Cumulative Distribution Functions and Moments of Weighted Lattice Polynomials

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Sketch of the Presentation

Part I : Weighted lattice polynomials

- Definitions
- Representation and characterization

Part II : Cumulative distribution functions of aggregation operators

- Weighted lattice polynomials
- Applications

Part I : Weighted lattice polynomials

Lattice polynomials

Let L be a lattice with lattice operations $\wedge$ and $\vee$

We assume that L is

- bounded (with bottom 0 and top 1)
- distributive

Definition (Birkhoff 1967)

An n-ary *lattice polynomial* is a well-formed expression involving n variables $x_1, \ldots, x_n \in L$ linked by the lattice operations $\wedge$ and $\vee$ in an arbitrary combination of parentheses

Example.

$$p(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$$

Lattice polynomial functions

Any lattice polynomial naturally defines a *lattice polynomial function* (l.p.f.) $p : L^n \to L$.

Example.

$$p(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$$

If p and q represent the same function, we say that p and q are equivalent and we write $p = q$

Example.

$$x_1 \vee (x_1 \wedge x_2) = x_1$$

Disjunctive and conjunctive forms of l.p.f.'s

Notation. $[n] := \{1, \ldots, n\}$.

Proposition (Birkhoff 1967)

Let $p : L^n \to L$ be any l.p.f.
Then there are nonconstant set functions $v, w : 2^{[n]} \to \{0,1\}$, with $v(\varnothing) = 0$ and $w(\varnothing) = 1$, such that

$$p(x) = \bigvee_{\substack{S \subseteq [n] \\ v(S)=1}} \bigwedge_{i \in S} x_i = \bigwedge_{\substack{S \subseteq [n] \\ w(S)=0}} \bigvee_{i \in S} x_i .$$

Example.

$$(x_1 \wedge x_2) \vee x_3 = (x_1 \vee x_3) \wedge (x_2 \vee x_3)$$

$$v(\{3\}) = v(\{1,2\}) = 1$$
$$w(\{1,3\}) = w(\{2,3\}) = 0$$

The set functions v and w, which generate p, are not unique :

$$x_1 \vee (x_1 \wedge x_2) = x_1 = x_1 \wedge (x_1 \vee x_2)$$

Notation. $\mathbf{1}_S$:= characteristic vector of $S \subseteq [n]$ in $\{0,1\}^n$.

Proposition (Marichal 2002)

From among all the set functions v that disjunctively generate the l.p.f. p, only one is isotone :

$$v(S) = p(\mathbf{1}_S)$$

From among all the set functions w that conjunctively generate the l.p.f. p, only one is antitone :

$$w(S) = p(\mathbf{1}_{[n]\setminus S})$$

Consequently, any n-ary l.p.f. can always be written as

$$p(x) = \bigvee_{\substack{S\subseteq[n] \\ p(\mathbf{1}_S)=1}} \bigwedge_{i\in S} x_i = \bigwedge_{\substack{S\subseteq[n] \\ p(\mathbf{1}_{[n]\setminus S})=0}} \bigvee_{i\in S} x_i$$

Example. $p(x) = (x_1 \wedge x_2) \vee x_3$

S	$p(\mathbf{1}_S)$	$p(\mathbf{1}_{[n]\setminus S})$
$\varnothing$	0	1
$\{1\}$	0	1
$\{2\}$	0	1
$\{3\}$	1	1
$\{1,2\}$	1	1
$\{1,3\}$	1	0
$\{2,3\}$	1	0
$\{1,2,3\}$	1	0

$$p(x) = x_3 \vee (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$$

$$p(x) = (x_1 \vee x_3) \wedge (x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

Particular cases : order statistics

Denote by $x_{(1)},\ldots,x_{(n)}$ the *order statistics* resulting from reordering $x_1,\ldots,x_n$ in the nondecreasing order : $x_{(1)}\leqslant\cdots\leqslant x_{(n)}$.

Proposition (Ovchinnikov 1996, Marichal 2002)

p is a symmetric l.p.f. $\iff$ p is an order statistic

Notation. Denote by $os_k : L^n \to L$ the kth order statistic function.

$$os_k(x) := x_{(k)}$$

Then we have

$$os_k(\mathbf{1}_S) = 1 \iff |S| \geqslant n-k+1$$
$$os_k(\mathbf{1}_{[n]\setminus S}) = 0 \iff |S| \geqslant k$$

Weighted lattice polynomials

We can generalize the concept of l.p.f. by regarding some variables as parameters.

Example. For $c \in L$, we consider

$$p(x_1, x_2) = (c \vee x_1) \wedge x_2$$

Definition

$p : L^n \to L$ is an n-ary *weighted lattice polynomial* function (w.l.p.f.) if there exist parameters $c_1, \ldots, c_m \in L$ and a l.p.f. $q : L^{n+m} \to L$ such that

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n, c_1, \ldots, c_m)$$

Disjunctive and conjunctive forms of w.l.p.f.'s

Proposition (Lausch & Nöbauer 1973)

Let $p : L^n \to L$ be any w.l.p.f.
Then there are set functions $v, w : 2^{[n]} \to L$ such that

$$p(x) = \bigvee_{S\subseteq[n]} \Big[v(S) \wedge \bigwedge_{i\in S} x_i\Big] = \bigwedge_{S\subseteq[n]} \Big[w(S) \vee \bigvee_{i\in S} x_i\Big].$$

Remarks.

- p is a l.p.f. if v and w range in $\{0,1\}$, with $v(\varnothing) = 0$ and $w(\varnothing) = 1$.
- Any w.l.p.f. is entirely determined by 2^n parameters, even if more parameters have been considered to construct it.

Disjunctive and conjunctive forms of w.l.p.f.'s

Proposition (Marichal 2006)

From among all the set functions v that disjunctively generate the w.l.p.f. p, only one is isotone :

$$v(S) = p(\mathbf{1}_S)$$

From among all the set functions w that conjunctively generate the w.l.p.f. p, only one is antitone :

$$w(S) = p(\mathbf{1}_{[n]\setminus S})$$

Disjunctive and conjunctive forms of w.l.p.f.'s

Consequently, any n-ary w.l.p.f. can always be written as

$$p(x) = \bigvee_{S \subseteq [n]} \Big[p(\mathbf{1}_S) \wedge \bigwedge_{i \in S} x_i \Big] = \bigwedge_{S \subseteq [n]} \Big[p(\mathbf{1}_{[n]\setminus S}) \vee \bigvee_{i \in S} x_i \Big]$$

Example. $p(x) = (c \vee x_1) \wedge x_2$

S	$p(\mathbf{1}_S)$	$p(\mathbf{1}_{[n]\setminus S})$
$\varnothing$	0	1
$\{1\}$	0	c
$\{2\}$	c	0
$\{1,2\}$	1	0

$$\begin{aligned} p(x) &= (0 \wedge 1) \vee (0 \wedge x_1) \vee (c \wedge x_2) \vee (1 \wedge x_1 \wedge x_2) \\ &= (c \wedge x_2) \vee (x_1 \wedge x_2) \\ p(x) &= (1 \vee 0) \wedge (c \vee x_1) \wedge (0 \vee x_2) \wedge (0 \vee x_1 \vee x_2) \\ &= (c \vee x_1) \wedge x_2 \end{aligned}$$

Particular case : the Sugeno integral

Let us generalize the concept of discrete Sugeno integral in the framework of bounded distributive lattices.

Definition (Sugeno 1974)

An L-valued *fuzzy measure* on $[n]$ is an isotone set function $\mu : 2^{[n]} \to L$ such that $\mu(\varnothing) = 0$ and $\mu([n]) = 1$.

The *Sugeno integral* of a function $x : [n] \to L$ with respect to μ is defined by

$$\mathcal{S}_\mu(x) := \bigvee_{S \subseteq [n]} \Big[\mu(S) \wedge \bigwedge_{i \in S} x_i\Big]$$

Remark. A function $f : L^n \to L$ is an n-ary Sugeno integral if and only if f is a w.l.p.f. fulfilling $f(\mathbf{1}_\varnothing) = 0$ and $f(\mathbf{1}_{[n]}) = 1$.

Particular case : the Sugeno integral

Notation. The median function is the function $os_2 : L^3 \to L$.

Proposition (Marichal 2006)

For any w.l.p.f. $p : L^n \to L$, there is a fuzzy measure $\mu : 2^{[n]} \to L$ such that

$$p(x) = \text{median}\big[p(\mathbf{1}_\varnothing), \mathcal{S}_\mu(x), p(\mathbf{1}_{[n]})\big]$$

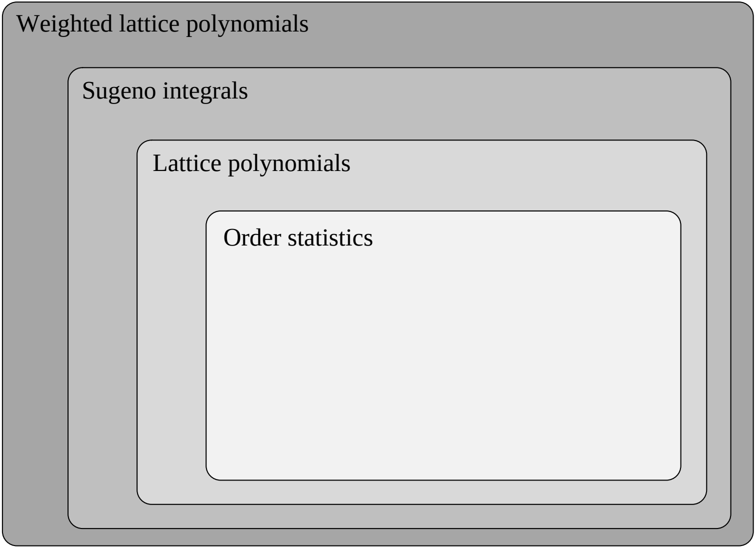
Corollary (Marichal 2006)

Consider a function $f : L^n \to L$.

The following assertions are equivalent :

- f is a Sugeno integral
- f is an idempotent w.l.p.f., that is such that $f(x, \dots, x) = x$
- f is a w.l.p.f. fulfilling $f(\mathbf{1}_\varnothing) = 0$ and $f(\mathbf{1}_{[n]}) = 1$.

Inclusion properties



The median based decomposition formula

Let $f : L^n \to L$ and $k \in [n]$ and define $f_k^0, f_k^1 : L^n \to L$ as

$$\begin{aligned} f_k^0(x) &:= f(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) \\ f_k^1(x) &:= f(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n) \end{aligned}$$

Remark. If f is a w.l.p.f., so are f_k^0 and f_k^1

Consider the following system of n functional equations, called the *median based decomposition formula*

$$f(x) = \text{median}\big[f_k^0(x), x_k, f_k^1(x)\big] \qquad (k = 1, \ldots, n)$$

The median based decomposition formula

Any solution of the median based decomposition formula

$$f(x) = \text{median}\big[f_k^0(x), x_k, f_k^1(x)\big] \qquad (k = 1, \ldots, n)$$

is an n-ary w.l.p.f.

Example. For $n = 2$ we have

$$f(x_1, x_2) = \text{median}\big[f(x_1, 0), x_2, f(x_1, 1)\big]$$

with

$$\begin{aligned} f(x_1, 0) &= \text{median}\big[f(0,0), x_1, f(1,0)\big] \quad \text{(w.l.p.f.)} \\ f(x_1, 1) &= \text{median}\big[f(0,1), x_1, f(1,1)\big] \quad \text{(w.l.p.f.)} \end{aligned}$$

The median based decomposition formula

The median based decomposition formula characterizes the w.l.p.f.'s

Theorem (Marichal 2006)

The solutions of the median based decomposition formula are exactly the n-ary w.l.p.f.'s

Part II : Cumulative distribution functions of aggregation operators

Cumulative distribution functions of aggregation operators

Consider

- an aggregation operator $A : \mathbb{R}^n \to \mathbb{R}$
- n independent random variables $X_1, \ldots, X_n$, with cumulative distribution functions $F_1(x), \ldots, F_n(x)$

$$\left.\begin{matrix} X_1 \\ \vdots \\ X_n \end{matrix}\right\} \quad \longrightarrow \quad Y_A = A(X_1, \ldots, X_n)$$

Problem. We are searching for the cumulative distribution function (c.d.f.) of Y_A :

$$\boxed{F_A(y) := \Pr[Y_A \leqslant y]}$$

Cumulative distribution functions of aggregation operators

From the c.d.f. of Y_A, we can calculate the expectation

$$\mathbf{E}\big[g(Y_A)\big] = \int_{-\infty}^{\infty} g(y)\,\mathrm{d}F_A(y)$$

for any measurable function g.

Some useful examples :

$g(x)$	$\mathbf{E}\big[g(Y_A)\big]$
x	expected value of Y_A
x^r	raw moments of Y_A
$\big[x - \mathbf{E}(Y_A)\big]^r$	central moments of Y_A
e^{tx}	moment-generating function of Y_A

Cumulative distribution functions of aggregation operators

If $F_A(y)$ is absolutely continuous, then Y_A has a probability density function (p.d.f.)

$$f_A(y) := \frac{\mathrm{d}}{\mathrm{d}y} F_A(y)$$

In this case

$$\mathbf{E}\big[g(Y_A)\big] = \int_{-\infty}^{\infty} g(y)\, f_A(y)\,\mathrm{d}y$$

Example : the arithmetic mean

$$AM(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$F_{AM}(y)$ is given by the convolution product of $F_1, \ldots, F_n$

$$F_{AM}(y) = (F_1 * \cdots * F_n)(ny)$$

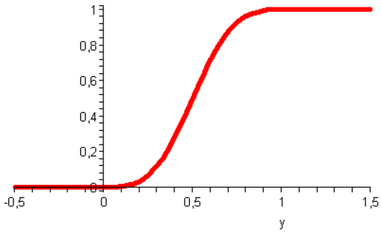
For uniform random variables $X_1, \ldots, X_n$ on $[0, 1]$, we have

$$F_{AM}(y) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (ny - k)_+^n \qquad (y \in [0, 1])$$

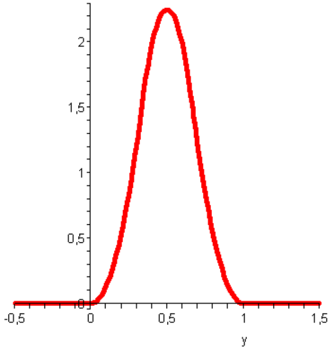
(Feller, 1971)

Example : the arithmetic mean

Case of $n = 3$ uniform random variables X_1, X_2, X_3 on $[0, 1]$



Graph of $F_{AM}(y)$



Graph of $f_{AM}(y)$

Example : Łukasiewicz t-norm

$$T_L(x_1,\ldots,x_n) = \max\left[0, \sum_{i=1}^{n} x_i - (n-1)\right]$$

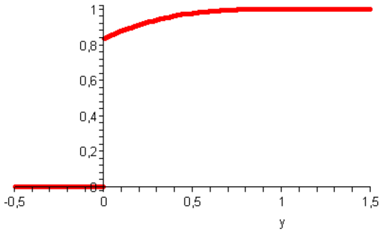
$$\begin{aligned}
F_{T_L}(y) &= \Pr\left[\max\left[0, \textstyle\sum_i X_i - (n-1)\right] \leqslant y\right] \\
&= \Pr\left[0 \leqslant y \quad \text{and} \quad \textstyle\sum_i X_i - (n-1) \leqslant y\right] \\
&= \Pr[0 \leqslant y]\, \Pr\left[\textstyle\sum_i X_i \leqslant y + n - 1\right] \\
&= H_0(y)\, F_{AM}\left(\tfrac{y+n-1}{n}\right)
\end{aligned}$$

where $H_c(y)$ is the *Heaviside* function

$$H_c(y) = \mathbf{1}_{[c,+\infty[}(y)$$

Example : Łukasiewicz t-norm

Case of $n = 3$ uniform random variables X_1, X_2, X_3 on $[0, 1]$



Graph of $F_{T_L}(y)$

Remark.
$F_{T_L}(y)$ is discontinuous
$\Rightarrow$ The p.d.f. does not exist

Example : order statistics on $\mathbb{R}$

$$\boxed{os_k(x_1,\ldots,x_n) = x_{(k)}}$$

$$F_{os_k}(y) = \sum_{\substack{S\subseteq[n]\\|S|\geqslant k}} \prod_{i\in S} F_i(y) \prod_{i\in[n]\setminus S} [1-F_i(y)]$$

(see e.g. David & Nagaraja 2003)

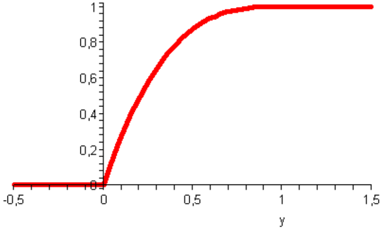
Examples.

$$F_{os_1}(y) = 1-\prod_{i=1}^{n}[1-F_i(y)]$$

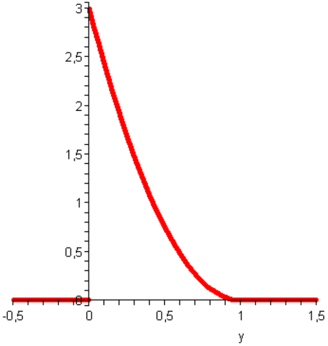
$$F_{os_n}(y) = \prod_{i=1}^{n} F_i(y)$$

Example : order statistics on $\mathbb{R}$

Case of $n = 3$ uniform random variables X_1, X_2, X_3 on $[0, 1]$



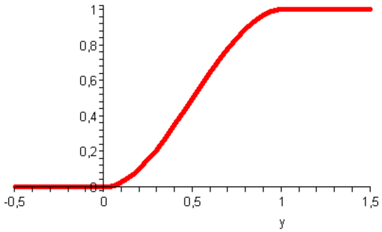
Graph of $F_{os_1}(y)$



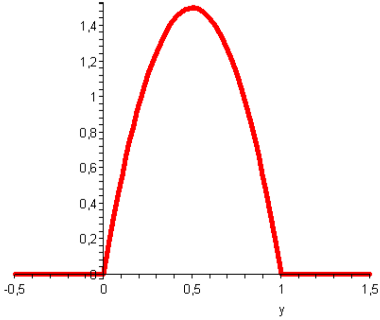
Graph of $f_{os_1}(y)$

Example : order statistics on $\mathbb{R}$

Case of $n = 3$ uniform random variables X_1, X_2, X_3 on $[0, 1]$



Graph of $F_{os_2}(y)$



Graph of $f_{os_2}(y)$

New results : lattice polynomial functions on $\mathbb{R}$

Let $p : L^n \to L$ be a l.p.f. on $L = [0,1]$
It can be extended to an aggregation function from $\mathbb{R}^n$ to $\mathbb{R}$.

$$p(x_1,\ldots,x_n) = \bigvee_{\substack{S\subseteq[n]\\ p(\mathbf{1}_S)=1}} \bigwedge_{i\in S} x_i = \bigwedge_{\substack{S\subseteq[n]\\ p(\mathbf{1}_{[n]\setminus S})=0}} \bigvee_{i\in S} x_i$$

Note. $\wedge = \min$, $\vee = \max$

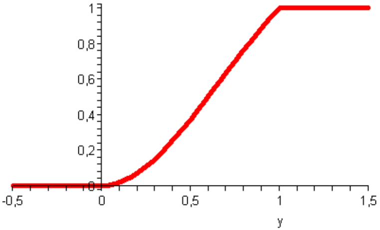
$$F_p(y) = 1 - \sum_{\substack{S\subseteq[n]\\ p(\mathbf{1}_S)=1}} \prod_{i\in[n]\setminus S} F_i(y) \prod_{i\in S} [1 - F_i(y)]$$

$$F_p(y) = \sum_{\substack{S\subseteq[n]\\ p(\mathbf{1}_{[n]\setminus S})=0}} \prod_{i\in S} F_i(y) \prod_{i\in[n]\setminus S} [1 - F_i(y)]$$

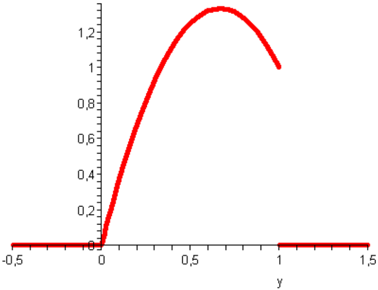
New results : lattice polynomial functions on $\mathbb{R}$

Example. $p(x) = (x_1 \wedge x_2) \vee x_3$

Uniform random variables X_1, X_2, X_3 on $[0, 1]$



Graph of $F_p(y)$



Graph of $f_p(y)$

New results : lattice polynomial functions on $\mathbb{R}$

Consider

- $v_p : 2^{[n]} \to \mathbb{R}$, defined by $v_p(S) := p(\mathbf{1}_S)$
- $v_p^* : 2^{[n]} \to \mathbb{R}$, defined by $v_p^*(S) = 1 - v_p([n] \setminus S)$
- $m_v : 2^{[n]} \to \mathbb{R}$, the Möbius transform of v, defined by

$$m_v(S) := \sum_{T \subseteq S} (-1)^{|S|-|T|}\, v(T)$$

Alternate expressions of $F_p(y)$

$$F_p(y) = 1 - \sum_{S \subseteq [n]} m_{v_p}(S) \prod_{i \in S} [1 - F_i(y)]$$

$$F_p(y) = \sum_{S \subseteq [n]} m_{v_p^*}(S) \prod_{i \in S} F_i(y)$$

New results : weighted lattice polynomial functions on $\overline{\mathbb{R}}$

Let $p : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ be a w.l.p.f. on $\overline{\mathbb{R}} = [-\infty, +\infty]$

Notation. $\mathbf{e}_S :=$ characteristic vector of S in $\{-\infty, +\infty\}^n$

$$p(x) = \bigvee_{S \subseteq [n]} \Big[p(\mathbf{e}_S) \wedge \bigwedge_{i \in S} x_i \Big] = \bigwedge_{S \subseteq [n]} \Big[p(\mathbf{e}_{[n] \setminus S}) \vee \bigvee_{i \in S} x_i \Big]$$

$$F_p(y) = 1 - \sum_{S \subseteq [n]} [1 - H_{p(\mathbf{e}_S)}(y)] \prod_{i \in [n] \setminus S} F_i(y) \prod_{i \in S} [1 - F_i(y)]$$

$$F_p(y) = \sum_{S \subseteq [n]} H_{p(\mathbf{e}_{[n] \setminus S})}(y) \prod_{i \in S} F_i(y) \prod_{i \in [n] \setminus S} [1 - F_i(y)]$$

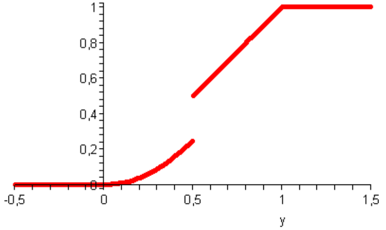
$+$ alternate expressions (cf. Möbius transform)

New results : weighted lattice polynomial functions on $\overline{\mathbb{R}}$

Example. $p(x) = (c \wedge x_1) \vee x_2$

Uniform random variables X_1, X_2 on $[0, 1]$
$F(y) = \text{median}[0, y, 1]$

S	$p(\mathbf{e}_S)$
$\varnothing$	$-\infty$
$\{1\}$	c
$\{2\}$	$+\infty$
$\{1,2\}$	$+\infty$



$$F_p(y) = F(y)\Big(F(y)+H_c(y)[1-F(y)]\Big)$$

Graph of $F_p(y)$ for $c = 1/2$

Application : computation of certain integrals

Example. Given a w.l.p.f. $p : [0,1]^n \to [0,1]$ and a measurable function $g : [0,1] \to \overline{\mathbb{R}}$, compute

$$\int_{[0,1]^n} g\left[p(x)\right] \mathrm{d}x$$

Solution. The integral is given by $\mathbf{E}\left[g(Y_p)\right]$, where the variables $X_1, \ldots, X_n$ are uniform on $[0,1]$

$$\mathbf{E}\left[g(Y_p)\right] = g(0) + \sum_{S \subseteq [n]} \int_0^{p(\mathbf{e}_S)} y^{n-|S|}(1-y)^{|S|} \,\mathrm{d}g(y)$$

Application : computation of certain integrals

Sugeno integral

$$\int_{[0,1]^n} \mathcal{S}_\mu(x)\,\mathrm{d}x = \sum_{S\subseteq[n]} \int_0^{\mu(S)} y^{n-|S|}(1-y)^{|S|}\,\mathrm{d}y$$

Example.

$$\int_{[0,1]^2} \big[(c\wedge x_1)\vee x_2\big]\,\mathrm{d}x = \frac{1}{2} + \frac{1}{2}\,c^2 - \frac{1}{3}\,c^3$$

Note. Recall the expected value of the Choquet integral

$$\int_{[0,1]^n} \mathcal{C}_\mu(x)\,\mathrm{d}x = \sum_{S\subseteq[n]} \mu(S)\,\int_0^1 y^{n-|S|}(1-y)^{|S|}\,\mathrm{d}y$$

(Marichal 2004)

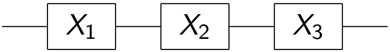
Application : reliability of systems

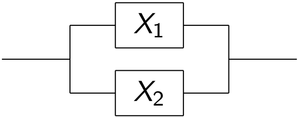
Consider a system made up of n indep. components $C_1, \ldots, C_n$

Each component C_i has

- a *lifetime* X_i
- a *reliability* $r_i(t)$ at time $t > 0$

$$r_i(t) := \Pr[X_i > t] = 1 - F_i(t)$$



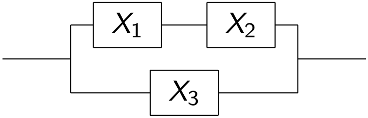


Assumptions :

- The lifetime of a series subsystem is the minimum of the component lifetimes
- The lifetime of a parallel subsystem is the maximum of the component lifetimes

Application : reliability of systems

Question. What is the lifetime of the following system ?



Solution. $Y = (X_1 \wedge X_2) \vee X_3$

Application : reliability of systems

For a system mixing series and parallel connections :

System lifetime :

$$Y_p = p(X_1, \dots, X_n)$$

where p is

- an n-ary l.p.f.
- an n-ary w.l.p.f. if some X_i's are constant

We then have explicit formulas for

- the c.d.f. of Y_p
- the expected value $\mathbf{E}[Y_p]$
- the moments

Application : reliability of systems

System reliability at time $t > 0$

$$R_p(t) := \Pr[Y_p > t] = 1 - F_p(t)$$

For any measurable function $g : [0, \infty[\to \mathbb{R}$ such that

$$g(\infty) r_i(\infty) = 0 \qquad (i = 1, \ldots, n)$$

we have

$$\mathbf{E}\big[g(Y_p)\big] = g(0) + \int_0^\infty R_p(t)\,\mathrm{d}g(t)$$

Mean time to failure :

$$\mathbf{E}[Y_p] = \int_0^\infty R_p(t)\,\mathrm{d}t$$

Application : reliability of systems

Example. Assume $r_i(t) = e^{-\lambda_i t}$ $(i = 1, \ldots, n)$

$$\mathbf{E}\big[Y_p\big] = \sum_{\substack{S \subseteq [n] \\ S \neq \varnothing}} m_{v_p}(S) \, \frac{1}{\sum_{i \in S} \lambda_i}$$

- Series system

$$\mathbf{E}\big[Y_p\big] = \frac{1}{\sum_{i \in [n]} \lambda_i}$$

- Parallel system

$$\mathbf{E}\big[Y_p\big] = \sum_{\substack{S \subseteq [n] \\ S \neq \varnothing}} (-1)^{|S|-1} \, \frac{1}{\sum_{i \in S} \lambda_i}$$

(Barlow & Proschan 1981)

Thanks for your attention !