A second infinite family of Steiner triple systems without almost parallel classes

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Abstract

For each positive integer n, we construct a Steiner triple system of order $v = 2(3^n) + 1$ with no almost parallel class; that is, with no set of $\frac{v-1}{3}$ disjoint triples. In fact, we construct families of (v, k, λ)-designs with an analogous property. The only previously known examples of Steiner triple systems of order congruent to 1 (mod 6) without almost parallel classes were the projective triple systems of order $2^n - 1$ with n odd, and 2 of the $11,084,874,829$ Steiner triple systems of order 19.

A (v, k, λ)-design is a pair $(V, \mathcal{B})$ where V is a v-set of points and $\mathcal{B}$ is a collection of k-subsets of V, called blocks, such that each (unordered) pair of points occurs in exactly λ blocks in $\mathcal{B}$. A $(v, 3, 1)$-design is called a *Steiner triple system* of order v, and the blocks of such a design are called *triples*. Kirkman proved in 1847 that there exists a Steiner triple system of order v if and only if $v \equiv 1$ or $3 \pmod 6$ [3]. A set of disjoint blocks in a (v, k, λ)-design is called a *partial parallel class*. Clearly any partial parallel class contains at most $\lfloor \frac{v}{k} \rfloor$ blocks. The maximum size of partial parallel classes in designs, and particularly in Steiner triple systems, has been well studied, see [7] for example. A partial parallel class containing all the points of a design is called a *parallel class* and a partial parallel class containing all but one of the points of a design is called an *almost parallel class*. Steiner triple systems of order $v \equiv 3 \pmod 6$ are candidates for having parallel classes, and Steiner triple systems of order $v \equiv 1 \pmod 6$ are candidates for having almost parallel classes.

This paper is concerned with designs having no almost parallel classes. We construct a $(q^n(q^d - 1) + 1, q^d, \binom{n-1}{d-1}_q)$-design with no almost parallel class when $1 \le d \le n$ and both q and $q^d - 1$ are prime powers (see Theorem 2). Here, $\binom{a}{b}_q$ denotes the Gaussian binomial coefficient. Taking $q = 3$ and $d = 1$ we obtain a Steiner triple system of order $2(3^n) + 1$ with no almost parallel class for each $n \ge 1$ (see Corollary 3). Our proof contains arguments that are somewhat reminiscent of those used to prove results concerning the non-existence of transversals in Latin squares [8].

Very few examples of a Steiner triple system without a partial parallel class having $\lfloor \frac{v}{3} \rfloor$ triples are known, and no example is known without a partial parallel class having $\lfloor \frac{v}{3} \rfloor - 1$ triples [1]. Nevertheless, it is conjectured that for all $v \equiv 3 \pmod 6$ except $v \in \{3, 9\}$, there exists a Steiner triple system of order v with no parallel class [5], and that for all $v \equiv 1 \pmod 6$ except $v = 13$ there exists a Steiner triple system of order v with no almost parallel class [7].

The unique Steiner triple system of order 9 has a parallel class, but the unique Steiner triple system of order 7 has no almost parallel class. Both Steiner triple systems of order 13 have almost parallel classes, all but 10 of the 80 Steiner triple systems of order 15 have parallel classes [1], and all but 2 of the 11,084,874,829 Steiner triple systems of order 19 have almost parallel classes [2]. Lo Faro [4] was the first to discover a Steiner triple system of order 19 without an almost parallel class. There are 12 Steiner triple systems of order 21 that are known to have no parallel class [5]. An argument credited to Wilson by Rosa and Colbourn [7] proves that for each odd n, the projective Steiner triple system of order $2^n - 1$ has no almost parallel class. Note that $2^n - 1 \equiv 3 \pmod 6$ when n is even, and $2^n - 1 \equiv 1 \pmod 6$ when n is odd. Up until now, these projective Steiner triple systems have provided the only examples of Steiner triple systems with orders greater than 21 which are known to have no partial parallel class containing $\lfloor \frac{v}{3} \rfloor$ triples.

Wilson's argument, which is given in [7], can be used to prove the following. Note that $\mathrm{PG}(n, q)$ denotes the projective geometry of dimension n over a field of order q.

Theorem 1. *If q is a prime power and n and d are integers such that $1 \leq d \leq n$ and $d + 1$ does not divide $n + 1$, then the (v, k, λ)-design with $v = \binom{n+1}{1}_q$, $k = \binom{d+1}{1}_q$ and $\lambda = \binom{n-1}{d-1}_q$ which is given by the d-dimensional subspaces of $\mathrm{PG}(n, q)$ has no partial parallel class containing $\lfloor \frac{v}{k} \rfloor$ blocks.*

Proof. Suppose for a contradiction that the (v, k, λ)-design $(V, \mathcal{B})$ given by the d-dimensional subspaces of $\mathrm{PG}(n, q)$ has a partial parallel class $\mathcal{P}$ containing $\lfloor \frac{v}{k} \rfloor$ blocks. Now, since the number t of points omitted by $\mathcal{P}$ is the least nonnegative residue of v modulo k, it can be seen that $t = \binom{x}{1}_q$ where x is the least nonnegative residue of $n+1$ modulo $d+1$. Since we are assuming $d+1$ does not divide $n+1$, $t \neq 0$ and it follows that $1 \leq t < q^d$. Now consider a subdesign $(U, \mathcal{A})$ of $(V, \mathcal{B})$ with $\binom{n}{1}_q$ points (a hyperplane of $\mathrm{PG}(n, q)$). There are q^n points in $V \setminus U$, and each block in $\mathcal{B}$ contains either 0 or q^d points in $V \setminus U$. Since q^d divides q^n, this implies that if any partial parallel class of $(V, \mathcal{B})$ omits any point in $V \setminus U$, then it omits at least q^d such points. Thus, since $t < q^d$, the t points omitted by $\mathcal{P}$ are all in U. This contradicts the fact that each point in V lies outside some subdesign with $\binom{n}{1}_q$ points. $\square$

Our theorem below provides (v, k, λ)-designs where $v \equiv 1 \pmod k$, so a partial parallel class containing $\lfloor \frac{v}{k} \rfloor$ blocks would necessarily be an almost parallel class. It gives an infinite family of $(v, k, 1)$-designs without almost parallel classes for each k such that k and $k-1$ are both prime powers (by putting $d=1$). For some such k, it also gives (v,k,λ)-designs without almost parallel classes for various values of $\lambda>1$. Since Mihăilescu has proved Catalan's conjecture [6], we know that this condition on k is satisfied exactly when $k=9$, when k is a Fermat prime, and when $k-1$ is a Mersenne prime. The first few values of k to which the theorem applies are $k=3,4,5,8,9,17,32$.

Theorem 2. *If q, n and d are integers with $1\leq d\leq n$ such that both q and q^d-1 are prime powers, then there exists a $(q^n(q^d-1)+1, q^d, \binom{n-1}{d-1}_q)$-design with no almost parallel class.*

Proof. Let q, n and d be as in the statement of the theorem, let $\mathbb{F}_q$ be a field of order q, and let $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$. We will construct a $(q^n(q^d-1)+1, q^d, \binom{n-1}{d-1}_q)$-design without an almost parallel class on point set $V=(\mathbb{F}_q^n\times\{1,\ldots,q^d-1\})\cup\{\infty\}$. Below, we will perform arithmetic on the first coordinates of points in $V\setminus\{\infty\}$ with the understanding that the operations take place in $\mathbb{F}_q^n$.

Let $(\mathbb{F}_q^n,\mathcal{A})$ be the $(q^n, q^d, \binom{n-1}{d-1}_q)$-design with blocks given by the translates of the d-dimensional subspaces in $\mathbb{F}_q^n$. This is an affine design derived from $\mathrm{AG}(n,q)$, the affine geometry of dimension n over a field of order q. It can be seen that for each block $A\in\mathcal{A}$, the sum (calculated in $\mathbb{F}_q^n$) of the points in A is 0. To see this, observe that each block $A\in\mathcal{A}$ can be written as $\{\alpha_1\boldsymbol{a_1}+\ldots+\alpha_d\boldsymbol{a_d}+\boldsymbol{b}:\alpha_1,\ldots,\alpha_d\in\mathbb{F}_q\}$ for some $\boldsymbol{a_1},\ldots,\boldsymbol{a_d},\boldsymbol{b}\in\mathbb{F}_q^n$ (noting that $(q,n)=(2,1)$ is excluded by our hypotheses).

For each block $A\in\mathcal{A}$, let $((A\times\{1,\ldots,q^d-1\})\cup\{\infty\},\mathcal{B}_A)$ be a $(q^d(q^d-1)+1,q^d,1)$-design such that $A\times\{1\}\in\mathcal{B}_A$ and $\{(\boldsymbol{a},1),\ldots,(\boldsymbol{a},q^d-1),\infty\}\in\mathcal{B}_A$ for each $\boldsymbol{a}\in A$ (this $(q^d(q^d-1)+1,q^d,1)$-design is a projective plane and exists because q^d-1 is a prime power). Note that for each block $B\in\mathcal{B}_A$ not containing ∞, the set of first coordinates of the points in B is A, which means that the sum of the first coordinates of the points in B is 0. Now let $\mathcal{B}$ be the collection of blocks

$$\{B: B\in\mathcal{B}_A,\infty\notin B, A\in\mathcal{A}\}\cup\mathcal{B}_\infty$$

where $\mathcal{B}_\infty$ is the collection of blocks which contains each block in $\{\{(\boldsymbol{a},1),\ldots,(\boldsymbol{a},q^d-1),\infty\}:\boldsymbol{a}\in\mathbb{F}_q^n\}$ exactly $\binom{n-1}{d-1}_q$ times. It is straightforward to verify that $(V,\mathcal{B})$ is a $(q^n(q^d-1)+1,q^d,\binom{n-1}{d-1}_q)$-design.

We claim that $(V,\mathcal{B})$ has no almost parallel class. Suppose for a contradiction that $\mathcal{P}$ is an almost parallel class of $(V,\mathcal{B})$. There is exactly one point $z\in V$ which is not in any block in $\mathcal{P}$. Let $W=\mathbb{F}_q^n\times\{1\}$. The proof splits into two cases according to whether $z\in W$.

Case 1: Suppose $z\notin W$. By the construction of $\mathcal{B}$, for each $B\in\mathcal{B}$ we have either $B\subseteq W$ or $|B\cap W|=1$, and hence we have $|B\cap(V\setminus W)|\in\{0,q^d-1\}$. It follows that q^d-1 divides $|V\setminus W|-1=q^n(q^d-2)$. Since $\gcd(q^d-1,q^d-2)=1$ and $\gcd(q^d-1,q^n)=1$, this implies $q^d-1=1$ which contradicts our hypothesis that q^d-1 is a prime power.

Case 2: Suppose $z \in W$. Thus, $z = (\boldsymbol{c}, 1)$ for some $\boldsymbol{c} \in \mathbb{F}_q^n$. Let $B_\infty = \{(\boldsymbol{b}, 1), \ldots, (\boldsymbol{b}, q^d - 1), \infty\}$ be the block in $\mathcal{P}$ containing ∞, where $\boldsymbol{b} \in \mathbb{F}_q^n \setminus \{\boldsymbol{c}\}$. The blocks in $\mathcal{P} \setminus \{B_\infty\}$ partition $V \setminus \{(\boldsymbol{b}, 1), \ldots, (\boldsymbol{b}, q^d - 1), \infty, (\boldsymbol{c}, 1)\}$ and we have already noted that the first coordinates of the points in any block in $\mathcal{B}$ not containing ∞ sum to zero. It follows that the first coordinates of the points in $V \setminus \{(\boldsymbol{b}, 1), \ldots, (\boldsymbol{b}, q^d - 1), \infty, (\boldsymbol{c}, 1)\}$ also sum to 0. That is,

$$\left((q^d - 1) \sum_{\boldsymbol{x} \in \mathbb{F}_q^n} \boldsymbol{x}\right) - (q^d - 1)\boldsymbol{b} - \boldsymbol{c} = 0.$$

Since $\sum_{\boldsymbol{x} \in \mathbb{F}_q^n} \boldsymbol{x} = 0$ (again noting that $(q, n) \neq (2, 1)$), we have $(q^d - 1)\boldsymbol{b} + \boldsymbol{c} = 0$. So we obtain $\boldsymbol{b} = \boldsymbol{c}$, a contradiction. $\square$

Corollary 3. *For each positive integer n there is a Steiner triple system of order $2(3^n) + 1$ with no almost parallel class.*

Of the two Steiner triple systems of order 19 without almost parallel classes, one is given by the construction used in the proof of Theorem 2 and the other is the Netto triple system of order 19 (see [1]). It is worth noting that a computer search by Ian Wanless has shown that, except those of order 7 and 19, every Netto triple system of order at most 300 has an almost parallel class.

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