CHARACTERS OF THE GENERIC HECKE ALGEBRA OF A SYSTEM OF BN-PAIRS

By ANDREW JOHN STARKEY

Mathematics Institute

University of Warwick

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For Joy

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ABSTRACT.

This thesis is concerned with the construction of the characters of the generic Hecke algebra of a system of BN-pairs of type (W,R). The approach used exploits the connection between these characters and those of the Coxeter group W.

Part I of the thesis gives definitions, results and conjectures relevant to calculating the characters of any generic Hecke algebra and Part II applies these results to the problem of calculating the characters of certain generic Hecke algebras.

In more detail: Chapter 1 gives basic definitions and describes the connection between the characters of the generic Hecke algebra and those of the Coxeter group W.

Chapter 2 divides the problem of calculating the characters into two parts and gives conjectures and results concerning one of these 'parts'. The results given solve this part of the problem (in a non-explicit manner) for some but not all generic Hecke algebras.

The last three chapters are all concerned with solving the other 'part' of the problem by an inductive method which uses the connection described in chapter 1.

Chapter 3 describes this method (the 'circle product'). Chapters 4 and 5 apply this method to a particular 'family' of generic Hecke algebras, culminating in an explicit formula (given in theorem 5.3.9) for certain character values.

The appendices contain tables of character values of some generic Hecke algebras.

STANDARD NOTATION.

Z the ring of rational integers

Q the field of rationals

C the complex field

Z_+ $\{ z \in Z \mid z > 0 \}$

Z_s $\{1,2,\dots,s\}$ for each $s \in Z_+$. ($Z_0 = 0$ see below)

k a subfield of C

u an indeterminate over C

$K_0 = Q(u)$ the field of rational functions in u over Q

K a finite field extention of K_0 (see also below definition 1.4.3)

$I_0 = Q[u]$ the ring of polynomials i u over Q

I the integral closure of I_0 in K

$\emptyset$ the empty set

$A \setminus B = \{a \in A \mid a \notin B\}$ (A and B sets)

$\langle \dots \rangle$ the group generated by a given set of elements

$(\alpha\cdot\beta)(d) = \alpha(\beta(d))$ where α and β are maps. d is in the domain of β and $\beta(d)$ is in the domain of α

$<\rho_1,\rho_2,\dots,\rho_t>$ a partition of $\sum_{i=1}^{t}\rho_i$ ($\rho_i \in Z_+\cup\{0\}$ for each $i \in Z_t$)

$<1^{\sigma_1}2^{\sigma_2}\dots n^{\sigma_n}>$ a partition of n. ($n \in Z_+$, $\sigma_i \in Z_+$ for each $i \in Z_n$) (see the beginning of §4.1)

PART I

CHAPTER I

THE GENERIC HECKE ALGEBRA AND ITS CHARACTERS.

Two equivalent definitions of the generic Hecke algebra are given in this chapter (see definition 1.1.8 and below lemma 1.3.4) and the significance of its characters is explained.

§1.1 COXETER GROUPS AND THE GENERIC HECKE ALGEBRA.

DEFINITION 1.1.1.

Given a group W and a subset R of involutions of W, the pair (W,R) is a Coxeter system if the following condition holds:

For each r,s in R let n_{rs} be the order of rs. Let T be the set of pairs (r,s) such that n_{rs} is finite. R is a set of generators for W with defining relations

$$(rs)^{n_{rs}} = 1_W \qquad \text{for all pairs } (r,s) \text{ in } T.$$

In which case W is a Coxeter group with distinguished generators R.

NOTE: The relations $(rs)^{n_{rs}} = 1_W$ can be rewritten as follows:

(1.1.2) $\qquad r^2 = 1_W \qquad$ for all r in R.

(1.1.3) $\qquad (rs\ldots\;)_{n_{rs}} = (sr\ldots\;)_{n_{rs}}$ for all pairs (r,s) in T with $r \neq s$, where $(ab\ldots\;)_n$ means the product of the first n terms in the alternating sequence a,b,a,b,...

Since every element $w \neq 1_W$ of W is a product of elements of R, we can make the following definition.

DEFINITION 1.1.4.

The length of the element w of W ($w \neq 1_W$), denoted by $l(w)$, is the least number of terms possible in an expression for w as a product of elements of R. Any such expression with this number of terms is said to be reduced. Conventionally $l(1_W) = 0$.

LEMMA 1.1.5.

Let $w \in W$ and $r \in R$. $l(wr) = l(w) \pm 1$.

Proof:

By equations (1.1.2) and (1.1.3) it is clear that $l(wr) \neq l(w)$. If $l(wr) > l(w)$ then clearly $l(wr) = l(w) + 1$. $l(wr.r) = l(w)$ thus if $l(wr) < l(w)$ we must have $l(wr) = l(w) - 1$.

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LEMMA 1.1.6.

Given two reduced expressions for w in $W(w \neq 1_W)$, one can be transformed into the other using only the relations (1.1.3).

Proof: [2,chapterIV,exercices §1.13b].

※

COROLLARY 1.1.7.

Given a set of positive integers $\{c_r | r \in R\}$ such that if r,s in R are conjugate in W then $c_r = c_s$, it follows that there exists a function $c: W \longrightarrow Z_+$ given by

$$c(w) = c(r_1) + c(r_2) + \dots\ .c(r_t)$$

where $r_1 r_2 \dots .r_t$ is any reduced expression for $w \in W$.

Proof:

If n_{rs} is odd then $(rs\dots .s)_{n_{rs}-1} r (sr\dots .r)_{n_{rs}-1} = s$

showing that r and s are conjugate in W. Thus the result now follows from lemma 1.1.6.

⋇

DEFINITION 1.1.8.

The set $\{c_r \mid r \in R\}$ described in corollary 1.1.7 is an indexing system for (W,R).

For the rest of this thesis we will consider only finite Coxeter groups.

DEFINITION 1.1.9.

The associative K-algebra with identity element h_1 generated by $\{h_r \mid r \in R\}$ with defining relations

$$(1.1.10) \qquad h_r^2 = u^{c_r} h_1 + (u^{c_r} - 1)h_r \qquad \text{for all } r \text{ in } R.$$

$$(1.1.11) \qquad (h_r h_s \ldots\ldots)_{n_{rs}} = (h_s h_r \ldots\ldots)_{n_{rs}} \qquad \text{for all}$$

r,s in R with $r \neq s$, is the generic Hecke algebra of type (W,R) with indexing system $\{c_r \mid r \in R\}$ and will be denoted by H(W,R,c,K,u) abbreviated to H(K,u).

THEOREM 1.1.12.

H(K,u) has K-basis $\{h_w \mid w \in W\}$ where

$$(1.1.13) \qquad h_w = h_{r_1} h_{r_2} \ldots .h_{r_t} \qquad \text{for any reduced}$$

expression $r_1 r_2 \ldots .r_t$ of w.

This theorem can be readily proved using a specialisation f of K with f(u) =1 (see §1.4), but as the given K-basis arises naturally in the alternative definition of H(K,u) given in §1.3 we omit the proof.

Note that lemma 1.1.6 shows that h_w is well defined.

§1.2 SYSTEMS OF BN-PAIRS.

A BN-pair is defined in [4, 8.2]. The definition of a system of BN-pairs is due to Curtis, Iwahori and Kilmoyer [5].

DEFINITION 1.2.1.

Let (W,R) be a Coxeter system with indexing system $\{c_r \mid r \in R\}$ (see definitions 1.1.1 and 1.1.8). A <u>system of BN-pairs of type (W,R)</u> is a set S of finite groups, indexed by an infinite set P of prime powers such that:

(i) Each $G(q)$ in S $(q \in P)$ has a BN-pair, say, $(B(q),N(q))$.

(ii) For each q in P there is a map $n_q : W \rightarrow N(q)$ such that $w \rightarrow n_q(w)(B(q) \cap N(q))$ defines an isomorphism from W onto the Coxeter group $W(q) = N(q)/(B(q) \cap N(q))$ which maps R onto the set of distinguished generators of $W(q)$.

(iii) For each q in P and r in R,

$$|B(q) : B(q) \cap (n_q(r))^{-1} B(q) n_q(r)| = q^{c_r}$$

Matsumoto [8, theorem 3] has shown that it is sufficient to specify that W is a finite group generated by a set R of involutions, since W is then necessarily a Coxeter group with distinguished generators R.

Each of the families of finite Chevalley groups of a fixed type form a system of BN-pairs. In these cases P is the set of all prime powers and the integers c_r all have value 1. [6, §1]

Each of the families of 'twisted Chevalley groups' form a system of BN-pairs. In these cases the integers c_r are not in general all equal and in some of these cases

P is a set of powers of a fixed prime. [10].

§1.3 HECKE ALGEBRAS.

Let $G(q)$ be an element of the system of BN-pairs S. Define in the group algebra $kG(q)$ the idempotent

$$e = \frac{1}{|B(q)|} \sum_{g \in B(q)} g$$

DEFINITION 1.3.1.

$ekG(q)e$ is a <u>Hecke algebra</u> and will be denoted by $E_k(q)$.

LEMMA 1.3.2.

The Hecke algebra $E_k(q) = ekG(q)e$ is isomorphic as a k-algebra to the endomorphism algebra $\mathrm{End}_{kG(q)}(V(q))$, where $V(q)$ is the left ideal $kG(q)e$ regarded as a left $kG(q)$-module.

Proof:

The map $\alpha : E_k(q) \longrightarrow \mathrm{End}_{kG(q)}(V(q))$ given by $\alpha(t): v \longmapsto vt$ for all t in $E_k(q)$ and v in $V(q)$, is an isomorphism.

⨳

THEOREM 1.3.3 ([8, theorem 4])

(i) $E_k(q)$ has k-basis $\{d_w \mid w \in W\}$ where $d_w = q^{c(w)} en_q(w)e$. (see definition 1.2.1(ii) and corollary 1.1.7).

(ii) Let $w \in W$ then $d_w = d_{r_1} d_{r_2} \ldots . d_{r_t}$ for any reduced expression $r_1 r_2 \ldots . r_t$ of w.

(iii) $E_k(q)$ is generated as a k-algebra with identity d_1 by $\{d_r \mid r \in R\}$ with defining relations

$$d_r^2 = q^{c_r} + (q^{c_r} - 1)d_r \qquad \text{for all } r \text{ in } R$$

$$(d_r d_s \dots .)_{n_{rs}} = (d_s d_r \dots .)_{n_{rs}} \quad \text{for all } r,s \text{ in } R$$

with $r \neq s$.

※

Tits (see [2, page 55]) is responsible for the idea of the generic Hecke algebra (see definition 1.1.9) which is seen to 'specialise' (see §1.4) to $E_k(q)$ on putting $u = q$ (k a suitable field of characteristic zero). This connection between $H(K,u)$ and $E_k(q)$ is described in detail below.

LEMMA 1.3.4.

For all x,y,z in W there exist polynomials $\sigma_{xyz}(u)$ in $Z[u]$ such that

(i) For any q in P, $d_x d_y = \sum_{z \in W} \sigma_{xyz}(q) d_z$

(ii) $\sum_{t \in W} \sigma_{xyt}(u)\sigma_{tzv}(u) = \sum_{s \in W} \sigma_{xsv}(u)\sigma_{yzs}(u)$ for all x,y,z,v in W.

Proof:

(i) This follows immediately from theorem 1.3.3.

(ii) Let $q \in P$. If u is replaced by q in the given equation it becomes equivalent to the associativity of $E_k(q)$. Since P is an infinite set the result follows.

※

Lemma 1.3.4 enables us to define $H(K,u)$ to be the K-algebra with K-basis $\{h_w \mid w \in W\}$ and multiplication given by

$$h_x h_y = \sum_{z \in W} \sigma_{xyz}(u) h_z \qquad \text{for all } x,y \text{ in } W.$$

Lemma 1.3.4(ii) shows that this algebra is associative. Theorem 1.3.3 clearly shows that this definition of $H(K,u)$ is equivalent to definition 1.1.9 and that theorem 1.1.12 is correct.

§1.4 CHARACTERS OF THE GENERIC HECKE ALGEBRA.

K_o and K are the fields of fractions of I_o and I respectively. Given a prime ideal D of I, let $K_D = \{ \gamma/\alpha | \gamma \in I, \alpha \in I \setminus D\}$; this is a subring of K containing I. Let $H(K_D,u)$ be the subring $\{ \sum_{w\in W} \lambda_w h_w | \lambda_w \in K_D\}$ of $H(K,u)$.

DEFINITION 1.4.1.

A <u>specialisation</u> f of K with <u>nucleus</u> D is a ring homomorphism $f: K_D \longrightarrow C$ with $f(1) = 1$ and $Ker(f) = DK_D$.

Note that D is determined by f since $D = I \cap Ker(f)$. The range $k = f(K_D)$ of f is a subfield of C (see [6, §4]).

LEMMA 1.4.2.

(i) For each q in P there exists a specialisation f_q of K with nucleus $D(q)$ such that $f_q(u) = q$.

f_q can be extended to a ring epimorphism

$f_q: H(K_{D(q)},u) \longrightarrow E_{k(q)}(q)$ where $k(q)$ is the range of f_q by setting $f_q(\sum_{w\in W} \lambda_w h_w) = \sum_{w\in W} f_q(\lambda_w) d_w$ for all λ_w in $K_{D(q)}$.

(ii) There is a specialisation f_1 of K with nucleus $D(1)$ and range $k(1)$ such that $f_1(u) = 1$.

f_1 can be extended to a ring epimorphism

$f_1: H(K_{D(1)},u) \longrightarrow k(1)W$ by setting

$f_1(\sum_{w\in W} \lambda_w h_w) = \sum_{w\in W} f_1(\lambda_w)w$ for all λ_w in $K_{D(1)}$.

Proof:

Follows immediately from [6, theorem 4.1 and lemma 4.2].

⋇

DEFINITION 1.4.3.

Given a specialisation f of K with nucleus D

(i) for $\alpha \in K$ we say that <u>'$f(\alpha)$ is defined'</u>

if and only if $\alpha \in K_D$.

(ii) for $\gamma \in H(K,u)$ we say that 'f(γ) is defined' if and only if $\gamma \in H(K_D,u)$.

Note that (ii) can be equivalently expressed by saying that for $\gamma = \sum_{w\in W} \alpha_w h_w$, $f(\gamma)$ is defined if and only if $f(\alpha_w)$ is defined for all w in W.

Tits has shown that H(K,u) is semi-simple and hence seperable because the characteristic of K is zero. (see [6, theorem 6.2]).Thus we can find a finite extension K of K_0 such that K is a splitting field for $H(K,u) \cong K \otimes_{K_0} H(K,u)$. We will from now on assume that K has this property.

A representation of H(K,u) over K is a K-algebra homomorphism $\xi : H(K,u) \longrightarrow \{\tau \mid \tau$ is a K-linear transformation from V to V$\}$ for some K-space V. The character of ξ is the K-linear map $\eta: H(K,u) \longrightarrow K$ given by $\eta(\gamma) = \text{trace}(\xi(\gamma))$ for all $\gamma \in H(K,u)$. η is an irreducible character if ξ is an irreducible representation. (see [3, chapters I and II]).

The set of all functions $\lambda : H(K,u) \longrightarrow K$ is an additive group with respect to the composition:

$$(\lambda_1 + \lambda_2)(\gamma) = \lambda_1(\gamma) + \lambda_2(\gamma) \quad \text{for all } \gamma \text{ in } H(K,u).$$

The subgroup generated by the characters of H(K,u) is called the character group of H(K,u) and is denoted by X(H(K,u)). The set of irreducible characters of H(K,u) is a free Z-basis for X(H(K,u)).

LEMMA 1.4.4.

If $\eta \in X(H(K,u))$ and $w \in W$ then

(i) $\eta(h_w) \in I$

(ii) for any specialisation f of K, $f(\eta(h_w))$ is defined.

Proof:

(i) follows from [6, lemma 7.2].

(ii) follows immediately from (i).

⋇

THEOREM 1.4.5 (Tits).

Let $X(E_C(q))$ be the character group of $E_C(q)$ over C. Let X(CW) be the character group of CW over C.

(i) For each q in P there exists a bijection from $X(H(K,u))$ to $X(E_C(q))$, which maps the irreducible characters of H(K,u) onto the set of irreducible characters of $E_C(q)$. If $\eta \in X(H(K,u))$ maps to $\mu \in X(E_C(q))$ then

$\mu(f_q(\gamma)) = f_q(\eta(\gamma))$ for all $\gamma \in H(K,u)$, in particular $\mu(d_w) = f_q(\eta(h_w)$ for all $w \in W$. (see lemma 1.4.2).

(ii) There exists a bijection from X(H(K,u)) to X(CW), which maps the set of irreducible characters of H(K,u) to the set of irreducible characters of CW. If $\eta \in X(H(K,u))$ maps to $\chi \in X(CW)$ then $\chi(f_1(\gamma) = f_1(\eta(\gamma))$ for all $\gamma \in H(K,u)$, in particular $\chi(w) = f_1(\eta(h_w)$ for all $w \in W$.

Proof:

This theorem follows from the proof of theorem 7.4 in [6].

⋇

J.A.Green has pointed out that the character group X(H(K,u)) is characterised in the following way:

THEOREM 1.4.6.

$X(H(K,u)) = \{ \eta : H(K,u) \longrightarrow K \mid \eta$ is K- linear, $\eta(h_w h_v) = \eta(h_v h_w)$ for all $w,v \in W \}$.

Proof:

Let F be a splitting field for the semi-simple F-algebra

A. Using Wedderburn's theorem and reducing to the case of a total matrix algebra one can readily show that the character group $X(A) = \{ \sigma : A \longrightarrow F \mid \sigma$ is F-linear, $\sigma(xy) = \sigma(yx)$ for all $x,y \in A \}$. Since $H(K,u)$ is semi-simple the theorem follows.

※

Note that for every pair α, β of linear transformations of some K-space that $\text{trace}(\alpha \circ \beta) = \text{trace}(\beta \circ \alpha)$. Hence it is clear that for any η in $X(H(K,u))$ and w,v in W that $\eta(h_w h_v) = \eta(h_v h_w)$.

The matirial in §1.2, §1.3 and §1.4 is nearly all in [6]. In that paper the generic Hecke algebra is denoted by $A_K(u)$ and the Hecke algebra by $H_k(q)$.

§1.5 THE ALGEBRA H_o AND ITS CHARACTERS.

The following chapters are independent of the material in this section.

By [6, theorem 4.1 and lemma 4.2] there exists a specialisation f_o of K with nucleus D_o and range k_o such that $f_o(u) = 0$.

Let H_o be the k_o-algebra with k_o-basis $\{b_w \mid w \in W\}$ and multiplication given by

$$b_x b_y = \sum_{z \in W} \sigma_{xyz}(0) b_z \qquad \text{for all } x,y \in W.$$

(see lemma1.3.4).

It is clear from theorem 1.3.3 that H_o is generated as a k_o-algebra with identity b_1 by $\{b_r \mid r \in R\}$ and that the following are defining relations for this set of generators:

(1.5.1) $$b_r^2 = -b_r \qquad \text{for all } r \in R.$$

(1.5.2) $(b_r b_s \ldots\ldots)_{n_{rs}} = (b_s b_r \ldots\ldots)_{n_{rs}}$ for all r,s in R with $r \neq s$.

It is also clear that $b_w = b_{r_1} b_{r_2} \ldots . b_{r_t}$ where $r_1 r_2 \ldots . r_t$ is any reduced expression for $w \in W$.

f_o can be extended to a ring epimorphism from $H(K_{D_o}, u)$ to H_o by setting

$$f_o\left(\sum_{w \in W} \lambda_w h_w\right) = \sum_{w \in W} f_o(\lambda_w) b_w \quad \text{for all } \lambda_w \in K.$$

(compare lemma 1.4.2).

Unlike $H(K,u)$, $E_k(q)$ ($q \in P$) and CW the algebra H_o is not semi-simple and there is hence no analogue of theorem 1.4.5 for this algebra. The irreducible characters of H_o are described in theorem 1.5.4.

LEMMA 1.5.3.

Let N_o be the nilpotent radical of H_o. H_o/N_o is commutative.

Proof:

We show that $b_r b_s - b_s b_r \in N_o$ for all $r,s \in R$ with $r \neq s$. It is well known that it is sufficient to show that $b_r b_s - b_s b_r$ is properly nilpotent, i.e. given any elements $\lambda_w \in k_o$ ($w \in W$) that

$$a = \sum_{w \in W} \lambda_w b_w (b_r b_s - b_s b_r) \text{ is nilpotent.}$$

Fix $w \in W$ and $r,s \in R$.

If $l(wr) < l(w)$ and $l(ws) < l(w)$

or $l(wr) < l(w)$, $l(ws) > l(w)$ and $l(wsr) < l(ws)$

or $l(ws) < l(w)$, $l(wr) > l(w)$ and $l(wrs) < l(wr)$

then using equation (1.5.1) one readily finds that

$b_w(b_r b_s - b_s b_r) = 0$.

Thus $b_w(b_r b_s - b_s b_r) \neq 0$ implies that $b_w(b_r b_s - b_s b_r) = b_{w_1} + b_{w_2}$ for some $w_1, w_2 \in W$ with $l(w_1), l(w_2) > l(w)$. So it is clear that there exists $t \in Z_+$ with $a^t = 0$.

$\times\!\!\times$

THEOREM 1.5.4.

Let $R = \{ r_1, r_2, \ldots r_m \}$. The set $\{\delta_{i_1 i_2 \ldots . i_m} \mid i_j \in \{0,1\},\ j \in Z_m\}$ of k_o-homomorphisms from H_o to k_o where

$$\delta_{i_1 i_2 \ldots . i_m}(b_{r_j}) = -i_j \qquad \text{for all } j \in Z_m$$

and $$\delta_{i_1 \ldots . i_m}(b_1) = 1$$

is the set of all irreducible characters of H_o over k_o.

Proof:

Let δ be an irreducible character of H_o. By lemma 1.5.3 $\delta(b_1) = 1$. Thus δ is an irreducible character if and only if

$$\delta(b_1) = 1$$

$$(\delta(b_r))^2 = -\delta(b_r) \qquad \text{for all } r \in R$$

and $$(\delta(b_r)\delta(b_s)\ldots\quad)_{n_{rs}} = (\delta(b_s)\delta(b_r)\ldots\quad)_{n_{rs}}$$

for all $r,s \in R$ with $r \neq s$.The result is now clear.

$\times\!\!\times$

COROLLARY 1.5.5.

$\dim_{k_o}(N_o) = |W| - 2^{|R|}$.

Proof:

$\dim_{k_o}(H_o) - \dim_{k_o}(N_o) = |\{ \delta \mid \delta \in X(H_o),\ \delta \text{ is irreducible}\}|$.

$\times\!\!\times$

Given $\eta \in X(H(K,u))$ it is clear that the k_o-linear map $f_o(\eta):H_o \longrightarrow k_o$ defined by

$$f_o(\eta): b_w \longrightarrow f_o(\eta(h_w)) \qquad \text{is a character of}$$

H_o. However, theorems 1.4.5(ii) and 1.5.4 show that even if η is irreducible, $f_o(\eta)$ will not in general be irreducible. We thus make the following definition.

DEFINITION 1.5.6.

Let $|R| = m$. Let $\{\eta_j \mid j \in Z_s\}$ be the set of all irreducible characters of $H(K,u)$. (By theorem 1.4.5(ii) s is the number of conjugacy class in W).

The decomposition matrix of H_o, denoted by D_{H_o}, is an $(s \times 2^m)$-matrix with columns indexed by the set $\Theta = \{(i_1, i_2, \ldots, i_m) \mid i_j \in \{0,1\},\ j \in Z_m\}$. The $(j,(i_1, \ldots, i_m))^{th}$ entry $d_{j,(i_1,\ldots,i_m)}$ of D_{H_o} is defined by the equations

$$f_o(\eta_j) = \sum_{(i_1,\ldots,i_m)\in\Theta} d_{j,(i_1,\ldots,i_m)} \delta_{i_1 i_2 \ldots i_m}$$

for all $j \in Z_s$.

Appendix 4 gives the decomposition matrix for some particular examples of H_o. The symmetry exhibited in these examples can be explained using the involutory semi-linear automorphism of $H(K,u)$ described in [6, §8].

CHAPTER 2

LINEAR DEPENDENCE OF THE CHARACTER VALUES.

The problem of evaluating the characters of the generic Hecke algebra is seperated into two parts(see below corollary 2.1.5). Some results and conjectures concerning one of these parts are then given(see §2.2).

§2.1 THE CHARACTER TABLE.

DEFINITION 2.1.1.

Let W have m_W conjugacy classes. The <u>character table,</u> T(H(K,u)), of H(K,u) is an $(m_W \times |W|)$-array of elements of K. Rows are indexed by the irreducible charaters of H(K,u) and the columns are indexed by the elements of W. Let η be an irreducible charater of H(K,u) and w an element of W. The $(\eta,w)^{th}$ entry of T(H(K,u)) is $\eta(h_w)$.
(Appendix 5 gives some examples of character tables).

REMARK 1. By lemma 1.4.4(i) all the entries of T(H(K,u)) are in I.

REMARK 2. Since any charater of H(K,u) is a K-linear function and $\{h_w | w \in W\}$ is a K-basis of H(K,u), we see that the charater table T(H(K,u)) completely determines the values of the irreducible characters of H(K,u).

THEOREM 2.1.2.

If B is a set of conjugacy class representatives for W then the columns of T(H(K,u)) indexed by B span the column space of T(H(K,u)).

Proof:

Let the column of T(H(K,u)) indexed by $w \in W$ be $\underset{\sim}{\eta}(h_w)$.

We can consider it to be an element of the K-space K^{m_W}.

Since the row rank of a matrix is equal to its column rank the theorem will follow if we can show that $\{\eta(h_b) | b \in B\}$ is a linearly independent set over K.

I_o is a principal ideal domain and hence is a Dedekind domain. K is a finite field extension of K_o, thus by [1, chapter VII, 2.5, prop^{n}5, corollary 3] I is a Dedekind domain.

Let f be a specialisation of K with the prime ideal D of I as its nucleus (see §1.4). [1, chapter VII, §2.2, theorem 1(g) and chapter II, §3.1, prop^n 2] show that K_D is a principal ideal domain with DK_D a prime ideal (since it is a maximal ideal).

Let $\sum_{b \in B} \sigma_b \eta(h_b) = \underline{0}$ where $\sigma_b \in K$ for all $b \in B$.

Since K is the field of fractions of I we can assume without loss of generality that each $\sigma_b \in I \subset K_D$. Further since we have shown that K_D is a principal ideal domain, it is a unique factorization domain and we can assume by cancelling out any common divisor of the elements σ_b $(b \in B)$ that either $\sigma_b = 0$ for all $b \in B$, or that there exists at least one $b \in B$ with $\sigma_b \notin DK_D$. $Ker(f) = DK_D$, so in this latter case

$\sum_{b \in B} f(\sigma_b) f(\eta(h_b))$ is a non-trival linear combination of $\{f(\eta(h_b)) | b \in B\} \subset C^{m_W}$. (f is applied component-wise to the vectors $\eta(h_b)$). In particular $\underline{0} = \sum_{b \in B} f_1(\sigma_b) f_1(\eta(h_b))$ is a non-trival linear combination over C of the columns of the character table of CW (see theorem 1.4.5(ii)), giving a contradiction since these columns are well known to be linearly independent over C.

Immediately we have

COROLLARY 2.1.3

Let B be a set of conjugacy class representatives for W. For each w in W there exist unique elements $\sigma(w,b)$ of K $(b \in B)$ such that

$$\eta(h_w) = \sum_{b \in B} \sigma(w,b) n(h_b) \quad \text{for all } \eta \in X(H(K,u)).$$

⨳

The coefficients $\sigma(w,b)$ are determined by the relations

$$\text{(2.1.4)} \qquad \eta(h_w h_v) = \eta(h_v h_w) \qquad \text{for all } \eta \in X(H(K,u))$$

and $w,v \in W$ (see theorem 1.4.6 and below), in the following way

LEMMA 2.1.5.

Let $H_o(K,u)$ be the K-space spanned by $\{h_w h_v - h_v h_w \mid w,v \in W\}$.

(i) Let $\gamma \in H(K,u)$. $\eta(\gamma) = 0$ for all $\eta \in X(H(K,u))$ if and only if $\gamma \in H_o(K,u)$.

(ii) The elements $\sigma(w,b)$ of K are uniquely determined by the equations

$$h_w = \sum_{b \in B} \sigma(w,b) h_b \bmod H_o(K,u) \qquad \text{for all } w \in W.$$

Proof:

(i) This follows readily from the proof of theorem 1.4.6.

(ii) By theorem 1.4.6 and (i) we see that

$$X(H(K,u)) \cong H(K,u)/H_o(K,u) \qquad \text{as K-spaces.}$$

So by theorem 2.1.2 $H(K,u)/H_o(K,u)$ has K-basis $\{h_b + H_o(K,u) \mid b \in B\}$.

⨳

Corollary 2.1.3 shows that we can divide the problem of evaluating the characters of $H(K,u)$ into two parts in the following way:

PART 1. Determine the elements $\sigma(w,b)$ of K defined in corollary 2.1.3 ($w \in W$, $b \in B$) for some suitable set B of conjugacy class representatives of W.

PART 2. Calculate the character values $\eta(h_b)$ for all irreducible characters η of $H(K,u)$ and all b in B, where B is the same set of class representatives as in 'Part 1'. (Note that the set of all irreducible charaters of $H(K,u)$ is a free Z-basis for $X(H(K,u))$.)

§2.2 $\sigma(w,b)$

NOTATION

$Y = \{w \in W \mid w \text{ is of minimal length in its conjugacy class}$

$Y^* = \{w \in W \mid l(w) \leq l(rwr) \text{ for all } r \in R\}$

B a set of conjugacy class representatives of W with $B \subset Y$.

$\{\sigma(w,b) \mid w \in W,\ b \in B\}$ the set of elements of K defined by corollary 2.1.3. (Appendix 7 gives some examples of these).

Note that clearly $Y \subset Y^*$, but in general $Y \neq Y^*$. For example in the Coxeter group S_5 (see Appendix 2) the 4-cycle $(1452) \in Y^* \setminus Y$.

CONJECTURE 2.2.1.

$$\sigma(w,b) \in Z[u] \quad \text{for all } w \in W \text{ and } b \in B.$$

CONJECTURE 2.2.2.

For each w in W and y in Y there exists $\alpha(w,y)$ in $Z[u]$

such that

(2.2.3) $$\eta(h_w) = \sum_{y\in Y} \alpha(w,y)\eta(h_y)$$ for all

$\eta \in X(H(K,u))$. Note that equations (2.2.3) do not in general determine the coefficients $\alpha(w,y)$ uniquely.

LEMMA 2.2.4.

For each w in W and y in Y^* there is an element $\delta(w,y)$ in $Z[u]$ such that

$$\eta(h_w) = \sum_{y\in Y^*}\delta(w,y)\eta(h_y) \quad \text{for all } \eta \in X(H(K,u)).$$

Proof:

We use induction on $l(w)$. If $w \in Y^*$ the result is trivial. Let $w \in W\setminus Y^*$. There exists $r \in R$ with $l(rwr) < l(w)$, infact by lemma 1.1.5, $l(rwr) = l(w) - 2$. Thus there exists $v \in W$ such that $l(v) = l(w) - 2$ and $w = rvr$. By (1.1.10), (1.1.13) and (2.1.4)

$$\begin{aligned}\eta(h_w) &= \eta(h_r h_v h_r)\\ &= \eta(h_v h_r h_r)\\ &= u^{c_r}\eta(h_r) + (u^{c_r} - 1)\eta(h_{vr})\end{aligned}$$

Since $l(v), l(vr) < l(w)$ the result follows by induction.

※

CONJECTURE 2.2.5.

If y and z in Y are conjugate in W then

$$\eta(h_y) = \eta(h_z) \quad \text{for all } \eta \in X(H(K,u)).$$

Note that in general the condition: '$y,z \in Y^*$, y conjugate to z in W' does not imply that $\eta(h_y) = \eta(h_z)$ for all η in $X(H(K,u))$. For example in the Coxeter group S_5 (see Appendix 2) we have (1452) and (1234) in Y^*. These two elements are clearly conjugate but $\mathfrak{J}(h_{(1452)} = u^5$ and $\mathfrak{J}(h_{(1234)} = u^3$ where $\mathfrak{J}$ is the unit character of $H(K,u)$. (see definition 4.1.3).

CONJECTURE 2.2.6.

Let w be an element of W. The following two statements are equivalent.

(i) $w \in Y$

(ii) If $w = tv$ $(t, v \in W)$ with $l(t) + l(v) = l(w)$ then $l(vt) = l(w)$.

(Clearly (i) implies (ii) as $vt = t^{-1}(tv)t$).

These four conjectures and lemma 2.2.4 are related in the following ways

LEMMA 2.2.7.

(i) Conjecture 2.2.6 implies conjecture 2.2.2.

(ii) Conjecture 2.2.2 together with conjecture 2.2.5 implies conjecture 2.2.1.

(iii) Conjecture 2.2.1 implies conjecture 2.2.2 which implies lemma 2.2.4.

Proof:

(ii) and (iii) are immediate.

(i) We use induction on $l(w)$. If $w \in Y$ result is trivial. Let $w \in W \setminus Y$. There exists a reduced expression $r_1 \cdots . r_t$ for w $(t = l(w))$ and $i \in Z_{t-1}$ such that $r_i r_{i+1} \cdots . r_t r_1 \cdots . r_{i-1}$ is reduced and $r_{i+1} \cdots . r_t r_1 \cdots . r_i$ is not reduced. By (1.1.13) and (2.1.4) we can without loss of generality take $i = 1$. Thus $l(rwr) = l(w) - 2$ where $r = r_1$. The proof can now be completed exactly as the proof of lemma 2.2.4 was.

⋇

The rest of this section is concerned with proving some special cases of the conjectures above.

As noted in §1.2 each of the families of finite Chevalley groups of a fixed type form a system of BN-pairs.

Carter's book [4] describes the Coxeter groups for these systems (Note that these Coxeter groups are Weyl groups and are refered to as such in [4]). In particular it shows that

(i) The Coxeter group of type A_{n-1} $(n > 1)$ is isomorphic to the symmetric group S_n ([4, page 124]) and an isomorphism can be found which maps the distinguished generators of W onto $\{u_i = (i\ i+1) \mid i \in Z_{n-1}\}$ (see theorem A2.1). We will identify the Coxeter group of type A_{n-1} with S_n and hence R with $\{u_i \mid i \in Z_{n-1}\}$.

(ii) The Coxeter group of type B_2 is isomorphic to the group $\langle a, g \mid a^2 = g^2 = (ag)^4 = 1 \rangle$ which has order 8.

(iii) The Coxeter group of type B_3 is isomorphic to the group $\langle a, g, d \mid a^2 = g^2 = d^2 = (ag)^3 = (gd)^4 = (ad)^2 = 1 \rangle$ which has order 48.

(iv) The Coxeter group of type G_2 is isomorphic to the group $\langle a, g \mid a^2 = g^2 = (ag)^6 = 1 \rangle$ which has order 12.

THEOREM 2.2.8.

Conjectures 2.2.1, 2.2.2, 2.2.5 and 2.2.6 are true for W of types A_t $(1 \leq t)$, B_2, B_3 and G_2.

Proof:

Lemma 2.2.7 shows that it is sufficient to prove that conjectures 2.2.5 and 2.2.6 hold. These can be checked for W of types B_2, B_3 and G_2 by listing all the elements of W in terms of reduced expressions and using the relations (2.1.4).

Let W be of type A_{n-1} $(n > 1)$. $W = S_n$ by part (i) of the discussion above this theorem. Thus conjecture

2.2.6 follows immediately from theorem A2.7(i) and since for x an indeterminate over C and z in Z_+ the field $Q(x)$ is a finite field extension of $Q(x^z)$ it is clear that conjecture 2.2.5 follows from corollaries A2.6 and 4.2.10(ii).

※

REMARK. For a group W for which conjectures 2.2.5 and 2.2.6 hold the proof of lemma 2.2.7(i) gives a method of calculating the coefficients $\sigma(w,b)$ ($w \in W$, $b \in B \subset Y$). This is illutrated in Appendix 1 for the case of W of type B_3.

LEMMA 2.2.9.

Let $r,s \in R$ and $\eta \in X(H(K,u))$.

(i) If n_{rs} is odd then r and s are conjugate in W.

(ii) If n_{rs} is odd then $\eta(h_r) = \eta(h_s)$.

Note that (ii) is a special case of conjecture 2.2.5.

Proof:

(i) Equations (1.1.2) and(1.1.3) show that

$$(rs\ldots s)_{n_{rs}-1}\, r\, (sr\ldots r)_{n_{rs}-1} = s$$

(ii) Since n_{rs} is odd we have

$$\eta(h_{(rs\ldots r)_{n_{rs}}}) = \eta(h_{(sr\ldots s)_{n_{rs}}})$$

By equations (1.1.13) and (2.1.4)

$$\eta(h_{(sr\ldots r)_{n_{rs}-1}} h_r) = \eta(h_s h_{(sr\ldots r)_{n_{rs}-1}})$$

By equations(1.1.10)and (1.1.13)

$$u^{c_r}\eta(h_{(sr\ldots s)_{n_{rs}-2}}) + (u^{c_r}-1)\eta(h_{(sr\ldots r)_{n_{rs}-1}})$$

$$= u^{c_s}\eta(h_{(rs\ldots r)_{n_{rs}-2}}) + (u^{c_s}-1)\eta(h_{(sr\ldots r)_{n_{rs}-1}})$$

By (i) r and s are conjugate in W hence $c_r = c_s$ and

$$\eta(h_{(rs\ldots r)_{n_{rs}-2}}) = \eta(h_{(sr\ldots s)_{n_{rs}-2}})$$

The result now clearly follows by 'decreasing induction'.

⁂

LEMMA 2.2.10.

Conjecture 2.2.6 is true for $w \in \{ w_\sigma \in W \mid \sigma \in \psi \}$, where W is a Weyl group with root system ψ (see [3, Chapter 2]).

Proof:

Since it is true that in conjecture 2.2.6 (i) implies (ii) for any Coxeter group W we need only prove that (ii) implies (i) in this case.

Let π be a fundamental system in ψ. $\{ w_\rho \mid \rho \in \pi \}$ is a set of distinguished generators of W (considered as a Coxeter group).

Fix w_σ $(\sigma \in \psi)$. Since $w_\sigma = w_{-\sigma}$ we can assume that $\sigma \in \psi^+$ the set of positive roots. Thus

$$\sigma = \sum_{\rho \in \pi} \lambda_\rho \rho \qquad \text{where } \lambda_\rho \in Z_+ \cup \{0\} \text{ for}$$

all $\rho \in \pi$.

There exists $\tau \in \pi$ such that the inner product $(\tau, \sigma) > 0$, otherwise $(\sigma,\sigma) = \sum_{\rho \in \pi} \lambda_\rho (\rho, \sigma) \leq 0$ a contradiction.

If w_σ is not of minimal length in its class then clearly $\sigma \notin \pi$, thus at least two elements of $\{ \lambda_\rho \mid \rho \in \pi \}$ are non-zero. Hence

$$w_\sigma(\tau) = \quad -\frac{2(\sigma,\tau)}{(\sigma,\sigma)}\sigma \in \psi^- = \psi \setminus \psi^+$$

and $$(w_\sigma w_\tau)^{-1} = w_\tau(w_\sigma(\tau))$$
$$= -2\frac{(\sigma,\tau)}{(\sigma,\sigma)}\sigma + \left\{ \frac{4(\sigma,\tau)^2}{(\sigma,\sigma)(\tau,\tau)} - 1 \right\} \quad \in \psi^-$$

By [4, lemma 2.2.1 and theorem 2.2.2]

$$l(w_\tau w_\sigma w_\tau) = l(w_\sigma) - 2$$

Thus there exists $x \in W$ with $l(x) = l(w_\sigma) - 2$ and $w_\sigma = w_\tau x w_\tau$. $l((xw_\tau)w_\tau) < l(w_\sigma)$ so the proof is complete. ⁂

We now give an explicit formula for the coefficients

$\sigma(w,b)$ for certain elements w of $W = S_n$.

DEFINITION 2.2.11.

Let $e,f \in Z$.

$$\binom{e}{f} = \begin{cases} 0 & \text{if } e < 0 \text{ or } f < 0 \\ 0 & \text{if } 0 \leq e \text{ and } e < f \\ 1 & \text{if } 0 \leq e \text{ and } f = 0 \\ e!/(f!(e-f)!) & \text{if } 0 < f \leq e \end{cases}$$

LEMMA 2.2.12.

Let W be of type A_{n-1} $(n > 1)$. By part (i) of the discussion above theorem 2.2.8, $W = S_n$. Corollary A2.6 shows that for any indexing system $\{c_r | r \in R\}$ (see definition 1.1.8) that $c_r = c_s$ for all $r,s \in R$. Let $c_r = c$ for $r \in R$. Let $\eta \in X(H(K,u))$ and $a,s,t \in Z_{n-1}$ with $a > s > t$. Then using cycle notation for elements of S_n we have

(i) $\eta(h_{(a\ s)}) =$

$$\sum_{j=0}^{a-1-s} \binom{a-1-s}{j} u^{cj}(u^c-1)^{a-1-s-j}\eta(h_{(a\ a-1\ a-2\ \ldots\ s+j)})$$

(ii) $\eta(h_{(a\ s\ s-1\ s-2\ \ldots\ t)}) =$

$$\sum_{j=0}^{a-1-s} \binom{a-1-s}{j} u^{cj}(u^c-1)^{a-1-s-j}\eta(h_{(a\ a-1\ a-2\ \ldots\ t+j)})$$

(iii) $\eta(h_{(a\ t\ t+1\ t+2\ \ldots\ s)})$

$$\sum_{j=0}^{a-1-t} \binom{a-1-t}{j} u^{cj}(u^c-1)^{a-1-t-j}\eta(h_{(s+j\ s+j+1\ \ldots\ a)})$$

Note that there exists a set of class representatives $B \subset Y$ for S_n such that $(a\ a-1\ \ldots\ s+j)$, $(a\ a-1\ \ldots\ t+j) \in B$ for all $j \in Z_{a-1-r} \cup \{0\}$ ($r = s$ or t as appropriate). Further, by theorem 2.2.8 (conjecture 2.2.5), $\eta(h_{(s+j\ s+j+1\ \ldots\ a)})$ $= \eta(h_{(a\ a-1\ \ldots\ s+j)})$.

Proof:

Let $u_i = (i\ i+1)$ for $i \in Z_{n-1}$. Note that

$$(a\ s) = u_s u_{s+1} \cdots u_{a-1} u_{a-2} \cdots u_s$$

$$(a\ s\ s-1\ \ldots\ t) = u_s u_{s+1} \cdots u_{a-1} u_{a-2} \cdots u_t$$

$$(a\ t\ t+1\ \ldots\ s) = u_t u_{t+1} \cdots u_{a-1} u_{a-2} \cdots u_s$$

Let $e,f \in Z_+$ with $a > e \geqslant f$. Using equations (A2.2), (A2.3), (A2.4), (1.1.13) and (2.1.4) we see that

$$\eta(h_{u_e u_{e+1} \cdots u_{a-1} u_{a-2} \cdots u_f}) = \eta(h_{u_{e+1} \cdots u_{a-1} u_{a-2} \cdots u_f} h_{u_e})$$

$$= \eta(h_{u_{e+1} \cdots u_{a-1} u_{a-2} \cdots u_e u_{e-1} u_e u_{e-2} \cdots u_f})$$

$$= \eta(h_{u_{e+1} \cdots u_{a-1} u_{a-2} \cdots u_{e-1} u_e u_{e-1} u_{e-2} \cdots u_f})$$

$$= \eta(h_{u_{e-1} u_{e+1} \cdots u_{a-1} u_{a-2} \cdots u_f})$$

$$= \eta(h_{u_{e+1} \cdots u_{a-1} u_{a-2} \cdots u_f} h_{u_{e-1}})$$

$$\cdots\cdots\cdots\cdots$$

$$= \eta(h_{u_{e+1} \cdots u_{a-1} u_{a-2} \cdots u_f} h_{u_f})$$

$$= u^c \eta(h_{u_{e+1} \cdots u_{a-1} u_{a-2} \cdots u_{f+1}})$$

$$+ (u^c - 1)\eta(h_{u_{e+1} \cdots u_{a-1} u_{a-2} \cdots u_f})$$

Using induction on (a-e) we prove that

$$\eta(h_{u_e \cdots u_{a-1} u_{a-2} \cdots u_f}) = \sum_{j=0}^{a-1-e} \binom{a-1-e}{j} u^{cj} (u^c - 1)^{a-1-e-j} \eta(h_{u_{a-1} u_{a-2} \cdots u_{f+j}}) \tag{2.2.13}$$

If $(a-e) = 1$ equation (2.2.13) is clearly valid. Using the inductive hypothesis and the equation derived above, we have

$$\eta(h_{u_e \cdots u_{a-1} u_{a-2} \cdots u_f}) =$$

$$u^c \sum_{i=0}^{a-2-e} \binom{a-2-e}{i} u^{ci} (u^c - 1)^{a-2-e-i} \eta(h_{u_{a-1} u_{a-2} \cdots u_{f+1+i}})$$

$$+ (u^c - 1) \sum_{i=0}^{a-2-e} \binom{a-2-e}{i} u^{ci} (u^c - 1)^{a-2-e-i} \eta(h_{u_{a-1} u_{a-2} \cdots u_{f+i}})$$

Since for $x,y \in Z_+$, $\binom{x}{y} + \binom{x}{y+1} = \binom{x+1}{y+1}$ this shows equation (2.2.13) to be valid.

(i) follows immediately from equation (2.2.13) by putting $e = f = s$.

(ii) follows immediately from equation (2.2.13) by putting $e = s$ and $f = t$.

(iii) can clearly be proved in a way analogous to that used to prove (ii).

※

CHAPTER 3

THE CIRCLE PRODUCT.

In this chapter the basis of an inductive method for calculating the characters of H(K,u) is described (see definition 3.2.5 and corollary3.2.10) and its connection with an analogous method for calculating the charaters of W is displayed in §3.3.

§3.1 GROTHENDIECK GROUPS.

No proofs are given in this section.

Let B be a ring and M a category of finitely generated B-modules with B-homomorphisms as the morphisms. For any finitely generated B-module V let $[V] = \{U \in M | U \cong V\}$.

DEFINITION 3.1.1.

The Grothendieck group $\mathcal{K}(M)$ of the category M is F/S where F is the free Z-module on $\{[V]\}_{V\in M}$ and S is the additive subgroup of F generated by $\{[U] - [V] + [Y] \mid 0 \to U \to V \to Y \to 0$ is a short exact sequence in M$\}$. Denote the element $[V] + S$ by $)V($.

We abbreviate 'short exact sequence' to s.e.s.

LEMMA 3.1.2.

(i) Every element of $\mathcal{K}(M)$ has the form $\sum_{V\in M} z_V)V($ where each $z_V \in Z$ and only a finite number of them are non-zero.

(ii) $)V(=)U(+)Y($ whenever $0 \to U \to V \to Y\ 0$ is a s.e.s. in M. In particular $)V \oplus U(=)V(+)U($.

(iii) $)0(= 0_{\mathcal{K}(M)}$

(iv) $U \cong V$ implies that $)U(=)V($.

※

LEMMA 3.1.3.

Given a map ψ from M to an abelian group L such that $\psi(V) = \psi(U) + \psi(Y)$ whenever $0 \to U \to V \to Y \to 0$ is a s.e.s in M then there exists a unique additive map

$$g : \mathcal{K}(M) \longrightarrow L$$
$$g : \ )V(\longmapsto \psi(V) \quad \text{for all } V \in M.$$

※

NOTATION

${}_B\mathcal{N}$ is the category with the set of all finitely generated left B-modules as objects and for $U,V \in {}_B\mathcal{N}$, $Mor(U,V) = \{ \theta : U \to V \mid \theta \text{ is a B-homomorphism} \}$.

LEMMA 3.1.4. (compare [9, page 130]).

If B is an algebra over some field and $U_1, U_2, \ldots U_t$ is a complete set (up to isomorphism) of irreducible left B-modules, then $\mathcal{K}({}_B\mathcal{N})$ is freely generated as an abelian group by $\{)U_i(\mid i \in Z_t \}$.

※

COROLLARY 3.1.5.

If B is a semi-simple F-algebra (F a field) with character group X then the map $\alpha : \ ({}_B\mathcal{N}) \longrightarrow X$ given by $\alpha :)V(\longmapsto$ character of V for all $V \in {}_B\mathcal{N}$ is an isomorphism of additive groups.

※

§3.2 THE CIRCLE PRODUCT.

NOTATION.

Let $J \subset R$. By lemma A3.2 (W_J, J) is a Coxeter system where W_J is the parabolic subgroup of W corresponding to J (see definition A3.1).

Clearly we can regard the generic Hecke algebra $H(W_J, J, c|_{W_J}, K, u)$ which we abbreviate to H_J as a subalgebra of $H(W,R,c,K,u) = H_R$. We abbreviate ${}_{H_J}\mathcal{N}$ to ${}_J\mathcal{N}$ and $\mathcal{K}({}_{H_J}\mathcal{N})$ to $\mathcal{K}(H_J)$. (see 3.1).

DEFINITION 3.2.1.

The statement $\perp(J_i | i \in S)$ where S is an indexing set and each J_i is a subset of R means that for $i \neq j$, $J_i \cap J_j = \emptyset$ and for all $r \in J_i$ and $t \in J_j$, $tr = rt$.

If $\perp(J_i | i \in S)$ we say that the sets J_i are mutually perpendicular.

LEMMA 3.2.2.

Let J and T be subsets of R.

(i) $\perp(J,T)$ implies that $W_J W_T = W_{J \cup T}$ and for $w \in W_J$ and $v \in W_T$, $l(wv) = l(w) + l(v)$.

(ii) $\perp(J,T)$ implies that $H_J \otimes H_T \cong H_{J \cup T}$ as K-algebras.

Proof:

(i) Trival.

(ii) By part (i) $H_{J \cup T} = H_J H_T$. Define a map $g : H_J \times H_T \longrightarrow H_{J \cup T}$ by $g : (\gamma_J, \gamma_T) \longmapsto \gamma_J \gamma_T$ for all $\gamma_J \in H_J$ and $\gamma_T \in H_T$. g is clearly a balanced map hence there exists a unique K-algebra homomorphism $g' : H_J \otimes_K H_T \longrightarrow H_{J \cup T}$ given by $g' : \gamma_J \otimes \gamma_T \longmapsto \gamma_J \gamma_T$

g' is clearly an epimorphism and since $\dim_K(H_J \otimes_K H_T) = \dim_K(H_{J \cup T}) = |W_J||W_T|$, g' is an isomorphism.

※

COROLLARY 3.2.3.

Let $\perp(J,T)$, $V_J \in {}_J\mathcal{N}$ and $V_T \in {}_T\mathcal{N}$. $V_J \otimes_K V_T$ can be made into an $H_{J \cup T}$-module by defining the following action

$(\gamma_J\gamma_T)(v_J\ v_T) = \gamma_J v_J\ \gamma_T v_T$ for all $\gamma_Y \in H_Y$ and $v_Y \in V_Y$ $(Y = J,T)$.

Proof:

It is well known that $V_J \otimes_K V_T$ is an $H_J \otimes_K H_T$-module with respect to the action $(\gamma_J \otimes \gamma_T)(v_J \otimes v_T) = \gamma_J v_J \otimes \gamma_T v_T$. Thus the result now follows immediately from the proof of lemma 3.2.2(ii).

※

LEMMA 3.2.4.

Let $J \subset R$. H_R is a right H_J-module with respect to right multiplication by elements of H_J.

Let X be a transversal for W_J in W then

$$H_R = \sum_{x \in X}^{+} h_x H_J \quad \text{and each } h_x H_J \text{ is a right } H_J\text{-module.}$$

Proof:

By equation (1.1.13) it is clear that $H_R = \sum^{+}_{d \in D_J^R} h_d H_J$ where D_J^R is the set of special coset representatives for W_J in W (see definition A3.4). Thus it is sufficient to show that if $xW_J = dW_J$ then $h_x H_J = h_d H_J$.

If $xW_J = dW_J$ then $x = dw$ for some $w \in W_J$ and $h_x = h_d h_w$. $h_w H_J = H_J$ since $h_1 \in h_w H_J$ by | , lemma 5.1|. So $h_x H_J = h_d H_J$.

※

Corollary 3.2.3 allows us to make the following definition

DEFINITION 3.2.5.

Let $T \subset R$ and $J_i \subset T$ for $i \in Z_s$ $(s > 1)$. Let $\perp(J_i | i \in Z_s)$. Given a module $V_i \in {}_{J_i}\mathcal{N}$ for each i in Z_s.

(i) $V_1 \circ_T V_2 \circ_T \cdots \circ_T V_s$ is the left H_R-module

$H_R \otimes_{H_{J_1 \cup J_2 \cup \ldots \cup J_s}} (\ldots ((V_1 \otimes_K V_2) \otimes_K V_3) \ldots \otimes_K V_s)$.

(H_R is regarded as a right $H_{J_1 \cup J_2 \cup \ldots \cup J_s}$-module. See lemma 3.2.4).

(ii) If η_i is the character of V_i then $\eta_1 \circ_T \eta_2 \circ_T \ldots \circ_T \eta_s$ is the character of $V_1 \circ_T V_2 \circ_T \ldots \circ_T V_s$.

In cases where no confusion will arise we will abbreviate $V_1 \circ_T \ldots \circ_T V_s$ to $V_1 \circ V_2 \ldots \circ V_s$ and $\eta_1 \circ_T \ldots \circ_T \eta_s$ to $\eta_1 \circ \ldots \circ \eta_s$ when $T = R$.

We have immediately

LEMMA 3.2.6.

In the notation of definition 3.2.5

(i) $\eta_1 \circ \eta_2 \circ \ldots \circ \eta_s$ is the induced character $(\eta_1 . \eta_2 \ldots . \eta_s)^{H_R}$ where $\eta_1 . \eta_2 \ldots . \eta_s$ is the character of $H_{J_1 \cup J_2 \cup \ldots \cup J_s}$ given by

$(\eta_1 \ldots . \eta_s)(\gamma_1 \gamma_2 \ldots \gamma_s) = \eta_1(\gamma_1) \ldots \eta_s(\gamma_s)$ for all $\gamma_i \in H_{J_i}$ $(i \in Z_s)$.

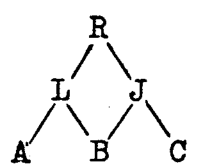
(ii) For any permutation $\sigma \in S_s$

$\eta_{\sigma(1)} \circ \eta_{\sigma(2)} \ldots \circ n_{\sigma(s)} = n_1 \circ n_2 \ldots \circ n_s$ and hence $V_{\sigma(1)} \circ \ldots \circ V_{\sigma(s)} \cong V_1 \circ V_2 \circ \ldots \circ V_s$.

⋇

LEMMA 3.2.7.

If



is a subset lattice diagram for R with $\perp(A,B,C)$, $\perp(L,C)$, $\perp(A,J)$ and if $V_Y \in {}_Y\mathcal{N}$ $(Y = A,B,C)$ then

(i) $(V_A \circ_L V_B) \circ_R V_C \cong V_A \circ V_B \circ V_C \cong V_A \circ_R (V_B \circ_J V_C)$ as left H_R-modules.

(ii) $(\eta_A \circ_L \eta_B) \circ_R \eta_C = \eta_A \circ \eta_B \circ \eta_C = \eta_A \circ_R (\eta_B \circ_J \eta_C)$

where η_Y is the character afforded by V_Y $(Y = A,B,C)$.

Proof:

(ii) follows immediately from (i).

(i) We have $W = D^R_{A\cup B\cup C} W_{A\cup B\cup C} = D^R_{I\cup C} D^L_{A\cup B} W_{A\cup B\cup C}$

(see definition A3.4). Using the length function one finds that $D^R_{A\cup B\cup C} = D^R_{I\cup C} D^L_{A\cup B}$.

By lemma 3.2.4

$$V_A \circ V_B \circ V_C = \sum_{d\in D^R_{A\cup B\cup C}} h_d \otimes ((V_A \otimes V_B) \otimes V_C)$$

$$= \sum_{s\in D^R_{I\cup C}} \sum_{g\in D^L_{A\cup B}} h_s h_g \otimes ((V_A \otimes V_B) \otimes V_C)$$

and

$$(V_A \circ_L V_B) \circ_R V_C = \sum_{s\in D^R_{I\cup C}} h_s \otimes ((\sum_{g\in D^L_{A\cup B}} h_g \otimes (V_A \otimes V_B)) \otimes V_C)$$

$$= \sum_s \sum_g h_s \otimes ((h_g \otimes (V_A \otimes V_B) \otimes V_C)$$

Clearly the map $\alpha : (V_A \circ_L V_B) \circ_R V_C \longrightarrow V_A \circ V_B \circ V_C$ given by $\alpha : h_s \otimes ((h_g \otimes (v_A \otimes v_B)) \otimes v_C \longmapsto h_s h_g \otimes ((v_A \otimes v_B) \otimes v_C)$ for all $v_Y \in V_Y$ $(Y = A,B,C)$ is a K-isomorphism and also an H_R-map. Hence it is an H_R-isomorhpism.

It is well known that the map $\sigma : (V_A \otimes V_B) \otimes V_C \longrightarrow V_A \otimes (V_B \otimes V_C)$ given by $\sigma : (v_A \otimes v_B) \otimes v_C \longmapsto v_A \otimes (v_B \otimes v_C)$ is an H_R-isomorphism hence the proof is complete.

※

THEOREM 3.2.8.

If $\perp(J_i \mid i \in Z_t)$ $(t > 1)$ where for each i in Z_t, $J_i \subset R$ then there exists a multi-Z-linear additive group monomorphism

$$\theta^R_{J_1 J_2 \ldots . J_t} : \mathcal{K}(H_{J_1}) \otimes_Z \ldots . \otimes_Z \mathcal{K}(H_{J_t}) \longrightarrow \mathcal{K}(H_R)$$

given by $\theta^R_{J_1 \ldots . J_t} :)V_1(\otimes \ldots . . \otimes)V_t(\longmapsto)V_1 \circ \ldots . \circ V_t($

for all $V_i \in {}_{J_i}\mathcal{K}$ $(i \in Z_t)$.

Proof:

STEP 1: Let $J,T \subset R$ with $\perp(J,T)$. It is sufficient to show that there exists a biadditive group monomorphism $\theta^R_{JT} : \mathcal{K}(H_J) \otimes_Z \mathcal{K}(H_T) \longrightarrow \mathcal{K}(H_R)$ given by $\theta^R_{JT} :)V_J(\otimes)V_T(\longmapsto)V_J \circ V_T($ for all $V_J \in {}_J\mathcal{N}$ and $V_T \in {}_T\mathcal{N}$.

Proof:

Immediate from lemma 3.2.7(i) and lemma 3.1.2(iv)

STEP 2: For each V_J in ${}_J\mathcal{N}$ there exists an additive group homomorphism $b_{V_J} : \mathcal{K}(H_T) \longrightarrow \mathcal{K}(H_R)$ given by $b_{V_J} :)V_T(\longmapsto)V_J \circ V_T($ for all $V_T \in {}_T\mathcal{N}$.

Proof:

Let $V_J \in {}_J\mathcal{N}$. Define a map $\alpha : {}_T\mathcal{N} \longrightarrow \mathcal{K}(H_R)$ by $\alpha(V_T) =)V_J \circ V_T($ for all $V_T \in {}_T\mathcal{N}$. By lemma 3.1.3 it is sufficient to show that if $0 \to U \to V \to Y \to 0$ is a s.e.s. in ${}_T\mathcal{N}$ then $\alpha(U) + \alpha(Y) = \alpha(V)$.

We show that there exists a s.e.s.

(3.2.9) $$0 \to V_J \circ U \to V_J \circ V \to V_J \circ Y \to 0$$

Using lemma 3.2.4 it can readily be shown that if $\gamma : U \to V$ and $\delta : V \to Y$ are the maps in the s.e.s. $0 \to U \to V \to Y \to 0$ then the maps

$$1_{H_R} \otimes (1_{V_J} \otimes \gamma) : V_J \circ U \longrightarrow V_J \circ V$$

$$1_{H_R} \otimes (1_{V_J} \otimes \delta) : V_J \circ V \longrightarrow V_J \circ Y$$

make (3.2.9) a s.e.s. in ${}_R\mathcal{N}$. Thus $\alpha(U) + \alpha(Y) = \alpha(V)$ as required.

STEP 3: There exists a biadditive group homomorphism $\theta^R_{JT} : \mathcal{K}(H_J) \otimes_Z \mathcal{K}(H_T) \longrightarrow \mathcal{K}(H_R)$ given by $\theta^R_{JT} :)V_J(\otimes)V_T(\longmapsto)V_J \circ V_T($ for all $V_J \in {}_J\mathcal{N}$ and $V_T \in {}_T\mathcal{N}$.

Proof:

By analogy with step 2, for each $V_T \in {}_T\mathcal{N}$

there exists an additive homomorphism $g_{V_T} : \mathcal{K}(H_J) \longrightarrow \mathcal{K}(H_R)$ given by $g_{V_T} :)V_J(\longmapsto)V_J \circ V_T($ for all $V_J \in {}_J\mathcal{N}$. Thus there exists a balanced map $b.g : \mathcal{K}(H_J) \times \mathcal{K}(H_T) \longrightarrow \mathcal{K}(H_R)$ given by $b.g : ()V_J(,)V_T() \longmapsto)V_J \circ V_T($ for all $V_J \in {}_J\mathcal{N}$ and $V_T \in {}_T\mathcal{N}$.

STEP 4: θ^R_{JT} is a monomorphism.

Proof:

Let $\{U_s \mid s \in S\}$ be a full set of irreducible H_J-modules.

Let $\{Y_t \mid t \in L\}$ be a full set of irreducible H_T-modules.

By corollary 3.2.3 it is clear that $\{U_s \otimes_K Y_t \mid (s,t) \in S \times L\}$ is a full set of irreducible $H_{J \cup T}$-modules.

Lemma 3.1.4 shows that $\mathcal{K}(H_J) \otimes \mathcal{K}(H_T)$ has Z-basis $\{)U_s(\otimes)Y_t(\mid (s,t) \in S \times L\}$. Suppose that

$$\theta^R_{JT}\left(\sum_{(s,t) \in S \times T} z_{st} ()U_s(\otimes)V_t()\right) = 0 \qquad \text{for some } z_{st} \in Z$$

then

$$\sum_{(s,t)} z_{st})U_s \circ V_t(= 0$$

Let U_s afford the character β_s and let Y_t afford the character λ_t then by lemma 3.2.6(i)

$$\sum_{(s,t)} z_{st} (\beta_s \lambda_t)^{H_R} = 0 \quad \text{and so} \quad \sum_{(s,t)} z_{st} (\beta_s \lambda_t) = 0.$$

Since $\{U_s \otimes Y_t \mid (s,t) \in S \times L\}$ is a full set of irreducible $H_{J \cup T}$-modules the characters $\beta_s \lambda_t$ ($s \in S$, $t \in L$) are linearly independent over K. Thus $z_{st} = 0$ for all $(s,t) \in S \times L$, and θ^R_{JT} is a monomorphism.

※

COROLLARY 3.2.10.

If $\perp(J_i \mid i \in Z_t)$ $(t > 1)$ where for each i in Z_t $J_i \subset R$ then there exists a multi-Z-linear additive group monomorphism

$$\underline{\theta}^R_{J_1 J_2 \ldots J_t}: X(H_{J_1}) \otimes_Z \ldots \otimes_Z X(H_{J_t}) \longrightarrow X(H_R)$$

given by

$$\underline{\theta}^R_{J_1 J_2 \ldots J_t}: \eta_1 \ldots \eta_t \longmapsto \eta_1 \circ \ldots \circ \eta_t \qquad \text{for}$$

all $\eta_i \in X(H_{J_i})$ $(i \in Z_t)$.

Proof:

Let $\rho_i: \mathcal{X}(H_{J_i}) \longrightarrow X(H_{J_i})$ $(i \in Z_t)$

and $\rho: (H_R) \longrightarrow X(H_R)$

be the isomorphisms described by corollary 3.1.5.

$$\underline{\theta}^R_{J_1 J_2 \ldots J_t} = \rho \cdot \theta^R_{J_1 J_2 \ldots J_t} \cdot (\rho_1 \otimes \ldots \otimes \rho_t)^{-1}$$

§3.3 CIRCLE PRODUCT AND THE COXETER GROUP.

NOTATION.

Let $J \subset R$. Abbreviate $\mathcal{X}({}_{CW_J}\mathcal{N})$ to $\mathcal{X}(W_J)$ and $X(CW_J)$ to $X(W_J)$.

Let $\rho_J: \mathcal{X}(H_J) \longrightarrow X(H_J)$

and $\lambda_J: \mathcal{X}(W_J) \longrightarrow X(W_J)$ be the isomorphisms described in corollary 3.1.5.

LEMMA 3.3.1.

(i) The map $\delta_J: X(H_J) \longrightarrow X(W_J)$ defined by $(\delta_J(\eta))(f_1(\delta)) = f_1(\eta(\delta))$ for all $\eta \in X(H_J)$ and $\delta \in H(K,u)$ is an additive group isomorphism.

(ii) The map $\sigma_J: \mathcal{X}(H_J) \longrightarrow \mathcal{X}(W_J)$ where $\sigma_J = \lambda_J^{-1} \cdot \delta_J \cdot \rho_J$ is an additive group isomorphism.

Proof:

(i) Immediate from theorem 1.4.5(ii)

(ii) Immediate from (i).

DEFINITION 3.3.2.

Let $\perp(J_i \mid i \in Z_t)$ $(t > 1)$ where for each i in Z_t, $J_i \subset R$.

(i) $\psi^R_{J_1 J_2 \cdots J_t} : \mathcal{X}(W_{J_1}) \otimes_Z \cdots \otimes_Z \mathcal{X}(W_{J_t}) \longrightarrow \mathcal{X}(W_R)$

is defined by

$$\psi^R_{J_1 \cdots J_t} = \sigma_R \cdot \theta^R_{J_1 \cdots J_t} \cdot (\sigma_{J_1} \cdots \cdot \sigma_{J_t})^{-1}$$

Let $\psi^R_{J_1 \cdots J_t}()U_1(\otimes \cdots \otimes)U_t() =)U_1 \circ \cdots \circ U_t($ for all $U_i \in {}_{CW_{J_i}}\mathcal{X}$ $(i \in Z_t)$.

(ii) The map

$$\underline{\psi}^R_{J_1 J_2 \cdots J_t} : X(W_{J_1}) \otimes_Z \cdots \otimes_Z X(W_{J_t}) \longrightarrow X(W_R)$$

is defined by

$$\underline{\psi}^R_{J_1 \cdots J_t} = \delta_R \cdot \underline{\theta}^R_{J_1 \cdots J_t} \cdot (\delta_{J_1} \otimes \cdots \otimes \delta_{J_t})^{-1}$$

Let $\underline{\psi}^R_{J_1 \cdots J_t}(X_1 \otimes \cdots \otimes X_t) = X_1 \circ \cdots \circ X_t$ for all $X_i \in X(W_{J_i})$ $(i \in Z_t)$.

From theorem 3.2.8 we have immediately

LEMMA 3.3.3.

(i) $\psi^R_{J_1 \cdots J_t}$ and $\underline{\psi}^R_{J_1 \cdots J_t}$ are multi-Z-linear additive group monomorphisms.

(ii) Let $\{\eta^i \mid i \in Z_{m_W}\}$ be the full set of irreducible characters of H_R, so that $\{X^i \mid i \in Z_{m_W}\}$ is the set of all irreducible characters of CW, where $X^i = \delta_R(\eta^i)$ for all i in Z_{m_W}. (m_W is the number of conjugacy classes in W).

If $\eta_j \in X(H_{J_j})$, $X_j = \delta_{J_j}(\eta_j)$ for all $i \in Z_t$ and $\eta_1 \circ \cdots \circ \eta_t = \sum_{i \in Z_{m_W}} z_i \eta^i$ $(z_i \in Z)$ then

$$X_1 \circ \ldots \circ X_t = \sum_{i \in Z_{m_W}} z_i X^i$$

※

THEOREM 3.3.4.

Let $\perp(J_i \mid i \in Z_t)$ $(t > 1)$ where for each i in Z_t, $J_i \subset R$.

For each $i \in Z_t$ let $U_i \in {}_{CW_{J_i}})($ and $X_i \in X(W_{J_i})$. Then

(i) $)U_1 \circ \ldots \circ U_t(=)CW \otimes_{CW_{J_1 \cup \ldots \cup J_t}} (\ldots (U_1 \otimes_C U_2) \ldots \otimes_C U_t)($

where CW is regarded as a right $CW_{J_1 \cup \ldots \cup J_t}$-module, the action being given by right multiplication.

(ii) $X_1 \circ \ldots \circ X_t = (X_1 . X_2 \ldots . X_t)^W$ where $X_1 . X_2 \ldots . X_t$ is the character of $W_{J_1 \cup \ldots \cup J_t}$ given by

$$(X_1 . X_2 \ldots . X_t)(w_1 w_2 \ldots . w_t) = X_1(w_1) . X_2(w_2) \ldots . X_t(w_t)$$

for all $w_i \in W_{J_i}$ $(i \in Z_t)$

(see definition 3.3.2 and lemma 3.2.2).

Proof:

(i) By lemma 3.1.2(iv) and the proof of corollary 3.2.10 this follows from (ii).

(ii) Lemma 3.2.7 shows that it is sufficient to prove the following two 'statements':

STATEMENT 1. Let $J, T \subset R$ with $\perp(J, T)$ and let $X_J \in X(W_J)$, $X_T \in X(W_T)$ then

$$(X_J \circ X_T) = (X_J . X_T)^W$$

STATEMENT 2. Let $J_i \subset R$ and $X_{J_i} \in X(W_{J_i})$ for i = 1,2 and 3. Let $\perp(J_1, J_2, J_3)$ then

$$(X_{J_1} . X_{J_2} . X_{J_3})^W = ((X_{J_1} . X_{J_2})^{W_{J_1 \cup J_2}} . X_{J_3})^W$$

This latter statement can readily be proved using the well known formula for induced group characters.

Proof of statement 1:

Let $\delta_J^{-1}(X_J) = \eta_J$ and $\delta_T^{-1}(X_T) = \eta_T$, so

$$X_J \circ X_T = \Psi_{JT}^R(X_J \otimes X_T)$$
$$= (\delta_R \cdot \theta_{JT}^R \ (\delta_J \otimes \delta_T)^{-1})(X_J \otimes X_T)$$
$$= \delta_R \cdot \theta_{JT}^R(\eta_J \otimes \eta_T)$$
$$= \delta_R(\eta_J \circ \eta_T)$$
$$= \delta_R((\eta_J \cdot \eta_T)^{H_R})$$

Clearly $\delta_{J \cup T}(\eta_J \cdot \eta_T) = X_J \cdot X_T$, thus it is sufficient to prove that if $S \subset R$ and $\beta \in X(H_S)$ then

$$\delta_R(\beta^{H_R}) = (\delta_S(\beta))^{W_R} \tag{3.3.5}$$

Let $h_w h_d = h_{d'(w)} \gamma_{w,d}$ where $w \in W$, $d, d'(w) \in D_S^R$ (see definition A3.4) and $\gamma_{w,d} \in H_S$.

Clearly $f_1(\gamma_{w,d})$ is defined and belongs to CW_S. So $f_1(h_w)f_1(h_d) = f_1(h_{d'(w)})f_1(\gamma_{w,d})$ which can be rewritten as $wd = d'(w)f_1(\gamma_{w,d})$. Thus by lemmas 3.2.4 and 3.3.1

$$(\delta_R(\beta^{H_R}))(w) = f_1(\beta^{H_R}(h_w))$$
$$= f_1\Big(\sum_{d \in D_S^R} \xi_{d,d'(w)} \beta(\gamma_{w,d})\Big)$$

where $\xi_{d,d'(w)} = \begin{cases} 1 & \text{if } d = d'(w) \\ 0 & \text{if } d \neq d'(w) \end{cases}$

$$= \sum_d \xi_{d,d'(w)} f_1(\beta(\gamma_{w,d}))$$
$$= \sum_d \xi_{d,d'(w)} (\delta_S(\beta))(f_1(\gamma_{w,d}))$$
$$= (\delta_S(\beta))^{W_R}(w)$$

which proves equation (3.3.5).

⋇

Clearly analogues of the results (3.3.1),(3.3.3) and (3.3.4) can be found with CW replaced by $E_C(q)$ for each $q \in P$ (see definition 1.2.1), but as these would be incidental to our study of the characters of H_R we omit them.

PART II

INTRODUCTION.

For the remaining chapters of this thesis we restrict our attention to a special case, namely that where W is a Weyl group of type A_{n-1} $(n > 1)$ and $c : W \longrightarrow Z_+$ (see corollary 1.1.7) coincides with the length function i.e. $c_r = 1$ for all $r \in R$. Denote the generic Hecke algebra by G_n in this case.

By part (i) of the discussion above theorem 2.2.8, we can identify W with the symmetric group S_n on n symbols and R with $\{\mu_i = (i\ i{+}1) \mid i \in Z_{n-1}\}$. (A2.2),(A2.3) and (A2.4) are a set of defining relations for this set R of generators.

We take G_n to be the K-algebra with K-basis $\{g_w \mid w \in S_n\}$. Thus denoting $g_{1_{S_n}}$ by g_o and g_{μ_i} by g_i we have that $g_w = g_{i_1} g_{i_2} \cdots g_{i_t}$ where $\mu_{i_1}\mu_{i_2}\cdots\mu_{i_t}$ is any reduced expression for w $(w \in S_n)$ and that G_n is generated as K-algebra with identity g_o by $\{g_i \mid i \in Z_{n-1}\}$ with defining relations

$$g_i^2 = ug_o + (u-1)g_i \qquad \text{for all } i \in Z_{n-1}$$

$$g_i g_j = g_j g_i \qquad \text{for all } i,j \in Z_{n-1} \text{ with } i+1 < j$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad \text{for all } i \in Z_{n-2}$$

Since the conjugacy classes of S_n can be indexed by the set of all partitions of n (see appendix 2 and above lemma 4.1.1) we can use the notation $\{X^\alpha \mid \alpha \vdash n\}$ for the set of all irreducible characters of S_n over C. Further by theorem 1.4.5(ii) we can denote the set of all irreducible characters of G_n over K by $\{\eta^\alpha \mid \alpha \vdash n\}$ and stipulate that for any $w \in S_n$, that $X^\alpha(w) = f_1(\eta^\alpha(g_w))$.

CHAPTER 4

INDUCTION FORMULAE

In this chapter 'part 2' (see below corollary 2.1.5) of the problem of evaluating the characters of the generic Hecke algebra G_n is solved (see theorems 4.2.8 and 4.3.8 and the discussion above definition 4.2.1) at least in a theoretical sense. (Chapter 5 gives a more practical method for calculating character values.)

§4.1 IRREDUCIBLE CHARACTERS.

NOTATION.

Let $n \in Z_+$. A partition γ of n (denoted $\gamma \vdash n$) is a finite sequence $<\gamma_1, \gamma_2, \ldots, \gamma_t>$ of non-negative integers such that

$$\sum_{i=0}^{t} \gamma_i = n \quad \text{and } 0 \leqslant \gamma_t \leqslant \gamma_{t-1} \leqslant \ldots \leqslant \gamma_1$$

We also use the alternative notation $\gamma = < 1^{a_1} 2^{a_2} \ldots n^{a_n}$ where $a_j = |\{i \mid i \in Z_t, \gamma_i = j\}|$. $\gamma_1, \ldots, \gamma_t$ are the parts of

We can assume $t = n$ if we wish since some of the γ_i can be equal to zero.

With each partition λ of n we associate a parabolic subgroup (see definition A3.1) S_n^λ of S_n as follows:

Let $\lambda = <\lambda_1, \lambda_2, \ldots, \lambda_r>$ with $0 < \lambda_r \leqslant \ldots \leqslant \lambda_1$.

Put $\lambda^t = \sum_{i=1}^{t} \lambda_i$ for $t \in Z_r$ and $J_\lambda = \{\mu_j = (j\ j{+}1) \mid j \in Z_{n-1},$ $j \neq \lambda^1, \lambda^2, \ldots, \lambda^{r-1}\} \subset R$

Then $S_n^\lambda = (S_n)_{J_\lambda}$ (see definition A3.1).

Also we denote the subalgebra $(G_n)_{J_\lambda}$ of G_n by G_n^λ.

(see above definition 3.2.1).

LEMMA 4.1.1.

With λ and notation as above; Let $J_t = \{\mu_j \mid \lambda^{t-1} < j < \lambda^t\}$ $(t \in Z_r)$ where convensionally $\lambda^0 = 0$. Let T_t be the parabolic subgroup $(S_n)_{J_t}$ and L_t be the subalgebra $(G_n)_{J_t}$ of G_n. Then for t in Z_r

(i) T_t is the subgroup of S_n consisting of all those elements fixing all the symbols other than $\{\lambda^{t-1}+1, \lambda^{t-1}+2, \ldots, \lambda^t\}$.

(ii) $T_t \cong S_{\lambda_t}$

(iii) $L_t \cong G_{\lambda_t}$ and the Coxeter group of L_t is T_t

(iv) $J = J_1 \cup \ldots \cup J_r$ and $\perp(J_1, \ldots, J_r)$

(v) $S_n = T_1 T_2 \ldots T_r$

(vi) $G_n = L_1 L_2 \ldots L_r$

Proof:

Clearly the image of the monomorphism $\pi : T_t \longrightarrow S_n$ defined by $\pi : \mu_j \longmapsto \mu_{j-\lambda^{t-1}}$ for all $j \in J_t$ is isomorphic to S_{λ_t}. Thus π induces an isomorphism $\pi' : T_t \longrightarrow S_{\lambda_t}$. The K-map $\pi'' : L_t \longrightarrow G_{\lambda_t}$ given by $\pi''(g_w) = g_{\pi'(w)}$ for all $w \in T_t$ is clearly a K-isomorphism. This proves (ii) and (iii). Parts (i),(iv),(v) and (vi) are readily seen to be true.

⋇

LEMMA 4.1.2.

(i) The map K-linear map $\mathcal{J}_n : G_n \longrightarrow K$ defined by $\mathcal{J}_n(g_w) = u^{l(w)}$ for all w in S_n is a character of G_n.

(ii) Under the bijection from $X(G_n)$ to $X(CS_n)$ described in theorem 1.4.5, $\mathcal{J}_n$ maps to 1_n the unit character of S_n.

(iii) Let $\lambda \vdash n$ be the partition in lemma 4.1.1. Denote the restriction of $\mathfrak{I}_n$ to G_n^λ by $\mathfrak{I}_n^\lambda$ and the restriction of 1_n to S_n^λ by 1_n^λ. Identify the character group $X(L_t)$ with $X(G_{\lambda_t})$ and the character group $X(T_t)$ with $X(S_{\lambda_t})$ for all t in Z_r (see lemma 4.1.1). Then

$$(\mathfrak{I}_n^\lambda)^{G_n} = \mathfrak{I}_{\lambda_1} \circ \mathfrak{I}_{\lambda_2} \circ \ldots \circ \mathfrak{I}_{\lambda_r}$$

and $$(1_n^\lambda)^{S_n} = 1_{\lambda_1} \circ 1_{\lambda_2} \circ \ldots \circ 1_{\lambda_r}.$$

Proof:

(i) That $\mathfrak{I}_n$ is a representation of G_n is clear from the defining relations given for G_n in the introduction to part II. Since $\mathfrak{I}_n(g_o) = 1$, $\mathfrak{I}_n$ is also a character.

(ii) Immediate.

(iii) Immediate from lemma 3.2.6(ii) and theorem 3.3.4(ii).

※

DEFINITION 4.1.3.

$\mathfrak{I}_n$ defined in the above lemma is the <u>unit character of G_n</u>.

NOTATION.

Let $\gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_n \rangle \vdash n$ and $w \in S_n$. Denote by γ_w the sequence whose terms are the elements of the following set arranged in decreasing order:

$\{ \gamma_1 + w(1) - 1, \gamma_2 + w(2) - 2, \ldots, \gamma_n + w(n) - n \}$.

Note; γ_w may or may not be a partition.

From [7, chapter 5] one can readily derive

THEOREM 4.1.4. (Frobenius).

$\{ X^\gamma = \sum_{w \in S_n} \text{sign}(w)(1_n^{\gamma_w})^{S_n} \mid \gamma \vdash n \}$ where $1_n^{\gamma_w} = 0$ if

δ_w is not a partition, is the set of all irreducible characters of S_n over C.
(sign(w) is +1 if w is an even permutation and is -1 if w is an odd permutation).

※

COROLLARY 4.1.5.

$$\{ \eta^{\delta} = \sum_{w \in S_n} \text{sign}(w) (\xi_n^{\delta_w})^{G_n} \mid \delta \vdash n\} \quad \text{where } \xi_n^{\delta_w} = 0$$

if δ_w is not a partition, is the set of all irreducible characters of G_n over K.

Proof:

Immediate from theorem 4.1.4, lemma 4.1.2(iii) and lemma 3.3.3(ii).

※

§ 4.2 $\alpha \circ \eta$

Fix n in Z_+ and t in Z_{n-1}. Assume that $t \geqslant n - t$ and let $\lambda = < t,n-t>$.

Put $J_1 = \{\mu_1, \mu_2, \ldots, \mu_{t-1}\}$ and $J_2 = \{\mu_{t+1}, \ldots, \mu_{n-1}\}$.

Set $T_i = (S_n)_{J_i}$ and $L_i = (G_n)_{J_i}$ for i = 1,2.

Lemma 4.1.1 shows that $S_n = T_1 T_2$, $G_n = L_1 L_2$, $T_1 \cong S_t$, $T_2 \cong S_{n-t}$, $L_1 \cong G_t$ and $L_2 \cong G_{n-t}$.

We identify $X(L_1)$ with $X(G_t)$ and $X(L_2)$ with $X(G_{n-t})$, thus if $\alpha \in X(G_t)$ and $\eta \in X(G_{n-t})$ we can form the character $\alpha \circ \eta$ of G_n.(see definition 3.2.5(ii)).

By lemmas 4.1.2(iii), 3.2.7(ii) and corollary 4.1.5 it is clear that in theory at least, we can evaluate all of the irreducible characters of G_n once we can evaluate all products of the form $\alpha \circ \eta$. Accordingly the evaluation of $(\alpha \circ \eta)(g_w)$ for certain w in S_n is the subject of the rest of this chapter (see theorems 4.2.8,4.3.8 and corollary

4.2.10). The results obtained together with the remark below theorem 2.2.8 enable $(\alpha \circ \eta)(g_w)$ to be evaluated for all w in S_n, α in $X(G_t)$ and η in $X(G_{n-t})$.

DEFINITION 4.2.1.

Let $m \in Z_+$.

(i) Given $\delta \vdash m$, the <u>subclass</u> $(\underline{\delta})$ of S_m is the subset of elements w of (δ) (the conjugacy class of S_m associated with δ. See appendix 2) such that if $w = c_1 c_2 \ldots c_r$ is the decomposition of w into disjoint cycles then each c_i $(i \in Z_r)$ is of the form

$$(j\ j-1\ j-2\ \ldots\ )\quad \text{for some } j \in Z_m.$$

(ii) A function $f: G_m \to K$ satisfying $f(g_w) = f(g_v)$ whenever w and v are in the same subclass of S_m is called a <u>subclass function on G_m</u>.

NOTATION.

Let $m \in Z_+$. Denote the subclass $(\underline{<1^{m-i}\ i^1>})$ of S_m by $\underline{i}$. Convensionally let $\underline{0} = (\underline{<1^m>})$

Given a subclass function $f: G_m \to K$ and $\delta \vdash m$, for any $w \in (\underline{\delta})$ <u>denote $f(g_w)$ by f_δ.</u>

Denote $f_{<1^{m-i}\ i^1>}$ by f_i .($f_0 = f(g_0)$).

LEMMA 4.2.2.

Let $m \in Z_+$, $\delta \vdash m$ and $w \in (\delta) \subset S_m$. Let w be of minimal length in its conjugacy class and $f: G_m \to K$ be a subclass function for which

$$f(g_x g_y) = f(g_y g_x) \quad \text{for all } x,y \in S_m$$

then $f(g_w) = f_\delta$. (Note that $\{w \in (\delta) \mid w \text{ is of minimal length}\}$ contains $(\underline{\delta})$ but is not in general equal to it

Proof: Let $z \in S_m$. If $z = xy$ $(x,y \in S_m)$ with $l(x) + l(y) =$

$l(z)$ and $l(yx) = l(y) + l(x)$ then we call yx a rotamer of z. Since $yx = x^{-1}xyx$, all rotamers of z are conjugates of z. $g_z = g_x g_y$ and $g_{yx} = g_y g_x$ thus if z' is any rotamer of z then $f(g_z) = f(g_{z'})$.

Let $w = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_r}$ $(r = l(w))$ be a reduced expression for w. Theorem A2.7(i) shows that all the elements of $\{i_1, i_2, \ldots, i_r\}$ are distinct. Let $\{j_1, \ldots, j_r\} = \{i_1, \ldots, i_r\}$ with $j_1 > j_2 > \ldots > j_r$ and let $w' = \mu_{j_1}\mu_{j_2}\cdots\mu_{j_r}$. Since $w' \in (\underline{\gamma})$ it is sufficient to prove that w' is a rotamer of w.

We use induction on $l(w)$. By relation (A2.3) it is clear that w has a reduced expression in terms of $\{\mu_j \mid j \in \{i_1, i_2, \ldots, i_r\}\}$ either beginning or ending with μ_{j_1}. Thus there exists $\sigma \in S_r$ with $\sigma(1) = 1$ such that

$$\mu_{j_1}\mu_{j_{\sigma(2)}}\cdots\mu_{j_{\sigma(r)}} = \mu_{j_1}v \quad \text{say}$$

is a rotamer of w. By the inductive hypothesis $\mu_{j_2}\mu_{j_3}\cdots\mu_{j_r}$ is a rotamer of v and as μ_{j_1} commutes with every element of the set $\{\mu_{j_3}, \mu_{j_4}, \ldots, \mu_{j_r}\}$ it is clear that $\mu_{j_1}\cdots\mu_{j_r}$ is a rotamer of w.

⁂

We see from theorem A3.8 that the elements of the special transversal $D^R_{J_1 \cup J_2}$ of S_n^λ in S_n can be indexed by the set of subsets of Z_n of order t:

$$D^R_{J_1 \cup J_2} = \{w_A \mid A \subset Z_n, |A| = t\} \quad \text{where if}$$

$A = \{a_1, a_2, \ldots, a_t\}$ with $a_1 < a_2 < \ldots < a_t$ and $Z_n \setminus A = \{b_1, \ldots, b_{n-t}\}$ with $b_1 < \ldots < b_{n-t}$ then

$$w_A = \begin{pmatrix} 1 & 2 & \cdots & t & t+1 & \cdots & n-t \\ a_1 & a_2 & \cdots & a_t & b_1 & \cdots & b_{n-t} \end{pmatrix}$$

Abbreviate g_{w_A} to g_A.

Using lemmas A2.2, A3.5 and corollary A3.6 we can readily prove

LEMMA 4.2.3.

Given $j \in Z_{n-1}$ and $A \subset Z_n$ with $|A| = t$

(i) if $j, j+1 \in A$ then

$$g_j g_A = g_A g_{w_A^{-1}(j)} \qquad \text{and } w_A^{-1}(j) \in T_1$$

(see above definition 4.2.1)

(ii) if $j, j+1 \in Z_n \setminus A$ then

$$g_j g_A = g_A g_{w_A^{-1}(j)} \qquad \text{and } w_A^{-1}(j) \in T_2$$

(iii) if $j \in A$ and $j+1 \in Z_n \setminus A$ then

$$g_j g_A = g_{(\mu_j w_A)} \qquad \text{and } \mu_j w_A \in D^R_{J_1 \cup J_2}$$

(iv) if $j \in Z_n \setminus A$ and $j+1 \in A$ then

$$g_j g_A = u g_{(\mu_j w_A)} + (u-1) g_A \quad \text{and } \mu_j w_A \in D^R_{J_1 \cup J_2}$$

※

COROLLARY 4.2.4.

Let $i \in Z_n \setminus \{1\}$ and $s \in Z_{n-i} \cup \{0\}$. Put $x = (i+s \;\, i-1+s \ldots \ldots 1+s) = \mu_{i-1+s}\mu_{i-2+s} \cdots \mu_{1+s} \in \underline{i} \subset S_n$ (see below definition 4.2.1). Let $A \subset Z_n$ with $|A| = t$, then

$$g_x g_A \in G_n \setminus g_A G_n \quad \text{unless}$$

either (i) $1+s, 2+s, \ldots, i+s \in A$ when

$$g_x g_A = g_A g_v \qquad \text{where } v \in \underline{i} \subset T_1$$

or (ii) $1+s, 2+s, \ldots, i+s \in Z_n \setminus A$ when

$$g_x g_A = g_A g_w \qquad \text{where } w \in \underline{i} \subset T_2$$

or (iii) There exists $r \in Z_{i-1}$ with $1+s, 2+s, \ldots, r+s \in Z_n \setminus A$ and $r+1+s, r+2+s, \ldots, i+s \in A$ when

$$g_x g_A = (u-1) g_A g_v g_w + q \qquad \text{where } v \in \underline{i-r} \subset T_1,$$

$w \in \underline{r} \subset T_2$ and $q \in G_n \;\; g_A G_n$

Proof:

Put $x = x_i$. We use induction on i. The case i = 2 follows immediately from lemma 4.2.2. Note that if p,p+1 $\in A$ then $w_A^{-1}(p+1) = w_A^{-1}(p) + 1$ and $w_A^{-1}(p)$, $w_A^{-1}(p+1) \in Z_t$. Similarly if $z, z+1 \in Z_n \setminus A$ then $w_A^{-1}(z+1) = w_A^{-1}(z) + 1$ and $w_A^{-1}(z)$, $w_A^{-1}(z+1) \in Z_n \setminus Z_t$. These facts together with the equation $g_{x_i} = g_{u_{i-1+s}} g_{x_{i-1}}$ $(i > 2)$ and lemma 4.2.3 are clearly sufficient to complete the proof.

※

THEOREM 4.2.5.

Let $i \in Z_n \setminus \{1\}$. If $\alpha \in X(G_t)$ and $\eta \in X(G_{n-t})$ are subclass functions(see definition 4.2.1) and if $x, y \in \underline{i} \subset S_n$ then

(i) $(\alpha \circ \eta)(g_x) = (\alpha \circ \eta)(g_y)$

(ii)

$$(\alpha \circ \eta)(g_x) = \binom{n-i}{t-i}\alpha_i\eta_1 + (u-1)\sum_{j=1}^{i-1}\binom{n-i}{t-j}\alpha_j\eta_{i-j} + \binom{n-i}{t}\alpha_1\eta_i \tag{4.2.6}$$

(see definition 2.2.11 and the notation below definition 4.2.1)

Note that $\alpha_1 = \alpha(g_o)$ and $\eta_1 = \eta(g_o)$.

Proof:

Let $A, B \subset Z_n$ with $|A| = |B| = t$. By lemma 3.2.4 there exist unique elements $\sigma_{x,B,A}$ in G_n such that

$$g_x g_A = \sum_{\substack{B \\ B \subset Z_n, |B|=t}} g_B \sigma_{x,B,A} \tag{4.2.7}$$

By lemmas 3.2.4 and 3.2.6(i)

$$(\alpha \circ \eta)(g_x) = \sum_{\substack{A \\ A \subset Z_n, |A|=t}} (\alpha . \eta)(\sigma_{x,A,A})$$

Since $x \in \underline{i} \subset S_n$, there exists $s \in Z_{n-i} \cup \{0\}$ with $x = (i+s \;\; i-1+s \; \ldots \; 1+s)$. Using corollary 4.2.4 to evaluate the algebra elements $\sigma_{x,A,A}$ and counting the number of sets A satisfying the various conditions it is clear that formula (4.2.6) is correct. Thus (ii) is proved.

(i) follows from (ii) since the right hand side of (4.2.6) is independent of x.

There is a sense in which we can 'multiply' together expressions of the type on the right hand side of equation (4.2.6) (see theorem 4.2.8(ii) and the example below theorem 4.2.12) to obtain a formula for $(\alpha\circ\eta)_\gamma$ where $\gamma \vdash n$. (see below definition 4.2.1). In order to be able to describe this 'multiplication' we introduce the following algebras and maps:

$B_{n,t}$ is the associative commutative free Q(u)-algebra with Q(u)-basis $\{x(e,f) \mid e,f \in Z\}$ multiplication given by

$$x(e,f).x(s,r) = x(e+s-n,f+r-t) \qquad \text{for all } e,f,s,r \in Z.$$

Let $T = \{\delta \mid \delta \vdash m,\ m \in Z_+\} \cup \{< 0 >\}$. Define a map from T to Z_+ by $\delta \mapsto n_\delta$ where for $\delta = < 1^{a_1}2^{a_2}\ldots m^{a_m}>$ $n_\delta = \sum_{i=2}^{m} ia_i$. ($n_{<1^1>} = n_{<0>} = 0$).

Define a map from $\{f \mid f:G_n \to K$ is a subclass function$\}$ to $\{r \mid r:T \to K\}$ by $f \mapsto f^T$ where for $\delta = < 1^{a_1}\ldots m^{a_m}>$

$$f^T(\delta) = \begin{cases} 0 & \text{if } n_o > n \\ f_{\delta'} & \text{if } n_\delta \le n, \text{ where } \delta' = < 1^{n-n_\delta}2^{a_2}3^{a_3}\ldots m^{a_m}> \end{cases}$$

and $f^T(< 0 >) = f_0 = f(g_o)$.

$M_{n,t}$ is the associative commutative free $B_{n,t}$-algebra with $B_{n,t}$-basis $T \times T$ and multiplication given by

$$(\delta_1,\delta_2)(\delta_3,\delta_4) = (\delta_1\delta_3,\delta_2\delta_4) \quad \text{for all } \delta_1,\delta_2,\delta_3,\delta_4 \in T$$

where for $\delta = < 1^{a_1}2^{a_2}\ldots >$ and $\tau = < 1^{b_1}2^{b_2}\ldots >$ in T $\delta\tau = < 1^{a_1+b_1}2^{a_2+b_2}\ldots >$ and $\delta< 0 > = < 0 >\delta = \delta$.

$\mathcal{J} : B_{n,t} \longrightarrow Q(u)$ is the $Q(u)$-linear map defined by $\mathcal{J} : x(e,f) \longmapsto \binom{e}{f}$ for all $e,f \in Z$ (see definition 2.2.11). Given subclass functions $\alpha \in X(G_t)$ and $\eta \in X(G_{n-t})$ we define the map $\mathcal{J}_{\alpha,\eta} : M_{n,t} \longrightarrow K$ by

$$\mathcal{J}_{\alpha,\eta} : \sum_{(\delta,\tau)} y_{\delta,\tau}(\delta,\tau) \longmapsto \sum_{(\delta,\tau)} J(y_{\delta,\tau})\alpha^T(\delta)\eta^T(\tau)$$

where (δ,τ) runs over any finite subset of $T \times T$ and each $y_{\delta,\tau} \in B_{n,t}$.

Using the above definitions and notation we are able to state and prove

THEOREM 4.2.8.

For each $i \in Z_n \setminus \{1\}$ define the element $m(n,t,i)$ of $M_{n,t}$ by

$$\begin{aligned} m(n,t,i) = {} & x(n-i,t-i)(<i>,<0>) \\ & + (u-1)\sum_{j=1}^{i-1} x(n-i,t-j)(<j>,<i-j>) \\ & + x(n-i,t)(<0>,<i>) \end{aligned}$$

Let $\alpha \in X(G_t)$ and $\eta \in X(G_{n-t})$ be subclass functions.

(i) If $y \in \underline{i} \subset S_n$ then $(\alpha \circ \eta)(g_y) = \mathcal{J}_{\alpha,\eta}(m(n,t,i))$

(ii) Let $\gamma = <\gamma_1,\gamma_2,\ldots,\gamma_s> \vdash n$ with $\gamma \neq <1^n>$. Let r be the greatest integer such that $\gamma_r > 1$. Then if $w \in (\underline{\gamma})$

$$(\alpha \circ \eta)(g_w) = J_{\alpha,\eta}(m(n,t,\gamma_1).m(n,t,\gamma_2)\ldots .m(n,t,\gamma_r))$$

(Note that the right hand side of this equation is independent of the choice of w in $(\underline{\gamma})$.)

(iii) $(\alpha \circ \eta)(g_o) = \binom{n}{t}\alpha_{<1^t>}\eta_{<1^{n-t}>}$

Proof:

(i) This is a restatement of theorem 4.2.5.

(iii) Immediate from definition 3.2.5(ii).

(ii) Let $z \in S_n$ be of minimal length in its class.

If $\mu_{i_1}\mu_{i_2}\ldots\mu_{i_s}$ is a reduced expression for z then $g_z = g_{i_1}g_{i_2}\ldots g_{i_s}$ and by theorem A2.7(i) $|\{i_1,i_2,\ldots,i_s\}| = s$.

Let $g_v g_A = \sum_{B\subset Z_n, |B|=t} g_B\, \sigma_{v,B,A}$ for all $A \subset Z_n$ with $|A|=t$ and $v\in S_n$, where $\sigma_{v,B,A} \in G_n$. (cf equation (4.2.7)).

By lemma 4.2.3 it is clear that

$$\sigma_{z,A,A} = \sigma_{u_{i_1},A,A}\sigma_{u_{i_2},A,A}\ldots\sigma_{u_{i_s},A,A} \qquad (4.2.9)$$

for all $A \subset Z_n$ with $|A| = t$.

Let $w = c_1c_2\ldots c_r$ be the decomposition of w into a product of disjoint cycles. We can assume that $c_j \in \underline{\gamma}_j$ for each j in Z_r.

By equation (4.2.9)

$\sigma_{w,A,A} = \sigma_{c_1,A,A}\sigma_{c_2,A,A}\ldots\sigma_{c_r,A,A}$ for all $A \subset Z_n$ with $|A| = t$.

Fix A. By corollary 4.2.4 for each $j \in Z_r$

$$\sigma_{c_j,A,A} = p_j(u)g_{v_j}g_{x_j}$$

where either $p_j(u) = 0$ or if $p_j(u) \neq 0$ then there exists $i_j \in Z_{\gamma_j} \cup \{0\}$ with $v_j \in \underline{i}_j \subset T_1$ and $x_j \in \underline{\gamma_j - i_j} \subset T_2$, in which case $p_j(u) = \begin{cases} 1 & \text{if } i_j = 0 \text{ or } \gamma_j \\ (u-1) & \text{otherwise.} \end{cases}$

$$g_{v_1}g_{x_1}\ldots g_{v_r}g_{x_r} = g_{v_1}g_{v_2}\ldots g_{v_r}g_{x_1}\ldots g_{x_r}$$

and by the proof of corollary 4.2.4 it is clear that $l(v_1v_2\ldots v_r) = l(v_1) + l(v_2) + \ldots + l(v_r)$ and $l(x_1\ldots x_r) = l(x_1) + \ldots + l(x_r)$. Thus

$$\sigma_{w,A,A} = p_1(u)\ldots p_r(u)g_{v_1\ldots v_r}g_{x_1\ldots x_r}$$

and $v_1\ldots v_r \in T_1$, $x_1\ldots x_r \in T_2$.

If $\sum_{j=1}^{r} i_j \leq t$ let τ be the partition of t who's parts are $i_1,i_2,\ldots,i_r$ with as many '1's added as necessary.

Similarly if $\sum_{j=1}^{r}(\gamma_j - i_j) \leq n-t$ let ξ be the partition of $n-t$ who's parts are $(\gamma_1-i_1),\ldots,(\gamma_r-i_r)$ with as many '1's added as is necessary.

We have that $|\{A \mid \sigma_{c_j,A,A} = q(u)g_{v_j}g_{x_j},\ q(u) \in Q(u),\ q(u) \neq 0,\ v_j \in \underline{i_j} \subset T_1,\ x_j \in \underline{\gamma_j - i_j} \subset T_2\}| = \binom{n-\gamma_j}{t-i_j}$ and if $S = \{A \mid \sigma_{w,A,A} = q'(u)g_v g_x,\ q'(u) \in Q(u),\ q'(u) \neq 0,\ v \in (\underline{\tau}) \subset T_1,\ x \in (\underline{\xi}) \subset T_2\}$ then

$$|S| = \binom{n-\gamma_1-\gamma_2-\ldots-\gamma_r}{t-i_1-i_2-\ldots-i_r}$$

Thus denoting $|\{j \mid j \in Z_r,\ i_j \neq 0 \text{ or } \gamma_j\}|$ by $m(\tau)$,

$$\sum_{A\in S}(\alpha.\eta)(\sigma_{w,A,A}) = \binom{n-\gamma_1-\ldots\ \gamma_r}{t-i_1-\ldots\ i_r}(u-1)^{m(\tau)}\alpha_\tau\eta_\xi$$

$$= \pi(x(n-\gamma_1,t-i_1).x(n-\gamma_2,t-i_2)\ldots$$

$$\ldots x(n-\gamma_r,t-i_r))(u-1)^{m(\tau)}\alpha^T(\tau)n^T(\xi)$$

(see the notation below definition 4.2.1 and theorem 4.2.5).

$(\alpha\circ\eta)(g_w) = \sum_{A\subset Z_n,|A|=t}(\alpha.n)(\sigma_{w,A,A})$ thus clearly from the above formula

$$(\alpha\circ\eta)(g_w) = \pi_{\alpha,\eta}(m(n,t,\gamma_1).m(n,t,\gamma_2)\ldots m(n,t,\gamma_r)).$$

※

COROLLARY 4.2.10.

(i) The characters of G_n $(n > 1)$ are subclass functions.

(ii) Given $\beta \in X(G_n)$, $\gamma \vdash n$ and w of minimal length in (γ), then $\beta(g_w) = \beta_\gamma$.

(iii) Let $\alpha \in X(G_t)$, $\eta \in X(G_{n-t})$ and let $\gamma = \langle\gamma_1,\gamma_2,\ldots,\gamma_s\rangle \vdash n$. Assume that $\gamma \neq \langle 1^n\rangle$ and let r be the greatest integer such that $\gamma_r > 1$. Then

(4.2.11) $$(\alpha\circ\eta)_\gamma = \pi_{\alpha,\eta}(m(n,t,\gamma_1)\ldots m(n,t,\gamma_r)).$$

Proof:

(ii) follows from (i), lemma 4.2.2 and relations (2.1.4).

(iii) follows from (i) and theorem 4.2.8(ii).

(i) The unit character $\mathcal{J}_m$ of G_m ($m \in Z_+$) is a subclass function (see definition 4.1.3), thus the result follows from lemmas 3.2.6(ii) 3.2.7(ii), 4.1.2(iii), corollary 4.1.5 and theorem 4.2.8(ii). (Note the remark below theorem 4.2.8(ii)).

※

Let $\lambda = \langle \lambda_1, \ldots, \lambda_s \rangle \vdash n$ with $0 < \lambda_s \leqslant \ldots \leqslant \lambda_1$. Let $^j\alpha \in X(G_{\lambda_j})$ for each j in Z_s. Lemma 4.1.1 shows that we can form the character $^1\alpha \circ {^2\alpha} \circ \ldots \circ {^s\alpha}$ of G_n. In a way analogous to that used to prove theorem 4.2.5 we can find a formula for $({^1\alpha} \circ \ldots \circ {^s\alpha})_i$ ($i \in Z_n \setminus \{1\}$). Since by corollary 3.2.7(ii) no 'new' information is gained we omit the proof and merely state the result

THEOREM 4.2.12.

Using the above notation,

$$({^1\alpha} \circ \ldots \circ {^s\alpha})_i = \Sigma\, (u-1)^{\prec \delta_1, \delta_2, \ldots, \delta_s \succ - 1} ({^1\alpha})_{\delta_1} ({^2\alpha})_{\delta_2} \cdots$$

$$\cdots ({^s\alpha})_{\delta_s} \frac{(n-i)!}{(\lambda_1 - \delta_1)! \ldots (\lambda_s - \delta_s)!}$$

where the summation is over all ordered s-tuples $\{\delta_1, \delta_2, \ldots, \delta_s\}$ such that $0 \leqslant \delta_j \leqslant \lambda_j$ for all $j \in Z_s$ and $\sum_{j=1}^{s} \delta_j = i$.

$$\prec \delta_1, \ldots, \delta_s \succ = |\{j \mid j \in Z_s, \delta_j \neq 0\}|.$$

※

Clearly an analogue of formula (4.2.11) must exist for $({^1\alpha} \circ \ldots {^s\alpha})_\tau$ ($\tau \vdash n$).

THEOREM 4.2.13.

$\eta(h_w) \in Z[u]$ for all $\eta \in X(G_n)$ and $w \in S_n$.

Proof:

By induction on n, it follows from lemma 3.2.7(ii), lemma 4.1.2(iii), corollary 4.1.5 and theorem 4.2.8 that $\eta_\gamma \in Z[u]$ for all $\eta \in X(G_n)$ and $\gamma \vdash n$. In the notation of §2.2 one readily finds from the remark below theorem 2.2.8 that $\sigma(w,b) \in Z[u]$ for all $w \in S_n$ and $b \in B$. Thus the result follows from corollary 4.2.10.

※

EXAMPLE ILLUSTATING THEOREM 4.2.8.

Let $n = 5$, $l = 3$, $\gamma = <2.2.1>$, $\alpha \in X(G_3)$ and $\eta \in X(G_2)$. By theorems 4.2.5 and 4.2.8:

$$(\alpha\circ\eta)_2 = \binom{5-2}{3-2}\alpha_2\eta_1 + (u-1)\binom{5-2}{3-1}\alpha_1\eta_1 + \binom{5-2}{3}\alpha_1\eta_2$$

$$= \mathcal{J}_{\alpha,\eta}(m(5,3,2))$$

$$(\alpha\circ\eta)_o = \mathcal{J}_{\alpha,\eta}((m(5,3,2))^2)$$

$$= \binom{5-2-2}{3-2-2}\alpha_{<2^2>}\eta_1 + (u-1)\binom{5-2-2}{3-2-1}\alpha_2\eta_1 + \binom{5-2-2}{3-2}\alpha_2\eta_2$$

$$+ (u-1)\binom{5-2-2}{3-1-2}\alpha_2\eta_1 + (u-1)^2\binom{5-2-2}{3-1-1}\alpha_1\eta_1 + (u-1)\binom{5-2-2}{3-1}\alpha_1\eta_2$$

$$+ \binom{5-2-2}{3\ \ -2}\alpha_2\eta_2 + (u-1)\binom{5-2-2}{3\ \ -1}\alpha_1\eta_2 + \binom{5-2-2}{3}\alpha_1\eta_{<2^2>}$$

$$= (u-1)\alpha_2\eta_1 + \alpha_2\eta_2 + (u-1)\alpha_2\eta_1 + (u-1)^2\alpha_1\eta_1 + \alpha_2\eta_2$$

$$= 2(u-1)\alpha_2\eta_1 + 2\alpha_2\eta_2 + (u-1)^2\alpha_1\eta_1$$

§4.3 (σ,δ,τ)-RECTANGLES.

Let $n \in Z_+$ and $t \in Z_{n-1}$. By equation (4.2.11) and lemma 3.2.6(ii) we see that given $\gamma \vdash n$ there exist elements $e(\gamma,\delta,\tau)$ of $Q(u)$ $(\delta \vdash t,\ \tau \vdash n-t)$ such that

(4.3.1) $$(\alpha\circ\eta)_\gamma = \sum_{\substack{\delta\vdash t\\ \tau\vdash n-t}} e(\gamma,\delta,\tau)\alpha_\delta\eta_\tau \quad \text{for all}$$

$\alpha \in X(G_t)$ and $\eta \in X(G_{n-t})$.

Corollary 4.3.9 gives a formula for $e(\gamma,\delta,\tau)$.

LEMMA 4.3.2.

The coefficients $e(\gamma,\delta,\tau)$ are uniquely determined by equation (4.3.1).

Proof:

Let $\{\alpha^a \mid a \vdash t\}$ and $\{\eta^b \mid b \vdash n-t\}$ be the sets of all irreducible G_t and G_{n-t} characters respectively. Fix $\gamma \vdash n$ and define the following three matrices:

$$M_1 = \{(\alpha^a)_\delta\}_{a,\delta} \qquad a,\delta \vdash t$$

$$M_2 = \{(\eta^b)_\tau\}_{\tau,b} \qquad \tau,b \vdash n-t$$

$$M_3 = \{(\alpha^a \circ \eta^b)_\gamma\}_{a,b} \qquad a \vdash t,\ b \vdash n-t.$$

By theorem 2.1.2 M_1 and M_2 are invertible. Equation (4.3.1) shows that $e(\gamma,\delta,\tau)$ is the $(\delta,\tau)^{th}$ entry of $M_1^{-1}M_3M_2^{-1}$.

⋇

DEFINITION 4.3.3.

Let
$$\sigma = <1^{\sigma_1}2^{\sigma_2}\ldots n^{\sigma_n}> \vdash n$$
$$\delta = <1^{\delta_1}2^{\delta_2}\ldots t^{\delta_t}> \vdash t$$
$$\tau = <1^{\tau_1}\ldots (n-t)^{\tau_{n-t}}> \vdash n-t$$

(i) A <u>(σ,δ,τ)-rectangle</u> is a $(t+1)\times(n-t+1)$ array of non-negative integers, say r_{ij} in the $(i,j)^{th}$ position ($i \in \{0,1,\ldots t\}$, $j \in \{0,1,\ldots n-t\}$) such that

(a) $\sum_{j=0}^{n-t} r_{ij} = \delta_i \quad$ for $i \in \{1,2,\ldots t\}$

(b) $\sum_{i=0}^{t} r_{ij} = \tau_j \quad$ for $j \in \{1,2,\ldots (n-t)\}$

(c) $\sum_{a=0}^{b} r_{a\ b-a} = \sigma_b \quad$ for $b \in \{1,2,\ldots n\}$

(d) $r_{oo} = 0$

(ii) The (σ,δ,τ)-rectangle $\mathbb{R}$ with entries r_{ij}

has <u>sum</u>

$$s(\mathbb{R}) = \sum_{i=1}^{t} \sum_{j=1}^{n-t} r_{ij}$$

and for each $a \in Z_n$ it has <u>a-value</u>

$$\mathbb{R}_a = \frac{\sigma_a}{r_{oa}! r_{1\ a-1}! \ldots r_{ao}!}$$

REMARK.

Given $\sigma \vdash n$, $\delta \vdash t$ and $\tau \vdash n-t$ there may exist none, one or more (σ,δ,τ)-rectangles.

DEFINITION 4.3.4.

Let $m \in Z_+$ and $d \in Z_{m-1}$. A $(d+1)\times(m-d+1)$ array of elements of Z is called an <u>(m,d)-rectangle</u> if it is a (β,ξ,ζ)-rectangle for some $\beta \vdash m$, $\xi \vdash d$ and $\zeta \vdash m-d$.

DEFINITION 4.3.5.

Let $m \in Z_+$, $d \in Z_{m-1}$ and $\beta = <\beta_1,\ldots,\beta_r>$ with $0 < \beta_r \leq \ldots \leq \beta_1$.

A <u>d-cutting</u> $\mathfrak{J}$ of β (denoted by $\mathfrak{J}(d)\beta$) is an ordered pair of ordered r-tuples $(\{z_1,\ldots,z_r\},\{y_1,\ldots,y_r\})$ of non-negative integers such that

$$\sum_{i=1}^{r} z_i = d \quad \text{and} \quad z_i+y_i = \beta_i \quad \text{for all } i \in Z_r.$$

LEMMA 4.3.6.

Given $m \in Z_+$ and $d \in Z_{m-1}$ there exists a surjection

$$\dagger_m : \{\aleph \mid \aleph(d)\mu\ ,\ \mu \vdash m\} \longrightarrow \{\mathbb{R} \mid \mathbb{R} \text{ is an } (m,d)\text{-rectangle}\}$$ given by:

If $\mathfrak{J}$ is as given in definition 4.3.5, then $\dagger_m(\mathfrak{J})$ is the

(β,z,y)-rectangle with entries $q_{ij} = |\{x \in Z_r |\ z_x=i,\ y_x=j\}|$ for all $(i,j) \in \{0,1,\ldots,d\}\times\{0,1,\ldots,m-d\} \setminus \{(0,0)\}$, $q_{oo} = 0$. z is the partition of d whose parts are $z_1,z_2,\ldots,z_r$ and y is the partition of m-d whose parts are $y_1,y_2,\ldots,y_r$.

Proof:

That $\{q_{ij}|\ (i,j) \in \{0,\ldots,d\}\times\{0,\ldots,m-d\}\}$ is an (m,d)-rectangle is readily proved by checking that conditions (i)(a),(b),(c)and (d) of definition 4.3.3 hold.

Let $\lambda = <\lambda_1,\ldots,\lambda_f> \vdash m$ $(\lambda_f > 0)$, $\xi \vdash d$, $\int \vdash m-d$ and let $\mathbb{R}$ be a $(\lambda,\xi,\int)$-rectangle with entries r_{ij}. For $v \in Z_m$ let x_v be the number of $i \in Z_f$ with $\lambda_i = v$ (so that $\lambda = <1^{x_1}\ldots$ $\ldots m^{x_m}>$). By definition 4.3.3(i)(c)

$$\sum_{a=0}^{v} r_{a\ v-a} = x_v$$ for each $v \in Z_m$. Thus it is

clear that there exists a d-cutting $\mathcal{K} = (\{d_1,\ldots,d_f\}\times\{b_1,\ldots$ $\ldots,b_f\})$ of λ such that for each $v \in Z_m$ and $a \in Z_v\cup\{0\}$ there are precisely $r_{a\ v-a}$ values of $i \in Z_f$ with $d_i = v$ and $d_i + b_i = v$. Clearly $\dagger_m(\mathcal{K}) = \mathbb{R}$, showing that $\dagger_m$ is a surjection.

✳

LEMMA 4.3.7.

Let $\mathbb{R}$ be an (m,d)-rectangle $(d \in Z_{m-1})$ then

$$|\{\mathcal{J}|\ \mathcal{J}(d)\beta \text{ for some } \beta \vdash m,\ \dagger_m(\mathcal{J}) = \mathbb{R}\}| = \sum_{a=1}^{m} \mathbb{R}_a$$

(see definition 4.3.3(ii)).

Proof:

Immediate for the part of the proof of lemma 4.3.6 which shows that $\dagger_m$ is surjective.

✳

THEOREM 4.3.8.

Let $\alpha \in X(G_t)$, $\eta \in X(G_{n-t})$ and $\delta \vdash n$ then

$$(\alpha \circ \eta)_\gamma = \sum_{\substack{\delta \vdash t \\ \tau \vdash n-t}} \left(\sum_{\mathbb{R} \text{ is } (\gamma,\delta,\tau)\text{-rectangle}} (u-1)^{s(\mathbb{R})} \prod_{a=1}^{n} \mathbb{R}_a \right) \alpha_\delta \eta_\tau$$

(see definition 4.3.3(ii)).

Proof:

If $\gamma = <1^n>$ the result is readily checked.

Let $\gamma = <\gamma_1,\gamma_2,\ldots,\gamma_s> \neq <1^n> \vdash n$ and r be the greatest integer such that $\gamma_r > 1$.

For each $b \in Z_r$ choose a term, say V_b, of $m(n,t,\gamma_b)$ (see theorem 4.2.8). So

$$V_b = p_b x(n-\gamma_b, t-z_b)(<z_b>,<\gamma_b-z_b>)$$

for some $z_b \in Z_{\gamma_b} \cup \{0\}$ and $p_b \in Q(u)$.

Let $\sum_{b=1}^{r} z_b = d$. We note that $\sum_{b=1}^{r} \gamma_b - z_b = n-d-(s-r)$.

Let z and y be the partitions of d and n-d-(s-r) with parts $z_1,z_2,\ldots,z_r$ and $\gamma_1-z_1,\ldots,\gamma_r-z_r$ respectively. Thus

$$V_1V_2\ldots V_r = p_1p_2\ldots p_r x(n-\gamma_1-\ldots-\gamma_r, t-z_1-\ldots z_r)(<z>,<y>)$$

$$= p_1\ldots p_r \binom{s-r}{t-d}(<z>,<y>)$$

Define

$$S_{V_1\ldots V_r} = \{ \mathfrak{J} = (\{a_1,a_2,\ldots,a_s\},\{b_1,\ldots,b_s\} \mid \mathfrak{J}(t)\gamma,\ a_i = z_i \text{ for all } i \in Z_r\}$$

Clearly $|S_{V_1\ldots V_r}| = \binom{s-r}{t-d}$ and if for each $b \in Z_r$ U_b is a term of $m(n,t,\gamma_b)$ then $S_{V_1\ldots V_r} \cap S_{U_1\ldots U_r} \neq \emptyset$ if and only if $V_b = U_b$ for all $b \in Z_r$.

Let $\mathfrak{K} = (\{d_1,\ldots,d_s\},\{i_1,\ldots,i_s\})(t)\gamma$. For each $b \in Z_r$ put $Y_b = q_b x(n-\gamma_b, t-d_b)(<d_b>,<i_b>)$ where

$$q_b = \begin{cases} 1 & \text{if } d_b = 0 \text{ or if } i_b = 0 \\ (u-1) & \text{otherwise} \end{cases}$$

Y_b is a term of $m(n,t,\gamma_b)$ and clearly $\mathfrak{K} \in S_{Y_1\ldots Y_r}$.

Thus $\{ \mathfrak{J} \mid \mathfrak{J}(t)\gamma \}$ is equal to the disjoint union:

$\cup S_{V_1 \ldots V_r}$ where the union runs over all possible choices of V_b for each $b \in Z_r$.

Let $\mathfrak{J} = (\{a_1, \ldots, a_s\}, \{b_1, \ldots, b_s\}) \in S_{V_1 \ldots V_r}$ and let a and b be the partitions of t and n-t with parts $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ respectively. By lemma 4.3.6 it is clear that

$$\mathcal{J}_{\alpha,\eta}(V_1 \ldots V_r) = (u-1)^{s(\dagger_n(\mathfrak{J}))} \binom{s-r}{t-d} \alpha^T(a) \eta^T(b)$$

Hence

$$\mathcal{J}_{\alpha,\eta}(m(n,t,\gamma_1) m(n,t,\gamma_2) \ldots m(n,t,\gamma_r)) =$$

$$\frac{1}{\binom{s-r}{t-d}} \left(\sum_{\mathfrak{J} = (\{a_1, \ldots, a_s\}, \{b_1, \ldots, b_s\})(t)\gamma} (u-1)^{s(\dagger_n(\mathfrak{J}))} \binom{s-r}{t-d} \alpha^T(a) \eta^T(b) \right)$$

where a and b are the partitions with parts $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ respectively.

Thus by lemmas 4.3.6 and 4.3.7, we see that

$$\mathcal{J}_{\alpha,\eta}(m(n,t,\gamma_1) \ldots m(n,t,\gamma_r)) =$$

$$\sum_{\substack{\delta \vdash t \\ \tau \vdash n-t}} \left(\sum_{\mathbb{R}} (u-1)^{s(\mathbb{R})} \prod_{a=1}^{n} \mathbb{R}_a \right) \alpha_\delta \eta_\tau$$

where the second sum is over all (γ, δ, τ)-rectangles. Equation (4.2.11) now shows that the proof is complete.

⋇

COROLLARY 4.3.9.

Let $e(\gamma,\delta,\tau)$ be defined by equation (4.3.1) (see lemma4.3.2) then

$$e(\gamma,\delta,\tau) = \sum_{\mathbb{R} \text{ is } (\gamma,\delta,\tau)\text{-rectangle}} (u-1)^{s(\mathbb{R})} \prod_{a=1}^{n} \mathbb{R}_a$$

Proof:

Immediate from theorem 4.3.8.

⋇

CHAPTER 5

CHARACTERISTICS.

In this chapter a formula is derived for the values of the irreducible characters of G_n in terms of the values of the irreducible characters of S_n. (see theorem 5.3.9).

§5.1 THE ALGEBRA F.

DEFINITION 5.1.1.

Let $m \in Z_+$. The function $\rho \mapsto |\rho|$ from S_m to Z_+ is defined as follows:

If $\rho = < 1^{\rho_1} 2^{\rho_2} \ldots m^{\rho_m} >$ then $|\rho| = \sum_{i=1}^{n} \rho_i$

DEFINITION 5.1.2.

F is the graded K-algebra $F_1 \oplus F_2 \oplus \ldots\ldots = \Sigma^{\oplus}_{m\geqslant 1} F_m$ where F_m is the K-space the K-basis $\{ x_\delta | \delta \vdash m\}$ and multiplication is given by:

If $\delta \vdash m$ and $\tau \vdash d$ $(m,d \in Z_+)$ then

(5.1.3) $$x_\delta x_\tau = \sum_{\sigma \vdash m+d} e(\sigma,\delta,\tau)x_\sigma$$

(recall that $e(\sigma,\delta,\tau)$ is defined by equation (4.3.1). See lemma 4.3.2).

NOTATION. We will abbreviate $x_{< r >}$ to x_r for all $r \in Z_+$.

DEFINITION 5.1.4.

Let $m \in Z_+$ and $\eta \in X(G_m)$. The following element of F_m

$$\phi_\eta = \sum_{\sigma \vdash m} \eta_\sigma x_\sigma$$ is the F-characteristic of η.

(recall that $\eta_\sigma = \eta(g_w)$ for any $w \in (\underline{\sigma})$. See definition

4.2.1 and below).

THEOREM 5.1.5.

(i) F is commutative.

(ii) F is associative.

(iii) If $\alpha \in X(G_m)$ and $\eta \in X(G_d)$ $(m,d \in Z_+)$ then

$$\phi_{\alpha \circ \eta} = \phi_\alpha \phi_\eta$$

Proof:

(i) Using the notation of definition 5.1.2: if $\mathbb{R}$ is a (σ,δ,τ)-rectangle then its transpose $\mathbb{R}^t$ is clearly a (σ,τ,δ)-rectangle. Further $s(\mathbb{R}) = s(\mathbb{R}^t)$ and for all $a \in Z_{m+d}$ $(\mathbb{R})_a = (\mathbb{R}^t)_a$ (see definition 4.3.3). Thus corollary 4.3.9 shows that F is commutative.

(iii)
$$\begin{aligned}
\phi_{\alpha\circ\eta} &= \sum_{\sigma \vdash m+d} (\alpha \circ \eta)_\sigma x_\sigma \\
&= \sum_{\sigma \vdash m+d} \sum_{\substack{\delta \vdash m \\ \tau \vdash d}} e(\sigma,\delta,\tau)\alpha_\delta \eta_\tau x_\sigma && \text{by equation (4.3.1)} \\
&= \sum_{\substack{\delta \vdash m \\ \tau \vdash d}} \alpha_\delta \eta_\tau x_\delta x_\tau && \text{by equation (5.1.3)} \\
&= \phi_\alpha \phi_\eta
\end{aligned}$$

(ii) By (iii) and lemma 3.2.7(ii), if $\xi \in X(G_r)$ $(r \in Z_+)$ then

$$\begin{aligned}
(\phi_\alpha \phi_\eta)\phi_\xi &= \phi_{\alpha\circ\eta}\phi_\xi = \phi_{(\alpha\circ\eta)\circ\xi} = \phi_{\alpha\circ(\eta\circ\xi)} = \phi_\alpha \phi_{\eta\circ\xi} \\
&= \phi_\alpha(\phi_\eta \phi_\xi)
\end{aligned}$$

From which equation and theorem 2.1.2 the associativite of F can be readily proved.

LEMMA 5.1.6.

Let $\sigma = \langle 1^{\sigma_1} \ldots m^{\sigma_m} \rangle \vdash m$ $(m \in Z_+)$ and let $Z(\sigma) = \{j \in Z_+ \mid \sigma_j \neq 0\}$. Then for each $d \in Z_+$

(5.1.7) $$x_\sigma x_d = (\sigma_d + 1)x_{\rho_0(d)} + (u-1)\sum_{j \in Z(\sigma)} (\sigma_{d+j}+1)x_{\rho_j(d)}$$

where

$$\rho_o(d) = < 1^{\sigma_1} 2^{\sigma_2} \ldots d^{\sigma_d + 1} \ldots > \vdash m+d$$

and

$$\rho_j(d) = < 1^{\sigma_1} \ldots j^{\sigma_j - 1} \ldots (d+j)^{\sigma_{d+j}+1} \ldots > \vdash m+d$$

for all $j \in Z(\sigma)$.

Note that $|\rho_o(d)| = |\sigma| + 1$ and that $|\rho_j(d)| = |\sigma|$ for all $j \in Z(\sigma)$.

Proof:

We use corollary 4.3.9. If $\gamma \vdash m+d$ and a $(\gamma, \sigma, < d >)$-rectangle exists then $\gamma = \rho_o(d)$ or $\rho_j(d)$ for some $j \in Z(\sigma)$. For each of these values for γ there exists a unique rectangle:

$$\begin{array}{cccccc} 0 & 0 & 0 & \ldots & 0 & 1 \\ \sigma_1 & 0 & \ldots & & & 0 \\ \sigma_2 & 0 & \ldots & & & \vdots \\ \vdots & & 0 & & & \\ \sigma_m & 0 & \ldots & & & 0 \end{array}$$

is the $(\rho_o(d), \sigma, < d >)$-rectangle.

$$\begin{array}{cccccc} 0 & 0 & 0 & \ldots & 0 & 0 \\ \sigma_1 & 0 & \ldots & & & 0 \\ \vdots & \vdots & & 0 & & \vdots \\ & 0 & & & & 0 \\ \sigma_{j-1} & & & \ldots & 0 & 1 \\ \vdots & 0 & & & & 0 \\ & \vdots & & 0 & & \vdots \\ \sigma_m & 0 & \ldots & & 0 & 0 \end{array}$$

is the $(\rho_j(d), \sigma, < d >)$-rectangle. $(j \in Z(\sigma))$.

⋇

LEMMA 5.1.8.

Let $\beta = < 1^{\beta_1} \ldots m^{\beta_m} > \vdash m \quad (m \in Z_+)$

There exist unique elements λ_ρ in K $(\rho \vdash m)$ such that

(i) $(x_1)^{\beta_1} (x_2)^{\beta_2} \ldots (x_m)^{\beta_m} = \sum_{\rho \vdash m} \lambda_\rho x_\rho$

(ii) $\{ \rho \mid \lambda_\rho \neq 0, \ |\beta| \leqslant |\rho| \} = \{\beta\}$

Proof:

(i) Immediate as F is a graded algebra.

(ii) We use induction on $|\beta|$. Let r be the greatest integer such that $\beta_r \neq 0$. Let $\sigma = < 1^{\beta_1} 2^{\beta_2} \cdots r^{\beta_r - 1} \cdots > \vdash m-r$.

By (i) there exist elements λ_π in K $(\pi \vdash m-r)$ with

$$(x_1)^{\beta_1}(x_2)^{\beta_2} \cdots (x_r)^{\beta_r - 1} = \sum_{\pi \vdash m-r} \lambda_\pi x_\pi$$

By the inductive hypothesis

$$\{ \pi \mid \lambda_\pi \neq 0, \ |\sigma| \leqslant |\pi| \} = \{\sigma\}$$

so the result now follows from lemma 5.1.6, since in the notation of that lemma $\beta = \rho_0(r)$.

※

NOTATION.

Let $m \in Z_+$ and $\sigma = < 1^{\sigma_1} \cdots m^{\sigma_m} > \vdash m$, then we denote the following element of F_m

$$(x_1)^{\sigma_1} \cdots (x_m)^{\sigma_m} \quad \text{by } \underline{x^\sigma}.$$

THEOREM 5.1.9.

(i) $\{ x_r \mid r \in Z_+ \}$ is a set of free generators for F.

(ii) $\{ x^\delta \mid \delta \vdash m \}$ is a K-basis for F_m. $(m \in \mathbb{Z}_+)$.

Proof:

(i) In the notation of lemma 5.1.8 and below

$$x_\beta = \frac{1}{\lambda_\beta}\Big(x^\beta - \sum_{\substack{\rho \vdash m \\ |\rho| < |\beta|}} \lambda_\rho x_\rho \Big)$$

Thus induction on $|\beta|$ shows that the given set generate F.

Suppose that $\sum \delta_\pi x_\pi = 0$ where π runs over a finite non-empty set T of partitions. Let $\rho \in \{ \sigma \mid \sigma \in T, \ |\pi| \leqslant |\sigma|$ for all $\pi \in T \}$, then it is clear by lemma 5.1.8 that $\delta_\rho = 0$. Hence $\delta_\pi = 0$ for all $\pi \in T$.

(ii) Immediate from (i) and its proof.

※

§5.2 THE MATRIX V_n.

NOTATION. Fix $n \in Z_+$. let $T_n = \{ \sigma_j \mid j \in J\}$ be the set of all partitions of n (J some indexing set) indexed in such a way that $i < j$ implies that $|\sigma_i| \geq |\sigma_j|$.

By theorem 5.1.9(ii) there exist elements v_{ij} of K ($i,j \in J$) such that

$$x^{\sigma_i} = \sum_{j \in J} v_{ij} x_{\sigma_j} \quad \text{for all } i \in J. \tag{5.2.1}$$

DEFINITION 5.2.2.

V_n is the $(|J| \times |J|)$-matrix with $(i\ j)^{th}$ entry v_{ij} . (Appendix 8 gives V_2, V_3 and V_4 explicitly).

DEFINITION 5.2.3.

Let $\beta = <\beta_1, \ldots, \beta_t>$ with $0 < \beta_t \leq \ldots \leq \beta_1$ and $\rho = <\rho_1, \ldots, \rho_s>$ with $0 < \rho_s \leq \ldots \leq \rho_1$ be elements of T_n. Then

(i) $\underline{\succcurlyeq}$ is the relation defined on T_n as follows: $\beta \succcurlyeq \rho$ if and only if there exist subsets I_j of Z_t ($j \in Z_s$) satisfying

(a) $I_j \cap I_i = \emptyset$ for all $i,j \in Z_s$ with $i \neq j$.

(b) $\bigcup_{j \in Z_s} I_j = Z_t$

(c) $\rho_j = \sum_{r \in I_j} \beta_r$ for all $j \in Z_s$.

(ii) $\underline{N(\beta,\rho)}$ is the number of sets $\{I_j \mid j \in Z_s\}$ of subsets of Z_t satisfying conditions (i)(a),(b) and (c).

We have immediately the following two lemmas

LEMMA 5.2.4.

Let $\beta, \rho \in T_n$. If $\beta \succcurlyeq \rho$ then $|\beta| \geq |\rho|$ with equality if and only if $\beta = \rho$.

REMARK. Lemma 5.2.4 shows that

$N(\sigma_i, \sigma_j) = 0$ if $i > j$.

LEMMA 5.2.5.

Let β and ρ be as in definition 5.2.3. Fix $r \in Z_t$. For each z in Z_s with $\rho_z \geqslant \beta_r$ define π_z to be the partition of $n-\beta_r$ with the same parts as ρ except that ρ_z is replaced by $\rho_z - \beta_r$. Let β' be the partition of $n-\beta_r$ with the same parts as β except that β_r is omitted. Then

$$N(\beta,\rho) = \sum_{\substack{z \in Z_s \\ \rho_z \geqslant \beta_r}} N(\beta', \pi_z) m_z$$

where $m_z = |\{i \mid i \in Z_s, \rho_i = \rho_z\}|$.

$\#$

THEOREM 5.2.6.

(i) $v_{ij} = N(\sigma_i, \sigma_j)(u-1)^{|\sigma_i| - |\sigma_j|}$

(ii) $v_{ii} = \sigma_{i1}!\sigma_{i2}! \ldots .\sigma_{in}!$ where $\sigma_i = < 1^{\sigma_{i1}} \ldots . n^{\sigma_{in}} >$

(iii) V_n is upper triangular, i.e. $v_{ij} = 0$ if $i > j$. (see the ordering of the partitions of n given at the beginning of this section.)

Proof:

(iii) Immediate from (i) and the remark below lemma 5.2.4.

(ii) Immediate from (i) and definition 5.2.3(ii).

(i) Let β and ρ be as in definition 5.2.3 then it is sufficient to show that $\lambda_{\beta,\rho} = N(\beta,\rho)(u-1)^{|\beta|-|\rho|}$ where

$$x^{\beta} = \sum_{\sigma \vdash n} \lambda_{\beta,\sigma} x_{\sigma} \quad (\lambda_{\beta,\sigma} \in K)$$

We use induction on $|\beta|$. In the notation of lemma 5.2.5: By theorem 5.1.5(i) $x^{\beta} = x^{\beta'} x_{\beta_r}$. So if

$x^{\beta'} = \sum_{\tau \vdash n-\beta_r} \lambda_{\beta'\tau} x_{\tau}$ then $\lambda_{\beta,\rho}$ is the coefficient of x_ρ in $\sum_{\tau \vdash n-\beta_r} \lambda_{\beta',\tau} x_\tau x_r$ which by equation (5.1.7) is the coefficient of x_ρ in $\sum_{\substack{z \in Z_s \\ \rho_z \geq \beta_r}} \lambda_{\beta' \pi_z} x_{\pi_z} x_{\beta_r}$

Thus by the inductive hypothesis and equation (5.1.7)

$$\lambda_{\beta,\rho} = \sum_{\substack{z \in Z_s \\ \rho_z \geq \beta_r}} N(\beta',\pi_z)(u-1)^{|\beta'|-|\pi_z|}(u-1)^{1-\xi_{o_z,r}} r_{m_z}$$

$$\text{where } \xi_{o_z,r} = \begin{cases} 0 & \text{if } \rho_z > \beta_r \\ 1 & \text{if } \rho_z = \beta_r \end{cases}$$

$$= N(\beta,\rho)(u-1)^{|\beta|-|\rho|} \quad \text{by lemma 5.2.5.}$$

$\#$

§5.3 V_n, $X(G_n)$ and $X(CS_n)$.

NOTATION.

Fix $n \in Z_+$. Let $\{ \eta^\alpha \mid \alpha \vdash n\}$ be the set of all irreducible characters of G_n over K and $\{ X^\alpha \mid \alpha \vdash n\}$ be the set of all irreducible characters of S_n over C. (see the introduction to Part II). By theorem 1.4.5(ii) we can assume that $X^\alpha(w) = f_1(\eta^\alpha(g_w))$ for all $w \in W$, where f_1 is the specialisation of K with $f_1(u) = 1$ which is shown to exist by lemma 1.4.2(ii).

Let $\mathcal{P}_n = \{ \sigma_j = \langle 1^{\sigma_{j1}} 2^{\sigma_{j2}} \ldots n^{\sigma_{jn}} \rangle \mid j \in J\}$ be the set of all partitions of n, indexed in such a way that $i < j$ implies that $|\sigma_i| \geq |\sigma_j|$ for all $i,j \in J$. (see definition 5.1.1).

Define two $(|J| \times |J|)$-matrices as follows:

Φ_n is the matrix with $(i\ j)^{th}$ entry $(\eta^{\sigma_i})_{\sigma_j}$. (see corollary 4.2.10(i) and the notation below definition 4.2.1).

Θ_n is the matrix with $(i\ j)^{th}$ entry θ_{ij} defined by the equations

$\phi_\eta \sigma_i = \sum_{j\in J} \theta_{ij} x^{\sigma_j}$ for all $i \in J$. (see definition 5.1.4 and theorem 5.1.9(ii)). Appendix 8 gives Θ_2, Θ_3 and Θ_4 explicitly.

DEFINITION 5.3.1.

let $\sigma = <1^{\sigma_1} \ldots n^{\sigma_n}> \vdash n$. The <u>polynomial of σ</u> is

$p(\sigma) = (u+1)^{\sigma_2}(u^2+u+1)^{\sigma_3} \ldots \ldots (u^{n-1}+u^{n-2}+\ldots+1)^{\sigma_n}$

NOTATION.

Let $m,d \in Z_+$. If $\lambda = <1^{\lambda_1}2^{\lambda_2}\ldots> \vdash m$ and $\tau = <1^{\tau_1}2^{\tau_2}\ldots> \vdash d$ then we denote the partition $<1^{\lambda_1+\tau_1}2^{\lambda_2+\tau_2}\ldots>$ of m+d by <u>$\lambda+\tau$</u>.

We have immediately

LEMMA 5.3.2.

Let $m,d \in Z_+$, $\lambda \vdash m$ and $\tau \vdash d$.

(i) $p(\lambda+\tau) = p(\lambda)p(\tau)$

(ii) $|\lambda+\tau| = |\lambda| + |\tau|$

LEMMA 5.3.3.

(i) $\Theta_n V_n = \Phi_n$

(ii) $f_1(\theta_{ij}) = (x^{\sigma_i})_{\sigma_j} / \sigma_{j1}!\sigma_{j2}!\ldots\sigma_{jn}!$

for all $i,j \in J$.

Proof:

(i) $\sum_{j\in J} (\eta^{\sigma_i})_{\sigma_j} x_{\sigma_j} = \phi_\eta \sigma_i$ by definition 5.1.4

$= \sum_{r\in J} \theta_{ir} x^{\sigma_r}$

$= \sum_{j\in J} \sum_{r\in J} \theta_{ir} v_{rj} x_{\sigma_j}$ by equation (5.2.1)

(ii) Let Y be the graded $\mathbb{C}$-algebra $\sum^{\oplus}_{m\geq 1} Y_m$ where Y_m is the

C-space with C-basis $\{ y_\delta \mid \delta \vdash m\}$ and multiplication is given by:

$$\text{For } \delta \vdash m \text{ and } \tau \vdash d \quad (m,d \in Z_+)$$

$$y_\delta y_\tau = \sum_{\sigma \vdash m+d} f_1(e(\sigma,\delta,\tau))y_\sigma$$

(Note that $e(\sigma,\delta,\tau)$ is defined by equation (4.3.1)).

Clearly f_1 can be extended to a ring homomorphism $f_1 : F_{D_1} \to Y$ where $F_{D_1} = \{ \sum_\delta \lambda_\delta x_\delta \mid \lambda_\delta \in K_{D_1}, \delta$ runs over any finite set of partitions$\}$ (see §1.4) by

$$f_1 : \sum_\delta \lambda_\delta x_\delta \longmapsto \sum_\delta f_1(\lambda_\delta) y_\delta$$

Let $\sigma = \langle 1^{\sigma_1} \ldots m^{\sigma_m} \rangle \vdash m$ $(m \in Z_+)$. By equation (5.1.7) $y_\sigma y_{\langle t \rangle} = (\sigma_t + 1) y_{\sigma + \langle t \rangle}$ for all $t \in Z_+$. Thus if we denote $(y_{\langle 1 \rangle})^{\delta_1} \ldots (y_{\langle m \rangle})^{\delta_m}$ by y^δ for any $\delta = \langle 1^{\delta_1} \ldots m^{\delta_m} \rangle \vdash m$,

(5.3.4) $$y^\delta = \delta_1 . \delta_2 \ldots \delta_m . y_\delta$$

Since $\emptyset_\eta \sigma_i = \sum_{j \in J} \theta_{ij} x^{\sigma_j} = \sum_{j \in J} (\eta^{\sigma_i})_{\sigma_j} x_{\sigma_j}$ we have that

$$\sum_{j \in J} f_1(\theta_{ij}) y^{\sigma_j} = \sum_{j \in J} (X^{\sigma_i})_{\sigma_j} y_{\sigma_j}$$

So by equation (5.3.4)

$$f_1(\theta_{ij}) = (X^{\sigma_i})_{\sigma_j} / \sigma_{j1}! \ldots \sigma_{jn}!$$

✻

To prove the main result in this section - theorem 5.3.9 - we need the following identity in the terms $N(\beta,\rho)$

LEMMA 5.3.5.

Let $\rho = \langle \rho_1 \ldots \rho_s \rangle$ and $\sigma = \langle 1^{\sigma_1} \ldots n^{\sigma_n} \rangle$ be elements of T_n. Denote the partition $\langle \rho_2, \ldots, \rho_s \rangle$ of $n-\rho_1$ by ρ', then

(5.3.6) $$N(\sigma,\rho) = \sum_{\substack{\tau \vdash \rho_1 \\ \lambda \vdash n-\rho_1 \\ \tau+\lambda = \sigma}} \frac{\sigma_1! \ldots \sigma_n!}{\tau_1! \ldots \tau_{\rho_1}! \lambda_1! \ldots \lambda_{n-\rho_1}!} N(\tau,\langle \rho_1 \rangle) N(\lambda,\rho')$$

where $\tau = < 1^{\tau_1} \ldots \rho_1^{\tau_{\rho_1}} >$ and $\lambda = < 1^{\lambda_1} \ldots (n-\rho_1)^{\lambda_{n-\rho_1}} >$

Proof:

Let $a = < 1^{a_1} \ldots m^{a_m} >$ and $b = < b_1, b_2, \ldots, b_t >$ be partitions of m ($m \in Z_+$). Define

$$S = \{ \{\delta_1, \delta_2, \ldots, \delta_t\} \mid \delta_j \vdash b_j \text{ for}$$

all $j \in Z_t$, $\delta_1 + \delta_2 + \ldots + \delta_t = a\}$ and put $M(a,b) = |S|$. Clearly
$N(a,b) = M(a,b) a_1! \ldots a_m! \sum_{\{\delta_1, \ldots, \delta_t\} \in S} \frac{1}{\delta_{11}! \delta_{12}! \ldots \delta_{21}! \ldots \delta_{t1}! \ldots}$
where $\delta_r = < 1^{\delta_{r1}} 2^{\delta_{r2}} \ldots >$ for all $r \in Z_t$ and if $b' = < b_2 \ldots b_t >$ then

$$M(a,b) = \sum_{\substack{\delta \vdash b_1 \\ \xi \vdash m-b_1 \\ \delta + \xi = a}} M(\delta, <b_1>) M(\xi, b')$$

Equation (5.3.6) can now readily be derived.

LEMMA 5.3.7.

$$\theta_{|J|j} = \frac{(X^{<n>})_{\sigma_j}}{\sigma_{j1}! \ldots \sigma_{jn}!} \cdot \frac{p(\sigma_j)}{1^{\sigma_{j1}} \ldots n^{\sigma_{jn}}}$$

Proof:

With the given ordering of the partitions we have $\sigma_{|J|} = < n >$ and by corollary 4.1.5 $\eta^{\sigma_{|J|}} = \mathfrak{I}_n$ the unit character of G_n. (see definition 4.1.3).

Thus by lemma 5.3.3(ii) and the fact that V_n is non-singular it is sufficient to show that

$$\sum_{j \in J} \frac{(X^{<n>})_{\sigma_j} \, p(\sigma_j)}{\sigma_{j1}! \ldots \sigma_{jn}! \; 1^{\sigma_{j1}} \ldots n^{\sigma_{jn}}} v_{ji} = (\mathfrak{I}_n)_{\sigma_i} \quad \text{for all } i \in J.$$

or equivalently (by theorem A2.7(ii) and lemma 4.1.2(ii)) that

$$\sum_{\sigma \vdash n} \frac{p(\sigma)\, N(\sigma,\rho)\, (u-1)^{|\sigma|-|\rho|}}{\sigma_1! \ldots \sigma_n! \; 1^{\sigma_1} \ldots n^{\sigma_n}} = u^{n-|\rho|} \quad \text{for all } \rho \in T_n$$

where $\sigma = < 1^{\sigma_1} \ldots n^{\sigma_n} >$

We use induction on n:

Fix $\rho = <\rho_1, \rho_2, \ldots, \rho_s>$ and let $\rho' = <\rho_2, .. , \rho_s>$

By equation (5.3.6) and Lemma 5.3.2

$$\sum_{\sigma \vdash n} \frac{p(\sigma)\ N(\sigma,\rho)\ (u-1)^{|\sigma|-|\rho|}}{\sigma_1! \ldots \sigma_n!\ 1^{\sigma_1} \ldots n^{\sigma_n}} =$$

$$\sum_{\substack{\tau \vdash \rho_1 \\ \lambda \vdash n-\rho_1}} \frac{p(\tau)N(\tau,<\rho_1>)(u-1)^{|\tau|-|<\rho_1>|}}{\tau_1! \ldots \tau_{\rho_1}!\ 1^{\tau_1} \ldots \rho_1^{\tau_{\rho_1}}} \cdot \frac{p(\lambda)N(\lambda,\rho')(u-1)^{|\lambda|-|\rho'|}}{\lambda_1! \ldots \lambda_{n-\rho_1}!\ 1^{\lambda_1} \ldots (n-\rho_1)^{\lambda_{n-\rho_1}}}$$

By the inductive hypothesis

$$\sum_{\tau} \frac{p(\tau)N(\tau,<\rho_1>)(u-1)^{|\tau|-|<\rho_1>|}}{\tau_1! \ldots \tau_{\rho_1}!\ 1^{\tau_1} \ldots \rho_1^{\tau_{\rho_1}}} = u^{\rho_1-|<\rho_1>|}$$

and $$\sum_{\lambda} \frac{p(\lambda)N(\lambda,\rho')(u-1)^{|\lambda|-|\rho'|}}{\lambda_1! \ldots \lambda_{n-\rho_1}!\ 1^{\lambda_1} \ldots (n-\rho_1)^{\lambda_{n-\rho_1}}} = u^{n-\rho_1-|\rho'|}$$

Since $u^{\rho_1-|<\rho_1>|} \cdot u^{n-\rho_1-|\rho'|} = u^{n-|\rho|}$ the proof is complete.

$\times\!\!\!\times$

LEMMA 5.3.8.

$$\theta_{ij} = \frac{(X^{\sigma_i})_{\sigma_j} p(\sigma_j)}{\sigma_{j1}! \ldots \sigma_{jn}!\ 1^{\sigma_{j1}} \ldots n^{\sigma_{jn}}}$$

Proof:

By lemma 5.3.3(ii) it is sufficient to show that $\theta_{ij} = q_{ij}p(\sigma_j)$ for some $q_{ij} \in Q$.

Lemmas 4.1.2(iii), 3.2.7(ii) and corollary 4.1.5 show that $\phi_{\eta}^{\sigma_i}$ is a linear combination over Z of characteristics of the type $\phi_{\alpha \circ \eta}$ where $\alpha \in X(G_t)$ and $\eta \in X(G_{n-t})$ for some $t \in Z_{n-1}$ and of $\phi_{\eta}^{\sigma_{|J|}}$ (note that $\sigma_{|J|} = <n>$). Thus by lemma 5.3.7 it is sufficient to prove that if $\phi_{\alpha\circ\eta} = \sum_{\sigma \vdash n} \theta_{\alpha\circ\eta\,\sigma} x^{\sigma}$ then $\theta_{\alpha\circ\eta\,\sigma} = q_{\alpha\circ\eta\,\sigma} p(\sigma)$ for some $q_{\alpha\circ\eta\,\sigma} \in Q$.

We use induction on n:

By the inductive hypothesis

$$\phi_{\alpha} = \sum_{\tau \vdash t} q_{\alpha\tau} p(\tau) x^{\tau} \text{ and } \phi_{\eta} = \sum_{\delta \vdash n-t} q_{\eta\delta} p(\delta) x^{\delta}$$

for some $q_{\alpha\tau}, q_{\eta\delta} \in Q$.

By theorem 5.1.5(iii) $\phi_{\alpha\circ\eta} = \phi_{\alpha}\phi_{\eta}$, so by lemma 5.3.2(i)

$$\phi_{\alpha\circ\eta} = \sum_{\substack{\tau \vdash t \\ \delta \vdash n-t}} q_{\alpha\tau} q_{\eta\delta} p(\tau+\delta) x^{\tau+\delta}$$

Thus $\theta_{\alpha\circ\eta\,\sigma} = \sum_{\substack{\tau \vdash t \\ \delta \vdash n-t \\ \delta+\tau=\sigma}} q_{\alpha\tau} q_{\eta\delta} p(\sigma)$

⋕

We are now able to state and prove a theorem which gives a complete solution to 'Part 2' (see below lemma 2.1.5) of the problem of evaluating the irreducible characters of G_n over K in terms of the values of the irreducible characters of S_n over C.

THEOREM 5.3.9.

$$(\eta^{\sigma_i})_{\sigma_j} =$$

$$\sum_{t \in J} \frac{(X^{\sigma_i})_{\sigma_t} \quad p(\sigma_t)}{\sigma_{t1}!\sigma_{t2}!\ldots\sigma_{tn}!\; 1^{\sigma_{t1}} 2^{\sigma_{t2}} \ldots n^{\sigma_{tn}}} N(\sigma_t,\sigma_j)\ (u-1)^{|\sigma_t|-|\sigma_j|}$$

for all $i,j \in J$.

Proof:

Immediate from lemmas 5.3.3(i), 5.3.7 and theorem 5.2.6(i).

⋕

The notation used to state theorem 5.3.9 can be found in theorem 4.1.4, corollary 4.1.5 and definitions 5.3.1(and above), 5.2.3(ii) and 5.1.1.

REMARK.

Theorem 5.3.9 and the remark below theorem 2.2.8 enable one to calculate the value of $\eta(g_w)$ for any irreducible character η of G_n and any element w of S_n, provided that the values of the irreducible characters of S_n over C are known.

Appendix 6 gives the values of η_σ for all irreducible characters η of G_n and all partitions σ of n for $n \in Z_6$.

Appendix 5 gives the values of $\eta(g_w)$ for all irreducible characters η of G_n and all elements w of S_n for $n \in Z_4$

Appendix 8 gives (implicitly) the values of $N(\beta,\rho)$ for all partitions β,ρ of 2,3 and 4.(see theorem 5.2.6(i)).

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AN EXAMPLE OF THE LINEAR DEPENDENCE OF CHARACTER VALUES. (See the remark below theorem 2.2.8).

Let W be of type B_3 (see [4, chapter 3]). Thus $W \cong \langle a,g,d \mid a^2 = g^2 = d^2 = (ag)^3 = (gd)^4 = (ad)^2 = 1_W \rangle$

We will identify W with this group. Let $c(a) = c(g) = c(d) = 1$ where c is the function from W to Z_+ described in Corollary 1.1.6.

The corresponding generic Hecke algebra $H(K,u)$ is thus generated as a K-algebra with identity h_1 by $\{h_a, h_g, h_d\}$ with the following defining relations;

(A.1.1) $h_x^2 = uh_1 + (u-1)h_x$ for all $x \in \{a,g,d\}$

(A.1.2) $h_a h_g h_a = h_g h_a h_g$

(A.1.3) $h_g h_d h_g h_d = h_d h_g h_d h_g$

(A.1.4) $h_a h_d = h_d h_a$

Also for any $w \in W$ we have

(A.1.5) $h_w = h_{r_1} h_{r_2} \ldots . h_{r_f}$ where $r_1 r_2 \ldots . r_f$ is any reduced expression for $w.(r_1, \ldots, r_f \in \{a,g,d\})$.

Let η belong to $X(H(K,u))$ then

$$\eta(h_{agdgag}) = \eta(h_{agdaga}) \quad \text{by (A.1.2)}$$
$$= \eta(h_a h_{gdag} h_a) \quad \text{by (A.1.5)}$$
$$= \eta(h_{gdag} h_a^2) \quad \text{by equations (2.1.4)}$$
$$= u\eta(h_{gdag}) + (u-1)\eta(h_{gdaga}) \quad \text{by}$$

(A.1.1)

In the same manner we find that

$$\eta(h_{gdag}) = u\eta(h_{da}) + (-1)\eta(h_{dag})$$
$$\eta(h_{gdaga}) = u\eta(h_{dag}) + (u-1)\eta(h_{dgag})$$

and $\eta(h_{dgag}) = u\eta(h_{dg}) + (u-1)\eta(h_{dag})$

Thus

$$\eta(h_{agdgag}) = u^2\eta(h_{da}) + (u-1)u\eta(h_{dg}) + (u-1)(u^2+1)\eta(h_{dag})$$

We note (without proof) that a set of class representatives of W consisting of elements of minimal length in their conjugacy classes can be chosen such that $\{da, dg, dag\}$ is a subset of it.

APPENDIX 2

THE SYMMETRIC GROUP

The symmetric group S_n on n symbols is the set of all permutations of the elements of Z_n with composition given by

$(\beta\sigma)(i) = \beta(\sigma(i))$ for all $\beta, \sigma \in S_n$ and $i \in Z_n$.

It is well known that each conjugacy class of S_n is the set of all elements of S_n of a fixed cycle-type. Denote the class of elements of cycle-type $1^{\alpha_1}2^{\alpha_2}\ldots n^{\alpha_n}$ by (α) where α is the partition $\langle 1^{\alpha_1}2^{\alpha_2}\ldots n^{\alpha_n}\rangle$ of n (see lemma 4.1.1).

THEOREM A2.1.

S_n is a Coxeter group. $\{\mu_i = (i\ i+1) \mid i \in Z_{n-1}\}$ is a set of distinguished generators and the following are defining relations for this set of generators.

(A2.2) $\mu_i^2 = 1_{S_n}$ for all $i \in Z_{n-1}$

(A2.3) $\mu_i\mu_j = \mu_j\mu_i$ for all $i,j \in Z_{n-1}$ with $i+2 \leq j$.

(A2.4) $\mu_i\mu_{i+1}\mu_i = \mu_{i+1}\mu_i\mu_{i+1}$ for all $i \in Z_{n-2}$.

Note that (A2.3) and (A2.4) can be expressed in the more usual froms; $(\mu_i\mu_j)^2 = 1_{S_n}$ and $(\mu_i\mu_{i+1})^3 = 1_{S_n}$.

Proof:

It is well known that S_n is generated by the set $\{\mu_i \mid i \in Z_{n-1}\}$. Thus it is sufficient to show that the group $U_n = \langle a_1, a_2, \ldots, a_{n-1} \mid a_i^2 = 1, (a_ia_j)^2 = 1$ for all $i,j \in Z_{n-1}, (a_ia_{i+1})^3 = 1$ for all $i \in Z_{n-2}\rangle$ has order less than or equal to n!. The set $\{a_ia_{i+1}\ldots a_{n-1}U_{n-1}\} \cup \{U_{n-1}\}$ is closed under left multiplication by elements of U_n, thus the result can be readily proved using induction on n.

※

LEMMA A2.5.

All the distinguished generators of S_n are conjugate in S_n.

※

COROLLARY A2.6.

If $\{c_{\mu_i} \mid i \in Z_{n-1}\}$ is an indexing system for S_n (see definition 1.1.8) then $c_{\mu_i} = c_{\mu_j}$ for all $i,j \in Z_{n-1}$.

※

Our main result in this appendix is

THEOREM A2.7.

(i) An element ρ in S_n is of minimal length in its conjugacy class if and only if in any reduced expression for ρ each μ_j ($j \in Z_{n-1}$) appears at most once.

(ii) The minimal length of an element of the conjugacy class (α) is $n - |\alpha|$. (see definition 5.1.1).

Proof:

Let $\rho \in (\alpha)$. Let ρ have a reduced expression in terms of the elements of $\{\mu_j \mid j \in J \subset Z_{n-1}\}$. Define an equivalence relation T on J by

iTj if and only if either $\{i,i+1,\ldots,j\} \subset J, (i < j)$ or $\{i,i-1,\ldots,j\} \subset J$, $(i \geqslant j)$.

and denote the equivalence classes with respect to this relation by $J_1, J_2, \ldots, J_t$.

Since for $i \in J_r$ and $j \in J_s$, $\mu_i\mu_j = \mu_j\mu_i$ provided that $r \neq s$, we can write the reduced expression of ρ in the form

$\gamma_1\gamma_2\ldots\gamma_t$ where γ_r is a product of elements of $\{\mu_i \mid i \in J_r\}$ $(r \in Z_t)$.

Clearly γ_r moves at most $|J_r|+1$ symbols (i.e. has at least $n-|J_r|-1$ fixed symbols). Thus if ρ moves λ points

$$\lambda \leq \sum_{r=1}^{t} (|J_r|+1) = \sum_{r=1}^{t} |J_r| + t = |J| + t$$

Clearly t is less than or equal to the number of non-1-cycles of ρ which equals $|\alpha| - \alpha_1$, where $\alpha = \langle 1^{\alpha_1} 2^{\alpha_2} \ldots n^{\alpha_n} \rangle$. Also $|J| \leq l(\rho)$. So

$$\lambda \leq l(\rho) + |\alpha| - \alpha_1$$

But $\lambda = n - \alpha_1$, thus $n - |\alpha| \leq l(\rho)$ with equality only if $|J| = l(\rho)$ i.e. each u_j appears at most once in the reduced expression for ρ.

(ii) and the 'only if' part of (i) now follow since δ in (α) given by

$$\delta = (1)(2)\ldots(\alpha_1)(\alpha_1+1\ \ \alpha_1+2)\ldots(\alpha_1+2\alpha_2-1\ \ \alpha_1+2\alpha_2)\ldots.$$

has length $\alpha_2+2\alpha_3+\ldots+(n-1)\alpha_n = n - |\alpha|$.

The proof is completed by showing that the product $\prod_{i\in J_r} u_i$ taken in any order is a $(|J_r|+1)$- cycle.

Clearly it is sufficient to prove

(A2.8) $\prod_{j=1}^{n-1} u_{\sigma(j)}$ is an n-cycle for any σ in S_{n-1}.

We proceed by induction on n.

Let $\rho_0 = \prod_{j=1}^{n-2} \mu_{\sigma(j)}$, $\rho_1 = \prod_{\substack{j=1 \\ \sigma(j)<\sigma(n-1)}}^{n-2} \mu_{\sigma(j)}$ and $\rho_2 = \prod_{\substack{j=1 \\ \sigma(j)>\sigma(n-1)}}^{n-2} \mu_{\sigma(j)}$

Clearly $\rho_0 = \rho_1\rho_2$ and by the inductive hypothesis ρ_1 is a $\sigma(n-1)$-cycle containing the symbol $\sigma(n-1)$ and ρ_2 is an $(n-\sigma(n-1))$-cycle containing the symbol $\sigma(n-1)+1$. Since $\rho = \rho_0\mu_{\sigma(n-1)}$, ρ is an n-cycle and (A2.8) is proved.

※

The length function on S_n can be evaluated using the following result

LEMMA A2.9.

Let $\rho = \begin{pmatrix} 1 & 2 & \ldots & n \\ \rho_1 & \rho_2 & \ldots & \rho_n \end{pmatrix} \in S_n$.

Put $m_i = |\{ j \mid j < i, \rho_j > \rho_i \}|$ for each $i \in Z_n$. Then

$$l(\rho) = \sum_{i=1}^{n} m_i$$

Proof:

S_n is isomorphic to the Weyl group of type A_{n-1} (see [4, page 124]). Using [4, lemma 2.2.1 and theorem 2.2.2] one can show that

$$l(\rho\mu_i) = \begin{cases} l(\rho) + 1 & \text{if } \rho_i < \rho_{i+1} \\ l(\rho) - 1 & \text{if } \rho_i > \rho_{i+1} \end{cases}$$

The lemma now follows by induction on $l(\rho)$.

⋇

APPENDIX 3

SPECIAL TRANSVERSALS IN COXETER GROUPS.

Let (W,R) be a Coxeter system (see definition 1.1.1).

DEFINITION A3.1.

Let $J \subset R$. The subgroup $W_J = \langle r \mid r \in J \rangle$ is the <u>parabolic subgroup of W</u> associated with J.

LEMMA A3.2.

(W_J, J) is a Coxeter system for each $J \subset R$.

Proof: [2, chapter 4, §1, theorem 2(i)].

※

THEOREM A3.3.

Let $J \subset R$. In each left coset of W_J in W there exists a unique element of minimal length. Further if d is such an element then

$$l(dw) = l(d) + l(w) \quad \text{for all } w \in W_J.$$

Proof:

It is sufficient to prove that $l(dw) = l(d) + l(w)$ for all $w \in W_J$, since this clearly implies that d is the unique element of minimal length in dW_J.

We use induction on $l(w)$. Let $w \in W_J$, $r \in R$ and assume that $l(dw) = l(d) + l(w)$ and $l(wr) = l(w) + 1$. Suppose that $l(dwr) \neq l(d) + l(wr)$. By lemma 1.1.4, $l(dwr) = l(dw) - 1$ and by the 'Exchange condition' (see [,chapter4,§1.5])

either $dwr = dw'$ where $w' \in W_J$ and $l(w') = l(w) - 1$, which gives a contradiction since $l(wr) = l(w) + 1$.

or $dwr = d'w$ where $l(d') = l(d) - 1$. In which case

$dW_J = d'W_J$ and we again get a contradiction since d is of minimal length in dW_J.

Thus $l(dwr) = l(d) + l(wr)$ and the theorem follows by induction.

※

DEFINITION A3.4.

The special transversal of W_J in W is the set of left coset representatives each of which is of minimal length in its coset (see theorem A3.3). This transversal will be denoted by D_J^R.

LEMMA A3.5.

Given $J \subset R$, $d \in D_J^R$ and $r \in R$ then

$$\text{either } rd \in dW_J \qquad \text{or } rd \in D_J^R$$

Proof:

Assume that $rd \notin D_J^R$. There exists $w \in W_J$ with $l(rdw) < l(rd) + l(w)$. Infact $l(rdw) = l(dw) - 1$ and by the 'Exchange condition'

either $rdw = dw'$ where $w' \in W_J$, so that $rd \in dW_J$.

or $rdw = d'w$ where $l(d') = l(d) - 1$. In which case $rd = d'$ and if $y \in W_J$ then

$$l(rdy) \geqslant l(dy) - 1 = l(d) - 1 + l(y) = l(d') + l(y) \geqslant l(d'y)$$

So $l(rdy) = l(rd) + l(y)$ showing that $rd \in D_J^R$, a contradiction.

※

COROLLARY A3.6.

Let $J \subset R$, $d \in D_J^R$ and $r \in R$. If $l(rd) < l(d)$ then $rd \in D_J^R$.

Proof:

Immediate from lemma A3.5 and definition A3.4.

※

LEMMA A3.7.

Fix $r \in R$ and $J \subset R$. Let $C(r)$ be the conjugacy class of r in W. Let

$$S_1 = \{ d \in D_J^R \mid rd \in dW_J \}$$

$$S_2 = \{ d \in D_J^R \mid rd \in D_J^R,\ l(rd) < l(d) \}$$

$$S_3 = \{ d \in D_J^R \mid rd \in D_J^R,\ l(rd) > l(d) \}$$

Then

$$|S_1| = \frac{|C(r) \cap W_J|\,|W|}{|C(r)|\,|W_J|}$$

$$|S_2| = 1/2\frac{|C(r) \setminus W_J|\,|W|}{|C(r)|\,|W_J|} = |S_3|$$

Proof:

By lemma A3.5 $|S_1| + |S_2| + |S_3| = |W|/|W_J|$. Since S_1 clearly has the given order it is thus sufficient to show that $|S_2| = |S_3|$. Let $d \in S_2 \cup S_3$. Since $r(rd) = d \in D_J^R$ we have that $rd \in S_2 \cup S_3$. Thus $r(S_2 \cup S_3) = S_2 \cup S_3$. By lemma 1.1.4, for any $w \in W$, $l(rw) = l(w) \pm 1$ and so $|S_2| = |S_3|$.

⋇

We now give an explicit description of D_J^R for a particular Coxeter group W.

THEOREM A3.8.

Let $W = S_n$ $(n \in Z_+)$ and $R = \{u_i = (i\ i{+}1) \mid i \in Z_{n-1}\}$ (see theorem A2.1). Given $\lambda = \langle \lambda_1, \lambda_2, \ldots, \lambda_r \rangle \vdash n$ with $0 < \lambda_r \leq \lambda_{r-1} \leq \ldots \leq \lambda_1$, let J_λ be the subset of R described above lemma 4.1.1 then

$$D_{J_\lambda}^R = \left\{ \begin{pmatrix} 1 & 2 & \ldots & \lambda_1 & \lambda_1+1 & \ldots & \lambda_1+\lambda_2 & \ldots & & \ldots n \\ a_{11} & a_{12} & \ldots & a_{1\lambda_1} & a_{21} & \ldots & a_{2\lambda_2} & \ldots & a_{r1} & \ldots a_{r\lambda_r} \end{pmatrix} \in S_n \;\middle|\; a_{i1} < a_{i2} < \ldots < a_{i\lambda_i} \text{ for all } i \in Z_r \right\}$$

Proof:

The given set of elements of S_n has order $|S_n|/|S_n^\lambda|$ (see lemma 4.1.1(ii) and (v)) and since for ρ , $\sigma \in S_n$, $\rho S_n = \sigma S_n$ is equivalent to $\rho^{-1}\sigma \in S_n$, it is clear that the given set is a transversal for S_n^λ in S_n. That it is the special transversal is clear from lemma A2.9.

APPENDIX 4.

THE DECOMPOSITION MATRIX OF H_o.

The decomposition matrix (see definition 1.5.6) of H_o (see 1.5) is given below in the cases where W is S_1, S_2, S_3, S_4, S_5 and S_6. (see theorem A2.1). We denote the algebra H_o by $\Gamma_1, \ldots, \Gamma_6$ in these cases respectively.

The notation for the irreducible characters of G_n (see the introduction to Part II) $(n \in Z_6)$ is that described by corollray 4.1.5.

In the matrices below all the omitted entries are zreo.

Γ_1	δ
$\eta^{\langle 1\rangle}$	1

Γ_2	δ_0	δ_1
$\eta^{\langle 2\rangle}$	1	
$\eta^{\langle 1^2\rangle}$		1

Γ_3	δ_{00}	δ_{01}	δ_{10}	δ_{11}
$\eta^{\langle 3\rangle}$	1			
$\eta^{\langle 12\rangle}$		1	1	
$\eta^{\langle 1^3\rangle}$				1

Γ_4	δ_{000}	δ_{001}	δ_{010}	δ_{100}	δ_{011}	δ_{101}	δ_{110}	δ_{111}
$\eta^{\langle 4\rangle}$	1							
$\eta^{\langle 13\rangle}$		1	1	1				
$\eta^{\langle 2^2\rangle}$			1			1		
$\eta^{\langle 1^2 2\rangle}$					1	1	1	
$\eta^{\langle 1^4\rangle}$								1

Γ_5	δ_{0000}	δ_{0001}	δ_{0010}	δ_{0100}	δ_{1001}	δ_{0011}	δ_{0101}	δ_{1001}	δ_{0110}	δ_{1010}	δ_{1100}	δ_{0111}	δ_{1011}	δ_{1101}	δ_{1110}	δ_{1111}
$\eta^{\langle 5\rangle}$	1															
$\eta^{\langle 14\rangle}$		1	1	1	1											
$\eta^{\langle 23\rangle}$			1	1			1	1		1						
$\eta^{\langle 1^2 3\rangle}$						1	1	1	1	1	1					
$\eta^{\langle 12^2\rangle}$							1		1	1			1	1		
$\eta^{\langle 1^3 2\rangle}$												1	1	1	1	
$\eta^{\langle 1^5\rangle}$																1

Γ_6	$\eta^{\langle 6\rangle}$	$\eta^{\langle 15\rangle}$	$\eta^{\langle 24\rangle}$	$\eta^{\langle 1^2 4\rangle}$	$\eta^{\langle 3^2\rangle}$	$\eta^{\langle 123\rangle}$	$\eta^{\langle 2^3\rangle}$	$\eta^{\langle 1^3 3\rangle}$	$\eta^{\langle 1^2 2^2\rangle}$	$\eta^{\langle 1^4 2\rangle}$	$\eta^{\langle 1^6\rangle}$
δ_{11111}											1
δ_{11110}										1	
δ_{11101}									1	1	
δ_{11011}							1		1	1	
δ_{10111}										1	
δ_{01111}										1	
δ_{11100}								1			
δ_{11010}						1		1	1		
δ_{10110}						1	1	1	1		
δ_{01110}								1	1		
δ_{11001}						1		1			
δ_{10101}					1	2	1	1	1		
δ_{01101}						1	1	1	1		
δ_{10011}						1		1			
δ_{01011}						1		1	1		
δ_{00111}								1			
δ_{11000}				1							
δ_{10100}			1	1		1					
δ_{01100}				1		1					
δ_{10010}			1	1	1	1					
δ_{01010}			1	1	1	2	1				
δ_{00110}				1		1					
δ_{10001}			1	1							
δ_{01001}			1	1	1	1					
δ_{00101}			1	1		1					
δ_{00011}				1							
δ_{10000}		1									
δ_{01000}		1	1								
δ_{00100}		1	1		1						
δ_{00010}		1	1								
δ_{00001}		1									
δ_{00000}	1										

APPENDIX 5

CHARACTER TABLES (Part 1).

The character tables (see definition 2.1.1) of the generic Hecke algebras G_1, G_2, G_3 and G_4 (see introdution to Part II) are given below. (That of G_4 is transposed). The notation used for the irreducible characters is that described in corollary 4.1.5 and cycle notation is used for elements of the Weyl groups.

The character table of the generic Hecke algebra whose Weyl group is of type B_2 (see part (ii) of the discusion above theorem 2.2.8) is also given. The algebra is denoted by H_{B_2} in this case.

H_{B_2}	1	a	gag	g	aga	ag	ga	agag
η^1	1	u	u^3	u	u^3	u^2	u^2	u^4
η^2	1	u	u	-1	$-u^2$	$-u$	$-u$	u^2
η^3	2	$u-1$	$u(u-1)$	$u-1$	$u(u-1)$	0	0	$-2u^2$
η^4	1	-1	$-u^2$	u	u	$-u$	$-u$	u^2
η^5	1	-1	-1	-1	-1	1	1	1

G_1	1
$\eta^{\langle 1 \rangle}$	1

G_2	1	(12)
$\eta^{\langle 2 \rangle}$	1	u
$\eta^{\langle 1^2 \rangle}$	1	-1

G_3	1	(12)	(23)	(13)	(123)	(132)
$\eta^{\langle 3 \rangle}$	1	u	u	u^3	u^2	u^2
$\eta^{\langle 12 \rangle}$	2	$u-1$	$u-1$	0	$-u$	$-u$
$\eta^{\langle 1^3 \rangle}$	1	-1	-1	-1	1	1

G_4	$\eta^{\langle 4\rangle}$	$\eta^{\langle 13\rangle}$	$\eta^{\langle 2^2\rangle}$	$\eta^{\langle 1^2 2\rangle}$	$\eta^{\langle 1^4\rangle}$
1	1	3	2	3	1
(12)	u	$2u-1$	$u-1$	$u-2$	-1
(23)	u	$2u-1$	$u-1$	$u-2$	-1
(34)	u	$2u-1$	$u-1$	$u-2$	-1
(13)	u^3	u^3	0	-1	-1
(24)	u^3	u^3	0	-1	-1
(14)	u^5	u^4	$-u^2(u-1)$	$-u$	-1
(123)	u^2	$u(u-1)$	$-u$	$-u+1$	1
(132)	u^2	$u(u-1)$	$-u$	$-u+1$	1
(243)	u^2	$u(u-1)$	$-u$	$-u+1$	1
(234)	u^2	$u(u-1)$	$-u$	$-u+1$	1
(143)	u^4	0	$-u^2$	0	1
(134)	u^4	0	$-u^2$	0	1
(142)	u^4	0	$-u^2$	0	1
(124)	u^4	0	$-u^2$	0	1
(1432)	u^3	$-u^2$	0	u	-1
(1234)	u^3	$-u^2$	0	u	-1
(1342)	u^3	$-u^2$	0	u	-1
(1243)	u^3	$-u^2$	0	u	-1
(1324)	u^5	$-u^3$	$-u^2(u-1)$	u^2	-1
(1423)	u^5	$-u^3$	$-u^2(u-1)$	u^2	-1
(12)(34)	u^2	$u(u-2)$	u^2+1	$-2u+1$	1
(13)(24)	u^4	$-u^2$	u^3+u	$-u^2$	1
(14)(23)	u^6	$-u^4$	$2u^3$	$-u^2$	1

APPENDIX 6

CHARACTER TABLES (Part 2).

Let η be an irreducible character of G_n (see the introduction to Part II) and let $\sigma \vdash \eta$. Then since η is a subclass function (see corollary 4.2.10(i) and definition 4.2.1(ii)) we can define

$\eta_\sigma = \eta(g_w)$ for any w in the subclass $(\underline{\sigma})$ (see definition 4.2.1(i)). Further by corollary 4.2.10(ii)

$\eta_\sigma = \eta(g_v)$ for any v of minimal length in the conjugacy class (σ).

The values of η_σ for all irreducible characters of G_n and all partitions σ of n are given below for $n \in Z_6$. The values are displayed in tables, one for each n, as follows: The rows are indexed by theirreducible characters of G_n (in the notation of corollary 4.1.5) and the columns are indexed with the partitions of n. The $(\eta\ \sigma)^{th}$ entry is an ordered tuple say $a_r, a_{r-1}, \ldots, a_o$ of non-negative integers with the following significance:

$$\eta_\sigma = \delta_r a_r u^r + \delta_{r-1} a_{r-1} u^{r-1} + \ldots + \delta_1 a_1 u + \delta_o a_o$$

where δ_i is the sign (+ or −) at the head of the column containing a_i $(i \in Z_r)$.

With this interpretation the columns of the tables are columns of the character table of G_n (see definition 2.1.1), infact they are the columns corresponding to a set of class representatives of S_n each of which is of minimal length in its class. Theorem 2.1.2 shows that these columns span the column space of the character table.

That the entris of the given tables are all integers follows from theorem 4.2.13.

There is an obvious symmetry in each of the given tables; namely that in each column the first entry is the reverse of the last entry and the second entry is the reverse of the penultimate entry etc..This symmetry can readily be explained using the involutory semi-linear automorphism of the generic Hecke algebra described in [6, §8].

G_1	$\langle 1\rangle$
	+
$\eta^{\langle 1\rangle}$	1

G_2	$\langle 1^2\rangle$	$\langle 2\rangle$	
	+	+	−
$\eta^{\langle 2\rangle}$	1	1	0
$\eta^{\langle 1^2\rangle}$	1	0	1

G_3	$\langle 1^3\rangle$	$\langle 12\rangle$		$\langle 3\rangle$		
	+	+	−	+	−	+
$\eta^{\langle 3\rangle}$	1	1	0	1	0	0
$\eta^{\langle 12\rangle}$	2	1	1	0	1	0
$\eta^{\langle 1^3\rangle}$	1	0	1	0	0	1

G_4	$\langle 1^4\rangle$	$\langle 1^2 2\rangle$		$\langle 13\rangle$			$\langle 4\rangle$				$\langle 2^2\rangle$		
	+	+	−	+	−	+	+	−	+	−	+	−	+
$\eta^{\langle 4\rangle}$	1	1	0	1	0	0	1	0	0	0	1	0	0
$\eta^{\langle 13\rangle}$	3	2	1	1	1	0	0	1	0	0	1	2	0
$\eta^{\langle 2^2\rangle}$	2	1	1	0	1	0	0	0	0	0	1	0	1
$\eta^{\langle 1^2 2\rangle}$	3	1	2	0	1	1	0	0	1	0	0	2	1
$\eta^{\langle 1^4\rangle}$	1	0	1	0	0	1	0	0	0	1	0	0	1

G_5	$\langle 1^5 \rangle$	$\langle 1^3 2 \rangle$		$\langle 1^2 3 \rangle$			$\langle 14 \rangle$				$\langle 1 2^2 \rangle$			$\langle 23 \rangle$				$\langle 5 \rangle$				
	+	+	−	+	−	+	+	−	+	−	+	−	+	+	−	+	−	+	−	+	−	+
$\eta^{\langle 5 \rangle}$	1	1	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0
$\eta^{\langle 14 \rangle}$	4	3	1	2	1	0	1	1	0	0	2	2	0	1	2	0	0	0	1	0	0	0
$\eta^{\langle 23 \rangle}$	5	3	2	1	2	0	0	1	0	0	2	2	1	1	1	1	0	0	0	0	0	0
$\eta^{\langle 1^2 3 \rangle}$	6	3	3	1	2	1	0	1	1	0	1	4	1	0	2	2	0	0	0	1	0	0
$\eta^{\langle 1 2^2 \rangle}$	5	2	3	0	2	1	0	0	1	0	1	2	2	0	1	1	1	0	0	0	0	0
$\eta^{\langle 1^3 2 \rangle}$	4	1	3	0	1	2	0	0	1	1	0	2	2	0	0	2	1	0	0	0	1	0
$\eta^{\langle 1^5 \rangle}$	1	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1

G_6	$\langle 1^6\rangle$	$\langle 1^4 2\rangle$		$\langle 1^3 3\rangle$			$\langle 1^2 4\rangle$				$\langle 1^2 2^2\rangle$			$\langle 123\rangle$			
	+	+	−	+	−	+	+	−	+	−	+	−	+	+	−	+	−
$\eta^{\langle 6\rangle}$	1	1	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0
$\eta^{\langle 15\rangle}$	5	4	1	3	1	0	2	1	0	0	3	2	0	2	2	0	0
$\eta^{\langle 24\rangle}$	9	6	3	3	3	0	1	2	0	0	4	4	1	2	3	1	0
$\eta^{\langle 1^2 4\rangle}$	10	6	4	3	3	1	1	2	1	0	3	6	1	1	4	2	0
$\eta^{\langle 3^2\rangle}$	5	3	2	1	2	0	0	1	0	0	2	2	1	1	1	1	0
$\eta^{\langle 123\rangle}$	16	8	8	2	6	2	0	2	2	0	4	8	4	1	4	4	1
$\eta^{\langle 2^3\rangle}$	5	2	3	0	2	1	0	0	1	0	1	2	2	0	1	1	1
$\eta^{\langle 1^3 3\rangle}$	10	4	6	1	3	3	0	1	2	1	1	6	3	0	2	4	1
$\eta^{\langle 1^2 2^2\rangle}$	9	3	6	0	3	3	0	0	2	1	1	4	4	0	1	3	2
$\eta^{\langle 1^4 2\rangle}$	5	1	4	0	1	3	0	0	1	2	0	2	3	0	0	2	2
$\eta^{\langle 1^6\rangle}$	1	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	1

G_6	$\langle 15\rangle$					$\langle 6\rangle$						$\langle 24\rangle$					$\langle 2^3\rangle$				$\langle 3^2\rangle$				
	+	−	+	−	+	+	−	+	−	+	−	+	−	+	−	+	+	−	+	−	+	−	+	−	+
$\eta^{\langle 6\rangle}$	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0
$\eta^{\langle 15\rangle}$	1	1	0	0	0	0	1	0	0	0	0	1	2	0	0	0	2	3	0	0	1	2	0	0	0
$\eta^{\langle 24\rangle}$	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0	0	3	3	3	0	1	2	1	0	0
$\eta^{\langle 1^2 4\rangle}$	0	1	1	0	0	0	0	1	0	0	0	0	2	2	0	0	1	6	3	0	0	2	3	0	0
$\eta^{\langle 3^2\rangle}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	3	0	1	1	0	1	0	0
$\eta^{\langle 123\rangle}$	0	0	1	0	0	0	0	0	0	0	0	0	1	2	1	0	2	6	6	2	0	2	2	2	0
$\eta^{\langle 2^3\rangle}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	3	1	0	0	1	0	1
$\eta^{\langle 1^3 3\rangle}$	0	0	1	1	0	0	0	0	1	0	0	0	0	2	2	0	0	3	6	1	0	0	3	2	0
$\eta^{\langle 1^2 2^2\rangle}$	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0	3	3	3	0	0	1	2	1
$\eta^{\langle 1^4 2\rangle}$	0	0	0	1	1	0	0	0	0	1	0	0	0	0	2	1	0	0	3	2	0	0	0	2	1
$\eta^{\langle 1^6\rangle}$	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1

APPENDIX 7.

LINEAR DEPENDENCE IN THE CHARACTER TABLE.

Let B be a set of conjugacy class representative in W. Corollary 2.1.3 shows how the linear dependence of the columns of the character table of the generic Hecke algebra (see definition 2.1.1) can be described in terms of the elements $\sigma(w,b)$ of K. The values of $\sigma(w,b)$ are given below for the cases where W is S_1, S_2, S_3 and S_4 (see theorem A2.1) thus the generic Hecke algebra is $G_1, \ldots, G_4$ respectively (see the introduction to Part II). In each case B is chosen to be a set of class representative such that each element of it is of minimal length in its class. By corollary 4.2.10(ii) the values of $\sigma(w,b)$ obtained are independent of the particular choice of B.

The values are given in the form of tables, one for each n ($n \in Z_4$). The rows are indexed by the elements of W and the columns are indexed by the elements of B. The $(w\ b)^{th}$ entry is $\sigma(w,b)$. (All omitted entries are zero).

G_1	1
1	1

G_2	1	(12)
1	1	
(12)		1

G_3	1	(12)	(123)
1	1		
(12)		1	
(23)		1	
(123)			1
(132)			1
(13)		u	u - 1

G_4	1	(12)	(123)	(1234)	$(12)(34)$
1	1				
(12)		1			
(23)		1			
(34)		1			
(13)		u	$u-1$		
(24)		u	$u-1$		
(14)		u^2	$2u(u-1)$	$(u-1)^2$	
(123)			1		
(132)			1		
(243)			1		
(234)			1		
(143)			u	$u-1$	
(134)			u	$u-1$	
(142)			u	$u-1$	
(124)			u	$u-1$	
(1432)				1	
(1234)				1	
(1342)				1	
(1243)				1	
(1324)			$u(u-1)$	u^2-u+1	
(1423)			$u(u-1)$	u^2-u+1	
$(12)(34)$					1
$(13)(24)$				$u-1$	u
$(14)(23)$			$u(u-1)^2$	$(u-1)(u^2+1)$	u^2

APPENDIX 8.

V_n and Θ_n

V_n is defined by 5.2.2. Θ_n is defined at the beginning of 5.3. They are both matrices over $Q[u]$ (see theorem 5.2.6 and lemma 5.3.8) and have the property that the $(i\ j)^{th}$ entry of their product $\Theta_n V_n$ is $(\eta^{\sigma_i})_{\sigma_j}$ (see lemma 5.3.3). η^{σ_i} is an irreducible character of G_n (see the introduction to Part II) and σ_j is a partition of n (see corollary 4.2.10(i) and below definition 4.2.1). Infact for each $n \in Z_6$ the matrix $\Phi_n = \{(\eta^{\sigma_i})_{\sigma_j}\}_{ij}$ is the 'table' for G_n given in appendix 6 after the correct rearrangement of rows and of columns (and interpreting the tables of appendix 6 as described there).

V_2, V_3, V_4 and $\Theta_2, \Theta_3, \Theta_4$ are given below and in each cases the order in which the partitions of n have been used to index the rows and columns is stated.

$$V_2 = \begin{pmatrix} 2 & u-1 \\ 0 & 1 \end{pmatrix} \qquad \Theta_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}(u+1) \\ \frac{1}{2} & -\frac{1}{2}(u+1) \end{pmatrix}$$

Order of partitions: $<1^2>, <2>$.

$$V_3 = \begin{pmatrix} 6 & 3(u-1) & (u-1)^2 \\ 0 & 1 & (u-1) \\ 0 & 0 & 1 \end{pmatrix} \qquad \Theta_3 = \begin{pmatrix} \frac{1}{6} & \frac{1}{2}(u+1) & \frac{1}{3}(u^2+u+1) \\ \frac{1}{3} & 0 & -\frac{1}{3}(u^2+u+1) \\ \frac{1}{6} & -\frac{1}{2}(u+1) & \frac{1}{3}(u^2+u+1) \end{pmatrix}$$

Order of partitions: $<1^3>, <12>, <3>$.

$$V_4 = \begin{pmatrix} 24 & 12(u-1) & 6(u-1)^2 & 4(u-1)^2 & (u-1)^3 \\ 0 & 2 & 2(u-1) & 2(u-1) & (u-1)^2 \\ 0 & 0 & 2 & 0 & (u-1) \\ 0 & 0 & 0 & 1 & (u-1) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Theta_4 = \begin{pmatrix} \frac{1}{24} & \frac{1}{4}(u+1) & \frac{1}{8}(u^2+2u+1) & \frac{1}{3}(u^2+u+1) & \frac{1}{4}(u^3+u^2+u+1) \\ \frac{1}{8} & -\frac{1}{4}(u+1) & -\frac{1}{8}(u^2+2u+1) & 0 & -\frac{1}{4}(u^3+u^2+u+1) \\ \frac{1}{12} & 0 & \frac{1}{4}(u^2+2u+1) & -\frac{1}{3}(u^2+u+1) & 0 \\ \frac{1}{8} & \frac{1}{4}(u+1) & -\frac{1}{8}(u^2+2u+1) & 0 & \frac{1}{4}(u^3+u^2+u+1) \\ \frac{1}{24} & -\frac{1}{4}(u+1) & \frac{1}{8}(u^2+2u+1) & \frac{1}{3}(u^2+u+1) & -\frac{1}{4}(u^3+u^2+u+1) \end{pmatrix}$$

Order of partitions:

$<1^4>, <1^2 2>, <2^2>, <13>, <4>$.