DUOTONE SURFACES: DIVISION OF A CLOSED SURFACE INTO EXACTLY TWO REGIONS

A Thesis

by

PRADEEP GARIGIPATI

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Approved by:

Chair of Committee,	Jianer Chen
Co-Chair of Committee,	Ergun Akleman
Committee Member,	John Keyser
Department Head,	Duncan M. Walker

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ABSTRACT

In this thesis work, our main motivation is to create computer aided art work which can eventually transform into a sculpting tool. The work was inspired after Taubin's work on constructing Hamiltonian triangle strips on quadrilateral meshes. We present an algorithm that can divide a closed 2-manifold surface into exactly two regions, bounded from each other by a single continuous curve. We show that this kind of surface division is possible only if the mesh approximation of a given object is a two colorable quadrilateral mesh. For such a quadrilateral mesh, appropriate texturing of the faces of the mesh using a pair of Truchet tiles will give us a Duotone Surface.

Catmull-Clark subdivision can convert any given mesh with arbitrary sided polygons into a two colorable quadrilateral mesh. Using such vertex insertion schemes, we modify the mesh and classify the vertices of the new mesh into two sets. By appropriately texturing each face of the mesh such that the color of the vertices of the face match with the colored regions of the corresponding Truchet tile, we can get a continuous curve that splits the surface of the mesh into two regions. Now, coloring the thus obtained two regions with two different colors gives us a Duotone Surface.

DEDICATION

This thesis is dedicated to my parents, Rama Krishna and Surya Kumari.

ACKNOWLEDGEMENTS

Dr. Ergun Akleman guided me well throughout my thesis work. He is very supportive and encouraging. Hadn't he given me the opportunity to work under him towards my thesis, I wouldn't have ventured into the arena of research. I am deeply indebted to him for giving me that chance. I am constantly surprised by his levels of enthusiasm, which is very inspiring for anyone. I would also like to thank Dr. Jianer Chen and Dr. John Keyser for extending their help and support all along my thesis work.

Having friends who boost your mental strength is always a perk. I have that privilege through Naga Raghuveer Modala and Santosh Anant Navale, who have given me invaluable advices every time i needed them. Though separated by large distances there are certain friends who will always influence you. One such friend of mine is Naveen Kumar Gollapudi. It is hard to go through every name because every friend I made has certain amount of influence on me, whether it is bad or good; my journey filled with them led me to this current stage. Hence, I would like to thank all of them.

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1. INTRODUCTION AND MOTIVATION*

In this thesis, we will show that it is always possible to construct a closed curve on a positive genus closed 2-manifold that divides the manifold surface into exactly two regions. We have developed an algorithm that can divide a boundaryless surface into exactly two regions bounded from each other by a single continuous curve. We have also shown that this kind of surface division is possible only if the mesh approximation of a given object is a two colorable quadrilateral mesh. For such a quadrilateral mesh, appropriate texturing of the faces of the mesh using a pair of truchet tiles will give us the Duotone surface of the corresponding object, see figure 1.1 for example.

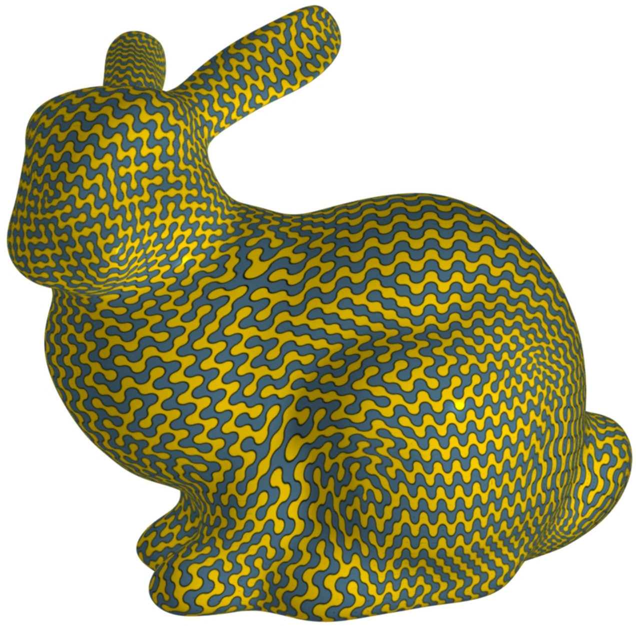

Figure 1.1: Duotone surfaces consist of two regions that are bounded by a single curve that covers the surface.

The key step for creating a Duotone surface is simple closed curve generation on the surface of the object where we want to create the dual toned texture. An example of such a curve is an unit circle. More complex and wiggled curves can also be used as long as the curve does not self-intersect. Such a non-self-intersecting closed curve is known as Jordan curve.

Definition 1.0.1. *A Jordan curve is a plane curve which is homeomorphic(topological equivalent) to the unit circle.*

In other words, the curve is closed and simple. A plane can always be divided into two regions by a Jordan curve, one region enclosed by the curve and the other region which lies outside the curve. This conjecture was first formalized by a french mathematician Camille Jordan as Jordan curve theorem[14]. He also provided a proof mathematically establishing the same conjecture.

The essence of this proposition is that every simple closed curve on a plane divides the plane into one interior region and one exterior region, each bounded from the other by the curve such that any continuous path connecting a point of one region to a point of the other intersects with that closed curve somewhere. Though the theorem statement might sound intuitive at first glance, it is only the case in simple situations where the closed curve is circle. The problem can quickly escalate in it's degree of complexity when the closed curve twists and turns a lot such as fractal curves. The situation becomes much more difficult when no-where differentiable curves like Koch snowflake are used for generating the closed curve. The proof of Jordan curve theorem establishes that there always exists an injective mapping between the complex non-self-intersecting closed curve and the unit circle on a plane, hence proving that a Jordan curve will always divide the plane into exactly two regions with the curve as the boundary.

Other mathematicians, Oswald Veblen and Thomas C. Hales also studied the Jordan curve theorem. Oswald Veblen had re-proved the theorem in 1905 for the original proof lacked an explanation for simple cases such as polygon. Thomas C. Hales stated that the original proof is essentially correct but lacks a detailed and satisfactory explanation, and hence some polishing would make it complete. Jordan curve theorem was also generalized to higher dimensions by Henri Lebesue and Luitzen Egbertus Jan Brouwer in 1911.

When the same problem is considered on three dimensional objects with no boundary instead of a plane, even a simple closed curve like circle cannot divide the surface of the object into exactly two regions. For example, look at the sphere and torus in figure 1.2. A simple circle drawn on the sphere divided it's surface into two regions, however a circle drawn on the surface of the torus couldn't accomplish the same. A surface with non-zero genus immediately changes the initial parameters of the problem. The problem difficulty increases with the increase of genus of the closed surface. The problem of constructing a Jordan curve on three dimensional closed surfaces has not been studied before, hence we intend to explore the topic.

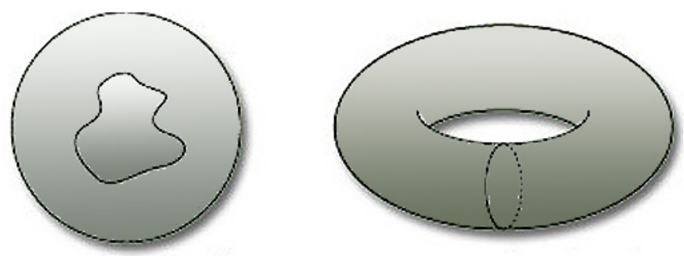

Figure 1.2: An example of how the choice of the closed curve effects the possibility of creation of a Duotone surface for a given object. The sphere surface is divided into two regions, however the torus's surface isn't.

1.1 Objective

Our aim is to show that we can always construct a closed curve on a positive genus closed 2-manifold that divides it's surface into two regions. In other words, we aim to develop a method that can generate a single-simple-closed surface filling curve that divides the given 2-manifold into two regions. One of the most recent works on surface filling curves by Xing et al.[24] is one of the inspirations for our work. Xing's work is built upon Tuabin's work on constructing Hamiltonian triangle strips on quadrilateral meshes[21]. Xing et al. created a single surface filling curve by joining the centers of the triangle strip cycle, however that curve does not necessarily divides the surface into two regions.

We will also analytically prove that constructing such a curve is always feasible. Another important thing we would like to mention is that we will not try to create the whole boundary curve explicitly, rather the boundary will appear by itself as we connect the disconnected regions by appropriately rotating the textures on the individual faces of the quad mesh. Hence, the final result should appear like an embedding of duotone Truchet tiling over surfaces[4]. Duotone truchet tiling is explained more in detail in the related works section.

2. PRELIMINARIES*

This chapter will provide some material on the definitions, notations and facts related to graphs, meshs and surfaces. Catmull-Clark subdivision algorithm and Graph Rotation Systems are also discussed in the sections 2.4 and 2.5, respectively. The concepts of parametric continuity and geometric continuity are also explained in the last section of this chapter.

2.1 Basics of Graph Theory

All the graphs considered in this thesis are simple-connected graphs. The meaning of the terms *simple* and *connected* with respect to graphs is explained in the following section.

2.1.1 Graphs

A graph denoted as $G = (V, E)$ consists of a set V of vertices, a set E of edges and an incidence map I which associates each edge of G with an unordered pair of vertices of G. Hence, a graph can be precisely denoted as $G = (V, E) \ni I(v_1, v_2) = e$ for all $e \in E$ where $(v_1, v_2) \in V$.

Based on the type of edges the graph has, graphs are divided into two types:

- Directed
- Undirected

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For a given graph $G = (V, E)$ and two vertices $v_1, v_2 \in V$ where $I(v_1, v_2) = e1$. If $I(v_1, v_2) = I(v_2, v_1)$, then G is *Undirected*, otherwise it is *Directed*.

In a graph $G = (V, E)$, for a given pair of vertices $v_1, v_2 \in V$ if $I(v_1, v_2) = e1$ where $e_1 \in E$, then v_1, v_2 are known as the End points of the edge e_1. If both the end points are same i.e. $v_1 = v_2$, then the edge e_1 is called a loop. If same end points index to two different edges i.e. $I(v_1, v_2) = e_1$ and $I(v_1, v_2) = e_2$, then the edges e_1, e_{12} are called parallel edges. A graph G which does not have loops and parallel edges is a Simple graph.

Any vertex of a graph which has zero edges incident on itself is known an Isolated vertex.

If v_0 and v_n are vertices of a graph G, then a *Walk of length n from* v_0 *to* v_n is an alternating sequence of vertices and edges,

$W = v_0, e_1, v_1, e_2, v_2, e_3, ..., e_n, v_n$

whose initial vertex is v_0 and the final vertex is v_n for $i = 1, ..n$. If $v_0 \neq v_n$, then W is called an *Open walk*, otherwise it is called a *Closed walk*. For a given open walk if all the vertices are distinct, then it is called a *Path*. Similarly, a closed walk is called a *Cycle* if every pair of vertices except the beginning and ending vertices are distinct. A path that includes every vertex of a given graph is known as a *Hamiltonian path* for that graph. A cycle that includes every vertex of a given graph is known as a *Hamiltonian cycle* for that graph.

A graph is called *Connected* if for every pair of vertices u and v, there is a path from u to v. This leads to a new concept of number of connected components in a given graph. Since there is no restriction on which vertices of the graph have to be connected, there is a possibility for a graph to have more than one connected components.

For a given graph G, if a path uses every edge of the graph exactly once, then such a path is called an *Eulerian path.* Similarly, a cycle that uses every edge of the graph is called an *Eulerian Circuit/Euler tour.* An undirected graph has an Eulerian cycle if and only if every vertex has even degree, and all of its vertices with nonzero degree belong to a single connected component.

2.1.2 Trees

A *Tree* is a connected graph with no cycles.

A Tree T of a undirected connected graph G is called a *Spanning Tree* if the tree consists of all the vertices of the given graph G and a subset of edges of the graph. Intuitive definition is that a spanning tree T of graph G is a selection of edges of G such that resulting tree spans every vertex of the graph. Examples of different spanning trees that can be generated from a graph are shown in the figure 2.1.

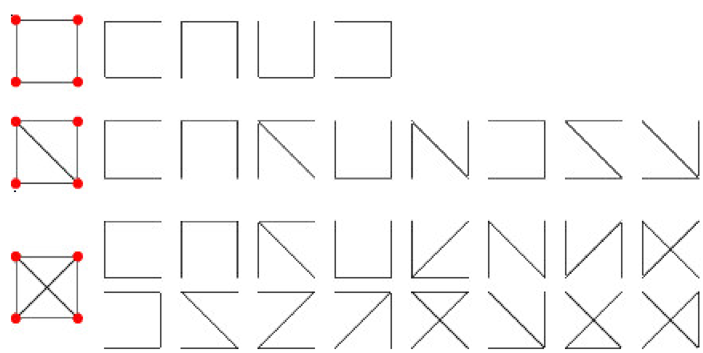

Figure 2.1: Spanning trees of a graph with same number of vertices but different edge bindings.

Addition of one edge from the original graph to the spanning tree creates a cycle known as *Fundamental cycle.*

In the previous section, we have learned that a graph need not be of a single connected component always. So, for example if a given graph has n connected components, then the collection of spanning trees of each connected component of the graph is called a *Spanning forest* of the graph G.

2.1.3 Graph Colorings

An *n-coloring* of graph G, is a mapping from the vertices of G to a set of n colors, such that no two adjacent vertices have same color. Obviously, if the graph have loops, they are not colorable. A 1-colorable graph is a graph with only no edges at all. An example of 1-colorable graph is point cloud. A graph is 2-colorable if and only if it is a bipartite graph. Proof of this statememt can be found as theorem 1.5.3 in [10]. One does not need a mathematical explanation to understand why a bipartite graph is 2-colorable. As we know, vertices of bipartite graph can be separated into two sets such that the edges of the given graph only pass from vertices of one set to another but not among the sets. Hence, by definition each set can be colored by using a single color. Therefore, if for a given graph has one-to-one mapping with a bipartite graph of same number of vertices, then the graph is 2-colorable. See figure 2.2 for an example.

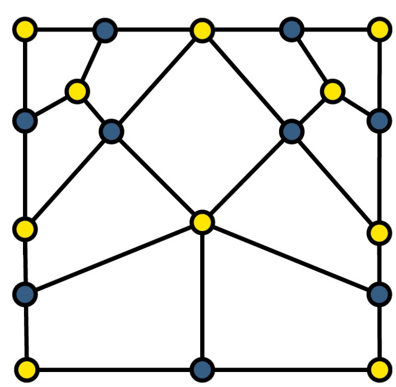

Figure 2.2: Blue vertices belong to one set, and Yellow vertices belong to the other set of given graph G.

In this thesis, 2-colorability of a given mesh is very important to achieve our target duotone art. We induce this property into the mesh by sub-dividing the given original mesh using Catmull-Clark subdivision described in section 2.4. How this particular subdivision method induces the required property into the mesh is also explained in the section 2.4.1.

2.1.4 Connectivity

An edge cannot exist without end points, therefore removal of a vertex from a graph implicitly indicates the removal of the edges incident on that vertex. If removal of a vertex can increase the number of connected components in the graph, then such a vertex is called a *Cut point*. The *Connectivity* of a graph G is defined as the minimum number of points whose removal results in a disconnected graph. Say the number of such points is denoted by K, then the graph is called k-connected. A k-connected graph is also $k+i$-connected where $i = 1, 2, 3....$

2.2 Surfaces and Imbeddings

A *Closed surface* is a compact boundaryless 2-dimensional manifold, which can be either orientable or non-orientable [10]. A surface is orientable if there is a consistent local frame of reference at every point on the surface. In other words, a surface is orientable if a consistent concept of clockwise rotation can be defined on the surface in a continuous fashion.

Closed surfaces like sphere, torus, double torus, and so on, as illustrated in figure 2.3, are orientable. These surfaces are denoted as S_i where $i = 0, 1, 2..$ indicating the number of holes in the closed surface. It had been proved that every closed connected orientable surface is homeomorphic to one of the S_i, one can find the proof of this in chapter 3 of the book [10]. One important aspect of closed orientable surfaces is that they can be obtained by adding handles to sphere in 3-space.

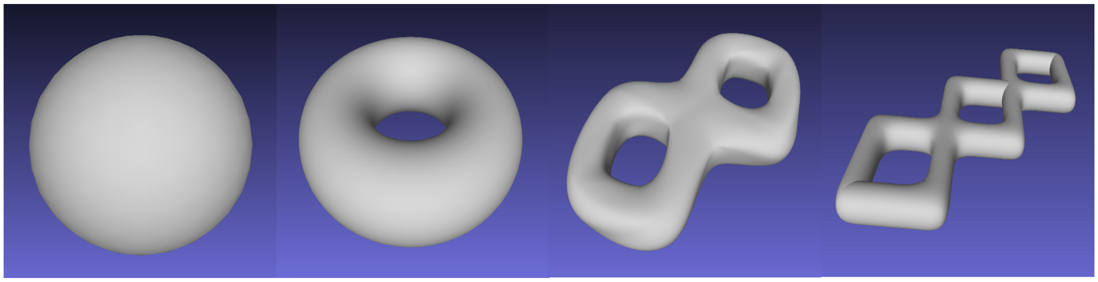

Figure 2.3: (a) Sphere - S_0, (b) Torus - S_1, (c) Double torus - S_2, and (d) Triple torus S_3 and so on ....

This leads us to the concept of *Genus* of an orientable surface. It is defined as the number of handles required to be added to a sphere in 3-space to create the homeomorphism type of the given orientable closed surface. Thus, the subscript i in S_i which we used to represent the i-tori surface is the genus of the orientable surface S_i.

A graph can be drawn without edge crossings on a closed surface. This is done by creating handles for every potential edge crossing on the surface. For a detailed process on how to create these handles refer to section 1.4.1 in [10]. If a connected graph is imbedded in a orientable closed sphere, then the complement of its image is a family of regions/faces, each with a potential of being homeomorphic to an open disk. If all such regions are homeomorphic to open disks, then such an imbedding is called a *2-cell imbedding*. For a given graph G and closed surface S, the imbedding is representated as $i : G \rightarrow S$. Intuitively the mapping indicates one-on-one relation between the regions on surface and the faces of the graph.

All the imbeddings considered in this thesis are *2-cell imbeddings* i.e. the graph that represents the mesh approximation of a given closed $3D$ surface is a 2-cell imbedding of the graph on the surface.

2.2.1 Principle of Duality

Given a connected graph G, a closed surface S, and 2-cell imbedding $i : G \to S$, for each region f of the imbedding, place a vertex f^* in its interior. Now, for each edge e of graph G, create a new edge e^* connecting the vertices in the regions edge e is present. The resulting graph $G^* = \{f^*, e^*\}$ is called the *Dual graph* for the imbedding $i : G \to S$. The new imbedding due to G^* on the surface S is called the *Dual imbedding*. Here, since we started with graph G and ended up with G^*, G, and G^* are known as *Primal graph* and *Dual graph*, respectively. The roles of these graphs can be easily reversed because we will end up with G if we start with G^*. An example of Duality is shown in the figure 2.4

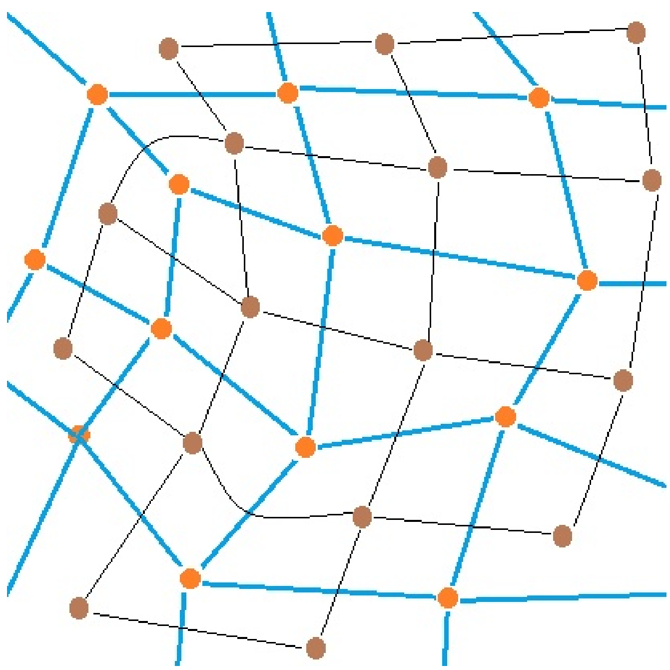

Figure 2.4: Graph with orange vertices and blue edges is the *Primal graph*, and the graph with brown vertices and black vertices is the *Dual graph*.

2.3 Meshes

A polygon mesh is defined by the position of the vertices(geometry); by the association between each face and its sustaining vertices (connectivity); and optional colors, normals and texture co-ordinates (properties). The graph of a mesh is the graph defined by the mesh vertices as graph vertices, and the mesh edges as graph edges. The meshes considered in our work have neither isolated vertices nor multiple connected faces (faces with holes). Two faces of a graph representing given mesh are connected if they are the ends of a path in the dual graph of the given graph.

2.4 Catmull-Clark Algorithm

Recursive patch subdivision algorithms are used extensively in computer graphics to convert a given polygonal mesh into a much coarser mesh that better approximates a smooth surface. Catmull-Clark subdivision [8] is one of the such methods used to generate subdivision surfaces. It is a recurvise refinement process in which a each face of the mesh is further divided into more sub-polygons such that the resulting mesh smoothly represents the orignal face across the edges of the face. Given below are the rules which guide the Catmull-Clark subdivision process. We start with an arbitrary polyhedron mesh, all the vertices of this mesh will be called original points.

1. Add a face point for each face and compute the face point as centroid of all original points for the respective face.

2. Add an edge point for each edge and compute the edge point as average of the two neighbouring face points and its two original endpoints.

3. Add an edge from face point to every edge point of the respective face.

4. Move each original point to the point $(F + 2R + (n - 3)P)/n$ where

Point F is average of all(say n) face points for faces touching P

Point R is average of all n edge midpoints for edges touching P, where each edge midpoint is the average of its two endpoint vertices.

The new mesh will consist only of quadrilaterals. Repeated subdivision results in smoother meshes.

2.4.1 2-Colorability of Subdivided Meshes

Catmull-Clark subdivision is fundamentally an interative process which operates on faces of the mesh i.e. it is a local operation that results in global change. Each face be it any n-gon, is replaced by a single vertex. A new vertex is created for every edge of the original face as computed in previous section. Now, if we do a walk covering the face-vertex, modified original vertex, and two edge-vertices which lie on adjacent edges in original graph, we essentially covered the vertices of a new face, which is one among the other faces created due to application of subdivision. Observe that the face-vertex and the modified vertices of the original face are not connected by paths of length 1. Now, put the new edge-vertices into $bin1$ and rest of the new vertices into $bin2$, repeat the same for rest of the original faces. Finally, we will have all the vertices of new graph divided into two sets, thus converting the original mesh graph into a bipartite graph. We know already know from 2.1.3 that all bipartite graphs are 2-colorable, hence the new subdivided mesh is also 2-colorable.

2.5 Graph Rotation System

Definition 2.5.1. *Let G be a graph with n vertices. A rotation at a vertex v of G is a cyclic ordering of the oriented edges originating at v. A (pure) rotation system of the graph G consists of a set of n rotations, one for each vertex of G.*

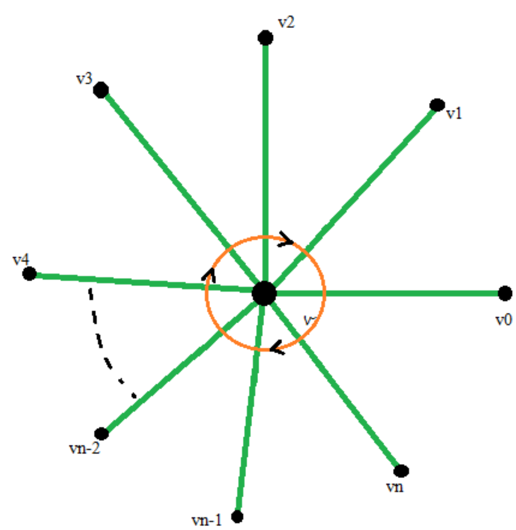

Figure 2.5: Clockwise ordering of edges incident on a given vertex

According to theorem 3.2.2 in [10], every rotation system on a graph G defines a unique locally oriented graph imbedding $G \to S$. A simple example of what graph rotation system represents is shown in the figure 2.5 An illustration of how the imbedding of a graph G on a surface S is extracted given a graph rotation system on a graph G is shown in figure 2.6

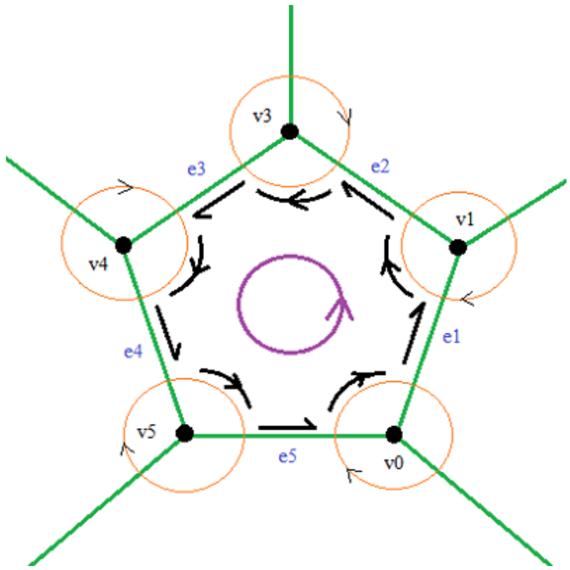

Figure 2.6: Visual explanation of how the boundary walk of a face is achieved using graph rotation systems.

Given a single vertex(say v_0 as shown in figure 2.6) that belongs to some face, by simply tracing along the edges as shown in the figure 2.6 i.e. $v_0 \rightarrow e_1 \rightarrow v_1 ...$, we can go to all the other vertices of the same face. Using this method, for every edge that is out-going from the vertices of the current face, if we choose a face which is not the current face then we essentially move to the faces which are adjacent to the given face. Thus, this process when applied recursively will help us move over the entire mesh.

2.6 Geometric Continuity and Parametric Continuity

Studying the continuity of derivatives is the conventional method of analyzing the smoothness of curves and surfaces.

A curve has C^n continuity if the value of $\frac{d^n s}{dt^n}$ is continuous where s is the arc length and t is the parameterization. n is called the order of parametric continuity. To understand what happens where the curves join with C^n continuity, some examples are provided below.

- C^0 means, the curves join at the point and they don't have any derivative continuity what so ever.
- C^1 means, the first order derivatives $\frac{ds}{dt}$ are equal at the point where curves join.
- C^2 means, the second order derivatives $\frac{d^2s}{dt^2}$ are equal at the point where curves join.

Geometric continuity is more relaxed form on smoothness measure which emphasizes more on the matching of shape of the curves where they join. For example, if two curves are G^0 continuous, they curves meet at a join point. If two curves are G^1 continuous, then the curves respective tangents have same direction at the join point. For G^2 continuity, it is the match of center of curvature at the join point.

A curve is G^n continous if it can be reparameterized to have C^n continuity, but the converse is not true. The reason behind it is trivial. Reparameterization of the curve does not alter the curve shape i.e. there is no geometrical change in the curve.

2.6.1 Geometric Continuity of Subdivided Meshes

We use Catmull-Clark subdivision for our purposes in the thesis, hence the continuity of Catmull-Clark subdivided meshes will only be discussed here.

Catmull-Clark subdivision is generalization of bi-cubic B-Splines, therefore the resulting mesh will be

- C^1 continuous at vertices of valence$\neq 4$(known as extraordinary vertices).
- C^2 continuous everwhere else.

Hence, the subdivided mesh will be G^2 continous everywhere except at the extraordinary vertices where it is G^1 continous. For more details on how this continuity is achieved by Catmull-Clark subdivision, please refer to [8].

3. LITERATURE REVIEW AND RELATED WORK*

In this chapter, I will discuss the earlier works which are related to our work. The discussion is split into three sections, each presenting one of the following works.

- Space Filling Curves
- TSP Art
- Truchet Tiling

3.1 Space Filling Curves

Space-filling curves, which are discovered by Giuseppe Peano[18] by his construction of a continuous mapping from the unit interval onto the unit square, are a mapping from one-dimensional space to a multi-dimensional space. A space filling curve is like a thread that passes through every cell element in the multi-dimensional space so that every cell is visited at least once.

Peano's goal was to illuminate Cantor's earlier counterintuitive result that set of points in a unit interval has the same cardinality as the set of points in the unit square. Peano's ground-breaking work did not contain any illustration of his construction [18]. David Hilbert provided first visual presentation of space filling curve construction [11]. His construction is now known as the Hilbert curve [6].

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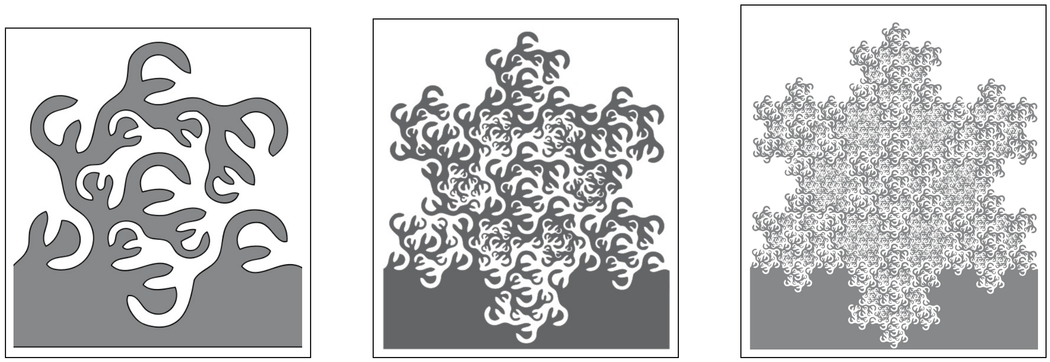
Space filling curves have became very well-known among mathematician/artists after Benoit Mandelbrot's seminal work on Fractal Geometry [15]. In his book, he categorized space filling curves as fractals since they can be constructed using a replacement algorithm starting from a simple shape. Mathematician and artist Douglas McKenna [16], who also created many images in Mandelbrot's Fractal Geometry of Nature, also discovered one space-filling curve. Wittin and Wyvill [23] presented an algorithm to generate space filling curves. Velho and Gomes [22] used space filling curves to obtain halftoning. Space filling curves are not only popular in art and mathematics, because of their clustering properties they are used in wide variety of applications such as context sensitive scanning, multidimensional indexing and spatio-temporal databases [9, 17].

Hilbert, Peano or McKenna curves are not the only space filling curves. McKenna [16] enumerated over 20 million new space-filling recursive designs. Ken Knolton [13] created a portrait of Douglas McKenna using one of the first space filling curves McKenna discovered. Most of existing examples of space filling curves are in 2D. A remarkable exception is Carlo Sequin's stainless steel and bronze sculpture called Hilbert Cube 512 [20]. This sculpture is a closed (thickened) curve that fills the volume of a cube.

The rest of this section contains the discussion of Mandelbrot's work and Peano's work with illustrated examples



(a) (b) (c)

Figure 3.1: An example of Mandelbrot's Duotone space filling curve art. This particular sequence of images are created by Alexis Monnerot-Dumaine under the pseudoname Prokofiev in 24 January 2010.

To understand how Mandelbrot created the above shown art work, it is important know about the fractal curve, Koch snowflake. Koch snowflake starts as an equilateral triangle. At each iteration of the fractal curve generation, every straight line is divided into three equal parts and the middle segment is replaced with an equilateral triangle of side length equal to this segment. Now, the middle segment is removed, creating a pointy head kind of shape as shown in figure 3.2(b). Repeating the same procedure to every straight line we see in each iteration will give us the Koch snowflake shown in figure 3.2(f). The iterations need not stop here and can be carried out infinitely.

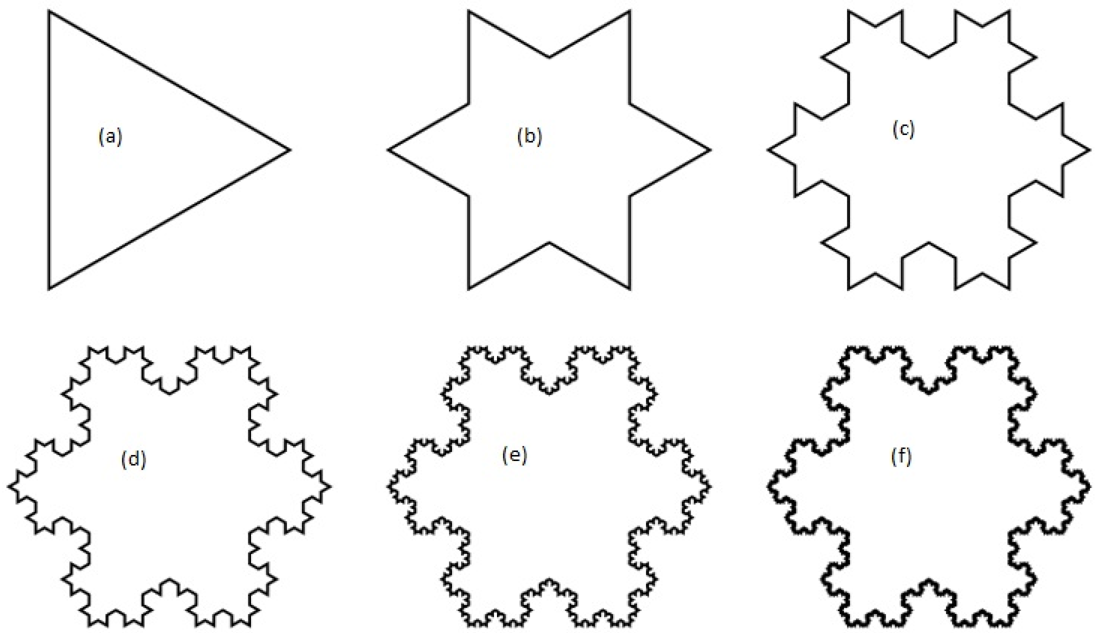


Figure 3.2: Koch snowflake generation. The picture shows iterations 0-5 in figures $(a), (b), (c), (d), (e), and(f)$

Replace hard straight lines with smooth curves as shown in figure 3.3(b) and embed the shape shown in figure 3.3(b) inside the new snowflake. This will be used a base shape for further iterations. Look at figure 3.4(a) to have an idea of how the next iteration with the new snow flake would look like. Shade the different regions of this snowflake using black, gray and white as shown in the figure 3.3(b).

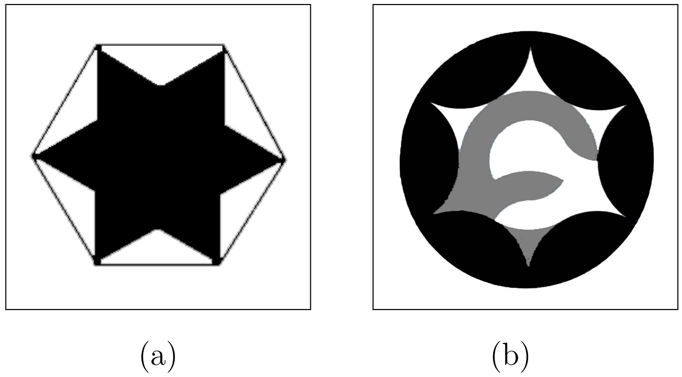


Figure 3.3: First two steps of Mandelbrot's surface filling curve

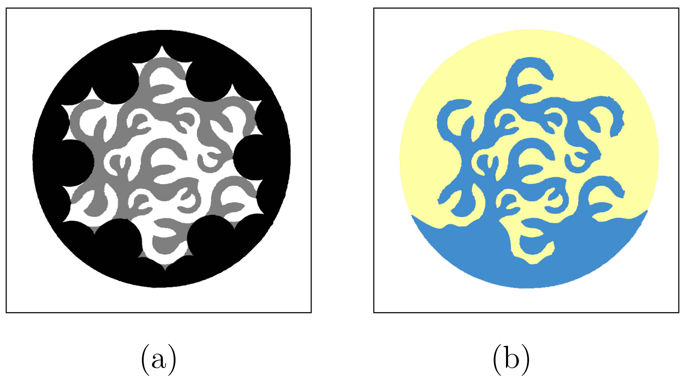


Figure 3.4: Final stages of Mandelbrot's surface filling curve

A simple trick was developed by Mandelbrot to create closed space filling curves on plane by assuming they are created on a sphere. Instead of using the gray scale colors to shade the regions of the snowflake, we shall use two colors blue and yellow to represent inside and outside as shown in figure 3.4(b). Now, run sufficiently large number of iterations to get a complex snowflake pattern. At the end of each iteration, color the region not occupied by hook like shape with yellow color. At the end of iterative process, place the curve generated on a sphere which was already colored in solid yellow color. The resulting texture on the sphere is a complex pattern of blue region enclosing the yellow region or vice versa. Refer to figure 3.4(b) to visualize the whole idea of how his technique works. I personally used the blue and yellow colors here to provide better visual cues, however the original work of Mandelbrot has gray scale colors to shade the snowflake regions.

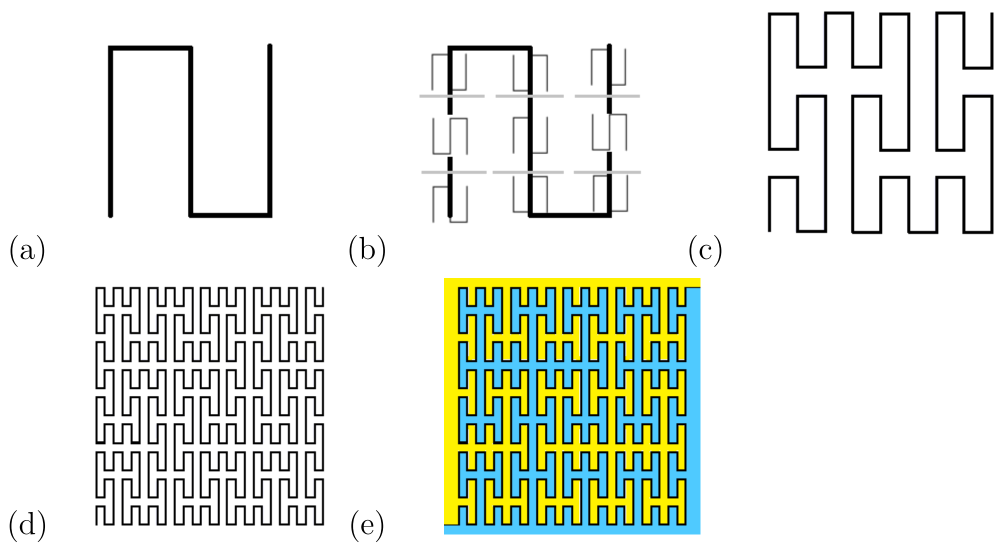


Figure 3.5: (a), (b), and (c) shows the process of mirroring which is used to create this Peano curve. (e) shows how a Peano curve divides the plane into two regions.

The following pictures show how a Peano curve can be constructed from a basic shape shown in figure 3.5(a). Peano provided a very rigorous proof without any graphical illustrations on how a curve constructed in specific fashion can occupy the whole two dimensional space. The fundamental idea is to mirror a given basic shape along the edges of the basic shape, then applying same operation on the result of each iteration recursively. This process is shown in the figures 3.5(a), 3.5(b), and 3.5(c) in respective order. Figure 3.5(d) shows the another iteration of the same process. One more important aspect to notice is that, Peano curves implicitly divide the plane into two regions, thus two colorable as shown in figure 3.5(e).

Another artist cum Mathematician, Douglas McKenna[16] created over 20 million new space-filling recursive designs using Mandelbrot's Fractal Geometry of Nature. Ken Knolton[13] used one of initial space filling curves McKenna discovered to create a portrait of Douglas McKenna.

Advantage of space filling curves over other fractal curves, which we are interested in, is that space filling curves inherently divides the space into indistinguishable inside and outside structures thereby creating Duotone art, as shown in Figure 3.1.

3.2 TSP Art

Robert Bosch and Adrianne Herman invented a different curve generation method known as Traveling Salesman Problem(TSP) art[2], see figure 3.6. In TSP art, a set of points are created to represent cities. Now, a traveling salesman has to visit every city once and then return to home finally. The salesman might want to minimize his tour length which in fact is a closed curve that divides the planar space into two regions. Similar to space filling curves, these curves also cannot guarantee two regions on closed 2-manifold surfaces.

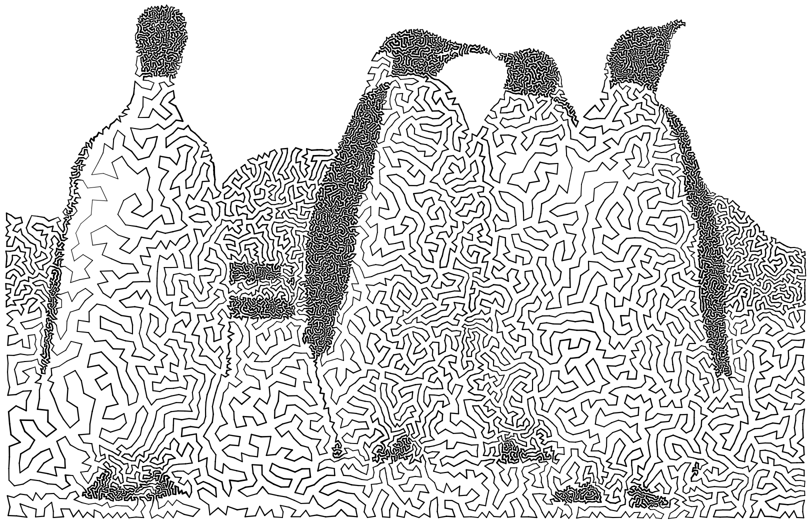


Figure 3.6: An example of TSP Art.

Bosch and Herman[3] observed that for a particular arrangement of the city positions, the piecewise curve that joins the cities on the salesman's itinerary has an aesthetic outlook. This method of creating art work has its own set of disadvantages. For example, the method requires a grid of points to create an original artwork, and it needs humongous number of points to create a good quality art work.

Craig S. Kaplan[12] introduced a different approach of choosing the city locations which uses weighted Voronoi stippling. The usage of weighted Voronoi stippling allowed TSP art to circumvent the requirement of huge point data set to create a better quality art work. This method of choosing city positions has an added advantage: the optimal tours are now are guaranteed to be closed simple curves. Therefore, TSP art which uses weighted Voronoi stippling for creating the city positions can always be colored using two colors to create Duotone plane art. See figure 3.7 for an example of it.

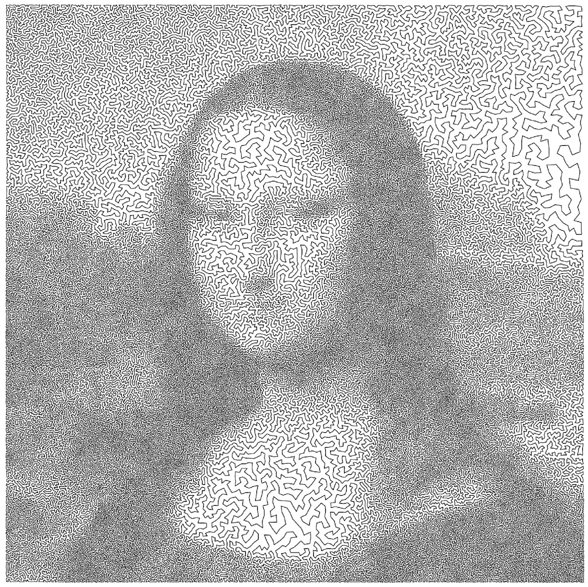


Figure 3.7: February 2009: Robert Bosch created a 100,000-city instance of the traveling salesman problem (TSP) that provides a representation of Leonardo da Vinci's Mona Lisa as a continuous-line drawing.

3.3 Truchet Tiling

One of the oldest and most related works was carried out by Sebastien Truchet. His work involves usage of Truchet tiles (see figure 3.8) to create art work that consists of all possible patterns formed by tiling (see figure 3.9) of right triangles oriented at the four corners of a square.

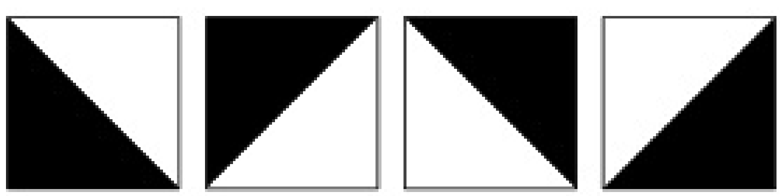


Figure 3.8: All possible orientations of a Truchet Tile

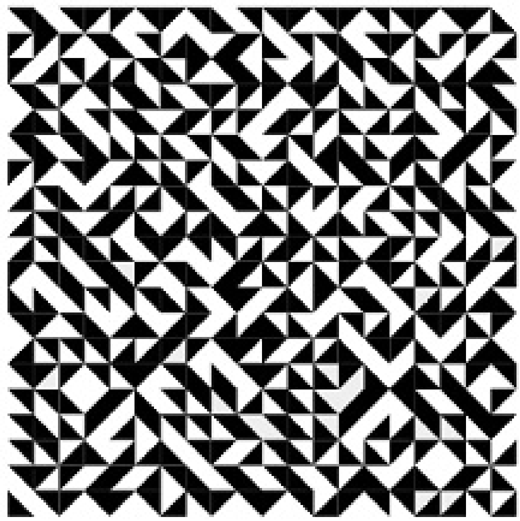
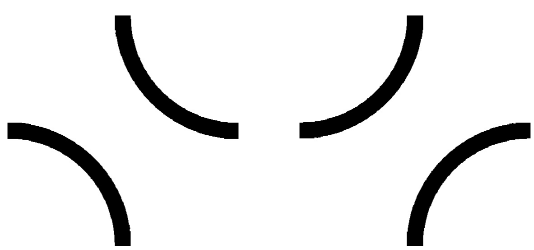


Figure 3.9: An example of Truchet tiling.

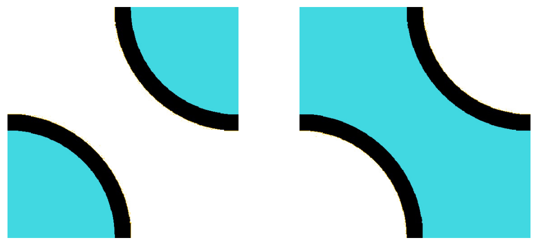
A modified version of the Truchet tile (see figure 3.10) which is orientable was used by Pickover[19] to create duotone art on planar surfaces. This new tile constitutes of two circular arcs centered at the opposite corners of the square tile with radius equal to half the tile edge length. Tiling of the plane randomly using this tile gives visually interesting patterns[5, 4]. Cameron B. Browne is another computer graphics researcher who used this modified Truchet tile to create interesting art work[5, 4]. We plan to use this modified tile in our work to texture the faces of the mesh to create duotone surfaces.



(a) (b)

Figure 3.10: Modified version of Truchet tile with two possible orientations.

Some examples of the art works of Cameron Browne, which use the modified Truchet tile(see figure 3.11) are given below. He colored the tiles with two colors and tiled the plane in various patterns to create Duotone art.



(a) (b)

Figure 3.11: Square Truchet tiles used by Cameron Browne.

Browne also adapted the square Truchet tile pattern to hexagonal faces and used it for creating Duotone art work. An example of hexagonal Duotone Truchet tiling is shown in figure 3.13.

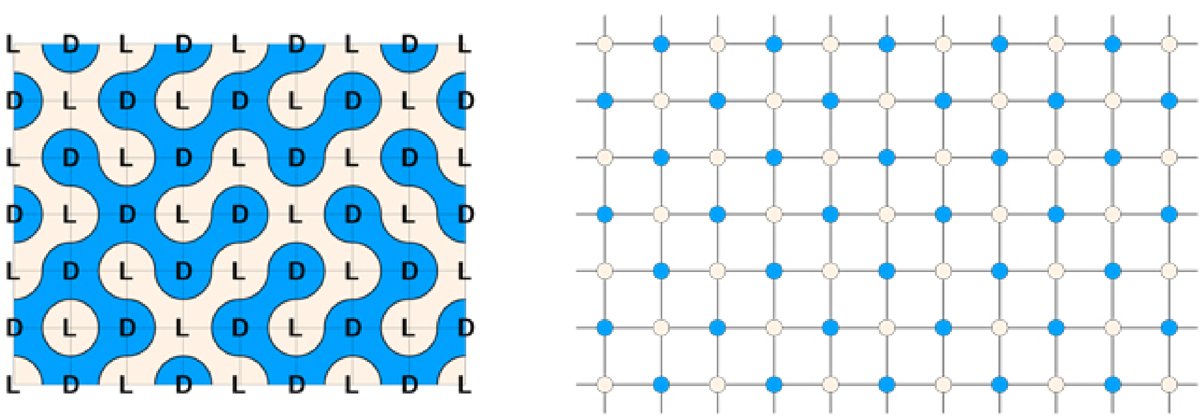


Figure 3.12: Square Truchet Tiling.

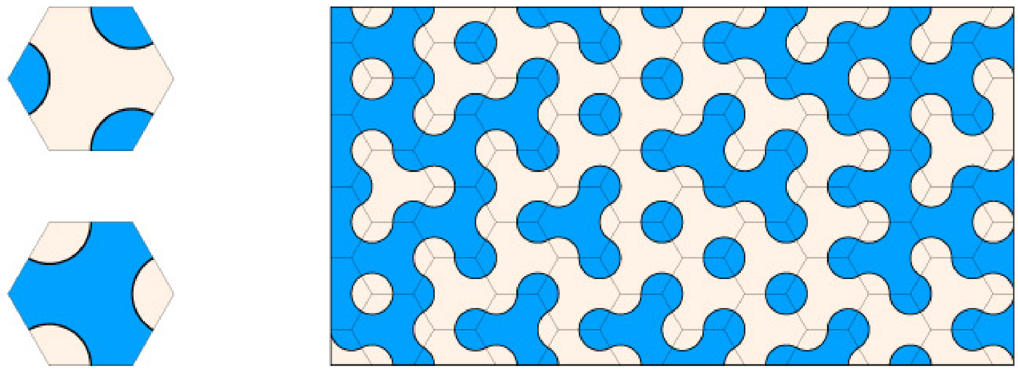


Figure 3.13: Hexagonal Truchet Tiling.

When we apply any of the above discussed art works as textures to three dimensional objects, the mapped textures are either stretched-out or squeezed-in due to the disparate scales of the texture and the object. Also, matching the texture patterns accross the edges for such art works is a very painstakingly labrious job. Using our method, the polygons of the mesh are individually textured using specially designed tiles, and then the textures mapped to the polygons are rotated appropriately to create a pattern on the surface of the object. Thus, our method doesn't suffer from any of the previously stated problems.

4. METHODOLOGY AND RESULTS*

This chapter covers the core portion of the procedure required to create Duotone Surfaces. It is divided into four sections, each explaining the steps in process of Duotone Surface construction in detail. The third section has a subsection titled "Analytical Proof" which mathematically establishes the feasibility of creating a Duotone Surface for a given closed 2-manifold surface. Below are the brief descriptions of the four stages of our algorithm for generating duotone surfaces.

- Convert the input mesh to a 2-colorable quadrilateral mesh.
- Color the quadrilateral mesh with two colors and assign textures to its faces.
- Connect the disconnected regions on the surface.
- Convert the modified two-colorable mesh into a subdivision surface to obtain G^1 continuity.

Refer to Algorithm 1 for a formal algorithm procedure.

The key idea of our process to be able to construct a Jordan curve on closed 2-manifold is to convert the mesh to a 2-colorable graph. This is achieved by subdividing the original mesh using Catmull-Clark subdivision. Once subdivided, the mesh is two colorable as explained in section 2.4.1 and hence a bipartite graph. Now, we choose edges of one color(BLUE or YELLOW, explained in detail in section 4.3)

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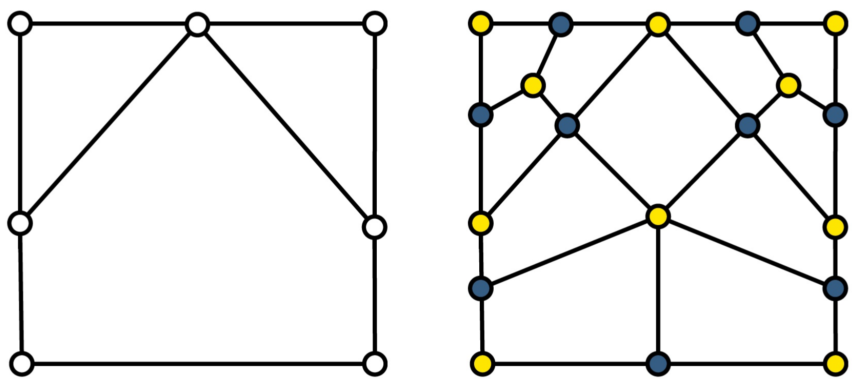
and form a spanning tree that connects all the vertices of corresponding color. While connecting the edges during spanning tree construction, we assign textures to every face we visit and rotate them accodring a predefined criteria, thereby creating a curve(the curve is a part of the texture we put on each face) that bounds the vertices of same color into one region. Irrespective of the genus of the object, we are guaranteed to be able to construct one spanning tree that connects all the vertices of one color thereby leaving all the other color vertices connected but with some closed loops.

Algorithm 1 Two Region Duotone Surface Construction.

```
Convert input mesh into a 2-colorable quadrilateral mesh, G = (V, E), by using
a subdivision schemes such as Catmull-Clark.
Color the vertices in V to either BLUE or YELLOW such that no edge exists in
E whose end vertices have same color. Say, U_0 = {v ∈ V : color = YELLOW}
and U_1 = {v ∈ V : color = BLUE}.
Assign a Truchet tile(texture) to each quadrilateral face of G such that the tex-
ture is consistent with vertex colors
if All faces in G are now like 4.4(a) then
   Mark the mesh indicating the same.
else if All faces in G are now like 4.4(b) then
   Mark the mesh indicating the same.
end if
while Any disconnected vertices in U_0 or U_1 do
   Pick a face which is not of target triangulation(marked in previous step).
   Assign the other Truchet tile to this face such that the texture map is consistent
   with its vertex colors.
end while
Convert the polygonal mesh into a subdivision surface to obtain G^1 continuity
for final rendering.
```

4.1 Convert Input Mesh into a 2-Colorable Quadrilateral Mesh

We plan to use Catmull-Clark subdivision scheme to obtain a 2-colorable quadrilateral mesh variation of the original mesh. Refer to figures 4.1(a) & 4.1(b) for a quick review of how Catmull-Clark subdivision process works.



(a) (b)

Figure 4.1: Conversion of a given mesh into a 2-colorable quadrilateral mesh by using a subdivision schemes such as Catmull-Clark.

Two important modifications happen to mesh after subdivision:

- Though original vertices are preserved, new vertices are inserted on the faces and the edges of the given mesh. Edge vertices are created by averaging the end points of the corresponding edge. Face vertices are created by averaging the vertices that form the face.

- New set of edges are created over each face by joining the face vertices to the edge vertices of the corresponding face.

This process converts each face into a bunch of quadrilaterals which are 2-colorable. Every such face when adjacent to one another are still 2-colorable, thus making the whole mesh 2-colorable with all faces as quadrilaterals.

4.2 Color the Vertices of Quadrilateral Mesh and Assign Textures

Assume the two colors used to color the mesh are Yellow(Y) and Blue(B). Since the output of previous step is a 2-colorable quadrilateral mesh, we can readily color it's vertices using Y & B colors.

(a) (b)

Figure 4.2: Colored Truchet tiles. The black boundary between the two colored regions will form a closed curve that bounds the regions from one another.

Now, we can assign a texture 4.2(a) or 4.2(b) to each face of the quadrilateral mesh such that the vertex color matches the color of the arc-region on the texture for all the faces incident on that particular vertex. This results in a surface with a texture resembling that shown in the figure 4.3.

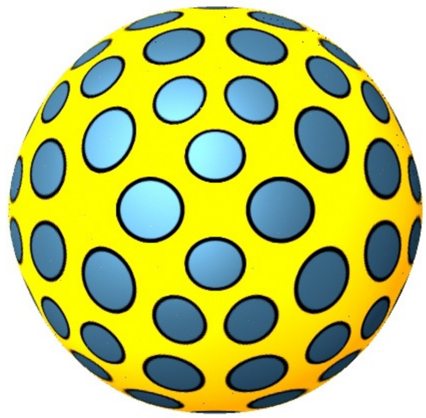
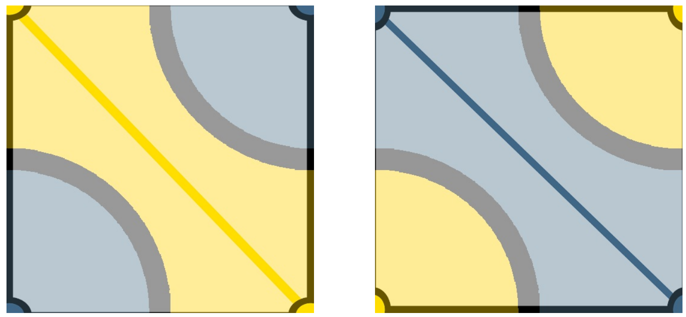


Figure 4.3: First look of a mesh after 2-coloring and texture assignment.

4.3 Connect the Regions of Same Color

If one observes carefully, we can find two potential graphs joining the same colored vertices of the quadrilateral mesh. The edges of this graph are the diagonals of the faces of the quadrilateral mesh. Another interesting observation is that the quadrilateral mesh is a bipartite graph with vertices of each color as the edge disjoint vertex sets. Let's say, $U0$ is the set of all yellow vertices and $U1$ is the set of all blue vertices. As we have noticed earlier, we have two more graphs apart from the given quadrilateral mesh(which is also a graph) and these graphs are called Yellow and Blue graphs since each graph has only edges of it's own color. See figure 4.4 to get idea of how the truchet textures correspond to a yellow or blue edge on a face. To connect the regions of same color, we need the blue and yellow graphs to span all the vertices of their respective color. For example, if yellow graph is disconnected, then it means region of blue is surrounding the isolated region of yellow which in turn means that we have more than two regions though they are of two colors only. See the algorithm 1 for overall procedure of construction of duotone surfaces. In the following section, we analytically prove the feasbility of construction of Duotone Surfaces.



(a) (b)

Figure 4.4: Correspondence of a Truchet tile with yellow or blue edge.

4.3.1 Convergence of the Algorithm

We already known through Taubin's work [21] that we can always represent a connected manifold quadrilateral mesh without boundary using a single Hamiltonian triangle strip. This is done by splitting each face along its diagonal and then connecting the resulting triangles along original mesh edges. This work was later used by Xing et. al.[24] to construct surface covering curves. Following sequence of images show how they created surface covering curves. Then we show how we adapted the process to our purpose to create Duotone Surfaces.

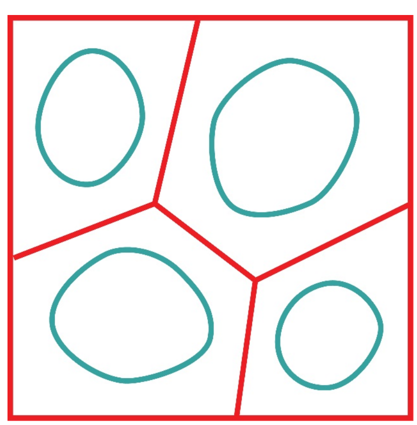


Figure 4.5: Each face of the original mesh turns to closed curve.

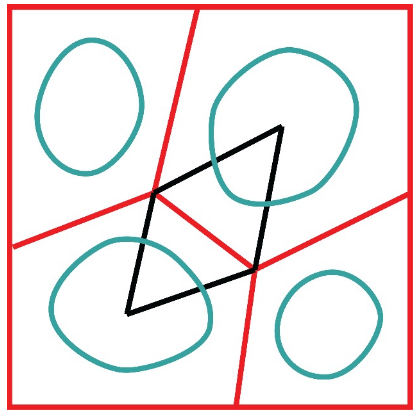


Figure 4.6: We select an old edge and flip it.

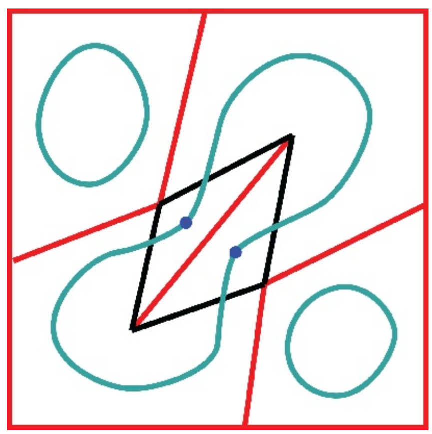
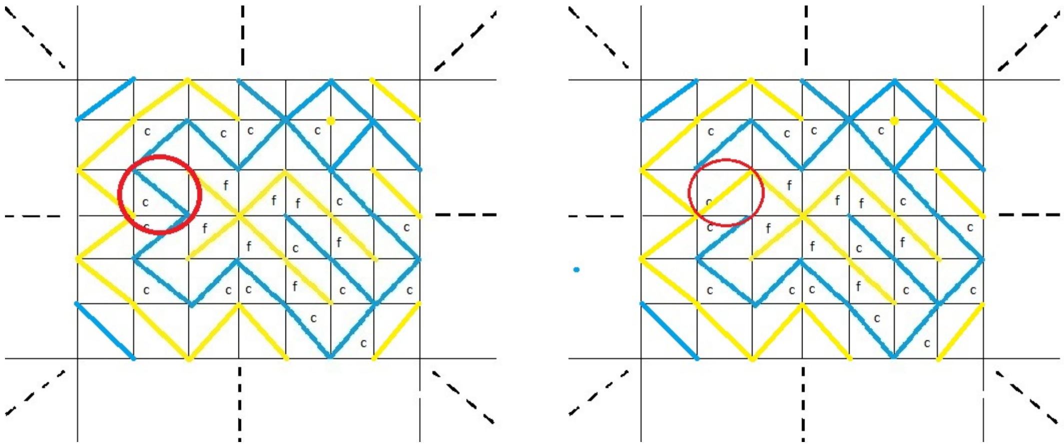


Figure 4.7: Each flip either connects/disconnects two curves, here it connected previously disconnected curves.

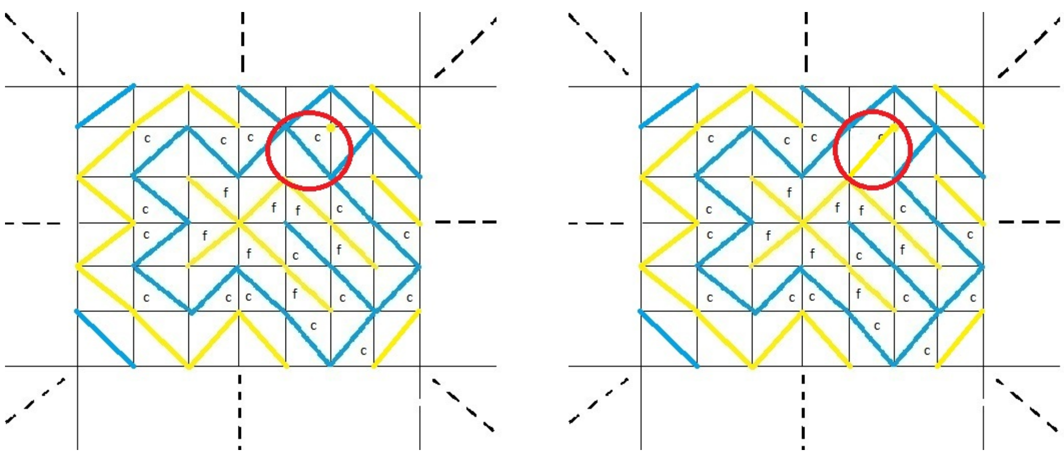
As shown in the figures 4.5, 4.6, and 4.7, by flipping the edges each closed curve inside a face is connected to it's adjacent closed curve. By keeping track of all connected faces, they build a single closed curve that covers the surface.

We have observed that if the initial mesh is two colorable then by coloring the vertices with two colors and then texturing the faces using Truchet tiles created by Cameron Browne [5] we can create Duotone look on the surface of the closed 2-manifold. We show in the figures 4.8, 4.9, and 4.10 how to connect the disconnected vertices of same color.



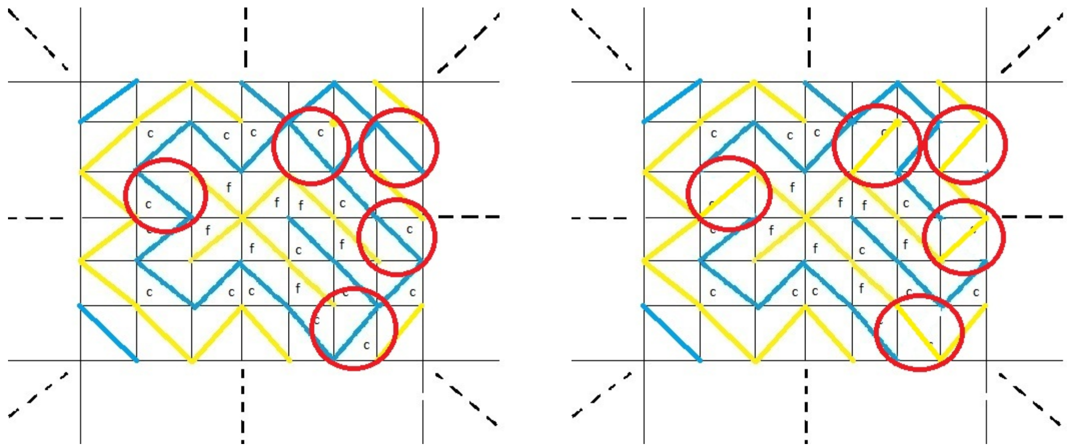
(a) (b)

Figure 4.8: The yellow and blue graphs are connected/disconnected by replacing an edge of one color with an edge of the other color on a face.



(a) (b)

Figure 4.9: Isolated vertices of yellow/blue graphs can be connected/disconnected by replacing an edge of one color with an edge of the other color on any of the faces incident of the isolated vertex.



(a) (b)

Figure 4.10: One way of connecting the disconnected components of yellow/blue graph.

Now, the trick is to start with a specific configuration if we want to end up with a Duotone surface. Once we colored the mesh with two colors, the textured quadrilateral mesh looks like 4.11 as already explained in 4.2 which is our initial configuration.

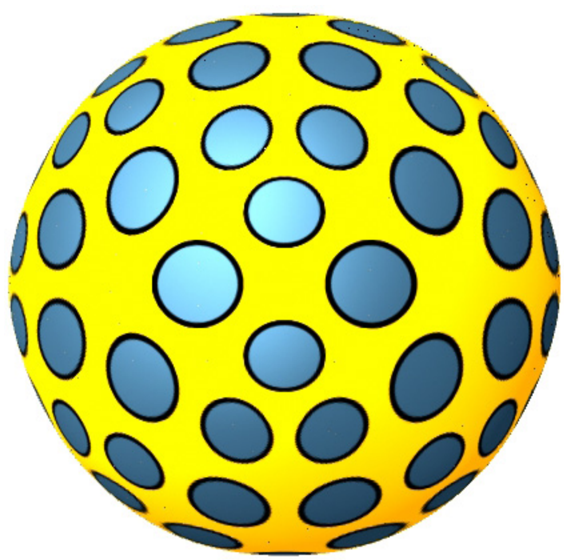
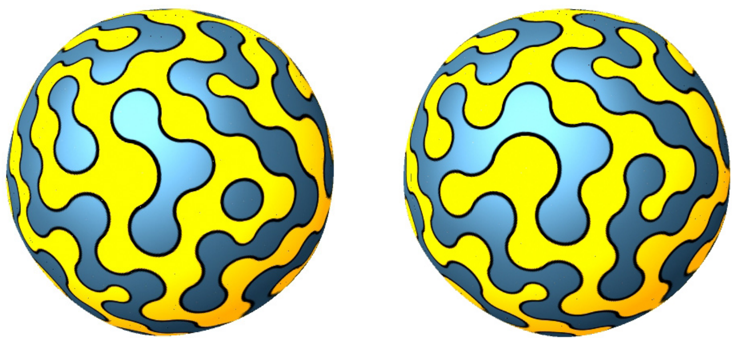


Figure 4.11: Each vertex of the original mesh forms a closed curve.

In the next step, we connect all the disconnected graphs as much as possible. If we have any isolated vertices, we can connect them as well as shown in 4.9. The results of these two steps look as shown in figures 4.12(a) & 4.12(b).

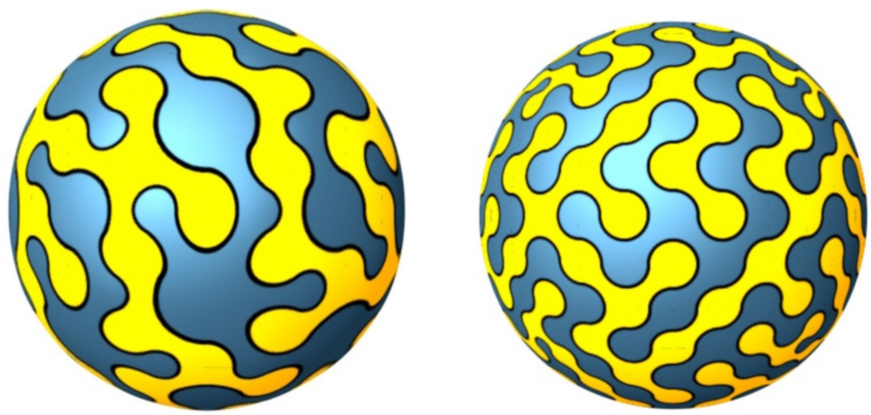


(a) (b)

Figure 4.12: Intermediate - (a) and final - (b) results of the algorithm.

4.4 Conversion to a Subdivision Surface to Obtain G^1 Continuity

Polygonal meshes are not G^1 continuous across edges of the faces i.e. they are not differentiable along the edges and vertices. One way to turn the G^1 discontinuous mesh into a G^1 continuous is to use subdivision schemes. One such method which we already employed in our algorithm to obtain 2-colorable quadrilateral meshes is Catmull-Clark subdivision. Catmull-Clark subdivision surfaces are G^2 continuous except at extraordinary vertices. Since our textures have same color around the vertices, discontinuities at the extraordinary vertices will not be visible to human eye. So, it is ok to use Catmull-Clark subdivision. Also, Truchet textures are also G^1 continuous at edge boundaries because the two arcs subtended by any corner meet at the same point along the edge (see figure 4.2). An example of final result is shown in figure 4.13.



(a) (b)

Figure 4.13: Duotone surfaces of a sphere mesh approximation which uses (a) hexagons and (b) octagons.

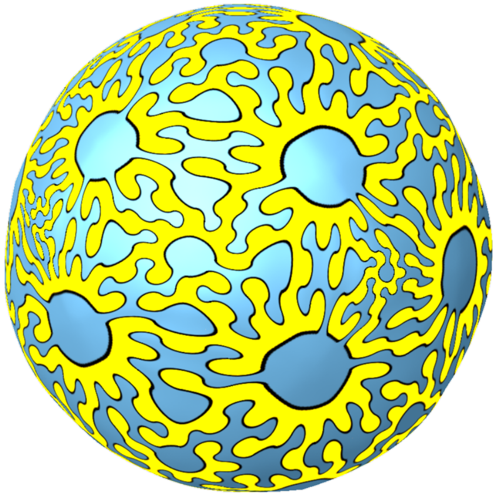
4.5 Discussion and Results

We observed that the below two factors of the mesh approximation of the closed 2-manifold effect the aestheticism of the Duotone Surface.

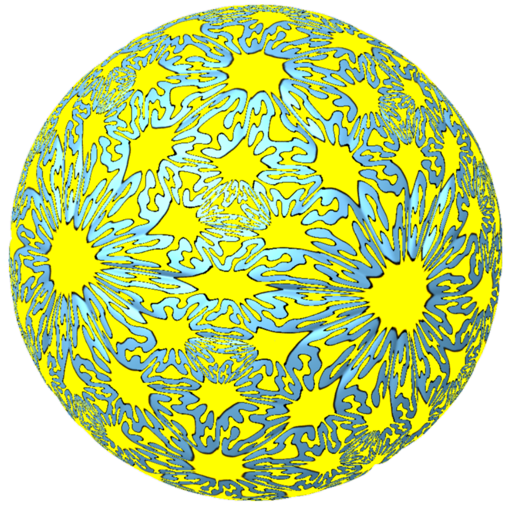
1. Vertex density distribution of the mesh.

2. Aspect ratio of the faces of the mesh.

In fact, these are the parameters which determine the quality of a given polygonal mesh. Vertex density distribution describes the pattern of distribution of the vertices of a mesh. The effect of vertex density distribution over the mesh is intuitive, for example high curvature regions of an object require larger number of vertices to accurately represent the geometry of the object in that local neighbourhood and vice versa.



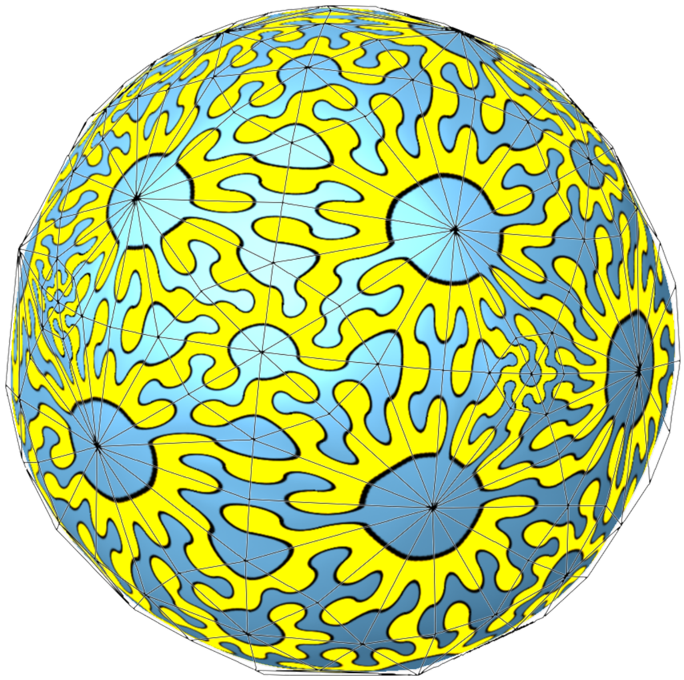
(a)



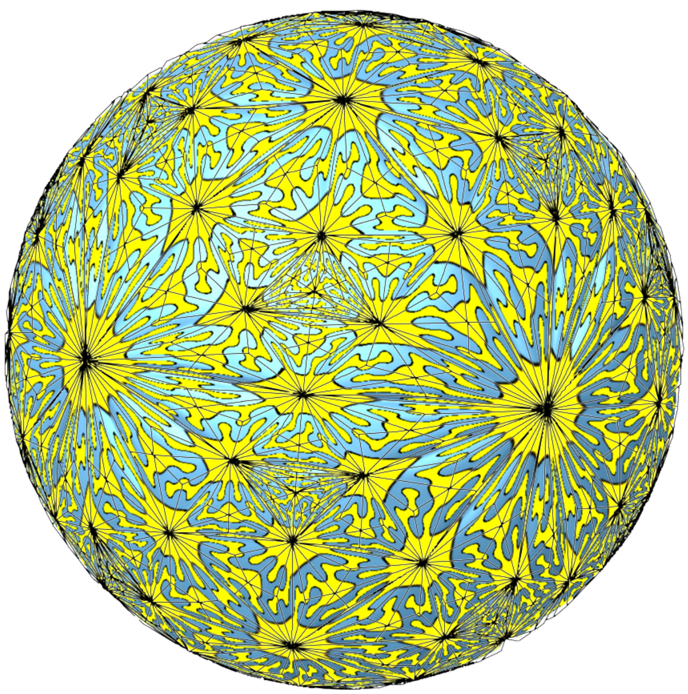
(b)

Figure 4.14: Examples of Duotone surfaces created using the high aspect ratio quadtrilateral meshes

Aspect Ratio can be intuitively understood as a measure of stretching of the polygon. There are multitude of formal definitions for Aspect Ratio and in all cases it is the ratio of maximum and minimum values of the respective metric. Hence, the ratio can be either 1 or greater than 1. Ideally aspect ratio of 1 is preferred but for general geometry, it is not possible to create a mesh of perfect aspect ratio due to non-uniformities in the shape of the objects such as curved geometry, thin features, and sharp corners. Therefore, only a close to 1 value is usually possible. As you can see in figure, unless the the original mesh polygons are too skinny(see figure 4.14(b) for skinny example), the resulting duotone surface looks reasonably good, see figure 4.14(a).



(a)

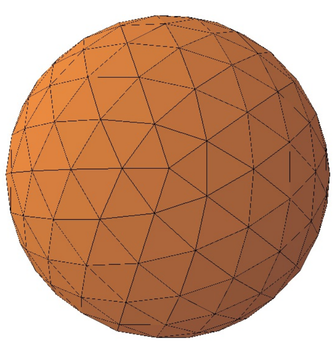


(b)

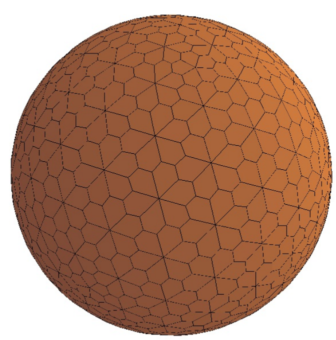
Figure 4.15: Examples of high aspect ratio quadtrilateral meshes. The figures show both the duotone surface and the underlying mesh structure.

Examples shown in figure 4.14 are showed along with the underlying mesh structure in figure 4.15 for your reference. Hence, it is better to pre-process the mesh to remove any of the above mentioned problems. Very recently, Campen et al.[7] developed a method that guarantees construction of quadrilateral meshes with a high-level patch structure for any given closed 2-manifold. Bommes et.al.[1] and Zhang et.al. [25] have also developed methods that can guarantee high fidelity quad layouts. Any of the above mentioned or unmentioned methods which remove the problems stated can be used in the preprocessing stage.

Apart from the above two mesh parameters, the mesh topology is another important parameter which is helpful in controlling the outcome of our process. For example, refer to the figures 4.16 & 4.17 to see how a mesh with 3-sided polygons looks completely different from a mesh with 6-sided polygons after application of one step of Catmull-Clark subdivision.

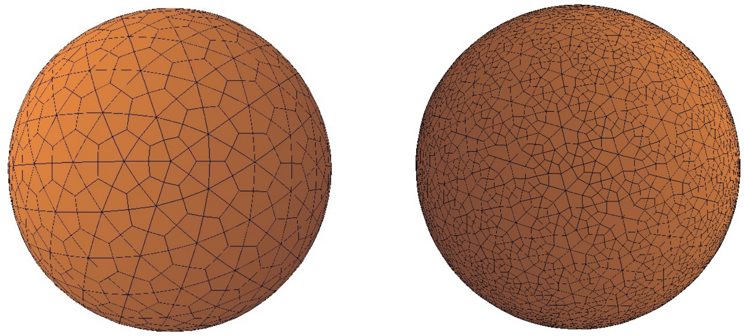


(a)



(b)

Figure 4.16: Sphere mesh with two different types of faces. (a) only triangle faces and (b) only pentagon faces.



(a) (b)

Figure 4.17: Sphere meshes after one step of Catmull-Clark subdivision. Observe that different type of polygons give different type of quadrangulations.

Also, as discussed in section 2.6.1, Catmull-clark subdivided meshes have G^2 continuity except near the extraordinary vertices. Hence, our Duotone surfaces also have G^2 continuity except near extraordinary vertices. However, visual continuity at those vertices is not disturbed in our Duotone surfaces since truchet tiles, shown in figure 4.2, have same color around the vertex and are borderless, thus creating the illusion of visual continuity.

Following two sections presents examples of Duotone Surfaces of various positive genus closed 2-manifolds.

4.5.1 Duotone Surfaces with Truchet Tiles of Two Regions

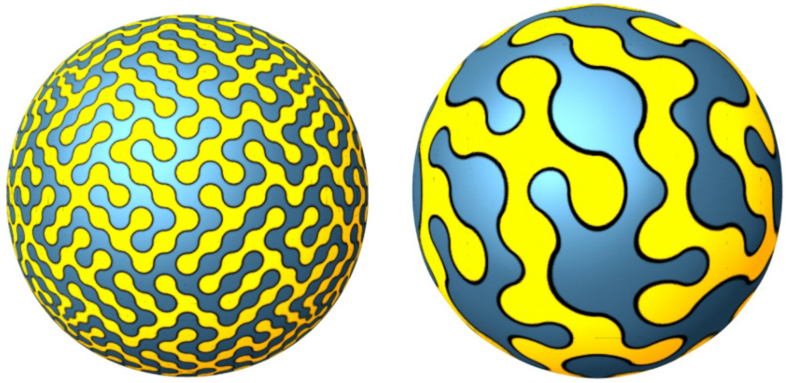


Figure 4.18: Genus 0 Duotone Surfaces

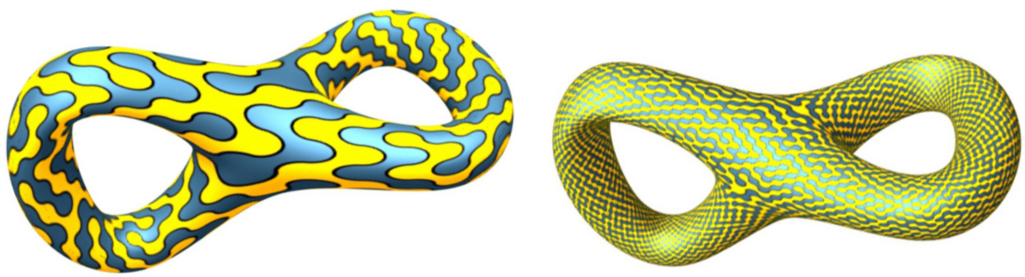


Figure 4.19: Genus 2 Duotone Surfaces

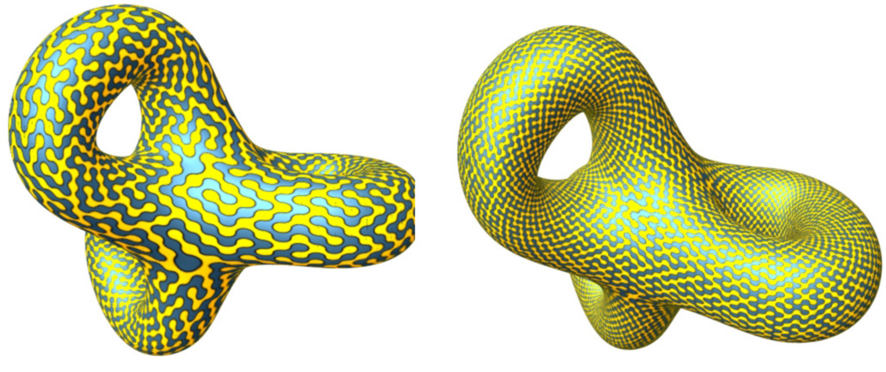


Figure 4.20: Genus 3 Duotone Surfaces

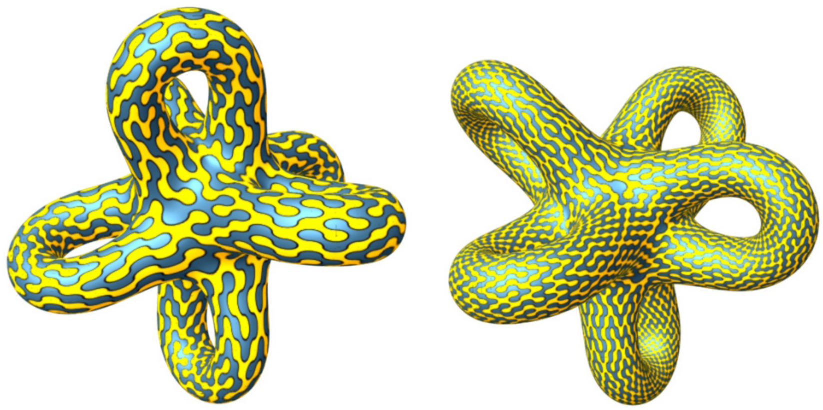


Figure 4.21: Genus 6 Duotone Surfaces

4.5.2 Duotone Surfaces with Truchet Tiles of Multiple Regions

Apart from the traditional two region Truchet tiles, we have also created new tiles with multiple regions as shown in figure 4.22;

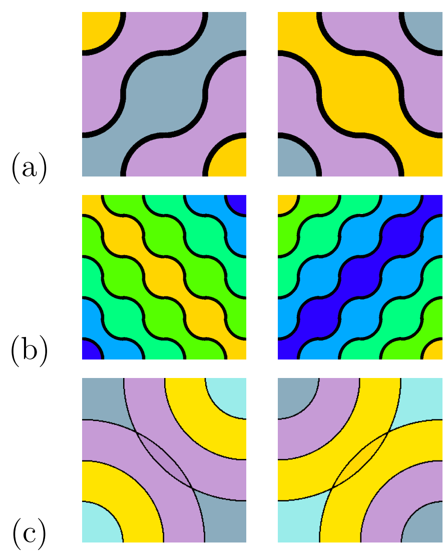


Figure 4.22: Multiple region Truchet tiles

It may be more appropriate to call the surfaces textured with the multi-region

Truchet tiles as Multitone Surfaces. Figure 4.23 shows examples of Multitone surfaces created using the tiles in figure 4.22(a). Figure 4.24 shows examples of Multitone surfaces created using the tiles in figure 4.22(b). Figure 4.25 shows examples of Multitone surfaces created using the tiles in figure 4.22(c).

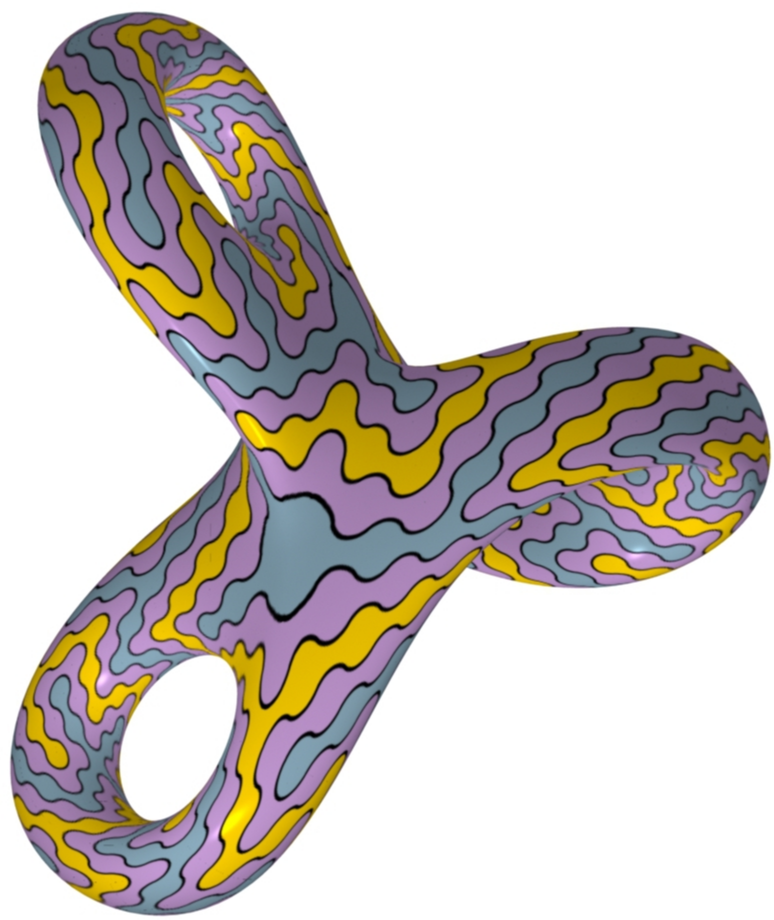


Figure 4.23: Multitone surface using tiles shown in figure 4.22(a)

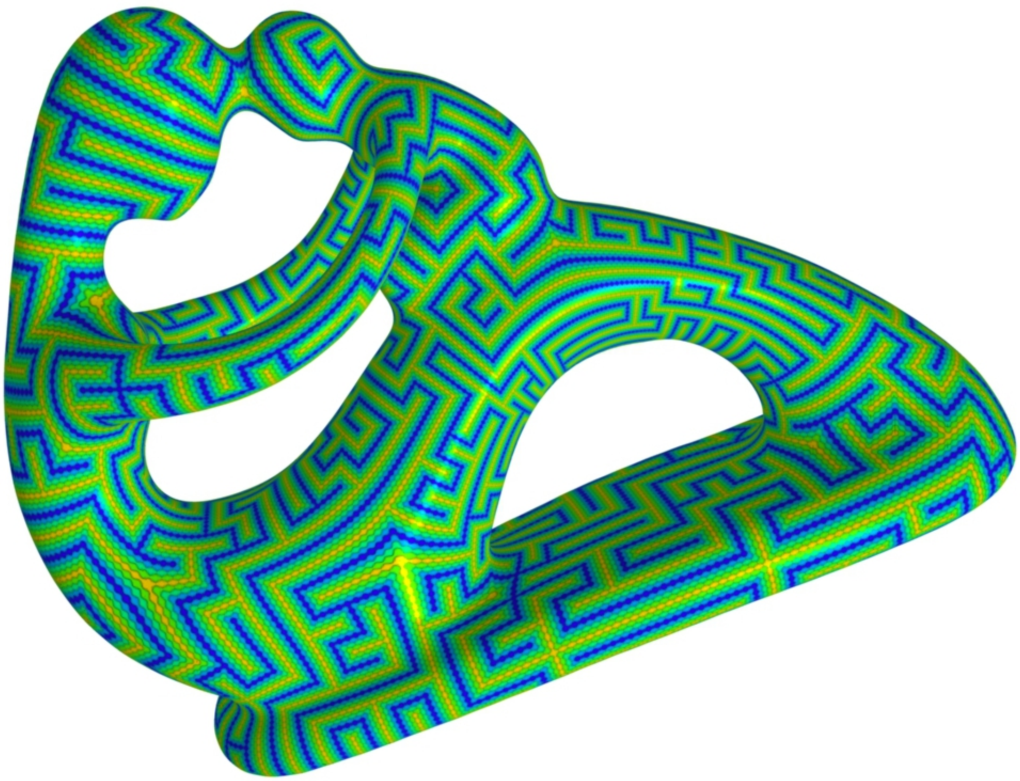


Figure 4.24: Multitone surface using tiles shown in figure 4.22(b)

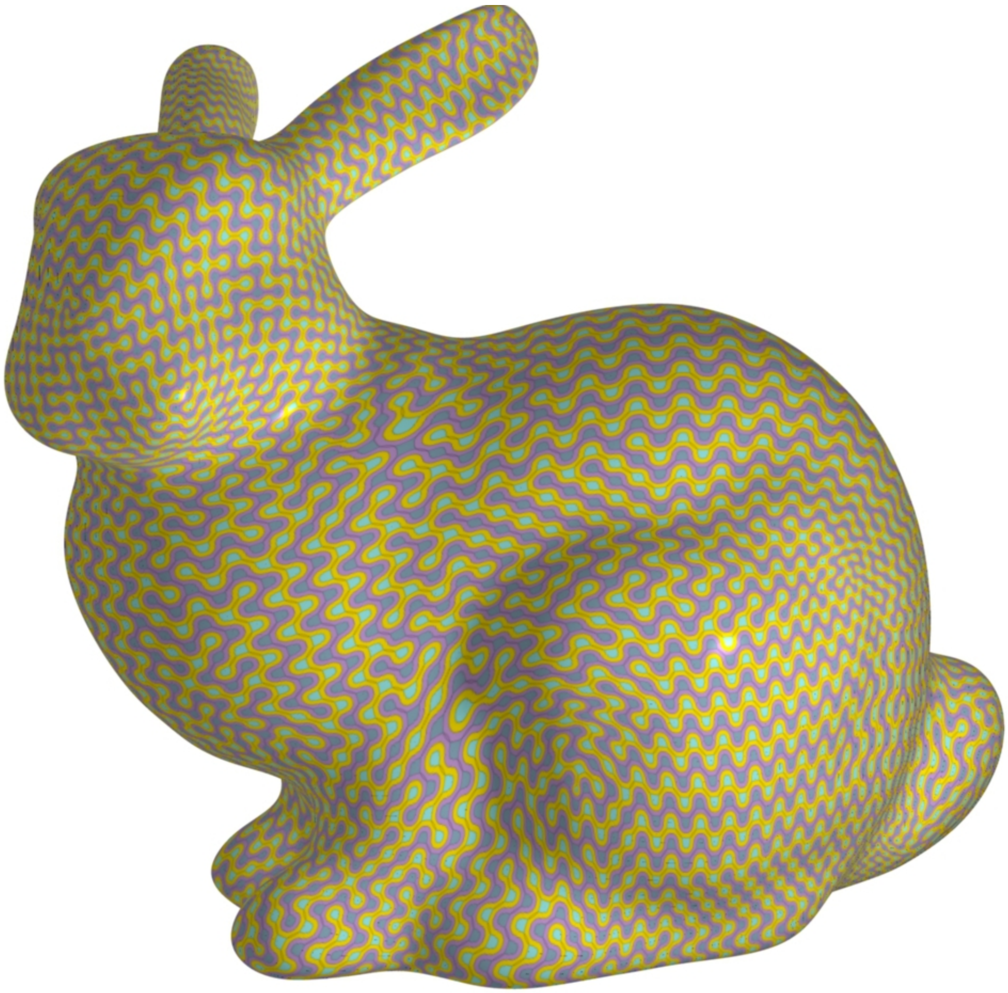


Figure 4.25: Multitone surface using tiles shown in figure 4.22(c)

5. CONCLUSION AND FUTURE WORK

In this thesis, we have presented the concept of duotone surfaces: the surface of a closed 2-manifold is divided into exactly two regions, which can be obtained from any manifold mesh that represents a given shape. These two regions which constitute the texture of the duotone surface are visually interlocked, the boundary curve between the two regions is the only separation between the two regions.

We can control how the boundary curve turns and there by control how the duotone pattern looks. Also, the areas of these two regions are approximately the same, provided the initial distribution of the mesh vertices is uniform.

The current process constructs the Jordan curve by creating a spanning tree using the vertices of one set of the bipartite graph created as a result of subdivision on the original input mesh. As a result of this, though we are able to contruct a complex Jordan curve that helps create a duotone surface, we can guarantee to have no closed loops in only one region of the duotone surface. Further research might help devise a better algorithm that can scan the region with closed loops and remove the loops without creating any new loops in either of the regions.

The duotone surfaces can also provide sculpting opportunities. For instance, the two regions on the duotone surface can be obtained by cutting the surface into two 2-manifolds with boundaries. To create a sculpture, these 2-manifold with boundaries can be turned to solid shapes which can be interlocked together to form the original shape.

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