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Symbolic Manipulations Related to Certain Aspects Such as Interpretations of Graphs

Maselan Bin Ali

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ABSTRACT

This thesis describes an investigation into university students' manipulation of symbols in solving calculus problems, and relates this to other aspects such as drawing and interpretation of graphs. It is concerned with identifying differences between students who are successful with symbol manipulation and those who are less successful.

It was initially expected that the more successful would have flexible and efficient symbolic methods whilst the less successful would tend to have single procedures which would be more likely to break down. Krutetskii (1976) noted that more successful problem-solvers curtail their solutions whilst the less able are less likely to acquire that ability even after a long practice. This suggested a possible correlation between success and curtailment. An initial pilot study with mathematics education students at a British University showed that in carrying out the algorithms of the calculus, successful students would often work steadily in great detail, however, they were more likely to have a variety of approaches available and were more likely to use conceptual ideas to simplify their task. However, the efficiency in handling symbolic manipulation may not be an indication that the students are able to relate their computational outcome to graphical ideas.

A modified pilot test was trialed at the Universiti Teknologi Malaysia before a main study at the same university in which 36 second year students were investigated in three groups of twelve, having grades A, B, C respectively in their first year examination.

The findings of this research indicate that there is no significant correlation between ability and curtailment, but ability correlates with conceptual preparation of procedures where there is an appropriate simplification to make the application of the algorithm simpler. The more able students may have several flexible strategies and meaningful symbolic mathematical representations but these may not always relate to visual and graphical ideas. On the other hand the less able students are less likely to break away from the security of a single procedure and liable to breakdown in getting the solutions for the calculus problems.

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Declaration

I declare that the material in this thesis has not been previously presented for any degree at any university. I further declare that the research presented in this thesis is entirely my unaided work. The only exception is that some of the parts in Chapter 8 is a modified version of joint paper Prof. David Tall and myself which is to be published in the Proceedings of the twentieth International Conference for the Psychology of Mathematics Education, Valencia (Spain).

CHAPTER 1

OVERVIEW

This thesis is focused on second year university students' manipulation of symbols in solving calculus problems, related to other aspects such as drawing and interpretation of graphs. It is concerned with identifying differences between students who are successful with symbol manipulation and those who are less successful.

Krutetskii (1976) observed that gifted students often solved problems in a very few steps, the capable students tend to *curtail* their solutions effectively while the incapable tend to fail in using curtailed solutions even after a long practice. This not only makes the solutions more efficient for the more successful, it makes them easier to manipulate mentally. The variation in the number of steps used by the students indicates that some compression takes place. In curtailment of solutions, the capable students use their strong conceptual linkages to produce effective solutions. The less able students have fragile conceptual and cognitive linkages. Thus they tend to face breakdown of procedures as they fail to link meaningfully between bits of relevant mathematical information in their long solutions.

When looking at students' different mathematical abilities, it is apparent that there is a range of different procedural lengths in the number of steps performed by these students. In other words, some students miss out certain steps in getting the answers for certain problems. I called this phenomenon *procedural compression* to indicate whenever two or more steps being compressed to a single step. Krutetskii (1976) used the term *curtailment.*

Students may have several meaningful representations of mathematical concepts. If the representations are rightly and strongly linked these students have no difficulty in switching rapidly and freely from one representation of a mathematical concept to another and are able to look for the one which is comfortable to perform in order to get to the solution of the mathematical problem more easily. In other words these students are able to move easily from one way of thinking to another which greatly increases their chances of solving a given mathematical problem. Such links of representations are less apparent in weaker students. To acquire such ability is likely to overburden them as they have to rely upon other links of representation which have already burdened them. As a result of that these students often restrict themselves to a single secure representation.

Another type of compression which I also suspected I initially termed *conceptual compression* which involves conceptual linkages to simplify technical complexity. In this type of compression, the students are reorganizing material, so when it comes to carrying out regular procedures, the actual techniques in the procedures are simplified. For example, when finding $\frac{dy}{dx}$ for the function $y = \frac{1+x^2}{x^2}$, conceptually more able students may simplify the expression before carrying out the differentiation, but less successful students may attack it straight away as the derivative of a quotient. In this example the initial simplification reduces the number of steps used, but it may happen that using conceptual linkages may improve the security of the solution without necessarily reducing the number of steps involved. Hence this type of simplification prior to using a standard procedure will be called *conceptual preparation* of the procedure rather than conceptual compression.

Other studies on students abilities and difficulties in manipulation of calculus symbols are widespread (Orton, 1983a, 1983b; Selden, Mason & Selden, 1989, 1990; Shin 1993 and Mundy, 1984); none of the studies indicate compression of students' procedures in trying to get the answer for a calculus problem.

In other areas, there are studies that indicate some younger students compress their symbolised procedures when performing arithmetic operations (Krutetskii. 1976; Gray & Tall, 1994).

In Chapter 2, studies on young students' performance in using mathematical symbolism are described. According to Thurston (1990), it is the compression of mathematical information that make successful students able to see relationships between notions at the university level.

In younger students' mathematics, there were at least two kinds of procedures available: visually moderating sequences (VMS) and integrated sequences (Davis 1983). According to him, in VMS, symbols are manipulated and written down and this cues the next symbolic manipulation. In an integrated sequence, the mathematical algorithm is conceived as whole and may be broken down systematically into smaller component sequences.

Hiebert and Lefevre (1986) distinguished the between procedural and conceptual approaches. They suggest that a procedural approach involves appropriateness of mathematical symbolism, representation and suitable procedures for solving mathematical problems. On the other hand, a conceptual approach involves relationship between individual bits of mathematical information.

Gray & Tall (1991, 1994) focused on the flexible way in which children use symbolism in arithmetic and algebra. They noted that many symbols in these subjects evoke both a process (such as the addition of two numbers 5+3) and also the concept produced (the sum 5+3, which is 8). They used the term *procept* to refer to a symbol which represents both process and concept. They hypothesised that students who thought in a flexible way which utilised symbols as both process to *do* mathematics and concept to *think* about would find mathematical problems easier to solve than procedural students who fail to develop the same conceptual richness and remain at a more routine procedural level.

Aware of Krutetskii's work with small children, it was initially suspected that university students with a strong calculus background would be likely to solve problems using less symbol manipulation and thus likely to show subtle curtailment in their solution. Based on this fact, I initially hypothesised that there is a correlation between curtailment of procedures and students' ability at university level.

Chapter 3 and 4 describe the pilot test at the University of Warwick. The purpose of this test is to obtain some notions about the second year students' symbolic manipulation at the Universiti Teknologi Malaysia. To serve the purpose, eight BA(QTS) (prospective British teachers) students were chosen. Analysis of their performance show that two selected students with grade A at A-level failed to curtail the solution whereas some students with lower grades did curtail. In this sample the B students often performed better than the A students, who not only failed in curtailment, but also sometimes failed to execute the integral and differentiation processes correctly or could not begin to a provide solution. It was realised that the more successful students (which in this case were the B students) would often write out algorithms in great detail to make certain that each step was correctly executed. Hence in this case the two grade B students possess strong conceptual structures to enable them to join reasonably isolated bits of procedures. But the other students fail to do so, because they do not possess such richness in conceptual knowledge.

This phenomenon may have occurred because there is only a small sample involved in the study, so the more and the less successful are not widely spaced as in the whole population. However, it may also suggest that there may be little correlation between the students' ability and curtailment. So what differences between success and failure may be identified?

In the pilot test at the University of Warwick, there was a qualitative difference between "more" and "less" successful students in which the "more" successful solve certain calculus problems by using general strategies which are less algorithmic. By doing so the students tend to reduce cognitive strain in solving a certain type of problem.

Such a possibility may be revealed when a student is given a problem which looks like a straight algorithm but is actually simplified further by doing some preliminary non-algorithmic simplification. For instance, to find $\frac{dy}{dx}$ when $y=\frac{1+x^2}{x^2}$, a perceptive student may perform some conceptual preparation by changing the expression $\frac{1+x^2}{x^2}$ into its equivalent

$$\frac{1}{x^2}+\frac{x^2}{x^2},$$

and then differentiating $y=x^{-2}+1$ to immediately obtain $\frac{dy}{dx}=-2x^{-3}$. In order to develop conceptual preparation of procedures, the students have to have a flexible conceptual structure to be able to see a subtle simplification that will reduce the complexity of the procedure.

In contrast with conceptual preparation of procedure, a student may use the quotient rule or product rule to differentiate the expression $\frac{1+x^2}{x^2}$. Using such a formula will involve a complicated computation and lengthy algorithm which need stages of simplification later on termed *post-algorithmic simplification.* For example in finding $\frac{dy}{dx}$, when $y=\frac{1+x^2}{x^2}$, post-algorithmic simplification is printed in bold as shown below:

$$\begin{aligned} y &= \frac{1+x^2}{x^2}, \\ \frac{dy}{dx} &= \frac{(2x)(x^2)-(2x)(1+x^2)}{(x^2)^2} \\ &= \mathbf{\frac{2x^3-2x-2x^3}{x^4}} \\ &= \mathbf{-\frac{2x}{x^4}} \\ &= \mathbf{-\frac{2}{x^3}}. \end{aligned}$$

Another possible indication of flexible thinking is the ability of a student to be able to use several different approaches to solve a problem. Analysis from the pilot test at the University of Warwick reveals that most students offered at most one method of solving each problem, with only one student offering two approaches for the same problem.

Chapter 5 and 6 report a pilot at the Universiti Teknologi Malaysia was carried out with the aim to identify the relationship between students' ability and conceptual preparation of procedures. Conceptual preparation is not done by using a specific formula or rule. The methods used will differ from one mathematical context to the other. It was found that the nature of questions plays an important role in determining whether the use of conceptual preparation of a procedure has an advantage or otherwise. For example, the problem

$$\text{Find } \frac{dy}{dx}, \text{ when } y = \left(x + \frac{1}{x}\right)^n$$

may be solved more easily if one uses the chain rule. Nevertheless, in finding $\frac{dy}{dx}$ when $y = \left(x + \frac{1}{x}\right)^2$, the simpler method is using conceptual preparation of procedure by multiplying out the bracket. There is a conflict in choosing either the generalisable chain rule method or the simpler method by expanding the expression $\left(x + \frac{1}{x}\right)^2$ prior to differentiation.

The main study in Chapter 7 considered several aspects which may distinguish *why* some students are more successful in using symbolism than others. It is hypothesised

(a) That there is a correlation between ability and curtailment (which is expected to be false).

(b) That there is a correlation between ability and conceptual preparation of procedures (which is expected to be true).

(c) That there is a correlation between ability and flexibility of process (several procedures for the same desired outcome).

(d) Students who are good at symbolic manipulation may fail to link this to visual ideas.

(e) Students who are flexible at performing familiar processes may fail to reverse them.

Thirty six students were involved in this study twelve of them with grade A, twelve with grade B and twelve with grade C. These grades were based on the first year mathematics examination result at the Universiti Teknologi Malaysia. An overall grade C is required for the student to be allowed to continue in the course.

Chapter 8 shows there is no clear relationship between attainment and curtailment. This is may due to the population under consideration. It consists only those Malaysian students following degrees involving mathematics taken from the 50th to the 90th percentile of the total population. The only clear phenomenon is that the number of lower attainers successfully solving the problem greatly decreases and it is thus statistically significant. However, for hypothesis (b), making use of a χ^2 test with Yates correction, the difference between grade A and grade C is significant at the 5% level.

In this study, the nature of conceptual preparation of procedures is also described. For example in finding $\frac{dy}{dx}$ when $y = \left(x + \frac{1}{x}\right)^2$, out of twelve students with grade A, six of them expand the bracket prior to differentiation whilst the other six carry out the chain rule. In the interview, four of the six using the chain rule could see a possible advantage in the alternative method but preferred to use the more general strategy and trust their facility in manipulation.

However, those high attainers have flexible alternatives in tackling calculus problems. The less successful who are unlikely to break away from using increasingly complicated symbol manipulation tend to experience breakdown in their procedures.

Chapter 8 reveals the student flexibility of using different approaches in solving the same mathematical problem. Aware that the problem, find $\frac{dy}{dx}$ when $y = \frac{1+x^2}{x^2}$ has at least four rather simple alternatives that can be used to solve it, I consider this question is most appropriate to serve the above purpose. More able students have strong conceptual linkages. Hence they are able to demonstrate more flexible strategies in symbolic manipulation. This phenomenon can be seen when the students develop meaningful relationship between symbolism and show the ability to interchange symbolism freely in different number of ways. The number of lower attainers successfully solving the same problem by different approaches greatly decreases. Using the χ^2 test with Yates correction, the difference between grade A and grade C is significant at the 5% level. Given several methods available for tackling a calculus problem, the more able students use their conceptual knowledge to retrieve an easier method that needs less cognitive strain. Hence, a calculus course does not cause a great problem for them. Lacking such conceptual quality, the less able students face considerable difficulties in performing calculus tasks.

Good symbolic manipulation does not necessarily imply reasonable interpretations for the same object in different forms. The occurrence of this phenomenon can be seen when certain more able students conceive the solutions of $\int (x+1)^2 dx$ obtained by different methods are not the same because of the different ways in which the added arbitrary constant is displayed.

This chapter reports that some students can link their computational outcomes to geometrical representations. For instance, the computational outcome obtained is finite, hence the more able students tend to draw graphs which are finite. The less able students are less likely to have good visual images and fail to obtain the reasonable graph. For instance, when asked to calculate the area between two graphs some of them shade infinite rather than finite regions. Using the χ^2 test with Yates correction, the difference between grade A and grade C students in this respect is significant at the 10% level.

Some students shows flexibility in using the integral notion to find the area of regions under the graph, but the not the reverse, i.e. to find the integral by using area under the graph. Such phenomenon occurs because all the separate processes in the brain operate in a single direction. To reverse a process cannot be done as with a video, by "running it in reverse", it requires the development of a new process in the reverse direction. Thus when students develop a procedure to do something in a certain way, may not be able to reverse the procedure easily. Hence it may be obvious that the integral may be used to determine the area under the graph, but it may not be obvious that the area under the graph (visualisation) may sometimes be used to determine the value of the corresponding integral. This phenomenon is described in Chapter 8.

Chapter 9 summarises the results and offers suggestion to fill gaps in the limitations of this study, together with ideas for future developments in research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Some students are efficient in handling symbolic manipulation when solving calculus problems. However, in performing the tasks these students tend to face some cognitive obstacles. Thus the literature review will set this in a wider context by considering cognitive research into the learning of calculus on the one hand and the wider issues of symbolism on the other.

2.2 Cognitive difficulties in the calculus

Research on the learning of calculus is fairly recent. Initially several studies focused on the difficulties with calculus concepts, often involving limits. For example, Orton (1983a, 1983b) interviewed 110 students over a wide range of calculus topics to seek the origins of various difficulties. His thesis (1980a) originally focused on whether these were related to Piagetian stages at the concrete operational and formal operational levels, but by the time it was published (1983a, 1983b) it had changed to focusing on whether errors made by students were structural (related to the conceptual ideas) executive (errors in performing the tasks) or arbitrary.

From the analysis of students' errors and misconceptions, Orton pointed out that most of the students, including those following mathematics degrees in colleges of education, demonstrated structural weaknesses; they gave a poor performance on conceptual tasks such as conceptualization of the limit processes related to the derivative and integral.

Robert (1982) was concerned with the students conception of the limit concept, whether it was *dynamic* with the variable *tending* to a limit, or *static*, in terms of neighbourhoods or the formal $\varepsilon - N$ definition.

Cornu (1981, 1983) considered the cognitive obstacles that students faced in handling calculus concepts such as limits of sequences and the concept of differentiation. These included:

> Genetic and psychological obstacles which occur as a result of the personal development of the student, didactical obstacles which occur because of the nature of the teaching and the teacher, and epistemological obstacles which occur because of the nature of the mathematical concepts themselves. (Cornu, 1991, p.158).

According to Cornu, the mathematical concept of limit is not easy for the students to grasp. He identified several obstacles which affected the students' perceptions, including:

- *The metaphysical aspect of the limit*, which uses completely new methods that do not rely on familiar arithmetic and algebra and seem "shrouded in mystery".
- *Is the limit attained?* The limit process seems to go on forever and the limit value may not be reached.
- *The notion of the infinitely large and infinitely small.* The idea of ε being "arbitrarily small" suggests the existence of arbitrarily small, "infinitesimal" quantities. Similarly "arbitrarily large *N*" suggests the existence of infinite quantities.

Insights into the nature of these difficulties came through later research into the relationship between the process of tending to a limit and the concept of the limiting value. The notion of a process being encapsulated (or reified) as a concept (Dubinsky 1991; Sfard 1991) had arisen earlier in the theory of Piaget (1952). Gray & Tall (1991) introduced the term *procept* to describe the combination of a symbol, process and concept where the symbol stood for both a process to be carried out and the concept output by that process. Tall (1991) suggested that $\lim_{n \to \infty} a_n$ is a procept i.e. the same notation represents both the *process of tending to a limit* and the *concept of the value of the limit*. This has the potential of shedding light on all three of the obstacles of Cornu mentioned above. According to him:

> Now we have the phenomenon that Cornu (1981, 1983, 1991) identified as an obstacle ...: understanding the dynamics of the process does not lead directly to the calculation of the limit. Instead indirect alternative methods of computation must be devised.
>
> (Tall, 1991, pp. 255).

Because of the strangeness of the process of computing the limit, the result is not encapsulated as an object in the same way as earlier experiences, so it is "shrouded in mystery" and may not be attained. Instead the process of the variable tending to zero is itself encapsulated as a cognitive infinitesimal.

The lack of encapsulation of the limit process into a limit concept is hypothesised by Tall to cause students to be locked in the procedural aspects of calculating the limit. But even here the structure of computation is different from their previous experiences:

> Just as with arithmetic, the theory of limits has a structure for devising new facts from old. But in arithmetic the new facts are derived from old using the calculation processes of arithmetic and the new facts have the same status as the old: they can be calculated by the processes of arithmetic in the same way. In the case of theory of limits, the "known facts" are one or two elementary deductions from the definition: that $\lim_{n \to \infty} \frac{1}{n}$ is zero,...

derived from the definition in a singularly peculiar way which can cause the initial confusion. The fact that $\frac{1}{n}$ tends to zero might be deduced from Archimedes' axiom, or perhaps by some heuristic appeal to the fact that: "I can make $\frac{1}{n}$ smaller than ε by making n bigger than the integer part of $1/\varepsilon$ plus one", both of which are strange ways of asserting $\frac{1}{n}$ gets small as n gets large...

> Thus, it is that the procepts in advanced mathematics work in a totally different and completely enigmatic way compared with the procepts in elementary mathematics. It is no wonder that, faced with this confusion, so many students end up conceiving the limit either as an (unencapsulated) process or in terms of meaningless rote-learned symbol pushing. (Tall, 1991, pp. 255-256).

Artigue (1991) noted the natural way in which students seem to think in infinitesimal terms. She claimed that

> Non-standard definitions are closer to the descriptions of differential and integral problems in physics than standard analysis. They also have fewer quantifiers and do not require the reversing of direction of the standard ε–d and ε–N formulations: for example, a sequence u_N is convergent to a limit l if and only if for every infinitely large N, $u_N - l$ is infinitesimal. Perhaps the definitions are more usable by students and the chasm between concept image and concept definition may be diminished by permitting a more gradual initiation to formalization. (Artigue, 1991, p. 198)

Sullivan (1976), Artigue (1991) and Frid (1992) claim that by using this approach the students were able to interpret the mathematical formalism but Tall (1980b) cautioned that there are significant differences between cognitive infinitesimals and mathematical non-standard infinitesimal. For instance, the sequence 0.9, 0.99, ... with nth term $1-\left(\frac{1}{10}\right)^n$ tends to 1. Cognitively students often believe that the limit is 0.999..., which is the "largest number less than 1", whereas $1-\left(\frac{1}{10}\right)^N$ for infinite N is infinitely close to 1, but $1-\left(\frac{1}{10}\right)^{N+1}$ is even larger.

2.3 What the students do to avoid the difficulties

Students apply different arguments suitable for each case instead of considering the whole argument globally. The benefit of using the argument globally is to prevent conflicts arising in separate compartments. For instance, a student might use different conceptions of limit selected according to a particular context being considered without being concerned about possible overall consistencies:

> And I thought about all the definitions that we deal with, and I think they're all right - they're all correct in a way and they're all incorrect in a way because they can only apply to a certain number of functions, while other apply to other functions, but it's like talking

about infinity or God, you know. Our mind is so limited that you don't know the real answer, but part of it. (William, 1991, p. 232)

There is an evidence that the students learn those materials that enable them to pass the examinations:

> Much of our students have actually learned...- more precisely, what they have invented for themselves - is a set of coping "skill" for getting past the next assignment, the next quiz, the next exam. When their coping skills fail them, they invent new ones. The new ones don't have to be consistent with the old ones; the challenge is to guess right among the available options and not to get faked out by teacher's tricky questions. ... We see some of the "best" students in the country; what makes them the "best" is that their coping skills have worked better than most for getting them past the various test barriers by which we sort students. We can assure you that does not necessarily mean our students have any real advantage in terms of understanding calculus.
>
> (Smith & Moore, 1991)

In order to be able to avoid the possible difficulties in dealing with conceptual questions , the students tend to emphasise more on the procedural-oriented materials that are most often asked in examinations. This is possible because the teacher prefers to set procedural questions rather than the conceptual questions. However, this type of approaches may influenced students learning at college or tertiary level. Ferrini-Mundy & Gaudard (1992) pointed out that:

> It is possible that procedural, technique-oriented secondary school courses in calculus may predispose students to attend more to the procedural aspects of the college course.
>
> (Ferrini-Mundy & Gaudard 1992, p.68)

2.4 Computer approaches

Computer algebra systems are now being used extensively in teaching calculus. In relation to computers, Tall (1991) note that:

> The first computer applications in calculus were numerical - using numerical algorithms to solve equations, calculate rates of change (differentiation), cumulative growth (integration and summation of series) and the solution of differential equations. All of these can be performed in a straightforward, but sometimes inaccurate, manner using simple algorithms, and then improved dramatically by using higher order methods...
>
> One method to improve matters is to engage the student in appropriate programming activities so that the act of programming requires the student to think through the process involved. (Tall,1991, p.16)

Small, Hosack & Lane (1986) report the effects of using a computer algebra system in college mathematics. The activities often encourage the students to apply a technique, already understood in a simple case, to more complicated cases where a symbolic manipulator can cope with the difficult symbolic computations.

Tall (1986) reports the building and testing of a graphical approach to the calculus, using software designed to allow the user to play with examples of a concept, to enable the abstraction of the underlying principle embodied by the software.

Tall (1986) suggests graphics alone were unsatisfactory in developing versatile movement between representations. Graphics give qualitative global insight where numerics give quantitative results and symbolics give powerful manipulative ability.

Heid (1984, 1988) used graphical software to study calculus concepts and the early computer algebra system Mu Math to perform symbolic manipulation. Her study reveals that

> Students showed better understanding of course concepts and performed almost as well on a final exam of routine skills as a class of 100 students who had practiced the skills for entire 15 weeks. (Heid, 1988, p. 7)

According to her,

> The students felt the computer aided in their conceptual understanding by refocusing their attentions in three ways:
>
> 1. Students felt the computer relieved them some of the manipulative aspects of calculus work...
>
> 2. They felt it gave them confidence in the results on which they based their reasoning....
>
> 3. They felt it helped them focus attention on more global aspects of problem solving....
>
> (ibid, p. 22)

Palmiter (1991) used the symbolic software MACSYMA to teach one cohort of students a first course in integration for five weeks whilst a parallel cohort studied a traditional course for a full of ten weeks. The MACSYMA students use the software to carry out routine computations whilst the traditional were taught the techniques. Both groups took a conceptual examination and a computational examination at the end. The conceptual examination was taken by both groups under identical condition, the experimental students were allowed to use MACSYMA in the computational examination but had only one hour whilst the control students were given two hours. The results showed that a student using MACSYMA can be more successful in conceptual and computational tasks than that of a traditional student.

2.5 Why not use computers?

Computers can provide a lot of facilities which include developing students' understanding of limit concept by offering a new approach of proving limit involving notation $\varepsilon - \delta$, illustrating basic derivative and integral concepts. Yet, there is a disadvantage in too much reliance on computers.

Coulombe & Mathews (1995) compare students in a computer laboratory using Derive with a traditional course. Their study reveals that there is no significant difference in knowledge, paper and pencil manipulation, conceptual understanding or higher order thinking skills.

According to Tall (1995)

> ...some "conceptually oriented" courses have shown students able to respond well to conceptual questions, able to perform manipulations better using technology and performing no worse at paper and pencil skills with a little practice, the knowledge being obtained is certainly different and is likely to have new strengths and also hidden flaws.
>
> The use of software with graphical facilities and symbol-manipulation changes conceptions of the calculus and their abilities to carry out the related skill.
>
> (Tall, 1995, p. 29)

Such phenomena are described by Hunter, Monaghan & Roper (1992). The students were asked to say something about u if $u = v + 3$ and $v = 1$.

Hunter et al, (1992) note that since

> The relationship $u = v + 3$ is a functional one, one which CAS effectively masks, and to respond correctly, an understanding of the role of substitution is required - a skill which the CAS has removed.
>
> (Hunter et al, 1992, p. 7)

According to Tall (1994), the use of symbol manipulators can cause students to go through very different processes for computing a result. Calculation of derivatives no longer requires working through first principles or using the formulae for the derivative such as product and composite. Monaghan, Sun & Tall, (1994) noted that some students using a computer algebra system to carry out the process of differentiation responded to a request for explanation of differentiation by describing the sequence of key strokes that were necessary to get the result. It appears that some students may simply replace one procedure which has little conceptual meaning with another.

2.6 Students of different abilities

Some studies focus on students having conceptual understanding of calculus topics, for instance, solving non-routine problems. Selden, Mason & Selden (1989) investigate the ability of average students on certain topics in calculus. In this study the students from a traditional course were posed with non routine conceptual calculus problems. Analysis from the students' test result show that none of them was able to complete and get the correct solutions to any non trivial problems and the highest score obtained was only 35%. Selden, Mason & Selden (1994) also observed that none out nineteen of grade A and B students were able to solve the non routine problems such as:

$$\text{Let } f(x) = \begin{cases} ax, & x \leq 1 \\ bx^2 + x + 1, & x > 1 \end{cases}$$

Find a and b so that f is differentiable at 1.

As a result of analyses of Selden, Mason & Selden (1989) and Orton (1983a, 1983b), Amit & Vinner (1990) assert that many students have impoverished conceptual knowledge of calculus at the university level. They carried out a detailed analysis of undergraduate students' notions of derivative in an attempt to locate and identify the source of misconceptions. Analysis of the answers reveals that misconceptions in calculus related to the derivative occur at certain crucial points. For example, in explaining the concept of the derivative, the teacher usually used a typical drawing of a curve with just one tangent shown. Such a static representation tend to mislead the students into a situation to conceive the derivative as the equation of the tangent. There is a possibility that a wrong implicit idea might affect the student's line of thought in non-routine conceptual problems even though the student seems to have correct ideas when solving routine derivative problems.

Shin (1993) has carried out an experiment to investigate the types of error and thinking strategies of Korean students in working out the answer to the calculus problems. These students have been divided into three groups – A, B and C with each group having a certain trait. Group A consists of "average" students from the standardized high school third grade. Group B consists of "excellent" students from selected high school second grade while in group C were teacher's college freshmen taking mathematics. He noted that both group A and B have difficulties in understanding the limit process and found that only 20% of students from both groups A and B had a vague idea about the complex symbol calculus symbols such as "$f(x^*)\Delta x$ for an independent choice of $x^* \in [x_{i-1}, x_i]$".

Studies performed by Selden, Mason & Selden (1994), and Shin (1993), give no indication of how different group of students with grade A, B and C carried out the procedures in getting the solution to a mathematical problem.

Krutetskii (1976) performed a wide range of studies on younger children. He classified the students selected as "very capable" (or "mathematically gifted"), "capable", "average" and "incapable".

According to him, the gifted/more capable children are less likely to find difficulty in generalisation of mathematical strategies. In solving mathematical problems they give curtailed solutions rather than detailed ones. These students also possess a number of flexible strategies which enable them to choose the more appropriate procedures to be executed in getting solutions to mathematics problems. This will promote their chances of solving mathematical problems posed to them more easily and efficiently.

For average children generalisation of strategies and curtailment of solutions are not immediately apparent. They are likely to acquire abilities to generalise strategies and to curtail solutions of mathematical problems after practising several problems of the same type. This means the average students have a limited number of methods in working out the solutions and the solutions obtained are often in more detailed form.

Incapable children are less likely to curtail their solutions. Their long-drawn-out solutions are often erroneous and continually include irrelevant mathematical information. In generalizing mathematical strategies, these students find great difficulty in perceiving the general features in the mathematical structures even after along practice.

The marked difference in ability between those three groups of students may be related to the strength of conceptual links formed by the more successful students in their cognitive structure (Hiebert and Lefevre, 1986) which helps the individual utilise knowledge in an efficient and powerful way. Hiebert & Lefevre discussed two types of mathematical knowledge which are mutually benefit:

> Procedural knowledge ... is made up two distinct parts. One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of algorithms, or rules, for completing mathematical tasks.
>
> (Hiebert & Lefevre, 1986, p.6).

> Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. In fact, a unit of conceptual knowledge cannot be isolated piece of information; by definition it is a part of the conceptual knowledge only if the holder recognizes its relationship to other pieces of information. (ibid., pp. 3-4).

Conceptual knowledge is not easy to achieve. For, in attempting to consciously link two pieces of knowledge there are *three* things to think about, two concepts and a connection. The "concepts" are in fact highly complex concept images and the "connection" needs to be made between appropriate parts of the concept images, provided that they exist. A learner with a rich and well-focused concept image may find conceptual learning an easy way of integrating new knowledge. Such a successful person may have many ways available to make connections that enable her or him to be even more successful. But the learner with inadequate images to cope with the new knowledge may find such an approach too demanding of the limited processing power of her or his focus of attention and seek the comfort and security of familiar procedures.

Out of 34 "gifted" students in Krutetskii's study, 6 were classified as "analytic", 5 as "geometric" and 23 as "harmonic".

According to Krutetskii, analytical thinking

> ...is characterized by an obvious predominance of a very well developed verbal-logical component over a weak visual-pictorial one. They operate easily with abstract schemes; they have no need for visual supports for visualizing objects or pattern in problem-solving, even when they mathematical relations given in the problem "suggest" visual concepts. (Krutetskii, 1976, p. 317).

In contrast to analytical thinking, he notes that students' geometric thinking

> ... is characterised by a very well developed visual-pictorial component, and we can tentatively speak of its predominance over a well developed verbal-logical component. These pupils a need to interpret visually an expression of an abstract mathematical relationship and demonstrate great ingenuity in this regard: in this sense, relatively speaking, figurativeness often replaces logic for them. But if they do not succeed in creating visual supports, in visualizing objects or diagrams to solve problems, then they have difficulty operating with abstract schemes They persist in trying to operate with visual schemes, images, and concepts even when a problem is easily solved by reasoning and the use of visual devices is superfluous or difficult. (Ibid, 1976, p. 321)

Hence given a variation of approaches by such a range of students, it becomes evident that methods that may be essential to some may be inappropriate for others. For example, the average tends to curtail solutions after routinising problems of the same type. Such repetition of routine problems may be less important for the gifted. But the others may develop the preference of clinging on to inflexible procedures in attacking mathematical problems.

In order to be successful in mathematics, the students should possess a number of meaningful representations of mathematical concepts that are strongly bonded together. In Krutetskii's study these students are named as gifted "harmonic" thinkers. To him gifted harmonic thinking is characterised by

> ...a relative equilibrium of well developed verbal-logical and visual-pictorial components with the former in the leading role. Spatial concepts are well developed in representatives of this type...They are successful at implementing both an analytic and a pictorial-geometric approach to solving many problems.
> (Ibid, 1976, p. 326).

These students have the flexibility in switching at speed from one representation of mathematical concept to another. They tend to move freely and effectively from one mode of thinking to another. The average or less successful students do not have such quality and tend to seek the security of a single representation. To them in order to form another representation require detailed work with concepts and operations. This tends to overburden them as they already experience a great task in trying all the possible variants and combinations in getting a single representation.

2.7 Compression of Information

Thurston (1990) suggests that the successful mathematics student is able to compress mathematical information. According to him,

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics.
>
> (Thurston, 1990, p.847).

According to Tall (1994), there are various method of compression of knowledge in mathematics. For instance, the method of *chunking* data together into meaningful chunks is a fairly primitive method of compression. If objects are collected together as a set, then one can conceive of the set itself as a unit and move this around physically, or "see" it as a unit to be manipulated. Chunking is a rudimentary method of compression, which can be performed even without language by putting objects into visible collections. But it can also be used in a sophisticated manner in formal mathematics to chunk together a collection of elements into a set.

A second, more powerful method, is to use language to give a *name* to something. Then it is possible to manipulate the name rather than the object itself. This is the widely used method which gives so much power to thinking *homo sapiens*. By using a single word to say something, it is easy *think* about it in relation to other things. We can communicate to each other in words, with the words evoking linkages in the brain that give them personal meaning. Mathematically, according to Tall (1994)

> Symbols such as $Ax = z$ for a system linear equations express relationship in a far more compact form than any corresponding use of natural language. But there is a common use of symbols in mathematics which introduces compression in a subtle way rarely used in ordinary language. It is a method of compression that mathematicians are aware of intuitively but do not articulate in any formal sense... (Tall, 1994, p.4).

Tall (1991) based his interpretation on the theory of procepts – a theory for a process which is symbolized by the same symbol as the product. This gives a great flexibility in thinking –

using the *process* to *do* mathematics and get answers, or using the *concept* as a compressed mental object to *think* about mathematics.

Related to symbolism, Tall (1992) claim that it is used flexibly by the good mathematician. According to him:

> Symbols allow mathematical thinking to be compressible, so that the same symbol can represent a process, or even a wide complex of related ideas, yet be conceived also as a single manipulable mental object. This flexibility is stock-in-trade for the mathematician. But it is not for the average student, who seeks a shorter term goal: to be to do mathematics by carrying out the necessary processes. It is this relationship between procedures to do mathematics and encapsulation of these concepts as single objects represented by manipulable symbols that is at the heart of mathematical success and its absence is a root cause of failure.
>
> We therefore see the use of symbols in a wider sense, dually representing process or concepts, linked with other representations including visualisation, gives a flexible view of mathematics that makes the subject easier for the more able. The less successful tend to cling more to a single representation, often a procedurally driven symbolic approach, which is inherently less flexible and imposes greater cognitive strain on the user. The short term gain of showing a student the procedure to be able to do a piece of mathematics may, for these students, lead to a cul-de-sac in which security in the procedure prevents the flexible use of symbolism as both process (to obtain a result) and object (to be able to manipulate as part of higher level thinking). (Tall, 1992, p. 66)

According to Thurston (1990):

> I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is 134/29 (and so forth). what a tremendous-labour saving device! To me, '134 divided by 29' meant a certain tedious chore, while 134/29 was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, a/b and a divided by b are just synonyms. To him it was a small variation in notation. (Thurston, 1990, p. 847).

A third method is to *represent* the object by some visuo-spatial configuration. It is said that "a picture is worth a thousand words". A picture can represent a large quantity of information and the viewer has the possibility of seeing it as a whole, or scanning it, to focus on selected detail in any order desired. In mathematics a diagram may represent information as a table of values or as a graph, or in some other way. Some students are not able to interpret the graphs they have drawn (Karplus, 1979; Dugdale, 1982; Goldenberg et al 1988; Schwartz, 1990). However, research shows that students do not always interpret graphs in the manner expected by mathematicians.

Linking symbolism to graphical representations can also produce unforeseen results. For instance, Caldwell (1995) expected students to find the roots and asymptotes of the rational function

$$f(x) = \frac{x(x-4)}{(x+2)(x-2)}$$

by algebraic means, only to be given a substantial number of approximate solutions such as 0.01 and 3.98 using a graphing calculator. Here a link to a graphical representation was made, without relating back to the precision of the algebra.

Boers & Jones (1993) report students use of a graphic calculator to draw a graph of

$$f(x) = \frac{x^2 + 2x - 3}{2x^2 + 3x - 5}$$

which has a removable discontinuity at $x = 1$. They found that more than 80% of the students had difficulty reconciling the graph with the algebraic information, for example, drawing in an asymptote suggested by the zero in the denominator, despite the graphic evidence of the calculator.

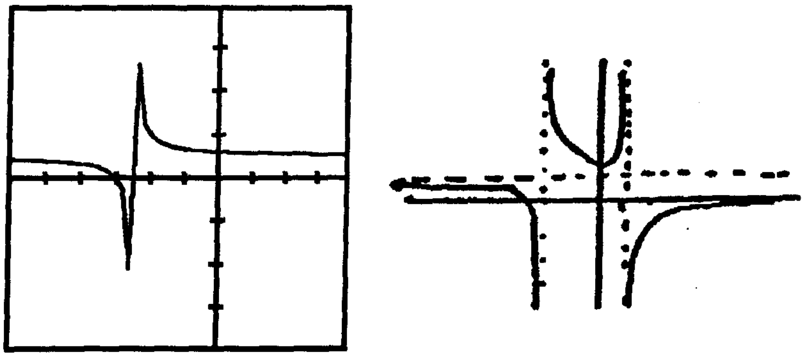


Figure 2.1

Graphic calculator display and student graph

In calculus there are studies (Schoenfeld, 1985; Dreyfus & Eisenberg, 1987, 1991) showing that the students are less likely to link their analytic ideas to graphical representations. For example these students were able to draw correctly the graph of certain integral problems but avoid using them in obtaining the value of the corresponding integral more trivially.

2.8 Compression of processes

Another important theme is the way in which the brain operates to handle mathematical *processes*. Because action schemas are fundamental tools of the brain, they have evolved to make links between successive actions to build a sequence that can be repeated as a single unit. In mathematics a simple thing to do is to *practise* a sequence of action until it becomes a routine process and the brain can carry it out as a single unit with little conscious intervention.

Operations of arithmetic of a young child benefit from being *symbolised*. These symbols enable a subtle form of compression that is not immediately apparent. An example of compression is seen in the addition of two numbers, say 3 + 2, which requires counting out 3 objects, then 2 objects, then putting the objects together and counting them all to get the answer 5. Initially therefore, the sum 3 + 2 is performed by "count all", involving three distinct counting processes which take up considerable processing space in conscious attention. When it is realised that it is not necessary to count everything twice, children may compress the schema to "count-both" numbers, first (often quickly) counting "one, two, three" then two more, "four, five". A further compression involves the realisation that it is not necessary to count the first number, but to just "count-on" the second, "(three), four, five".

Tall (1994), demonstrates symbol processing by making use the equation $3x + 4 = 2(x + 4)$. According to him,

> This equation drastically compresses the information into a compact form which can be solved algebraically by a sequence of manipulations:
>
> $$\begin{aligned} & 3x+4=2(x+4) \\ \textit{expand brackets:}\quad & 3x+4=2x+2\times 4 \\ \textit{simplify:}\quad & 3x+4=2x+8 \\ \textit{subtract 4 from both sides:}\quad & 3x+4-4=2x+8-4 \\ \textit{simplify:}\quad & 3x=2x+4 \\ \textit{subtract } 2x \textit{ from both sides:}\quad & 3x-2x=2x+4-2x \\ \textit{simplify:}\quad & x=4. \end{aligned}$$
>
> Experience leads to compression of several steps into one, such as:
>
> $$\begin{aligned} & 3x+4=2(x+4) \\ \textit{expand brackets:}\quad & 3x+4=2x+8 \\ \textit{subtract 4 from both sides:}\quad & 3x=2x+4 \\ \textit{subtract } 2x \textit{ from both sides:}\quad & x=4. \end{aligned}$$

The solution process is a visually moderated sequence (VMS) of symbol manipulations (Davis, 1984), where each manipulation is performed and written down to be used as a cue for the next one. When the procedure is comprehended as a whole, it becomes an integrated sequence. The procedure has a strategy in which both sides are simplified by multiplying out brackets and collecting together like terms before getting all the numbers one side and all the *x*s on the other, then divide by the coefficient of x (if it is not one). A VMS uses mental resources to good advantage, focusing successively on salient features to make best use of our short term attention.

2.9 Misconceptions and errors in handling standard differentiation and integration in elementary calculus

2.9.1 Overgeneralisation of basic rules in algebra

Recent reports and studies had shown students find difficulties in handling algebraic manipulation (Radatz, 1979; Vinner et al., 1981; Kuchëmann, 1981; Confrey, 1982; Booth, 1984; Tall & Thomas, 1991; Tall, 1992; Anibal, 1993; Fischbein and Barash, 1993). Such deficiency might result in overgeneralisation in algebraic rules that might affect calculus performance. As for example, Orton (1983b) found that most students expanded the expression of the form $(a+h)^2$ as a^2+h^2. This misconception is an example of students' over generalisation of previously learned rules in algebra (Hiebert and Carpenter, 1992). The students viewed the expression $(a+h)^2$ to be associated and having the similar properties as $(ab)^n$ and $\sqrt{ab}$. Since the last two expressions can be decomposed as $a^n b^n$ and $\sqrt{a}\sqrt{b}$ respectively, Hiebert & Carpenter assumed there were possibilities that the students tend to write $(a+b)^n$ as a^n+b^n and $\sqrt{a+b}$ as $\sqrt{a}+\sqrt{b}$.

Another example of such algebraic overgeneralisation is shown as follows:

> ..., since from $(x-3)(x-4)=0$ one deduce $x=3$ and $x=4$, the student writes that from $(x-\mathrm{A})(x-\mathrm{B})=\mathrm{K}$ follows $x-\mathrm{A}=\mathrm{K}$ and $x-\mathrm{B}=\mathrm{K}$.
>
> (Fischbein & Barash, 1993, p.162)

Fischbein & Barash (1993) have a different view related to overgeneralisation. They hypothesised that the students overgeneralisation were inspired by an algebraic model of distributive law $m(a+b)=ma+mb$ that the students have strongly developed in their mind. According to them, errors might be as a result from the misapplication of such a model.

2.9.2 Overgeneralisation in basic derivative and integral rules in calculus

Since the introduction of symbols dx and dy in 1677, Leibniz had shown quite a number of over generalisation in the derivation of such formulae like product and quotient rules without giving any rationale for neglecting the infinitesimal of higher order in the formulae. For example, by considering dx and dy as the difference in x and y respectively, the difference dxy in xy was initially considered to be $dxdy$ in Leibniz's manuscripts. It was only by considering "infinitesimals of different orders" and using

$$d(xy)=(x+dx)(y+dy)-xy=xdy+ydx+dxdy$$

to neglect the higher order of infinitesimal $dxdy$ to obtain the formula $d(xy) = xdy + ydx$.

Norman and Prichard (1992) in their studies to identify cognitive obstacles in college students' learning of calculus, identified students' thinking processes in elementary calculus related to derivatives and integration. They made a detailed analysis of students' problem-solving behaviours. Their study reveals that some of the students over generalised the previously learned rules and algorithms in solving calculus problems. For example, in finding the derivative of $f(x) = \frac{1}{2}.x\sqrt{\frac{3}{2}}$, they found that the student had used the product rule after regarding the function as a product of two factors $\frac{1}{2}x$ and $\sqrt{\frac{3}{2}}$.

Mundy (1984) describes such overgeneralisation in her study as *misidentification*—a term used by Vinner, Hershkowitz & Bruckheimers (1981). She notes that 34.1% of the students have misidentified the rule $\frac{d}{dx}(x^n) = nx^{n-1}$ when the students gave $y' = (x+1)x^x$ as the derivative of $y = x^{x+1}$. According to her, without understanding the related concepts, the students perceive the equation $y = x^{x+1}$ as $y = x^n$ and apply a well-rehearsed derivative rule and hence the obtained answer are inappropriate. Similarly in the case of integration, the students after perceiving the expression $\int_{-3}^{3} |x+2|\, dx$ as $\int_{-3}^{3} (x+2)\, dx$ start to apply the rule of $\int x^n dx = \frac{x^{n+1}}{n+1} + c$.

Algebraically, related to the fact of exponent rule $(ab)^n = a^n b^n$, another overgeneralisation occur in calculus when the student applied inappropriately the exponent rule in derivative problem as pointed out by Norman & Prichard (1992). According to them, in finding $(fg)'$, for example, some students with good knowledge of the exponent rule tend to write $(fg)' = f'g'$, instead of differentiating the expression fg by means of the product rule. The same phenomenon occurs when the students try to evaluate $(fg)''$.

In a differential equation class, they note that some students find the solution for $y'' + 2y' = 0$ by performing procedures as follows:

$$y'' + 2y' = 0 \rightarrow y'(y'+2) = 0$$
$$\Rightarrow y = c,\ y = -2x + c$$

Tall (1986) found Adam had made a similar overgeneralisation. He tried to differentiate $V = \frac{1}{3}\pi r^2(6-r)$ by differentiating each part of the product to give $\frac{dV}{dx} = \frac{2}{3}\pi r(-1)$. This type of over generalisation was first made by Leibniz, whose first idea of the product rule as $dxy = dxdy$ was inspired by his discovery of the formula $d(x+y) = dx + dy$, but soon corrected to the usual formula.

Tall (1986) also had shown an example the overgeneralisation of the quotient rule $\left(\frac{f}{g}\right)' = \frac{f'}{g'}$. In his study, Adam tried to differentiate

$$\frac{(300T - 2400) - 100 - 5T}{T}$$

"top and bottom". On the other hand, Norman & Prichard (1992) observed that one of their respondents use quotient rule in order to find the derivative of $\frac{\sqrt{3}x^2}{2}$.

For simplicity, in this study the rules $\frac{d}{dx}x^n = nx^{n-1}$ and $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ will be respectively named the *basic rule of differentiation* and *basic rule of integration.*

2.10 Summary

According to Krutetskii, the more capable students are likely to curtail their solutions. In order to have the such curtailment these students must have a strong conceptual images, rich in relationships. In addition to the ability to curtail the solution, the more capable students have a number of strategies when dealing with mathematical problems. They are able to interchange freely and meaningfully mathematical symbolism and hence they are able to choose those methods which are easier. Since these students have strong conceptual linkages, they are able to generate successfully new knowledge.

The less able students have inadequate conceptual knowledge. The lacking of such knowledge means a deficit in conceptual relationships. This phenomenon may hampers the less able students to curtail their procedures when handling mathematical problems. So, in order to solve mathematical problems they rely upon detailed solutions whereby each bits of mathematical information in the procedures may be erroneous, not related or may be dispensable. These students have limited number of approaches in dealing with mathematical problems. They face great difficulties in interchanging the symbolism systematically and correctly. Development of new knowledge are less likely to occur. Thus, in order to be secure they are satisfied with familiar procedures.

More able students have a number of reasonable and strongly linked representations. They can switch easily from one representation to another and are always looking for a simpler and effective representation to be used in solving mathematical problems.

The less able students have a limited number of such representations. For example, they find great difficulty in linking the graphical representations with the algebraic information. The reversal of processes are less likely to occur in weaker students. Hence they manage to work just in one direction.

CHAPTER 3

PILOT STUDY AT THE UNIVERSITY OF WARWICK

3.1 Introduction

This thesis is focused on the second year Universiti Teknologi Malaysia students' manipulation of symbols in solving calculus problems, relating this to other aspects, such as the drawing and interpretation of graphs. These students had undergone a calculus course in the first year. They are prospective teachers and engineers. The first year calculus course at the Universiti Teknologi Malaysia is equivalent to British A-level calculus. Thus, in order to obtain some notions about the thinking of the second year students at the Universiti Teknologi Malaysia in the above area, a pilot study on symbolic calculus was initially conducted with a group of eight first year BA(QTS) students (prospective teachers) at the University of Warwick in July 1993.

The aim of this pilot study was to see how the more successful students at tertiary level differ from less successful students in the way they solve calculus problems. Krutetskii (1976) studied younger children solving problems, classifying them into four groups with different mathematical abilities: gifted, capable, average and incapable. In terms of carrying out the procedures, he noted that there is a marked difference between those performed by the gifted/capable students compared with those performed by the incapable ones:

> Able pupils are distinguished by a rather pronounced tendency for the rapid and radical curtailment of reasoning and of the corresponding system of mathematical operations also "on the spot," in certain sense, since it even begins to appear in the first problem of a type new to them.
>
> (Krutetskii, 1976, p. 263).

He added that

> No appreciable curtailment was observed in incapable pupils , even as a result of many exercises... The reasoning of incapable pupils was always marked by superfluous comprehensiveness, detail, and unnecessary activity... At the same time, their reasoning was not distinguished by accuracy, consistency, or a proper logical decomposition into parts.
>
> (ibid, 1976, p. 265).

Based on Krutetskii's study, it was hypothesised that at university level, the more successful students will tend to use the information in a more compact way so that there is a less strain on their cognitive structure and thus it is hypothesised that:

(a) there will be a correlation between curtailment of procedure and success.

In the pilot study other phenomena will be considered. For instance, it is expected that:

(b) Good manipulation does not necessarily imply the ability to interpret the same objects in different forms.

(c) There exists students who are good at symbolic manipulation yet fail to visualise.

(d) The more successful students will show greater flexibility in using different approaches to tackle the same calculus problem.

3.2 Research instruments

In this study, students' thinking in calculus is revealed by questionnaires administered in interviews. This combined approach is used to take account of individual strengths and limitations of the two methods. According to Amit & Vinner (1990):

> The common belief is that an interview is a better instrument than the questionnaire. This is because many ambiguities can be resolved in an interview that cannot be resolved in a questionnaire. Also, some spontaneous reactions in an interview can be extremely illuminating, much more than the controlled or even inhibited reactions one can get in a questionnaire. This might be true in many cases but there are also many cases in which the situation is more delicate. Assume that a student makes an ambiguous statement in an interview and the interviewer wants to ask a question which is supposed to clarify this ambiguity. Of course, this must be done in such a way that the student will not change his mind as a result of the question posed to him. Practically, however, this might be impossible. There are situations in which any reconsideration of a given answer causes a critical analysis. This analysis will lead to a clarification in a direction different from the one in the original answer. Everybody with minimal self awareness knows that very often he has vague ideas which he believes in, but the moment he formulates them in words or even listens to somebody else's formulation he realizes that these are faulty ideas. So, there are cases in which an interview will not lead to clear and unambiguous information but even to distorted information. Also the belief that in an interview we can obtain more spontaneous reactions is not necessarily true. It depends on the student and on the interviewer and on the relations between them formed before and during the interview. (Amit & Vinner, 1990, pp. 4-5)

3.3 Sampling

3.3.1 The Subjects

Eight first year BA(QTS) students (prospective teachers) following the same mathematics course were selected. Based on their mathematics grades in the British A-level examination, the sample comprised two students having grade A, two with grade B, two with grade C while the other two scored grade D. The two students with grade D took mathematics as second subject while the rest had mathematics as their main subject. For simplicity in analysing the data in this study, the students were simply numbered. For example the two A students were marked as A1 and A2.

3.3.2 Rationale behind choosing the students

(i) Since the study is involving students' thinking in calculus, the students chosen are those who had experienced a calculus course at British A-level.

(ii) So as to see the students' range of performance in the first year calculus course, the students with various grades (A, B, C and D) in mathematics of the British A-level examination are preferable.

(iii) Since some of the students involved in the main study at the Universiti Teknologi Malaysia are prospective teachers, it is considered appropriate to see the extent of performance in calculus of their British counterparts at the University of Warwick.

3.4 Interview

3.4.1 Procedure

The interviews were carried out on the 3rd, 4th, 10th and 17th November 1993. For each of the corresponding days, two students were welcomed into the room. Before answering any of the questions in the interview the students were put at their ease. At this time the researcher introduced himself to the students. The students were told that the meeting was not an examination and the researcher explained his intentions of carrying out the study. All their responses would be kept confidential.

Before proceeding with the calculus questionnaires, the students were asked general questions with the intention to identify the way they conceptualise and learn calculus topics.

After getting this information each student was given a set of calculus questions that related to differentiation and integration. The whole process of interviewing and answering all the questions lasted for about one hour. After each question had been solved, the researcher discussed the solution with the student concerned. If the solution was correct, some appreciation was given. If the solutions were incorrect or unreasonable, the students were not told that the solutions were *wrong* or *incorrect.* Instead they were asked whether they were able to make some adjustment to what they had done or to suggest other alternatives to tackle the problem.

Each student was given a chance to explain the procedures or solutions without any intermediate interference from the researcher. If the students were silent, they were asked what they were thinking. If they could not proceed with a problem, the researcher guided them in reaching the required solution.

Since two students were involved in each meeting, the interview was carried out alternately so that each student had the same chance as the other. The interviews were video-taped and subsequently transcribed. Only seven students underwent the whole interview and questionnaires because one D-grade student had to leave the interview before its end.

3.4.2 Interview Questions

The interview consists of two parts.

The first part of the Interview:

> The questions posed are open-ended and informal with the intention of getting general information on how students conceptualise and study calculus topics.
> *Rationale: To get some insight into how students learn calculus and to get some information on whether the students find calculus difficult to understand, interesting, challenging, confusing or otherwise.*

The second part of the Interview

The questions are open-ended and based on the questionnaires. By considering

(a) The role of good symbolic manipulation in interpretation of the computational outcomes.
Rationale: To observe whether the students with good symbolic manipulation have the ability to interpret reasonably the same mathematical concepts in different forms

(b) The geometrical interpretation of the computational result.
Rationale: To see whether the students who are good at computation may easily visualise its representation.

(c) The number of approaches available in dealing with the same calculus problem.
Rationale: To identify students' flexibility in tackling the same calculus problem.

3.5 The questionnaires

3.5.1 The nature of the questionnaires

All the questions related to simple integration and differentiation. The questions are presented in a hierarchical form from simple to more difficult. The aim of the questions is to identify at some stage the level of students' understanding in calculus from the aspect of their

Ability to handle the basic differentiation and integration,

Ability to curtail or detail solutions,

Flexibility in using different approaches to tackle the same calculus problem,

Misconceptions and errors in calculus.

3.5.2 Sources and rationales of the questionnaires

The questions are selected from various references:

SECTION I(a)

Question 1.

$$\text{Find } \frac{dy}{dx}, \text{ when } y = 2x^4 - x^2 - 6$$

is taken from Universiti Teknologi Malaysia *Tutorial Sheet* 1992/1993. This question is used to identify the students' basic idea and knowledge of symbolic differentiation in calculus.

Question 2.

$$\text{Find } \frac{dy}{dx}, \text{ when } y = \frac{1}{(x+1)^{\frac{1}{2}}}$$

is taken from Universiti Teknologi Malaysia *Tutorial Sheet* 1992/1993. This question is used to identify the students' procedural knowledge of symbolic differentiation in calculus involving fractional indices.

Question 3 consisted of two parts. 3(a) is taken from Kolman and Denlinger's *Calculus for the Management, Life and Social Sciences* (1988, p.143) and question 3(b) is a modification of question 3(a).

Question 3(a).

$$\text{Find } \frac{dy}{dx}, \text{ when } y = 5.$$

Question 3(b).

$$\text{Find } \frac{dy}{dx}, \text{ when } y = \sqrt[3]{4}.$$

This problem is relatively harder than the previous one. These questions are used to identify students' knowledge of differentiating constant terms and to test the students' geometrical interpretation of the computational result.

Question 4.

$$\text{Find } \frac{dy}{dx}, \text{ when } y = \sqrt{2x^3}.$$

is taken from Universiti Teknologi Malaysia *Tutorial Sheet* 1992/1993. This question is used to identify students' preference of methods (either curtailed or detailed procedures) in solving the given problem. It is considered that this question is likely to provide a sense of the measure that is being considered for different number of steps in the solution in differentiation.

Question 5.

$$\text{Find } \frac{dy}{dx}, \text{ when } y = \left(x + \frac{1}{x}\right)^2$$

is taken from Dakin and Porter's *Elementary Analysis* (1980, p.41) and is used to identify students' different techniques in differentiation of the same expression. For example by expansion of the squares in the expression prior to differentiation, by the chain rule, or possibly by the product rule.

Question 6.

$$\text{Find } \frac{dy}{dx}, \text{ when } y = \frac{1+x^2}{x^2}$$

is taken from De Sapio's *Calculus for the Life Sciences* (1976, p.124). This question is used to identify students'

(a) preference of methods (either by first simplifying the expression $\frac{1+x^2}{x^2}$ before carrying out the actual differentiation or applying the standard rules of differentiation) in solving the given problem.

(b) the number of methods that can be used to tackle this problem.

SECTION I(b)

Question 1.

$$\text{Evaluate } \int (x+1)^2 dx$$

is taken from Elliot, Fryer, Gardner & Hill's *Calculus* (1966, p.150).

(a) This question is used to identify the students' basic idea and knowledge of the integral in symbolic calculus.

(b) There are two possible ways of solving this problem:

(i) By expanding the square before actual differentiation,
(ii) By means of substitution.

By the above methods the answer obtained for the above problem will be in two different forms. Thus, this question is also used to observe whether the students with good symbolic manipulation have the ability to make a reasonable interpretation of the same objects in different forms.

Question 2.

$$\text{Evaluate} \int \frac{1}{(x-3)^5}dx$$

is taken from Universiti Teknologi Malaysia *Tutorial Sheet* 1992/1993. This question is used to identify students' procedural knowledge on integration involving negative integral indices.

Question 3.

$$\text{Evaluate} \int (6x)^2 dx$$

is taken from Universiti Teknologi Malaysia *Tutorial Sheet* 1992/1993. This problem is used to identify students' different techniques for integration of the same expression. For example by expansion of the squares in an expression prior to integration or possibly by substitution.

Question 4.

$$\text{Evaluate} \int \sqrt{3x^3}\,dx$$

This question is designed to identify the extent within which the students are able to compress algebraic procedures in integration. It is a modification of a question

$$\int \frac{1}{2\sqrt{2x^3}}dx$$

taken from *Calculus* (Abbot, 1970, p. 161) and the problem

$$\int \sqrt{3t}\,dt$$

taken from Edward and Penney's, *Calculus and Analytic Geometry* (1990, p. 246). It is considered that these two questions are likely to provide a sense of the measure that is being considered for different number of steps in the solution. The former question is considered to be lengthy in its procedural implication, whilst the latter is considered relatively straight forward. The final question contained components from each of the two questions in an attempt to establish the procedural implication. This question will be used to identify whether the students apply curtailed or detailed procedure in solving the problem.

CHAPTER 4

ANALYSIS OF DATA IN THE PILOT TEST AT THE UNIVERSITY OF WARWICK

4.1 Interview

4.1.1 General information about how students feel about and learn calculus.

Eight students took part in this part of the interview. Seven of them expressed sentiments indicating that the calculus course is difficult to understand, boring, complicated, involving too much computation. These students have to rely upon standard rules for solving the calculus problems. For example, as one B student said:

> Calculus is difficult because a lot of theorems, laws and formulae that have to be remembered. Normally, the theorems are explained in sentences which are too difficult to be understood. There are a lot of proofs which I don't understand.
>
> (BA(QTS) student, 1993)

The other B student (B1) found that learning calculus is very interesting and does not stress much upon the reliance of standard rules and formulae in calculus. According to her:

> Sometimes the rules are not necessary because some of the calculus problems can be solved without using such rules...By doing so, sometimes the problem might be easier to be solved. (BA(QTS) student, 1993)

4.1.2 The ability of students who are good at symbolic manipulation in interpreting the same objects in different form

The students are asked to solve integral problem 1 from section I(b), i.e.

Find the integral $\int (x+1)^2 dx$.

Before reaching this part of the interview one of the students (D2) had left the room. Thus only responses of seven of them are recorded. Out of seven students attempting the above problem, only one (B1) is able to respond that the integration process could be performed in two ways. According to her:

This problem can be solved by two methods. The first one by expanding the square of (x+1) before carrying out the integration. The other one by means of substitution of u = x + 1. (BA(QTS) student, 1993)

Her performance in carrying out the process of integration in this problem are shown in the table below:

Performance of B1	
By first expanding the square	By substitution of $u = x + 1$
$\int(x+1)^2dx = \int(x+1)(x+1)dx$ $= \int(x^2+2x+1)dx$ $\Rightarrow \frac{x^3}{3} + \frac{2x^2}{2} + x + c$ $\Rightarrow \frac{x^3}{3} + x^2 + x + c$	$u = x + 1$ $\frac{du}{dx} = 1$ $\int(x+1)^2dx \Rightarrow \int u^2du$ $= \frac{u^3}{3} + c = \frac{(x+1)^3}{3} + c$

Table 4.1

Since the problem solved is the same, B1 stresses that the solution obtained should be the same irrespective of the methods used. In explaining the sameness of the solution, she expands the cube of the expression in the second solution, i.e. $\frac{(x+1)^3}{3} + c$ as $\frac{x^3}{3} + x^2 + x + \frac{1}{3} + c$. But after looking at the expanded solution of the second one, she changes her mind. This is because of the constant $\frac{1}{3}$ in the second solution which is absent in the first solution. She conceives the constant c in the first solution to be the same as that in the second solution.

By this observation there is a possibility that good symbolic manipulation does not necessarily imply the ability to interpret reasonably the same object in different forms.

4.1.3 Geometrical interpretation of the computational result

The questions for this interview are 3(a) and 3(b) from Section I(a), i.e. Find $\frac{dy}{dx}$, when

(a) y = 5

(b) $y = \sqrt[3]{4}$.

All selected and attempted one or the other of these questions. Out of eight students, only two students (one with grade A and one with grade B) obtained the right answers and were able to interpret the results geometrically,

Gradient of a straight line... It is parallel to the x axis,.

Three students (one with grade A, one with grade B and one with grade C) obtained the correct answer but failed to respond to the geometrical interpretation of the result, one (with grade C) carried out wrong procedures because of the absence of the variable x in the equation and two students (both of them D students) made no progress at all. The students' performance in solving problem 3 can be summarised as below:

Students performance			
A1 & B2	A2, B1 & C2	C1	D1 & D2
(a) $y = 5$ $\frac{dy}{dx} = 0$	(b) $y = \sqrt[3]{4}$ $\frac{dy}{dx} = 0$	(b) $y = \sqrt[3]{4}$ $= 4^{\frac{1}{3}}$ $dy = 4^{\frac{1}{3}}dx$	Cannot proceed
Gradient of a straight line... It is parallel to the x axis	No verbal response	No verbal response	

Table 4.2

From the above analysis, there is an indication that a successful student in this case (B1) may fail to relate her computational outcomes to visual ideas.

4.2 To identify the students' ability in handling the basic differentiation and integration

Realising that some of the BA(QTS) students involved in this study had learned calculus long before joining the university, four questions were designed to test their procedural knowledge of basic differentiation and integration. For differentiation, two problems have been chosen from Section I(a), i.e.

Question 1 — Find $\frac{dy}{dx}$, when $y = 2x^4 - x^2 - 6$.

Question 2 —Find $\frac{dy}{dx}$, when $y = \frac{1}{(x+1)^{\frac{1}{2}}}$.

In the case of integration, two problems are taken from Section I(b), i.e.

Question 1—Evaluate $\int (x+1)^2 dx$.

Question 2—Evaluate $\int \frac{1}{(x-3)^5} dx$.

All students could solve problem 1 in both sections. When attempting to solve question 2 some students found difficulty and started to make errors. There were two types of errors found in this study —algebraic manipulation and inability to carry out the algorithms of differentiation or integration. This will be discussed later in the analysis of errors.

4.3 To identify whether students curtail or detail their procedures in solving calculus problems

When looking at students' different abilities in performing calculus tasks, there are variations of procedural length in term of number of steps performed by these students. Some students prefer detailed solutions while some go for curtailment. In order to see this phenomenon Question 4 in section I(a), is chosen. The problem is

Find $\frac{dy}{dx}$, when $y = \sqrt{2x^3}$,

Out of seven students attempting this problem, only one student displayed a curtailed procedure by using three steps in order to get to the solution. The solution obtained was in a conventional form. One student demonstrated detailed procedure. Both of these were grade B students. The remaining five students, after changing the equation to $y = (2x^3)^{1/2}$ started to use a formula and did not show either a curtailed or detailed procedure.

A simple count of the steps used in each procedure does not give a true measure of the actual steps involved. Some students begin by writing the question down and then performing simplification, others start immediately with a form different from the format of the question. In counting the number of steps in a procedure, the first step counted is the first change in the format of the question. Re-writing the question is not counted as a step.

At the end of the question some students made further simplifications not done by others. Hence in order to gain a better measure of uniformity, the idea of a conventional solution is introduced. A note was then made as to whether the solution was given in what may be considered a conventional form, or not, to give further information to distinguish between the degree to which one solution might be curtailed or detailed compared with another. The idea is illustrated in the detailed solution displayed in the following:

Given the equation $y = \sqrt{2x^3}$, the student manipulates it to give its equivalent $y = (2x^3)^{1/2}$. Knowing that the expression $(ab)^n = (a \times b)^n$ for $n = \frac{1}{2}$, this student uses the power rule to give the expression $2^{1/2} \times (x^3)^{1/2}$ and then uses $(x^m)^n = x^{mn}$, so that the original equation becomes

$$y = 2^{1/2} \times x^{3/2}.$$

The process from $y = (2x^3)^{1/2}$ to $y = 2^{1/2} \times x^{3/2}$ is performed in a single step. Since the number $2^{1/2}$ is separated from the power of x, this number can be changed into its conventional form $\sqrt{2}$, the equation becomes

$$y = \sqrt{2} \times x^{3/2}$$

but this equation can be written as $y = \sqrt{2}x^{3/2}$. Now, by applying $\frac{d(cf)}{dx} = c\frac{df}{dx}$ and using the formula nx^{n-1} for $n = \frac{3}{2}$, this student obtains

$$\frac{dy}{dx} = \sqrt{2} \times 3/2\, x^{1/2}$$

The above result can be written in its equivalent form as

$$\frac{dy}{dx} = \frac{3\sqrt{2}x^{1/2}}{2}.$$

The answer obtained by this student will be considered in a conventional form. Another conventional form of the solution could be

$$\frac{3}{2}\sqrt{2}x^{1/2}.$$

Analysis of the performance reveals that grade B students produce more satisfactory solutions than the two selected grade A students. This means that, in this small study, grade B students are more successful than grade A students.(see Table 4.3 and 4.4).

The B students are able to curtail solutions. In contrast, the grade A students not only fail to curtail solutions, but also fail to carry out the integral processes correctly. Students' performance in this problem can be summarised as follows:

Solution of the second B student	Solution of the first B student	Erroneous solutions
$y=(2x^3)^{\frac{1}{2}}=\sqrt{2}\sqrt{x^3}$ $\frac{dy}{dx}=\frac{3}{2}\sqrt{2}x^{\frac{1}{2}}$	$y=\sqrt{2x^3}$ $y=(2x^3)^{1/2}$ $y=2^{1/2}\times(x^3)^{1/2}$ $\Rightarrow y=\sqrt{2}\times x^{3/2}$ $y=\sqrt{2}x^{3/2}$ $\Rightarrow\frac{dy}{dx}=\sqrt{2}\times 3/2\,x^{1/2}$ $=\frac{3\sqrt{2}x^{1/2}}{2}$	[Erroneous solutions are given in the table below]
Curtailed procedure (3 steps) (B2)	Detailed procedure (6 steps) (B1)	A1, A2, C1, C2 & D1

Table 4.3

Students' errors in solving problem 3 in section I(a)			
A1	A2	*C1&C2	D1
$y=(2x^3)^{\frac{1}{2}}$ $\frac{dy}{dx}=\frac{1}{2}(6x^2)^{-\frac{1}{2}}$ $=\frac{1}{2(6x^2)^{\frac{1}{2}}}$	$y=\sqrt{2x^3}$ $=(2x^3)^{\frac{1}{2}}$ $\frac{dy}{dx}=\frac{1}{2}.6x(2x^3)^{-\frac{1}{2}}$ $=3x(2x^3)^{-\frac{1}{2}}$ $=\frac{3x}{\sqrt{(2x^3)}}$	$y=\sqrt{2x}=(2x)^{1/2}$ $=4x^{1/2}$ $\frac{dy}{dx}=\frac{4x^{-1/2}}{2}$ $=\frac{2}{x^{1/2}}$	$y=(2x^3)^{\frac{1}{2}}$ $\frac{dy}{dx}=1/2(2x^3)^{-\frac{1}{2}}\times 6x^2$ $=12x^2(2x^3)^{-\frac{1}{2}}$
Mixture of rule for $\frac{d}{dx}x^n$ and chain rule	Misapplication of chain rule	Error in manipulating the indices	Error in simplification the product of correct chain rule

Table 4.4

Key: * — example displayed in the table indicates the work of the corresponding student. For instance in this case, it is the solution of C1.

The ability to obtain more satisfactory solutions indicate that these grade B students may have stronger conceptual structures than the grade A students. In this problem B1 carries out the procedures in great detail. Such a performance indicates that this student is probably making sure that each step is correctly carried out. Hence in this case B1 has sound conceptual linkages and is more likely able to join isolated bits of information to carry out the procedures in a meaningful manner. But the other students fail to do so, presumably because they do not possess such richness of conceptual knowledge. Their links are often flawed and they tend to break down in their procedures in getting the solution. The reversal in performance of the A and B students may be due to the small sample involved in the study, so the more and the less successful are not as widely spaced as they might be in the whole population. However, it may also suggest that there may be little correlation between the students' grades and their tendency to curtail solutions.

4.4 Breakdown of procedures

The students who demonstrated detailed solution may have experienced a breakdown in their procedures. The breakdown of procedures in the solution may be due to involvement of less appropriate linkages in certain areas. For example in this pilot study, lack of strong conceptual knowledge produce phenomena such as algebraic misconceptions and inappropriate overgeneralisation of a standard rule in differentiation and integration. To see a breakdown of detailed procedures in integral problem, Question 4 in section I(b) is chosen, i.e. evaluate $\int \sqrt{3x^3}\,dx$. Out of seven students who attempted this question, six of them broke down.

One B student, B2, faced breakdown of procedures after successfully performing three steps while two students A1 and C1 experienced breakdown of procedures due to incorrect algebraic manipulation, i.e. after changing $\int (3x^3)^{\frac{1}{2}}dx$ into $\int 9(x^3)^{\frac{1}{2}}dx$. One student C2, after manipulating the term $\int \sqrt{3x^3}\,dx$ to give $\int (3x^3)^{\frac{1}{2}}dx$ started to apply an overgeneralisation of a mixture the basic rule of integration and possibly the product rule (differentiation) to give an answer as $\frac{2}{3}(3x^3)^{\frac{3}{2}} + \frac{3}{4}x^4$. Two students A2 and D1 got stuck right from the beginning. Only B1 successfully use detailed procedures to reach final answer. She carried out her detailed solution as shown below:

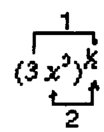
Given the integral

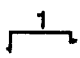
$$\int \sqrt{3x^3}\,dx,$$

she changed it into its equivalent form

$$\int \left(3x^3\right)^{1/2} dx,$$

and possibly manipulated it mentally thus :



In other words,  gives $3^{1/2}$ or $\sqrt{3}$ whereas  gives $x^{3/2}$. The above integral becomes

$$\int ((3^{1/2}) \times (x^{3/2}))dx$$

Knowing that the multiplicative sign need not to be written, the above integral can be changed into equivalent form thus

$$\int \sqrt{3}x^{3/2} dx.$$

$\sqrt{3}$ can be placed outside the integral sign and applying the formula $\frac{x^{n+1}}{n+1}$ for $n = 3/2$ gives

$$\frac{\sqrt{3}x^{5/2}}{5/2} + c.$$

She performed these operations in a single step. The term $\frac{\sqrt{3}x^{5/2}}{5/2}$ is not in a form which would normally be considered conventional because the denominator is in fractional form. Thus, the whole term is rewritten as

$$\frac{2\sqrt{3}x^{5/2}}{5}$$

to give the answer as

$$\frac{2\sqrt{3}x^{5/2}}{5} + c.$$

This is a final solution given by the students. It is a matter of convention that the radical $\sqrt{3}$ is placed *after* the fraction such as $\frac{2\sqrt{3}x^{5/2}}{5}$, so that some students may perform a further modification to give a final result as $\frac{2\sqrt{3}x^{5/2}}{5}$, $\frac{2}{5}\sqrt{3}x^{5/2}$, $\frac{2}{5}\sqrt{3}\sqrt{x^5}$ or equivalent.

The solutions of all seven students in this problem may be summarised as follows:

Solutions of students				
Detailed procedure	Breakdown of procedures			
$\int\sqrt{3x^3}dx = \int(3x^3)^{1/2}dx$ $= \int((3^{1/2})\times(x^{3/2}))dx$ $= \int\sqrt{3}x^{3/2}dx$ $= \frac{\sqrt{3}x^{5/2}}{5/2} + c$ $= \frac{2\sqrt{3}x^{5/2}}{5} + c$	$\int\sqrt{3x^3}dx$ $= \int(3x^3)^{\frac{1}{2}}dx$ $= \int\left(3^{\frac{1}{2}}x^{\frac{3}{2}}\right)dx$ $= \int 3^{\frac{1}{2}}\times(x^3)dx$	$\int\sqrt{3x^3}dx$ $= \int(3x^3)^{\frac{1}{2}}dx$ $= \int 9(x^3)^{\frac{1}{2}}dx$ $= \int 9x^{\frac{3}{2}}dx$ $= \frac{9x^{\frac{5}{2}}}{\frac{5}{2}}$ $= \frac{18x^{\frac{5}{2}}}{5} + c$	$\int\sqrt{3x^3}dx$ $= \int\left(3x^3\right)^{\frac{1}{2}}dx$ $= \frac{2}{3}\left(3x^3\right)^{\frac{3}{2}} + \frac{3}{4}x^4$	Cannot proceed right from the beginning
(5 steps) B1	Incomplete procedure B2	Algebraic misconception A1 & *C1	Mixture of the basic rule of integration and the product rule C2	A2 &D1

Table 4.5

4.5 Comparison of solutions to problems in differentiation and integration

For the purpose of comparing students' performance in differentiation and integration, two problems were chosen one from each of section I(a) and I(b). They are respectively

Find $\frac{dy}{dx}$, when $y = \sqrt{2x^3}$

and

Evaluate $\int\sqrt{3x^3}dx$.

4.5.1 Comparison of solutions of B1 and B2 to problems in differentiation and integration

B1 is able to display procedures in great detail while solving both differentiation and integral problems . She uses six steps in differentiating the first problem while in the integral problem she uses five steps. From this analysis, it is apparent that B1 has powerful conceptual structures and is more likely to build sound relationships between bits of information so as to obtain a logical sequence of procedures. With this quality, she is more likely to reflect on her previous steps during working out solution for the problem. By doing so, she can detect whether the procedure used is reasonable or otherwise. Her performance in the two problems are tabulated thus:

Solutions of the B1	
Differentiation	Integration
$y=\sqrt{2x^3}$ $y=(2x^3)^{1/2}$ $y=2^{1/2}\times(x^3)^{1/2}$ $\Rightarrow y=\sqrt{2}\times x^{3/2}$ $y=\sqrt{2}x^{3/2}$ $\Rightarrow \frac{dy}{dx}=\sqrt{2}\times \frac{3}{2}x^{1/2}$ $=\frac{3\sqrt{2}x^{1/2}}{2}$	$\int\sqrt{3x^3}dx=\int(3x^3)^{1/2}dx$ $=\int((3^{1/2})\times(x^{3/2}))dx$ $=\int\sqrt{3}x^{3/2}dx$ $=\frac{\sqrt{3}x^{5/2}}{5/2}+c$ $=\frac{2\sqrt{3}x^{5/2}}{5}+c$
Detailed solution 6 steps	Detailed solution 5 steps

Table 4.6

In the case of B2, she curtails the procedures in doing the first question. She uses three steps to get to the solution. In the second problem she tries to write a detailed procedure but fails. This phenomenon may occur due to her failure in coping with the long sequence of procedures. In other words, her conceptual structures may not be strong enough to recall the next reasonable steps. Hence she faces breakdown in the procedure in solving the integral problem. In this case the breakdown of procedures occurs after performing three steps. B2's solutions in both problems are shown below:

Solutions of B2	
Differentiation	Integration
$y=(2x^3)^{\frac{1}{2}}=\sqrt{2}\sqrt{x^3}$ $\frac{dy}{dx}=\frac{3}{2}\sqrt{2}x^{\frac{1}{2}}$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int(3^{\frac{1}{2}}x^{\frac{1}{2}})$ $=\int 3^{\frac{1}{2}}\times(x^3)^{\frac{1}{2}}dx$
Curtailed procedure 3 steps	Incomplete procedure

Table 4.7

4.5.2 Comparison of solutions of A1 and A2 to problems in differentiation and integration

In the case of the A grade students, both fail to get the solutions when dealing with these two problems. In other words these students experience breakdown in the solutions. The causes of breakdown of procedures in both problems are not the same. In the case of A1, the breakdown in the differentiation problem occurs due to an overgeneralisation of a mixture of the basic rule of differentiation and the chain rule. It appears after the second step. Breakdown in the second problem occurs due to an algebraic error. Here, the phenomenon occurs also after the second step. From the analysis of A1's performance in both problems, it seems that the mathematical information retained by her may not linked by conceptual linkages. So, when it comes to retrieve the necessary information she may tend to get the wrong or inappropriate one. The presence of unreasonable mathematical facts in the solutions indicates the starting point for the occurrence of breakdown of procedures. A1's performance in both problems can tabulated as shown on page 45.

Solutions of A1	
Differentiation	Integration
$y=(2x^3)^{\frac{1}{2}}$ $\frac{dy}{dx}=\frac{1}{2}(6x^2)^{-\frac{1}{2}}$ $=\frac{1}{2(6x^2)^{\frac{1}{2}}}$	$\int(3x^3)^{\frac{1}{2}}$ $\int(3x^{\frac{3}{2}})$ $=\frac{3x^{\frac{5}{2}}}{\frac{5}{2}}$ $=\frac{6x^{\frac{5}{2}}}{5}+c$
Overgeneralisation of the basic rule of differentiation and the chain rule	Algebraic Misconception

Table 4.8

Like A1, A2 also fails in getting the solutions for both problems. However, the causes of breakdown are entirely different. In the first problem, the cause of breakdown is due to misapplication of chain rule. This phenomenon occurs after the first step. In the second problem, A2 cannot proceed from the beginning. From this analysis, it is found that A2's conceptual structures seem weaker than those of A1. Performance of A2 in both problems is tabulated as follows:

Solutions of A2	
$y=\sqrt{2x^3}$ $=(2x^3)^{\frac{1}{2}}$ $\frac{dy}{dx}=\frac{1}{2}.6x(2x^3)^{-\frac{1}{2}}$ $=3x(2x^3)^{-\frac{1}{2}}$ $=\frac{3x}{\sqrt{(2x^3)}}$	$\int\sqrt{3x^3}dx$
Misapplication of chain rule	Cannot proceed

Table 4.9

4.5.3 Comparison of solutions of C1 and C2 to problems in differentiation and integration.

In the case of C students none of them get to the solutions for both problems. Similar to grade A students, the grade C students face breakdown. C1 experiences breakdown of procedures in both problems because of algebraic errors. In both problems, the breakdown occurs after the first step. When comparing the conceptual linkages of B1 to that of C1, it seems C1 possesses less appropriate linkages in this area. The performance of C1 in both problems can be seen in the table below:

Solutions of C1	
Differentiation	Integration
$y=\sqrt{2x}=(2x)^{1/2}$ $=4x^{1/2}$ $\frac{dy}{dx}=\frac{4x^{-1/2}}{2}$ $=\frac{2}{x^{1/2}}$	$\int\sqrt{3x^3}\,dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int 9(x^3)^{\frac{1}{2}}dx$ $=\int 9x^{\frac{3}{2}}dx$ $=\frac{9x^{\frac{5}{2}}}{\frac{5}{2}}$ $=\frac{18x^{\frac{5}{2}}}{5}+c$
Algebraic Error (Error in manipulating the indices before differentiation)	Algebraic Error (Error in manipulating the indices before differentiation)

Table 4.10

C2 experiences breakdown of procedures due to algebraic error. The breakdown occurs in the first step. In the second problem she overgeneralises the basic rule of integration and the product rule of differentiation or possibly the chain rule. This phenomenon occurs after getting the first step correct. Performance for C2 is tabulated on page 47.

Solutions of C2	
Differentiation	Integration
$y = \sqrt{2x}$ $= 2x^{\frac{1}{2}}$ $\frac{dy}{dx} = \frac{2x^{-\frac{1}{2}}}{2}$ $= \frac{1}{x^{\frac{1}{2}}}$	$\int \sqrt{3x^3}\,dx$ $= \int (3x^3)^{\frac{1}{2}} dx$ $= \frac{2}{3}(3x^3)^{\frac{3}{2}} + \frac{3}{4}x^4$
Algebraic error	Overgeneralisation of basic rule of integration and product rule in differentiation

Table 4.11

In the case of D students, a comparison of their solutions in differentiation and integration cannot be made. This is because one of the D students, D2, left the interview room before answering both questions. D1 is able to use the chain rule but fails to simplify the final answer. The work of D2 is shown in the table below:

Solutions of D1	
Differentiation	Integration
$y = (2x^3)^{\frac{1}{2}}$ $\frac{dy}{dx} = \frac{1}{2}(2x^3)^{-\frac{1}{2}} \times 6x^2$ $= 12x^2(2x^3)^{-\frac{1}{2}}$	$\int (6x)^2 dx = \frac{1}{3}(6x)^3 \times 6$ $= 18(6x)^3$
Chain rule	Mixture of the basic rule of integration and the chain rule

Table 4.12

From the above observations it is found that the more able student, in this case B1, has strong conceptual structures which enable her to retain and retrieve effectively the necessary mathematical information when required. In performing detailed procedures she tends to reflect the previous steps as they proceed to the next steps by joining the existing information to the new mathematical facts. By this means, her procedures are more likely to be connected in reasonable sequence and hence breakdown in solutions is noticeable and can be avoided. In contrast, the conceptual linkages for the less able students are less well connected. Their bits of mathematical information may not connected. So, when recalling the required mathematical facts, they usually get the inappropriate bits. These inappropriate bits will affect the reasonableness of procedures of the less able students.

4.6 Flexibility of method in solving same calculus problems

Certain calculus problem can be solved in a number of ways. These methods are not necessarily common rules in differentiation or integration. Sometimes they are more efficient ways for the students to obtain solutions. For example in differentiating $\frac{1+x^2}{x^2}$ the problem can be solved easier by first simplifying the expression as $\frac{1}{x^2}+1$ before carry out the actual process of differentiation. Out of seven students attempting this problem, only B1 uses this method. This student is the more successful one on several items in the test. She uses her strong conceptual knowledge and links to interchange the symbolism in a meaningful manner and displays her procedure thus:

$$y=\frac{1+x^2}{x^2}$$

$$y=\frac{1}{x^2}+\frac{x^2}{x^2}$$

$$\Rightarrow \quad y=\frac{1}{x^2}+1$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{2}{x^3}$$

This student can also see the expression $\frac{1+x^2}{x^2}$ as a quotient. Thus the quotient rule can be used to differentiate the expression directly. According to this student she preferred the former method. Her richness in conceptual links makes her more likely able to choose the easier method to be employed.

In contrast less successful students may not easily realise the existence of the first method and their narrower procedural knowledge fails to help them in switching the calculus symbolism in getting the easier approach. As these students cling onto specific strategies, they find great difficulty in handling the calculus task. In this problem the less able students see the expression to be differentiated just as a quotient which leads them into more complicated symbol manipulations.

Based on the analysis of students' performance in this question, six out of seven students tried to solve this problem by a complicated quotient rule method. Out of six students, only two of them, A1 and C1, managed to get the correct solution.

$$y = \frac{1+x^2}{x^2}$$

$$u = 1 + x^2 \qquad v = x^2$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{x^2.2x - (1+x^2)(2x)}{x^4}$$

$$= \frac{2x^3 - 2x - 2x^3}{x^4}$$

$$= -\frac{2x}{x^4}$$

$$= -\frac{2}{x^3}.$$

Four of them failed to obtain the simplified form of a solution. Of these four students, two of them, A2 and B2, displayed incomplete procedures while the other two, D1 and C2, failed to simplify the solution. As the less able students seem to be lacking in conceptual structures, they are less likely to reflect back on all the steps they have displayed and fail to check whether the solutions they have obtained are reasonable or otherwise. The performance of these four students can be tabulated as below:

Solutions of Students	
Incomplete procedure	Non-simplified answer
$u = 1 + x^2 \quad v = x^2$ $\frac{du}{dx} = 2x \quad \frac{dv}{dx} = 2x$ $\frac{(x^2 \times 2x) - [(1+x^2) \times 2x]}{x^4} =$ $2x^3 - 2x(1 - x^2) =$ $2x(x^2 - 1 -$	$u = 1 + x^2 \quad v = x^2$ $\frac{du}{dx} = 2x \quad \frac{dv}{dx} = 2x$ $\frac{dy}{dx} = \frac{x^2(2x) - (1+x^2)(2x)}{x^4}$ $= \frac{2x^3 - 2x - 2x^3}{x^4}$ $= -\frac{2x}{x^4}$
2 students (B2 & *A2)	2 students (D1 & *C2)

Table 4.13

By these observations, it seems that the less able students believe the quotient rule to be the only means to get to the solution for this problem. They fail to see that this problem can be solved by at least four methods namely — by first simplifying the expression prior to actual differentiation, by using the quotient rule, by using the product rule, and by rearranging the equation and using implicit differentiation. In this study none of the students was able to carry out more than two approaches named above in tackling this problem. None of them used the product rule or the less familiar implicit differentiation. Only one student (B1) in the interview responded that this problem can be solved by two methods.

> In solving this problem, I can use the quotient rule. But I can also solve this problem by another method. (BA(QTS) student, 1993)

This phenomenon may occur as only seven students were involved in the study. This number is very small and hence possibly their abilities are not widely spaced as in the whole population.

However, there is an indication that the more able students have flexibility in using approaches with more than one strategy for solving the same calculus problem and have the ability to switch easily from one representation to the another, looking for the simpler method. Such a student is more likely to succeed in working out the solution for a calculus task. The less successful students lack flexible strategies and may find great difficulties in solving various types of calculus problems.

4.7 Algebraic misconception

One cause of an algebraic misconception results from the failure in handling indices during manipulation. For example, question 5, in section I(a),

$$\text{Find } \frac{dy}{dx} \text{ when } y = \left(x + \frac{1}{x}\right)^2.$$

Some students seem to prefer to expand the square of the expression prior to differentiation. Out of seven students attempting this problem, five students tried to get the solution by using this approach. Because of their weaknesses in handling indices during algebraic symbolic manipulation, all five failed completely or managed to expand the expression partially. Students' errors in handling indices in this problem can be tabulated as on page 51.

Errors of students		
$y = \left(x+\frac{1}{x}\right)^2$ $= \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)$ $= x^2 + \frac{2}{x} + \frac{1}{x^2}$ $\frac{dy}{dx} = 2x - 2\ln x - 2x^{-3}$	$y = \left(x+\frac{1}{x}\right)^2$ $\left(x+x^{-1}\right)^2$ $x^2 + 2x.x^{-1} + (x^{-1})^2$ $x^2 + 2x^{-1} + x^{-2}$ $\frac{dy}{dx} = 2x - 2x^{-2} - 2x^{-3}$	$y = \left(x+\frac{1}{x}\right)^2$ $\left(x+x^{-1}\right)^2$ $\left(x+x^{-1}\right)\left(x+x^{-1}\right)$ $x^2 + x^{-2}$ $\frac{dy}{dx} = 2x - \frac{2}{x^3}$
A2	B2	A1, *C1 & D1

Table 4.14

From the above analysis, A2 made an algebraic error when she expanded the expression $\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)$ to get $x^2+\frac{2}{x}+\frac{1}{x^2}$. Perhaps, in the expansion A2 conceived all the expressions must contain x. So, instead of writing 2 in the expanded expression she writes $\frac{2}{x}$. This expression may seem appropriate because the next expression is $\frac{1}{x^2}$.

B2 makes an error when she simplifies the expression $x^2+2x.x^{-1}+(x^{-1})^2$ as $x^2+2x^{-1}+x^{-2}$. In this case, perhaps this student multiplies the power of x in the term $2xx^{-1}$ as $1\times(-1)$, leaving the term in the answer as $2x^{-1}$.

A1, C1 and D1 who made the same errors expanded the square $\left(x+x^{-1}\right)\left(x+x^{-1}\right)$ as $\left(x^2+x^{-2}\right)$. These students failed to see the existence of 2 as the middle term. Such a type of error perhaps arises from over generalisation of previously learned procedures which is in accordance with view of Hiebert & Carpenter, (1992), i.e. if $(xy)^n$ is $x^n y^n$, then many students will think $(x+y)^n$ is x^n+y^n.

4.8 Other form of algebraic misconception

In problem 2 from section I(a), i.e.

Find $\frac{dy}{dx}$ when $y=\frac{1}{(x+1)^{\frac{1}{2}}}$, two students B2 and C2 were able to carry the procedural differentiation but had misconceptions in their final answers. Their performance in this problem can be tabulated as below:

Solutions of students	
B2	C2
$y=(x+1)^{-\frac{1}{2}}$ $\frac{dy}{dx}=-\frac{1}{2}(x+1)^{-\frac{3}{2}}$ $=-\frac{2}{(x+1)^{\frac{3}{2}}}$	$y=\frac{1}{(x+1)^{\frac{1}{2}}}$ $y=(x+1)^{-\frac{1}{2}}$ $\frac{dy}{dx}=-\frac{1}{2}(x+1)^{-\frac{3}{2}}$ $=-\frac{1}{\frac{1}{2}(x+1)^{\frac{3}{2}}}$

Table 4.15

In this case, C2 produces an answer $-\frac{1}{\frac{1}{2}(x+1)^{\frac{3}{2}}}$ possibly after conceiving the expression $-\frac{1}{2}(x+1)^{-\frac{3}{2}}$ as $-[\frac{1}{2}(x+1)^{\frac{3}{2}}]^{-1}$.

In problem 4 from Section I(a),

Find $\frac{dy}{dx}$, when $y=\sqrt{2x^3}$,

D1 is able to carry out the process of differentiation but has a misconception in her final answer.

Solutions of D1	
Differentiation	Integration
$y=(2x^3)^{\frac{1}{2}}$ $\frac{dy}{dx}=\frac{1}{2}(2x^3)^{-\frac{1}{2}}\times 6x^2$ $=12x^2(2x^3)^{-\frac{1}{2}}$	$\int(6x)^2dx=\frac{1}{3}(6x)^3\times 6$ $=18(6x)^3$

Table 4.16

Here, D1 finds difficulty in simplifying the complex expression. She makes the error after conceiving the product of $\frac{1}{2} \times 6$ as 12 to yield an answer for the above problem as $12x^2(2x^3)^{-\frac{1}{2}}$.

In the integration problem

Evaluate $\int (6x)^2 dx$

she makes a similar kind of mistake in simplification of the answer.

4.9 Errors in performing integration and differentiation

Some students overgeneralised the calculus rule or wrongly applied the existing formulae when trying to work out the answer for certain problems. For example, A1 overgeneralised the basic rule of integration when she gave the answer for problem 3 from section I(b),

$$\int (6x)^2 dx$$

as

$$\frac{1}{3}(6x)^3 + c.$$

In solving problem 2,

$$\int \frac{1}{(x-3)^5} dx,$$

A2 over generalises the mixture of integration of reciprocal function $\frac{1}{x}$ and the basic rule of integration as she proceed as follows:

$$\int \frac{1}{(x-3)^5} dx = \int (x-3)^{-5}$$
$$= \ln(x-3)^{-5} + c$$
$$= -\frac{\ln(x-3)^{-4}}{4} + c$$

B1 over generalises the basic rule of differentiation when she tries to solve problem 5 in Section I(a) as follows:

$$y=\left(x+\frac{1}{x}\right)^2$$
$$\frac{dy}{dx}=2\left(x+\frac{1}{x}\right)$$
$$=2x+\frac{2}{x}.$$

B2 over generalises the basic rule of integration in problem 3 in Section I(b), when she demonstrates her procedure in trying to get the solution as shown below:

$$\int(6x)^2dx=\frac{(6x)^3}{3}+c.$$

C2 makes an error when she wrongly applies chain rule to problem 5 in Section I(a). Her work is as shown below:

$$y=\left(x+\frac{1}{x}\right)^2$$
$$\frac{dy}{dx}=2\times -x^{-2}\left(x+\frac{1}{x}\right)$$
$$=-2x^{-2}\left(x+\frac{1}{x}\right).$$

C2 also overgeneralises the basic rule of integration possibly with the product rule or chain rule in integrating expression in problem 4 in Section I(b), when she tries to solve the problem thus:

$$\int\sqrt{3x^3}dx$$
$$=\int\left(3x^3\right)^{\frac{1}{2}}dx$$
$$=\tfrac{2}{3}\left(3x^3\right)^{\frac{3}{2}}+\tfrac{3}{4}x^4.$$

D1 has a mixture of the basic rule of integration and possibly the chain rule in differentiation when trying to solve problem 3 from section I(b)

Evaluate $\int(6x)^2\,dx$

as follows:

$$I_n=\frac{1}{3}(6x)^3\times 6$$
$$=18(6x)^3$$

4.10 Summary

From the above observations, it is found that the grade A student failed to see the sameness of a solution when it is displayed in different mode. Thus, it is hypothesised that students who are good at symbolic manipulation may not necessarily be able to reasonably interpret the same objects in different forms.

The more able student, B1, displays detailed procedures in problems 4 in Section I(a) and in problem 4 in Section I(b). This phenomenon suggests that there is no correlation between curtailment of procedures and success. The failure to get a wide spectrum of students' performance may be due to the limited size of the sample involved. In this study, only eight students (one left before interviews ended) participated. Hence, it is possible this phenomenon might affect the study. Thus, it become more obvious that some students who are successful may wish to include a detailed solution to make sure they are correct. Thus more successful students might include both curtailed and detailed solutions.

Here, the more able student possesses a number of reasonable representations and tend to show flexibility in switching from one representation to the other. Thus, it is hypothesised that students with higher grades may have more strategies to tackle the same calculus problem and are likely to choose the easier method. The students with lower grades, with inadequate conceptual knowledge are more likely to be satisfied with the security of a single familiar procedure.

4.11 Implication for the main study

4.11.1 Limitations of the pilot test

1. The pilot study was designed to identify range of students' performance in the first year calculus course. Thus, the most appropriate students selected must be based on

their achievement of calculus course at A-level. But the students here are selected on the basis of their achievement in mathematics in general, which is less appropriate.

2. The students involved in the study are prospective teachers. Since in the main study at the Universiti Teknologi Malaysia, engineering students are also involved, the result of the pilot study might be giving a very vague notion about all students' thinking and their performance in a calculus course. These two group of students might have different levels of achievement and thinking in calculus.

3. During an interview, two students are chosen and are interviewed openly one after the other in the same room. Since the questions asked are the same, the second student might consider and imitate the first student responses and thus fail to express her/his own personal views or responses. In other words, the interviews in the pilot study are not independent and confidential.

4. The same phenomenon might be seen during answering the questionnaires. Since the students sat very closely together, one could copy the solution of the other. In addition to that, good student' s achievement might have some effect on the feeling of their weaker friends. Thus, the weaker students might present not their actual work but only one which they consider is desired by the interviewer.

4.11.2 Comments about the questionnaires

1. All the integral problems are of indefinite form. Thus, students' knowledge and ideas regarding the definite integral can not be identified.

2. There are no graphs or illustrations in the pilot test.

In order to design the questionnaires for the pilot test at the Universiti Teknologi Malaysia, the questions are carefully re-selected from the set of the pilot test at the University of Warwick. The questions in pilot questionnaires at the Universiti Teknologi Malaysia will be discussed in Chapter 5.

CHAPTER 5

PILOT STUDY AT THE UNIVERSITI TEKNOLOGI MALAYSIA

5.1 Introduction

Based on Krutetskii's (1976) findings on younger children, it was initially hypothesised that there would be a correlation between university student's mathematical ability and his/her curtailment of solution. However, in the pilot test at the University of Warwick, some successful students preferred to write out algorithms in detail, so this is likely to be less clear. What was revealed in the pilot test at the University of Warwick was a qualitative difference between "more" and "less" successful students in which the "more" successful solve certain calculus problems by using general strategies which are less algorithmic.

Such a possibility may be revealed when a student is given a problem which looks like a straight algorithm but is actually simplified further by doing some preliminary non-algorithmic simplification. For example:

$$\text{Find } \frac{dy}{dx}\text{, whenever } y = \frac{1+x^2}{x^2},$$

In determining the derivative of $\frac{1+x^2}{x^2}$, using the standard algorithm for the derivative of a quotient involves the student needing to use the formula in a cumbersome way and then simplifying the result:

$$y = \frac{1+x^2}{x^2}$$

$$u = 1+x^2 \qquad v = x^2$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{x^2.2x - (1+x^2)(2x)}{x^4}$$

$$= \frac{2x^3 - 2x - 2x^3}{x^4}$$

$$= -\frac{2x}{x^4}$$

$$= -\frac{2}{x^3}$$

Using such a formula or method may involve a complicated computation and lengthy algorithm which need stages of simplification later on, which will be termed *post-algorithmic simplification.*

However, if the expression is first simplified as $x^{-2}+1$ then its derivative is straight away seen to be $-2x^{-3}$, affording a considerable reduction in processing.

Students may shorten their solutions in various ways. For instance, the initial simplification in this problem might be conceived as a succession of formal manipulations:

$$\frac{1+x^2}{x^2} = \frac{1}{x^2} + \frac{x^2}{x^2} = x^{-2} + 1.$$

However, several students compress this further to a single written step:

$$\frac{1+x^2}{x^2} = x^{-2} + 1.$$

A possible reason for this may involve first processing $\frac{1}{x^2}$ as x^{-2}, then $\frac{x^2}{x^2}$ as 1. Some students do this by reading the symbol $\frac{1+x^2}{x^2}$ as two fractions in the form

$$\frac{1}{x^2} + \frac{x^2}{x^2},$$

scanning it in such a way as to read $\frac{1}{x^2}$, translating it and writing it down immediately as x^{-2}, then the rest of the expression is seen and written down as $+1$. By operating in this way, the simplification may be written down in a single composite step. Such simplification of an algebraic expression before carrying out the actual process of differentiation is termed *conceptual preparation.*

For the purpose of the main study, it is hypothesised that there is a relationship between ability and curtailment (which is expected to be false) and a relationship between ability and conceptual preparations of procedures (which is expected to be true).

Graphs contain a considerable quantity of mathematical information. But there is an evidence that students find great difficulty in linking the analytical use of symbolism to graphical ideas. For example these students were able to draw the graph of certain integral problems correctly but ignored them in obtaining the value of the corresponding integral which would help them to get the solutions more easily. Based on this evidence, it is also hypothesised in the main study that:

(a) Students who are good at symbolic manipulation yet fail to link to visual ideas.

(b) Students who are flexible at performing familiar processes but fail to reverse them.

5.2 Questionnaires in the pilot study at the Universiti Teknologi Malaysia

The questions in the pilot test at Warwick university were modified. Several questions were deleted because they were repetitions and involved just using procedures in order to get the correct answers. The questionnaire used in the pilot study at the Universiti Teknologi Malaysia are as follows. This questionnaire was in interview format, with the student doing one question at a time followed by a discussion of what had been done.

PILOT STUDY AT THE UNIVERSITI TEKNOLOGI MALAYSIA

SECTION I

(i) Evaluate $\int \sqrt{3x^3}\,dx$.

(ii) Find $\frac{dy}{dx}$ when

(a) $y = \frac{1}{\sqrt{x}}$,

(b) $y = \frac{1+x^2}{x^2}$,

(c) $y = \left(x + \frac{1}{x}\right)^2$.

SECTION II

(i) Find the area of the shaded region.

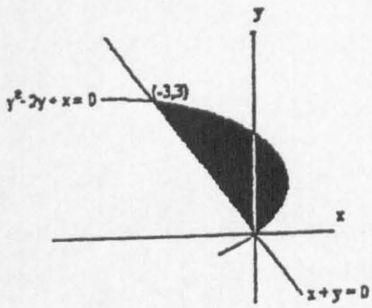


Figure 5.1

Please explain graphically, how this might be done.

(ii) By considering the function $y=\sqrt{4-x^2}$, sketch and shade the region of the graph given by the following integral

$$\int_0^2 \sqrt{4-x^2}\,dx$$

What is the value of the above integral?

The questionnaires in the pilot test at the UTM comprised two sections. Section I consists of simple indefinite integral and differentiation while Section II dealt with visual ideas. All questions except problem ii (a) i.e. find $\frac{dy}{dx}$ when $y=\frac{1}{\sqrt{x}}$, in section I have been taken from the pilot study at the University of Warwick. The purpose of selecting those problems have been stated in the pilot study. Problem ii(a) is taken from Love and Rainville's *Differential and Integral Calculus* (1958, p. 35) and is used to identify the extent of students' ability in curtailment of solution in differentiation. Questions 4 from section I(a) in the pilot study at the University of Warwick, i.e. find $\frac{dy}{dx}$ when $y=\sqrt{2x^3}$ is omitted. The reason for omission is that problem 4 from I(a) is similar to problem 4 in section I(b), i.e. to evaluate $\int \sqrt{3x^3}\,dx$. The only major difference is that one is an integral question while the other one in the pilot test at the University of Warwick is differentiation.

Other questions which are used to identify the basic knowledge and ideas in differentiation and integration in the Warwick pilot test are also omitted. They are too trivial or can be solved by using a standard rule in calculus. When the students faced questions for example find $\frac{dy}{dx}$, when $y=2x^4-x^2-6$, they just applied the basic rule of differentiation or when they came to evaluate $\int \frac{1}{\sqrt{x+1}} dx$, they either use the substitution method or changed it its equivalent $\int (x+1)^{-\frac{1}{2}} dx$ before integrating it by using familiar basic rules of integration.

In Section II, two new questions are added. The first question aims to identify those students who are good at symbolic manipulation yet fail to link to visual ideas. In that case questions 3(a) and 3(b) in Section I(a) in the pilot study at the University of Warwick are replaced with a problem (i) in Section II as shown above. This question is a modification of example 6 from the text used by the first year students (Common First Year Mathematics Group of the University Teknologi Malaysia, 1993/1994, pp. 141-142). The second question in Section II is used to identify those students who have a number of representations but fail to link between them. It is stated as number (ii) in section II above. This question is taken from *Introductory Calculus Unit Guide* (SMP, p. 26).

5.3 The sampling of the subjects

For the purpose of a pilot study at the Universiti Teknologi Malaysia, a set of twelve second year students were chosen at random, six of them from SPK (prospective computer science teachers) while the other six from SKA course (prospective civil engineers), based on their grades achieved in first year mathematics examination at the Universiti Teknologi Malaysia. Their scores in mathematics vary: three with grade A, three with grade B, three with grade C, three with grade D. However one of the D grade students failed to turn up in the interview.

5.4 The procedure of the interview

The questionnaires administered in the interview in the pilot test at the UTM commenced on the 6th July 1994. Altogether there were 6 interviews. The second, third, fourth, fifth and sixth were carried out respectively on the 7th , 9th, 12th, 14th and 17th of July 1994. For each of the corresponding days, there were 2 students interviewed at different times. The interviews were held either in the morning, afternoon or night depending on the availability of the students. These interviews were carried out either in the researcher's office or in a room (specially arranged for the interviews) in the students' hostel. The idea of using this combined method was already stated in Chapter 3.

Before the interview each student was invited into the room. He/she was allowed for sometime to relax with some refreshment given. The idea of doing this was to enable the student to regain confidence and build a rapport with the researcher. Thus the researcher was able to obtain the actual student's responses to the questions posed in the interviews.

At this time the researcher introduced himself to the student. In this introduction the researcher explained his post as a lecturer in the Mathematics Department at the Universiti Teknologi Malaysia and perhaps he could be teaching them in 1994 if he was not away at the University of Warwick in England for a PhD.

The researcher also told the students about his interest in identifying the extent of students' understanding in the calculus course and the researcher was aware that the students had completed a calculus course last session. It was emphasised to the students that this is not an examination. The result would be confidential to the researcher. Lastly, before the interview started the researcher thanked the student for giving him some of his/her time in participating the interview.

Each student was given a set of questions and was asked to write his/her name at the top of the front page. The whole process of interviewing and answering all the questions lasted for about one hour. After each question had been solved, the researcher carried out a similar procedures and strategy to that in the pilot study at the University of Warwick. However, in this case all responses were tape recorded and transcribed.

CHAPTER 6

ANALYSIS OF DATA IN THE PILOT STUDY AT THE UNIVERSITI TEKNOLOGI MALAYSIA

6.1 A study of the extent to which students compress algebraic procedures in terms of the number of steps used to carry them out

Faced with algorithmic calculus problems, some students preferred curtailed solutions while some preferred detailed solutions. However, many faced breakdown of procedures in their solutions. In order to see this phenomenon, the extent to which students compress algebraic procedures, in terms of the number of steps used, is analysed. For this purpose question (i) from Section I,

$$\textit{Evaluate} \int \sqrt{3x^3}\,dx$$

is used. Eleven students — three with grade A, three with grade B, three with grade C and two with grade D attempted this question. The performance of all three grade A students may be summarised as on page 64:

Solutions of grade A students		
All responses correct (3)		
A1	A2	A3
$\int\sqrt{3}x^{\frac{3}{2}}dx$ $=\sqrt{3}\frac{x^{\frac{5}{2}}}{5}.2$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}$	$\int\sqrt{3x^3}dx=\int\sqrt{3}x^{\frac{3}{2}}dx$ $\sqrt{3}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+c$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$\int\sqrt{3x^3}dx=\int(3x^3)^{\frac{1}{2}}dx$ $=\int\sqrt{3}x^{\frac{3}{2}}dx$ $=\sqrt{3}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right)$ $=\sqrt{3}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$
3 steps Curtailed solution	4 steps	6 steps Detailed solution

Table 6.1

Out of three A students attempting this question, A1 uses three steps, A2 uses four steps while A3 uses six steps in order to get the solution.

The A student who performed six steps (the more detailed solution) might have carried out the procedures as already described in Chapter 4. From the above observation, it is found that A1 uses the same general method as that used by A2 or A3. Comparison of performance of grade A2 and A3 students in doing this question is described thus:

In getting to $\int\sqrt{3}x^{\frac{3}{2}}dx$ from the original question, A3 uses more detailed procedures (2 steps) than that of A2 (only one step). Similarly in getting to $\sqrt{3}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)$ from $\int\sqrt{3}x^{\frac{3}{2}}dx$, A3 has included the arithmetic operation in the step immediately after the integration process but A2 just uses one step in getting to the above stage. Since A3 performs extra steps, A3's solution is more detailed than that of A2. In A1 and A3's solutions the constant c has been left out.

However, in building up detailed procedures one may hypothesise that these more able students are more likely to make use of their strong conceptual structures to link between bits of procedures. They are more likely to use their rich conceptual knowledge to reflect the reasonableness of their procedures as they proceed. In this way, these students are able to display a long systematic sequence of meaningful procedures in reaching the final answer (in conventional form) for calculus problems. Thus, the breakdown in the procedures are less likely to occur in solutions performed by the more able students.

Out of three B grade students attempting this problem, two of them, B1 and B2, are able to reach the final solution while one of them, B3, has an error in his procedure. The performance of all three B grade students can be seen in the table below:

Solutions of grade B students		
Correct responses (2)		Error (1)
B1	B2	B3
$\int\sqrt{3x^3}dx$ $\int(3x^3)^{\frac{1}{2}}dx$ $\sqrt{3}\int x^{\frac{3}{2}}dx$ $\sqrt{3}\left[\frac{2}{5}x^{\frac{5}{2}}\right]+c$ $\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int 3^{\frac{1}{2}}.x^{\frac{3}{2}}dx$ $=\sqrt{3}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}.\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+c$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$\int\sqrt{3x^3}dx$ $\int(3x^3)^{1/2}dx$ $\frac{(3x^3)^{3/2}}{3/2}$ $=\frac{2(3x^3)^{3/2}}{3}$
4 steps	5 steps	Breakdown in the second step Misapplication of basic rule of integration

Table 6.2

The successful B students in this problem use either four or five steps in order to get the solutions. B3 faces breakdown after successfully performing the first step. She makes an error in integrating

$$\int(3x^3)^{1/2}dx$$

as

$$\frac{(3x^3)^{\frac{1}{2}+1}}{\frac{1}{2}+1},$$

In other words, she overgeneralises the basic rule of integration.

In the case of C grade students, all of them demonstrate erroneous procedures. Their performances are as in the table below:

Solutions of grade C students	
Errors in all responses	
*C1 & C2	C3
$\int\sqrt{3x^3}\,dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int 3x^{\frac{3}{2}}dx$ $=\frac{3x^{\frac{5}{2}}}{\frac{5}{2}}+c$ $=\frac{2}{5}.3x^{\frac{5}{2}}+c$ $=\frac{6}{5}x^{\frac{5}{2}}+c$	$\int\sqrt{3x^3}\,dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\frac{(3x^3)^{\frac{3}{2}}}{\frac{1}{2}}$
Breakdown in the second step Error due to algebraic misconceptions	Breakdown in the second step Error due to direct application of basic rule of integration

Table 6.3

All face breakdown after the first step. There are two types of errors display by the C students. Two of them have algebraic misconceptions as a result of changing the integral $\int(3x^3)^{\frac{1}{2}}dx$ to $\int 3x^{\frac{3}{2}}dx$ prior to integration whilst the other one considers the integral expression $\int(3x^3)^{\frac{1}{2}}dx$ as $\int x^n dx$ before erroneously applying the basic rule of integration.

The two D students also experience breakdown of procedures. D1's error is due to an algebraic misconception whilst D2's error is due to misapplication of substitution method of integration. As in the case of grade C students, the breakdown of procedures occur just after the first step.

Solutions of grade D students	
Errors in all responses	
D1	D2
$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{1/2}dx$ $=\int 3x^{3/2}dx$ $=3\int x^{3/2}dx$ $=3\frac{2}{5}x^{5/2}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{1/2}dx$ $=\int(3u)^{1/2}\frac{du}{dx}$
Breakdown in the second step Error due to algebraic misconceptions	Breakdown in the second step Error due to misapplication of substitution method of integration

Table 6.4

The less able students in trying to perform or to cope with detailed procedures probably have to rely most of the time upon their memorization of numerous isolated bits of mathematical information. Without strong conceptual structures, the less able students may fail to retrieve all the necessary information and find great difficulties in using them effectively. Hence, the procedures and solutions displayed by these students are not satisfactory. In other words, the less able students fail to reflect on the appropriateness of their procedures and solutions. Hence, they are likely to experience breakdown in their procedures. From the above analysis, it is found that one of the grade B students, all the three C grade students and both D grade students exhibit such a phenomenon. All the breakdown of procedures occur at a very early stage in their solutions.

6.2 To identify the extent of students' ability in curtailment of solution

For this purpose, question ii(a),

$$\text{find } \frac{dy}{dx} \text{ when } y=\frac{1}{\sqrt{x}}$$

is considered. All seven students attempted this problem.

The grade A students performance in this problem are as follows:

Solutions of grade A students	
A3	A1 & A2
$y = \frac{1}{\sqrt{x}}$ $= x^{-\frac{1}{2}}$ $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$	$y = x^{-\frac{1}{2}}$ $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$ $= -\frac{1}{2\sqrt{x^3}}$
2 steps	3 steps

Table 6.5

All three A students attempted this problem. All of them used two steps in reaching the answer for this problem. In the case of A1 and A2, they have written one extra step, i.e. $-\frac{1}{2\sqrt{x^3}}$. But based on idea in the text written (Moore, 1973; Kruglak & Moore, 1973,), the answer of the above question can be rewritten as

$$-\frac{1}{2}x^{-\frac{3}{2}}, -\frac{x^{-\frac{3}{2}}}{2}, -\frac{1}{2x^{\frac{3}{2}}}, -\frac{1}{2\sqrt{x^3}}$$

or equivalent which may all be considered *conventional form.* Thus, A1 and A2' have written the same answer but in different form.

The grade three B students performance on this problem are as shown below:

Solutions of grade B students	
B1 & B3	B2
$y = x^{-\frac{1}{2}}$ $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$	$y = x^{-\frac{1}{2}}$ $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}-1}$ $= -\frac{1}{2}x^{-\frac{3}{2}}$ $= -\frac{x^{-\frac{3}{2}}}{2}$
2 steps	4 steps

Table 6.6

In solving this problem two of B students (B1 & B3) use two steps while the other one (B2) uses three steps to reach the solution. B2 includes arithmetic operations in the second step. This is why B2's solution is more detailed than that of B1 and B3. Here student B2 has displayed one extra step in the solution by displaying $-\frac{x^{-\frac{3}{2}}}{2}$ which is another form of the same solution.

From the above observation, the presence of extra steps in a solution does not necessarily imply that the solution is *essentially* more detailed than that of another.

Solutions from grade C students are as follows:

Solutions of grade C students	
C1 & *C2	C3
$y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$	$y = \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}$ $= x^{-1/2}$ $\frac{dy}{dx} = -1/2\, x^{-\frac{3}{2}}$
2 steps	3 steps

Table 6.7

C1 and C2 use more curtailed solutions than that of C3. In other words, C1 and C2 perform two steps while C3 performs three steps in getting the solution for this problem. C3 starts to detail the solution when she tries to convert the expression $\frac{1}{\sqrt{x}}$ to its equivalent form $x^{-1/2}$; she demonstrates the intermediate step $\frac{1}{x^{1/2}}$ which C1 and C2 miss out. Performance of students with D grade are tabulated as below:

Solutions of grade D students	
D2	D1
$y=(x)^{-\frac{1}{2}}$ $\frac{d}{dx}(x)^{-1/2} = -\frac{1}{2}(x)^{-3/2}$	$y = x^{-1/2}$ $\frac{dy}{dx} = x^{-3/2}$
2 steps	Breakdown after the first step

Table 6.8

One grade D student (D2) performed the procedures in two steps in reaching the solution while another (D1) faced breakdown as she failed to differentiate the expression $x^{-1/2}$.

6.3 Comparison of solutions to problems in differentiation and integration

Question (i), i.e. to evaluate $\int\sqrt{3x^3}\,dx$ and (ii)(a), i.e. to find $\frac{dy}{dx}$, when $y=\frac{1}{\sqrt{x}}$, are used to compare students' solution in differentiation and integration. All questions are from Section I. The performance of A1 and A3 in these problems can tabulated thus:

Solution of A1		Solution of A3	
Problem (i)	Problem (ii)(a)	Problem (i)	Problem (ii)(a)
$\int\sqrt{3}x^{\frac{3}{2}}dx$ $=\sqrt{3}\frac{x^{\frac{5}{2}}}{5}.2$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}$	$y=x^{-\frac{1}{2}}$ $\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{3}{2}}$ $=-\frac{1}{2\sqrt{x^3}}$	$\int\sqrt{3x^3}dx=\int(3x^3)^{\frac{1}{2}}dx$ $=\int\sqrt{3}x^{\frac{3}{2}}dx$ $=\sqrt{3}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right)$ $=\sqrt{3}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$y=\frac{1}{\sqrt{x}}$ $=x^{-\frac{1}{2}}$ $\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{3}{2}}$
3 steps Curtailed solution as compared to solution of A3	2 steps	6 steps Detailed solution	2 steps

Table 6.9

Student A1 uses a curtailed solution to solve problem (i)(a) while student A3 applied detailed procedures in getting the answer for the same problem. Student A3 included arithmetic operations during performing the integration process. In other words, this student wrote $\frac{3}{2}+1$ in the solution. Both answers to this questions are in conventional form. For problem (ii)(a) student A3 stopped the procedure after getting $\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{3}{2}}$, but A1 continued to the next step to get the same solution of different form. However both solutions (A1 &A3) are in conventional form.

Solution of B1		Solution of B2	
$\int\sqrt{3x^3}\,dx$ $\int(3x^3)^{\frac{1}{2}}dx$ $\sqrt{3}\int x^{\frac{3}{2}}dx$ $\sqrt{3}\left[\frac{2}{5}x^{\frac{5}{2}}\right]+c$ $\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$y=x^{-\frac{1}{2}}$ $\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{3}{2}}$	$\int\sqrt{3x^3}\,dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int 3^{\frac{1}{2}}.x^{\frac{3}{2}}dx$ $=\sqrt{3}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}.\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+c$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$y=x^{-\frac{1}{2}}$ $\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{1}{2}-1}$ $=-\frac{1}{2}x^{-\frac{3}{2}}$ $=-\frac{x^{-\frac{3}{2}}}{2}$
4 steps Less detailed solution as compared to solution of B2	2 steps Curtailed solution as compared to that of B2	5 steps Detailed solution	4 steps Detailed solution

Table 6.10

In doing problems (i) and (ii)(a) from Section I(a), student B2 applied detailed solutions when compared to that of B1. Student B2 displayed arithmetic operations in her procedures. All solutions displayed are in conventional form.

Solution of C1		Solution of C3	
$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{1/2}dx$ $=\int 3x^{3/2}dx$ $=\frac{3x^{5/2}}{\frac{5}{2}}+c$ $=\frac{2}{5}.3x^{5/2}+c$ $=\frac{6}{5}x^{5/2}+c$	$y=\frac{1}{\sqrt{x}}$ $=x^{-1/2}$ $\frac{dy}{dx}=-1/2\,x^{-3/2}$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{1/2}dx$ $=\frac{(3x^3)^{3/2}}{1/2}$	$y=\frac{1}{\sqrt{x}}=\frac{1}{x^{1/2}}$ $=x^{-1/2}$ $\frac{dy}{dx}=-1/2\,x^{-\frac{3}{2}}$
Breakdown after the first step Error due to an algebraic misconception	2 steps	Breakdown after the first step Error due to direct application of basic rule of integration	3 steps

Table 6.11

Both students have an error in solving problem (i). C1's error arises due to an algebraic misconception by writing $(3x^3)^{1/2}$ as $3x^{3/2}$ whilst C3 made an error when overgeneralising the basic rule of integration. But in solving problem (ii)(a), C1 used two steps whilst C3 used three steps. Both answers to this problem are in conventional form.

Solution of D1		Solution of D2	
$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int 3x^{\frac{3}{2}}dx$ $=3\int x^{\frac{3}{2}}dx$ $=3\frac{2}{5}x^{\frac{5}{2}}+c$	$y=x^{-\frac{1}{2}}$ $\frac{dy}{dx}=x^{-\frac{3}{2}}$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int(3u)^{\frac{1}{2}}\frac{du}{dx}$	$y=(x)^{-\frac{1}{2}}$ $\frac{d}{dx}(x)^{-\frac{1}{2}}$ $=-\frac{1}{2}(x)^{-\frac{3}{2}}$
Breakdown after the first step Error due to algebraic misconceptions	Breakdown after the second step Overgeneralisation of basic rule of differentiation	Breakdown after the first step Error due direct application of substitution method of integration	2 steps

Table 6.12

In the case of D students, both students experienced breakdown of procedures in doing problem (i) from Section I. In this problem D1 has an algebraic misconception by writing $(3x^3)^{\frac{1}{2}}$ as $3x^{\frac{3}{2}}$ whilst D2 overgeneralised the substitution method. He correctly substituted $u = x^3$ in $(3x^3)^{\frac{1}{2}}$ but failed to handle dx term satisfactorily. In problem (ii)(a) from Section I, D2 use 2 steps and give the answer in conventional form, while student D1 had a breakdown when he tried to differentiate the expression.

By looking at the solutions of B2, it seems that a student who used a detailed solution in one problem may use a detailed solution in another. (See page 72). In general, this is not the case since A3 gave a detailed solution for problem (i) from Section I but a curtailed solution for problem (ii)(a) from the same Section.

6.4 To identify the relationship between ability and conceptual preparation of procedure

Question (ii)(b) from Section I serves this purpose

Find $\frac{dy}{dx}$, when $y = \frac{1+x^2}{x^2}$.

Out of eleven students attempting this problem, five of them—three with grade A and two with grade B solved this problem by conceptual preparation of procedure. The way in which students may carry out symbolic manipulation in this aspect has already been fully described at the beginning of Chapter 5.

The other six students failed to perform conceptual preparation of procedures but were successful with a more complex version of the algorithm.

The students' performance and preference of conceptual preparations of procedures in this problem may be summarised as follows:

Students with grade	Conceptual preparation of procedures	Post-algorithmic simplification
A	3	0
B	2	1
C	0	3
D	0	2

Table 6.13

From these observations, it is suspected that there is a relationship between students' ability and conceptual preparation of procedures in a larger size of population. Based on this fact, it is hypothesised that there is a correlation between students' ability and conceptual preparation of procedure.

6.5 Various alternatives in tackling one particular problem

Some students have the ability to use effectively more than one method of solving calculus problems. This phenomenon can be seen when these students work out solutions for problem (ii)(b), i.e. find $\frac{dy}{dx}$ when $y = \frac{1+x^2}{x^2}$. This problem is chosen because it can be solved by using at least four possible alternatives:

(a) conceptual preparation of procedures,

(b) quotient rule,

(c) product rule

and

d) implicit differentiation, if the equation $y = \frac{1+x^2}{x^2}$ is changed into the equivalent form $x^2 y = 1 + x^2$ prior to the actual differentiation.

Of all seven students attempting this problem, none could solve this problem by using four methods or carry out rather complicated implicit differentiation. However, from this group of students it is found that, one A grade student, A2 is able to use three methods in differentiating the expression $\frac{1+x^2}{x^2}$. According to her:

> The expression $\frac{1+x^2}{x^2}$ is a quotient. Thus it can be solved by means of quotient rule. But I think I can go for a simpler method... I can transform $\frac{1+x^2}{x^2}$ to $x^{-2}+1$ first before performing the differentiation.
>
> (Second year SPK students, 1994).

In other words, the first method chosen is conceptual preparation of procedures after seeing that the expression can be differentiated easier if it is simplified prior to the actual differentiation. The second method exhibited by this student is the normal quotient rule. This expression can be expressed as a product of two factors, i.e.

$$\frac{1+x^2}{x^2} = \frac{1}{x^2}(1+x^2).$$

Hence, this expression can be differentiated by means of product rule. Her various approaches in getting the solutions for this problem are tabulated on page 77.

An A grade student with 3 approaches		
$y = \dfrac{1+x^2}{x^2}$ $= \dfrac{1}{x^2} + 1$ $\dfrac{dy}{dx} = \left(-\dfrac{2}{x^3}\right) + 0$ $= -\dfrac{2}{x^3}$	$\dfrac{dy}{dx} = \dfrac{x^2 \dfrac{d}{dx}(1+x^2) - (1+x^2)\dfrac{d}{dx}(x^2)}{(x^2)^2}$ $= \dfrac{x^2(2x) - (1+x^2)(2x)}{(x^2)^2}$ $= \dfrac{2x^3 - 2x - 2x^3}{x^4}$ $= \dfrac{2}{x} - \dfrac{2}{x^3} - \dfrac{2}{x}$ $= -\dfrac{2}{x^3}$	$y = \dfrac{1}{x^2}(1+x^2)$ $\dfrac{dy}{dx} = (1+x^2)\left(-\dfrac{2}{x^3}\right) + \left(\dfrac{1}{x^2}\right)(2x)$ $= \left(-\dfrac{2}{x^3} - \dfrac{2}{x}\right) + \dfrac{2x}{x^2}$ $= -\dfrac{2}{x^3} - \dfrac{2}{x} + \dfrac{2}{x}$ $= -\dfrac{2}{x^3}$
Conceptual preparation of procedure	Quotient rule	Product rule

Table 6.14

The other two A students (A1 & A3) manage to use two alternatives in tackling this problem. Both A1 and A3 showed a preference for conceptual preparation of procedure over the quotient rule.

Out of three grade B students attempting this problem, B2 and B3 show the ability of using two methods of solving the problem. B3 shows a preference for conceptual preparation of procedure over the quotient rule while B2 goes for the reverse. B1 has only one method of tackling this problem. He uses conceptual preparation of procedure.

Out of three C students attempting this problem, only C1 can solve this problem by means of two methods—the quotient rule and conceptual preparation of procedures (in that order of preference) while the other two grade C students (C2 & C3) manage to work out this problem only by means of the quotient rule.

In the case of two D grade students, one (D2) demonstrates two ways of getting the solution for this problem. She prefers the quotient rule over conceptual preparation of procedures. The other one (D1) just applies the quotient rule for this problem.

The various approaches used by the students in tackling problem (ii)(c) can be summarised as follows:

Student Performance (in order of preference)			
Grade	3 App.	2 App.	1 App.
A	3 1 CP/QR/PR	2 2 CP/QR	0
B	0	2 1 CP/QR, 1 QR/CP	1 CP
C	0	1 QR/CP	2 2QR
D	0	1 QR/CP	1 QR

Table 6.15

Key: App.—Approaches/Approach; CP—Conceptual preparation of procedure; QR—Quotient rule; PR—Product rule

From the above observation, it is possible that the students with higher grades have more flexible strategies in dealing with symbolic manipulation. They show the ability to relate meaningfully between symbols. For example, A2 is able to interchange the symbolism in the equation $y = \frac{1+x^2}{x^2}$ so that the equation can be represented as $y = \frac{1}{x^2}(1+x^2)$. They tend to use those approaches which involve less cognitive strain. Such an ability may not be fully developed in less successful students who are more likely to cling to specific rules.

6.6 Effect of the nature of the problem on conceptual preparation of procedure

For this purpose question (ii)(c) from Section I, i.e. Find $\frac{dy}{dx}$ when $y = \left(x + \frac{1}{x}\right)^2$ is used. Out of eleven students attempted this problem, four students (A2, B1, B3 and C1) expand the bracket before carrying out the differentiation. For example

$$
\begin{aligned}
y &= \left(x + \frac{1}{x}\right)^2 \\
&= x^2 + 2 + \frac{1}{x^2} \\
&= x^2 + 2 + x^{-2} \\
\frac{dy}{dx} &= 2x - 2x^{-3} \\
&= 2x - \frac{2}{x^3}.
\end{aligned}
$$

Four of them (A1, A3, B3, C2) successfully use the chain rule. For example, thus

$$
\begin{aligned}
y &= \left(x + \frac{1}{x}\right)^2 \\
\frac{dy}{dx} &= 2\left(x + \frac{1}{x}\right)\left[1 + \left(-\frac{1}{x^2}\right)\right] \\
&= 2\left(x + \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right) \\
&= 2\left(x - \frac{1}{x} + \frac{1}{x} - \frac{1}{x^3}\right) \\
&= 2x - \frac{2}{x^3}.
\end{aligned}
$$

The other three students (C3, D1 and D2) failed to give the correct final solution. C3 used the chain rule but displayed an incomplete post algorithmic simplification. D1 wrongly applied the chain rule whilst D2 correctly applied the quotient rule but failed to carry out the immediate post algorithmic simplification.

Errors performed by these students can be tabulated as below:

Students' solution		
C3	D1	D2
$y=\left(x+\frac{1}{x}\right)^2$ $\frac{dy}{dx}=2\left(x+x^{-1}\right)\times\left(1-x^{-2}\right)$ $=2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)$	$\frac{dy}{dx}=2\left(x+\frac{1}{x}\right).\frac{1}{x^2}$ $=\left(2x+\frac{2}{x}\right).\frac{1}{x^2}$ $=\frac{2}{x}+\frac{2}{x^3}$	$y=\left(x+\frac{1}{x}\right)^2$ $=\left(\frac{x^2+1}{x}\right)^2$ $\frac{dy}{dx}=\frac{(x)^2(4x^3+4x)-(x^4+2x^2+1)(2x)}{(x^2)^2}$ $=\frac{4x^6+4x^3-2x^5-4x^3-2x}{(x)^4}$ $\frac{4x^6-2x^5-2x}{(x)^4}$
Incomplete post algorithmic simplification	Wrong application of chain rule	Correct quotient rule but error in post algorithmic simplification

Table 6.16

Conceptual preparation of procedures varies from case to case and is not given by a single algorithm, so students may use some form of conceptual preparation of procedures in some calculus problems, but not in others.

Sometimes it may not even be clear whether some form of conceptual preparation may be advantageous. For instance, the problem

Find $\frac{dy}{dx}$, when $y=\left(x+\frac{1}{x}\right)^n$

is best solved by using the chain rule with $u=x+\frac{1}{x}$ to obtain the derivative in the form $nu^{n-1}\frac{du}{dx}$. However the problem

Find $\frac{dy}{dx}$, when $y=\left(x+\frac{1}{x}\right)^2$

happens to be more curtailed by expanding the bracket to differentiate $x^2 + 2 + x^{-2}$. In this case there is a choice between using the generalisable chain rule method and the particular method expanding the bracket, which happens to be marginally shorter. This is reflected in the performance of the grade A students where one of three A students expanded the bracket prior to differentiation.

6.7 Students who are good at symbol manipulation yet fail to link this to visual ideas

Question (i) in section II,

Find the area of the shaded region.

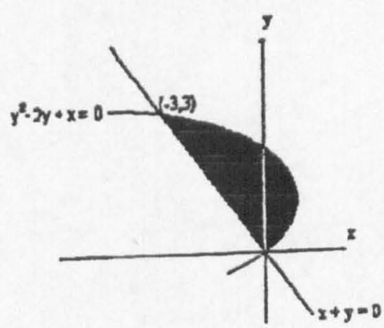


Figure 6.1

Please explain graphically, how this might be done.

is used for this purpose. Out of eleven students, only one of three grade A students managed to solve this problem. An example of the grade A performance is as follows:

$$y^2 - 2y + x = 0$$

$$x = 2y - y^2$$

Luas →
[Area]

$$\int_0^3 -y\,dy + \int_0^3 2y - y^2\,dy$$

$$= -\left[\frac{y^2}{2}\right]_0^3 + \left[y^2 - \frac{y^3}{3}\right]_0^3$$

$$= -\left(\frac{9}{2}\right) + \left[9 - \frac{27}{3}\right]$$

$$= -\frac{9}{2} + 9 - 9$$

$$= -\frac{9}{2}$$

$$= \frac{9}{2}$$

Although this student succeeded in computing the area, he failed to explain graphically how this might be done. This phenomenon occurred because the process of integration was carried out with respect to the variable y.

> In the class most of the times we were given questions, with the integrals that can be evaluated with respect to the variable x.. But if we were given the integrals than can be evaluated with respects to variable y, the related questions are not as difficult as this one.
>
> (Second year SPK student, 1994).

However, from the above observation there is a possibility that there are students who are good at symbolic manipulation yet fail in linking up their computational outcomes to visual ideas.

6.8 Students who can compute the area by using an integral notation but fail to use the area to compute the value of the integral

Some students may have several representations of mathematical concepts. But when it comes to solving certain calculus problems, they fail to switch from one representation to another. When this inflexibility occurs these students have great difficulties in finding the solution for a certain calculus problem. For example in problem (ii) from Section II, i.e.

By considering the function $y = \sqrt{4-x^2}$, sketch and shade the region of the graph given by the following integral

$$\int_0^2 \sqrt{4-x^2}\,dx$$

What is the value of the above integral?

In this problem A2 and A3 is able to draw the graph $\int_0^2 \sqrt{4-x^2}\,dx$ which represents a region within the circle in the first quadrant.

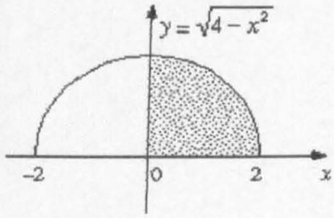


Figure 6.2

They succeed in using the integral notation to calculate the area under the graph. However they are probably unaware that the area under the graph can be used to find the corresponding value of the integral.

Two students, A1 and B2 try to get the solution by means of trigonometric substitution but can only proceed halfway due to the failure of retrieving the trigonometric identity involved in the procedure.

The rest of the students cannot proceed at all or soon break down.

In this problem each student is required to draw the graph $\int_0^2 \sqrt{4-x^2}\,dx$ before proceeding with the computation. If they fail, they are guided until the correct graph is drawn.

The performance of students in this problem can be summarised as below:

Solutions of student in problem (ii) in section II		
Trigonometric substitution	Incomplete procedure	Error
$\int_0^2 \sqrt{4-x^2}dx$ $x = 2\sin\theta$ $dx = 2\cos d\theta$ $When\, x = 0; \theta = 0$ $x = 2; \theta = \pi/2$ $\int_0^{\pi/2} \sqrt{4-4\sin^2\theta}\, 2\cos\theta d\theta$ $= \int_0^{\pi/2} 2\sqrt{1-\sin^2\theta}\, 2\cos\theta d\theta$ $= \int_0^{\pi/2} 4\cos^2\theta\, d\theta$ $2\cos^2\theta - 1 = \cos 2\theta$ $4\cos^2\theta = 2 + 2\cos 2\theta$ $\int_0^{\pi/2} 2 + 2\cos 2\theta\, d\theta$ $= 2\int_0^{\pi/2}(1+\cos 2\theta)d\theta$ $= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$ $= 2\left[(\pi/2 + 0) - (0+0)\right]$ $= \pi$	$\int_0^2 \sqrt{4-x^2}dx$ $x = 2\sin\theta$ $dx = 2\cos d\theta$ $\int \sqrt{4-4\sin^2\theta}\, 2\cos\theta d\theta$ $= 4\int\sqrt{1-\sin^2\theta}\, \cos\theta d\theta$ $= 4\int\cos^2\theta d\theta =$	$\int_0^2 (4-x^2)^{\frac{1}{2}}dx$ $= \left[\frac{2(4-x^2)^{\frac{3}{2}}}{3}\right]_0^2$ $= \left[\frac{2(4-0)^{\frac{3}{2}}}{3}\right] - \left[\frac{2(4-0)^{\frac{3}{2}}}{3}\right]$ $= 0 - \frac{2(4)^{\frac{3}{2}}}{3}$ $= \frac{2\sqrt{4^3}}{3}$ $= \frac{2(8)}{3}$ $= -\frac{16}{3}$
*A2 & A3	A1 &*B2	

Table 6.17

From the above observation A2 and A3 students show flexibility in using the integral notation in finding the area of a region under the graph, but fail to use the area under the curve to find the value of the integral. This is because of the lack of a process in the reverse direction when these students have already become fixed in solving certain problems in a particular way.

6.9 Summary of the Pilot Test at the Universiti Teknologi Malaysia

From the above analysis, the students with higher grades can curtail and detail their solutions. For example, in problem (i) from Section I, in reaching the solution for this problem, A1 used three steps whilst A3 used six steps. A3 performed a detailed solution may be because she wanted to make sure that every step carried out was right. In order to perform long detailed solutions, A3 must have strong conceptual and cognitive links to enable them to join all the necessary bits of mathematical information in a meaningful manner. A3 may have the ability to reflect on the previous steps while performing the new one. Hence by doing so, A3 is less likely to experience break down in her procedures. The students with lower grades have more fragile conceptual knowledge. They are more likely find great difficulty in making cognitive connections and thus they are more likely to make errors in their procedures and thus exhibit breakdown of procedures.

In this study A3 has three strategies whilst A1 and A2 have two ways in tackling the same mathematical problem. From here it is observable that A3 has three representations (all symbolic manipulation) of the particular mathematical concepts. Out of variation of methods or representations, there is an indication that the students with higher grades prefer to use the conceptual preparation of procedures in solving certain calculus problems so as to make their computation easier. From the above analysis the students with lower grades are less likely to switch freely and easily from one representation to another. Hence if they work using a single representation they are reluctant to break away from it.

This evidence suggests that there is a correlation between students' success and flexibility of process (several procedures for the same desired outcome).

6.10 Limitations of the Pilot Test and Suggestions for the Main Study at the Universiti Teknologi Malaysia

6.10.1 Question in the pilot test at the Universiti Teknologi Malaysia

The problem on graphical ideas in the pilot test at the University Teknologi Malaysia is too difficult for many students. Only one with grade A was able to get to the computational solution. The rest could not proceed. So, for the purpose of the main study, another question was carefully selected so that the value of the definite integral could be evaluated with respect to the variable x. Such a question is considered appropriate because much of the time, the students face integration problems with respect to that variable.

6.10.2 The findings in the pilot tests at the University of Warwick and Universiti Teknologi Malaysia.

Based on results obtained in the pilot test at the University of Warwick and the pilot test at the Universiti Teknologi Malaysia, it was hypothesised that:

(a) There is a correlation between students' ability and their curtailment of solution (which is expected to be false).

(b) There is a correlation between students' ability and conceptual preparation of solutions.

(c) There is a correlation between students' ability and flexibility of process (several procedures for the same desired outcome).

(d) Students may be good at symbolic manipulation yet fail to link to visual ideas.

(e) Students may be flexible at performing familiar processes but fail to reverse them.

6.10.3 The place to carry out the interview

In order to obtain uniform data, all the interview were carried out in the researcher's office so that all the students will experience the same learning environment in answering the questions posed to them.

CHAPTER 7

METHODOLOGY OF THE MAIN STUDY AT THE UNIVERSITI TEKNOLOGI MALAYSIA

7.1 Introduction

In order to get a wide spectrum of students in dealing with calculus symbolic manipulation, a study which involved a group of 36 selected students was carried out. This study commenced on 9th of August and ended on 25th of September 1994.

7.2 Sampling

7.2.1 The subjects

The students chosen were those second years undergraduates who had undergone the same calculus and mathematics courses in the first year. They were from group SPK course (prospective computer-science teachers), and group SKA courses (prospective civil engineers). The number in each group initially were 83 and 95 respectively. The other engineering students following the same calculus course or students taking different mathematics courses in the first year were left out. Based on grades and marks of mathematics obtained in the first year examination, the students' ability involved in this study range from 50% up to 90% of the total population The top 10% were not included because they were sent overseas to further their study. The bottom 50% percent have taken different mathematics course in the first year at the Universiti Teknologi Malaysia.

Each group of students was divided into three categories according to their grades A, B and C. Students having grade E and some with grade D were considered not to reach the minimum requirement in the first year examination. Since some of them were not given chances to continue or to repeat the course, the number of these categories of students in the class was very small and insufficient for this study. For example, from the SPK course, only four students scored grade D and two had grade E in mathematics (with overall grade C compensated by better performances in other subjects). Thus, the investigations of mathematical thinking of the second year students are restricted up to only those students having A, B and C grades.

7.2.2 The procedures

In the main study, the students (different from those in phase one) are grouped according to their grades A, B and C. Their names are arranged in alphabetical order. To ensure the randomness of the selection of the students, the first nine students were chosen from each group. Altogether fifty four students were selected: twenty seven students from SPK course and twenty seven students from SKA course.

In this study it is suspected that students may have different attitudes towards calculus. This notion is based on Yusof's (1995) finding that students have different attitudes towards mathematics. Thus in order to obtain students' homogeneity from this aspect, each of them was given an attitudinal questionnaires towards calculus. The questions (as stated below) are based on Yusof's (1995) thesis on problem solving:

Q1. The calculus topics we study at the university make sense to me.

Q2. I learn calculus through memory.

Q3. I usually understand a new idea in calculus quickly.

Q4. I am able to relate calculus ideas learned.

Q5. Calculus is abstract at University.

In addition students were asked:

Q6. In few sentences describe your feeling about calculus.

Questions one to five are based upon a five point scale: **Y**, **y**, **–**, **n**, **N** (i.e. definitely yes, yes, no opinion, no and definitely no). Questions Q1, Q3 and Q4 are considered positive while questions Q2 and Q5 to be negative. Based on Yusof & Tall (1995) the score recorded on responses of the former is the reverse of that recorded on the latter. In this study responses of positive questions, thus—the response **N** is equivalent to score 0, response **n** is equivalent to score 1, response **–** is equivalent to score 2, response **y** is equivalent to score 3 while response **Y** is equivalent to score 4. Thus, score 0 implies the most negative attitude, score 2 indicates that the students has no opinion regarding the course and 4 implies the most positive attitude towards the subject. In general, a score of below 2 represents a negative attitude and a score above 2 represents a positive one.

The total score of each student was computed. Three students who showed more positive attitude and three students with more negative attitude towards calculus were identified and selected from each group of nine students. Altogether thirty six students were selected with their total attitudinal scores on calculus were as follows:

Student No.	Maths Grade	Attitudinal Scores Q1	Q2	Q3	Q4	Q5	Total Score
13	A	4	3	3	3	3	16
18	A	3	4	3	3	3	16
19	A	4	3	1	3	4	15
21	A	3	3	2	3	3	14
17	A	4	3	3	3	1	14
22	A	4	1	1	3	3	12
23	A	3	3	1	1	3	11
24	A	3	3	3	1	1	11
14	A	3	1	1	1	0	6
15	A	2	0	1	1	1	5
20	A	2	1	1	1	0	5
16	A	0	0	1	2	0	3
7	B	4	3	4	3	3	17
8	B	4	3	3	3	2	15
9	B	4	3	3	1	3	14
1	B	3	3	1	3	3	13
10	B	4	1	3	3	1	12
6	B	4	3	1	3	1	12
5	B	3	3	1	3	1	11
11	B	4	3	1	1	1	10
12	B	3	1	3	2	1	10
2	B	3	0	1	1	2	7
3	B	3	1	0	1	0	5
4	B	1	1	0	1	1	4
28	C	3	4	3	3	3	16
29	C	3	4	3	3	3	16
30	C	3	3	2	3	3	14
31	C	1	3	3	3	4	14
33	C	3	3	3	1	3	13
32	C	4	0	4	3	1	12
36	C	4	1	1	3	0	9
35	C	4	1	1	1	1	8
34	C	2	2	1	1	0	6
27	C	3	0	1	2	0	6
25	C	1	0	1	1	1	4
26	C	1	0	1	1	1	4

Table 7.1

7.3 Aim of the study

Students in the main study at the Universiti Teknologi Malaysia were grouped according to their mathematics grade—A, B and C, obtained in the first year mathematics examination. The aims of this study are (i) to identify some possible qualitative differences and (ii) to test for their significant difference.

(i) Qualitative differences which are likely to occur have been stated at the end of the last Chapter.

(ii) Test of significance

The intention of this study is to look for significant difference of the above qualitative differences at 1% and 5% level by making use of a χ^2 test with Yates correction. The number of students in each group involved in the study is very small as low as 12 students, a slight change in the figure might affect the statistics greatly. Hence the statistic used is not stable. Thus, in this study qualitative differences at 10% level are also included. For instance, suppose we have two groups of art students — X (experienced artists) and Y (non-experienced artist) and there are twelve students in each group. Let 3 be the number of students in group X prefer light colour —LC while 9 students in this group like heavy colour— HC. Let the number of students in group Y be the reverse.

Group	LC	HC
X	3	9
Y	9	3
Total	12	12

Table 7.2

By making use of a χ^2 test with Yates correction, there is a relationship between artists' experience and preference of colour and it is significant at the 5% level. Since the number in each group is as small as 12 students, a slight change in the figure might have a considerable influence on the statistics used. Based on the above table, if the number of students who prefer heavy colour in group X is changed to eight, and those who prefer light colour to be four, it is found that relationship between artists' experience and preference of colour is no longer significant at 5% level . However, the statistic is significant at 10% level.

7.4 Calculus questionnaires in the main study

7.4.1 The calculus questions

In this study, the questionnaires are administered in interviews similar to that performed in the pilot studies at the University of Warwick and the Universiti Teknologi Malaysia. The reasons for using a combined approach is already mentioned in the pilot study at the University of Warwick. The calculus questions used in this study are as follows:

Question 1.

Evaluate

(a) $\int \sqrt{3x^3}\,dx.$

(b) $\int (x+1)^2 dx$

Question 2.

Find $\frac{dy}{dx}$, when

(a) $y = \frac{1}{\sqrt{x}}$

(b) $y = \frac{1+x^2}{x^2}$

(c) $y = \left(x + \frac{1}{x}\right)^2$

(d) $y = \frac{1}{1+x^2} - \frac{x^4}{1+x^2}$

Question 3.

Based on the diagram below, compute the area of the shaded region. Please explain graphically how this might be done.

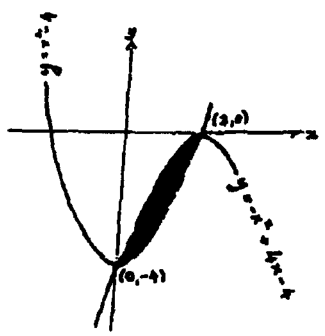


Figure 7.1

Question 4

By considering the function $y = \sqrt{4 - x^2}$, sketch and shade the region of the graph given by the following integral

$$\int_0^2 \sqrt{4 - x^2}\, dx$$

What is the value of the above integral?

7.4.2 Source and rationale of calculus questionnaires

The questions in the pilot test at the Universiti Teknologi Malaysia were modified for the second phase (main part) of the study. A few questions are added to the pilot test at the Universiti Teknologi Malaysia. Those questions added are

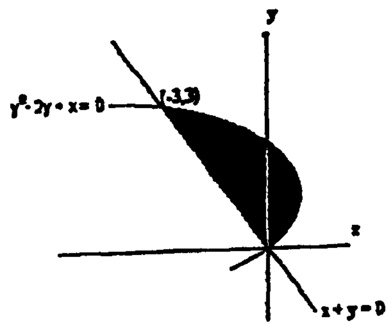
1(b) $\int (x+1)^2 dx$

2 (a) $y = \dfrac{1}{\sqrt{x}}$

2(d) $y = \dfrac{1}{1+x^2} - \dfrac{x^4}{1+x^2}$

The following question (i) in Section II on graph in the pilot test proved too difficult for most students.

Find the area of the shaded region.



Please explain graphically, how this might be done.

Figure 7.2

This graph has been replaced by the graph in question 3 above.

Questions 1(a), 2(b), 2(c) and 4 are taken from questions of the pilot study at the Universiti Teknologi Malaysia. The rationales for choosing these problems and their sources have been stated in the pilot study at the Universiti Teknologi Malaysia.

Question 1(b) is taken from the pilot test at the University of Warwick. Its source and rationale has been stated in the pilot study at the University of Warwick.

Question 2(a) is designed to study the extent to which students compress algebraic procedures in terms of number of steps used to carry them out. Its source and rationale has been stated in the pilot test at the Universiti Teknologi Malaysia.

Question 2(d) is used to identify the fragility of conceptual preparation of procedure. This problem is taken from *Calculus Tutorial Questions for the First Year Students at the Malaysia* (Common First Year Mathematics Group of the Malaysia, 1993/1994).

Question 3 is constructed to identify those students who are good at symbolic manipulation yet fail to visualise a representation. This question is taken from the text used by first year students (Common First Year Mathematics Group of the Malaysia, 1993/1994, p. 139).

7.5 Interviews

In the main study students' thinking in calculus is revealed by questionnaires administered in the interviews similar to that performed in the pilot study at the University of Warwick and the pilot study at the Universiti Teknologi Malaysia. The reasons for choosing this combined method have been described in Chapter 3. Open-ended questionnaires are used in the interviews consisting of two parts. The first part of the interview mainly focused on how the students learned calculus. The second part of the interviews in this study is based on the questionnaires and deals with symbolic manipulation relating this to other aspects such as drawing and interpretation of graphs. There are several reasons for choosing such questions:

> ...they are flexible; they allow the interviewer to probe so that he may go into more depth if he chooses, or clear up any misunderstandings: they enable the interviewer to test the limits of the respondent's knowledge; they encourage co-operation and help to establish rapport; and they allow the interviewer a truer assessment of what the respondent really believes.
>
> (Cohen & Manion, 1989, p. 313).

7.5.1 Interview questions

The first part of the interview:

The questions posed are informal with the intention of getting general information on how the students study calculus. For example:

General idea on how students find calculus topics .

Rationale:

To get some insight whether calculus subjects interest the students and to get some information on whether the students find the calculus subject difficult to understand, challenging, confusing or otherwise when compared to other subjects.

The method used by students in solving calculus problem.

Rationale:

To observe whether the students merely rely upon the rule to get to the solution or otherwise. Also, to get some insight into what extent basic rules and theorems in integration and differentiation influence the students performance.

Students' ability in memorizing without understanding the various rules and theorems related to integration and differentiation.

Rationale:

To get some insight into whether students really understand the various rules and theorems related to integration and differentiation and are able to relate them to problem in different context.

Relationship of the calculus which has been taught to the activities in every day life?

Rationale:

To see whether the students visualize the needs of study and understanding calculus.

The second part of the interview:

The questions in this part of the interview are based on the calculus questionnaires. They are used to test several aspects:

(a) The number of alternatives students used in tackling the same calculus problem.

Rationale:

(i) *To identify the extent of students' flexibility in using different approaches in tackling a calculus problem. By displaying the various methods applied in solving the same problem, the students can be categorised as having the idea of conceptual preparation of procedures or otherwise. The phenomenon can be seen from their order of preference of procedures.*

(ii) *To identify the relationship between conceptual preparation of procedures and the nature of the problem posed.*

(iii) *To identify students who are good at symbolic manipulation but do not necessarily have a reasonable interpretation for the same object in different forms.*

(b) Geometrical interpretation of the computational result.

Rationale:

To see whether the students who are good at symbolism have the ability to visual representations more easily.

(c) Reversal of processes.

Rationale:

To see whether those students with flexibility of using the integral notion to find the area under also have the ability to use the area the curve to find the value of the corresponding integral.

7.5.2 The procedures

Before the interview the researcher introduced his name and position and in fact, the method applied at the starting of the interview is based on that is used by Tuckman (1978).

> At the meeting, the interviewer,...should brief the respondent as to the nature or purpose of the interview...and attempt to make the respondent feel at ease. He or she should explain the manner in which responses will be recorded, and if a tape recording is to be made, the respondent's assent should be obtained. At all times the interviewers must remember that they are data collection instruments and must try not to let their own biases, opinions, or curiosity affect their behavior. It is important that the interviewers not deviate from their format and interview schedules although many schedules will permit some flexibility in choice of questions. The respondent should be kept from rambling away from the essence of a question, but not at the sacrifice of courtesy. (Tuckman, 1978).

The second part of the interviews is conducted immediately after each problem being solved. Each interview together with answering the questionnaire lasts for about one hour. Each interview is tape-recorded. Out of thirty six students, twelve of them — four from each group A, B and C are fully transcribed. Out of these four students from each group, two of them obtain the highest attitudinal score whilst the other two obtained the lowest attitudinal score. These students' attitudinal scores can be summarised as in Table 7.3 below:

Student No.	Maths Grade	Attitudinal Scores Q1	Q2	Q3	Q4	Q5	Total Score
13	A	4	3	3	3	3	16
18	A	3	4	3	3	3	16
20	A	2	1	1	1	0	5
16	A	0	0	1	2	0	3
7	B	4	3	4	3	3	17
8	B	4	3	3	3	2	15
3	B	3	1	0	1	0	5
4	B	1	1	0	1	1	4
28	C	3	4	3	3	3	16
29	C	3	4	3	3	3	16
25	C	1	0	1	1	1	4
26	C	1	0	1	1	1	4

Table 7.3

The students were interviewed individually so as to keep the information confidential.

CHAPTER 8

DATA ANALYSIS OF THE MAIN STUDY

8.1 Introduction

In this chapter we consider the way in which students carried out standard algorithms of differentiation and integration, with possible links to visual representations of calculus problems.

Students were given standard problems to see if they *curtailed* their solutions (in the sense of Krutetskii), reducing the number of steps to produce a shortened solution, or if they gave *detailed* solutions involving a larger number of steps. Krutetskii (1976) suggested that curtailment of solutions was an indication of capability — that gifted and capable students were likely to have powerful links which enabled them to make short connections between ideas, average students were likely to curtail solutions only after practice, and incapable students were characterised by longer solutions which often contained redundant or repeated steps, with a high chance of error and likelihood of breakdown. The students concerned here are in the 50th to 90th percentile, which suggests the most gifted have been taken away and these are more likely to be capable or average students. On previous examinations these students have been graded at level A, B, C. It would be expected to find that the grade A students are more capable than B, who are in turn more capable than the C students. Two questions to be investigated are therefore whether the A students produce more curtailed solutions than the B and the C in turn.

8.2 A study of the extent to which students compress algebraic procedures in terms of the number of steps used to carry them out

To serve the above purpose Question 1(a)

Find $\int \sqrt{3x^3}\,dx$.

is used. All thirty six students attempted this problem. In this problem all A grade students gave correct responses. The number of steps performed by these students in reaching the solution varied. Out of twelve grade A students one student performed two steps, two students performed three steps, five students used four steps, two students used five steps and two students performed six steps. The two students with six steps gave solutions for this problem in non conventional form. Hence from these observations the most curtailed solution is the one with two steps while the most detailed solution is the one with six steps. A typical detailed solution (given by an A grade student) had already been described in Chapter 4.

The solutions of the 12 A grade students may be summarised as follows:

Typical solutions of grade A students				
1 student	2 students	5 students	2 students	2 students
$\int\sqrt{3x^3}dx$ $=\sqrt{3}\int x^{3/2}dx$ $=\frac{2\sqrt{3}}{5}\left(x^{5/2}\right)+c$	$\int\sqrt{3}x^{3/2}dx$ $=\sqrt{3}\frac{x^{5/2}}{5}(2)+c$ $=\frac{2\sqrt{3}}{5}x^{5/2}+c$	$\int(3x^3)^{\frac{1}{2}}dx$ $=\sqrt{3}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}\left[x^{\frac{5}{2}}.\frac{2}{5}\right]+c$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$\int\sqrt{3x^3}dx$ $=\int\sqrt{3}(x^3)dx$ $=\sqrt{3}\int x^{3/2}dx$ $=\sqrt{3}\left[\frac{x^{3/2+1}}{3/2+1}\right]$ $=\sqrt{3}\left(\frac{2}{5}\right)x^{5/2}$ $=\frac{2\sqrt{3}}{5}x^{5/2}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int 3^{\frac{1}{2}}(x^3)^{\frac{1}{2}}dx$ $=3^{\frac{1}{2}}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+\mathrm{c}$ $=\sqrt{3}\times\frac{2}{5}x^{\frac{5}{2}}+\mathrm{c}$ $=\sqrt{3}\frac{2}{5}x^{\frac{5}{2}}+\mathrm{c}$
2 steps	3 steps (including unwritten first line)	4 steps (one solution non-conventional)	5 steps	6 steps (Both in non -conventional form)

Table 8.1

These students do not usually curtail solutions in the manner suggested by Krutetskii for gifted students. Since the top ten percent of Malaysian students have been transferred to study abroad, those that remain show a wider array of response including the characteristics more likely of capable students who learn to carry out routine procedures after practice and then curtail in routine ways. The interesting cases here are the students who take a larger number of steps, who may be doing so in a methodical way to make sure that each small step is performed correctly.

Because of their greater grasp of the procedures these A students are indeed successful in these more detailed solutions.

The B grade students show a considerable number who fail to obtain the correct answer:

Solutions of grade B students		
3 students	2 students	7 students
$\int(3x^3)^{\frac{1}{2}}dx$ $=\sqrt{3}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}\left[x^{\frac{5}{2}}.\frac{2}{5}\right]+c$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int 3^{\frac{1}{2}}x^{\frac{3}{2}}dx$ $=3^{\frac{1}{2}}\int x^{\frac{3}{2}}dx$ $=3^{\frac{1}{2}}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)+c$ $=3^{\frac{1}{2}}.\frac{2}{5}.x^{\frac{5}{2}}+c$ $=\frac{2}{5}\sqrt{3}\left(x^{\frac{5}{2}}\right)+c$	Wrong application of procedures [see analysis of errors later]
4 steps	6 steps	

Table 8.2

Here we see those who are successful use either four or six steps with the solution. More than half (seven out of twelve) make errors.

The case of the C grade students show a similar number of failures, there are more curtailed solutions amongst those successful.

Solutions of grade C students			
1 student	1 student	2 students	8 students
$\int\sqrt{3x^3}dx$ $=\int\sqrt{3}x^{3/2}dx$ $=\sqrt{3}\frac{2}{5}x^{5/2}+c$	$\int\sqrt{3x^3}dx$ $=\int\sqrt{3}x^{3/2}dx$ $=\frac{\sqrt{3}x^{5/2}}{5/2}+c$ $=\frac{2}{5}\sqrt{3}x^{5/2}+c$	$\int(3x^3)^{\frac{1}{2}}dx$ $=\sqrt{3}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}\left[x^{\frac{5}{2}}.\frac{2}{5}\right]+c$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	Wrong application of procedures [see analysis of errors later]
2 steps Non-conventional solution	3 steps	4 steps	

Table 8.3

Seven B students and eight C students give incorrect solutions, so that the predominant phenomenon amongst these students is the inability to complete the algorithm correctly. Of the correct solutions, the B are more detailed (three with four steps and two with six steps) whilst the C students are more curtailed (one with two steps, one with three steps and two with four steps). However, these represent the minority. The essential difference is that two of the B students gave detailed solutions in six steps whilst two of the C students gave curtailed solutions with two or three steps. In both cases, the remaining students either use four steps, or fail.

From these solutions of students in grades A, B, C we note that the higher attainers in grade A are all successful but vary considerable in the number of steps taken. Grade B students are less successful (5 out of 12 students) and the correct solutions vary from 4 to 6 steps. The grade C students are even less successful (4 out of 12 students) and the four successful students have solutions varying in length from 2 to 4 steps. It cannot be asserted that there is any clear pattern between curtailment and attainment. However, there is a clear diminution in lower attaining students successfully completing the problem. The difference between the performance of Grade A (12 out of 12 students) and Grade B (5 out of 12 students) is statistically significant using the χ^2-test with Yates correction ($p<0.01$), and between Grade A and Grade C (4 out of 12 students) even more so ($p<0.0025$). The zero entry in the Grade A failures greatly biases these results, nevertheless the cognitive differences are clearly striking.

8.3 Breakdown of procedures

In problem 1(a), it is apparent that students with grade A are efficient in handling correct symbolic manipulation. None of them make errors in performing curtailed or detailed solutions. Hence, they do not experience breakdown of procedures. But some B and C grade students face breakdown in their solutions. This phenomenon can be seen from errors performed by these students. In this problem, there are two types of errors identifiable— an algebraic error (handling of indices) and inability to carry out the actual process of integration.

Example of errors performed by seven grade B students are summarised below:

Error of B students		
2 students	3 students	2 students
$\int\sqrt{3x^3}\,dx$ $=\int(3x^3)^{1/2}dx$ $=\frac{(3x^3)^{3/2}}{3/2}$ $=\frac{2(3x^3)^{3/2}}{3}$	$\int\sqrt{3x^3}\,dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $Let\quad u=3x^3$ $\frac{du}{dx}=9x^2$ $dx=\frac{du}{9x^2}$ $\therefore\int(u)^{\frac{1}{2}}dx=\int u^{1/2}\frac{du}{9x^2}$ $=\frac{1}{9x^2}\int u^{1/2}du$ $=\frac{1}{9x^2}\left[\frac{2}{3}u^{3/2}\right]+c$ $=\frac{2u^{3/2}}{27x^2}+c=\frac{2(3x^3)}{27x^2}$ $=\frac{6x^3}{27x^2}=\frac{2}{9}x$	$\int\sqrt{3x^3}\,dx$ $=\int 3x^{3/2}dx$ $=3\int x^{3/2}dx$ $=3x^{5/2}.2/5$ $=6/5\,x^{5/2}+c$
Overgeneralisation of the basic rule of integration	Mixture of substitution and the basic rule of integration	Algebraic Error (Indices manipulation)

Table 8.4

Two B grade students have overgeneralised the basic rule of integration. They considered the expression $(3x^3)$ in the integral $\int(3x^3)^{1/2}dx$ as a single term and conceived the whole integral as $\int x^n\,dx$, whereby n is a positive fraction. So, in order to get solution for the problem, these students used the formula $\frac{x^{n+1}}{n+1}$ for $n=\frac{1}{2}$ to give $\frac{2(3x^3)^{3/2}}{3}$. Three students tried to solve the problem using the mixture of substitution method and the basic rule of integration. Two students have algebraic misconceptions. These students neglected the square root for the number three. Based on the above example, it is possible that the students had an intuition that the fractional power can be only be applied to the variable and not to the constant.

The breakdown of procedures experienced by these students occurs at an early stage. In other words, these students either make errors right from the beginning or fail after successfully performing the first step.

Similarly, in the case of eight C grade students whereby three overgeneralise the basic rule of integration, two apply the mixture of substitution method $u = 3x^3$ and the basic rule of integration whilst three students have algebraic misconceptions. Examples of errors performed by the C grade students are summarised below:

Errors of C grade students		
3 students	2 students	3 students
$\int\sqrt{3x^3}dx$ $= \int(3x)^{1/2}dx$ $= 2/3(3x^3)^{3/2} + c$	$\int\sqrt{3x^3}dx$ $= (3x^3)^{\frac{1}{2}}$ $u = 3x^3$ $= \int(u)^{\frac{1}{2}}$ $= \frac{u^{3/2}}{3/2} + c$ $= \frac{2}{3}u^{3/2} + c$ $= \frac{2}{3}(3x^3)^{3/2} + c$ $= 2x^{9/2} + c$	$\int\sqrt{3x^3}dx$ $= \int(3x^3)^{1/2}$ $= 9\int(x^3)^{1/2}$ $= 9\left[\frac{(x)^{3/2}}{\frac{3}{2}}\right] + c$ $= 9\left[\frac{(x)^{3/2}}{\frac{9}{2}}\right] + c$ $= 2\left[(x^3)^{3/2}\right] + c$
Overgeneralisation of the basic rule of integration	Mixture of substitution method $u = 3x^3$ and the basic rule of integration	Algebraic error (indices manipulation)

Table 8.5

8.4 Comparison of solutions to problems in differentiation and integration

By comparing students' performance in differentiation and integration, there is an indication that grade A students with detailed solutions in one problem can carry out curtailed solutions in other problem. Problem 1(a) as above and problem 2(a), i.e.

Find $\frac{dy}{dx}$, whenever

$$y = \frac{1}{\sqrt{x}}$$

are used for such a comparison. The table below shows that student A1 performed problem 1(a) by making use of five steps while in solving problem 2(a) he used two steps. Both A1's solutions are in conventional form.

A2 used six steps in dealing with problem 1(a) but used two steps to work out the solution for problem 2(a). A2's first solution is in non-conventional form but his second solution is in conventional form.

The performances of A1 and A2 in these two problems can tabulated thus:

Performance of A1 student		Performance of A2 student	
Problem 1(a)	Problem 2(a)	Problem 1(a)	Problem 2(a)
$\int\sqrt{3x^3}dx$ $=\int\sqrt{3}(x^3)dx$ $=\sqrt{3}\int x^{3/2}dx$ $=\sqrt{3}\left[\frac{x^{3/2+1}}{3/2+1}\right]$ $=\sqrt{3}\left(\frac{2}{5}\right)x^{5/2}$ $=\frac{2\sqrt{3}}{5}x^{5/2}+c$	$y=\frac{1}{\sqrt{x}}$ $y=(x)^{-\frac{1}{2}}$ $\frac{dy}{dx}=-\frac{1}{2}x^{-3/2}$	$\int\sqrt{3x^3}dx$ $=\int 3^{1/2}x^{3/2}dx$ $=\int 3^{1/2}x^{3/2}dx$ $=3^{1/2}\int x^{3/2}dx$ $=3^{1/2}.\frac{x^{5/2}}{5/2}+c$ $=3^{1/2}.\frac{2}{5}x^{5/2}+c$ $=\frac{\sqrt{3}.2}{5}x^{5/2}+c$	$y=x^{-1/2}$ $\frac{dy}{dx}=-\frac{1}{2x^{3/2}}$
5 steps Conventional solution	2 steps Conventional solution	6 steps Non-conventional solution	2 steps Conventional solution

Table 8.6

In the case of two grade B students who used six steps in getting the solution for problem 1(a) used three or four steps in getting solutions for problem 2(a). Hence, these students demonstrated detailed solutions (when compared to those of A1 & A2) in problem 2(a). From these observations, it is apparent that these grade B students prefer detailed solutions rather than curtailed solutions. Probably, they are unable to curtail solutions. However, all solutions obtained are in conventional form. Their performance in these two problems can be summarised as below:

Performance of B1 student		Performance of B2 student	
Problem 1(a)	Problem 2(a)	Problem 1(a)	Problem 2(a)
$\sqrt{3x^3}\,dx$ $=\int(3x^3)^{1/2}dx$ $=3^{1/2}\int x^{3/2}dx$ $=\sqrt{3}\left(\frac{x^{3/2+1}}{3/2+1}\right)+c$ $=\sqrt{3}\left(\frac{x^{5/2}}{5/2}\right)+c$ $=\sqrt{3}\left(\frac{2}{5}x^{5/2}\right)+c$ $=\frac{2\sqrt{3}}{5}x^{5/2}+c$	$y=\frac{1}{\sqrt{x}}$ $=x^{-1/2}$ $\frac{dy}{dx}=-\frac{1}{2}x^{-1/2-1}$ $=-\frac{1}{2}x^{-3/2}$ $=-\frac{1}{2x^{3/2}}$	$\sqrt{3x^3}\,dx$ $=\int(3x^3)^{\frac{1}{2}}dx$ $=\int 3^{\frac{1}{2}}x^{\frac{3}{2}}dx$ $=3^{\frac{1}{2}}\int x^{\frac{3}{2}}dx$ $=3^{\frac{1}{2}}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)+c$ $=3^{\frac{1}{2}}.\frac{2}{5}.x^{5/2}+c$ $=\frac{2}{5}\sqrt{3}\left(x^{5/2}\right)+c$	$y=\frac{1}{\sqrt{x}}$ $=x^{-1/2}$ $\frac{dy}{dx}=-\frac{1}{2}x^{-3/2}$ $=-\frac{1}{2\sqrt{x^3}}$
6 steps Conventional solution	4 steps Conventional solution	6 steps Conventional solution	3 steps Conventional solution

Table 8.7

In the case of C grade students, those successful displaying curtailed solution for problem 1(a) also manage to curtail their solution in problem 2(a). For example C1 manages to solve problem 1(a) and 2(a) by using two steps. His first solution obtained is in non-conventional form whilst his second solution is in conventional form. C2 successfully solved problem 1(a) by using three steps and problem 2(a) by using two steps. Both answers are in conventional form. Their performance can be tabulated thus:

Performance of C1		Performance of C2	
Problem 1(a)	Problem 2(a)	Problem 1(a)	Problem 2(a)
$\int\sqrt{3x^3}\,dx$ $=\int\sqrt{3}x^{3/2}dx$ $=\sqrt{3}\frac{2}{5}x^{5/2}+c$	$y=\frac{1}{\sqrt{x}}$ $y=x^{-1/2}$ $\frac{dy}{dx}=-\frac{1}{2}x^{-3/2}$	$\int\sqrt{3x^3}\,dx$ $=\int\sqrt{3}x^{3/2}dx$ $=\frac{\sqrt{3}x^{5/2}}{5/2}+c$ $=\frac{2}{5}\sqrt{3}x^{5/2}+c$	$y=\frac{1}{\sqrt{x}}=x^{-1/2}$ $\frac{dy}{dx}=-\frac{1}{2\sqrt{x^3}}$
2 steps Non-conventional solution	2 steps Conventional solution	3 steps Conventional solution	2 steps Conventional solution

Table 8.8

From these observations, we see that some grade A students may detail many steps. Such detailed procedures may not be a sign of inability to curtail solutions, perhaps representing the desire of the students to write out the full steps carefully to be sure of making a correct deduction. In order to build up such a considerable number of meaningful steps, the students have to use strong conceptual knowledge and cognitive linkages so that links between bits of relevant procedures can be established more easily. With such linkages, the A students are more likely to reflect the reasonableness of previous steps in the solutions while solving mathematical problems. By doing so, the A students are less likely to experience breakdown of procedures in their solutions.

Most grade B and C may not have such qualities. Thus, they are more likely to make errors either by misapplication of procedures or by using irrelevant mathematical information in their solutions. In other words, these students are far more prone to break down. However, in the above problem, some grade B students succeed by detailed procedures but some grade C students succeed by curtailed procedures.

8.5 The use of conceptual preparation for algorithms

In order to see the extent in which the students use conceptual preparation of procedure, question 2(b),

$$\text{Find } \frac{dy}{dx}, \text{ whenever } y = \frac{1+x^2}{x^2},$$

is used. The students' performance in the pilot study in this problem has already been dicussed in Chapter 5.

Out of thirty six students attempting this problem, twenty of them simplified the expression $\frac{1+x^2}{x^2}$ before carrying out the differentiation. For example by writing:

$$y = x^{-2} + 1$$
$$\frac{dy}{dx} = -2x^{-3} = \frac{-2}{x^3}$$

Fifteen students failed to conceptually prepare and so led to a more complex version of the algorithm and the need to perform more simplification afterwards. All but one student was successful in this task, the remaining student producing the following erroneous solution:

$$\frac{dy}{dx} = \frac{2x(x^2) + 2x(1+x^2)}{(x^2)^2}$$
$$= \frac{2x^3 + 2x + 2x^3}{x^4}$$
$$= \frac{4x^3 + 2x}{x^4}$$
$$= \frac{4}{x} + \frac{2}{x^3}$$

The students in the various grades performed as follows:

Students with grade	Conceptually prepared	Post-algorithmic simplification	No further simplification
A	10	2	-
B	6	6	-
C	4	7	1
Total	20	15	1

Table 8.9

Here the number carrying out conceptual preparation reduces from 10 out of 12 in grade A to only 4 out of 12 in grade C. Using a χ^2 test with Yates correction, this is significant at the 5% level (with $p = 0.038$). The numbers involved are small and the differences between groups A and B and between B and C are *not statistically significant.*

8.6 The fragility of conceptual preparation

The conceptual preparation for a solution depends very much on the nature of the problem. There is no obvious algorithm to cover all possible cases. For instance the derivative of $y = \frac{1+x^2}{x^2}$ is simplified by separating the expression into two parts, but the derivative of

$$y = \frac{1}{1+x^2} - \frac{x^4}{1+x^2}$$

is found more easily by adding the two expressions together and the numerator.

$$\begin{aligned} y &= \frac{1}{1+x^2} - \frac{x^4}{1+x^2} \\ &= \frac{1-x^4}{1+x^2} \\ &= \frac{(1-x^2)(1+x^2)}{(1+x^2)} \\ &= 1-x^2, \\ \frac{dy}{dx} &= -2x. \end{aligned}$$

In this example, only six of the twelve Grade A students added the terms together and factorised the numerator. Conceptual preparation therefore varies considerably from case to case and is not given by a single algorithm, so students may use some form of conceptual preparation in some problems, but not in others.

Sometimes it may not even be clear whether some form of conceptual preparation may be advantageous. For instance, the problem

$$\text{Find } \frac{dy}{dx}, \text{ when } y = \left(x + \frac{1}{x}\right)^n$$

is best solved by using the chain rule with $u = x + \frac{1}{x}$ to obtain the derivative in the form $nu^{n-1}\frac{du}{dx}$. However the problem

$$\text{Find } \frac{dy}{dx}, \text{ when } y = \left(x + \frac{1}{x}\right)^2$$

happens to be more curtailed by expanding the bracket to differentiate $x^2 + 2 + x^{-2}$. In this case there is a tension between using the generalisable chain rule method and the particular method expanding the bracket, which happens to be marginally shorter. This is reflected in the performance of the grade A students where six used the chain rule and six expanded the bracket. In the interview, four of the six using the chain rule could see a possible advantage in the alternative method but preferred to use the more general strategy and trust their facility in manipulation. An example of a grade A student's preference for the chain rule in this problem is:

Preference of method	
First choice	Second choice
Chain rule	Expanding the bracket first
$y = \left(x + \frac{1}{x}\right)^2$ $\frac{dy}{dx} = 2(x + \frac{1}{x}).(1 - \frac{1}{x^2})$ $= 2(x - \frac{1}{x} + \frac{1}{x} - \frac{1}{x^3})$ $= 2(x - \frac{1}{x^3})$	$y = x^2 + 2 + \frac{1}{x^2}$ $\frac{dy}{dx} = 2x - \frac{2}{x^3}$

Table 8.10

According to this student:

> Actually the chain rule is more appropriate to be used in solving problems of this form According to my the first year calculus lecturer, it is favourable to have a method that can be applied in

many cases. So, in this problem I've the chain rule as my first choice instead of the second method by expanding the brackets.

(Second year SKA student, 1994).

8.7 Flexibility of using approaches to tackle a same problem

Some students have a certain flexibility in their choice of methods of solving calculus problems. In order to identify the extent of students' flexibility in dealing with calculus problems, question 2(b), i.e. find $\frac{dy}{dx}$ when $y = \frac{1+x^2}{x^2}$, is chosen. This problem can be solved by using at least four possible methods, namely (a) conceptual preparation of procedures, (b) quotient rule, (c) product rule and (d) implicit differentiation.

Out of thirty six students attempting this problem, none solved this problem by using four methods. This may be due to the absence of those mathematically gifted 10% of the population. However, from this group of students it is found that, three A students are able to use three methods in differentiating the expression $\frac{1+x^2}{x^2}$. Two of them used conceptual preparation of procedures after seeing the expression can be simplified prior to differentiation, so as to make the computation easier. This expression is a quotient, thus it can be differentiated by means of the quotient rule. These students also see this expression can be expressed as a product of two factors, i.e.

$$\frac{1+x^2}{x^2} = (1+x^2)\frac{1}{x^2} = (1+x)x^{-2}.$$

As a result of that, these students applied the product rule in order to get the solution. An example of a typical A student who used three methods in getting a solution for problem 2(b) is as follows:

Typical A student with 3 approaches		
$y = x^{-2} + 1$ $\frac{dy}{dx} = -2x^{-3}$ $= \frac{-2}{x^3}$	$\frac{dy}{dx} = \frac{x^2(2x) - (1+x^2)(2x)}{x^4}$ $= \frac{2x^3 - 2x - 2x^3}{x^4}$ $= -\frac{2}{x^3}$	$y = (1+x^2)(x^{-2})$ $\frac{dy}{dx} = x^{-2}(2x) + (1+x^2)\left(-\frac{2}{x^3}\right)$ $= \frac{2}{x} - \frac{2}{x^3} - \frac{2}{x}$ $= -\frac{2}{x^3}$
Conceptual preparation of procedure	Quotient Rule	Product Rule

Table 8.11

The other A student who also managed to use three alternatives in tackling this problem, instead of using the product rule for the third method, used implicit differentiation. This particular student first changed the equation $y = \frac{1+x^2}{x^2}$ to

$$x^2 y = 1 + x^2$$

and performed the differentiation thus:

$$x^2 y = 1 + x^2$$

$$x^2 \frac{dy}{dx} + y \frac{d}{dx}(x^2) = 2x$$

$$x^2 \frac{dy}{dx} = 2x - 2xy$$

$$\frac{dy}{dx} = \frac{2x - 2xy}{x^2}$$

$$= \frac{2 - 2y}{x}$$

Out of thirty six students who used two methods in solving this problem, six with Grade A used conceptual preparation of procedure and quotient rule.

Five with grade B also applied two methods. Out of these five students, two preferred conceptual preparation of procedure over quotient rule, one preferred the reverse, one went for the product rule first rather than conceptual preparation of procedure whilst the other one used the quotient rule and product rule.

Three students with grade C also had two ways of solving problem 2(c). Out of these three students only one showed preference for conceptual preparation of procedure over the quotient or the product rule.

It is also found that there were three students with grade A, seven with grade B and eight with grade C who had only one method to deal with the problem.

Out of the three grade A students here, two used conceptual preparation of procedure while the other applied the quotient rule.

In the case of grade B students, four out of seven students used conceptual preparation of procedure whilst the other three used the quotient rule.

From eight students with grade C five applied the quotient rule, one applied the product rule whilst the other two used conceptual preparation of procedure. Out of twelve C grade students, one of them had an error due to misapplication of the quotient rule. Students' performance (in order of preference) in dealing with problem 2(b) from the above aspect can be tabulated as below:

Student Performance with				
Grade	3 App.	2 App.	1 App.	Wrong App.
A	3 2 CP/QR/PR 1 CP/QR/ID	6 6 CP/ QR	3 2 CP, 1 QR	0
B	0	5 2 CP/QR, 1 QR/CP 1 PR/CP, 1 QR/PR	7 4 CP, 3 QR	0
C	0	3 1 CP/QR, 1 QR/CP 1 PR/CP	8 2 CP, 5 QR 1 PR	1 Misapplication of quotient rule
Total	3	14	18	1

Table 8.12

Key: CP– Conceptual preparation of procedure, QR–Quotient rule, PR–Product rule, ID–Implicit differentiation.

The number of lower attainers successful in solving the same problem by different approaches greatly decreased. In order to find the relationship between students' ability and flexibility in tackling a certain problem, the students are divided into two groups— those students who applied two or more methods in one group whilst those students who used only one approach or misapply the standard rule in another as tabulated below:

Student Performance with		
Grade	3 or 2 App.	1 or wrong App.
A	9	3
B	5	7
C	3	9
Total	17	19

Table 8.13

Key:

App.—Approach/approaches

Here the number of students who were able to perform two or more methods reduced from nine out of twelve in grade A to three out of twelve in grade C. By making use of a χ^2 test with Yates correction, the difference in flexibility between grade A and grade C is significant at the 5% level but the differences between groups A and B and between B and C are not statistically significant.

From the above analysis, it was found that the grade A students demonstrated more flexible strategies in dealing with symbolic manipulation. These students developed meaningful relationship between symbols. They interchanged the symbolism freely in the equation so as to get an expression to be represented in different form. However, they were aware that the new expression still yielded the same answer after carrying out the differentiation process. As an example, one of the A students was able to perform the implicit differentiation. This method is rather complicated especially in differentiating the part of the expression which involves x^2y. Although the final answer still contained the variable y, he stressed in the interview that the final solution is the same as the those obtained from the first two methods. He verified his answer by replacing the variable y with $\frac{1+x^2}{x^2}$ thus:

We know the variable y = $\frac{1+x^2}{x^2}$. By changing the y with $\frac{1+x^2}{x^2}$ in the last solution we will get the solution as above.

$$\frac{2-2y}{x} = \frac{2}{x} - \frac{2y}{x}$$
$$= \frac{2}{x} - \frac{2}{x}\left(\frac{1+x^2}{x^2}\right)$$
$$= \frac{2}{x} - \frac{2}{x^3} - \frac{2}{x}$$
$$= -\frac{2}{x^3}$$

(Second year SKA student, 1994)

In spite of such variation of methods of solving problem 2(b), the more successful students with conceptual richness are more likely to choose approaches that involve less cognitive strain. This flexibility greatly increases their chances of solving mathematical problems. Thus mathematical problems given to them are not difficult tasks to be performed:

> In solving this type problem, the first thing I look is whether the expression in the equation can be simplified or not. By such simplification we can make our computation task easier.
>
> (Second year SKA student, 1994).

In contrast, the less successful students may fail to develop such conceptual richness and appropriate linkages. Thus they are less likely to possess a number of strategies or tend to develop an error in tackling problem posed to them. This phenomenon can be seen in the performance of one C grade student in problem 2(b). A deficit in his conceptual networks probably caused this student to fail in checking whether the quotient rule is completely used or otherwise. In this case, he failed to multiply $-2x$ in the numerator by x^2, as shown below:

$$y = \frac{1+x^2}{x^2}$$
$$\frac{dy}{dx} = \frac{x^2(2x)-2x}{(x^2)^2}$$
$$= \frac{x^2 2x - 2x}{x^4}$$

Hence, by clinging on to specific strategies the less successful students may find calculus problems are not an easy task to be performed.

8.8 Good symbolic manipulation does not necessarily imply reasonable interpretation for the same object in different forms.

Solutions for certain calculus problems can be represented in different forms. It depends on the method by which the problem is being solved. Some students view such solutions to be different. For example the solution for question 1(b), i.e. find the value of $\int (x+1)^2 dx$, can be represented as

$$\frac{(x+1)^3}{3} + c$$

or

$$\frac{x^3 + 3x^2 + 3x + 1}{3} + c = \frac{x^3}{3} + x^2 + x + \frac{1}{3} + c$$

if the expression $(x+1)^2$ is integrated directly by means of standard basic rule of integration. The solution of the above integral can also be represented as

$$\frac{x^3}{3} + x^2 + x + C,$$

with $C = \frac{1}{3} + c$, obtained by expanding the square of the brackets prior to the actual integration. However, some students conceived the constant c and C in the solutions of integral problem as the same number—the data to justify this claim will be discussed in the last paragraph. This phenomenon of conceiving the sameness of c and C could be seen from students' solutions thus:

$$\frac{x^3}{3} + x^2 + x + \frac{1}{3} + c \text{ and } \frac{x^3}{3} + x^2 + x + c$$

Here the students have used the letter c to symbolise the last constant in each of the solutions. Confusion arises when the student see the constant term in the first solution as $\frac{1}{3} + c$ and that in the second is only c. By displaying such solutions, it may seem that the two answers are different.

From the analysis, it is found that some students who are good at symbolic manipulation are not necessarily able to point out whether the two solutions obtained are the same or otherwise. This phenomenon can be seen from grade A performance in this problem. Out of twelve grade A students, only five of them demonstrated two ways of getting a solution to this problem. Of these five students, four of them are able to explain why the two solutions are the same.

The remaining seven grade A students managed to solve this problem by using either one of the methods discussed above. However they were guided to use alternatives to enable them to obtain the solution of the same problem in another form. Since the question is the same, the other four of the seven A students at first confidently expressed that the two solutions are the same no matter what methods are used. But after looking at the final result these students changed their mind—one of them was doubtful about the correctness of his computation:

> The solutions should be the same because we integrate the same problem. But after expanding the cube in the solution obtained by the first method I've got $\frac{x^3}{3} + x^2 + x + \frac{1}{3} + c$ ... This is slightly different from the solution obtained by the second method. May be something wrong with my calculation.
>
> (Second year SPK student, 1994)

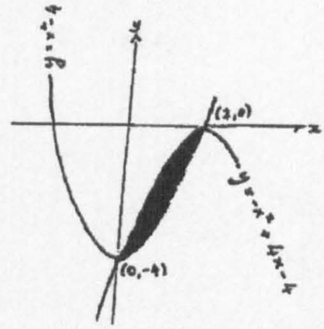
The other three students pointed out that the two solutions from the same calculus problem could be different if it is integrated by two different methods. One of them comments

> I never come across a problem like this one. If we solve it by different methods, its solutions will be different.
>
> (Second Year SPK student, 1994)

8.9 Students who are good at symbolic manipulation yet fail to visualize representation.

Some students can perform symbolic manipulation effectively but fail to relate their computational outcomes to geometrical representations. This phenomenon can be seen whenever they are asked to explain graphically how the area of the required region to be found as in problem 3:

Based on the diagram below, compute the area of the shaded region.



Please explain graphically, how this might be done.

Figure 8.1

Out of thirty six students attempting this question, fifteen of them managed to explain reasonably—eight with grade A, four with grade B, and three with grade C. According to them the area of the shaded region can be computed if they subtract the area enclosed by the positive x-axis, the negative y-axis and the curve $y = x^2 - 4$ from the area of the region which is bounded by positive x-axis, the negative y-axis and the curve $y = -x^2 + 4x - 4$. These students are able to make use of the correct limits. Such phenomenon can be seen from the interview and graphs drawn by the students. For example, the graph drawn by one the successful students in this problem thus:

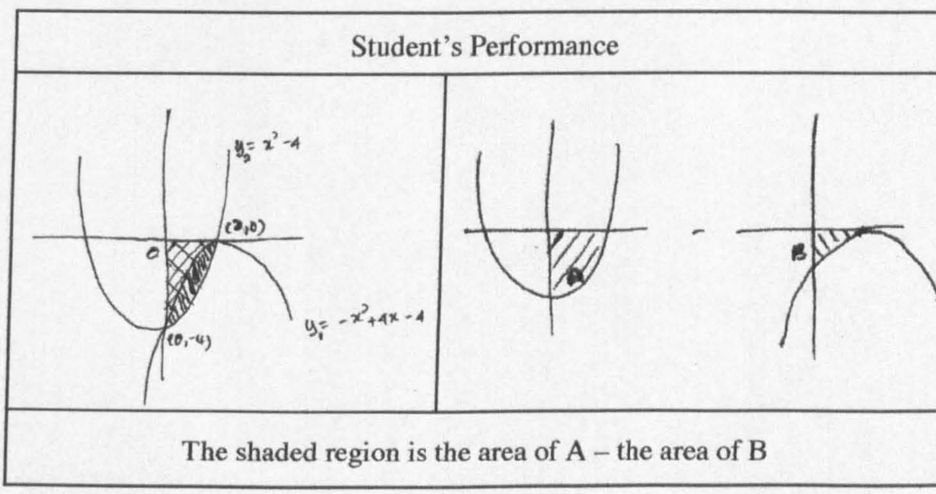


Figure 8.2

According to him:

> We can find the area of the shaded region, if we minus the area of B from the area of A . The area of A can be obtained by making use of the integral $\int_0^2 (x^2 - 4)dx$ and the area of region B can be obtained by making use of the integral $\int_0^2 (-x^2 + 4x - 4)dx$.
>
> (Second year SKA student 1994)

Twelve students — three with grade A, six with grade B and three with grade C are able to compute the area of the required region but responded to the geometrical question differently from those above. In other words, these students failed in linking the computational outcome to the graphical representation. The graphical responses observable in this case can be divided into groups—incorrect graphical notion of area between two curves and notion of intersection of sets.

Notion of area of a region between two curves.

This notion, if not fully understood, may lead the students into error. Such phenomenon occurs when some students conceive the area enclosed between the two regions as the area under the upper graph minus the area of the lower graph.

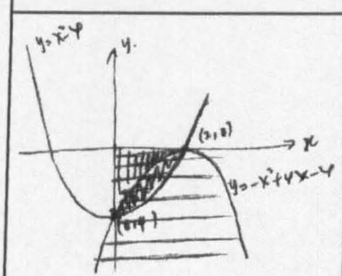
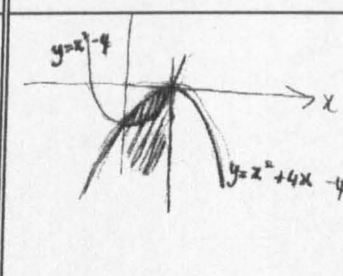
Response of students	
A_1	A_2, B_1, C_1, C_3
	

Figure 8.3

Having that notion in mind, A_1 shades the whole region under the upper curve $y = -x^2 + 4x - 4$ without taking into account the limits involved. However, she successfully shaded the area of the second region, i.e. the area enclosed by the negative y-axis, the positive x-axis and the graph $y = x^2 - 4$, with limits of x from 0 to 2.

According to her :

> The area of the shaded region can be obtained if we minus the area of the region under the graph $y=x^2-4$ from the area under the curve $y=-x^2+4x-4$. (Second year SPK student, 1994).

Out twelve students, three of them A_2, B_1, C_1 and C_3 conceive that the area of the required region can be obtained if they subtract the area below the graph $y=x^2-4$ for limits of x from 0 to 2 from the area enclosed by the graph $y=-x^2+4x-4$ with the same x limits.

One student B_5 explains that the area can be obtained first finding the sum of area under the curve $y=-x^2+4x-4$ from limit 0 to 2 and partly under the curve $y=x^2-4$. From this, according to her, subtract the area under the curve $y=x^2-4$ to obtain the area of the shaded region.

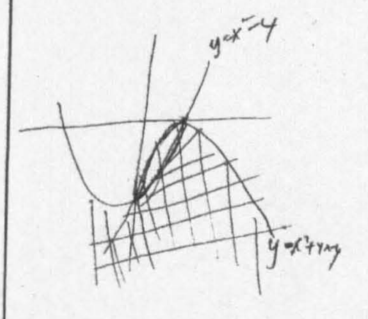
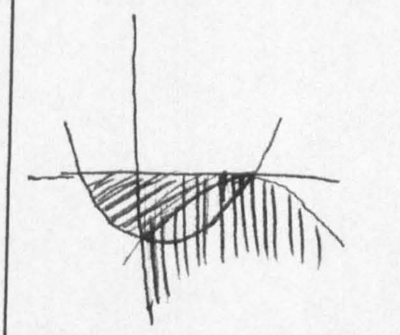
Graph drawn by B_5	Graph drawn by B_6
	

Figure 8.4

B_6 confuses in distinguishing graphically between a region under the upper graph and that under the lower graph.

According to him:

> I was not being taught how to obtain the area between two curves graphically. I wonder whether this can be explained graphically as the area under any of the curves is indefinite. But the computed area is definite. (Second year SPK student, 1994).

Notion of intersection of sets

In explaining graphically how to obtain the required region, it is observable that some students' line of thought may have been influenced by the notion of intersection of sets. In this study five students are in that situation. They are A_3, B_2, B_3, B_4 and C_2.

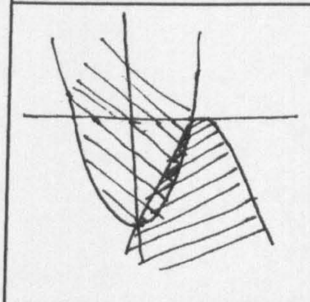
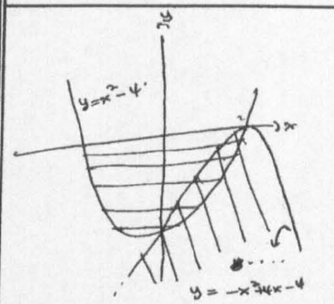
Graph	
A_3, B_2, B_3	B_4, C_2
	

Figure 8.5

A_3, B_2 and B_3, shade the whole region enclosed by the curve $y = -x^2 + 4x - 4$ and the whole region enclosed by the curve $y = x^2 - 4$. Thus:

> Intersection of the two areas is the area of the required region.
> (Second year SPK student, 1994).

In the case of B_4 and C_2, the idea of how to obtain the area of the enclosed region graphically is similar. But in here, they have shaded the region enclosed by the graph $y = x^2 - 4$ just up to the x-axis.

From the above observations some students are able to link their computational knowledge to the graphical task. For example, the required area obtained from computation is definite. So, the required regions drawn by these students lie within limited scope. The performance of the students in dealing with the graph can be summarised as on page 121

Students' ability in linking computation to the graph			
Grade	CC\CG	CC\WG	WC\WG
A	8	3	1
B	4	6	2
C	3	3	6

Table 8.14

Key: CC-Correct calculation; CG-Correct graphical explanation; WC-Wrong calculation; WG-Wrong graphical explanation.

Here the number of students with the ability CC\CG reduces from eight out twelve in grade A to only 3 out of twelve in grade C. By classifying those who perform CC\CG as one group and those who perform CC\WG and WC\WG as another, the information in Table 8.14 can be tabulated thus:

Students' ability in graph		
Grade	CC\CG	CC\WG+WC\WG
A	8	4
B	4	8
C	3	9
Total	15	21

Table 8.15

By making use of a χ^2-test with Yates correction, the difference in ability (CC/CG) between grade A and grade C is statistically significant at 10% level. The differences between groups A and B and between B and C are not statistically significant.

However for some students, the area under the graphs have infinite values which are impossible to calculate.

There exist a number of possible reasons for the occurrence of this phenomenon:

(i) The students fail to understand the meaning of *area under the curve*. To them such area is the whole area under the curve, no matter whether the curve lie below x-axis. To some of them the area of regions involved are enclosed by the graph.

(ii) In explaining how to obtain the required region, the students' line of thought have been influenced by

(a) The idea of *intersection* of sets. This can be seen when the students shade the two regions within the graph and point out the intersection of the two graphs to be the required region.

(b) The fact that

if f and g are continuous in $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area enclosed by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ is given by $L = \int_b^a [f(x) - g(x)]dx$.

In this case $x^2 - 4 \leq -x^2 + 4x - 4$ for all x in [0, 2]. Based on this fact the students shade the region under the curve $y = -x^2 + 4x - 4$ and that below the curve $y = x^2 - 4$ in the same interval [0, 2]. According to them the area under the former minus that under the latter is the area of the required region.

(c) Calculus instructors emphasise the symbolic solution for this type of problem. All students were told that the area of any region is always positive. Thus, the students with the notion that an area of any region is always positive tend to subtract the smaller area under graph from that under the bigger one. From the graph drawn in this problem, it appears that the area under the curve $y = -x^2 + 4x - 4$ is bigger than that below the curve $y = x^2 - 4$ in the interval [0, 2]. So, the area of the required region is the area of the former subtract the area of the latter.

(iii) The inefficiency and difficulty in explaining the nature of the graphical representations may be due to lack of emphasis of calculus instructors in the subject. In fact, one of the instructors comments

For this type of question, it is advisable to put the absolute value of the function to avoid the negative sign.

This statement may lead the students into confusion about which area is to be subtracted from the other. Out of eight lecturers attempting this problem, only two of them give the required explanations graphically. For example:

Calculus instructor method in finding the area of the shaded region (computation and graphically)

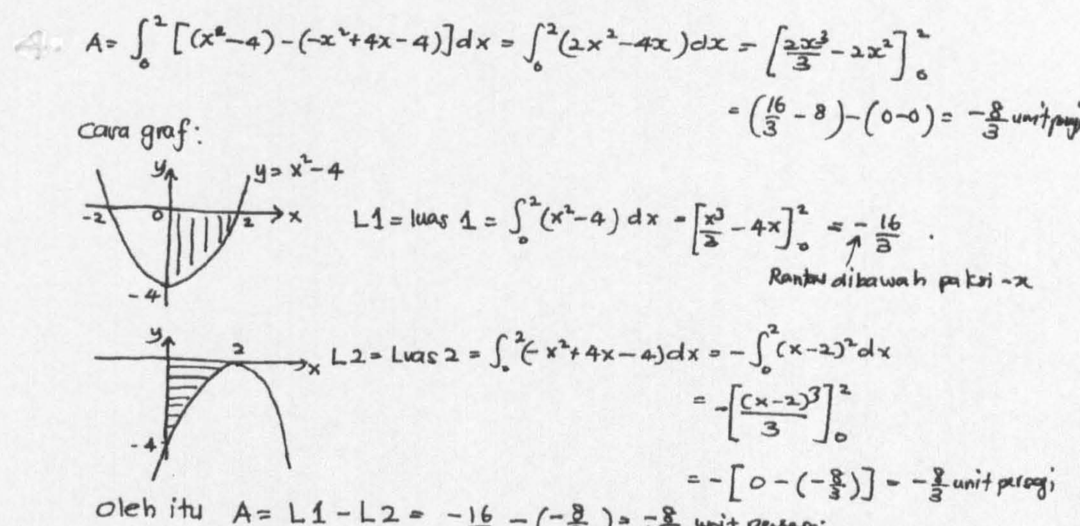


Figure 8.6

(iv) The examples and problems in the text used by the students emphasise computational skill. Thus, none of the problems in the text need the students to explain verbally how the required region is obtained graphically.

8.10 Students' flexibility in using integral notation to find the area under the graph but not the reverse

More able students possess reasonable representations that are strongly linked together. These students are more likely to show flexibility by switching from one representation to the others. Hence they are more likely to have the ability to reverse the process. In contrast, the less able students have limited representations and they may find great difficulty in reversing a process.

Reversal of a process is not a simple matter because all the separate processes in the brain operate in a single direction (Crick, 1994). A reversal cannot be done as with a video, by "running it in reverse", it requires the development of a new process in the reverse direction. For the development of new process, considerable meaningful conceptual linkages may be involved which may not be developed in some students. Thus when these students develop a procedure to do something in a certain way, they may not be able to easily reverse the procedure. Hence in this case it may not be obvious to them that the area under the graph (visualisation) may sometimes be used to determine the value of the corresponding integral.

The most appropriate problem to serve the purpose is question 4 , i.e.

By considering the function $y = \sqrt{4-x^2}$, *sketch and shade the region of the graph given by the following integral*

$$\int_0^2 \sqrt{4-x^2}\,dx$$

What is the value of the above integral?

The function $y = \sqrt{4-x^2}$ represents the graph of a circle above the x-axis and the integral $\int_0^2 \sqrt{4-x^2}\,dx$ represents the part of a circle in the first quadrant. Hence, by using the familiar rule of finding an area of a circle, i.e. area = πr^2, whereby r is the radius of the circle, the value of the integral $\int_0^2 \sqrt{4-x^2}\,dx$ can be determined almost trivially. In this problem, quite a number of students failed to draw the graph but they were encouraged and guided to do so. The aim is not to see the students' ability in drawing a graph but to identify whether the students were able to use the area under the graph to determine the value of a corresponding integral.

Thirty six students attempted this problem, twelve grade A, twelve grade B and twelve grade C. Out of twelve grade A students, five of them were able to use visualisation in finding the value of an integral $\int_0^2 \sqrt{4-x^2}\,dx$

with one of them failing to draw the appropriate graph at first. However, this student was able to draw the required graph after some guidance. Three of them managed to use lengthy trigonometric substitutions. The other four failed to obtain the answer. In this case one had an incomplete trigonometric substitution while the other three misapplied the substitution methods or used inappropriate methods.

Some examples of performance of A grade students in this problem can be seen in the table below:

Grade A students performance in evaluating an integral		
Visualisation	Trigonometric substitution	Error
For example $\frac{1}{4}\pi(2)^2$ $= \frac{1}{4}\pi(4)$ $= \pi$	For example $\int_0^2 \sqrt{4-x^2}dx$ *Let* $x = 2\sin\theta$ $dx = 2\cos d\theta$ $\sqrt{4-x^2} = \sqrt{4-2\sin^2\theta}$ $= 2\sqrt{1-\sin^2\theta}$ $= 2\cos\theta$ *Then* $\int_0^2 \sqrt{4-x^2}dx = \int_0^{\pi/2} 4\cos^2\theta\, d\theta$ $= 4\int_0^{\pi/2} \frac{1}{2}(2\cos^2\theta - 1) + \frac{1}{2}\, d\theta$ $= 4\int_0^{\pi/2} \left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) d\theta$ $= 2\int_0^{\pi/2} (\cos 2\theta d\theta + 1)\, d\theta$ $= 2(\frac{1}{2}[\sin 2\theta]_0^{\pi/2} + [\theta]_0^{\pi/2})$ $= [\sin 2\theta]_0^{\pi/2} + 2[\theta]_0^{\pi/2}$ $= (\sin\pi - \sin 0) + 2(\pi/2 - 0)$ $= \pi$	For example $\int_0^2 \sqrt{4-x^2}dx$ $= \int_0^2 (4-x^2)^{\frac{1}{2}}dx$ *Let* $u = x^2; \frac{du}{dx} = 2x$ $dx = \frac{du}{2x}$ $\int_0^2 (4-u)^{\frac{1}{2}} \frac{du}{2x}$
5 students	3 students	4 students

Table 8.16

Five with grade B were able to use visualisation, but out of these five only one of them was able to draw an appropriate graph for the integral $\int_0^2 \sqrt{4-x^2}\,dx$ and to compute its value by means of visualisation. The other four students at first had a different interpretation for symbolic representation of the integral $\int_0^2 \sqrt{4-x^2}\,dx$ such as the region of the first half of a circle. However, after some guidance, they managed to draw the required graph and hence solve the problem. Seven of the B grade students failed to compute the value of the integral. Here, two of the students had incomplete trigonometric substitutions while the other five could not proceed or misapplied the substitution method. Some examples of performance of B grade students in this problem can be seen in the table below:

Grade B students' performance in evaluating an integral		
Visualisation	Incomplete trigonometric substitution	Error
For example The area of the above diagram $\frac{1}{4}(\pi r^2) = \frac{1}{4} \times \frac{22}{7} \times 4$ $= \frac{22}{7}$	For example Let $x = 2sin\theta$; $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$ $\sin\theta = 0 \Rightarrow \theta = 0$ $dx = 2\cos d\theta$ $\sqrt{4-x^2} = 2\cos\theta\,(\textit{formula})$ $\int_0^2 \sqrt{4-x^2}\,dx = 4\int_0^{\pi/2} \cos^2\theta\; d\theta$	For example $\int_0^2 \sqrt{4-x^2}$ let $u = 4-x^2$ $\frac{du}{dx} = -2x$ $\int_0^2 u^{\frac{1}{2}} dx$ $= \int_0^2 u^{\frac{1}{2}} \frac{du}{-2x}$
5 students	2 students	5 students

Table 8.17

Out of twelve C grade students, none was able to use visualisation in getting the solution for this problem. However, this student was unable to get the correct graph before the interviewer offered guidance. None demonstrated the trigonometric substitution. The whole twelve C grade students could not proceed or applied inappropriate methods. An example of an error performed by a C grade student in dealing with this problem:

Grade C students' performance in evaluating an integral	
Visualisation	Error
	$\int_0^2 (4-x^2)^{\frac{1}{2}}dx$ $u = 4 - x^2$ $du = -2xdx$ $\frac{du}{-2x} = dx$ $\int_0^2 (4-x^2)^{\frac{1}{2}}dx$ $= \int_0^2 u^{\frac{1}{2}} \frac{du}{-2x}$ $= \left[\frac{1}{-2}\left(\frac{2}{3}u^{\frac{3}{2}} \ln x\right)\right]_0^2$ $= \left[-\frac{1}{3}(4-x^2)^{\frac{3}{2}} \ln x\right]_0^2$ $= -\frac{1}{3}$
0 student	12 students

Table 8.18

Although some students have good visual images as above, they are unlikely to use them in their calculus exercises and examinations. In this study there was only one student who showed flexibility in using visualisation over the standard rule of integration. This student is an A grade student. For the rest of them, the use of standard formulae are preferable. This is revealed in the interview. According to them, problem 4 may better be solved by using the integral technique and the rather trivial formula $Area = \frac{1}{4}\pi r^2$ was not acceptable. There are a few possible reasons observable in the interview for the occurrence of this phenomenon:

(i) To the students word *integration* in the last sentence of the question implies that the problem has to be solved by means of integral technique and not by other methods.

(ii) When facing this type of problem in the examination, the formula will be given to the students to enable them to compute the integral.

(iii) In fact, in the text used by the students there is an indication that the use of visualisation is not preferable. It seems to provide the simplest algorithmic method for the students. Consider the following statement in the text:

If u is a differentiable function in x, then the following formula can be used in substitution in order to carry out the integration ***easier****. For the integral which involved expression* $\sqrt{a^2 - u^2}$, *let* $u = a\sin\theta$; *then* $\sqrt{a^2 - u^2} = a\cos\theta$.

(iv) The students' methods of solving calculus problems are greatly influenced by lecturers' techniques during instruction. In other words, if the lecturers fail to use visualisation, then the students are reluctant to accept visualisation as another reasonable method that can be used to solve their calculus task. These students believe that if they use visualisation some marks may be deducted in their examination or the lecturers might considered such methods as mistakes because such methods are too elementary to be used in calculus class. As a result of that, they try to use algorithmic alternatives.

8. 11 The lecturers' performance in evaluating the integral

Out of eight lecturers attempting this problem only two of them used visualisation to get to the solution. The other six although, they obtained the correct graphs, overlooked the use of visualisation in determining the value of the integral. Instead they used the rather complicated trigonometric substitution method. Their performance can be seen on page 129.

Instructors' method in evaluating the integral	
Visualisation	Trigonometric substitution
For example $\int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}(\text{area of a circle with radius } 2)$ $= \frac{1}{4}(\pi(2)^2) = \pi$ square units	For example $\int_0^2 \sqrt{4-x^2}\,dx$ let $x = 2\sin t$ $dx = 2\cos t\,dt$ and $\sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = 2\cos t$ Therefore $\int_0^2 \sqrt{4-x^2}\,dx = \int_0^{\pi/2} 4\cos^2 t\,dt$ $= 4\int_0^{\pi/2} \frac{(1+\cos 2t)}{2}dt$ $= 2\left[t + \frac{\sin 2t}{2}\right]_0^{\pi/2}$ $= 2\left[\pi/2\right]$ $= \pi$
2 lecturers	6 lecturers

Table 8.19

8.12 Summary

Krutetskii (1976) suggests that curtailment of solutions is an indication of capability but in this study it is not the case. However, cognitive differences between the performance of grade A and grade B and grade A and grade C is statistically significant.

In this study some successful students preferred detailed solutions. These students can also performed curtailed procedures in working out answers for calculus problems. Thus detailed procedures may not be an indication of inability to curtail solutions. Why did some students preferred detailed solution when they could curtail? As these students possess strong conceptual and cognitive linkages they are more likely to have a desire to write out correctly all the steps.

These students performed correct steps in the detailed solution by joining meaningfully new bits of mathematical facts to the existing information. In performing such solutions they are more likely to reflect the reasonableness of the preceding steps. Such linkages may not be well developed in less successful students. Thus they are likely to make errors by writing inappropriate information in trying to get to the solutions. In other words solutions of the less able students are far more susceptible to breakdown.

In performing the procedures, the more capable students would be more likely to carry out conceptual preparation of procedures while the less capable students are more likely to plunge straight into the differentiation procedure, so that students already having greater difficulties are setting themselves a more complex task which is more likely to lead to error. In other words students who fail to carry out conceptual preparation are also more likely to increase their problems at this later stage. In this study it is found that, there is a strong correlation between students' success and conceptual preparation of procedures.

The more successful students may have several flexible strategies in dealing with symbolic manipulation. They are efficient in interchanging calculus symbolism in obtaining simpler alternatives for solving the problems. In other words, they are more likely to switch from one representation to another and this may increase their chances of solving the problem. The less able students may not have such qualities and tend to cling onto the security of a specific strategy. There is a strong correlation between ability and flexibility in using possible approaches in solving the same calculus problem. However, it is found that some students who are good at symbolic manipulation fail to interpret the same object in different form.

Efficiency in symbolic manipulation is not necessarily an indication of an ability to establish links between computational outcomes with graphical or geometrical representations as some students find great difficulty in relating their computational outcomes to visual ideas. For instance, in the problem of area under the curve whereby the students' computational area is finite but the graph drawn is infinite.

Another problem faced by the students is reversal of process. Once the students have developed a procedure to do something in a certain way, they may not be able to reverse the procedure easily. To reverse a process requires the development of a new process to operate in the reverse direction. Such development needs strong conceptual linkages which is less developed in some students.

CHAPTER 9

SUMMARY

9.1 An overview of the findings

Here I shall review the main result of the research. Krutetskii (1976) observed that gifted students/capable students tend to *curtail* their solutions effectively while the incapable tend to fail in using curtailed solutions even after a long practice.

It is not in the case for this group of students. It is found that the more successful students may include in their mathematical work both form of solutions—curtailed and detailed solutions. They are more likely to display detailed solutions by making use of their sound conceptual and cognitive linkages. Such linkages may be fragile in the less able students. Thus they tend to face breakdown of procedures as they fail to develop meaningful connections between different bits of relevant mathematical information in their long solutions.

In this study, it is found that the more successful students develop more flexible approaches in tackling calculus problems. They have strong conceptual linkages. Hence they are more likely to demonstrate flexibility in handling mathematical symbolic manipulation. This phenomenon can be seen when the students develop meaningful relationship between symbolism and show the ability to interchange symbolism freely in a different number of ways. Given several methods available for tackling a calculus problem, the more successful students probably make use of their strong conceptual knowledge to choose an easier method that needs less cognitive strain in execution. Lacking such conceptual quality, the less able students are more likely to face considerable difficulties in performing calculus tasks.

In other words the more successful students have several meaningful representations of mathematical concepts. Such representations (all involved symbolic manipulation) seem to be rightly and strongly linked as these students have less difficulty in switching from one representation of a mathematical concept to another. In other words these students are able to move easily from one way of thinking to another which greatly increases their chances of solving a given mathematical problem.

As the conceptual and cognitive links in weaker students is fragile, such connection of representations may not very well developed in these students. For each development of new links between representations may require new mathematical pieces of information to be linked with the existing links of representation. This is less likely to develop in less able students. As a result of that these students much of the time restricted themselves to a single secure representation.

Although the more able students may have several flexible strategies and meaningful symbolic mathematical representations, these may not always relate to visual and graphical ideas: the graphs drawn by these students may not be reasonable. Even when the graphs are correctly drawn these students are less likely to use them in getting solutions for the calculus problems.

As a result of that we have the less able students who are less likely to break away from the security of a single procedure and liable to breakdown and the more able students with several strategies in symbolic manipulation but fail to link them to visual geometrical representation.

The findings in this thesis fit with the findings in theses of Bakar (1991) and Yusof (1995). The thesis of Bakar shows that curriculum designers show concept of functions as fundamental. But the teachers taught the students those materials that they thought the students needed to pass the examination. As a result of that the students learn to do the procedures in order to solve the examination questions given to them.

Yusof's study reveals that the students are very dutiful and attempt to learn the procedures they are taught. When they were given problem solving ideas, they changed their attitudes to become more self-confident. However, the actual technique involve were often less demanding than the content of the mathematical courses. When these students went back to normal mathematics courses, in the main they revert to the procedural approaches.

9.2 Suggestions for further research

The students involved in this study form only a small fraction of the whole population in the university. They are prospective civil engineers and prospective teachers. These students are selected on their general mathematics performance in the first year mathematics examination and not from the calculus achievement in particular. Thus their way of thinking may be restricted to a certain extent to reflect wholly the calculus thinking. In order to get the wider spectrum of students thinking in calculus those diploma students should also be included in a study. By doing so, it is hoped to get clearer differences in thinking between the more successful and the less successful students.

Although the sample in this study is small, the findings indicate that there is a difference between the more successful and the less successful students in the way they solve calculus problems. For example, the more able students may have different strategies to tackle the same calculus problem but the less successful students are more comfortable with the security of a single procedure. The more able students if they are procedural may still successful in solving the problem as they are very efficient in handling the symbolic manipulation but the less able students tend to make error and breakdown in their solutions in coping up the long solution.

From the above studies, the procedural students will remain procedural and tend to breakdown in their solutions. It is therefore essential to consider whether a curriculum which causes this to happen is appropriate. Some changes have to be made in the pedagogical aspects and curriculum as a whole. The learning materials in the curriculum have to undergo trial until obtaining required appropriate topics fit to be taught to the students. So, research in this area with the intention to identify the effectiveness of the materials has to be conducted.

Similarly from the pedagogical aspects, apart from teaching efficiency in symbolic manipulations, the teachers should encourage the students to use diagrams and graphs wherever possible in the learning of calculus. The intention is to promote the building of connections between analytical ideas and visual geometrical representations. Hence the students tend to have more representations in facing certain calculus problems.

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Appendix I

PILOT STUDY AT THE UNIVERSITY OF WARWICK

SECTION I(a).

Find $\frac{dy}{dx}$, when

1. $y = 2x^4 - x^2 - 6$

2. $y = \frac{1}{(x+1)^{\frac{1}{2}}}$

3 (a) $y = 5.$

(b) $y = \sqrt[3]{4}.$

4. $y = \sqrt{2x^3}.$

5. $y = \left(x + \frac{1}{x}\right)^2.$

6. $y = \frac{1+x^2}{x^2}.$

SECTION I(b)

Evaluate the following integral:

1. $\int (x+1)^2 dx$

2. $\int \frac{1}{(x-3)^5} dx$

3. $\int (6x)^2 dx$

4. $\int \sqrt{3x^3} dx$

Appendix 2

PILOT STUDY AT THE UNIVERSITI TEKNOLOGI MALAYSIA

SECTION I

(i) Evaluate $\int \sqrt{3x^3}\,dx$.

(ii) Find $\frac{dy}{dx}$ when

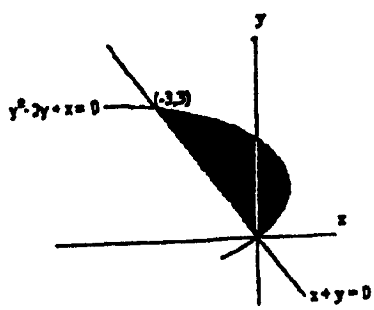
(a) $y = \frac{1}{\sqrt{x}}$,

(b) $y = \frac{1 + x^2}{x^2}$,

(c) $y = \left(x + \frac{1}{x}\right)^2$.

SECTION II

(i) Find the area of the shaded region.



Please explain graphically, how this might be done.

(ii) By considering the function $y = \sqrt{4 - x^2}$, sketch and shade the region of the graph given by the following integral

$$\int_0^2 \sqrt{4 - x^2}\,dx$$

What is the value of the above integral?

Appendix 3

THE MAIN STUDY AT THE UNIVERSITI TEKNOLOGI MALAYSIA

1. Evaluate

(a) $\int \sqrt{3x^3}\,dx.$

(b) $\int (x+1)^2 dx$

2. Find $\dfrac{dy}{dx}$, when

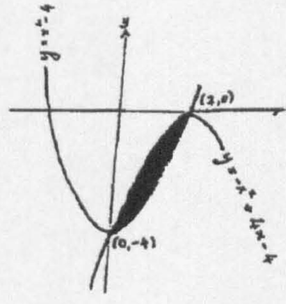
(a) $y = \dfrac{1}{\sqrt{x}}$

(b) $y = \dfrac{1+x^2}{x^2}$

(c) $y = \left(x + \dfrac{1}{x}\right)^2$

(d) $y = \dfrac{1}{1+x^2} - \dfrac{x^4}{1+x^2}$

3. Based on the diagram below, compute the area of the shaded region. Please explain graphically how this might be done.



4. By considering the function $y = \sqrt{4 - x^2}$, sketch and shade the region of the graph given by the following integral

$$\int_0^2 \sqrt{4 - x^2}\,dx$$

What is the value of the above integral?

Appendix 4

My feelings about calculus

1. **The calculus topics we study at the university make sense to me.**

	Definitely Yes	Yes	No opinion	No	Definitely No
Calculus					

2. **I learn calculus through memory.**

	Definitely Yes	Yes	No opinion	No	Definitely No
Calculus					

3. **I usually understand a new idea in calculus quickly.**

	Definitely Yes	Yes	No opinion	No	Definitely No
Calculus					

4. **I am able to relate calculus ideas learned.**

	Definitely Yes	Yes	No opinion	No	Definitely No
Calculus					

5. **Calculus is abstract at the university .**

	Definitely Yes	Yes	No opinion	No	Definitely No
Calculus					

6. **In a few sentences describe your feelings about calculus**

....

....

....

Appendix 5(i)

SKA (A) -no. 18

Questions	Spoken dialogue	Written symbol manipulation	Comment/Observation
(1) (i) Find (a) $\int\sqrt{3x^3}dx$ (b) $\int(x+1)^2dx$	I. Please do for me no 1.(a). S. . OK. I. Have you finish? S. Yes, I do.	$(a)\int\sqrt{3x^3}dx$ $=\sqrt{3}\int x^{3/2}dx$ $=\frac{2\sqrt{3}}{5}(x^{5/2})+c$	To identify the extent to which students compress algebraic procedures in terms of the number of steps used to carry them. The number of steps is 2 and the answer is in conventional form
	I. Wonderful. Now let's go on to no 1(b)? I. You have solve this problem using direct rule of integration. Is it right? S. Yes. I. Do you think any other way of doing this problem? S. I can also solve this problem by expanding the bracket first. I. Can show me, please? S. OK. I. Fine. Now, can you tell me why in the first solution you write c_1 but in the second solution you just write c? S. This shows the c_1 and c are not the same. I. Now, is the first and second solution the same? S. Yes, it is the same. I. Will you please show me? S. In this case we have to expand the cube in the first solution.	$(b)\int(x+1)^2dx$ $=\frac{1}{3}[(x+1)^3]+c_1$ *or* $\int x^2+2x+1dx$ $=[\frac{1}{3}x^3+x^2+x+c]$	To identify the ability of students in distinguishing the same solution for a problem. manage to explain why the two solutions are the same.
(2) Find $\frac{dy}{dx}$, whenever (a) $y=\frac{1}{\sqrt{x}}$.		$2(a\ y=(x)^{-1/2}$ $\frac{dy}{dx}=(-\frac{1}{2})x^{-3/2}$	To identify the extent to which students compress algebraic procedures in terms of the number of steps used to carry them. The number of steps is 2 and the answer is in conventional form

(b) $y = \frac{1+x^2}{x^2}$

(c) $y = \left(x + \frac{1}{x}\right)^2$

(d) $y = \frac{1}{1+x^2} - \frac{x^4}{1+x^2}$

I. Now, please do 2(b).
S. O. K.
I. Is there any other method of solving this problem?
S. I think there is ? Quotient rule
I. Please show me.
S. Yes.
I. Are you done?
Good work. Is it possible to work out the solution by other alternative?
S. Let me try.
I. What method you have used?
S. Implicit differentiation.
I. Now. In your last solution there is a variable y? Do you think your last solution in the same with the first two?
S. Yes.
I. Please show me.
S. We know the variable y = $\frac{1+x^2}{x^2}$.
By changing the y with $\frac{1+x^2}{x^2}$ in the last solution we will get the solutions as above.

$$\frac{2-2y}{x} = \frac{2}{x} - \frac{2y}{x}$$
$$= \frac{2}{x} - \frac{2}{x}\left(\frac{1+x^2}{x^2}\right)$$
$$= \frac{2}{x} - \frac{2}{x^3} - \frac{2}{x}$$
$$= -\frac{2}{x^3}$$

$(b) y = \frac{1+x^2}{x^2}$

$$y = \frac{1}{x^2} + 1$$
$$\frac{dy}{dx} = -2\frac{1}{x^3}$$

or

$$y = \frac{1+x^2}{x^2}$$
$$\frac{dy}{dx} = \frac{x^2(2x) - 2x(1+x^2)}{x^4}$$
$$= \frac{2x^3 - 2x - 2x^3}{x^4}$$
$$= -2\frac{1}{x^3}$$

or

$$x^2 y = 1 + x^2$$
$$x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2) = 2x$$
$$x^2\frac{dy}{dx} = 2x - 2xy$$
$$\frac{dy}{dx} = \frac{2x - 2xy}{x^2}$$
$$= \frac{2-2y}{x}$$

To identify the number of methods that can be used in tackling the same calculus problem.
Using three ways in getting the answer to this problem . with preference of conceptual preparation of procedures over standard rule of differentiation.. i.e. Simplification of the expression before differentiation, the quotient rule and the product rule.

I. Do you think there is another method of doing this problem?
S. I don't think so.
[*At this point the interviewer told the student that this problem can be solved using the product rule*].
I. Can we go on to the next one?
S. Let me have a try?
I. Please do so.
S. I think, I've got the answer.
I. What method are tou using?
S. The chain rule.
I. Do you think any other nethod of doing this problem?
S. I can expand the btacket and then I differentiate the expression.
I. That's real wonderful. Anyway, please show me your suggested method.
S. OK.
I. Let's go on to the next problem.
I.. Please tell me how to do 2(d)?
S. The numerator can be factorised. This problem can also be solved using quotient rule . But it is too long

$$(c)\, y = \left(x + \frac{1}{x}\right)^2$$

$$\frac{dy}{dx} = 2(x + \frac{1}{x})(1 - x^{-2})$$

$$= 2(x + x^{-1} - x^{-1} - x^{-3})$$

$$= 2x - 2x^{-3}$$

or

$$y = x^2 + 2 + \frac{1}{x^2}$$

$$\frac{dy}{dx} = 2x - 2\frac{1}{x^3}$$

$$(d)\, y = \frac{1 - x^4}{1 + x^2}$$

$$= \frac{(1 + x^2)(1 - x^2)}{(1 + x^2)}$$

$$= 1 - x^2$$

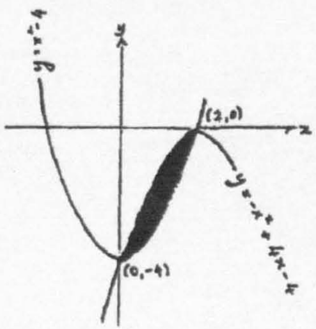
$$\frac{dy}{dx} = -2x$$

Two approaches were:
1. Preference of using the standard rule of differentiation (Chain rule).
2. Expanding the brackets first.

Ability to use simplification prior to actual differentiation.

(3) Based on the diagram below, compute the area of the shaded region.

Please explain graphically how this might be done.



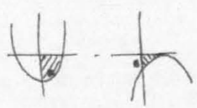
I. Now let's do no 3.

I. Have you got the answer for the first part.

S. Yes, I do.

I. Now,. can you explain graphically how this might be done?

S. The two areas can be computed separately. Then the area of the enclosed region is the area of A minus the area of B



$$(3)\int_0^2 (-x^2+4x-4)-(x^2-4)dx$$

$$=\int_0^2 4x-2x^2dx=\left[2x^2-\frac{2}{3}x^3\right]_0^2$$

$$=8-\frac{16}{3}=\frac{8}{3},$$

Relationship with the graph

Able to link the computational outcome to graphical ideas.

(4) By considering the function $y = \sqrt{4 - x^2}$, sketch and shade the region of the graph given by the following integral

$$\int_0^2 \sqrt{4 - x^2}\, dx$$

What is the value of the above integral?

I. Now, let's proceed to the move to the next number.
S. May I use the graph to get the value of the integral?
I. Please do so.
S. The graph which I've got is quarter of a circle. Then the value of the integral is the area of a circle in the first quadrant

Able to draw the region correctly

$A = \frac{1}{4}(2)^2$

$= \frac{1}{4}\pi(4)$

$= \pi$

To identify the reversal of process.
Able to use the area to find the value of the integral

Appendix 5(ii)

SPK (B) -no.7

Questions	Spoken dialogue	Written symbol Manipulation	Comment/Observation
(1) Find (a) $\int\sqrt{3x^3}\,dx$ (b) $\int(x+1)^2dx$	I. Can you show me how to do no 1? The answer $\frac{2}{5}\sqrt{3}x^{5/2}+c$ is true. I. Can you solve problem 1 (b) ? S. I can solve this problem using substitution method? I. That's good. I. Anyway is there any other method to solve this problem? S. There is no alternative to be used to solve this problem. *[At this point the interviewer guided the student to solve the problem by expanding the brackets first]*. I. Now you have two solutions. Do you think it is the same? Please expand the cube in the first solution. S. I'm not very sure. In the first solution there is an extra number $\frac{1}{3}$.. *[At this point the interviewer gave a brief explanation of the difference between the constant in the two cases]*	$(a)\int\sqrt{3x^3}dx$ $=\sqrt{3}\int\sqrt{x}^3\,dx$ $=\sqrt{3}\int x^{3/2}\,dx$ $=\sqrt{3}\,\frac{x^{5/2}}{5/2}+c$ $=\frac{2\sqrt{3}}{5}x^{5/2}+c$ $(b)\int(x+1)^2dx$ $Substitue\ u=x+1$ $du=dx$ $=\int u^2\,du$ $=\int\frac{u^3}{3}+c$ $=\frac{(x+1)^3}{3}+c$	To identify the extent to which students compress algebraic procedures in terms of the number of steps used to carry them. The number steps used is 4 and the answer obtained is in conventional form. To identify the ability to distinguish the same solution for a problem. In this case the student is not sure whether the two solutions are same. or otherwise..

I.Can you simplify the answer for problem 2(c)

I. Is there any other method that can be used to solve this problem?

S. The quotient rule

S. Oh! yes I forget. If I expand the brackets, the method would be easier than the quotient rule..

$$(c)y = \left(x + \frac{1}{x}\right)^2$$

$$= 2\left(x + \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right)$$

$$= 2\left(\frac{x^2+1}{x}\right)\left(\frac{x^2-1}{x^2}\right)$$

$$= \frac{2(x^2+1)(x^2-1)}{x^3}$$

$$= \frac{2(x^4-1)}{x^3}$$

or

$$y = \left(\frac{x^2+1}{x}\right)^2$$

$$\frac{(x^2+1)^2}{x^2}$$

$$= \frac{x^4+2x^2+1}{x^2}$$

$$\frac{dy}{dx} = x^2(4x^3+4x) - (x^4+2x^2+1)$$

Two methods of solving the problem exhibited:

1. By the chain rule.

2. Trying to tuse he quotient rule.

I. Can we proceed to no 2(d)?
S. Yes.
I. Very nice. You have done this problem using the quotient rule.
I. Is there any simplest method to be used to solve this problem?
S. I don't know.
[At this point the interviewer gave a brief explanation that the numerator can be factorised and the whole expression can simplified before carrying out the actual differentiation.]

$$(d) y = \left(\frac{1}{1+x^2}\right) - \left(\frac{x^4}{1+x^2}\right)$$

$$= \frac{1-x^4}{1+x^2}$$

$$\frac{dy}{dx} = \frac{(1+x^2)(-4x^3)-(1-x^4)(2x)}{(1+x^2)^2}$$

$$\frac{-4x^3-4x^5-2x+2x^5}{(1+x^2)^2}$$

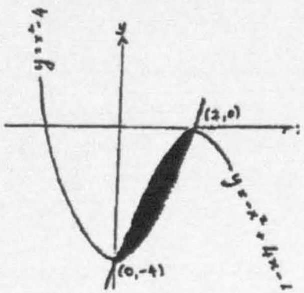
$$= \frac{-2x(1+2x^2+x^4)}{(1+x^2)^2}$$

$$= \frac{-2x(1+x^2)^2}{(1+x^2)^2}$$

$$= -2x$$

Using the quotient rule to solve the problem

(3) Based on the diagram below, compute the area of the shaded region. If you cannot do it, please explain graphically how this might be done.



I. Please tell me how to do no. 3? The area lies below the x axis?
I. The area lies under the axis, the area should be negative. How come you get it positive?
S. I just take its numerical value.
I. Now, can you explain graphically how the area of the shaded region to be found.
S. I've no idea. We never come across such a problem before.

3. $\int_0^2 (-x^2+4x-4)dx - \int_0^2 (x^2-4)dx$

$= \left[\frac{-x^3}{3} + 2x - 4x\right]_0^2 - \left[\frac{x^3}{3} - 4x\right]_0^2$

$\left(-\frac{8}{3}+8-8\right) - \left(\frac{8}{3}-8\right)$

$-\frac{16}{3} = 8$

$= \frac{8}{3}$

Cannot explain graphically how this might be done.

Inability to relate the computational outcome with graphical ideas.

(4) By considering the function $y = \sqrt{4-x^2}$, sketch and shade the region of the graph $\int_0^2 \sqrt{4-x^2}\,dx$ What is the value of the above integral?	I. Can you show me how to do no. 4? S. I try I. What you have to do for the first part of the question? S. To sketch the graph> I. Have you drawn. S. No. I. $y^2 = 4 - x^2$ $x^2 + y^2 = 4$ This is a circle. Can you proceed from there? S. Yes, using trigonometric substitution. I. Geometrically what is meant by that integral? S. Area. I. What is the value of the above integral? Usually how you do it? S. By using formula. I. Can you do it without using formula? S. I don't know I. This is an area of a quarter of a circl. The area lies below the x axis. $A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(4) = \pi$. Do you think this method will be accepted by your lecturer? S. I don't think so. I. Why? S. Because the question ask us to find the value of integral, so we must carry out the integration.	4 (a) $y = \sqrt{4-x^2}dx$ $\int \sqrt{4-x^2}dx$ *cannot proceed* (b) $\sqrt{4-x^2}$ $x = 2sin\theta$; When $x = 2; \theta = \frac{\pi}{2}$ $x = 0; \quad \theta = 0$ $\frac{dx}{d\theta} = 2kos\theta$ $\therefore \int_0^2 \sqrt{4-x^2}dx = \int_0^{\pi/2} 4kos^2\theta\, d\theta$	Inability to use geometrical representation to find the value of the integral.

Appendix 5(iii)

SPK(C)-no 28

Questions

(1) (i) Find

(a) $\int \sqrt{3x^3}\,dx$

(b) $\int (x+1)^2 dx$

(2) Find $\frac{dy}{dx}$, whenever

(a) $y = \frac{1}{\sqrt{x}}$.

Spoken dialogue

I. Will you plesae do no 1(a) ?
S. Yes.
I. Can we slight adjusment to the solution..
S. Adjustment?
I. Yes.
S. Please texplain tome?
[At this point the interviewer, explain the way to integrate the expression].
I. Now, let's do 1(b).
S. OK.
I. Is there any other method to be use to solve 1(b)?
S. Usually, I do by this way. Other method, I don't know.
[At this point the interviewer, explain the way to integrate the expression].
I. Now, you have two answers. Is it the same?
S. We need to expand the cube in the first solution.
I. Please expand it?
I. Now how is it?
S. No, because there is $\frac{1}{3}$ in the first solution.
[At this point the interviewer, explain briefly the fiffrence between the constant in both solutions].
I. Can you proceed to 2(a)?
S. OK.
I. Can we make slight adjustment to the solution.

Written symbol manipulation

$$(a)\int \sqrt{3x^3}\,dx$$
$$= \int (3x^3)^{1/2} dx$$
$$= \frac{2(3x^3)^{3/2}}{3} + c$$
$$(b)\int (x+1)^2 dx$$
$$= \frac{(x+1)^3}{3} + c$$
$$2(a)\quad y = \frac{1}{\sqrt{x}}$$
$$y = (x)^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = x^{-3/2}.(-3/2)$$
$$= -\frac{3}{2(x)^{3/2}}$$

Comment/Observation

To identify the exttent to which students compress algebraic procedures in terms of number of steps used to carry it out. Breakdown of procedure in the secod step.

Unable to see the same object in the same form.

Beakdown of procedure in the thid step.

(b) $y = \left(x + \frac{1}{x}\right)^2$

(c) $y = \frac{1+x^2}{x^2}$

(d) $y = \frac{1}{1+x^2} - \frac{x^4}{1+x^2}$

. All right. Let's poceed to 2(b).

S. All right.

I. The method is very simple. Is there any other method to solve this problem?

S. Let me try first.

I. Very good. Is there any other method to solve this problem?

S. No.

[At this point the interviewer, briefly explained other ways of doing this problem- implicit differentiation and the product rule].

I. Now, how many ways we can solve the same problem?

S.4

$$(b)y = \frac{1+x^2}{x^2}$$

$$y = \frac{1}{x^2} + 1$$

$$= -\frac{2}{x^3}$$

or

$$(b)y = \frac{1+x^2}{x^2}$$

$$v = x^2 \qquad u = 1 + x^2$$

$$\frac{dv}{dx} = 2x \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2)(2x) - (1+x^2)(2x)}{(x^2)^2}$$

$$= \frac{2x^3 - (2x + 2x^3)}{x^4}$$

$$= \frac{2x^3 - 2x - 2x^3}{x^4}$$

$$= -\frac{2}{x^3}$$

Exhibited two approaches of solving the same mathematical problem

(i) Conceptual preparation of procedure

(ii) the quotient rule.

I. Can you proceed to 2(c)?
S. OK, I will do using the chain rule.
I. How is it?
S. The answer is $-\frac{3}{2(x)^2}$
[At this point the interviewer guided the student to get the correct solution]
I. Is there any other method of doing 2(c)?
S. Let me think first.
I. Wonderful. Now, which is easier, the first method or the second?
S. The second one.
I. Why you prefer the first one?
S. No time to think of the second mehod.

$$(c)y = \left(x + \frac{1}{x}\right)^2$$

$$= 2(x + \frac{1}{x}).(-\frac{1}{x^2})$$

$$= 2(x + \frac{1}{x})(-\frac{1}{x^2})$$

After some guidance

$$= 2(x + \frac{1}{x})(1 - \frac{1}{x^2})$$

$$= 2(x - \frac{1}{x} + \frac{1}{x} - \frac{1}{x^3})$$

$$= 2(x - \frac{1}{x^3})$$

$$= 2(\frac{x^4 - 1}{x^3})$$

$$(c)y = \left(x + \frac{1}{x}\right)^2$$

$$= (x + \frac{1}{x}).(1 + \frac{1}{x})$$

$$= x + 1 + 1 + \frac{1}{x^2}$$

$$= x^2 + 1 + 1 + x^{-2}$$

$$\frac{dy}{dx} = 2x - 2x^{-3}$$

Preference of the chain rule over expanding the brackets first

I. Can we proceed to 2(d)?
S. All right.
I. Wonderful.

$$(d) y = \frac{1}{1+x^2} - \frac{x^4}{1+x^2}$$

$$= \frac{1-x^4}{1+x^2}$$

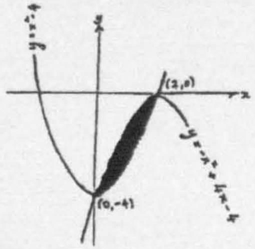
$$= \frac{(1^2 - x^2)(1+x^2)}{1+x^2}$$

$$= 1 - x^2$$

$$\frac{dy}{dx} = -2x$$

Able to use simplification in solving this problem.

(3) Based on the diagram below, compute the area of the shaded region.
Please explain graphically how this might be done.



I. Can youproceed to no 3?
S. Sorry, I can't proceed.
[At this point, the interviewer guided the student to get an answer]

$$(3) area = \left|\int_0^2 (x^2-4)dx - \int_0^2 (-x^2+4x-4)dx\right|$$

$$= \left|\left[\frac{x^3}{3} - 4x\right]_0^2 - \left[\frac{-2x^3}{3} + \frac{4}{2}x^2 - 4x\right]_0^2\right|$$

$$= \left|\left[\frac{x^3}{3} - 4x + \frac{2x^3}{3} - 2x^2 + 4x\right]_0^2\right|$$

$$= \left|\left[x^3 - 2x^2\right]_0^2\right|$$

$$= 8 -$$

Unable to compute the area of the sha region

(4) By considering the function $y = \sqrt{4-x^2}$, sketch and shade the region of the graph

$$\int_0^2 \sqrt{4-x^2}\,dx$$

What is the value of the above integral?

I. Can you proceed to no. 4?
S. Sorry, it is too hard for me.
[*At this point the interviewer guided the student to draw the correct graph and show to the students that the value of the integral could be found by the area*]
I. Do you think will be accepted by your lecturer.
S. I don't think so.
I. Why?
S. Because we to perform the integration.

$$\int_0^2 \sqrt{4-x^2}\,dx$$

$$= \int_0^2 (4-x^2)dx$$

$$= \frac{2[4-x^2]^{\frac{3}{2}}}{3}$$

Reversal of process.

Unable to draw the graph.

Unable to use the area under the curve evaluater the value of he integral.