X) = k.$$

On the other hand, if there are only $d < n$ algebraically independent generators, then $\dim(X) = d$.

The following observations resume relevant conclusions from the examples:

- If $X = \overline{\mathrm{Im}(\varphi)} = \mathrm{Im}(\varphi)$ is smooth, then $\dim(X) = \dim_{\mathrm{Imm}}(X)$.
- If $\mathrm{Im}(\varphi) \subsetneq X = \overline{\mathrm{Im}(\varphi)}$ and X is smooth, the image of the parametrization is an open subset of an affine algebraic variety, or a quasi-affine variety of dimension m. But if we consider the closure of X, we must immerse it in a higher dimensional space.

- IF X is a singular variety of dimension m, it cannot be an affine algebraic subvariety of $\mathbb{A}^m$: the only closed affine subset of $\mathbb{A}^m$ of dimension m is the whole $\mathbb{A}^m$. As it is smooth, it cannot be isomorphic to a singular variety X. Therefore, in this situation $\dim_{\mathrm{Inm}}(X) > m$.

Chapter 5

APENDIX

5.1 Affine varieties

Let K be an algebraically closed field of characteristic zero and consider the n-dimensional affine space $\mathbb{A}^n_K$ (we will denote it simply by $\mathbb{A}^n$ unless it is necessary to specify the field. An *affine algebraic variety* $X \subset \mathbb{A}^n_K$ is the zero locus of a prime ideal $I \subset K[x_1, \ldots, x_n]$. That is, the set of points $p \in \mathbb{A}^n_K$ for which $f(p) = 0$ for any polynomial f in I. In other words, an algebraic affine variety is an irreducible closed set in the Zariski topology in $\mathbb{A}^n_K$ (see [22, sec. 1.1]). An open set of an affine variety is a *quasi-affine variety.*
We write $X = V(I)$, and the ideal $I = I(X)$ is said to be the defining ideal of the variety X. If I is a prime ideal which is also principal, that is, generated by a unique non constant polynomial $f \in K[x_1, \ldots, x_n]$, then X is called a *hypersurface.*

We define the *affine coordinate ring* of an affine algebraic variety $X = V(I) \subseteq \mathbb{A}^n_K$ as the quotient ring $K[x_1, \ldots, x_n]/I = K[X]$. The coordinate ring is the algebraic object containing the information of X, which is the geometric object. Algebraic geometry studies this possibility of associating algebraic objects to geometric objects. A closer view of X is provided by the local rings which, instead of general data about the variety, contain local information for each point. Let p be a point of X. Denote the ideal defining p as $m_p = I(p) \subset K[X]$. The Nullstellensatz ensures m_p is a maximal ideal. The *local ring of X at p*, denoted by $K[X]_{m_p}$, is the localization of $K[X]$ at m_p.

We define the notion of dimension of an algebraic affine variety through a property of its coordinate ring. The *Krull dimension* of a noetherian ring R is defined as the supremum of the heights of all prime ideals $\mathcal{P} \subset R$. That is, the length n of the largest possible chain of prime ideals in $\mathcal{P}$

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \ldots \subsetneq \mathcal{P}_n = \mathcal{P}$$

for some prime ideal $\mathcal{P} \subset R$. The *dimension of an affine variety* X is the Krull dimension of its coordinate ring $K[X]$. We will denote it by $\dim(X)$.

For a given affine algebraic variety, we want to find a local description attending to the regularity of the variety in each point. By regularity we mean that there is not a strong change in the geometry of the variety in that point compared to the points around it, and also that the variety intersects this

point of the ambient space only once.
The *Zariski tangent space* to a variety X at a point p is defined in terms of the maximal ideal (see [37, section 1.3, Corollary 1]) as the K-vector space

$$T_pX = \left(m_p/m_p^2\right)^*, \tag{5.1}$$

that is, the dual space to the quotient m_p/m_p^2. This vector space is isomorphic to the tangent space defined in differential geometry, which consists on the space of the tangent vectors to a manifold at a given point.

An affine algebraic variety X is said to be *singular* at a point $p \in X$, and p is then said to be a singular point of the variety, if the local ring $K[X]_{m_p}$ is not a regular ring.
Geometrically, $p \in X$ is singular if the dimension of the Zariski tangent space to X at $p \in X$, $\dim(T_pX)$ is not equal to $\dim(X)$. The point $p \in V(I)$ of the algebraic affine variety X defined by the ideal $I = (f_1, \dots, f_m)$ is singular if and only if the Jacobian matrix

$$J_{m\times n} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\delta x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{5.2}$$

is not of maximal rank at p.
If X is a hypersurface defined by f, then $p \in X$ is a singular point if and only if all partial derivatives of f vanish at p. Any point of X which is not singular is called *smooth*. An affine algebraic variety X is a smooth variety if it has no singular points. By saying that B is a *smooth* K-algebra, we mean it is the coordinate ring of a smooth variety X.
We call the set of singular points in X its *singular locus*. The singular locus of an algebraic affine variety X is a (proper) closed subset of X, see [22, Theorem 5.3].

A necessary property for smoothness is the concept of normality, which we study next.
A set $f_1, \dots, f_d$ of elements in a ring R, is said to be *algebraically independent* if there exists no polynomial $h(x_1, \dots, x_d)$ in $R[x_1, \dots, x_n]$, $h \neq 0$ such that $h(f_1, \dots, f_d) = 0$.

Lemma 69. *If the set $\{f_1, \dots, f_d\} \subset K[x_1, \dots, x_n]$ of polynomials in n variables is algebraically independent, then the K-algebra*

$$B = K[f_1, \dots, f_d]$$

is smooth. Note that necessarily $d \leq n$, since no more than n elements of the ring can be algebraically independent.

Proof. The K-algebra $K[f_1, \dots, f_d]$ is isomorphic to $K[x_1, \dots, x_d]$. That is clear by the surjective ring homomorphism:

$$\begin{aligned} F : K[x_1, \dots, x_n] &\longrightarrow K[f_1, \dots, f_d] \\ x_i &\longmapsto f_i \ , \ i = 1, \dots, d, \\ x_j &\longmapsto 0 \ , \ j = d+1, \dots, n, \end{aligned}$$

whose kernel is the ideal $(x_{d+1}, \dots, x_n)$. □

Let B be an integral domain, and let $B \subset A$ be a finite ring extension (see [35, sec. 4.1]). An element $f \in A$, where A is finite over B is *integral over B* if there exists a monic polynomial h in $B[x]$ such that $h(f) = 0$. The ring B is said to be a *normal ring* if it is integrally closed in its field of fractions $\mathrm{Q}(B)$, that is, if every element in $\mathrm{Q}(B)$ which is integral over B belongs to B.

An affine variety X is said to be *normal at a point* $p \in X$ if its local ring $K[X]_{m_p}$ is normal. The variety X is *normal* if it is normal at each point $p \in X$, see [22, exercise 3.17]. An affine variety is normal if and only if its coordinate ring is a normal ring, see [37, section 5.1].

Remark 70. The singular locus of a normal variety S has always codimension at least 2 in X, that is, $\dim(X) - \dim(\mathrm{Sing}(X)) \geq 2$ see [38], page 243. From this follows that normal curves are smooth.

Examples 71. Now we give some examples to ilustrate the concepts of smoothness and normality for surfaces in $\mathbb{A}^3_{\mathbb{C}}$.

1. One of the simplest smooth surfaces is the cylinder X, which is shown in figure 5.1. Its equation is $X : \left\{x^2 + y^2 - 1 = 0\right\}$.

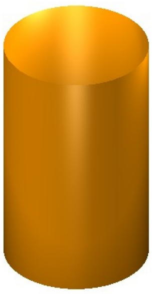

Figure 5.1: A smooth surface: Cylinder.

2. The cone V, with equation $V : \left\{xy - z^2 = 0\right\}$, is a normal variety, but it has a singular point at the origin, as the Jacobian criterion on the matrix (5.2) shows. This singularity has codimension 2, and it can be seen in figure 5.2.

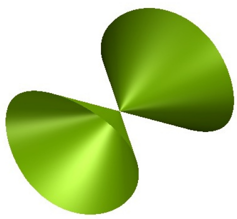

Figure 5.2: A normal but non smooth surface: Cone.

3. The surface given by the equation $Z : \left\{x^2 + z^3 - y^2z^2 = 0\right\}$ is singular along the y-axis, as it is shown in figure 5.3. The singular locus of Z has

codimension 1, and therefore Z is not normal. Take for example the element $f = \frac{x}{z} \in \mathrm{Q}(K[Z])$. There exists a monic polynomial in $K[Z][T]$ which is zero in f, namely $h(T) = T^2 - y^2 + z$. This means $f \notin K[Z]$ is integral over $Q(K[Z])$.

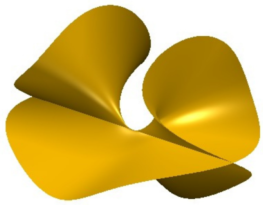

Figure 5.3: A non smooth and non normal surface.

There is a analogue notion to affine variety for subsets of projective space which are the zerosets of some arbitrary collection of polynomials. However, an additional condition must be satisfied by such polynomials to give a well defined variety: they must be homogeneous.
A *projective algebraic variety* $Y \subseteq \mathbb{P}^n$ is the common zero set of an arbitrary collection of homogeneous polynomials in $n+1$ variables, that is, $Y = V(\{h_i\}_{i\in\mathcal{I}})$ with $f_i \in K[x_0, \ldots, x_n]$ homogeneous for all $i \in \mathcal{I}$.
A *quasi-projective variety* is a locally closed subset of $\mathbb{P}^n$ with the Zariski topology induced from $\mathbb{P}^n$. That is, the intersection of an open and a closed subset of $\mathbb{P}^n$. In particular, affine and projective varieties are quasi-projective varieties.

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An alternative approach to the resolution of singularities of toric varieties.

Beatriz Pascual Escudero

November 20, 2012

Abstract

The problem of Resolution of Singularities consists of interpreting an algebraic variety X with singular points as the image by a birational proper morphism of some smooth algebraic variety X'. This morphism must define an isomorphism outside the singular locus of X.

The present work analyzes this problem for those affine algebraic varieties whose coordinate ring is an algebra over an algebraically closed field of characteristic zero, generated by a finite set of monomials. The aim is to construct an algorithm that, given such a variety, finds a resolution of its singularities. To do so, we translate the effect of a blowup into a transformation of the exponents of the monomials generating the coordinate ring of X. After that, we deal with a combinatorial problem. The result is a combinatorial procedure which works for curves and for hypersurfaces of dimension 2. A discussion about the difficulties that appear for higher dimensional varieties and how could the problem be addressed in this case follows.

Ein alternativer Zugang zur Auflösung von Singularitäten torischen Varietäten.

Beatriz Pascual Escudero

November 20, 2012

Abstract

Das Problem des Auflösens von Singularitäten besteht daraus, eine singuläre algebraische Varietät als Bild einer anderen glatten algebraischen Varietät X' unter einem eigentlichen birationalen Morphismus zu interpretieren. Dieser birationale Morphismus muss ein Isomorphismus außerhalb der singulären Punkte von X definieren.

Diese Arbeit analysiert das Problem des Auflösens von Singularitäten für algebraische Varietäten, die einen Koordinatenring mit speziellen Eigenschaften besitzen: der Koordinatenring muss eine Algebra über einem algebraisch abgeschlossenen Körper sein und von einer endlichen Menge von Monomen erzeugt sein. Das Ziel ist es, einen Algorithmus zu konstruieren, der für eine solche Varietät eine Auflösung von Singularitäten findet. Um dies zu erreichen, übersetzen wir die Effekte eines Blowups in eine Transformation der Exponenten der Monome, die die Koordinatenring von X erzeugen. Dann betrachten wir ein kombinatorisches Problem. Das Ergebnis dieser Arbeit ist ein kombinatorisches Verfahren, das für Kurven und Hyperflächen von Dimension 2 funktioniert. Weiters folgt eine Diskussion über Schwierigkeiten, die in höherer Dimension auftreten, und wie das Problem angegangen werden könnte.

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