Diagonal Resolutions for the Metacyclic Groups $\mathrm{G}(\mathrm{pq})$

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I, Jonathan Joseph Remez, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

We study the notion of a free resolution. In general a free resolution can be of any length depending on the group ring under investigation. We consider the metacyclic groups $G(pq)$ which admit periodic resolutions. In such circumstances it is possible to achieve fully *diagonalised resolutions*.

By discussing the representation theory over integral group rings we obtain a complete list of *indecomposable modules* over $\mathbb{Z}[G(pq)]$. Such a list aids the decomposition of the augmentation ideal (the first *syzygy*) into a direct sum of indecomposable modules. Therefore we are able to achieve a diagonalised map here. From this point it is possible to decompose all of the remaining syzygies in terms of indecomposable modules, leaving a diagonal resolution in principle.

The existence of these diagonal resolutions significantly simplify a problem in low-dimensional topology, namely the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem. There are two stages to verifying this problem, and we prove the first stage using cohomological properties of the syzygy decompositions. The second stage is realising the Swan map. Although we do not manage to realise it fully, we are able to realise certain terms.

Finally this thesis includes an in depth exposition of the $\mathcal{R}(2)$-$\mathcal{D}(2)$ for the non-abelian group of order 21. In this case a positive result has been achieved using an explicitly calculated diagonal resolution.

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Dedicated to my parents

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Contents

Chapter 1

Introduction

1.1 Motivation

There are two motivations for this thesis, one being of algebraic, and the other of topological nature. Throughout, our concern is to look at group rings $\mathbb{Z}[G]$ of certain finite fundamental groups G. Specifically, we look at the metacyclic groups $G(pq)$:

$$G(pq) = \langle x, y \mid x^p = y^q = 1,\ yx = x^r y \rangle$$

where $r^q \equiv 1 \pmod p$, and $p,\ q$ are prime with $q|(p-1)$. Algebraically we look at the construction of free resolutions and show that in the case $G(pq)$ we are able to formulate completely *diagonalised* minimal free resolutions. These diagonal resolutions, whose existence was quite unexpected, significantly simplify the discussion. In particular, they greatly aid our attack on a problem in low-dimensional topology, namely the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem. We make significant progress towards proving the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem for $G(pq)$, and in the case of $G(21)$ a complete proof is achieved.

The notion of a free resolution was established by Hilbert's work on Invariant Theory [10]. Let Λ be the group ring of a finite group G. Any Λ-module M has a free resolution

$$\begin{tikzcd}
\cdots \arrow[r] & F_{n+1} \arrow[rr, "\partial_{n+1}"] \arrow[dr] & & F_n \arrow[r, "\partial_n"] & \cdots \arrow[r, "\partial_3"] & F_2 \arrow[rr, "\partial_2"] \arrow[dr] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr] & & F_0 \arrow[r, "\epsilon"] & M \arrow[r] & 0 \\
& & K(n+1) \arrow[ur] & & & & K(2) \arrow[ur] & & K(1) \arrow[ur] & & &
\end{tikzcd}$$

where F_m is a finitely generated free module over Λ. The $K(m)$ represents the m^{th} *syzygy*. Given two resolutions $\mathfrak{F}, \mathfrak{F}'$, we have exact sequences at each kernel, namely $K(m), K(m)'$. Then by Schanuel's Lemma there exists an isomorphism $K(m) \oplus F' \cong K(m)' \oplus F$ for finitely generated free modules F, F'. We therefore say that $K(m)$ and $K(m')$ are *stably equivalent*. We denote the class of all modules stably equivalent to $K(m)$ by $\Omega_m(M)$.

Constructing explicit free resolutions is not a straightforward task. Most groups do not have finite periodicity. Artin-Tate-Zassenhaus showed conditions for when a free resolution may be periodic. In the case of the metacyclic groups $G(pq)$, the free resolutions have period $2q$. More details on groups of periodic cohomology can be found in Swan's paper [28] and Wolf [31].

In Fox's paper [7], he describes a general geometric method in obtaining the first two $\partial_i s$ that relies on the generators and relators of a group G. However there are no such clear methods to describe a free resolution which has a period greater than two. Instead of using Fox's method, we look directly at the syzygies at the minimal level. We decompose the mystery of each syzygy into a direct sum of indecomposable modules. Thus, by describing ∂_1 as a direct sum of polynomials, we show that there is a completely diagonalised minimal free resolution for any group $G(pq)$.

Although the construction of free resolutions is an interesting problem in its own right, we turn our attentions to the use of resolutions in a topological setting. The $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem originates from Wall's $\mathcal{D}(n)$ problem.

The problem arises from an attempt to investigate the effectiveness of cohomology to determine the minimal dimension of a cell complex (CW space) $\mathcal{X}$. We say that $\widetilde{\mathcal{X}}$ is the universal cover of $\mathcal{X}$. By definition, it is true that if $\mathcal{X}$ is of geometrical dimension n, then $\mathcal{X}$ satisfies $H^m(\mathcal{X};\mathbb{Z}) = 0$ for all $m > n$. However, it is not necessarily true that if $H^m(\mathcal{X},\mathbb{Z}) = 0$ for all $m > n$ and $H^n(\mathcal{X},\mathbb{Z}) \neq 0$ then $\mathcal{X}$ is geometrically n-dimensional. For example, if $n \geq 1$, consider the cohomology of an $n + 1$ dimensional thickened out n-sphere. Such a cell complex is homotopy equivalent to a complex of dimension n. Thus,

Wall's $\mathcal{D}(n)$-problem. Let $\mathcal{X}$ be a connected cell complex of geometric dimension $n + 1$ such that:

$$H_{n+1}(\widetilde{\mathcal{X}},\mathbb{Z}) = H^{n+1}(\mathcal{X},\mathcal{B}) = 0$$

for all coefficient systems $\mathcal{B}$ on $\mathcal{X}$. Is $\mathcal{X}$ homotopy equivalent to a finite complex of dimension n?

In Wall's paper [30], he describes the $\mathcal{D}(n)$-Problem and solves it for all cases $n \in \mathbb{N}$ except when $n = 2$. Hence the only case left to be proved:

Wall's $\mathcal{D}(2)$-problem. Let $\mathcal{X}$ be a connected cell complex of geometric dimension 3 such that:

$$H_3(\widetilde{\mathcal{X}},\mathbb{Z}) = H^3(\mathcal{X},\mathcal{B}) = 0$$

for all coefficient systems $\mathcal{B}$ on $\mathcal{X}$. Is $\mathcal{X}$ homotopy equivalent to a finite complex of dimension 2?

In recent years there has been progress on the $\mathcal{D}(2)$ problem with the development of a theory linking the $\mathcal{D}(2)$ problem to another problem; realising algebraic 2-complexes.

Let $\mathcal{G} = \langle x_1, \ldots, x_g \mid W_1, \ldots, W_r \rangle$ be a presentation for a group G, and let $K_{\mathcal{G}}$ be the Cayley complex (a two dimensional CW complex) of $\mathcal{G}$, such that the fundamental group $\pi_1(K_{\mathcal{G}}) \cong G$. The cellular chain complex of the universal cover $\widetilde{K}_{\mathcal{G}}$ gives rise to:

$$C_*(\mathcal{G}) = \left(0 \to \pi_2(K_{\mathcal{G}}) \to C_2(\widetilde{K}_{\mathcal{G}}) \xrightarrow{\partial_2} C_1(\widetilde{K}_{\mathcal{G}}) \xrightarrow{\partial_1} C_0(\widetilde{K}_{\mathcal{G}}) \xrightarrow{\epsilon} \mathbb{Z} \to 0\right)$$

as an exact sequence of right $\mathbb{Z}[G]$-modules. The second homotopy group is defined by $\pi_2(K_{\mathcal{G}}) = \mathrm{Ker}(\partial_2)$. By an *algebraic 2-complex* over G we mean an exact sequence of right $\mathbb{Z}[G]$ modules, which mirrors the chain complex of a Cayley complex

$$\mathbf{F} = \left(0 \to J \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0\right)$$

where F_i is a finitely generated free module over $\mathbb{Z}[G]$. So for two algebraic 2-complexes **F**, and **F'** arising geometrically from cell complexes $C_*(\mathcal{G})$ and $C_*(\mathcal{G})'$ respectively, there exists a chain map making the algebraic 2-complexes equivalent, if and only if the cell complexes are *homotopically equivalent*. In terms of syzygies J acts as our $K(3)$ in a free resolution. The decomposition of $K(3)$ into indecomposable modules gives a structured approach to look at *geometric realisability* of algebraic 2-complexes. This leads us to the Realisation problem.

The Realisation $\mathcal{R}(2)$-problem. Let G be a finitely presented group. Is every algebraic 2-complex over G realised up to chain homotopy in the form $C_*(\mathcal{G})$?

The $\mathcal{R}(2)$ problem stems from a more general question, namely the $\mathcal{R}(n)$ problem. Johnson has given an affirmative answer for the $\mathcal{R}(n)$ problem when $n \geq 3$ ([15], chapter 9).

Johnson has shown in ([13], p. 256) that the $\mathcal{D}(2)$ problem is equivalent to the Realisation problem subject to mild conditions. The result was then extended to hold for all finitely presented fundamental groups by Mannan [19]. Thus, the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem.

1.2 Statement of results

The reader can look at this thesis in two different ways. Reading through in numerical order gives the preliminaries, followed by the general theory and the full example of $G(21)$ to finish. However, $G(21)$ was the original stimulus for this thesis. So alternatively, the reader may want to look at Chapter 7 first and see the explicit example, then return to Chapter 2 and digest the general theory.

In Chapter 2 we begin by recalling some basic algebraic preliminaries such as the Eckmann Shapiro Lemma, and the cyclic algebra construction $\mathscr{C}_n(\mathcal{R}, \theta, a)$. In section 2.3 we give an in depth discussion about discriminants based on a series of lectures by Johnson [12]. We describe in detail the discriminant of the cyclic algebra, as well as that of the quasi-triangular algebra $\mathscr{T}_n(S, \pi)$. This kind of discussion is not readily found in the literature.

Chapter 3 introduces more advanced preliminaries. Here we define G to be a finite group, and $\Lambda = \mathbb{Z}[G]$. A Λ-module that is finitely generated and free over $\mathbb{Z}$ is called a Λ-lattice. All modules in this thesis are Λ-lattices unless otherwise stated. We explain free resolutions and the existence of stable classes (syzygies). We describe the potential 'tree' structures of a stable module, followed by an explanation of certain cancellation properties.

Section 3.2 gives an overview of certain concepts already proven from homological algebra. Hence we state many results without proof. We take only the concepts necessary for this thesis, omitting many interesting points, and so a more in depth discussion of these topics, as well as free resolutions, can be found in various publications, such as Cartan and Eilenberg [2], and Maclane [17].

The first Theorem of this thesis arises in Chapter 4. We look at Milnor squares of the metacyclic groups, where we now restrict our group ring $\Lambda = \mathbb{Z}[G(pq)]$. The most direct fibre square for such groups is:

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \mathcal{I}^*_{\mathcal{G}} \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}/|G(pq)| \end{tikzcd}$$

However this does not give us any substantial information about Λ. Instead, we use the Milnor square of $\mathbb{Z}[C_p]$, and use the cyclic algebra to achieve Λ. As a first approximation, we obtain the Wedderburn decomposition for $\mathbb{Q}[G(pq)]$. Then we have the following square,

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \mathscr{C}_q(R, \theta_{[q]}, 1) \arrow[d] \\ \mathbb{Z}[C_q] \arrow[r] & \mathbb{F}[C_q] \end{tikzcd}$$

We are left with the question, what is $\mathscr{C}_q(R, \theta_{[q]}, 1)$? In Rosen's thesis he gives an explicit description, but with an opaque proof. Here, we re-interpret Rosen's Theorem [25] in a manner that hopefully is more accessible.

Theorem I (Rosen). *We take a group of the form $C_p \rtimes C_q$. Let $R = \mathbb{Z}[\zeta_p]$, q is a divisor of $(p-1)$. We then have $\theta_{[q]} = \theta^{(p-1/q)}$ with order q. Therefore the cyclic algebra $\mathscr{C}_q(R, \theta_{[q]}, 1)$ decomposes in the following manner:*

$$\mathscr{C}_q(R, \theta_{[q]}, 1) \cong \mathcal{J}_1 \oplus \ldots \oplus \mathcal{J}_q$$

where each $\mathcal{J}_i$ is an ideal. Furthermore $\mathcal{J}_i \not\cong \mathcal{J}_k$ if $i \neq k$. In fact each of the ideals are of the form $P^e = (\zeta_p - 1)^e \mathbb{Z}[\zeta_p]$ where $0 \leq e \leq q-1$.

The above formulation allows us to give a clear account of the duality relations which hold amongst the summands, a topic which is entirely absent from the literature.

Theorem II. *For any metacyclic group $G(pq)$, the following duality relations hold for the ideals contained in $\mathscr{C}_q(R, \theta_{[q]}, 1) \cong R \oplus P \oplus \ldots \oplus P^{q-1}$.*

$$P^e \cong_\Lambda (P^{q-e+1})^*$$

where $0 \leq e \leq q-1$. As a result we always have $P \cong R^*$

We now investigate the indecomposable modules that exist over Λ. Pu lists all the indecomposable modules in her paper [23], but her proofs in general are uninformative, and so using the re-interpretation of Rosen's Theorem in terms of the quasi-triangular matrix, we give a more elementary and hopefully clearer approach.

In what follows we employ the conventional term 'genera' in the enumeration of isomorphism classes of modules. In view of this it seems appropriate to include a brief exposition of the idea. Denote $\mathbb{Z}_{(p)}$ to be the localisation of $\mathbb{Z}$ at $\mathbb{Z} - p\mathbb{Z}$. Then, for a module M we say that $M_{(p)} = M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Thus, we say that two modules $M, N \in \Lambda$ belong to the same *genus*, denoted by $M \vee N$, when

$$M_{(p)} \cong N_{(p)}$$

as Λ-modules for all primes p. The same therefore applies for p-adic completions. So if we let $\widehat{\mathbb{Z}}_{(p)}$ be the ring of p-adic integers, that is, the completion of $\mathbb{Z}_{(p)}$ at p, then $M \vee N$ if and only if $\widehat{M_{(p)}} \cong \widehat{N}_{(p)}$, where $\widehat{M_{(p)}} = M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_{(p)}$. An in depth discussion of genus is found in Curtis and Reiner ([3], section 31). In the case where the projective class group $\widetilde{K}_0(\mathbb{Z}[G]) = 0$ membership of the same genus is identical with isomorphism over $\mathbb{Z}[G]$. This is the case, for example when $G = D_6, D_{10}$, but not generally.

We proceed by describing the *basic* genera of indecomposable modules. We prove some cohomological properties that exist between certain basic indecomposable modules to assist us in obtaining the genera of non-split extension indecomposable modules. We verify Pu's claim that the number of genera of indecomposable modules is $2 + q + 2^{q-1} + 2^q$.

To conclude this chapter we explain that the genera of indecomposable modules give a set of indecomposable modules over the free group ring $\mathbb{Z}[G(pq)]$. Any additional indecomposable modules then arise from projective elements in the projective class group $\widetilde{K}_0([\mathbb{Z}[G(pq)])$.

Chapter 5 deals with obtaining diagonalised free resolutions for the non-abelian groups $G(pq)$, discussing the importance of the free resolutions being periodic. We give an explicit example $\mathbb{Z}[G(21)]$ of the Fox method, showing the simplicity of it, but also to highlight the fact that using the Fox method we cannot obtain a diagonalised resolution.

Utilising the list of indecomposable modules obtained in Chapter 4, we determine at the minimal level that each syzygy decomposes into a direct sum of two indecomposable components. From here, we investigate the decomposition of the augmentation ideal (the first syzygy) explicitly.

Theorem III. $\Omega_1(\mathbb{Z})$ *at the minimal level can be described as* $\mathcal{I}_{\mathcal{G}} = P \oplus X$.

To verify we can split the augmentation ideal, we need to obtain polynomial interpretations for P and X that correspond with a change of basis to $(x-1, y-1)$ from the Fox method.

Theorem IV. $\mathcal{I}_{\mathcal{G}}$ *splits as a direct sum of* Λ*-modules*

$$\mathcal{I}_{\mathcal{G}} \;\cong\; [x-1+(y-1)\alpha) \;\dot{+}\; [y-1)$$

for some suitable $\alpha \in \Lambda$

The Fox method gives a neater interpretation of the augmentation ideal, but it is not possible to split this map as in Theorem IV. Note that Theorem IV gives a rank of $pq-1$ as desired, while splitting $(x-1, y-1)$ to give $[x-1)\dot{+}[y-1)$ leaves us with $\mathrm{rank}_{\mathbb{Z}}([x-1)\dot{+}[y-1)) = q(p-1)+p(q-1)$, which is greater than the rank of augmentation ideal. $[x-1)$ is in fact a polynomial representation of the cyclic algebra $\mathscr{C}_q(R, \theta_{[q]}, 1)$.

We finish the chapter by describing all the syzygies at the minimal level in terms of indecomposable modules. We therefore show the existence of fully diagonalised free resolutions, although not all of the indecomposable modules used in the free resolutions appear to have clear polynomial interpretations. Thus, we present the polynomials for whichever indecomposable modules currently have a clear polynomial interpretation.

Theorem V. *For any metacyclic group* $\Lambda = \mathbb{Z}[G(pq)]$*, we describe the syzygies at the minimal level of its free resolution by:*

$$\Omega_m(\mathbb{Z}) = \begin{cases} P^{(m+1)/2} \oplus X & \textit{when } 2 \nmid m \\ \mathcal{Q}(P^{(m/2)+1}) \oplus \overline{\mathbb{Z}[C_p]} & \textit{when } 2 \mid m \end{cases}$$

where $1 \leq m \leq 2q-1$.

In Chapter 6 we consider the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem for metacyclic groups, using the theory developed in Chapter 3 and the free resolutions from Chapter 5. To give a positive result for the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem we need to prove two conditions.

i) The tree structure of $\Omega_3(\mathbb{Z})$ is straight, implying only a single homotopy type exists.

ii) If **i)** holds, then we need to show fullness of the homotopy type via the Swan map. Thus, if we can realise all elements of the Swan map, we have a positive result for the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem.

Clearly condition **ii)** relies on condition **i)** being true. So using the indecomposable modules from Section 4.3 we can show straightness of $\Omega_3(\mathbb{Z})$.

Theorem VI. *Let $\Lambda = \mathbb{Z}[G(pq)]$. $\Omega_3(\mathbb{Z})$ has no branching at the minimal level, and so $\Omega_3(\mathbb{Z})$ is straight.*

As a corollary to Theorem VI, we notice that the tree structure of any odd syzygy $\Omega_{2m+1}(\mathbb{Z})$ must be straight. Henceforth we can proceed to condition **ii)**. At this point we manage to realise a couple of the elements of the Swan map in general. However this is where the exposition ends, and we are left with the task of realising the remainder of the Swan map in order to give a positive result to the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem for the groups $G(pq)$. We leave this problem in terms of a conjecture.

Conjecture VII (The Swan map for metacyclic groups $G(pq)$)**.** *When q is odd, all the elements in the reduced Swan map:*

$$\bar{s} : C_{q-1} \times C_{n/2} \twoheadrightarrow \widetilde{K}_0(\Lambda)/C_q$$

where $n = (p-1)/q$, can be realised.

We can realise the Swan map by brute force as achieved in Chapter 7 for $G(21)$, but this is not a feasible approach when dealing with large groups. We refer the reader to Bentzen and Madsen [1], and Milgram [20] for more details on realising Swan maps for various groups.

Chapter 7 is dedicated to verifying the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem for the non-abelian group of order 21. $G(21)$ is just about a small enough group where we can calculate the properties described in the other chapters explicitly. It is the smallest group of periodic cohomology six. As a result $G(21)$ was the original group looked at, and most of the theory developed has come as a consequence of investigating $G(21)$. To illustrate how the general theory for $G(pq)$ works out in practice we include full details in this particular case.

Each section here relates to its respective chapter in the general theory. Apart from detailed calculations, the one major addition in Chapter 7 is that the Swan map is fully realised for $G(21)$. Thus $G(21)$ gives a positive result for the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem.

Chapter 2

Preliminaries

2.1 Basic algebraic concepts

Let $\mathcal{A}$ be an arbitrary ring. An abelian group M is called a right $\mathcal{A}$-module if, for each $a \in \mathcal{A}$ and $m \in M$, a product $mr \in M$ is defined such that

$$\begin{array}{ll} (m_1 + m_2)a = m_1 a + m_2 a, & m(a_1 + a_2) = ma_1 + ma_2 \\ m(a_1 a_2) = (ma_1)a_2, & m \cdot 1 = m \end{array}$$

Throughout this thesis we work with right modules unless specified otherwise. We say that a module M is *simple* when the only $\mathcal{A}$-submodules of M are $\{0\}$ and M itself. Therefore we take M to be *semisimple* when:

$$M \cong M_1 \oplus \ldots \oplus M_m$$

where M_i is simple. An $\mathcal{A}$-module M which has an $\mathcal{A}$-basis is a *free* $\mathcal{A}$-module. These modules behave very much like vector spaces. More generally there is the notion of a *projective module.* An $\mathcal{A}$-module M is called a projective module when it is a direct summand of a free module. In other words there exists a module N such that $M \oplus N$ is a free module over $\mathcal{A}$. More details are available in Johnson's book ([13], chapter 1), leading to the following classification theorem.

Theorem 2.1.1 (Wedderburn)**.** *Let $\mathcal{A}$ be a semisimple algebra of finite dimension over a field k. Then there exists an isomorphism of k-algebras such that:*

$$\mathcal{A} \cong \prod_{i=1}^{m} M_{n_i}(\mathfrak{D}_i)$$

where m, n_i are natural numbers, $\mathfrak{D}_i$ are division algebras over k determined uniquely up to order and isomorphism.

With these basics in place we look at the representation theory of the group algebra $\mathbb{A}[G]$, where G is a finite group, and $\mathbb{A}$ is a commutative ring. Define a group representation by a homomorphism $\rho : G \to GL_n(\mathbb{A})$, where $GL_n(\mathbb{A})$ is the group of $n \times n$ matrices over $\mathbb{A}$. The group algebra construction $\mathbb{A}[G]$ is the set of all formal sums:

$$\mathbb{A}[G] = \sum_{g \in G} \alpha_g g$$

where $\alpha_g \in \mathbb{A}$. In fact $\mathbb{A}[G]$ is a free module over $\mathbb{A}$. The sum is given by:

$$\sum_{g \in G} \alpha_g g + \sum_{g \in G} \beta_g g = \sum_{g \in G} (\alpha_g + \beta_g) g$$

and the product is given by:

$$\sum_{g \in G} \alpha_g g \cdot \sum_{h \in G} \beta_h h = \sum_{g \in G} \left(\sum_{h \in G} \alpha_h \beta_{h^{-1}g} \right) g.$$

Theorem 2.1.2 (Maschke)**.** *Let G be a finite group and k a field whose characteristic is coprime to the order of G, then $k[G]$ is semisimple.*

We concentrate mainly on the integral group ring $\mathbb{Z}[G]$, as this will be the main aim of study in the thesis. A $\mathbb{Z}[G]$ module M is said to be *indecomposable* if $M \neq 0$ and if it is impossible to express M as a direct sum of two non-trivial submodules, $M \neq M_1 \oplus M_2$.

Define the dual of a $\mathbb{Z}[G]$-module M by

$$M^* = \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$$

where the G-action is given by $\rho^*(g) = \rho(g^{-1})^t$. We can see that this naturally transforms a right $\mathbb{Z}[G]$-module into a left $\mathbb{Z}[G]$-module. In most of the representation theory we describe such dual modules as modules over $\mathbb{Z}$. We will show below that these two cases are isomorphic here. Define the dual of a $\mathbb{Z}$-module M by

$$M^\star = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$$

where the G-action is given by $(fg)(m) = f(mg^{-1})$. The Eckmann-Shapiro Lemma will be used here in order to show $M^* = M^\star$. It will be used again later when dealing with the indecomposable modules of metacyclic groups.

Lemma 2.1.3 (Eckmann Shapiro)**.** *Let G be a finite group. Then let H be a subgroup $H \subset G$, and take the inclusion map to be $i : \mathbb{Z}[H] \hookrightarrow \mathbb{Z}[G]$. Then take M as a $\mathbb{Z}[G]$-module and N as an $\mathbb{Z}[H]$-module, so that $i^*(M)$ is the restriction of scalars,* *while $i_*(N) = N \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G]$ is the extension of scalars. Hence there exist two isomorphisms. The first isomorphism is called Frobenius Reciprocity:*

$$\mathrm{Hom}_{\mathbb{Z}[G]}(i_*(N), M) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(N, i^*(M))$$

and the second isomorphism is Shapiro's Lemma:

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M, i_*(N)) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(i^*(M), N)$$

Proposition 2.1.4. *Let M be a $\mathbb{Z}[G]$-module, then as $\mathbb{Z}[G]$-modules $M^* \cong M^\star$.*

Proof. Take i to be the inclusion map of the trivial group into G.

$$M^* = \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G]) = \mathrm{Hom}_{\mathbb{Z}[G]}(M, i_*(\mathbb{Z}))$$

Now using the Eckmann Shapiro lemma we see that

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M, i_*(\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(i^*(M), \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) = M^\star$$

Hence $M^* = M^\star$. □

By invoking the natural involution $\tau : \mathbb{Z}[G] \to \mathbb{Z}[G]$ where $\tau(g) = g^{-1}$, we transform M^* into a right module. Hence when discussing duality we can use M^* to denote a right $\mathbb{Z}[G]$-module. We say a module is *self-dual* when:

$$M^* \cong M$$

Furthermore define a polynomial interpretation of a module M by $p(x)$. The dual of $p(x)$ is defined by $p(x)^* = p(x)^{-1}$. Thus, $p(x)$ is self-dual when $p(x)^* = p(x)$. Lastly, such a polynomial is *anti self-dual* when:

$$p(x)^* = -p(x)$$

Cyclic algebra construction

The metacyclic groups $G(pq)$ give rise to cyclic algebras. We start by defining a commutative involuted ring $(\mathcal{R}, \theta, a)$ where $a \in \mathcal{R}$. By this we fix the following notation:

a) $\mathcal{R}$, a commutative ring,

b) a natural number $n \geq 2$,

c) an automorphism $\theta : \mathcal{R} \to \mathcal{R}$ satisfying $\theta^n = Id$,

d) an element $a \in \mathcal{R}$ such that $\theta(a) = a$

The cyclic algebra $\mathscr{C}_n(\mathcal{R}, \theta, a)$ of an involuted ring $(\mathcal{R}, \theta, a)$ is the two-sided free $\mathcal{R}$ module of rank n with basis $\mathcal{B} = \{1, y, \ldots, y^{n-1}\}$:

(2.i) $$\mathscr{C}_n(\mathcal{R}, \theta, a) = \mathcal{R} \dot{+} \mathcal{R}y \dot{+} \ldots \dot{+} \mathcal{R}y^{n-1}$$

subject to the following relation:

$$y^i \lambda = \theta^i(\lambda) y^{-i} \quad \text{for } 0 \leq i \leq n-1$$

The cyclic algebra construction commutes with direct products:

Proposition 2.1.5 ([13], p. 44)**.**

$$\mathscr{C}_n(\mathcal{R}_1 \times \mathcal{R}_2, \theta_1 \times \theta_2, a_1 \times a_2) \cong \mathscr{C}_n(\mathcal{R}_1, \theta_1, a_1) \times \mathscr{C}_n(\mathcal{R}_2, \theta_2, a_2)$$

Take two cyclic algebras of rank n, namely $\mathscr{C}_n(\mathcal{R}_1, \theta_1, a_1)$, $\mathscr{C}_n(\mathcal{R}_2, \theta_2, a_2)$ and a ring-homomorphism:

$$\rho : \mathcal{R}_1 \to \mathcal{R}_2$$

such that $\rho \circ \theta_1 = \theta_2 \circ \rho$, and $\rho(a_1) = a_2$, then we say that ρ is a *cyclic ring homomorphism*. Furthermore $\mathscr{C}_n(\mathcal{R}, \theta, a)$ may be regarded as an algebra over the fixed point ring in the following manner:

(2.ii) $$\mathcal{R}^\theta = \{x \in \mathcal{R} : \theta(x) = x\}$$

We define the rule of multiplication by:

$$y.y^r = \begin{cases} y^{r+1} & 0 \leq r < n-1 \\ a.1 & r = n-1 \end{cases}$$

Therefore $\mathcal{R}$ is free of rank n over $\mathcal{R}^\theta$, which implies $\mathscr{C}_n(\mathcal{R}, \theta, a)$ is free of rank n^2 over $\mathcal{R}^\theta$.

If we take $\mathcal{R} = \mathbb{F}$ to be a field, and $\theta : \mathbb{F}[G] \to \mathbb{F}[G]$ to be of order n, then we can take the fixed point field $\mathbb{F}^\theta$ to be the centre of the cyclic algebra construction $\mathscr{C}_n(\mathbb{F}, \theta, a)$. This result will be useful when looking at $\mathbb{Q}[G(pq)]$. More details on the cyclic algebra construction is found in Pierce [22].

2.2 Examples of Wedderburn decompositions

Take G to be a finite group and $\mathbb{F}$ to be a field of characteristic zero. We eventually want to discuss modules over the group ring $\mathbb{Z}[G]$. From Wedderburn's Theorem we have the following decomposition for the group algebra

$$\mathbb{F}[G] \cong \prod_{i=1}^{m} M_{n_i}(\mathfrak{D}_i)$$

where each $\mathfrak{D}_i$ is a finite dimensional division algebra over $\mathbb{F}$. We can always take $n_1 = 1$, as this represents the trivial one-dimensional representation of G which occurs with multiplicity$= 1$.

The easiest case is when $\mathbb{F} = \mathbb{C}$. As $\mathbb{C}$ is an algebraically closed field, the only division algebra over $\mathbb{C}$ is $\mathbb{C}$ itself. So the Wedderburn decomposition looks like

$$\mathbb{C}[G] \cong \mathbb{C} \times M_{d_2}(\mathbb{C}) \times \ldots \times M_{d_m}(\mathbb{C})$$

Furthermore in complex representation theory m is equal to the number of conjugacy classes, and $|G| = \sum_{i=1}^{m} d_i^2$. This makes complex Wedderburn decompositions straightforward to calculate.

The next field is $\mathbb{F} = \mathbb{R}$, the real field. There are now three division rings that can arise for any group G. They are $\mathbb{R}, \mathbb{C} = \mathbb{R}[x]/(x^2+1)$, and $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$, the quaternion algebra. It is the $\mathbb{H}$ that we will be most interested in, due to its relevance to the Eichler condition. So the Wedderburn decompositions for $\mathbb{R}[G]$ are only a little more complicated than $\mathbb{C}[G]$. Subsequently $\mathbb{R}[G]$ is just a small stepping stone towards $\mathbb{Q}[G]$, which is the closest approximation we have to help look at $\mathbb{Z}[G]$. In fact, if we can obtain a $\mathbb{Q}[G]$ Wedderburn decomposition, then it becomes apparent whether there exists an $\mathbb{H}$ factor in the corresponding $\mathbb{R}[G]$. Over $\mathbb{R}[G]$, m is equal to the number of conjugacy classes of subsets of G of the form $\{g, g^{-1}\}$.

Over $\mathbb{Q}[G]$ there are infinitely many $\mathfrak{D}_i$ that can arise. So, in general we do not know which $\mathfrak{D}_i$ occur. Over $\mathbb{Q}[G]$, m is equal to the number of conjugacy classes of cyclic subgroups of G. Below are a couple of groups which have fully decomposed representations. The details of the calculations can be found in Remez [24]. First the dihedral group of order ten $D_{10} = \langle x, y \mid x^5 = y^2 = 1, yx = x^4y \rangle$:

$$\begin{aligned}
\mathbb{C}[D_{10}] &\cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}) \times M_2(\mathbb{C}) \\
\mathbb{R}[D_{10}] &\cong \mathbb{R} \times \mathbb{R} \times M_2(\mathbb{R}) \times M_2(\mathbb{R}) \\
\mathbb{Q}[D_{10}] &\cong \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q}(\sqrt{5}))
\end{aligned}$$

This shows how rational representation differs. Now, we look at the Wedderburn decomposition of the quaternion group Q_8 in order to see a group that contains an $\mathbb{H}$ factor. $Q_8 = \langle x, y \mid x^4 = 1, y^4 = 1, x^2 = y^2, yx = xy \rangle$:

$$\begin{aligned}
\mathbb{C}[Q_8] &\cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}) \\
\mathbb{R}[Q_8] &\cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H} \\
\mathbb{Q}[Q_8] &\cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \left(\frac{-1,-1}{\mathbb{Q}}\right)
\end{aligned}$$

With the differences seen in the above examples, we see that rational representation $\mathbb{Q}[G]$ is the most complicated. Yet it gives a good first approximation to the module theory of $\mathbb{Z}[G]$. For that reason we obtain the rational Wedderburn decomposition for the cyclic groups C_n, followed by the dihedral groups D_{2n}. The cyclic groups of prime order will be needed when discussing the metacyclic groups $G(pq)$ as p, q prime. Within the dihedral groups, the metacyclic groups $G(2p)$ exist.

Cyclic groups

The cyclic group C_n is described as

$$C_n = \langle x | x^n = 1 \rangle$$

We can identify the group algebra $\mathbb{Q}[C_n]$ with the quotient $\mathbb{Q}[x]/(x^n - 1)$. It is well known that

$$x^n - 1 = \prod_{d|n} c_d(x)$$

where $c_d(x)$ is the d^{th} cyclotomic polynomial. Put $\zeta = e^{2\pi i/n}$, such that $\zeta^n = 1$. Therefore we can factorise

$$(x^n - 1) = (x - 1)(x - \zeta) \dots (x - \zeta^{n-1})$$

Clearly $C_n = \{1, \zeta, \dots, \zeta^{n-1}\}$. So all we need to know is the order of each ζ^r to obtain each $c_d(x)$

$$ord(\zeta^r) = \frac{n}{HCF(r, n)}$$

This means that for each $d|n$

$$c_d(x) = \prod_{ord(\zeta^r)=d} (x - \zeta^r)$$

and that each $c_d(x)$ is an irreducible polynomial over $\mathbb{Q}$. Hence

$$\mathbb{Q}(d) = \mathbb{Q}[x]/(c_d(x))$$

is a field. So it is now possible to decompose $\mathbb{Q}[C_n]$ into simple factors

$$\mathbb{Q}[C_n] \cong \prod_{d|n} \mathbb{Q}[x]/c_d(x) \cong \prod_{d|n} \mathbb{Q}(d)$$

Furthermore if ζ_d is a primitive d^{th} root of unity then the fixed field of $\mathbb{Q}(d)$ under complex conjugation is $\mathbb{Q}(\mu_d)$ where $\mu_d = \zeta_d + \zeta_d^{-1}$.

Dihedral groups

For $n \geq 3$, define the dihedral group D_{2n} by the presentation:

$$D_{2n} = \langle x, y | x^n = y^2 = 1, \quad yx = x^{n-1}y \rangle$$

Take the cyclic algebra over $\mathbb{Q}[C_n]$, with θ acting as the natural involution

$$\mathbb{Q}[D_{2n}] \cong \mathscr{C}_2(\mathbb{Q}[C_2], \theta, 1)$$

where $\theta : \mathbb{Q}[C_n] \to \mathbb{Q}[C_n]$ is the involution given on the basis elements, $\theta(g) = g^{-1}$. Knowing $\mathbb{Q}[C_n]$ we want to see the effect of the involution on each simple term $\mathbb{Q}(d)$. When $d = 1, 2$, $\theta : \mathbb{Q}(d) \to \mathbb{Q}(d)$ induces the identity, while when $d \geq 3$, θ induces complex conjugation $\theta(g) = g^{-1} = \bar{g}$. Hence

$$\mathscr{C}_2(\mathbb{Q}(d), \theta, 1) = \begin{cases} \mathbb{Q} \times \mathbb{Q} & d = 1, 2 \\ M_2(\mathbb{Q}(\mu_d)) & d \geq 3 \end{cases}$$

where $\mu = \zeta_d + \bar{\zeta}_d$ and ζ_d is a primitive d^{th} root of unity. So in order to obtain the Wedderburn decomposition of $\mathbb{Q}[D_{2n}]$ we just need to add the simple factors from

$$\mathbb{Q}[D_{2n}] \cong \prod_{d|n} \mathscr{C}_2(\mathbb{Q}(d), \theta, 1)$$

Hence the Wedderburn decomposition for $\mathbb{Q}[D_{2n}]$ when $n \geq 2$

$$\mathbb{Q}[D_{2n}] \cong \begin{cases} \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \prod\limits_{d|n, d\geq 3} M_2(\mathbb{Q}(\mu_d)) & if 2 \mid n \\ \mathbb{Q} \times \mathbb{Q} \times \prod\limits_{d|n, d\geq 3} M_2(\mathbb{Q}(\mu_d)) & if 2 \nmid n \end{cases}$$

See [24] for more explicit examples, as well as decompositions of the quaternion groups Q_{4n}. Note that if n is a prime number p, then we have a complete Wedderburn decomposition for all the metacyclic groups of the form $G(2p)$.

$$G(2p) \cong \mathbb{Q}[D_{2p}] \cong \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q}(\mu_p))$$

2.3 Discriminants

The following section is taken from a series of lectures by Johnson [12]. Not all of the discussion of discriminants described here is readily found in the literature, and so we give a detailed analysis. We commence by looking at the bilinear form of an associative algebra.

Let S denote a commutative ring, and A an S-algebra which is free of finite rank over S. Considering A as a right S-module, there exists a mapping, the *adjoint representation*, $\mathrm{Ad}_A : A \to \mathrm{End}_S(A)$ given by:

$$Ad_A(x)(z) = xz$$

From this we can see:

$\mathrm{Ad}_A : A \to \mathrm{End}_S(A)$ is a homomorphism of S-algebras.

Composing with the trace map $\mathrm{Tr}_S : \mathrm{End}_S(A) \to S$ we construct a mapping:

$$\beta : A \times A \to S \quad ; \quad \beta(x, y) = \mathrm{Tr}_S(Ad_a(x)Ad_A(y))$$

Hence β is a symmetric bilinear form of A.

If $\mathcal{E} = \{e_1, \ldots, e_n\}$ is an S-basis for A define the *discriminant* $\mathscr{D}\mathrm{isc}_S(A, \mathcal{E})$ of A *relative to* $\mathcal{E}$ by:

$$\mathscr{D}\mathrm{isc}_S(A, \mathcal{E}) = \det(\beta(e_i, e_j)_{1\leq i,j\leq n}$$

To inspect the effect of basis change suppose that $\Phi = \{\phi_1, \ldots, \phi_n\}$ is also an S-basis for A and let $\beta^{\mathcal{E},\Phi}$ denote the $n \times n$ matrix:

$$\left(\beta^{\mathcal{E},\Phi}\right)_{ij} = \det(\beta^{\mathcal{E},\mathcal{E}})$$

Re-interpreting this definition we get:

$$\mathscr{D}\mathrm{isc}_S(A, \mathcal{E}) = \det(\beta^{\mathcal{E},\mathcal{E}})$$

Take $Q = (q_{kj})$ to denote the invertible matrix which expresses Φ in terms of $\mathcal{E}$; $\phi_j = \sum\limits_{k=1}^{n} e_k q_{kj}$. As a result $\beta(e_i, \phi_j) = \sum\limits_{k=1}^{n} \beta(e_i, e_k)q_{kj}$ and:

$$\beta^{\mathcal{E},\Phi} = \beta^{\mathcal{E},\mathcal{E}}Q$$

The determinant of $\beta^{\mathcal{E},\Phi}$ gives:

(2.iii) $$\det(\beta^{\mathcal{E},\Phi}) = \det(Q)\mathscr{D}\mathrm{isc}_S(A, \mathcal{E})$$

Likewise $\beta(\phi_i, e_j) = \sum\limits_{k=1}^{n} q_{ki}\beta(e_k, e_j)$, so:

$$\beta^{\Phi,\mathcal{E}} = Q^t\beta^{\mathcal{E},\mathcal{E}}$$

So putting the two together we get:

$$\beta^{\Phi,\Phi} = Q^t\beta^{\mathcal{E},\mathcal{E}}Q$$

So by taking determinants we achieve:

$$\mathscr{D}\mathrm{isc}_S(A,\Phi) = \det(Q)^2 \mathscr{D}\mathrm{isc}_S(A,\mathcal{E})$$

The matrix Q is invertible over S so that $\det(Q) \in S^*$. We denote $[\lambda]$ as the class of $\lambda \in S$ in the quotient monoid $(S/S^*)^2$ then we may define the discriminant $\mathscr{D}\mathrm{isc}_S(A)$ as an element of $S/(S^*)^2$ by:

$$\mathscr{D}\mathrm{isc}_S(A) = [\mathscr{D}\mathrm{isc}_S(A,\mathcal{E})]$$

for any S-basis $\mathcal{E}$.

This gives a relative interpretation. Thus suppose A is an S-algebra which is free of finite rank n over S and that Φ is an S-basis for B. Then we can express Φ in terms of $\mathcal{E}$:

$$\phi_j = \sum_{k=1}^{n} e_k q_{kj}$$

Again it follows $\beta^{\Phi,\Phi} = Q^t \beta^{\mathcal{E},\mathcal{E}} Q$, and so taking determinants:

(2.iv) $$\mathscr{D}\mathrm{isc}_S(B,\Phi) = \det(Q)^2 \mathscr{D}\mathrm{isc}_S(A,\mathcal{E})$$

However in general $\det(Q) \notin S^*$; and in fact $\det(Q) \in S^*$ if and only if $A = B$.

The discriminant of a cyclic algebra

Take the cyclic algebra $\mathscr{C}_n(\mathcal{R},\theta,1)$ as explained in (2.i). Let S be defined as the fixed point ring $S = \mathcal{R}^\theta = \{x \in \mathcal{R} : \theta(x) = x\}$. S holds the same properties described in (2.ii).

We make the abbreviation $\mathscr{C} = \mathscr{C}_n(\mathcal{R},\theta,1)$. In this subsection we shall compute the discriminant of $\mathscr{C}_n(\mathcal{R},\theta,1)$ in terms of $\mathcal{R}, n$ and θ. Explicitly we show:

(2.v) $$\mathscr{D}\mathrm{isc}_S(\mathscr{C}) = \sigma_n \det(\theta)^{n(n-1)/2} \mathscr{D}\mathrm{isc}_S(\mathcal{R})^n n^{n^2}$$

Here we have $\det(\theta) \in S^*$ as the determinant of θ considered as an S-linear map $\mathcal{R} \to \mathcal{R}$ and σ_n is taken to be the sign ± 1. Explicitly we have:

(2.vi) $$\sigma_n = \begin{cases} -1 & \text{if } n \equiv 3 \bmod 4 \\ 1 & \text{otherwise} \end{cases}$$

When **w**,**x** $\in \mathscr{C}$ write:

$$\mathbf{w} = \sum_{k=0}^{n-1} \widetilde{w_k} y^k, \qquad \mathbf{x} = \sum_{l=0}^{n-1} \widetilde{x}_l y^l$$

where $\widetilde{w_k}, \widetilde{x_l} \in \mathcal{R}$. To multiply in $\mathscr{C}$ we add indices mod n:

$$\mathbf{w}\,\mathbf{x} = \sum_{k,l=0}^{n-1} \widetilde{w_k}\theta^k(\widetilde{x_l})y^{k+l}$$

To compute $\mathscr{D}\mathrm{isc}_S(\mathscr{C})$ we must first consider the bilinear form $B : \mathscr{C} \times \mathscr{C} \to S$:

$$B(\mathbf{w,x}) = \mathrm{Tr}_{\mathscr{C}/S}[Ad_{\mathscr{C}}(\mathbf{wx})]$$

We shall also consider the bilinear form $\beta : \mathcal{R} \times \mathcal{R} \to S$:

$$\beta(\phi, \psi) = \mathrm{Tr}_{\mathcal{R}/S}[Ad_{\mathcal{R}}(\phi, \psi)]$$

We note that:

$$\mathrm{Tr}_{\mathscr{C}/S}[AD_{\mathscr{C}}(\sigma y^k)] = \begin{cases} n\mathrm{Tr}_{\mathcal{R}/S}[Ad_{\mathcal{R}}(\sigma)] & k = 0 \\ 0 & k \neq 0 \end{cases}$$

If $\Phi = (\phi)_{1 \leq s \leq n}$ is an S-basis for $\mathcal{R}$ we obtain an S-basis $\mathcal{E} = (e_j)_{1 \leq j \leq n^2}$ for $\mathscr{C}$ on putting

$$e_{kn+s} = \phi_s y^k \quad (0 \leq k \leq n-1, 1 \leq s \leq n)$$

We let $B^{k,l}$ denote the $n \times n$ matrix over S given by $(B^{k,l})_{s,r} = B(e_{kn+s,lm+r})$. Evidently:

$$(B^{k,l})_{s,r} = B(\phi_s y^k, \phi_r y^l) = \mathrm{Tr}_{\mathscr{C}}[Ad_{\mathscr{C}}(\phi_s\theta^k(\phi_r)y^k + l)]$$

Let ρ denote the permutation of the index set $\{0, 1, \ldots, n-1\}$ given by

$$\rho(k) = -k \text{ (mod } n)$$

Then it follows that:

$$(B^{k,l})_{s,r} = \begin{cases} n\mathrm{Tr}_{\mathcal{R}/S}[Ad_{\mathcal{R}}(\phi_s\theta^k(\phi_r)] & l = \rho(k) \\ 0 & l \neq \rho(k) \end{cases}$$

If we regard the indices k, l as belonging to the group $\mathbb{Z}/n$ and replace each occurrence of the value 0 by n we obtain a decomposition of the matrix

$$\widetilde{B} = (B(e_i, e_j)) \quad (1 \leq i, j \leq n^2)$$

into $n \times n$ blocks $B^{k,l}$ $(1 \leq k, l \leq n)$ where from above it follows that:

$$(B^{k,l}) = 0 \text{ if } l \neq \rho(k)$$

and

$$(B^{k,\rho(k)})_{s,r} = n\beta(\phi_s \theta^k(\phi_r))$$

From earlier this becomes

$$B^{k,\rho(k)} = n\beta^{\Phi,\theta^k(\Phi)}$$

By taking determinants (2.iii), we deduce:

$$\det(B^{k,\rho(k)}) = \det(\theta^k)\mathscr{D}\mathrm{isc}_S(\mathcal{R})n^n$$

If σ is a permutation of the indices $1 \leq i \leq n^2$ denote $Q(\sigma)$ as the corresponding permutation matrix of size $n^2 \times n^2$. Left multiplication by $Q(\sigma)$ then performs the corresponding permutation to the rows of $\widetilde{B}$. We denote by $[Q(\sigma)\widetilde{B}]^{k,l}$ the decomposition of $Q(\sigma)\widetilde{B}$ into $n \times n$ blocks. It follows that we may choose σ such that

$$[Q(\sigma)\widetilde{B}]^{k,l} = \begin{cases} B^{k,\rho(k)} & \text{if } k = l \\ 0 & \text{if } l \neq k \end{cases}$$

Thus sign$(\sigma)\det(\widetilde{B}) = \prod_{k=1}^{n} \det(B^{k,\rho(k)})$ and we obtain:

$$\text{sign}(\sigma)\mathscr{D}\mathrm{isc}_S(\mathscr{C}) = \left(\prod_{k=1}^{n} \det(\theta^k)\right) \mathscr{D}\mathrm{isc}_S(\mathcal{R})^n(n^n)^n$$

Recall that $\theta^n = Id$. We therefore calculate that $\prod_{k=1}^{n} \det(\theta^k) = \det(\theta)^{n(n-1)/2}$. Multiplying across by $\sigma_n = sign(\sigma)$ gives

$$\mathscr{D}\mathrm{isc}_S(\mathscr{C}) = \sigma_n \det(\theta)^{n(n-1)/2}\mathscr{D}\mathrm{isc}_S(\mathcal{R})^n n^{n^2}$$

Hence we obtain (2.v) provided we can show that σ_n =sign(σ) has the form shown in (2.vi).

Let f denote the number of indices in $\mathbb{Z}/n$ fixed under ρ. If we view this as a permutation, $\{0, 1, 2, \ldots, n-1\}$, then ρ interchanges the elements of precisely $m = (n-f)/2$ pairs, leaving the remaining indices fixed. Thus the permutation σ on $I = \{i \mid 1 \leq i \leq n^2\}$ may be written as

$$\sigma = \tau_1 \circ \ldots \circ \tau_m$$

where each τ_r swaps over a pair of disjoint subsets of size n taken from I. To swap each such pair requires n transpositions; that is, for each r, sign$(\tau_r) = (-1)^n$ and so

$$\sigma_n = (-1)^{n(n-f)/2}$$

When n is even the only indices fixed by ρ are $0(= n)$ and $n/2$ so that $f = 2$. In this case $\sigma_n = 1$. When n is odd the only index fixed by ρ is $0(= n)$ so that $f = 1$. Thus

$\sigma_n = (-1)^{n(n-1)/2}$. So, as required in (2.vi), $\sigma_n = -1$ when $n \equiv 3 \pmod 4$, and $\sigma_n = 1$ otherwise. Consequently:

(2.vii)
$$\sigma_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ (-1)^{n(n-1)/2} & \text{if } n \text{ is odd} \end{cases}$$

The discriminant of a quasi-triangular algebra

Let $M_n(S)$ be the algebra of $n \times n$ matrices over a commutative ring S. If $\pi \in S$ we denote by $\mathscr{T}_n(S, \pi)$ the following subalgebra of $M_n(S)$:

$$\mathscr{T}_n(S, \pi) = \{X = (X_{ij}) \in M_n(S) \mid X_{ij} \equiv 0 \ (\mathrm{mod}\ \pi) \text{ for } i > j\}$$

This subsection is devoted to showing the discriminant $\mathscr{D}\mathrm{isc}_S(\mathscr{T}_n(S, \pi))$ is given by:

(2.viii)
$$\mathscr{D}\mathrm{isc}_S(\mathscr{T}_n(S, \pi)) = (-1)^{n(n-1)/2} \pi^{n(n-1)} n^{n^2}$$

In fact, the general case follows directly from the special case where $\pi = 1$; then $\mathscr{T}_n(S, \pi) = M_n(S)$ and the discriminant is:

(2.ix)
$$\mathscr{D}\mathrm{isc}_S(M_n(S)) = (-1)^{n(n-1)/2} n^{n^2}$$

We begin by describing $M_n(S)$ as a cyclic algebra. Let c denote the cyclic permutation of $\{1, \ldots, n\}$

$$c(r) = \begin{cases} r+1 & r < n \\ 1 & r = n \end{cases}$$

and let $c_* : S^{(n)} \to S^{(n)}$ denote the ring automorphism of the n-fold direct product which permutes the co-ordinates via c

$$c_*(x_1, \ldots, x_n) = x_{c(1)}, \ldots, x_{c(n)})$$

Then S is isomorphic to the subring of $S^{(n)}$ fixed under c_* via the diagonal imbedding $S \to S^{(n)}; x \mapsto (x, \ldots, x, x)$. In particular, $S^{(n)}$ is an S-algebra.

Proposition 2.3.1. *There is a ring isomorphism $\nu : \mathscr{C}_n(S^{(n)}, c_*) \to M_n(S)$.*

Proof. There is a ring homomorphism $\nu : S^{(n)} \to M_n(S)$ given by:

$$\nu(x_1, x_2, \ldots, x_n) = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$$

Take δ to be the Kronecker delta, consider the permutation matrix $\eta \in M_n(S)$ given by:

$$\eta_{s,r} = \delta_{c(s),r}$$

We see that ν extends to a ring homomorphism $\nu : \mathscr{C}_n(R^{(n)}, c_*) \to M_n(S)$ on assigning $\nu(y) = \eta$. We claim that ν is bijective.

First note, as both $\mathscr{C}_n(R^{(n)}, c_*)$ and $M_n(S)$ are free of rank n^2 over the commutative ring S, then any S-linear surjection $\mathscr{C}_n(R^{(n)}, c_*) \to M_n(S)$ is necessarily injective. It therefore suffices to show that ν is surjective. To see this, let $\mathcal{E} = \{\epsilon(p,q)\}(1 \leq p, q \leq n)$ denote the canonical S-basis for $M_n(S)$:

$$\epsilon(p,q)_{s,r} = \delta_{p,s}\delta_{q,r}$$

We can see that $\epsilon(p,q) = \eta^{(1-p)}\epsilon(1,1)\eta^{q-1)}$. Taking $E = (1, 0, \ldots, 0)$ in $S^{(n)}$ we have $\nu(E) = \epsilon(1,1)$. So:

$$\epsilon(p,q) = \nu[y^{(1-p)}Ey^{(q-1)}]$$

Hence ν is surjective. $\square$

We have shown that $\mathscr{D}\mathrm{isc}_S(S^{(n)}) = 1$. Moreover, $\det(c_*) = \text{sign}(c)$, thus using Proposition 2.3.1:

$$\mathscr{D}\mathrm{isc}_S(M_n(S)) = \sigma_n sign(c)^{n(n-1)/2}n^{n^2}$$

Since sign$(c) = (-1)$ if n is even, whilst sign$(c) = 1$ if n is odd, we see that:

$$\text{sign}(c)^{n(n-1)/2} = \begin{cases} (-1)^{n(n-1)/2} & \text{if n is even} \\ 1 & \text{if n is odd} \end{cases}$$

So for all n,

$$\sigma_n \text{sign}(c)^{n(n-1)/2} = (-1)^{n(n-1)/2}$$

The special case (2.ix) now follows.

To consider the general case we abbreviate $\mathscr{T}_n(S, \pi) = \mathscr{T}_n$. If $\mathcal{F}$ is an S-basis for $\mathscr{T}_n$ then from (2.iv) we have:

$$\mathscr{D}\mathrm{isc}_S(\mathscr{T}_n, \mathcal{F}) = \det(Q)^2\mathscr{D}\mathrm{isc}_S(M_n(S), \mathcal{E})$$

where Q is the $n^2 \times n^2$ matrix which expresses $\mathcal{F}$ in terms of the canonical basis $\mathcal{E}$ of $M_n(S)$. As $\mathscr{D}\mathrm{isc}_S(M_n(S)) = \mathscr{D}\mathrm{isc}_S(M_n(S), \mathcal{E})$ it follows from the special case (2.ix) that:

$$\mathscr{D}\mathrm{isc}_S(\mathscr{T}_n, \mathcal{F}) = (-1)^{n(n-1)/2}\det(Q)^2 n^{n^2}$$

However the specific basis $\mathcal{F} = \{\phi(p,q)\}(1 \le p, q \le n)$ for $\mathscr{T}_n(S,\pi)$ is given by:

$$\phi(p,q) = \begin{cases} \epsilon(p,q) & \text{if } p \le q \\ \pi\epsilon(p,q) & \text{if } q < p \end{cases}$$

Therefore we see that $\det(Q) = \pi^{n(n-1)/2}$. Hence

$$\mathscr{D}\mathrm{isc}_S(\mathscr{T}_n, \mathcal{F}) = (-1)^{n(n-1)/2}\pi^{n(n-1)}n^{n^2}$$

The general case (2.viii) now follows as $\mathscr{D}\mathrm{isc}_S(\mathscr{T}_n(S,\pi)) = \mathscr{D}\mathrm{isc}_S(\mathscr{T}_n, \mathcal{F})$.

Chapter 3

Free resolutions and module extensions

3.1 Free resolutions and stable modules

Let G be a finite group, and $\Lambda = \mathbb{Z}[G]$ be the integral group ring. We define a Λ-lattice to be a Λ-module that is finitely generated and free over $\mathbb{Z}$. Throughout this thesis when we discuss modules we automatically imply lattices unless clearly stated otherwise. We denote $\mathcal{F}(\Lambda)$ as the category of all Λ-lattices. It can be regarded as a full subcategory of Λ-modules.

We require that Λ possesses the IBN property (Invariant Basis Property), that is, for positive integers a, b

$$\Lambda^a \xrightarrow{\simeq} \Lambda^b \Longrightarrow a = b$$

The IBN property is automatic for group algebras. We define a Λ-module M to be *free* when there exists a basis $\{e_r\}_{1\leq r\leq n}$ for M over Λ. So if Λ is a field then every module $M \cong \Lambda^n$ for some n. However, let M not be free, and let $\{\mu\}_{1\leq r\leq n}$ be the minimal generating set such that for all $x \in M, x = \Sigma_{r=1}^{n}\mu_r\lambda_r$, where $\lambda_r \in \Lambda$. Let $\{e_r\}_{1\leq r\leq n}$ be a standard basis for Λ^{n_0}. We then have the following surjective map:

$$\mu : \Lambda^{n_0} \twoheadrightarrow M$$

where $\mu(e_r) = \mu_r$. This implies that if M is not free, then μ has a kernel. This gives a short exact sequence:

$$0 \to K_1 \overset{i}{\hookrightarrow} \Lambda^{n_0} \overset{\mu}{\twoheadrightarrow} M \to 0$$

where $K_1 = \mathrm{Ker}(\mu)$. Then there exists a Λ^{n_1} which has a surjective ring homomorphism to K_1, and a kernel of K_2.

$$\begin{tikzcd}
 & & K_2 \arrow[dr, "i_2"] & & & & & \\
\dots \arrow[r, "\partial_3"] & \Lambda^{n_2} \arrow[ur, "p_2"] \arrow[rr, "\partial_2"] & & \Lambda^{a_1} \arrow[rr, "\partial_1"] \arrow[dr, "p_1"'] & & \Lambda^{a_0} \arrow[r, two heads, "\mu"] & M \arrow[r] & 0 \\
 & & & & K_1 \arrow[ur, "i_1"'] & & &
\end{tikzcd}$$

It is clear to see that $K_2 = \mathrm{Ker}(\partial_1) = \mathrm{Ker}(p_1)$. We can keep doing this process inductively to obtain the notion of a *free resolution*, where all modules apart from M are free:

$$\dots F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{(\mu=\partial_0)} M \to 0$$

Here each F_n is finitely generated and free over Λ, and $K_m = \mathrm{Ker}(\partial_{m-1})$. As a result $K_m \in \mathcal{F}(\Lambda)$. However the $K_m s$ in a free resolution are not unique. It is possible to find a different free resolution to M:

$$\dots F'_m \xrightarrow{\partial'_m} F'_{m-1} \xrightarrow{\partial'_{m-1}} \dots \xrightarrow{\partial'_2} F'_1 \xrightarrow{\partial'_1} F'_0 \xrightarrow{(\mu'=\partial'_0)} M \to 0$$

Here we have $K'_m = \mathrm{Ker}(\partial'_{m-1})$. We can also take $K_0 = K'_0 = M$. This shows that the kernels are not uniquely defined. However they are stably related. We say two modules $K, K' \in \mathcal{F}(\Lambda)$ are *stably equivalent*, $K \sim K'$, if for some a, b:

$$K \oplus \Lambda^a \cong K' \oplus \Lambda^b$$

and we can denote this category by:

$$\mathrm{Stab}(\Lambda) = \mathcal{F}(\Lambda)/\sim$$

In order to show the stability relation for free resolutions we need the following result.

Proposition 3.1.1 (Schanuel's Lemma,[13], p.52)**.** *If we have two short exact sequences:*

$$0 \to K \to F \to M \to 0$$
$$0 \to K' \to F' \to M \to 0$$

and F, F' are free, then:

$$K \oplus F' \cong K' \oplus F$$

Hence $K \sim K'$.

As a result we prove that the two free resolutions are stably equivalent:

Proposition 3.1.2. *Take two free resolutions:*

$$\ldots F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{(\mu=\partial_0)} M \to 0$$
$$\ldots F'_m \xrightarrow{\partial'_m} F'_{m-1} \xrightarrow{\partial'_{m-1}} \ldots \xrightarrow{\partial'_2} F'_1 \xrightarrow{\partial'_1} F'_0 \xrightarrow{(\mu'=\partial'_0)} M \to 0$$

and as before $K_m = \mathrm{Ker}(\partial_{m-1})$, $K'_m = \mathrm{Ker}(\partial'_{m-1})$. *Then* $K_m \sim K'_m$ *for each* m.

Proof. The proof follows by induction. It clearly works when $i = 1$ by using Schanuel's Lemma, as $K_0 = K'_0 = M$. Now suppose it works when $n = m-1$ such that:

$$K_{m-1} \oplus \Lambda^a \cong K'_{m-1} \oplus \Lambda^b$$

Looking at when $n = m$:

$$0 \to K_m \to F_{m-1} \to K_{m-1} \to 0$$
$$0 \to K'_m \to F'_{m-1} \to K'_{m-1} \to 0$$

Using the stabilisation of when $n = m-1$ we can obtain the following sequences:

$$0 \to K_m \to F_{m-1} \oplus \Lambda^a \to K_{m-1} \oplus \Lambda^a \to 0$$
$$0 \to K'_m \to F'_{m-1} \oplus \Lambda^b \to K'_{m-1} \oplus \Lambda^b \to 0$$

This satisfies the conditions of Schanuel's Lemma:

$$K_m \oplus F'_{m-1} \oplus \Lambda^b \cong K'_n \oplus F_{m-1} \oplus \Lambda^a$$

Thus $K_m \sim K'_m$. □

The above result shows us that although K_m is not uniquely determined by M for different resolutions, K_m is unique up to stable equivalence.

We define the *stable class* $[K] \in \mathrm{Stab}(\Lambda)$ as the class of Λ-lattices equivalent to K by:

$$[K] = \{K' : K' \oplus \Lambda^a \cong K \oplus \Lambda^b \text{ for some } a, b\}$$

This leads us to the concept of the *syzygy* operator. We write

$$\Omega_m(M) = [K_m]$$

as the m^{th} syzygy of the module M. Syzygies play an integral role in this thesis, and Chapter 5 is dedicated to understanding $\Omega_m(\mathbb{Z})$ for the metacyclic groups $\mathbb{Z}[G(pq)]$. This will be done by decomposing $\Omega_m(\mathbb{Z})$ at the minimal level into indecomposable modules, helping us simplify the calculation in obtaining free resolutions for metacyclic groups.

For each module $M \in \mathcal{F}(\Lambda)$, there exists a dual resolution such that $\Omega_{-m}(M)$ is the stable class $[K_m]$ of a module $K_m \in \mathcal{F}(\Lambda)$:

$$0 \to M \to F_0 \to F_1 \to \ldots \to F_{n-1} \to K_m \to 0$$

where each F_n is finitely generated and free over Λ. This leads to the idea of dual stable classes, more specifically syzygies. Hence:

Proposition 3.1.3. *For any module $M \in \mathcal{F}(\Lambda)$ the following syzygy relations exist:*

a) $\Omega_m(\Omega_n(M)) = \Omega_{m+n}(M)$

b) $\Omega_n(M^*) = \Omega_{-n}(M)^*$

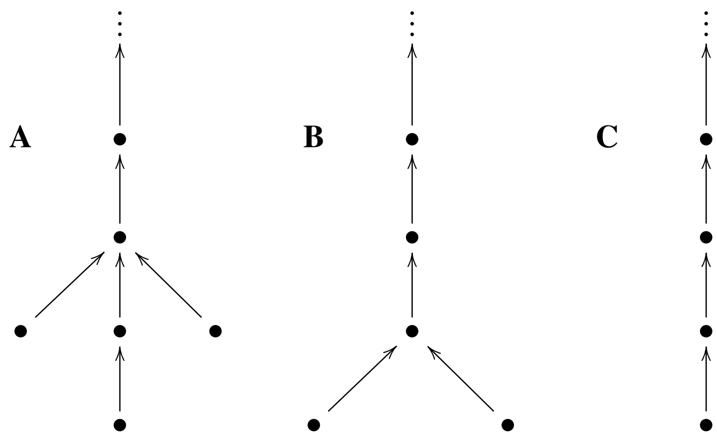
We look now at a cancellation theorem for Λ-lattices. Let M be a Λ-lattice. We say that M satisfies the *cancellation property* when for any $N \in \mathcal{F}(\Lambda)$ such that $\mathrm{rank}_{\mathbb{Z}}(M) \leq \mathrm{rank}_{\mathbb{Z}}(N)$:

$$N \oplus \Lambda^a \cong M \oplus \Lambda^b \Longrightarrow N \cong M \oplus \Lambda^{b-a}$$

In fact any stable module $[M]$ has a representation as a directed graph in which the vertices are the modules $N \in [M]$, and where we draw an arrow $N \to N \oplus \Lambda$ for each isomorphism type N. Such directed graphs have no cycles assuming the ring Λ satisfies the *surjective rank property*. We define Λ to have the surjective rank property if, given positive integers N, n and a surjective Λ-homomorphism:

$$\varphi : \Lambda^N \longrightarrow \Lambda^n, \quad \text{then } N \geq n.$$

We can also say that a ring Λ is *weakly finite* if any surjective Λ-homomorphism $\varphi : \Lambda^n \to \Lambda^n$ is necessarily bijective. We can therefore portray a stable module $[M]$ graphically as a 'tree with roots'. This concept was first introduced by Dyer and Sieradski [5]. The tree structure does not extend infinitely downwards, and so define a *minimal module* M_0 to be a module that does not contain a summand isomorphic to Λ. There are three types of structures that can be found for $[M]$ when G is finite:



A occurs as the stable class of $[0]$ when $\Lambda = \mathbb{Z}[Q_{24}]$, where the presentation of Q_{24} is defined by $Q_{24} = \langle x, y | x^6 = y^2, y^4 = 1, yxy^{-1} = x^{-1} \rangle$. **B** arises as the stable class $\Omega_3(\mathbb{Z})$ of $\mathbb{Z}[Q_{32}]$. Lastly **C** arises as the stable class $\Omega_3(\mathbb{Z})$ for the dihedral groups D_{4n+2}. As we will see in Chapter 6, all metacyclic groups have $\Omega_3(\mathbb{Z})$ of type **C**.

To investigate properties for cancellation over Λ, we move back to the case $\mathbb{R}[G]$. So take $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ which is semisimple over $\mathbb{R}$. Using Wedderburn's Theorem we have a decomposition of $\Lambda_{\mathbb{R}} \cong \mathrm{End}_{\Lambda_{\mathbb{R}}}(\Lambda_{\mathbb{R}})$ into a product of finite-dimensional division algebras. We discussed in Section 2.2 that over $\mathbb{R}$ there exist three types finite-dimensional division algebras, namely $\mathbb{C}, \mathbb{R}, \mathbb{H}$. So the general Wedderburn decomposition of $\Lambda_{\mathbb{R}}$ takes the form:

$$\Lambda_{\mathbb{R}} \cong M_{r_1}(\mathbb{R}) \times \ldots \times M_{r_k}(\mathbb{R}) \times M_{c_1}(\mathbb{C}) \times \ldots \times M_{c_l}(\mathbb{C}) \times M_{h_1}(\mathbb{H}) \times \ldots \times M_{h_m}(\mathbb{H})$$

for $r_i, c_i, h_i \geq 1$. Non-cancellation is determined by the quaternion factor $\mathbb{H}$.

The Eichler Condition. We say that Λ satisfies the Eichler condition if no $h_i = 1$.

This means that we can have no simple factor $\mathbb{H}$ in its Wedderburn decomposition. However we note that $h_i \geq 2$ does not affect the Eichler condition. Following the treatment in Johnson [13], we obtain the following:

Proposition 3.1.4 ([13], p57)**.** *If Λ satisfies the Eichler condition, then every module $M \in \mathcal{F}(\Lambda)$ satisfies the Eichler condition.*

The Eichler condition leads to the following important form of the Swan-Jacobinski Theorem:

Theorem 3.1.5 (Swan Jacobinski,[14], p. 233)**.** *Let $M \in \mathcal{F}(\Lambda)$ where G is a finite group. If $M \oplus \Lambda$ satisfies the Eichler condition , then $M \oplus \Lambda$ has the cancellation property.*

Our main interest is to look at $\Omega_3(\mathbb{Z})$. So when looking at $\mathbb{R}[G]$ we can parametrise the odd and even syzygies. There exists a direct sum decomposition $\mathbb{R}[G] \cong I_{\mathbb{R}} \oplus \mathbb{R}$, where $I_{\mathbb{R}}$ is the augmentation ideal of $\mathbb{R}[G]$. Subsequently we consider the extensions:

$$0 \to I_{\mathbb{R}} \to \mathbb{R}[G] \to \mathbb{R} \to 0$$
$$0 \to \mathbb{R} \to \mathbb{R}[G] \to I_{\mathbb{R}} \to 0$$

This results in a setting for the odd and even syzygies:

$$\Omega_m(\mathbb{R}) = \begin{cases} [\mathbb{R}] & \text{if } 2|m \\ [I_{\mathbb{R}}] & \text{if } 2 \nmid m \end{cases}$$

We revert back to $\mathbb{Z}[G]$. By a *fork* we mean a tree structure with a finite number of 'prongs' at the minimal level, and no branching above level 1. So a stable module $[M]$ is said to have a fork structure if it looks like either **B** or **C**. Johnson then proved the following for the odd syzygies:

Proposition 3.1.6 ([13], p. 120)**.** *$\Omega_{2m+1}(\mathbb{Z})$ has the tree structure of fork.*

This result means that the odd syzygies must be of the form **B** or **C**. The even syzygies $\Omega_{2m}(\mathbb{Z})$ are more complicated, and it is here where structure **A**, the *crow's foot* can be found. We are uninterested in the structure of $\Omega_{2m}(\mathbb{Z})$ in this thesis, and so **A** was described for completeness. Thus at this point we may leave the tree structure for the even syzygies aside.

We say that the stable module $[M]$, where $M \in \mathcal{F}(\Lambda)$ has the *weak cancellation property* if and only if $[M]$ is a fork. However there is a stronger condition than this that we are interested in. We want a condition to satisfy a structure that can only be represented by **C**. So we say that a stable module $[M]$ is *straight* when its tree structure looks like **C**.

We define a finitely generated module S to be *stably free* when $S \oplus \Lambda^a \cong \Lambda^b$ for positive integers a, b. We say S is stably free precisely when S belongs to the stable class of the zero module, $[0]$. Denote $\mathcal{SF}(\Lambda)$ to be the category of finitely generated stably free modules over Λ. Hence we arrive at the following property shown in Johnson ([14], p. 234):

The Cancellation Property for free modules. We say that a finite group G has the cancellation property for free modules when finitely generated stably free modules $S \in \mathcal{SF}(\Lambda)$ are actually free. That is

$$S \oplus \Lambda^a \cong \Lambda^b \Longrightarrow S \cong \Lambda^{b-a}$$

This is otherwise known as Stably Free Cancellation (SFC). From here we see that

Proposition 3.1.7. *If Λ satisfies the Eichler condition, then Λ has the cancellation property for free modules.*

We finally observe that under duality straightness is preserved such that:

$$[M] \text{ is straight } \Longleftrightarrow [M^*] \text{ is straight}$$

The last discussion in this section is the description of the projective class group. There exists a classical invariant of rings obtained from the equivalence classes of projective Λ-modules. Let $\mathcal{P}(\Lambda)$ denote the collection of finitely projective modules over Λ. $\mathcal{P}(\Lambda)$ is closed with direct sum, and so the set $\mathcal{P}(\Lambda)/\sim$ of stable classes in $\mathcal{P}(\Lambda)$ forms a commutative monoid

$$[P_1] + [P_2] = [P_1 \oplus P_2]$$

in which the class of any free module represents zero. This monoid is in fact a group known as the *reduced projective class group* of Λ, namely $\widetilde{K}_0(\Lambda)$. It used to be denoted by $\mathcal{C}(\Lambda)$, and so some of the references referred to later use $\mathcal{C}(\Lambda)$. The subsequent two results were proved by Swan [27]. Firstly, if G is finite, $\widetilde{K}_0(\Lambda)$ is finite. Secondly any class in $\widetilde{K}_0(\Lambda)$ can be represented by a module J with $\mathrm{rank}_{\mathbb{Z}}(J) = |G|$.

When we look later at indecomposable Λ-lattices we will see that the main indecomposable modules looked at will be over the free module Λ. The remaining indecomposable Λ-lattices are obtained from looking at the other projective elements in $\widetilde{K}_0(\Lambda)$. For free resolutions we are only interested in the indecomposable Λ-lattices over the free module Λ.

3.2 Module extensions and the derived module category

We include only the main results without proof that we need here. The details can be found in Johnson ([15], chapter 4). We denote the collection of exact sequences of Λ-modules by:

$$\mathcal{E}\mathrm{xt}^1_\Lambda(M, N) = (0 \to N \to ? \to M \to 0)$$

where $M, N \in \mathcal{F}(\Lambda)$. Take two such short exact sequences:

$$\mathcal{E} = \left(0 \to N \xrightarrow{i} X \xrightarrow{p} M \to 0\right)$$
$$\mathcal{E}' = \left(0 \to N \xrightarrow{p} X' \xrightarrow{q} M \to 0\right)$$

We write $\mathcal{E} \equiv \mathcal{E}'$ to mean a *congruence* when there exists a commutative diagram of Λ-homomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "Id"] & X \arrow[r] \arrow[d, "\nu"] & M \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & N \arrow[r] & X' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

So by the 5-Lemma $\nu : X \to X'$ is an isomorphism. Hence we have an equivalence relation on $\mathcal{E}\mathrm{xt}^1_\Lambda(M, N)$. Define:

$$C = \mathcal{E}\mathrm{xt}^1_\Lambda(M, N)/\equiv$$

as the collection of equivalence classes in $\mathcal{E}\mathrm{xt}^1_\Lambda(M, N)$ under the congruence relation. There is always a *trivial extension*:

$$\mathcal{T} = \left(0 \to N \xrightarrow{i} N \oplus M \xrightarrow{p} M \to 0\right)$$

where $i(n) = (n, 0)$ and $p(n, m) = m$. We say that a $\mathcal{E}$ *splits* when $\mathcal{E} \equiv \mathcal{T}$, meaning $\nu : X \to N \oplus M$ is an isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "Id"] & X \arrow[r] \arrow[d, "\nu"] & M \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & N \arrow[r] & N \oplus M \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

We shall define the pullback and pushout constructions on $\mathrm{Ext}^1_\Lambda(M, N)$.

Pullback: Let M_1, M_2, N be Λ-modules, and we define a Λ-homomorphism f by $f : M_1 \to M_2$. We take $\mathcal{E} = (0 \to N \xrightarrow{i} X \xrightarrow{p} M_2 \to 0) \in \mathcal{E}\mathrm{xt}^1_\Lambda(M_2, N)$ then we take:

$$f^*(\mathcal{E}) = \left(0 \to N \xrightarrow{j} \varprojlim(p, f) \xrightarrow{q} M_1 \to 0\right)$$

where $\varprojlim(p, f) = X \times_{p,f} M_1 = \{(x, m) : p(x) = f(m)\}$ is the fibre product, and q is the projection $q(x, y) = y$. Hence there exists the pullback functor $f^* : \mathcal{E}\mathrm{xt}^1_\Lambda(M_2, N) \to \mathcal{E}\mathrm{xt}^1_\Lambda(M_1, N)$. Thus, there is a natural transformation $\psi_f : f^* \to Id$:

$$\begin{tikzcd}
f^*(\mathcal{E}) \arrow[d, "\psi_f"] & 0 \arrow[r] & N \arrow[r, "j"] \arrow[d, "Id"] & \varprojlim(p, f) \arrow[r, "q"] \arrow[d, "\psi"] & M_1 \arrow[r] \arrow[d, "f"] & 0 \\
\mathcal{E} & 0 \arrow[r] & N \arrow[r, "i"] & X \arrow[r, "p"] & M_2 \arrow[r] & 0
\end{tikzcd}$$

where $\psi : X \to \varprojlim(p, f)$ is the projection $\psi(x, y) = x$. Furthermore, if we take $f' : M_2 \to M_3$ we see:

$$(f' \circ f)^*(\mathcal{E}) = f^* \circ (f')^*(\mathcal{E})$$

and so the pullback construction *reverses* direction. It is otherwise known as *contravariant*.

Pushout: Let M, N_1, N_2 be Λ-modules, and we define a Λ-homomorphism g by $g : N_1 \to N_2$. We take $\mathcal{E} = (0 \to N_1 \xrightarrow{i} X \xrightarrow{p} A \to 0) \in \mathcal{E}\mathrm{xt}^1_\Lambda(M, N_1)$ then we take:

$$g_*(\mathcal{E}) = \left(0 \to N_2 \xrightarrow{j} \varinjlim(g, i) \xrightarrow{q} M \to 0\right)$$

where $\varinjlim(g,i) = (N_2 \oplus X)/\mathrm{Im}(g \times -i)$ denotes the colimit and j is the injection where $j(x) = [x, 0]$. This results in the existence of a pushout functor $g_* : \mathcal{E}\mathrm{xt}^1_\Lambda(M, N_1) \to \mathcal{E}\mathrm{xt}^1_\Lambda(M, N_2)$. Thus, there is a natural transformation $\varphi_g : Id \to f_*$:

$$\begin{tikzcd}
\mathcal{E}: \arrow[d, "\varphi_g"] & (0 \arrow[r] & N_1 \arrow[r, "i"] \arrow[d, "g"] & X \arrow[r, "p"] \arrow[d, "\varphi"] & M \arrow[r] \arrow[d, "Id"] & 0) \\
g_*(\mathcal{E}): & (0 \arrow[r] & N_2 \arrow[r, "j"] & \varinjlim(g,i) \arrow[r, "q"] & M \arrow[r] & 0)
\end{tikzcd}$$

where $\varphi : X \to \varinjlim(g,i)$ is the mapping $\varphi(x) = [0, x]$. Furthermore, if we take $g' : N_2 \to N_3$ it is clear:

$$(g' \circ g)_*(\mathcal{E}) = g'_* \circ g_*(\mathcal{E})$$

and so the pushout construction *preserves* direction. It is alternatively called *covariant*.

At this point we refer the reader to Johnson ([15], chapter 4) which discusses extensions of modules in further detail. We will take some results from here without proof.

Proposition 3.2.1. $\mathrm{Ext}^1_\Lambda(M, N)$ *is naturally an abelian group.*

Proposition 3.2.2. *Given an exact sequence* $\mathcal{E} = (0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0)$ *of* Λ*-modules, and a coefficient module* N*, then if we apply the contravariant functor* $\mathrm{Hom}_\Lambda(-, N)$ *we get an exact sequence:*

$$0 \to \mathrm{Hom}_\Lambda(C,N) \xrightarrow{p^*} \mathrm{Hom}_\Lambda(C,N) \xrightarrow{i^*} \mathrm{Hom}_\Lambda(C,N) \xrightarrow{\delta} \mathrm{Ext}^1_\Lambda(C,N) \ldots$$

The above proposition can be applied to the dual result using the covariant functor. From here we exclude the detail and just describe group cohomology from the Eilenberg Maclane approach. Again take the following to be a free resolution:

$$\ldots F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{(\mu=\partial_0)} M \to 0$$

Abbreviate this by $F_* \to M$. The more general concept of a projective resolution exists with $P_* \to M$, and the argument for projective resolutions follow in a similar manner to free resolutions. Then we can apply the contravariant functor $\mathrm{Hom}_\Lambda(F_*, N)$ for a coefficient module N. From this we can take cohomology.

Theorem 3.2.3. *The n^{th} cohomology group of M with coefficients in N is defined by:*

$$H^n(M,N) = \begin{cases} \mathrm{Ker}(\partial_1^*) = \mathrm{Hom}_\Lambda(M,N) & n=0 \\ \frac{\mathrm{Ker}(\mathrm{Hom}_\Lambda(F_n,N) \xrightarrow{\partial_{n+1}^*} \mathrm{Hom}_\Lambda(F_{n+1},N))}{\mathrm{Im}(\mathrm{Hom}_\Lambda(F_{n-1},N) \xrightarrow{\partial_n^*} \mathrm{Hom}_\Lambda(F_n,N))} & n \geq 1 \end{cases}$$

Thus, cohomology groups are independent of the particular chain $F_* \to M$.

The derived module category

The study of stable modules provides motivation to look at a category where modules that are stably equivalent are isomorphic objects. In Johnson ([15], chapter 5) he provides plenty of detail and proves the stated results below on the construction and use of such a category.

Let M, N be Λ-modules contained in the category $\mathcal{F}(\Lambda)$. Define:

$$\mathrm{Hom}_\Lambda(M,N) = \{f : M \to N, f \text{ is a } \Lambda \text{ homomorphism}\}$$

If we take $f \in \mathrm{Hom}_\Lambda(M,N)$, we write $f \approx 0$ if and only if there exists a commutative diagram:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "a"'] & & N \\ & P \arrow[ur, "b"'] & \end{tikzcd}$$

with P being a projective module. This means that the map α factors through a projective. So define:

$$\mathrm{Hom}_0(M,N) = \{\alpha \in \mathrm{Hom}_\Lambda(M,N),\ f \approx 0\}$$

$\mathrm{Hom}_0(M,N)$ is a well-defined subgroup of $\mathrm{Hom}_\Lambda(M,N)$. We can obtain the following quotient category $\mathcal{D}er = \mathcal{D}er(\Lambda)$ to be the category in which objects are right Λ-modules, and so:

$$\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M,N) = \mathrm{Hom}_\Lambda(M,N)/\mathrm{Hom}_0(M,N)$$

We observe that $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M,N)$ has the natural structure of an abelian group. We can extend the above to the functor Ext^1. Take $f, g : M_1 \to M_2$ to be Λ-homomorphisms such that $f \approx g$. Let

$$\begin{tikzcd} M_1 \arrow[rr, "f-g"] \arrow[dr, "\alpha"'] & & M_2 \\ & Q \arrow[ur, "\beta"'] & \end{tikzcd}$$

be a factorisation of $f-g$ through a projective Q. We apply $\mathrm{Ext}^1(-,N)$ to the above factorisation

$$\begin{tikzcd} Ext^1(M_2,N) \arrow[rr, "f^*-g^*"] \arrow[dr, "\beta^*"'] & & Ext^1(M_1,N) \\ & Ext^1(Q,N) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

This gives a factorisation of f^*-g^* through $\mathrm{Ext}^1(Q,N)$. As Q is projective then $\mathrm{Ext}^1(Q,N)$ =0. Hence if $f \approx g$ then $f^* = g^* : Ext^1(M_2,N) \to Ext^1(M_1,N)$. So this implies that for any Λ-module N the correspondence $M \to Ext^1(M,N)$ defines a contravariant functor $\mathrm{Ext}^1(-,N) : \mathcal{D}er \to Ab$.

Theorem 3.2.4. *Let M_1, M_2 be Λ-modules, then:*

$$M_1 \cong_{\mathcal{D}\mathrm{er}} M_2 \Longleftrightarrow M_1 \oplus P_1 \cong_\Lambda M_2 \oplus P_2$$

for some projective modules P_1, P_2.

We say that M is *coprojective* when $\mathrm{Ext}^1_\Lambda(M,P) = 0$, where P is a projective module. So in the derived module category we have the following

Proposition 3.2.5. *Suppose we have an exact sequence of modules over Λ and G is a finite group:*

$$0 \to J \to F_m \to F_{m-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

where each F_n is finitely generated and free. Then $\mathrm{End}_{\mathcal{D}\mathrm{er}}(J) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}}(M)$.

Now take the case when $M = \mathbb{Z}$ so we can obtain more explicit results:

Proposition 3.2.6. $\mathrm{End}_{\mathcal{D}\mathrm{er}}(J) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathbb{Z}) \cong \mathbb{Z}/|G|$

Proof. The only homomorphisms $\Lambda \to \mathbb{Z}$ are $k\varepsilon$ where $k \in \mathbb{Z}$. The only homomorphisms $\mathbb{Z} \to \Lambda$ are $k'\varepsilon^*$ where $k' \in \mathbb{Z}$. So if we have:

$$\begin{tikzcd} \mathbb{Z} \arrow[rr, "(kk')\varepsilon\varepsilon^*"] \arrow[dr] & & \mathbb{Z} \\ & \mathbb{Z}[G]^m \arrow[ur] & \end{tikzcd}$$

then $\varepsilon\varepsilon^* = |G|$. So we get:

$$\begin{aligned} \mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathbb{Z}) &\cong \mathrm{Hom}_\Lambda(\mathbb{Z},\mathbb{Z})/\mathrm{Hom}_0(\mathbb{Z},\mathbb{Z}) \\ &\cong \mathbb{Z}/|G| \end{aligned}$$

From Proposition 3.2.5, we also attain $\mathrm{End}_{\mathcal{D}\mathrm{er}}(J) \cong \mathbb{Z}/|G|$ $\square$

Corollary 3.2.7. $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J) \cong (\mathbb{Z}/|G|)^*$

The last concept we will state here without proof is the co-representability theorem of cohomology.

Theorem 3.2.8 (Co-representability of cohomology, [9], p. 37)**.** *Let M, N be lattices and let $n \geq 1$, then there exists an isomorphism:*

$$\mathrm{Ext}^n_\Lambda(M, N) \cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_n(M), N)$$

and the reverse functor gives:

$$\mathrm{Ext}^n_\Lambda(M, N) \cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M, \Omega_{-n}(N))$$

Lastly note that using the derived module category it has been established that group cohomology is represented by:

$$\mathrm{Ext}^n_\Lambda(M, N) = H^n(M, N)$$

Chapter 4

Representation theory of metacyclic groups

4.1 Fibre squares

We commence by defining a *fibre square*. Let:

(4.i)
$$\begin{tikzcd} \mathcal{R} \arrow[r, "\eta_-"] \arrow[d, "\eta_+"'] & \mathcal{R}_- \arrow[d, "\varphi_-"] \\ \mathcal{R}_+ \arrow[r, "\varphi_+"] & \mathcal{R}_0 \end{tikzcd}$$

be a commutative square of ring homomorphisms. We say (4.i) is a fibre square if it satisfies the following condition:

- **i)** $\mathcal{R}$ is the *fibre product* of $\mathcal{R}_-$ and $\mathcal{R}_+$ over $\mathcal{R}_0$. In other words, if $\lambda_- \in \mathcal{R}_-$, $\lambda_+ \in \mathcal{R}_+$ and $\varphi_-(\lambda_-) = \varphi_+(\lambda_+)$ then there exists a unique $\lambda \in \mathcal{R}$ such that $\eta_-(\lambda) = \lambda_-$ and $\eta_+(\lambda) = \lambda_+$.

We say (4.i) is a *Milnor square* if the square satisfies another condition, namely:

- **ii)** At least one of the maps φ_-, φ_+ is surjective.

Apply the cyclic algebra $\mathscr{C}_n(\mathcal{R}, \theta, a)$ as described in (2.i) to the fibre square of cyclic ring homomorphisms in (4.i). Using Proposition 2.1.5 we say that a fibre square of cyclic ring-homomorphisms induces a fibre square of the associated cyclic algebras, and so we obtain the following square:

$$\begin{tikzcd} \mathscr{C}_n(\mathcal{R}, \theta, a) \arrow[r, "\eta_-"] \arrow[d, "\eta_+"'] & \mathscr{C}_n(\mathcal{R}_-, \theta_-, a_-) \arrow[d, "\varphi_-"] \\ \mathscr{C}_n(\mathcal{R}_+, \theta_+, a_+) \arrow[r, "\varphi_+"] & \mathscr{C}_n(\mathcal{R}_0, \theta_0, a_0) \end{tikzcd}$$

More detail on this fibre square construction is found in Kamali ([16], chapter 5). For our study of finite group rings the above information on fibre squares is sufficient. The reader is referred to Milnor [21], for an in depth discussion of fibre squares and their applications. In addition Johnson [15] discusses the uses of fibre squares for the study of stably free modules over infinite group rings such as $\mathbb{Z}[G \times C_\infty]$, where G is a finite group. He uses tools such as Milnor patching and Karoubi squares.

The main type of fibre square that we use to describe the integral representation of metacyclic groups is:

Proposition 4.1.1. *If I and J are ideals over a ring $\mathcal{R}$, the following commutative square*

$$\begin{tikzcd} \mathcal{R}/(I \cap J) \arrow[r] \arrow[d] & \mathcal{R}/J \arrow[d] \\ \mathcal{R}/I \arrow[r] & \mathcal{R}/(I+J) \end{tikzcd}$$

is a fibre square. Furthermore all maps are surjective.

4.2 Wedderburn decompositions of $G(pq)$

Throughout this chapter we take the following presentation for the metacyclic groups $G(pq)$:

$$G(pq) = \langle x, y \mid x^p = y^q = 1, yx = x^r y \rangle$$

where p, q are distinct primes, $q|(p-1)$ and $r^q \equiv 1 \pmod p$. When discussing the integral group ring of $G(pq)$ in the next three Chapters, we restrict $\Lambda = \mathbb{Z}[G(pq)]$ for convenience.

In the following we give a full rational Wedderburn decomposition for the groups $G(pq)$. Before looking at $\mathbb{Q}[G(pq)]$, we describe the Wedderburn decomposition of $\mathbb{C}[G(pq)]$. It is straightforward to obtain as $\mathbb{C}$ is an algebraically closed field. We just need to obtain the conjugacy classes of $\mathbb{C}[G(pq)]$, and work out the unique decomposition. So the general complex Wedderburn decomposition for metacyclic groups of order pq is:

$$\mathbb{C}[G(pq)] \cong \mathbb{C}^q \times \prod_{1}^{(p-1)/q} M_q(\mathbb{C})$$

In Section 6.2 we discuss the Eichler condition for metacyclic groups. For the Eichler condition to hold we need to show that these metacyclic groups contain no $\mathbb{H}$ factor over $\mathbb{R}[G(pq)]$. However it will become apparent in the rational representation of $G(pq)$, that these groups do not contain an $\mathbb{H}$ factor over $\mathbb{R}[G(pq)]$. Thus, the real representation of metacyclic groups will be omitted here.

We turn our attention to $\mathbb{Q}[G(pq)]$. Take the cyclic group $C_p = \langle x \mid x^p = 1\rangle$, then the following is a commutative square for the rational group ring $\mathbb{Q}[C_p]$:

$$\begin{tikzcd}
\mathbb{Q}[x]/(x^p-1) \arrow[r] \arrow[d] & \mathbb{Q}[x]/(1+x+\ldots+x^{p-1}) \arrow[d] \\
\mathbb{Q}[x]/(x-1) \arrow[r] & 0
\end{tikzcd}$$

Observe that $\mathbb{Q}[x]/(1+x+\ldots+x^{p-1}) \cong \mathcal{I}_p^*$, the dual of the augmentation ideal. Consequently we have the following square:

$$\begin{tikzcd}
\mathbb{Q}[C_p] \arrow[r] \arrow[d] & \mathcal{I}_p^* \arrow[d] \\
\mathbb{Q} \arrow[r] & 0
\end{tikzcd}$$

Evidently, $\mathbb{Q}[C_p] \cong \mathbb{Q} \times \mathcal{I}_p^*$. Moreover, we have $\mathcal{I}_p^* = \mathbb{Q}(\zeta_p)$, where $\zeta_p = e^{2\pi i/p}$, the primitive p^{th} root of unity. We shall call $\mathcal{I}_p^* = K$, and $\dim_{\mathbb{Q}}(K) = (p-1)$.

Let θ_p be a generator of Gal$(K/\mathbb{Q})$. We have q, a divisor of $(p-1)$, so take $\theta_{[q]} \mapsto \theta_p^r$. Hence ord$(\theta_{[q]}) = q$. This gives the relation needed to form $\mathbb{Q}[G(pq)]$ from $\mathbb{Q}[C_p]$. Finally take the fixed field to be

$$K_0 = K^{\theta_{[q]}} = \{x \in K : \theta_{[q]}(x) = x\}$$

It is clear from the definition that $\dim_{\mathbb{Q}}(K_0) = (p-1)/q$. In order to obtain the above commutative diagram over $\mathbb{Q}[G(pq)]$, take the cyclic algebra over the prime q, $\mathscr{C}_q(\mathcal{R}, \theta_{[q]}, 1)$ where $\theta_{[q]} \mapsto \theta_p^r$:

$$\begin{tikzcd}
\mathscr{C}_q(\mathbb{Q}[C_p], \theta_{[q]}, 1) \arrow[r] \arrow[d] & \mathscr{C}_q(K, \theta_{[q]}, 1) \arrow[d] \\
\mathscr{C}_q(\mathbb{Q}, \theta_{[q]}, 1) \arrow[r] & 0
\end{tikzcd}$$

Observe that $\mathscr{C}_q(\mathbb{Q}[C_p], \theta_{[q]}, 1)$ is represented by $\mathbb{Q}[G(pq)]$ where the cyclic algebra induces $\theta_{[q]}(x) \mapsto x^r$. Furthermore $\mathscr{C}_q(\mathbb{Q}, \theta_{[q]}, 1)$ represents $\mathbb{Q}[C_q]$, as the cyclic algebra induces the identity over $\mathbb{Q}$. We obtain:

$$\begin{tikzcd}
\mathbb{Q}[G(pq)] \arrow[r] \arrow[d] & \mathscr{C}_q(K, \theta_{[q]}, 1) \arrow[d] \\
\mathbb{Q}[C_q] \arrow[r] & 0
\end{tikzcd}$$

This gives the decomposition $\mathbb{Q}[G(pq)] \cong \mathbb{Q}[C_p] \times \mathscr{C}_q(K, \theta, 1)$. We know that $\mathscr{C}_n(K, \theta_{[q]}, 1)$ is a simple algebra over its centre, and the fixed field K_0 is the centre of the cyclic algebra. Take the following structure theorem:

Proposition 4.2.1 ([22], p. 235)**.** *Let A be a finite dimensional simple algebra with centre C. Then:*

- ***i)*** *$A \cong M_n(\mathfrak{D})$ where $\mathfrak{D}$ is a division algebra with centre C.*
- ***ii)*** *$dim_C(\mathfrak{D}) = d^2$ for some d.*

*If **i)** and **ii)** are satisfied then $dim_C(A) = (dn)^2$.*

It is possible now to determine a representation for $\mathscr{C}_q(K, \theta_{[q]}, 1)$.

Proposition 4.2.2. *$\mathscr{C}_q(K, \theta_{[q]}, 1)$ is a simple algebra over its centre K_0, then*

$$\mathscr{C}_q(K, \theta_{[q]}, 1) \cong M_q(K_0)$$

Proof. $\dim_{K_0}(\mathscr{C}_q(K, \theta_{[q]}, 1)) = (dn)^2$. We know $\dim_{\mathbb{Q}}(\mathscr{C}_q(K, \theta_{[q]}, 1)) = (p-1)q$, and $\dim_{\mathbb{Q}}(K_0) = (p-1)/q$. From this $\dim_{K_0}(\mathscr{C}_q(K, \theta_{[q]}, 1) = q^2$. All that is left is $dn = q$. As q is prime, there are only two possibilities.

- **i)** Let $d = q, n = 1$. In this case $\mathscr{C}_q(K, \theta_{[q]}, 1) \cong \mathfrak{D}$ where $\mathfrak{D}$ is a division algebra. However $\mathscr{C}_q(K, \theta_{[q]}, 1)$ is clearly not a division algebra $\mathfrak{D}$. So we must have
- **ii)** Let $d = 1, n = q$. Then

$$\mathscr{C}_q(K, \theta_{[q]}, 1) \cong M_q(K_0)$$

□

Thus we are able to give a complete rational Wedderburn decomposition of the metacyclic groups $G(pq)$:

$$\mathbb{Q}[G(pq)] \cong \mathbb{Q}[C_q] \times M_q(K_0)$$

where $K_0 = \{x \in K : \theta_{[q]}(x) = x\}$ is the fixed field (and centre) of $\mathscr{C}_q(K, \theta_{[q]}, 1)$.

4.3 Integral representation theory of $G(pq)$

To understand the integral group ring $\mathbb{Z}[G(pq)]$, we need to use the fibre square construction explained at the beginning of the chapter. There are similarities between the squares of $\mathbb{Z}[G(pq)]$ and $\mathbb{Q}[G(pq)]$. Take the following to be a commutative square for $\mathbb{Z}[C_p]$:

$$\begin{tikzcd} \mathbb{Z}[C_p] \arrow[r] \arrow[d] & \mathcal{I}_p^* \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

More detail about this square can be seen in Milnor ([21], chapter 3). As in the rational case, we can take the dual of the augmentation ideal to be

$$\mathcal{I}_p^* = \mathbb{Z}[\zeta_p] = R$$

where, as before $\zeta_p = e^{2\pi i/p}$. Thus, R is the ring of algebraic integers of K. Let θ_p and $\theta_{[q]}$ be defined as above. Let R_0 to be the algebraic integers of the fixed field K_0:

$$R_0 = \{x \in R : \theta_{[q]}(x) = x\}$$

Hence R_0 is the subring of R fixed by $\theta_{[q]}$. Furthermore, similar to the rational case $\dim_{\mathbb{Z}}(R) = (p-1)$ and $\dim_{\mathbb{Z}}(R_0) = (p-1)/q$. Taking the cyclic algebra $\mathscr{C}_q(\mathcal{R}, \theta_{[q]}, 1)$ where $\theta_{[q]} \mapsto \theta_p^r$ over the fibre square we get:

$$\begin{tikzcd}
\mathbb{Z}[G(pq)] \arrow[r] \arrow[d] & \mathscr{C}_q(R, \theta_{[q]}, 1) \arrow[d] \\
\mathbb{Z}[C_q] \arrow[r] & \mathbb{F}_p[C_q]
\end{tikzcd}$$

The problem here, as in the rational case, is how to describe $\mathscr{C}_q(R, \theta_{[q]}, 1)$. In Rosen's thesis [25] he describes a decomposition of this construction into ideals.

Theorem I (Rosen). *We take a group of the form $C_p \rtimes C_q$. Let $R = \mathbb{Z}[\zeta_p]$, q is a divisor of $(p-1)$. We then have $\theta_{[q]} = \theta^r$ with order q. Therefore the cyclic algebra $\mathscr{C}_q(R, \theta_{[q]}, 1)$ decomposes in the following manner:*

$$\mathscr{C}_q(R, \theta_{[q]}, 1) \cong \mathcal{J}_1 \oplus \ldots \oplus \mathcal{J}_q$$

where each $\mathcal{J}_i$ is an ideal. Furthermore $\mathcal{J}_i \ncong \mathcal{J}_k$ if $i \neq k$. In fact each of the ideals are of the form $P^e = (\zeta_p - 1)^e \mathbb{Z}[\zeta_p]$ where $0 \leq e \leq q-1$.

We shall give an alternative approach to Rosen's Theorem via the quasi-triangular algebra $\mathscr{T}_q(R_0, \pi)$. Rosen's proof is rather opaque, and this method gives a clearer picture. Denote $M_q(R_0)$ to be the algebra of $q \times q$ matrices over the commutative ring R_0, where R_0 has a unique prime π over p. Denote by $\mathscr{T}_q(R_0, \pi)$ the following subalgebra of $M_q(R_0)$:

$$\mathscr{T}_q(R_0, \pi) = \{X = (X_{ij}) \in M_q(R_0) | X_{ij} \equiv 0 \pmod{\pi} \text{ for } i > j\}$$

We observe that this construction has the right sort of ideal decomposition required by Rosen's Theorem. To see this more explicitly, take $q = 3$ for example:

$$\mathscr{T}_3(R_0, \pi) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \pi a_{21} & a_{22} & a_{23} \\ \pi a_{31} & \pi a_{32} & a_{33} \end{pmatrix} ; a_{ij} \in R_0 \right\}$$

Each row here is an ideal. Take:

$$\mathfrak{R}(1) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, a_i \in R_0 \right\}$$
$$\mathfrak{R}(2) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \pi b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{pmatrix}, b_i \in R_0 \right\}$$
$$\mathfrak{R}(3) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \pi c_1 & \pi c_2 & c_3 \end{pmatrix}, c_i \in R_0 \right\}$$

Hence we can interpret $\mathscr{T}_3(R_0, \pi) \cong \mathfrak{R}(1) \oplus \mathfrak{R}(2) \oplus \mathfrak{R}(3)$. So to prove Rosen's Theorem we need to show:

Proposition 4.3.1. $\mathscr{C}_q(R, \theta_{[q]}, 1) \cong \mathscr{T}_q(R_0, \pi)$

We prove Proposition 4.3.1 in two stages. Firstly, we will give an algorithm that shows there exists an injective ring homomorphism $i : \mathscr{C}_q(R, \theta_{[q]}, 1) \to \mathscr{T}_q(R_0, \pi)$. Secondly, we show that if such a ring homomorphism exists, it must then be an isomorphism. There is an explicit example of Stage 1 in Section 7.1 for $G(21)$.

Before investigating Stage 1, we describe π explicitly for the metacyclic groups. In Johnson ([15], chapter 10) he shows certain number theoretic properties for dihedral groups with results from Hasse ([11], p. 525). We extend these results for $\mathbb{Z}[G(pq)]$. We therefore have the following identities:

i) $\text{rank}_{\mathbb{Z}}(R) = p - 1$ in $\mathbb{Q}[\zeta_p]$

ii) $(\zeta_p - 1)R$ has index p in R

iii) $(\zeta_p - 1)^{p-1} = pu$ for some unit $u \in R^*$

R/p is a finite local ring. So we have $R/p \to R/(\zeta_p - 1)R \cong \mathbb{F}_p$ as the canonical surjection. From the above, complete ramification is obtained for R over prime p by using the correspondence $t \mapsto \zeta_p - 1$. Thus we have the following isomorphism:

iv) $\mathbb{F}_p[t]/t^{p-1} \cong R/pR$

There are corresponding statements for R_0:

i) $\mathrm{rank}_{\mathbb{Z}}(R_0) = (p-1)/q$ in K_0

ii) πR_0 has index p in R_0

iii) $\pi^{(p-1)/q} = pv$ for some unit $v \in R_0^*$

R_0/p is a finite local ring. We have $R_0/p \to R_0/\pi R_0 \cong \mathbb{F}_p$ as the canonical surjection. Again we can see that over prime p we have complete ramification for R_0. Hence we induce the following isomorphism:

iv) $\mathbb{F}_p[\pi]/\pi^{(p-1)/q} \cong R_0/pR_0$

In this case we have shown the existence of π, but have not yet shown what the correspondence π is explicitly. Thus,

Proposition 4.3.2. *Define m_α as the minimal polynomial of R_0. Then, it is clear that $dim_{\mathbb{Z}}(m_\alpha) = dim_{\mathbb{Z}}(R_0) = (p-1)/q$. So if we take m_α over $\mathbb{F}_p$ we get:*

$$\begin{aligned} m_\alpha &\equiv \pi^{(p-1)/q} \ (mod\ p) \\ &\equiv (\alpha - q)^{(p-1)/q} \ (mod\ p) \end{aligned}$$

Hence $\pi \equiv (\alpha - q)$ mod p.

Proof. m_α is completely ramified over $\mathbb{F}_p$. Hence,

$$m_\alpha = \alpha^{(p-1)/q} + a_1\alpha^{((p-1)/q)-1} + \ldots + a_{(p-1)/q} = (\alpha - b)^{(p-1)/q} \pmod p$$

Then we can take $a_1 = -\mathrm{Tr}(\alpha)$. Observe that $-\mathrm{Tr}(\alpha) = -((p-1)/q)b \bmod p$. Thus,

$$b \equiv (q/(p-1))\mathrm{Tr}(\alpha) \bmod p$$

We know that $\mathrm{Tr}(\alpha) = \sum_{i=1}^{p-1} \zeta_p^i$. Over $\mathbb{F}_p$ we have $\sum_{i=1}^{p-1} \zeta_p^i \equiv p - 1$. So we are left with

$$b \equiv q \pmod p$$

Hence we arrive at the required result $\pi \equiv (\alpha - q)$ over $\mathbb{F}_p$. □

Below are three explicitly calculated examples of π:

Example 4.3.3. *Let $p = 7$, $q = 3$. This gives the non-abelian group $G(21)$. The minimal polynomial of R_0 here is*

$$\begin{aligned} m_\alpha &= \alpha^2 + \alpha + 2 \\ &= (\alpha - 3)^2 \ mod\ 7 \end{aligned}$$

So $\pi = (\alpha - 3)$ mod 7.

Example 4.3.4. *Let $p = 11$, $q = 5$. This gives the non-abelian group $G(55)$. The minimal polynomial of R_0 here is*

$$\begin{aligned} m_\alpha &= \alpha^2 + \alpha + 3 \\ &= (\alpha - 5)^2 \text{ mod } 11 \end{aligned}$$

So $\pi = (\alpha - 5)$ mod 11.

Example 4.3.5. *Let $p = 13$, $q = 3$. This gives the non-abelian group $G(39)$. The minimal polynomial of R_0 here is*

$$\begin{aligned} m_\alpha &= \alpha^4 + \alpha^3 + 2\alpha^2 - 4\alpha + 3 \\ &= (\alpha - 3)^4 \text{ mod } 13 \end{aligned}$$

So $\pi = (\alpha - 3)$ mod 13.

Moving back to Rosen's Theorem.

Proposition 4.3.6 (Stage 1)**.** *There exists an injective ring homomorphism*

$$i : \mathscr{C}_q(R, \theta_{[q]}, 1) \to \mathscr{T}_q(R_0, \pi)$$

We commence by recalling that over $\mathbb{Q}[G(pq)]$ there exists a surjective ring homomorphism, $\rho_\alpha : \mathbb{Q}[G(pq)] \to M_q(K_0)$. So an integral surjective ring homomorphism also exists. Let m_α be the minimal polynomial for R_0. So:

$$\rho_\alpha : \mathbb{Z}[G(pq)] \to M_q(R_0)$$

We want to conjugate this to a representation of the form:

$$\tilde{\rho} : \mathbb{Z}[G(pq)] \to \mathscr{T}_q(R_0, \pi)$$

where π is a unique prime in R_0 over p. Begin by reducing the above representation mod p:

$$\mathbb{F}_p[G(pq)] \to M_q(R_0 \otimes \mathbb{F}_p)$$

We know that $R_0 \cong \mathbb{Z}[\alpha]/m_\alpha$. We use Proposition 4.3.2 to obtain m_α over $\mathbb{F}_p$, so $m_\alpha = (\alpha - q)^{(p-1)/q}$ (mod p). Thus:

$$R \otimes \mathbb{F}_p \cong \mathbb{F}_p[\alpha]/(\alpha - q)^{(p-1)/q} \cong \mathbb{F}_p[\pi]/\pi^{(p-1)/q}$$

where $\pi = (\alpha - q)$. We are left with the following intermediate step:

$$\mathbb{F}_p[G(pq)] \to M_q(\mathbb{F}_p[\pi]/\pi^{(p-1)/q})$$

From here take $\pi \to 0$ to obtain $M_q(\mathbb{F}_p)$.

$$\bar{\rho} : \mathbb{F}_p[G(pq)] \to M_q(\mathbb{F}_p)$$

At this point we want to conjugate $\bar{\rho}(x)$ to its Jordan Normal Form over $\mathbb{F}_p$. Formally we define:

$$\begin{aligned} \hat{\rho} : \mathbb{F}_p[G(pq)] &\to M_q(\mathbb{F}_p) \\ \hat{\rho}(x) &= \bar{Q}\bar{\rho}(x)(\bar{Q})^{-1} \end{aligned}$$

where $\bar{Q} \in GL_q(\mathbb{F}_p)$, and $\det(\bar{Q}) = 1$. We know that y acts by the Galois action $\theta_{[q]}$, and so y normalises $\operatorname{span}\{1, x, \ldots, x^{p-1}\}$. Subsequently $\bar{\rho}(y)$ normalises $\operatorname{span}\{1, \bar{\rho}(x), \ldots, \bar{\rho}(x)^{p-1}\}$. The normaliser of a Jordan Block is upper triangular, and so $\hat{\rho}(y) = \bar{Q}\bar{\rho}(y)(\bar{Q})^{-1}$ is upper triangular.

Now lift $\bar{Q}$ back to $Q \in GL_q(R_0)$. So all that remains is to conjugate the original representation by Q, and replace α with the identity $\pi = \alpha - q$. This leaves a presentation in the form $\mathscr{T}_q(R_0, \pi)$, namely:

$$\begin{aligned} \tilde{\rho}(x) &= Q\rho_\alpha(x)Q^{-1} \\ \tilde{\rho}(y) &= Q\rho_\alpha(y)Q^{-1} \end{aligned}$$

This verifies that we have a surjective homomorphism $\tilde{\rho} : \mathbb{Z}[G(pq)] \to \mathscr{T}_q(R_0, \pi)$. Therefore the injective ring homomorphism $i : \mathscr{C}_q(R, \theta_{[q]}, 1) \to \mathscr{T}_q(R_0, \pi)$ exists.

To verify Rosen's Theorem, we must show that $i : \mathscr{C}_q(R, \theta_{[q]}, 1) \to \mathscr{T}_q(R_0, \pi)$ is in fact an isomorphism. To do this, we apply a discriminant argument, using the concepts from Section 2.3. The following discriminant is known ([29], p. 9):

$$\mathscr{D}\mathrm{isc}_{\mathbb{Z}}(R) = \pm p^{p-2}$$

where $\mathscr{D}\mathrm{isc}_{\mathbb{Z}}(R)$ is negative when $p \equiv 3 \pmod 4$, and positive otherwise.

Proposition 4.3.7. *Let $N_{R_0/\mathbb{Z}}$ define the Norm of $R_0/\mathbb{Z}$. Then we have:*

$$N_{R_0/\mathbb{Z}}[\mathscr{D}\mathrm{isc}_{R_0}(R)] = p^{q-1}$$

Proof. R is free of rank q over R_0 and it follows from the composition formula that $\mathscr{D}\mathrm{isc}_{\mathbb{Z}}(R) = \mathscr{D}\mathrm{isc}_{\mathbb{Z}}(R_0)^q N_{R_0/\mathbb{Z}}[\mathscr{D}\mathrm{isc}_{R_0}(R)]$. Thus:

$$\mathscr{D}\mathrm{isc}_{\mathbb{Z}}(R_0)^q N_{R_0/\mathbb{Z}}[\mathscr{D}\mathrm{isc}_{R_0}(R)] = \pm p^{p-2}$$

It follows that $\mathscr{D}\mathrm{isc}_{\mathbb{Z}}(R_0) = \pm p^a$, and $N_{R_0/\mathbb{Z}}[\mathscr{D}\mathrm{isc}_{R_0}(R)] = p^b$ for some non-negative integers a, b. Hence $p^{qa}p^b = p^{p-2}$. Looking at the exponents:

$$p - 2 = qa + b \qquad (*)$$

We write $p-1 = nq$. If $q = p-1$ then $R_0 = \mathbb{Z}$, then $\pi = p$ and the result just follows from above to be p^{p-2}. This does not occur except when $p = 3$, so we assume that $1 \leq q \leq p-2$. So we can apply the division algorithm to $q \leq p-2$ to obtain:

(**) $$p - 2 = q(n-1) + (q-1)$$

Comparing (*) and (**), we have $a = n-1$, and so $b = (q-1)$ as wanted. $\square$

From here it follows that for some unit $u \in R_0^*$:

$$\mathscr{D}\mathrm{isc}_{R_0}(R) = u\pi^{q-1}$$

If we compare (2.v) with (2.ix) and observe that $\det(\theta_{[q]})$ is a unit in R_0 we obtain the following equation as an equation in $R_0/(R_0^*)^2$:

$$\frac{\mathscr{D}\mathrm{isc}_{R_0}[\mathscr{C}_q(R, \theta_{[q]}, 1)]}{\mathscr{D}\mathrm{isc}_{R_0}[M_q(R_0)]} = \pm\pi^{q(q-1)}$$

In the same manner compare (2.viii) and (2.ix):

$$\frac{\mathscr{D}\mathrm{isc}_{R_0}[\mathscr{T}_q(R_0, \pi)]}{\mathscr{D}\mathrm{isc}_{R_0}[M_q(R_0)]} = \pm\pi^{q(q-1)}$$

It follows immediately as an equation in $R_0/(R_0^*)^2$ that:

$$\frac{\mathscr{D}\mathrm{isc}_{R_0}[\mathscr{C}_q(R, \theta_{[q]}, 1)]}{\mathscr{D}\mathrm{isc}_{R_0}[\mathscr{T}_q(R_0, \pi)]} = \pm 1$$

Proposition 4.3.8 (Stage 2)**.** *The injective ring homomorphism defined by* $i : \mathscr{C}_q(R, \theta_{[q]}, 1) \to \mathscr{T}_q(R_0, \pi)$ *is an isomorphism.*

Proof. Put $B = \mathrm{Im}(i)$. We claim that $B = \mathscr{T}_q(R_0, \pi)$. We know that i is injective and $\mathscr{C}_q(R, \theta_{[q]}, 1)$ is free of rank q^2 over R_0, then B is free of rank q^2 over R_0. Hence $\mathscr{D}\mathrm{isc}_{R_0}(B) = \mathscr{D}\mathrm{isc}_{R_0}[\mathscr{C}_q(R, \theta_{[q]}, 1)]$. In addition $\mathscr{T}_q(R_0, \pi)$ is defined to be free of rank q^2 over R_0, and so there exists a $q^2 \times q^2$ matrix Q over R_0 that expresses some basis for B in terms of the standard basis for $\mathscr{T}_q(R_0, \pi)$. Utilising (2.iv) we view the following as an equation in the quotient monoid $R_0/(R_0^*)^2$:

$$\det(Q)^2 = \frac{\mathscr{D}\mathrm{isc}_{R_0}(B)}{\mathscr{D}\mathrm{isc}_{R_0}[\mathscr{T}_q(R_0, \pi)]} = \frac{\mathscr{D}\mathrm{isc}_{R_0}[\mathscr{C}_q(R, \theta_{[q]}, 1)]}{\mathscr{D}\mathrm{isc}_{R_0}[\mathscr{T}_q(R_0, \pi)]} = \pm 1$$

Therefore it follows that $\det(Q) \in R_0^*$. Thus Q is invertible and so $B = \mathscr{T}_q(R_0, \pi)$ as claimed. $\square$

Having verified $\mathscr{C}_q(R, \theta_{[q]}, 1) \cong \mathscr{T}_q(R_0, \pi)$, we want to show the ideal decomposition $\mathscr{T}_q(R_0, \pi) \cong R \oplus P \oplus \ldots \oplus P^{q-1}$, where each $P^e = (\zeta_p - 1)^e \mathbb{Z}[\zeta_p]$, $0 \leq e \leq q-1$. Moreover in Rosen's thesis neither does he specify what the P^es are isomorphic to when $e \geq q$, nor does he discuss the duality relations of the P^es.

To look at the duality relations within $\mathscr{T}_q(R_0, \pi)$ we consider two forms of the quasi triangular matrix. For clarity we shall rename $\mathscr{T}_q(R_0, \pi) = \mathscr{T}_q^-(R_0, \pi)$, and describe the other form by:

$$\mathscr{T}_q^+(R_0, \pi) = \{X = (X_{ij}) \in M_q(R_0) | X_{ij} \equiv 0 \text{ (mod } \pi) \text{ for } i < j\}$$

If we conjugate by a $q \times q$ matrix Q where $Q^2 = 1$, we have:

$$Q(\mathscr{T}_q^-(R_0, \pi))Q^{-1} = \mathscr{T}_q^+(R_0, \pi)$$

Thus $\mathscr{T}_q^-(R_0, \pi) \cong \mathscr{T}_q^+(R_0, \pi)$.

Define $\mathfrak{R}(j) = \mathfrak{R}^-(j)$ to be the j^{th} row of $\mathscr{T}_q^-(R_0, \pi)$, and let $\mathfrak{R}^+(j)$ be the j^{th} row of $\mathscr{T}_q^+(R_0, \pi)$. There are corresponding definitions for the columns of $\mathscr{T}_q^-(R_0, \pi)$ and $\mathscr{T}_q^+(R_0, \pi)$, namely $\mathfrak{C}^-(j)$ and $\mathfrak{C}^+(j)$ respectively. Thus

(4.ii) $$\mathfrak{R}^-(j) \cong \mathfrak{R}^+(q + 1 - j)$$

Moreover we define the dual of any ideal by

$$\mathfrak{R}^-(j)^* = \text{ Hom } (\mathfrak{R}^-(j), \mathscr{T}_q^-(R_0, \pi))$$

Proposition 4.3.9. *The duality relations for the ideals in $\mathscr{T}_q^-(R_0, \pi)$ are:*

$$(\mathfrak{R}^-(j))^* \cong \mathfrak{R}^-(q + 1 - j)$$

Proof. Let $\epsilon(j) = \{X = X_{ij} \in M_q(R_0) | X_{ij} = 1 \text{ if } i = j, \ 0 \text{ otherwise}\}$. So we can generalise $\mathfrak{R}^-(j)$ by:

$$(\epsilon(j))\mathscr{T}_q^-(R_0, \pi) = \mathfrak{R}^-(j)$$

We note that $\epsilon(j)^2 = \epsilon(j)$. So let $T \in (\mathfrak{R}^-(j))^*$. Hence $T(\epsilon(j)) \in \mathscr{T}_q^-(R_0, \pi))$. So we have:

$$T(\epsilon(j)) = T(\epsilon(j)^2) = T(\epsilon(j))\epsilon(j) \in \mathfrak{C}^-(j)$$

The dual of a right module is a left module, so we transpose $\mathfrak{C}^-(j)$ to obtain a right module again. We know that $\mathfrak{C}^-(j) \cong \mathfrak{R}^+(j)$. By dualising and transposing we obtain:

$$(\mathfrak{R}^-(j))^* \cong \mathfrak{R}^+(j)$$

By (4.ii), we have $\mathfrak{R}^+(j) \cong \mathfrak{R}^-(q + 1 - j)$ giving the required result. $\square$

Finally, before identifying what each row represents, we explain that distinct rows are not isomorphic. We return to our previous notation where

$$\mathscr{T}_q(R_0, \pi) = \{X = (X_{ij}) \in M_q(R_0) | X_{ij} \equiv 0 \text{ (mod } \pi) \text{ for } i > j\}$$

and $\mathfrak{R}(j)$ is the j^{th} row of $\mathscr{T}_q(R_0, \pi)$. If we take $\mathscr{T}_q(R_0, \pi)$ mod π, we obtain an upper triangular matrix over $R_0/\pi = \mathfrak{u}$. Thus, if we tensor any row by $\mathfrak{u}$ we obtain distinct ranks for each row over R_0/π, namely

$$rank(\mathfrak{R}(j) \otimes \mathfrak{u}) = (q + 1 - j) \text{ over } R_0/\pi$$

Hence distinct rows in $\mathscr{T}_q(R_0, \pi)$ are not isomorphic.

Example 4.3.10. *Let $q = 3$. From earlier we saw:*

$$\mathscr{T}_3(R_0, \pi) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \pi a_{21} & a_{22} & a_{23} \\ \pi a_{31} & \pi a_{32} & a_{33} \end{pmatrix} ; a_{ij} \in R_0 \right\}$$

If we tensor $\mathscr{T}_3(R_0, \pi)$ with $\mathfrak{u}$ we have:

$$\mathscr{T}_3(R_0, \pi) \otimes \mathfrak{u} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} ; a_{ij} \in \mathfrak{u} \right\}$$

Looking at the ranks of each row individually we get,

$$\begin{aligned} \textit{rank}(\mathfrak{R}(1) \otimes \mathfrak{u}) &= 3 \\ \textit{rank}(\mathfrak{R}(2) \otimes \mathfrak{u}) &= 2 \\ \textit{rank}(\mathfrak{R}(3) \otimes \mathfrak{u}) &= 1 \end{aligned}$$

The rows have different ranks mod 3. Consequently distinct rows are not isomorphic.

With the above discussion on $\mathscr{T}_q(R_0, \pi)$ we proceed in realising what each row represents explicitly. We can establish R straightaway.

Proposition 4.3.11. $\mathfrak{R}(q) \cong R$

Proof. $\mathfrak{R}(1)$ is the natural map from the cyclic algebra. Hence the augmentation ideal, $\mathcal{I}_{\mathcal{G}} \cong \mathfrak{R}(1)$. Furthermore when discussing $\mathbb{Z}[C_p]$, R is the dual of the augmentation ideal, $\mathcal{I}_{\mathcal{G}}^*$. From Proposition 4.3.9,

$$\mathcal{I}_{\mathcal{G}}^* \cong \mathfrak{R}(1)^* \cong \mathfrak{R}(q)$$

Thus $R \cong \mathfrak{R}(q)$ □

Proposition 4.3.12. *For $1 \leq j \leq q$ let $\mathfrak{R}(j)$ denote the j^{th} row. Then the following decomposition exists:*

$$\begin{aligned}\mathscr{T}_q(R_0, \pi) &\cong \mathfrak{R}(1) \oplus \mathfrak{R}(2) \oplus \ldots \oplus \mathfrak{R}(q)\\ &\cong R \oplus P \oplus \ldots \oplus P^{q-1}\end{aligned}$$

with $\mathfrak{R}(j) \cong P^j$ where $1 \leq j \leq q$, and $P^q \cong R$.

Proof. There exists a chain of inclusions with index p at each inclusion, as each row differs by a value of π ($R_0/\pi \cong \mathbb{F}_p$):

$$\mathfrak{R}(1) \supset \mathfrak{R}(2) \supset \ldots \supset \mathfrak{R}(q) \supset \mathfrak{R}(q+1) \ldots$$

So as a chain we see that $\mathfrak{R}(q+j) \cong \mathfrak{R}(j)$, as $\mathfrak{R}(q+j) = \pi\mathfrak{R}(j)$. Furthermore there exists a similar chain to the above using P^e ideals. We know that $R/(\zeta_p - 1)R \cong \mathbb{F}_p$, thus P^e/P^{e+1} has index p:

$$R \supset P \supset P^2 \supset \ldots \supset P^q \supset P^{q+1}$$

From proposition 4.3.11, $R \cong \mathfrak{R}(q)$. So applying $\mathfrak{R}(q+q) \cong \mathfrak{R}(q) \cong R$, we see that $R \cong P^q$. Now both R/P and $\mathfrak{R}(q)/\mathfrak{R}(q+1)$ have index p, so as a result we can observe that $\mathfrak{R}(q+1) \cong P$. Thus $P \cong \mathfrak{R}(1)$.

Any other option for P would mean that there would be less than q distinct non-isomorphic ideals. This is not possible as the rows $\mathfrak{R}(1), \mathfrak{R}(2), \ldots, \mathfrak{R}(q)$ in $\mathscr{T}_q(R_0, \pi)$ are distinct non-isomorphic ideals. So

$$\mathfrak{R}(j) \cong P^j$$

Hence $\mathscr{T}_q(R_0, \pi) \cong R \oplus P \oplus \ldots \oplus P^{q-1}$.

To verify this, we can look at the opposite chain of inclusions:

$$P \subset P^2 \subset \ldots \subset P^{q-1} \subset R \subset P \subset P^2$$

Evidently the index between the inclusions are p^{q-1}. Define:

$$T = \begin{pmatrix} 0 & 0 & 0 & \ldots & \ldots & 1 \\ \pi & 0 & 0 & \ldots & \ldots & 0 \\ 0 & \pi & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \ddots & \ldots & \ldots & 0 \\ \vdots & \vdots & & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ldots & \pi & 0 \end{pmatrix} \in \mathscr{T}_q(R_0, \pi)$$

From this, we see that $T(P) \subset P^2, T(P^2) \subset P^3, \ldots, T(R) \subset P$. Each inclusion has index p^{q-1}, as wanted, due to $T(P^e) \subset P^{e+1}$ having an extra π^{q-1} factor. $\square$

Corollary 4.3.13. *Let P^e have any value for e. Then P^e is isomorphic to $P^{e \,(mod\, q)}$*

Proof. This is an extension of the previous proposition. We saw that $P^q \cong R$, $P^{q+1} \cong P$. Using the chain of inclusions within $\mathscr{T}_q(R_0, \pi)$, let $e \in \mathbb{N}$ and reduce it over q. This gives the required result. So if $e \equiv 0 \pmod{q}$, then $P^e \cong R$. □

We arrive at Theorem II, the duality relations of the ideals contained in the cyclic algebra $\mathscr{C}_q(R, \theta_{[q]}, 1)$.

Theorem II. *For any metacyclic group $G(pq)$, the following duality relationships hold for the ideals contained in $\mathscr{C}_q(R, \theta_{[q]}, 1) \cong R \oplus P \oplus \ldots \oplus P^{q-1}$.*

$$P^e \cong_\Lambda (P^{q-e+1})^*$$

where $0 \leq e \leq q-1$. As a result we always have $P \cong R^*$

Proof. From proposition 4.3.12, we have $\mathfrak{R}(j) \cong P^j$ in $\mathscr{T}_q(R_0, \pi)$, and $P^q \cong R$. Using proposition 4.3.9, the duality relations for P^e give the required result.

$P \cong R^*$ is always true as $\mathfrak{R}(1) \cong P$, $\mathfrak{R}(q) \cong R$. From proposition 4.3.9 these two rows are always the dual of one another irrespective of q. □

In Chapter 7, there is an explicit calculation showing the duality relations using matrix representations of R, P, P^2 over $\mathbb{Z}[G(21)]$. This is an alternative approach for working out the duality relations, and was the original method used. However this brute force method is cumbersome, and more awkward to generalise than the above.

It is clear to see that the decomposition of the cyclic algebra only relies on q, and that p only determines the nature and rank of these ideals.

Example 4.3.14. *Let $q = 5$, then we have the group $G(5p)$. The cyclic algebra decomposes:*

$$\mathscr{C}_5(R, \theta_{[5]}, 1) \cong R \oplus P \oplus P^2 \oplus P^3 \oplus P^4$$

The duality relations for these ideals are:

i) $P \cong R^*$

ii) $P^2 \cong (P^4)^*$

iii) $P^3 \cong (P^3)^*$

This comes in particular use when looking at the free resolutions of $G(pq)$, as we can almost ignore the p when looking purely at the decomposition of syzygies $\Omega_m(\mathbb{Z})$. The complication is when looking at the differentials, where p plays an important, yet problematic role as will become apparent in Chapter 5.

Remark 4.3.15. *The main interest in this thesis is to look at groups $G(pq)$. However it should be noted that Rosen's Theorem, as well as the duality arguments work for the Affine groups $(G(p(p-1)) \cong C_p \rtimes C_{p-1}$, where p is prime. Similar to when $q = 2$, there are an even number of ideals produced from Rosen's Theorem, and as a result the duality arguments contain no self-dual ideal. This can be verified with a little effort by repeating this section with the replacement $q = p - 1$.*

The reader is warned that we can only take $q = p - 1$ for this section (Section 4.3), and that for the remainder of the thesis we rely on q being a prime.

4.4 The indecomposable modules over $\mathbb{Z}[G(pq)]$

In order to obtain a complete description of all the genera of indecomposable modules over $\mathbb{Z}[G(pq)]$, we must first describe the 'basic' indecomposable modules. With these basic indecomposable modules it is possible to construct all the other genera of indecomposable modules that exist over $\mathbb{Z}[G(pq)]$. We described most of the basic indecomposable modules in the previous section, but we shall restate them here in a concise manner. We look at the integral commutative square for $\mathbb{Z}[G(pq)]$ again:

$$\begin{tikzcd}
\mathbb{Z}[G(pq)] \arrow[r] \arrow[d] & \mathscr{C}_q(R, \theta_{[q]}, 1) \arrow[d] \\
\mathbb{Z}[C_q] \arrow[r] & \mathbb{F}_p[C_q]
\end{tikzcd}$$

From this square arise basic indecomposable modules over $\mathbb{Z}[C_q]$ and $\mathscr{C}_q(R, \theta_{[q]}, 1)$. There are three indecomposable modules over $\mathbb{Z}[C_q]$:

i) The trivial module: $\mathbb{Z}$

ii) The augmentation ideal, namely $\mathcal{J} = \mathrm{Ker}(\mathbb{Z}[C_q] \to \mathbb{Z})$

iii) The group ring itself $\mathbb{Z}[C_q]$.(This may be obtained by the non-split extension $0 \to \mathcal{J} \to \mathbb{Z}[C_q] \to \mathbb{Z} \to 0$.)

The non-isomorphic distinct indecomposable modules over $\mathscr{C}_q(R, \theta_{[q]}, 1)$ were established earlier. They are

iv) $P^e = (\zeta_p - 1)^e \mathbb{Z}[\zeta_p]$, where $0 \leq e \leq (q-1)$. Like before $P^0 = R = \mathbb{Z}[\zeta_p]$, and ζ_p is the primitive p^{th} root of unity.

Cohomological relations

To describe the other genera of indecomposable $\mathbb{Z}[G(pq)]$ modules, consider certain cohomological relations between the two types of 'basic' indecomposable modules. The remaining genera of indecomposable modules can be formed by the non-split extensions $0 \to \mathcal{A} \to ? \to \mathcal{B} \to 0$ where $\mathcal{A}$ is a direct sum of a combination of the modules P^e, and $\mathcal{B} = \mathbb{Z}, \mathcal{J}$ or $\mathbb{Z}[C_q]$,and so the cohomological interpretations of these extensions are necessary.

In L.C. Pu's paper [23] she describes half of the cohomological relations. The proofs here for $\mathrm{Ext}^1_\Lambda(\mathcal{A}, \mathcal{B})$ give an alternative method to Pu's proofs. In order to understand the duality relations, $\mathrm{Ext}^1_\Lambda(\mathcal{B}, \mathcal{A})$ are also considered.

First we explain what happens when we have an $\mathrm{Ext}^1_\Lambda(\mathcal{A}, \mathcal{B})$ that is cyclic of order p. If we localise Λ so as to make p a unit, then this Ext-group vanishes, and there is only one extension with kernel the localisation of $\mathcal{B}$ and quotient the localisation of $\mathcal{A}$. If on the other hand we localise at the prime p, then the integers k with $1 \leq k < p$ become units in the ring, and so all non-zero elements of the Ext-group yield isomorphic modules in this case, resulting in why we get just two isomorphism types of module expressible as extensions with kernel the localisation of $\mathcal{B}$ and quotient the localisation of $\mathcal{A}$.

We begin by looking at the sequence: $0 \to \mathcal{I}_p \to \mathbb{Z}[C_p] \to \mathbb{Z} \to 0$. Here $\mathcal{I}_p$ is the augmentation ideal, and over $\mathbb{Z}[C_p]$, $\mathcal{I}_p^* = \mathcal{I}_p = R$ as $\mathbb{Z}[C_p]$ has cohomological period 2.

Proposition 4.4.1. *Over Λ, the augmentation ideal $\mathcal{I}_p = P$.*

Proof. Take the sequence $0 \to \mathcal{I}_p \to \mathbb{Z}[C_p] \to \mathbb{Z} \to 0$, over Λ. One gets:

$$0 \to P \to \overline{\mathbb{Z}[C_p]} \to \mathbb{Z} \to 0$$

where $\overline{\mathbb{Z}[C_p]}$ is $\mathbb{Z}[C_p]$ by conjugation. Over Λ, R is still the dual of the augmentation ideal, but to obtain the augmentation ideal $\mathcal{I}_p$ this time, we can take the dual of $\mathcal{I}_p^* = R$, giving R^*. From earlier we have seen that over Λ, $R^* \cong P$. $\square$

From Proposition 4.4.1 it can be established that the above sequences are non-split extensions, a fact which will be needed when describing $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P)$.

Proposition 4.4.2. *Let $i_* : \mathbb{Z}[C_p] \to \mathbb{Z}[G(pq)]$. Then $i_*(\mathcal{I}_p) \cong R \oplus P \oplus \ldots \oplus P^{q-1}$.*

Proof. Take $i_*(M) = M \otimes_{\mathbb{Z}[C_p]} \mathbb{Z}[G(pq)]$. Then the non-split extension that forms the group ring $\mathbb{Z}[G(pq)]$ is obtained:

$$\begin{tikzcd}
\varepsilon : & (0 \arrow[r] & \mathcal{I}_p \arrow[r] & \mathbb{Z}[C_p] \arrow[r] \arrow[d, squiggly, "i_*"] & \mathbb{Z} \arrow[r] & 0) \\
\imath_*(\varepsilon) : & (0 \arrow[r] & \imath_*(\mathcal{I}_p) \arrow[r] & \mathbb{Z}[G(pq)] \arrow[r] & \mathbb{Z}[C_q] \arrow[r] & 0)
\end{tikzcd}$$

We have seen the dual scenario using the cyclic algebra:

$$\begin{array}{lc}
\varepsilon^* : & (0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[C_p] \longrightarrow \mathcal{I}_p^* \longrightarrow 0) \\
& \big\downarrow \mathscr{C}_q \\
\mathscr{C}_q(\varepsilon^*, \theta_{[q]}, 1) : & (0 \longrightarrow \mathbb{Z}[C_q] \longrightarrow \mathbb{Z}[G(pq)] \longrightarrow \mathscr{C}_q(\mathcal{I}_p^*, \theta_{[q]}, 1) \longrightarrow 0)
\end{array}$$

This in turn implies that $\mathscr{C}_q(\mathcal{I}_p^*, \theta_{[q]}, 1) \cong i_*(\mathcal{I}_p)$. Hence using proposition 4.3.12 $i_*(\mathcal{I}_p) = R \oplus P \oplus \ldots \oplus P^{q-1}$. This is the only option as any other possibility for $i_*(\mathcal{I}_p)$ splits. □

Proposition 4.4.3. $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P) \neq 0$

Proof.

$$0 \to P \to \overline{\mathbb{Z}[C_p]} \to \mathbb{Z} \to 0$$

This does not split over $\mathbb{Z}[C_p]$. Hence it does not split over Λ. So this defines a non-trivial class in $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P)$. □

Corollary 4.4.4.

$$\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P^e) \cong \begin{cases} \mathbb{Z}/p & e = 1 \\ 0 & 0 \leq e \leq q-1, e \neq 1 \end{cases}$$

Proof. Using the Eckmann Shapiro lemma:

$$\mathrm{Ext}^1_\Lambda(\mathbb{Z}, i_*(\mathcal{I}_p)) \cong \mathrm{Ext}^1_{\mathbb{Z}[C_p]}(i^*(\mathbb{Z}), \mathcal{I}_p) \cong \mathrm{Ext}^1_{\mathbb{Z}[C_p]}(\mathbb{Z}, \mathcal{I}_p) \cong \mathbb{Z}/p$$

As $i_*(\mathcal{I}_p) \cong R \oplus P \oplus \ldots \oplus P^{q-1}$, then $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, i_*(\mathcal{I}_p))$ splits up into:

$$\mathrm{Ext}^1_\Lambda(\mathbb{Z}, i_*(\mathcal{I}_p)) \cong \mathrm{Ext}^1_\Lambda(\mathbb{Z}, R) \oplus \mathrm{Ext}^1_\Lambda(\mathbb{Z}, P) \oplus \ldots \oplus \mathrm{Ext}^1_\Lambda(\mathbb{Z}, P^{q-1}) \cong \mathbb{Z}/p$$

However we have shown that $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P) \neq 0$. As $\mathbb{Z}/p$ has no non-trivial subgroups the only option is that $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P) \cong \mathbb{Z}/p$. Subsequently, we are left with $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, R) = \mathrm{Ext}^1_\Lambda(\mathbb{Z}, P^2) = \ldots = \mathrm{Ext}^1_\Lambda(\mathbb{Z}, P^{q-1}) = 0$. □

Proposition 4.4.5. $\mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_q], P^e) \cong \mathbb{Z}/p$ *for all* $0 \leq e \leq q-1$.

Proof. Again use the Eckmann Shapiro Lemma:

$$\begin{aligned} \mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_q], P^e) &\cong \mathrm{Ext}^1_{\mathbb{Z}[C_p]}(\mathbb{Z} \otimes_{\mathbb{Z}[C_p]} \mathbb{Z}[C_q], P^e) \\ &\cong \mathrm{Ext}^1_{\mathbb{Z}[C_p]}(\mathbb{Z}, P^e) \end{aligned}$$

It is enough to check $\mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_q], R) \cong \mathbb{Z}/p$ (as $R \cong P \cong \ldots \cong P^{q-1}$ over $\mathbb{Z}[C_p]$). Over $\mathbb{Z}[C_p]$ it is also know that $R \cong \mathcal{I}_p^*$, the dual augmentation ideal, so $R \cong \Omega_{-1}^{C_p}(\mathbb{Z})$. So using the derived module category:

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{Z}[C_p]}(\mathbb{Z}, R) &\cong \mathrm{Ext}^1_{\mathbb{Z}[C_p]}(\mathbb{Z}, \Omega_{-1}^{C_p}(\mathbb{Z})) \\ &\cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\mathbb{Z}, \Omega_1^{C_p}\Omega_{-1}^{C_p}(\mathbb{Z})) \\ &\cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z}/p\end{aligned}$$

□

With all this information we solve the last set of cohomological relations.

Proposition 4.4.6.

$$\mathrm{Ext}^1_\Lambda(\mathcal{J}, P^e) \cong \begin{cases} 0 & e = 1 \\ \mathbb{Z}/p & 0 \le e \le q-1, e \neq 1 \end{cases}$$

Proof. Take the following exact sequence $0 \to \mathcal{J} \to \mathbb{Z}[C_q] \to \mathbb{Z} \to 0$. Suppose that N is any of $R, P, \ldots, P^{q-1}$. By taking a long exact sequence in cohomology we obtain:

$$\mathrm{Hom}_\Lambda(\mathcal{J}, N) \to \mathrm{Ext}^1_\Lambda(\mathbb{Z}, N) \to \mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_q], N) \to \mathrm{Ext}^1_\Lambda(\mathcal{J}, N) \to \mathrm{Ext}^2_\Lambda(\mathbb{Z}, N) \ldots$$

Over $\mathbb{Z}[C_p]$, $\mathcal{J}$ is trivial. So $\mathrm{Hom}_{\mathbb{Z}[C_p]}(\mathcal{J}, N) = 0$. As a result $\mathrm{Hom}_\Lambda(\mathcal{J}, N) = 0$. One can also see:

$$\mathrm{Ext}^2_\Lambda(\mathbb{Z}, i_*(N)) \cong \mathrm{Ext}^2_{\mathbb{Z}[C_p]}(\mathbb{Z}, N) = 0$$

This leaves $\mathrm{Ext}^2_\Lambda(\mathbb{Z}, N) = 0$. Eliminating these factors from the long exact sequence, we are left with the exact sequence:

$$0 \to \mathrm{Ext}^1_\Lambda(\mathbb{Z}, N) \to \mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_q], N) \to \mathrm{Ext}^1_\Lambda(\mathcal{J}, N) \to 0$$

We proved in proposition 4.4.5 that $\mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_q], N) \cong \mathbb{Z}/p$. So all that remains is to take each N in turn, put in the value known for $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, N)$, and the value for $\mathrm{Ext}^1_\Lambda(\mathcal{J}, N)$ falls out. For example, when $N = R$, $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, R) = 0$, so that $\mathrm{Ext}^1_\Lambda(\mathcal{J}, R) \cong \mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_q], R) \cong \mathbb{Z}/p$. □

With all these results known, the duality arguments can be looked at. Using the same principles as in the previous cases, the following results fall out without much effort.

Corollary 4.4.7. *The duality arguments are as follows:*

$$\mathrm{Ext}^1_\Lambda(P^e,\mathbb{Z}) \cong \begin{cases} 0 & e=1 \\ \mathbb{Z}/p & 0 \le e \le q-1, e \neq 1 \end{cases}$$

$$\mathrm{Ext}^1_\Lambda(P^e,\mathcal{J}) \cong \begin{cases} \mathbb{Z}/p & e=1 \\ 0 & 0 \le e \le q-1, e \neq 1 \end{cases}$$

$$\mathrm{Ext}^1_\Lambda(P^e,\mathbb{Z}[C_q]) \cong \mathbb{Z}/p \textit{ for all } 0 \le e \le q-1$$

A complete list of the indecomposable genera of $\mathbb{Z}[G(pq)]$

Proposition 4.4.8. *There are a total of* $2+q+2^{q-1}+2^q$ *distinct non-isomorphic genera of indecomposable modules for* $\mathbb{Z}[G(pq)]$.

We describe the genera of indecomposable modules for the non-abelian group $G(pq)$, which form a complete class of $\mathbb{Z}$-free modules over the group ring $\mathbb{Z}[G(pq)]$. We proceed by stating the basic indecomposable modules.

I. There are three indecomposable modules over $\mathbb{Z}[C_q]$:

- **i)** The trivial module: $\mathbb{Z}$. $(\mathrm{rank}=1)$
- **ii)** The augmentation ideal, namely $\mathcal{J}=\mathrm{Ker}(\mathbb{Z}[C_q]\to\mathbb{Z})$. $(\mathrm{rank}=q-1)$
- **iii)** The group ring itself $\mathbb{Z}[C_q]$. $(\mathrm{rank}=q)$

They are modules over $\mathbb{Z}[G(pq)]$ via the quotient map $G(pq)\to C_q$

II. There are q distinct indecomposable modules over $\mathscr{C}_q(R,\theta_{[q]},1)$. They are all of rank $p-1$.

- **iv)** $P^e=(\zeta_p-1)^e\mathbb{Z}[\zeta_p]$, where $0\le e\le (q-1)$. Like before $P^0=R=\mathbb{Z}[\zeta_p]$.

They are all distinct $\mathbb{Z}[G]$ modules via the twisting relation $y\zeta_p=\zeta_p^r y$.

These are all the basic indecomposable modules over $\mathbb{Z}[G(pq)]$. We shall consider the non-split extensions.

Proposition 4.4.9 ([3], p. 750-751)**.** *Take the non-split extensions of indecomposable modules as:*

$$0\to\mathcal{A}\to ?\to\mathcal{B}\to 0$$

Then $\mathcal{A}$ *is a direct sum of modules of particular combinations of* $R,P,\ldots,P^{q-1}$, *where each the modules* $R,P,\ldots,P^{q-1}$ *can be used at most once in any extension, and* $\mathcal{B}=\mathbb{Z},\mathcal{J}$ *or* $\mathbb{Z}[C_p]$ *where only one of the three modules can be used in any extension.*

Curtis and Reiner show proposition 4.4.9 is true by localising Λ at p and q, and by using p-adic completions they show that any potential extra terms in an extension would have to split. A similar argument can be found in Pu's paper. Below is an alternative approach without localisation. We shall show this alternative approach for a single example.

Proposition 4.4.10. *The extension* $\mathcal{E} = (0 \to R \oplus R \to W \to \mathcal{J} \to 0)$ *splits.*

Proof. W determines the isomorphism class of this extension. $\mathrm{Ext}^1_\Lambda(\mathcal{J}, R \oplus R)$ is acted upon naturally by $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{J})$ and by $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(R \oplus R)$. We write the natural action as:

$$\begin{aligned} \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{J}) \times \mathrm{Ext}^1_\Lambda(\mathcal{J}, R \oplus R) \times \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(R \oplus R) &\to \mathrm{Ext}^1_\Lambda(\mathcal{J}, R \oplus R) \\ (\mathfrak{a}, \mathcal{E}, \mathfrak{b}) &\to \mathfrak{b}^{*-1}\mathfrak{a}_*^{-1}(\mathcal{E}) \end{aligned}$$

With this notation we represent

$$\mathrm{Ext}^1_\Lambda(\mathcal{J}, R \oplus R) \cong \mathrm{Ext}^1_\Lambda(\mathcal{J}, R) \oplus \mathrm{Ext}^1_\Lambda(\mathcal{J}, R)$$

as a (1×2) matrix. Let $(\delta_1, \delta_2) \in \mathrm{Ext}^1_\Lambda(\mathcal{J}, R) \oplus \mathrm{Ext}^1_\Lambda(\mathcal{J}, R)$. Thus we look at the map

$$(\delta_1, \delta_2) \to [\delta_1, \delta_2] \in \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{J}) \setminus \mathrm{Ext}^1_\Lambda(\mathcal{J}, R \oplus R)/\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(R \oplus R)$$

If we can show that $[\delta_1, \delta_2] = [0, \gamma]$ then the extension splits. $\mathrm{Ext}^1_\Lambda(\mathcal{J}, R) \cong \mathbb{F}_p$, so

$$[\delta_1, \delta_2] \in \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{J}) \setminus M_2(\mathbb{F}_p)/\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(R \oplus R)$$

If $\delta_1 = 0$ or $\delta_2 = 0$ then we have no problem and the extension must split. However suppose $\delta_1 \neq 0$, $\delta_2 \neq 0$. The mapping $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(R) \to \mathbb{F}_p^*$ is surjective and so we shall lift δ_2 back to $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(R)$. We define $\alpha_2 \in \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(R)$ where $|\alpha_2| = \delta_2$. Operate on the extension by

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha_2^{-1} \end{pmatrix}$$

to replace (δ_1, δ_2) by $(\delta_1, 1)$. If we act on the extension by

$$D = \begin{pmatrix} 1 & 0 \\ -\delta_1 & 1 \end{pmatrix} \in GL_2(R)$$

then $(\delta_1, 1)D = (0, 1)$. Hence $[\delta_1, \delta_2] = [0, 1]$. This implies we can split the extension for W to get

$$0 \to R \to W' \to \mathcal{J} \to 0$$

where $W' \cong W \oplus R$. □

This method with some adjustments in each case can be used for any of the potential non-split extensions to show that proposition 4.4.9 holds. Proposition 4.4.10 uses the concept of generalised Swan modules which are not discussed in this thesis. For details on generalised Swan modules see Edwards [6], and Johnson ([15], chapter 15).

Using Proposition 4.4.9 and the cohomological relations established earlier for $0 \to \mathcal{A} \to ? \to \mathcal{B} \to 0$, we describe all the remaining genera of indecomposable modules that exist.

III. There is only one extension for when $\mathcal{B} = \mathbb{Z}$. This is a result of the cohomological properties, where $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P^e) = 0$ when $e \neq 1$, and so such extensions split. Hence

v) $0 \to P \to \overline{\mathbb{Z}[C_p]} \to \mathbb{Z} \to 0$

where $\overline{\mathbb{Z}[C_p]}$ is $\mathbb{Z}[C_p]$ acting by conjugation. $\mathrm{rank}_\mathbb{Z}(\overline{\mathbb{Z}[C_p]}) = p$.

IV. There exist $2^{q-1} - 1$ indecomposable non-split extensions for when $\mathcal{B} = \mathcal{J}$. $\mathrm{Ext}^1_\Lambda(\mathcal{J}, P^e) = 0$ when $e = 1$, so such extensions cannot contain P. There are a total of $q - 1$ indecomposable modules in $\mathcal{A}$. Let k_1 define the number of distinct type **II.** modules combined in $\mathcal{A}$ that is contained in an extension with $\mathcal{J}$. There exist:

vi) $\displaystyle\sum_{k_1=1}^{q-1} \binom{q-1}{k_1}$ extensions of the form $0 \to \mathcal{A} \to V_c \to \mathcal{J} \to 0$ where $1 \leq c \leq 2^{q-1} - 1$.

The rank of such a module is $\mathrm{rank}_\mathbb{Z}(V_c) = (p-1)k_1 + (q-1)$.

V. There exist $2^q - 1$ indecomposable non-split extensions for when $\mathcal{B} = \mathbb{Z}[C_q]$. Here there are no split extensions, and so there are a total of q indecomposable modules in $\mathcal{A}$. Let k_2 define the number of distinct type **II.** modules combined in $\mathcal{A}$ that is contained in an extension with $\mathbb{Z}[C_q]$. Then there exist:

vii) $\displaystyle\sum_{k_2=1}^{q} \binom{q}{k_2}$ extensions of the form $0 \to \mathcal{A} \to Y_d \to \mathbb{Z}[C_q] \to 0$ where $1 \leq d \leq 2^q - 1$.

The rank of such a module is $\mathrm{rank}_\mathbb{Z}(Y_d) = (p-1)k_2 + q$.

We have obtained a full list of the possible indecomposable genera over $\mathbb{Z}[G(pq)]$. This verifies Proposition 4.4.8 that there are $2 + q + 2^{q-1} + 2^q$ genera of indecomposable modules. Importantly, there are a limited number of ranks that an indecomposable module can be in any given group ring $\mathbb{Z}[G(pq)]$.

There are certain indecomposable modules of non-split extension type that we are particularly interested in when discussing free resolutions. We state them here in order to gain familiarity, as they exist in all the metacyclic groups. We commence with the only non-split extension of type **III.**:

(4.iii) $0 \to P \to \overline{\mathbb{Z}[C_p]} \to \mathbb{Z} \to 0$. This has $\operatorname{rank}_{\mathbb{Z}}(\overline{\mathbb{Z}[C_p]}) = p$. This module is self-dual as there does not exist another indecomposable module with the same rank, $\overline{\mathbb{Z}[C_p]} \cong (\overline{\mathbb{Z}[C_p]})^*$.

There is one indecomposable module of type **IV.** that we will be interested in:

(4.iv) Let $k_1 = q-1$. Then there is only one extension of this type. We call it X. As $k_1 = q-1$, $\mathcal{A}$ here consists of all P^e modules except for P. So the extension looks like:

$$0 \to \bigoplus_{i=0, i\neq 1}^{q-1} P^i \to X \to \mathcal{J} \to 0$$

This extension is of $\operatorname{rank}_{\mathbb{Z}}(X) = (q-1)p$. This module is unique in the set of indecomposable modules as another indecomposable module with the same rank does not exist. Thus it is self-dual, $X \cong X^*$.

There are two types of indecomposable modules of type **V.** we are interested in:

(4.v) Let $k_2 = q-1$. Then $\binom{q}{q-1} = q$. There are q possibilities of choosing $q-1$ indecomposable modules for $\mathcal{A}$. Rather than looking at these as extensions, look at them as quotients. Define:

$$\mathcal{Q}(P^e) = \mathbb{Z}[G(pq)]/P^e$$

Each of these modules have $\operatorname{rank}_{\mathbb{Z}}(\mathcal{Q}(P^e)) = pq - (p-1)$.

(4.vi) Let $k_2 = q$. Then there exists only one extension of this type. This is just the group ring itself:

$$0 \to \bigoplus_{i=0}^{q-1} P^i \to \mathbb{Z}[G(pq)] \to \mathbb{Z}[C_q] \to 0$$

This obviously has $\operatorname{rank}_{\mathbb{Z}}(\mathbb{Z}[G(pq)]) = pq$.

Below is the simplest example possible. It gives a complete list of indecomposable genera for dihedral metacyclic groups. The modules of the form (4.iv), (4.v) and (4.vi) are interpreted on the side of any such genera.

Example 4.4.11. *Let $q = 2$. The dihedral groups D_{2p} have a total of 10 indecomposable genera. There are the three modules of type* ***I.****:*

i) *The trivial rank one module* $\mathbb{Z}$

ii) *The non-trivial rank one module* $\mathbb{Z}'$*.(This is* $\mathcal{J}$ *for* $\mathbb{Z}[C_2]$*).*

iii) *The integral group ring* $\mathbb{Z}[C_2]$

There are two indecomposable modules of type ***II.****:*

iv) $R = \mathbb{Z}[\zeta_p]$

v) $P = (\zeta_p - 1)\mathbb{Z}[\zeta_p]$

There is the one non-split extension of type ***III.****:*

vi) $0 \to P \to \overline{\mathbb{Z}[C_p]} \to \mathbb{Z} \to 0$

There is only one extension of type ***IV.*** *as* $q - 1 = 1$*, and so* $k_1 = 1$ *only:*

vii) $0 \to R \to V_1 \to \mathbb{Z}' \to 0$*. Here* $V_1 = X$*.*

There exist three extensions of type ***V.****, as* $q = 2$*, and so* $k_2 = 1, 2$*.*

viii) $0 \to R \to Y_1 \to \mathbb{Z}[C_2] \to 0$*. We can take* $Y_1 = \mathcal{Q}(R)$

ix) $0 \to P \to Y_2 \to \mathbb{Z}[C_2] \to 0$*. We can take* $Y_2 = \mathcal{Q}(P)$

x) $0 \to R \oplus P \to Y_3 \to \mathbb{Z}[C_2] \to 0$*.* Y_3 *is the group ring* $\mathbb{Z}[D_{2p}]$*.*

The projective class group and its link to indecomposable modules

We complete this chapter by explaining where all the remaining possible indecomposable modules arise from. The genera of indecomposable modules gives a complete class of $\mathbb{Z}$ free indecomposable modules over the free group ring $\mathbb{Z}[G(pq)]$. However in many groups, $\widetilde{K}_0 \neq 0$, and so there are plenty more indecomposable modules arising from the non-trivial elements in the projective class group.

The projective class group for metacyclic groups is given in a paper by Galovich, Reiner and Ullom [8]. The paper is dedicated to proving the result, and so the proof is omitted here. First we state a result from Milnor:

Proposition 4.4.12 ([21], p. 28)**.** *Let* $\widetilde{K}_0$ *stand for ideal class group. Then if* q *is prime:*

$$\widetilde{K}_0(\mathbb{Z}[C_q]) \cong \widetilde{K}_0(\mathcal{J})$$

where $\mathcal{J}$ *is the dual of the augmentation ideal of* $\mathbb{Z}[C_q]$*.*

Theorem 4.4.13 ([8], p. 105)**.** *Let $\widetilde{K}_0(R_0)$ define the ideal class group of R_0, and $\widetilde{K}_0(\mathbb{Z}[C_q])$ denote the ideal class group of $\mathbb{Z}[C_q]$, then there exists an epimorphism for the projective class group of $\mathbb{Z}[G(pq)]$:*

$$\widetilde{K}_0(\mathbb{Z}[G(pq)]) \to \widetilde{K}_0(R_0)\dot{+}\widetilde{K}_0(\mathbb{Z}[C_q])$$

whose kernel $D_0(\mathbb{Z}[G(pq)])$ is a finite cyclic group of order q, q odd, and of order $q/2$, q even.

The identity element in $\widetilde{K}_0(\mathbb{Z}[G(pq)]$ represents the free group ring $\mathbb{Z}[G(pq)]$, and the indecomposable modules here form a complete class of $\mathbb{Z}$ free indecomposable modules over the free group ring. Each other element in $\widetilde{K}_0(\mathbb{Z}[G(pq)])$, namely $\mathcal{P}_i$ contains an additional class of indecomposable modules. These additional indecomposable modules are determined by whether $\mathcal{P}_i \in \widetilde{K}_0(R_0)$ or $\mathcal{P}_i \in \widetilde{K}_0(\mathbb{Z}[C_q])$.

The remaining $\mathcal{P}_i$ elements are contained in the kernel, and indecomposable modules over the kernel elements are obtained by unit arguments over the group ring. The kernel size of $\widetilde{K}_0(\mathbb{Z}[G(pq)])$ is the main result of Galovich, Reiner and Ullom's paper. Pu's paper discusses the indecomposable modules that can arise from such unit arguments ([23], chapter 4). Under certain conditions many potential indecomposable modules from the kernel $\mathcal{P}_i$ elements are isomorphic to the indecomposable modules in the free group ring $\mathbb{Z}[G(pq)]$, namely when $q^2 \nmid p$.

Pu discusses the existence of such potential projective modules in the kernel of $\widetilde{K}_0(\mathbb{Z}[G(pq)])$, however she does not give a formula to obtain the number of these elements. Theorem 4.4.11 was published years after Pu's paper, and so it is noted that $D_0(\mathbb{Z}[G(pq)])$ is the unit argument Pu tries to describe in her paper.

However, for investigationg free resolutions, we may ignore indecomposable modules that do not arise from the identity element in $\widetilde{K}_0(\mathbb{Z}[G(pq)])$. Hence we only need to look at the genera of indecomposable modules.

Chapter 5

Syzygies and free resolutions for $\mathbf{G(pq)}$

5.1 Periodic resolutions and the fox method

We begin by giving the general condition for when a finite group G has a periodic resolution. It depends on the Sylow subgroup structure of G.

Theorem 5.1.1. *(Artin-Tate-Zassenhaus Theorem) Let G be a finite group, and let $p_1, \ldots, p_N$ be the distinct primes dividing $|G|$. Then a resolution $\mathbb{Z}$ over the group ring $\mathbb{Z}[G]$ has a finite cohomological period if and only if:*

i) *For every odd prime p_i, every p_i-subgroup is cyclic.*

ii) *If $p_i = 2$, then every Sylow 2-subgroup is either cyclic or generalised quaternion.*

This theorem implies that all groups $G(pq)$ have finite cohomological period, and so have periodic free resolutions. However groups that contain a $C_p \times C_p$ subgroup, where p is prime, do not have periodic resolutions. Groups such as the dihedral groups D_{4n} have this subgroup property (i.e. D_8 contains a $C_2 \times C_2$ subgroup), and so do not have a periodic resolution. In addition such groups have infinite families of indecomposable modules, complicating the integral representation theory.

Proposition 5.1.2. *([13], p. 163) The groups $G(pq)$ have cohomological period $2q$.*

An important property for periodic resolutions is that if $M \in \mathcal{F}(\mathbb{Z}[G])$ has period $2q$, then for $0 \leq m \leq 2q$ there exists a duality relation between the syzygies:

$$\Omega_m(\mathbb{Z})^* \cong \Omega_{2q-m}(\mathbb{Z})$$

where $\Omega_m(\mathbb{Z})^*$ denotes the dual stable class of $\Omega_m(\mathbb{Z})$.

To investigate the free resolutions of metacyclic groups, we first look at the geometric interpretation, before moving back into the abstract algebraic theory.

We define a *Cayley complex* $K_{\mathcal{G}}$ to be a 2-dimensional cell complex (*CW complex*) arising from a presentation $\mathcal{G}$ of a finitely presented group G, by starting with a 0-cell and attaching a 1-cell to this 0-cell for every generator in $\mathcal{G}$, and use each relation in $\mathcal{G}$ as an attaching map for a 2-cell. By construction, the resulting cell complex has fundamental group G, $\pi_1(K_{\mathcal{G}}) \cong G$. Let:

$$\mathcal{G} = \langle x_1, \ldots, x_g | W_1, \ldots, W_r \rangle$$

We say $\mathcal{G}$ has a *balanced presentation* if $g = r$. So the metacyclic groups form balanced presentations in the following form:

$$\mathcal{G}(pq) = \langle x, y | x^p = y^q, yx = x^r y \rangle$$

where r is a primitive q^{th} root of unity mod p. To see clearly what we are trying to do we restrict ourselves to the case $\mathcal{G}(21)$:

$$\mathcal{G}(21) = \langle x, y | x^7 = y^3, yx = x^2 y \rangle$$

Before moving on, we check that $\mathcal{G}(21)$ defines the same group as the standard presentation $G(21)$ that we use earlier.

Proposition 5.1.3. *$\mathcal{G}(21)$ is a balanced presentation for the metacyclic group $G(21)$.*

Proof. All we need to show is either $x^7 = e$ or $y^3 = e$. Take x, and manipulate it using the second relation:

$$\begin{aligned} y^3 x y^{-3} &= y^2 x^2 y^{-2} \\ &= y(x^4) y^{-1} \\ &= (yxy^{-1})^4 \\ &= x^8 = x \end{aligned}$$

Hence $x^7 = e$ □

With such a balanced presentation of $\mathcal{G}(21)$ it is possible to construct a corresponding *Cayley complex* $K_{\mathcal{G}}$ with $\pi_1(K_{\mathcal{G}})$.

We use the formal method of *free differential calculus* given by Fox [7]. At the universal covering level, we have two generators x, y that give rise to two 'lifted' 1-cells ϵ_1, ϵ_2. We may display them as:

$$[1] \xrightarrow{e_1} [x] \qquad\qquad [1] \overset{e_2}{\rightsquigarrow} [y]$$

If this is orientated from $[1]$ to $[x]$, then the boundaries are given by:

$$\partial(\epsilon_1) = x - 1 \qquad\qquad \partial(\epsilon_2) = y - 1$$

Observe that $\partial(\epsilon_1), \partial(\epsilon_2)$ together generate a map for the augmentation ideal:

$$\partial_1 = (x - 1, y - 1)$$

In a similar manner we can lift the 2-cells to the universal cover. We use the relations of $\mathcal{G}(21)$ to obtain the 2-cells. The first relation is $x^7 = y^3$, namely E_1. This becomes a 10-sided polygon bounded by using the basic 1-cells described above:

$$\begin{tikzcd}
 & {[1]} \arrow[dl, "\epsilon_1"'] \arrow[r, squiggly, "\epsilon_2"] & {[y]} \arrow[r, squiggly, "\epsilon_2 \cdot y"] & {[y^2]} \arrow[dr, squiggly, "\epsilon_2 \cdot y^2"] & \\
{[x]} \arrow[d, "\epsilon_1 \cdot x"] & & E_1 & & {[y^3 = x^7]} \\
{[x^2]} \arrow[dr, "\epsilon_1 \cdot x^2"] & & & & {[x^6]} \arrow[u, "\epsilon_1 \cdot x^6"'] \\
 & {[x^3]} \arrow[r, "\epsilon_1 \cdot x^3"] & {[x^4]} \arrow[r, "\epsilon_1 \cdot x^4"] & {[x^5]} \arrow[ur, "\epsilon_1 \cdot x^5"] &
\end{tikzcd}$$

Expressing the boundary of E_1 we obtain:

$$\partial(E_1) = \epsilon_1(1 + x + x^2 + x^3 + x^4 + x^5 + x^6) - \epsilon_2(1 + y + y^2)$$

We use the same method using the second relation $yx = x^2y$ to obtain the 2-cell E_2. This gives a 5-sided polygon when lifted to the universal covering:

$$\begin{tikzcd}
 & {[1]} \arrow[dl, "\epsilon_1"'] \arrow[rr, squiggly, "\epsilon_2"] & & {[y]} \arrow[dr, "\epsilon_1 \cdot y"] & \\
{[x]} \arrow[dr, squiggly, "\epsilon_2 \cdot x"] & & E_2 & & {[xy]} \arrow[dll, "\epsilon_1 \cdot xy"] \\
 & & {[yx = x^2y]} & &
\end{tikzcd}$$

Expressing the boundary of E_2 we obtain:

$$\partial(E_2) = \epsilon_1(1 - y - xy) + \epsilon_2(x - 1)$$

Taking these boundaries in matrix form, and we obtain a map for ∂_2:

$$\partial_2 = \begin{pmatrix} \sum_x & 1-y-xy \\ -\sum_y & x-1 \end{pmatrix}$$

where $\sum_x = (1 + x + \ldots + x^6)$, and $\sum_y = 1 + y + y^2$.

At present there is no elementary geometric interpretation to describe ∂_3 and beyond. Subsequently the only way to go further from here is to calculate $\mathrm{Ker}(\partial_2)$ explicitly in order to obtain ∂_3. We are left with the algebraic Cayley complex:

$$0 \to \mathrm{Ker}(\partial_2) \to \mathbb{Z}[G(21)]^2 \xrightarrow{\partial_2} \mathbb{Z}[G(21)]^2 \xrightarrow{\partial_1} \mathbb{Z}[G(21)] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

For the Realisation Problem, we need a description of $\pi_2(K_{\mathcal{G}}) = \mathrm{Ker}(\partial_2)$. We discuss $\pi_2(K_{\mathcal{G}})$ in more detail in the next chapter.

The other problem with this method is, that although it can be useful for free resolution of cohomological period 4, as one would only need to work out $\mathrm{Ker}(\partial_2)$ explicitly, any group with cohomological period greater than 4 quickly becomes tedious to deal with. As $\mathbb{Z}[G(21)]$ has cohomological period 6, there are four mappings that we must compute. Moreover, we know very little about each stability class except for the duality relation.

The one case where using the Fox method gives a complete free resolution is when G is a cyclic group, which has cohomological period 2. Hence an explicit resolution for such groups is easy to construct. Let $C_n = \langle x \mid x^n = 1 \rangle$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\sum_x} \mathbb{Z}[C_n] \xrightarrow{x-1} \mathbb{Z}[C_n] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Seeing the problem of the Fox method, we give an alternative method to obtaining free resolutions for metacyclic groups. This alternative method helps us learn more about each syzygy, as well as simplifies the calculations in obtaining a free resolution. Thus, we proceed onto looking at free resolutions by the decomposition of syzygies at the minimal level. We move back to the general case $\mathbb{Z}[G(pq)]$. Look at the following generic free resolution for Λ:

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda^{n_{2q-1}} \xrightarrow{\partial_{2q-1}} \ldots \xrightarrow{\partial_2} \Lambda^{n_1} \xrightarrow{\partial_{n_1}} \Lambda^{n_0} \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

We start by noting that we can use the Fox method to obtain the rank of the first three $\Lambda^{n_i}s$. Evidently, as we have two generators and two relations any free resolution of the group ring Λ begins with:

$$\ldots \Lambda^{n_3} \xrightarrow{\partial_3} \Lambda^2 \xrightarrow{\partial_2} \Lambda^2 \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

Furthermore as there exists a duality relation between syzygies,

$$\begin{aligned}\Lambda^{n_0} &= \Lambda^{n_{2q-1}} = \Lambda \\ \Lambda^{n_1} &= \Lambda^{n_{2q-2}} = \Lambda^2 \\ \Lambda^{n_2} &= \Lambda^{n_{2q-3}} = \Lambda^2\end{aligned}$$

So for any group of order $2p$, $3p$ we know the structure of the free resolution straight away. For groups where $q > 3$, the cohomological period is still even, but greater than 6, so it has the following structure:

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{\partial_{2q-1}} \Lambda^2 \to \Lambda^2 \to \Lambda^{n_{2q-4}} \to \ldots \to \Lambda^{n_3} \to \Lambda^2 \to \Lambda^2 \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

Define $K(m)$ to be a representative of the minimal level in its respective syzygy $\Omega_m(\mathbb{Z})$. Thus $K(1)$ is the augmentation ideal, namely $\mathcal{I}_{\mathcal{G}}$. We observe from the Fox method that the rank of the following minimal level representatives are:

(5.i) $\mathrm{rank}_{\mathbb{Z}}(K(1)) = pq - 1$

(5.ii) $\mathrm{rank}_{\mathbb{Z}}(K(2)) = pq + 1$

(5.iii) $\mathrm{rank}_{\mathbb{Z}}(K(3)) = pq - 1$

We establish how many indecomposable modules may exist in $K(1)$. For any module M, write $\mathcal{E}(M) = \mathrm{End}_{\mathcal{D}\mathrm{er}}(M)$. Then if M is any representative module for $\Omega_1(\mathbb{Z})$ we recall:

i) $\mathcal{E}(M) \cong \mathbb{Z}/pq \cong \mathbb{Z}/p \oplus \mathbb{Z}/q$

ii) $\mathcal{E}(N) \cong 0 \iff N$ is projective

Proposition 5.1.4. *If $K(1)$ is a minimal representative of $\Omega_1(\mathbb{Z})$ then either:*

a) *$K(1)$ is indecomposable or*

b) *$K(1) \cong K(1)_1 \oplus K(1)_2$ where each $K(1)_j$ is indecomposable.*

Proof. Suppose that $K(1) \cong M_1 \oplus M_2 \oplus M_3$ where each M_j is non-zero. Then $\mathcal{E}(K(1))$ contains $\mathcal{E}(M_1) \oplus \mathcal{E}(M_2) \oplus \mathcal{E}(M_3)$. However we know from **i)** that:

$$\mathcal{E}(K(1)) \cong \mathbb{Z}/p \oplus \mathbb{Z}/q$$

so that for some j, $\mathcal{E}(M_j) = 0$. Thus, using **ii)** is a nonzero projective for some j. Hence $\mathrm{rank}_{\mathbb{Z}}(K(1)) \geq pq$. However this is a contradiction, $\mathrm{rank}_{\mathbb{Z}}(K(1)) = pq - 1$. The result now follows. □

For the next result to hold recall Swan's result that over a finite group $\mathbb{Z}[G]$, projective modules can only have the same $\mathbb{Z}$ ranks as free modules [27].

Corollary 5.1.5. *If $K(1)$ is a minimal representative of $\Omega_1(\mathbb{Z})$ then $K(1)$ must split into two indecomposable modules,*

$$K(1) \cong K(1)_1 \oplus K(1)_2$$

Proof. From our complete list of indecomposable modules over Λ there does not exist an indecomposable module of rank $pq - 1$. Hence $K(1)$ must split into two indecomposable modules. □

It is clear that $K(3)$ follows the exact same argument and so also has rank $pq - 1$. $K(2)$ also follows the same argument above, but this time $\mathrm{rank}_{\mathbb{Z}}(K(2)) = pq+1$. All the steps remain the same and so $K(2)$ also splits into two indecomposable modules, as no indecomposable module of rank $pq + 1$ exists.

From the periodicity of the free resolutions we observe by induction that $K(4)$ also decomposes into two indecomposable modules. This holds for all $K(m)$ in the free resolution. We note at this stage:

$$\Omega_q(\mathbb{Z})^* = \Omega_{-q}(\mathbb{Z}) = \Omega_q(\mathbb{Z})$$

such that $K(q)^* = K(q)$. So we only need to express the structure of $K(m)$ where $m \leq q$, as taking the duals we obtain the structure of $K(2q - m)$.

Proposition 5.1.6. *In a minimal free resolution of a group $G(pq)$, each $K(m)$ decomposes into two indecomposable modules $K(m)_1 \oplus K(m)_2$. Hence in*

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{\partial_{2q-1}} \Lambda^{n_{2q-2}} \to \ldots \to \Lambda^{n_1} \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

each $n_i = 2$.

Proof. Suppose this is not true. Then take $n_1 = 3$, so $\mathrm{rank}_{\mathbb{Z}}(K(2)) = 2pq+1$. However it is not possible to construct $K(2)$ here using only 2 indecomposable modules, as the largest indecomposable module is of rank pq. Thus $n_1 < 3$. $n_1 \neq 1$ as this would give a free resolution of cohomological period 2. This is not possible as the only groups with cohomological period 2 are $\mathbb{Z}[C_n]$. Hence $n_1 = 2$.

Progressing to the next term, again take $n_2 = 3$. Then $\mathrm{rank}_{\mathbb{Z}}(K(3)) = 2pq-1$. It is not possible to obtain $K(3)$ using only 2 indecomposable modules. Hence $n_2 = 2$. By induction this is true up to n_{2q-2}. □

Thus using (5.i), (5.ii), (5.iii) and Proposition 5.1.6 we generalise the ranks of the even and odd syzygies.

(5.iv) $\mathrm{rank}_{\mathbb{Z}}(K(2m + 1)) = pq - 1$

(5.v) $\mathrm{rank}_{\mathbb{Z}}(K(2m)) = pq + 1$

5.2 The augmentation ideal

We describe the augmentation ideal of Λ by $\mathcal{I}_{\mathcal{G}}$. The augmentation ideal can be represented as the minimal level of the first syzygy. In other words, the stable class $[\mathcal{I}_{\mathcal{G}}] = \Omega_1(\mathbb{Z})$. There exists a sequence:

$$0 \to \mathcal{I}_{\mathcal{G}} \to \Lambda \to \mathbb{Z} \to 0$$

To simplify $\mathcal{I}_{\mathcal{G}}$, we describe it in terms of indecomposable modules from the list established in Section 4.3.

Theorem III. $\Omega_1(\mathbb{Z})$ *at the minimal level can be described as* $\mathcal{I}_{\mathcal{G}} = P \oplus X$.

Proof. Take the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigoplus_{i=0}^{q-1} P^i \arrow[r] \arrow[d, "id"] & \mathcal{I}_{\mathcal{G}} \arrow[r] \arrow[d] & \mathcal{J} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{q-1} P^i \arrow[r] & \Lambda \arrow[r] \arrow[dr, "\epsilon"] & \mathbb{Z}[C_q] \arrow[r] \arrow[d, "\eta"] & 0 \\
 & & & \mathbb{Z} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

We have established from the cohomological properties that $\mathrm{Ext}^1_\Lambda(\mathcal{J}, P) = 0$, so this term can split off from the extension. Thus the quotient $\mathcal{I}_{\mathcal{G}}/P$ is of dimension $p(q-1)$. X is the unique free indecomposable module of rank $p(q-1)$, implying that $\mathcal{I}_{\mathcal{G}}/P = X$. Hence, we see that $\mathcal{I}_{\mathcal{G}} = X \oplus P$. Moreover $\mathcal{I}_{\mathcal{G}}$ is minimal in its stability class as $\mathrm{rank}(\mathcal{I}_{\mathcal{G}}) = pq - 1 < pq = \mathrm{rank}(\Lambda)$. $\square$

We want to show that the differential ∂_1 can be described as a direct sum. We note that the mappings are not unique, and there are many options for what they can be. Below is a systematic argument to show that the polynomial mapping obtained using 'Fox's free calculus' can be split up into a direct sum of two independent polynomials for the augmentation ideal. We shall therefore prove:

Theorem IV. *$\mathcal{I}_{\mathcal{G}}$ splits as a direct sum of Λ-modules*

$$\mathcal{I}_{\mathcal{G}} \;\cong\; [x-1+(y-1)\alpha) \;\dot{+}\; [y-1)$$

for some suitable $\alpha \in \Lambda$

We note that $[y-1) \subset \mathcal{I}_{\mathcal{G}}$. Suppose that the module P is described in the usual way with $\mathbb{Z}$-basis $\{\epsilon_j\}_{1\leq j\leq p-1}$ where $\epsilon_j = (\zeta_p - 1)\zeta_p^{j-1}$ then:

Proposition 5.2.1. *There exists an exact sequence of Λ-modules*

$$\mathcal{S} \;=\; (0 \to [y-1) \to \mathcal{I}_{\mathcal{G}} \xrightarrow{\nu} P \to 0)$$

in which $\nu(x-1) = \epsilon_1$.

Proof. Put $W_j = (x-1)x^{j-1}$ for $1 \leq j \leq p-1$ and put $\mathcal{W} = \mathrm{span}_{\mathbb{Z}}\{W_1, \ldots W_{p-1}\}$. Then we see that $\mathcal{I}_{\mathcal{G}}$ decomposes as a direct sum of $\mathbb{Z}$-modules

$$\mathcal{I}_{\mathcal{G}} \;=\; \mathcal{W} \;\dot{+}\; [y-1)$$

We let $\natural : \mathcal{I}_{\mathcal{G}} \to \mathcal{I}_{\mathcal{G}}/[y-1)$ be the canonical map. Then $\{\natural(W_1), \ldots, \natural(W_{p-1})\}$ is a $\mathbb{Z}$-basis for $\mathcal{I}_{\mathcal{G}}/[y-1)$. We know that $[y-1)$ is a Λ-submodule of $\mathcal{I}_{\mathcal{G}}$, hence $\mathcal{I}_{\mathcal{G}}/[y-1)$ is naturally a Λ-submodule. Moreover, by computing the action of x, y in the quotient, it is possible to see that $\mathcal{I}_{\mathcal{G}}/[y-1) \cong P$ via the correspondence $\natural(W_j) \mapsto \epsilon_j$. □

By construction it is clear that $\mathcal{S}$ splits as a sequence of $\mathbb{Z}[C_p]$-modules. We need to show that $\mathcal{S}$ splits over Λ. In order to see that $\mathcal{S}$ splits we construct an auxiliary sequence. Let:

$$[x-1) = \mathrm{Ker}\left(\Sigma_x : \Lambda \to \Lambda, \quad \alpha \mapsto \sum_{j=0}^{p-1} x^j \alpha\right)$$

Clearly $\mathcal{W} \subset [x-1) \subset \mathcal{I}_{\mathcal{G}}$. This leads to a direct sum of $\mathbb{Z}[C_p]$-modules:

$$[x-1) \;=\; \mathcal{W} \;\dot{+}\; [x-1) \cap [y-1)$$

which, when restricting ν to $[x-1)$, we get an exact sequence of Λ-modules:

$$\mathcal{S}' \;=\; (0 \to [x-1) \cap [y-1) \to [x-1) \xrightarrow{\nu} P \to 0$$

We observe that $\mathscr{C}_q(R, \theta_{[q]}, 1) = [x-1)$.

Proposition 5.2.2. *$\mathcal{S}'$ splits over Λ.*

Proof. $\mathrm{rank}_{\mathbb{Z}}([x-1)) = (p-1)q$. Then $\mathcal{S}'$ is an exact sequence over the cyclic algebra. We know that R is the dual of the augmentation ideal $\mathbb{Z}[C_p]$. Using Rosen's Theorem, P is projective as a module over $\mathscr{C}_q(R, \theta_{[q]}, 1)$. Hence $\mathcal{S}'$ splits over $\mathcal{C}_q(R, \theta_{[q]}, 1)$, and so must also split over Λ. □

Corollary 5.2.3. *$\mathcal{S}$ splits over Λ.*

Proof. Let $\sigma : P \to [x-1)$ be a right splitting of $\mathcal{S}'$ over Λ, then $\nu \circ \sigma = Id$. Thus σ is also a right splitting of $\mathcal{S}$. □

With this established, $\mathcal{I}_{\mathcal{G}}$ splits as a direct sum of Λ-modules

$$\mathcal{I}_{\mathcal{G}} \;=\; [\sigma(\epsilon_1) \;\dot{+}\; [y-1))$$

where ν restricts to an isomorphism $[\sigma(\epsilon_1)) \overset{\simeq}{\to} P$. However, we have seen that $\nu(x-1) = \epsilon_1$, and $\nu \circ \sigma = Id$, so that $\sigma(\epsilon_1)$ takes the form $\sigma(\epsilon_1) = (x-1)+(y-1)\alpha$ for some $\alpha \in \Lambda$. This gives the required result proving Theorem IV. Hence:

$$\mathcal{I}_{\mathcal{G}} \;\cong\; [(x-1)+(y-1)\alpha) \;\dot{+}\; [y-1)$$

If not obvious already, we can now match up the indecomposable decomposition of $\mathcal{I}_{\mathcal{G}}$ with the polynomial interpretation. We have:

(5.vi) $X = [y-1)$

(5.vii) $P = [(x-1)+(y-1)\alpha)$

We also have the interpretation $\bigoplus_{i=0, i\neq 1}^{q} P^i = [x-1) \cap [y-1)$. We know as a module X is self-dual. So we just need to verify that $[y-1)$ is also (anti) self-dual, so that we can use $[y-1)$ as the polynomial for X over any group ring Λ.

Proposition 5.2.4. *Let $X = [y-1)$. The polynomial is (anti) self-dual over all group rings Λ.*

Proof. There are two cases to consider.

case 1 When $q = 2$, we have the groups $G(2p)$. In this case $[y-1)$ is self-dual itself, as $[y-1)^* = [y-1)$. Hence $[y-1) = X$.

case 2 When q is odd we have $[y-1)^* = [y^{q-1}-1)$. This at first glance does not look self-dual. However, if we multiply the polynomial by $y^{(q-1)/2}$ we get an isomorphism, as all that changes is the ordering of the basis elements in $[y-1)$:

$$[y-1) \;\cong\; [y-1)y^{(q-1)/2}$$

Now taking the dual of this polynomial we get it to be anti-self-dual:

$$[(y-1)y^{(q-1)/2})^* \quad = \quad -[(y-1)y^{(q-1)/2})$$

So we can see that $X = [y-1)$ is anti self-dual.

□

With X completely described we turn our attention to P. Notice that in the polynomial interpretation of P there is still an unknown α. Whereas, the above decomposition works for all $\mathcal{I}_{\mathcal{G}}$, at this point there does not seem to be a solution for α in general. One can see the difficulty in obtaining α by looking at two groups where an α has been calculated explicitly.

In Strouthos's thesis [26] he gives an interpretation for $P = (\zeta_3 - 1)\mathbb{Z}[\zeta_3]$ for $\mathbb{Z}[D_6]$. His polynomial does not use the general formula obtained above, but with a rearrangement it gives the required result. Strouthos takes $P = [(x^2y - xy)(1 - x^2y))$. Rearranging, it is a polynomial in the required form:

$$P = (\zeta_3 - 1)\mathbb{Z}[\zeta_3] = [(x-1) + (y-1)(x - x^2))$$

where $\alpha = (x - x^2)$. In Chapter 7, the P shown in the free resolution for $\mathbb{Z}[G(21)]$ was originally obtained by finding an explicit polynomial for α and then rearranged into canonical form. Before rearranging we have:

$$P = (\zeta_7 - 1)\mathbb{Z}[\zeta_7] = [(x-1) + (y-1)(x^2 - x^6 + (x^3 + x^4 - x^5 - x^6)y))$$

where $\alpha = (x^2 - x^6 + (x^3 + x^4 - x^5 - x^6)y)$. There are many alternatives for α, and from this example alone we can see how complicated it is to parametrise the α factor. At this stage, we know what P should look like, but there does not seem to be a feasible way of generalising a polynomial for α.

5.3 Free resolutions and syzygy decompositions

From Proposition 5.1.6 we see that any free resolution of a metacyclic group can be broken up into short exact sequences of the form:

$$0 \to K(m+1)_1 \oplus K(m+1)_2 \to \Lambda^2 \to K(m)_1 \oplus K(m)_2 \to 0$$

We have seen in the previous section that $\mathcal{I}_{\mathcal{G}}$ splits into two separate indecomposable modules with a ∂_1 that consists of two independent maps. This in turn implies we can 'untwist' the augmentation ideal to form two separate short exact sequences of the form:

$$0 \to K(2)_1 \to \Lambda \to P \to 0$$
$$0 \to K(2)_2 \to \Lambda \to X \to 0$$

So, if we take any metacyclic free resolution, we observe that it can be 'untwisted' to form two separate monogenic infinite resolutions. We note that the two infinite monogenic resolutions do not give the same resolution as the one obtained by splicing copies of the exact sequence as in Section 5.1. This method gives a different approach that simplifies our problem of obtaining explicit syzygy decompositions. Thus, the two monogenic resolutions are:

A

$$\begin{tikzcd}
 & & & & & K(2q-2)_1 \arrow[dr] & & & K(2)_1 \arrow[dr] & & & & & \\
{} \arrow[r, dotted] & \Lambda \arrow[r, "\partial(0)_1"] & \Lambda \arrow[rr, "\partial(2q-1)_1"] \arrow[dr] & & \Lambda \arrow[ur] \arrow[rr, "\partial(2q-2)_1"] & & \Lambda \arrow[r, dotted] & \Lambda \arrow[ur] \arrow[rr, "\partial(2)_1"] & & \Lambda \arrow[rr, "\partial(1)_1"] \arrow[dr] & & \Lambda \arrow[r, "\partial(0)_1"] & \Lambda \arrow[r, dotted] & {} \\
 & & & K(2q-1)_1 \arrow[ur] & & & & & & & P \arrow[ur] & & &
\end{tikzcd}$$

B

$$\begin{tikzcd}
 & & & & & K(2q-2)_2 \arrow[dr] & & & K(2)_2 \arrow[dr] & & & & & \\
{} \arrow[r, dotted] & \Lambda \arrow[r, "\partial(0)_1"] & \Lambda \arrow[rr, "\partial(2q-1)_2"] \arrow[dr] & & \Lambda \arrow[ur] \arrow[rr, "\partial(2q-2)_2"] & & \Lambda \arrow[r, dotted] & \Lambda \arrow[ur] \arrow[rr, "\partial(2)_2"] & & \Lambda \arrow[rr, "\partial(1)_2"] \arrow[dr] & & \Lambda \arrow[r, "\partial(0)_2"] & \Lambda \arrow[r, dotted] & {} \\
 & & & K(2q-1)_2 \arrow[ur] & & & & & & & X \arrow[ur] & & &
\end{tikzcd}$$

where $\partial(m) = \begin{pmatrix} \partial(m)_1 & 0 \\ 0 & \partial(m)_2 \end{pmatrix}$. We work out the ranks in resolutions **A** and **B**.

Start with **A**. We have $\mathrm{rank}(P) = p-1$. Looking at the short exact sequence $0 \to K(2)_1 \to \Lambda \to P \to 0$, we see that $\mathrm{rank}(K(2)_1) = pq-(p-1)$. Denote

$$\langle \mathscr{N} \rangle = rank(\mathscr{N}) \text{ where } \mathscr{N} \text{ is an indecomposable module}$$

From the periodicity of the free resolutions all indecomposable modules in **A** are either $\langle p-1 \rangle$ or $\langle pq-(p-1) \rangle$. We can look at **B** in the same manner and rewrite **A, B** as:

C

$$\begin{tikzcd}
 & & & & & \langle pq-(p-1) \rangle \arrow[dr] & & & \langle pq-(p-1) \rangle \arrow[dr] & & & & & \\
{} \arrow[r, dotted] & \Lambda \arrow[r, "\partial(0)_1"] & \Lambda \arrow[rr, "\partial(2q-1)_1"] \arrow[dr] & & \Lambda \arrow[ur] \arrow[rr, "\partial(2q-2)_1"] & & \Lambda \arrow[r, dotted] & \Lambda \arrow[ur] \arrow[rr, "\partial(2)_1"] & & \Lambda \arrow[rr, "\partial(1)_1"] \arrow[dr] & & \Lambda \arrow[r, "\partial(0)_1"] & \Lambda \arrow[r, dotted] & {} \\
 & & & \langle p-1 \rangle \arrow[ur] & & & & & & & P \arrow[ur] & & &
\end{tikzcd}$$

D

$$
\begin{tikzcd}
 & & & & & \langle p \rangle \arrow[dr] & & & \langle p \rangle \arrow[dr] & & & & \\
{} \arrow[r, dotted] & \Lambda \arrow[r, "\partial(0)_1"] & \Lambda \arrow[rr, "\partial(2q-1)_2"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(2q-2)_2"] \arrow[ur] & & \Lambda \arrow[r, dotted] & \Lambda \arrow[rr, "\partial(2)_2"] \arrow[ur] & & \Lambda \arrow[rr, "\partial(1)_2"] \arrow[dr] & & \Lambda \arrow[r, "\partial(0)_2"] & \Lambda \arrow[r, dotted] & {} \\
 & & & \langle (q-1)p \rangle \arrow[ur] & & & & & & & X \arrow[ur] & &
\end{tikzcd}
$$

It remains to be seen what indecomposable modules each term in **C, D** are represented by. We have the indecomposable modules for $\mathcal{I}_G$ already, and use them as a starting point. We treat **D** first,

Proposition 5.3.1. *The indecomposable modules in* ***D*** *are:*

a) $\langle (q-1)p \rangle = X$

b) $\langle p \rangle = \overline{\mathbb{Z}[C_p]}$

Proof. Looking at the list of indecomposable modules over Λ, (4.iv) shows X is the unique module of rank $(q-1)p$. Hence it is self-dual, and all modules of rank $(q-1)p$ must be equal to X.
In the same manner, looking at (4.iii), all modules $\langle p \rangle = \overline{\mathbb{Z}[C_p]}$. □

We already have a polynomial interpretation for X (5.vi), and we observe that there is a straightforward polynomial interpretation for $\overline{\mathbb{Z}[C_p]}$:

(5.viii) $\overline{\mathbb{Z}[C_p]} = \sum_y = 1 + y + \ldots + y^{q-1}$

Subsequently we have an explicit description of the infinite resolution **D**:

$$
\begin{tikzcd}
 & & & & & \overline{\mathbb{Z}[C_p]} \arrow[dr] & & & \overline{\mathbb{Z}[C_p]} \arrow[dr] & & & & \\
{} \arrow[r, dotted] & \Lambda \arrow[r, "y-1"] & \Lambda \arrow[rr, "\sum_y"] \arrow[dr] & & \Lambda \arrow[rr, "y-1"] \arrow[ur] & & \Lambda \arrow[r, dotted] & \Lambda \arrow[rr, "y-1"] \arrow[ur] & & \Lambda \arrow[rr, "\sum_y"] \arrow[dr] & & \Lambda \arrow[r, "y-1"] & \Lambda \arrow[r, dotted] & {} \\
 & & & X \arrow[ur] & & & & & & & X \arrow[ur] & &
\end{tikzcd}
$$

At this point we notice that **D** has period two. This implies that **C** must have period $2q$, and so it is the more complicated of the two resolutions. We look at **C** by first considering a couple of examples. We commence by investigating the most basic case, when $q = 2$.

Example 5.3.2. $\Lambda = \mathbb{Z}[G(2p)]$

In such cases, we have a free resolution of period four, and so the branch **A** *here looks like:*

$$\begin{tikzcd}
 & & K(4)_1 \arrow[dr] & & & & K(2)_1 \arrow[dr] & & & & K(0)_1 \arrow[dr] & & \\
{} \arrow[r, dotted] & \Lambda \arrow[ur] \arrow[rr, "\partial(4)_1=\partial(0)_1"] & & \Lambda \arrow[dr] \arrow[rr, "\partial(3)_1"] & & \Lambda \arrow[ur] \arrow[rr, "\partial(2)_1"] & & \Lambda \arrow[dr] \arrow[rr, "\partial(1)_1"] & & \Lambda \arrow[ur] \arrow[rr, "\partial(0)_1"] & & \Lambda \arrow[r, dotted] & {} \\
 & & & & K(3)_1 \arrow[ur] & & & & P \arrow[ur] & & & &
\end{tikzcd}$$

Observe that $K(4)_1 \cong K(0)_1$. *From the list of indecomposable modules there exist only two distinct modules of rank* $\langle (p-1) \rangle$, *namely* R, P. *Furthermore, from Theorem II,* $P^* \cong R$. *Hence in such resolutions* $R = K(3)_1$.

For the even syzygies, we use the short exact sequences at each stage. Thus we have:

$$0 \to P \to \Lambda \to K(0)_1 \to 0$$
$$0 \to R \to \Lambda \to K(2)_1 \to 0$$

These sequences are of the form (4.iv), and so we observe that $K(0)_1 = \mathcal{Q}(P)$, *and* $K(2)_1 = \mathcal{Q}(R)$. *The infinite resolution therefore has the following description:*

$$\begin{tikzcd}
 & & \mathcal{Q}(P) \arrow[dr] & & & & \mathcal{Q}(R) \arrow[dr] & & & & \mathcal{Q}(P) \arrow[dr] & & \\
{} \arrow[r, dotted] & \Lambda \arrow[ur] \arrow[rr, "\partial(4)_1=\partial(0)_1"] & & \Lambda \arrow[dr] \arrow[rr, "\partial(3)_1"] & & \Lambda \arrow[ur] \arrow[rr, "\partial(2)_1"] & & \Lambda \arrow[dr] \arrow[rr, "\partial(1)_1"] & & \Lambda \arrow[ur] \arrow[rr, "\partial(0)_1"] & & \Lambda \arrow[r, dotted] & {} \\
 & & & & R \arrow[ur] & & & & P \arrow[ur] & & & &
\end{tikzcd}$$

Before moving onto the next example, it is worth noting that although the above resolution is correct, it is worth looking at it in a different form:

$$\begin{tikzcd}
 & & \mathcal{Q}(P^3) \arrow[dr] & & & & \mathcal{Q}(P^2) \arrow[dr] & & & & \mathcal{Q}(P) \arrow[dr] & & \\
{} \arrow[r, dotted] & \Lambda \arrow[ur] \arrow[rr, "\partial(4)_1"] & & \Lambda \arrow[dr] \arrow[rr, "\partial(3)_1"] & & \Lambda \arrow[ur] \arrow[rr, "\partial(2)_1"] & & \Lambda \arrow[dr] \arrow[rr, "\partial(1)_1"] & & \Lambda \arrow[ur] \arrow[rr, "\partial(0)_1"] & & \Lambda \arrow[r, dotted] & {} \\
 & & & & P^2 \arrow[ur] & & & & P \arrow[ur] & & & &
\end{tikzcd}$$

This gives an infinite resolution where the odd syzygies are

(5.ix) $$K(2m+1)_1 = P^{((2m+1)+1)/2}$$

However over $\mathbb{Z}[G(2p)]$ any $P^{((2m+1)+1)/2}$ must be isomorphic to one of R, P. Using the identities $P^{2e} \cong R$, and $P^{2e+1} \cong P$ for all $e \in \mathbb{N}$ we can convert $P^{((2m+1)+1)/2}$ to either R, P.

Example 5.3.3. $\Lambda = \mathbb{Z}[G(3p)]$
We have free resolutions of period six, and so branch **A** *has the form:*

$$\begin{tikzcd}
& & & & & & K(4)_1 \arrow[dr] & & & & K(2)_1 \arrow[dr] & & & & K(0)_1 \arrow[dr] & & \\
\arrow[r, dotted] & \Lambda \arrow[rr, "\partial(6)_1"] & & \Lambda \arrow[rr, "\partial(5)_1"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(4)_1"] \arrow[ur] & & \Lambda \arrow[rr, "\partial(3)_1"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(2)_1"] \arrow[ur] & & \Lambda \arrow[rr, "\partial(1)_1"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(0)_1"] \arrow[ur] & & \Lambda \arrow[r, dotted] & {} \\
& & & & K(5)_1 \arrow[ur] & & & & K(3)_1 \arrow[ur] & & & & P \arrow[ur] & & & &
\end{tikzcd}$$

Here $\partial(6)_1 = \partial(0)_1$. In such cases there exist three modules of rank $(p-1)$, namely R, P, P^2. The duals from Theorem II give $P^2 \cong (P^2)^*$, and $P^* \cong R$. $K(3)_1$ in the free resolution is self-dual, thus $K(3)_1 = P^2$. In addition $K(5)_1 = R$. Hence looking at short exact sequences again we attain:

$$\begin{tikzcd}
& & & & & & \mathcal{Q}(R) \arrow[dr] & & & & \mathcal{Q}(P^2) \arrow[dr] & & & & \mathcal{Q}(P) \arrow[dr] & & \\
\arrow[r, dotted] & \Lambda \arrow[rr, "\partial(6)_1"] & & \Lambda \arrow[rr, "\partial(5)_1"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(4)_1"] \arrow[ur] & & \Lambda \arrow[rr, "\partial(3)_1"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(2)_1"] \arrow[ur] & & \Lambda \arrow[rr, "\partial(1)_1"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(0)_1"] \arrow[ur] & & \Lambda \arrow[r, dotted] & {} \\
& & & & R \arrow[ur] & & & & P^2 \arrow[ur] & & & & P \arrow[ur] & & & &
\end{tikzcd}$$

As in Example 5.3.2, we have an infinite resolution with the odd syzygies following (5.ix). However here we have three non-isomorphic distinct modules of rank (p-1). So applying the identities $R \cong P^{3e}, P \cong P^{3e+1}, P^2 \cong P^{3e+2}$ for all $e \in \mathbb{N}$, to convert $P^{((2m+1)+1)/2}$ to one of R, P, P^2.

From these examples, we recognize that equation (5.ix) gives a general formula for the odd indecomposable modules in **A**. We need to prove that this equation holds for all metacyclic groups.

Proposition 5.3.4. *For any metacyclic group Λ we can describe the odd syzygies in resolution* **A** *by:*

$$K(2m+1)_1 = P^{((2m+1)+1)/2}$$

Proof. We have explicitly shown this works for metacyclic groups where $q = 2, 3$. If true for all q, then $K(2m+1)_1 \cong K(2m+1)_1$ (mod $2q$). Thus it suffices to show the proposition is true up to $K(2q-1)_1$.

We established that all the $K(2m+1)_1 \leq K(2q-1)_1$ must be distinct and non-isomorphic. If this were not true, then the resolution would have cohomological period less than $2q$.

From Theorem III, $K(1)_1 = P$. Furthermore the dual gives $K(2q-1)_1 = R$. As $R \cong P^q$ the proposition holds here. When q is odd, $K(q)_1$ is self-dual, and so the only P^e where $e < q$ that is self-dual is $P^{(q+1)/2}$. Hence:

$$K(q)_1 = P^{(q+1)/2}$$

This term also holds in the proposition, and so we assume the remaining odd syzygies follow as shown in the proposition. In order to verify the remaining odd syzygies are correctly interpreted we look at the duals $K(2m+1)_1^*$. Using Theorem II, the duals for all $K(2m+1)_1$ are confirmed to be as expected in the resolution, and so the result follows. □

Corollary 5.3.5. $K(2m)_1 = \mathcal{Q}(P^{((2m+1)+1/2})$

Proof. We use short exact sequences from the resolution. From Proposition 5.3.4 $K(2m+1)_1 = P^{((2m+1)+1)/2}$, such that:

$$0 \to P^{((2m+1)+1)/2} \to \Lambda \to K(2m)_1 \to 0$$

Thus, we obtain $K(2m)_1 = \Lambda / P^{((2m+1)+1)/2} = \mathcal{Q}(P^{((2m+1)+1/2})$. □

With all of the above in place we 'retwist' **A** and **B**, the two infinite resolutions to obtain free resolutions for Λ. As a consequence we arrive at:

Theorem V. *For any metacyclic group $\Lambda = \mathbb{Z}[G(pq)]$, we describe the syzygies at the minimal level of its free resolution by:*

$$\Omega_m(\mathbb{Z}) = \begin{cases} P^{(m+1)/2} \oplus X & \text{when } 2 \nmid m \\ \mathcal{Q}(P^{(m/2)+1}) \oplus \overline{\mathbb{Z}[C_p]} & \text{when } 2 \mid m \end{cases}$$

where $1 \leq m \leq 2q-1$.

For the next chapter we are particularly interested in $\Omega_3(\mathbb{Z})$. At the minimal level, the third syzygy $\Omega_3(\mathbb{Z})$ of $\mathbb{Z}[G(pq)]$ is decomposed by Theorem V:

(5.x) $$\Omega_3(\mathbb{Z}) = P^2 \oplus X$$

Theorem V shows us that we have fully diagonal resolutions. To describe such resolutions explicitly we need polynomial interpretations for the indecomposable modules in each syzygy. Theorem IV gives a polynomial interpretation of the augmentation ideal using (5.vi) and (5.vii). Utilising (5.vii) and (5.iii) we can describe half of the mappings in a resolution to give:

$$\partial(m) = \begin{cases} \begin{pmatrix} P & , [y-1) \end{pmatrix} & \text{when } m = 1 \\ \begin{pmatrix} \mathcal{Q}(P^{(m/2)+1}) & 0 \\ 0 & \sum_y \end{pmatrix} & \text{when } 2 \mid m \\ \begin{pmatrix} P^{(m+1)/2} & 0 \\ 0 & [y-1) \end{pmatrix} & \text{when } 2 \nmid m,\ m \neq 1, 2q-1 \\ \begin{pmatrix} R \\ [y-1) \end{pmatrix} & \text{when } m = 2q-1 \end{cases}$$

where $1 \leq m \leq 2q-1$. Presently there are no obvious polynomial descriptions for the remaining indecomposable modules in the free resolution. For this reason we leave the indecomposable module interpretation for these terms in the mappings.

Chapter 6

The $\mathcal{R}(2)$-$\mathcal{D}(2)$ Problem for metacyclic groups

6.1 The $\mathcal{R}(2)$-$\mathcal{D}(2)$ Problem

Let G be a group with a finite presentation $\mathcal{G} = \langle x_1, \ldots, x_g | W_1, \ldots W_r \rangle$ and let $K_{\mathcal{G}}$ be the Cayley complex of $\mathcal{G}$., such that $\pi_1(K_{\mathcal{G}}) \cong G$. The cellular chain complex of the universal cover $\widetilde{K}_{\mathcal{G}}$ gives rise to:

$$C_*(\mathcal{G}) = \Big(0 \to \pi_2(K_{\mathcal{G}}) \to C_2(\widetilde{K}_{\mathcal{G}}) \xrightarrow{\partial_2} C_1(\widetilde{K}_{\mathcal{G}}) \xrightarrow{\partial_1} C_0(\widetilde{K}_{\mathcal{G}}) \xrightarrow{\epsilon} \mathbb{Z} \to 0\Big)$$

as an exact sequence of right $\mathbb{Z}[G]$-modules. The second homotopy group $\pi_2(K_{\mathcal{G}})$ is identified with Ker($\partial_2 : C_2(\widetilde{K}_{\mathcal{G}}) \to C_1(\widetilde{K}_{\mathcal{G}})$). Hence it is a free abelian group. More generally, by an *algebraic 2-complex* over G we mean an exact sequence of right $\mathbb{Z}[G]$-modules of the form:

$$\mathbf{F} = \Big(0 \to J \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0\Big)$$

where F_i is a finitely generated free module over $\mathbb{Z}[G]$. In the previous chapter we took $J = K(3)$, and so denote the class of all modules stably equivalent to J by $\Omega_3(\mathbb{Z})$.

The isomorphism class of $\pi_2(K_{\mathcal{G}}) \in \mathcal{F}(\mathbb{Z}[G])$ is not unique, but it is stably unique. Let the presentation $\mathcal{G}$ contain g generators and r relations. By *Tietze' Theorem* ([18] p. 48-53) we can relate any two presentations of a finitely presented group G with a finite chain of transformations of the following type:

I) Add a new generator g' and a relation of the form $g = w$, where w is a word in the existing generators.

II) Add a relation r' which is a word in the existing relations.

The inverse of these relations can also be applied. We observe that if we start with a finite presentation $\mathcal{G}$ of a group G with g generators and r relations, then we have a geometrically realised complex:

$$0 \to \pi_2(K_{\mathcal{G}}) \to \mathbb{Z}[G]^r \to \mathbb{Z}[G]^g \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

If we use a Tietze transformation of type **I)** to obtain a new group presentation, we obtain a new resolution but $\pi_2(K_{\mathcal{G}})$ remains the same:

$$0 \to \pi_2(K_{\mathcal{G}}) \to \mathbb{Z}[G]^{r+1} \to \mathbb{Z}[G]^{g+1} \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

However, when using a Tietze transformation of type **II)**, $\pi_2(K_{\mathcal{G}})$ adds on a direct summand $\mathbb{Z}[G]$, as we only change the $\mathbb{Z}[G]^r$ module:

$$0 \to \pi_2(K_{\mathcal{G}}) \oplus \mathbb{Z}[G] \to \mathbb{Z}[G]^{r+1} \to \mathbb{Z}[G]^g \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

Thus if we are able realise $\pi_2(K_{\mathcal{G}})$, then by **II)** we can also realise $\pi_2(K_{\mathcal{G}}) \oplus \mathbb{Z}[G]$. So if we can realise a 2^{nd} algebraic homotopy group J, then as a result $J \oplus \mathbb{Z}[G]^m$ can be realised for any $m \geq 1$. From this we look at the following question:

The Realisation $\mathcal{R}(2)$-Problem. Let G be a finitely presented group. Is every algebraic 2-complex over G realised up to chain homotopy in the form $C_*(\mathcal{G})$?

It is clear that the module J plays the role of an algebraic $\pi_2(K_{\mathcal{G}})$, and so the Realisation Problem is parametrised by the class $\Omega_3(\mathbb{Z})$, which contains all such possible algebraic homotopy groups J.

In Johnson's book he proves two results which helps simplify the Realisation Problem into a form we can then investigate for the metacyclic groups $G(pq)$. He proves:

Theorem 6.1.1 ([13], p. 213)**.** *The finite group G has the realisation property if and only if all minimal algebraic* 2*-complexes are realisable.*

This gives a general condition on homotopy types, but it is more useful to have a criterion for realisation in terms of the second homotopy group. So we say that a minimal module $J \in \Omega_3(\mathbb{Z})$ is realisable when for some finite 2-complex $K_{\mathcal{G}}$, where $\pi_1(K_{\mathcal{G}}) \cong G$, there is an isomorphism of $\mathbb{Z}[G]$-modules $J \cong \pi_2(K_{\mathcal{G}})$. We achieve this via the Swan map. For any minimal module $J \in \Omega_3(\mathbb{Z})$ the natural map exists:

$$\nu : \mathrm{Aut}_{\mathbb{Z}[G]}(J) \to \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J)$$

This leads to the Swan map:

$$s : \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J) \to \widetilde{K}_0(\Lambda)$$

Thus, we say J is *full* when $\mathrm{Im}(\nu) = \mathrm{Ker}(s)$. Intuitively fullness ensures that J can be attached in an appropriate manner to the preceding modules in an algebraic 2-complex, such that no 'twisting' exists, ensuring geometrical realisability of the whole complex. Thus we have the result:

Theorem 6.1.2 ([13], p. 216)**.** *If each minimal module $J \in \Omega_3(\mathbb{Z})$ is both realisable and full, then G has the realisation property.*

Therefore the $\mathcal{R}(2)$-Problem has two properties we need to investigate for the groups $G(pq)$. First we prove that $\Omega_3(\mathbb{Z})$ has a straight tree structure. This leaves only one minimal module $J \in \Omega_3(\mathbb{Z})$. Secondly we discuss the existence of the Swan map, and in the case of $G(pq)$ we explain how far we can go in showing whether the minimal module J is full.

Before attempting to prove the $\mathcal{R}(2)$-Problem, we remind ourselves of the original question of interest:

Wall's $\mathcal{D}(2)$ Problem: Let $\mathcal{X}$ be a connected cell complex of geometric dimension 3 such that $\mathrm{H}_3(\widetilde{\mathcal{X}}, \mathbb{Z}) = 0$ and $\mathrm{H}_3(\mathcal{X}, \mathcal{B}) = 0$ for all coefficient systems $\mathcal{B}$ on $\mathcal{X}$. Is $\mathcal{X}$ homotopy equivalent to a finite complex of dimension 2?

So when these two conditions are satisfied, we say that a 3-complex $\mathcal{X}$ is cohomologically two dimensional with $\pi_1(\mathcal{X}) \cong G$. We proceed in trying to prove the more accessible $\mathcal{R}(2)$ problem for metacyclic fundamental groups $G(pq)$, and consequently this would give an affirmative result to the more notorious $\mathcal{D}(2)$ problem.

6.2 The tree structure of $\Omega_3(\mathbb{Z})$

We concern ourselves here with proving the following:

Theorem VI. *Let $\Lambda = \mathbb{Z}[G(pq)]$. $\Omega_3(\mathbb{Z})$ has no branching at the minimal level, and so $\Omega_3(\mathbb{Z})$ is straight.*

We shall first prove that the stable class of the augmentation ideal $[\mathcal{I}_G] = \Omega_1(\mathbb{Z})$ is straight.

Proposition 6.2.1. *If Λ has the Eichler property, then Λ must have the cancellation property for free modules, so the tree structure of $[0]$ is straight.*

Proof. It is sufficient to show that in the Wedderburn decomposition of $\Lambda_{\mathbb{Q}}$, there does not exist a factor that could be represented by $\mathbb{H}$ when looking at $\Lambda_{\mathbb{R}}$. From Section 4.2 we have:

$$\Lambda_{\mathbb{Q}} \cong \mathbb{Q}[C_q] \times M_q(K_0)$$

where K_0 is the fixed field of $\mathscr{C}_q(K, \theta_{[q]}, 1)$. It is clear from this rational representation that an $\mathbb{H}$ factor does not exist in $\Lambda_{\mathbb{R}}$. $\square$

From here we can prove:

Proposition 6.2.2 ([14], p. 235-236)**.** *If Λ has the cancellation property for free modules then $\Omega_1(\mathbb{Z})$ is straight, meaning that if $J \oplus \Lambda \cong I \oplus \Lambda$, then $I \cong J$.*

From this, we can straight away see that $\Omega_{-1}(\mathbb{Z})$ is also straight. This comes from the fact that syzygies preserve their structure under duality. So:

$$[\mathcal{I}_{\mathcal{G}}] = [\mathcal{I}_{\mathcal{G}}{}^*]$$

In the special case when Λ admits a free resolution of period four, more specifically the dihedral group rings $\mathbb{Z}[D_{2p}]$, we can already describe $\Omega_3(\mathbb{Z})$. This work was done by Johnson.

Corollary 6.2.3. *For the group rings $\mathbb{Z}[D_{2p}]$, $\Omega_3(\mathbb{Z})$ is straight.*

Proof. We know that $\Omega_1(\mathbb{Z})$ is straight. As a result $\Omega_{-1}(\mathbb{Z})$ is straight. The group ring $\mathbb{Z}[D_{2p}]$ have a free resolution period four such that $\Omega_{-1}(\mathbb{Z}) = \Omega_3(\mathbb{Z})$. The result easily follows. $\square$

However what happens in the more general case when the free resolution is of period greater than four? In such circumstances, the tree structure of Ω_1 is no longer sufficient to describe the tree structure of $\Omega_3(\mathbb{Z})$, as $\Omega_{-1}(\mathbb{Z}) \neq \Omega_3$.

We move back to looking at indecomposable modules for Λ. Let J be a minimal representative of $\Omega_3(\mathbb{Z})$. Corollary 5.1.5 proved that J must decompose into two indecomposable modules. We take this further:

Proposition 6.2.4. *If J is a minimal representative of $\Omega_3(\mathbb{Z})$ then $J \cong J_1 \oplus J_2$ where $\mathrm{rank}_{\mathbb{Z}}(J_1) = p - 1$, and $\mathrm{rank}_{\mathbb{Z}}(J_2) = (q-1)p$.*

Proof. We know $\mathrm{rank}_{\mathbb{Z}}(J) = pq - 1$, and there does not exist any indecomposable module of $\mathrm{rank}_{\mathbb{Z}} = pq - 1$. Hence J must be of the form $J_1 \oplus J_2$. Having a look at the full list of indecomposable modules from Section 4.4, the only way we can obtain J by using two indecomposable modules is:

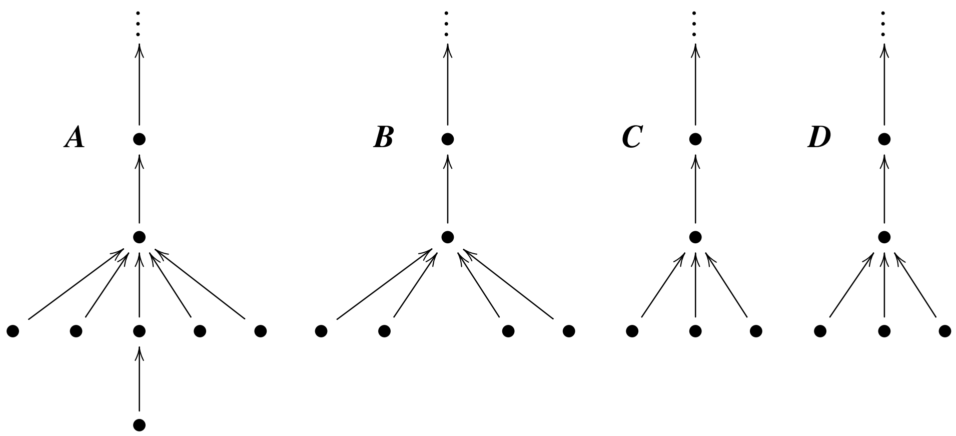
$$J \cong \langle p-1 \rangle \oplus \langle (q-1)p \rangle$$

$\square$

At this point we recall that for groups where $\widetilde{K}_0(\Lambda) = 0$, the genera of indecomposable modules forms a complete list of indecomposable modules over $\mathbb{Z}[G(pq)]$, the free group ring. However, when $\widetilde{K}_0(\Lambda) \neq 0$, there exist extra indecomposable modules arising from elements of $\widetilde{K}_0(\Lambda)$ other than the identity.

In Johnson's book he discusses that when $\widetilde{K}_0(\Lambda) \neq 0$ there exist so called 'cousin' tree structures for the projective elements within $\widetilde{K}_0(\Lambda)$.

Example 6.2.5 ([15], p.125)**.** *Take the quaternion group ring $\mathbb{Z}[Q(36)]$ presented by $Q(36) = \langle x, y \mid x^9 = y^2,\ xyx = y\rangle$. Then $\widetilde{K}_0(\mathbb{Z}[Q(36)]) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ so that $\pi_0([0])$ has precisely four connected components. Below are the tree structures for these components, where the height function is the normalised $\mathbb{Z}$-rank of the $\mathbb{Z}[Q(36)]$-modules involved:*



*The tree labelled **A** is simply the stable class of the zero module; namely the tree of stably free modules over $\mathbb{Z}[Q(36)]$. The remaining trees come from the other elements of the projective class group $\widetilde{K}_0(\mathbb{Z}[Q(36)])$.*

As we are only interested in the free group ring $\mathbb{Z}[G(pq)]$, we may restrict ourselves to the set of indecomposable modules that exist over the identity element of $\widetilde{K}_0(\Lambda)$. Any additional indecomposable modules that exist may be ignored.

Theorem VI. *Let $\Lambda = \mathbb{Z}[G(pq)]$. $\Omega_3(\mathbb{Z})$ has no branching at the minimal level, and so $\Omega_3(\mathbb{Z})$ is straight.*

Proof. Let J be a minimal representative of $\Omega_3(\mathbb{Z})$. We know from Section 5.3 that J must be of the form

$$J \cong J_1 \oplus J_2$$

We have seen that there is only one way it can decompose numerically:

$$J \cong \langle p-1\rangle \oplus \langle (q-1)p\rangle$$

In fact, from equation (5.x) we have seen that in free resolution constructions we get:

$$J \cong P^2 \oplus X$$

We have seen in (4.iv) that X is the unique indecomposable module of rank $(q-1)p$, and as a result is self-dual. All we therefore need to show is that P^2 is the only option for the module of rank $p-1$ in $\Omega_3(\mathbb{Z})$ at the minimal level.

Suppose that J' is another module, where $J' \cong \langle p-1 \rangle \oplus X$ such that J' is stably equivalent to J:

$$J \oplus \Lambda^a \cong J' \oplus \Lambda^a$$

As X is the unique element of order $(q-1)p$ we have:

$$P^2 \oplus X \oplus \Lambda^a \cong \langle p-1 \rangle \oplus X \oplus \Lambda^a$$

Now consider the following cohomological property:

$$\mathrm{Ext}^1_\Lambda(\overline{\mathbb{Z}[C_p]}, N) \begin{cases} = 0 & \text{if } N \cong P^2 \text{ or } \langle p-1 \rangle \\ = 0 & \text{if } N \cong \Lambda \\ \neq 0 & \text{if } N \cong X \end{cases}$$

Thus, as X is a finite abelian group we can cancel it from both sides to leave:

$$P^2 \oplus \Lambda^a \cong \langle p-1 \rangle \oplus \Lambda^a$$

From the list of indecomposable modules over the free group ring Λ, there exist q candidates for $\langle p-1 \rangle$, namely $P^e = (\zeta_p - 1)^e \mathbb{Z}[\zeta_p]$, where $0 \leq e \leq q-1$. However from Section 4.3 they are all cohomologically distinct and as stabilising preserves cohomological properties then the only possibility for $\langle p-1 \rangle$ is P^2 itself. As a result:

$$J \cong J'$$

Hence, there is no branching of $\Omega_3(\mathbb{Z})$ at the minimal level as J is the only minimal representative. So, by using the Swan-Jacobinski Theorem we determine that the stable class $\Omega_3(\mathbb{Z})$ is straight.

□

Using the same argument as in Theorem VI it follows that:

Corollary 6.2.6. *The odd syzygies $\Omega_{2m+1}(\mathbb{Z})$ have no branching at the minimal level.*

In consequence of the Swan-Jacobinski Theorem we now see that:

Corollary 6.2.7. *The odd syzygies $\Omega_{2m+1}(\mathbb{Z})$ are all straight.*

Thus Theorem VI shows that for any metacyclic group there exists a single homotopy module at the minimal level, $J \in \Omega_3(\mathbb{Z})$, so the groups $G(pq)$ are realisable by the standard presentation of the group. So no 'exotic' homotopy types exist.

6.3 The Swan map

To verify the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem for $G(pq)$, we use the *Swan map* to show that all homotopy types are geometrically realisable. A detailed exposition of k-invariants and the Swan map is found in Johnson ([13], chapter 6). Here we outline the main points. We first look at the Swan homomorphism for a finite group $\Lambda = \mathbb{Z}[G]$. We begin by looking at $0 \to I \to \Lambda \to \mathbb{Z} \to 0$.

Recall from Proposition 3.2.6 that

$$\mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathbb{Z}) \cong \mathbb{Z}/|G|$$

and as a corollary, $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(I) \cong (\mathbb{Z}/|G|)^*$. Now look at the following pushout construction:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "\alpha"] & \Lambda \arrow[r] \arrow[d, dotted] & \mathbb{Z} \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & I \arrow[r] & \varinjlim(\alpha, i) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

If we take $\alpha \in \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(I)$ then $\varinjlim(\alpha, i)$ is a finitely generated projective module. We can now look at the 'Swan mapping':

$$\begin{tikzcd}
\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(I) \arrow[rr, "s"] \arrow[dr, "\cong"] & & \widetilde{K_0}(\Lambda) \\
& (\mathbb{Z}/|G|)^* \arrow[ur, "s"] &
\end{tikzcd}$$

Proposition 6.3.1. *$s : (\mathbb{Z}/|G|)^* \to \widetilde{K}_0(\Lambda)$ is a homomorphism. So we have the following $s(ab) = s(a) + s(b)$.*

Proof. Looking at the following diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "Id"] & (I, ab) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[d, "Id"] & (I, a) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "a \in (\mathbb{Z}/|G|)^*"] & 0 \\
0 \arrow[r] & I \arrow[r] & \Lambda \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

we have

$$0 \longrightarrow (I,a) \longrightarrow \Lambda \longrightarrow \mathbb{Z}/a \longrightarrow 0$$
$$0 \longrightarrow (I,ab) \longrightarrow (I,a)\mathbb{Z}/b \longrightarrow 0$$
$$0 \longrightarrow (I,b) \longrightarrow \Lambda \longrightarrow \mathbb{Z}/b \longrightarrow 0$$

So if $a \in (\mathbb{Z}/|G|)^*$, then (I,a) is a projective. Hence:

$$\begin{aligned} \Lambda \oplus (I,ab) &\cong (I,a) \oplus (I,b) \\ s(ab) &= s(a) + s(b) \end{aligned}$$

□

Also a natural map $\nu : \mathrm{Aut}_{\Lambda}(I) \to \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(I)$ exists where $\alpha \mapsto [\alpha]$. So what is the relation with $\mathrm{Ker}(s)$?

Proposition 6.3.2. $\mathrm{Im}(\nu) \subset \mathrm{Ker}(s)$

Proof. We observe the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "\alpha"] & \Lambda \arrow[r] \arrow[d, "\kappa"] & \mathbb{Z} \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & I \arrow[r] & \varinjlim(\alpha, i) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

If $\alpha : I \to I$ is a Λ-isomorphism then κ is also a Λ-isomorphism by the 5 Lemma. So $\varinjlim(\alpha, i) \cong \Lambda$, leaving $s(\alpha) = 0 \in \widetilde{K}_0(\Lambda)$. □

This leads us to see that $\mathrm{Ker}(s)/\mathrm{Im}(\nu)$ is an invariant of G. However the above gives information for the augmentation ideal. For the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem we need to look at $\Omega_3(\mathbb{Z})$. The argument only needs to be extended a couple more steps. Take the following to be an algebraic 2-complex:

$$\begin{tikzcd}
 & & & K \arrow[dr] & & & & & \\
0 \arrow[r] & J \arrow[r] & F_2 \arrow[ur] \arrow[rr] & & F_1 \arrow[rr] \arrow[dr] & & F_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
 & & & & & I \arrow[ur] & & &
\end{tikzcd}$$

Importantly, when G is finite, $\mathbb{Z}, I, K, J$ are all coprojective. This solves our problem to give:

$$\mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathbb{Z}) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}}(I) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}}(K) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$$

So we can consider the Swan mapping as $s : \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J) \to \widetilde{K}_0(\Lambda)$, where now $J \in \Omega_3(\mathbb{Z})$. Finally we get $\mathrm{Ker}(s)/\mathrm{Im}(\nu)$ to be an invariant for the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem. This implies there is a one to one correspondence between $\mathrm{Ker}(s)/\mathrm{Im}(\nu)$ and the number of homotopy types of algebraic two-complexes.

Looking again at the pushout construction:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "i"] \arrow[d, "\alpha"] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & J \arrow[r] & \varinjlim(\alpha, i) \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

There are two scenarios that can arise from here. Firstly if α can be chosen to be an isomorphism over Λ then $\varinjlim(\alpha, i)$ is automatically stably free and so we just get a homotopy equivalence, and so nothing new. Conversely a problem can arise when $\varinjlim(\alpha, i)$ is stably free, but we do not know whether α can be chosen to be an automorphism over Λ. If no α can be chosen, then we are left with a new homotopy type.

The Swan map for the groups $G(pq)$

We turn our attention back to the metacyclic groups $G(pq)$. In Section 6.2 we showed that the tree structure for $\Omega_3(\mathbb{Z})$ is straight, and so to verify the Realisation problem, we need to show fullness, so that there only exists a single homotopy type for a group of the form $G(pq)$. In other words we need to show that all elements in $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J)$ can be either realised in the kernel of the Swan map, or inject into $\widetilde{K}_0(\Lambda)$.

For the groups $G(pq)$ we showed the existence of free resolutions with diagonal maps. It therefore suffices to look at the Swan map over the augmentation ideal, I, as there is no problem in lifting from the augmentation ideal back to the minimal homotopy type J. So the argument can be simplified.

We know that we can decompose the augmentation ideal $I = \mathcal{I}_{\mathcal{G}} \cong P \oplus [y-1)$. There exists $\alpha \in \Lambda$ such that the following commutes where $\lambda_\alpha : \Lambda \to \Lambda$ is taken as $\lambda_\alpha(x) = \alpha x$. Hence:

$$\begin{tikzcd}
0 \arrow[r] & P \oplus [y-1) \arrow[r, "i"] \arrow[d, "\alpha"] & \Lambda \arrow[r] \arrow[d, "\lambda_\alpha"] & \mathbb{Z} \arrow[r] \arrow[d, "\epsilon(\alpha)"] & 0 \\
0 \arrow[r] & P \oplus [y-1) \arrow[r] & \varinjlim(\alpha, i) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

So for metacyclic groups the problem now reduces to the task of realising all elements in $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{I}_{\mathcal{G}})$. In its own right this problem is a large task and apart from trying to realise each element explicitly by suitable α's, it is at this point the journey ends. Such explicit realisations are complicated, and at present there is no reasonable way to generalise.

Here we show which elements can be realised in general. In the next chapter we show by explicit calculation that for the group $G(21)$ the Swan map is full. Thus in this particular case we are able to obtain a positive result for the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem.

There are two elements that can be realised. We have

$$\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{I}_{\mathcal{G}}) \cong (\mathbb{Z}/pq)^* \cong C_{q-1} \times C_{p-1}$$

and so the Swan map for metacyclic groups is described as:

$$s : C_{q-1} \times C_{p-1} \twoheadrightarrow \widetilde{K}_0(\Lambda)$$

Proposition 6.3.3. $\mathrm{Ker}(s)$ *contains a copy of* C_2*.*

Proof. $p-1$ is even, as p is prime and $p \neq 2$. Hence $\mathrm{Ker}(s)$ contains a copy of C_2 that can be realised when $\alpha = \pm Id$. Consequently:

$$\hat{s} : (C_{q-1} \times C_{p-1})/\{\pm Id\} \twoheadrightarrow \widetilde{K}_0(\Lambda)$$

□

When $q = 2$, this is all we are able realise by general methods. But when q is odd, there also exists an injective map $i : C_q \hookrightarrow \widetilde{K}_0(\Lambda)$. Define $p - 1 = nq$ where n is a positive integer.

Proposition 6.3.4. *In the Swan map* $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{I}_{\mathcal{G}}) \to \widetilde{K}_0(\Lambda)$ *there exists an injective map* $i : C_q \hookrightarrow \widetilde{K}_0(\Lambda)$ *where* $C_q \subset \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{I}_{\mathcal{G}})$*, and* q *is odd.*

Proof. Recall from 4.4.13 that

$$\widetilde{K}_0(\mathbb{Z}[G(pq)]) \to \widetilde{K}_0(R_0) \dotplus \widetilde{K}_0(\mathbb{Z}[C_q])$$

whose kernel $D_0(\mathbb{Z}[G(pq)]$ is a finite cyclic group of order q, q odd, and of order $q/2$, q even. So when q is odd, the kernel is equal to C_q. Then in the Swan map we have:

$$s : C_{q-1} \times C_n \times C_q \twoheadrightarrow \widetilde{K}_0(\Lambda)$$

So as a copy of $C_q \cong \mathrm{Ker}(\widetilde{K}_0(\Lambda)) \subset \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{I}_{\mathcal{G}})$, then the injective map i exists. □

At present we cannot realise anymore of the Swan map by general methods. We would need to realise terms explicitly, and here lies the limit of the thesis. However we are very close to verifying the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem for the general metacyclic group $G(pq)$, and progress on realising the Swan map would help.

The belief at present is that the Swan map should be 'full' for all metacyclic groups, but we do not have a rigorous proof to verify this. Thus, using proposition 6.3.3 and proposition 6.3.4 we are able to reduce the Swan map and leave the following question:

Conjecture VII (The Swan map for metacyclic groups $G(pq)$)**.** *When q is odd, all the elements in the reduced Swan map:*

$$\bar{s} : C_{q-1} \times C_{n/2} \twoheadrightarrow \widetilde{K}_0(\Lambda)/C_q$$

where $n = (p-1)/q$, can be realised.

If this Conjecture can be proved, then the $\mathcal{R}(2)$-$\mathcal{D}(2)$ is verified for all groups $G(pq)$. The conjecture has excluded the case when $q = 2$, as this is dealt with by Johnson [13], where he verifies the $\mathcal{R}(2)$-$\mathcal{D}(2)$ for groups D_{4n+2}, which includes our study of D_{2p}.

Chapter 7

The $\mathcal{R}(2)$-$\mathcal{D}(2)$ Problem for $\mathbf{G(21)}$

7.1 Representation theory of $G(21)$

In this section, we follow the theory developed in Chapter 4 for the specific group $G(21)$. As a result we restrict the notation for Λ here, such that $\Lambda = \mathbb{Z}[G(21)]$. We investigate the integral representation of $\mathbb{Z}[G(21)]$, concluding with a description of all 21 indecomposable modules that exist over $\mathbb{Z}[G(21)]$. Throughout this chapter we take the following presentation for $G(21)$:

$$G(21) =< x, y \mid x^7 = y^3 = 1, yxy^{-1} = x^2 >$$

Rational representation theory of $G(21)$

Before we begin looking at $\mathbb{Q}[G(21)]$, we will describe the Wedderburn decomposition of $\mathbb{C}[G(21)]$. This is straightforward to obtain as $\mathbb{C}$ is an algebraically closed field. We need to obtain the conjugacy classes of $\mathbb{C}[G(21)]$, which are: $\{1\}$, $\{x, x^2, x^4\}$, $\{x^3, x^5, x^6\}$, $\{y, xy, \ldots, x^6y\}$, $\{y^2, xy^2, \ldots, x^6y^2\}$. In total, there are five conjugacy classes, leaving only one option for the Wedderburn decomposition:

$$\mathbb{C}[G(21)] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_3(\mathbb{C}) \times M_3(\mathbb{C})$$

We move on to $\mathbb{Q}[G(21)]$. Taking the cyclic group $C_7 = \langle x | x^7 = 1\rangle$, then the following is a square for the rational group ring $\mathbb{Q}[C_7]$:

$$\begin{tikzcd} \mathbb{Q}[C_7] \arrow[r] \arrow[d] & \mathcal{I}^* \arrow[d] \\ \mathbb{Q} \arrow[r] & 0 \end{tikzcd}$$

Therefore $\mathbb{Q}[C_7] \cong \mathbb{Q} \times \mathcal{I}^*$. Over $\mathbb{Z}[C_7]$, we clearly see that $\mathcal{I}^* = \mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i/7}$, the primitive seventh root of unity. We denote $\mathcal{I}^* = K$.

Using the cyclic algebra $\mathscr{C}_3(\mathbb{Q}[C_7], \theta_{[3]}, 1)$ where $y\zeta = \zeta^2 y$ to obtain the following square for $\mathbb{Q}[G(21)]$:

$$\begin{array}{ccc} \mathbb{Q}[G(21)] & \longrightarrow & \mathscr{C}_3(K, \theta, 1) \\ \downarrow & & \downarrow \\ \mathbb{Q}[C_3] & \longrightarrow & 0 \end{array}$$

This leaves us with the decomposition $\mathbb{Q}[G(21)] \cong \mathbb{Q}[C_3] \times \mathscr{C}_3(K, \theta_{[3]}, 1)$. We shall now obtain a description for $\mathscr{C}_3(K, \theta, 1)$.

Proposition 7.1.1. *The following surjective ring homomorphism exists:*

$$\rho_\alpha : \mathbb{Q}[G(21)] \to M_3(K_0)$$

where $K_0 = \mathbb{Q}(\alpha)$*, and* $\alpha = i\sqrt{7}$*.*

Proof. Return to $\mathbb{Q}[C_7]$. Here we have $\mathbb{Q}[C_7] \cong \mathbb{Q}[x]/(x-1) \times \mathbb{Q}[x]/c_7(x)$ where $c_7(x) = (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$. From the fact that $\mathbb{Q}[x]/c_7(x) \cong \mathbb{Q}(\zeta)$ we have:

$$\begin{aligned} c_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\ &= (x-\zeta)(x-\zeta^2)(x-\zeta^3)(x-\zeta^4)(x-\zeta^5)(x-\zeta^6) \\ &= (x^3 - \alpha x^2 + \overline{\alpha} x - 1)(x^3 - \overline{\alpha} x^2 + \alpha x - 1) \end{aligned}$$

where $\alpha = (\zeta + \zeta^2 + \zeta^4)$, $\overline{\alpha} = (\zeta^3 + \zeta^5 + \zeta^6)$. By equating coefficients, we obtain α and $\overline{\alpha}$ explicitly.

$$\begin{aligned} \alpha + \overline{\alpha} &= -1 \\ \alpha + \alpha\overline{\alpha} + \overline{\alpha} &= 1 \end{aligned}$$

This can be rearranged to give $\alpha^2 + \alpha + 2 = 0$. So, $\alpha = (-1/2 + \sqrt{-7}/2)$, and $\overline{\alpha} = (-1/2 - \sqrt{-7}/2)$. Thus $K_0 = \mathbb{Q}(i\sqrt{7})$. This leaves us with the following representation:

$$\rho_\alpha : \mathbb{Q}[G(21)] \to M_3(\mathbb{Q}(\alpha))$$

$$\rho_\alpha(x) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & (\alpha+1) \\ 0 & 1 & \alpha \end{pmatrix}, \quad \rho_\alpha(y) \mapsto \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

This representation holds the properties $x^7 = y^3 = 1$, and $yx = x^2$. □

We see from the above that $\dim_{K_0}(K) = 3$, and $\dim_{\mathbb{Q}}(K_0) = 2$. Hence we use Proposition 4.2.2 to show that $\mathscr{C}_3(K, \theta_{[3]}, 1) \cong M_3(\mathbb{Q}(i\sqrt{7}))$. This results in the full Wedderburn decomposition of $\mathbb{Q}[G(21)]$:

$$\mathbb{Q}[G(21)] \cong \mathbb{Q} \times \mathbb{Q}(\omega) \times M_3(\mathbb{Q}(i\sqrt{7}))$$

Integral representation theory of $G(21)$

In a similar manner to above, but now taking the integral group ring we get:

$$\begin{tikzcd} \mathbb{Z}[C_7] \arrow[r] \arrow[d] & \mathcal{I}^* \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{F}_7 \end{tikzcd}$$

We take the dual of the augmentation ideal to be $\mathcal{I}^* = \mathbb{Z}[\zeta] = R$. Thus R is the ring of algebraic integers of K. Taking the cyclic algebra of $\mathbb{Z}[C_7]$ we obtain:

$$\begin{tikzcd} \mathbb{Z}[G(21)] \arrow[r] \arrow[d] & \mathscr{C}_3(R, \theta_{[3]}, 1) \arrow[d] \\ \mathbb{Z}[C_3] \arrow[r] & \mathbb{F}_7[C_3] \end{tikzcd}$$

where $\mathbb{F}_7[C_3] \cong \mathbb{F}_7 \times \mathbb{F}_7 \times \mathbb{F}_7$. The problem here is how to describe $\mathscr{C}_3(R, \theta_{[3]}, 1)$. We use Rosen's Theorem for the specific case of $G(21)$.

Theorem 7.1.2 (Rosen)**.** *The cyclic algebra $\mathscr{C}_3(R, \theta_{[3]}, 1)$ decomposes in the following manner:*

$$\mathscr{C}_3(R, \theta_{[3]}, 1) \cong R \oplus P \oplus P^2$$

where each $P^e = (\zeta - 1)^e \mathbb{Z}[\zeta]$ is an ideal.

The main incentive for reproducing Rosen's Theorem again is that we calculate Stage 1 of Rosen's Theorem explicitly for $G(21)$ using the algorithm provided in Section 4.2.

As in the rational case for $G(21)$, $\mathscr{C}_3(R, \theta_{[3]}, 1)$ is an algebra, free of rank nine over $R_0 = R^{\theta_{[3]}} = \{x \in R; \theta_{[3]}(x) = x\}$, where R_0 = the ring of integers of $K_0 = \mathbb{Q}(i\sqrt{7})$. Furthermore R_0 has a unique prime π over 7.

Define $\mathscr{T}_3(R_0, \pi) = \{X \in M_3(R_0), X_{ij} = 0\ (mod \pi) \text{ when } i > j\}$ as the quasi-triangular matrix. Explicitly this looks like:

$$\mathscr{T}_3(R_0, \pi) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \pi a_{21} & a_{22} & a_{23} \\ \pi a_{31} & \pi a_{32} & a_{33} \end{pmatrix} ; a_{ij} \in R_0 \right\}$$

Each row is an ideal, hence $\mathscr{T}_3(R_0, \pi) \cong \mathfrak{R}(1) \oplus \mathfrak{R}(2) \oplus \mathfrak{R}(3)$. So to prove Rosen's Theorem we need to show:

Proposition 7.1.3. $\mathscr{C}_3(R, \theta_{[3]}, 1) \cong \mathscr{T}_3(R_0, \pi)$

Proposition 7.1.3 is shown in two stages as in Section 4.3.

Proposition 7.1.4 (Stage 1)**.** *There exists an injective ring homomorphism*

$$i : \mathscr{C}_3(R, \theta_{[3]}, 1) \to \mathscr{T}_3(R_0, \pi)$$

We commence by recalling that in the rational representation of $G(21)$ we have the following surjective ring homomorphism, $\rho_\alpha : \mathbb{Z}[G(21)] \to M_3(R_0)$.

$$\rho_\alpha(x) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & (\alpha+1) \\ 0 & 1 & \alpha \end{pmatrix}, \quad \rho_\alpha(y) \mapsto \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

We want to conjugate this to a representation of $\tilde{\rho} : \mathbb{Z}[G(21)] \to \mathscr{T}_3(R_0, \pi)$, where π is a unique prime in R_0 over 7. We reduce $\mathbb{Z}[G(21)] \to M_3(R_0)$ (mod 7):

$$\mathbb{F}_7[G(21)] \to M_3(R_0 \otimes \mathbb{F}_7)$$

From the rational representation theory we can take $R_0 \cong \mathbb{Z}/(\alpha^2 + \alpha + 2)$. Over $\mathbb{F}_7$, we see that $\alpha^2 + \alpha + 2 = (\alpha - 3)^2$. Thus $R \otimes \mathbb{F}_7 \cong \mathbb{F}_7[\alpha]/(\alpha - 3)^2 \cong \mathbb{F}_7[\pi]/\pi^2$, where $\pi = (\alpha - 3)$. We are left with the following mapping:

$$\mathbb{F}_7[G(21)] \to M_3(\mathbb{F}_7[\pi]/\pi^2)$$

$$x \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & \pi+4 \\ 0 & 1 & \pi+3 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 & \pi+3 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

From here take $\pi \to 0$ to obtain $M_3(\mathbb{F}_7)$

$$\overline{\rho} : \mathbb{F}_7[G(21)] \to M_3(\mathbb{F}_7)$$

$$\overline{\rho}(x) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}, \quad \overline{\rho}(y) \mapsto \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

At this point we conjugate $\overline{\rho}(x)$ to it Jordan Normal Form over $\mathbb{F}_7 \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Formally we define:

$$\begin{aligned} \hat{\rho} : \mathbb{F}_7[G(21)] &\to M_3(\mathbb{F}_7) \\ \hat{\rho}(x) &= \overline{Q}\overline{\rho}(x)(\overline{Q})^{-1} \\ \hat{\rho}(y) &= \overline{Q}\overline{\rho}(y)(\overline{Q})^{-1} \end{aligned}$$

We see that $\hat{\rho}(x)$ exists over $\mathbb{F}_7$ by taking $\overline{Q}$ and $\overline{Q}^{-1}$ as:

$$\overline{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \quad (\overline{Q})^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

From here we obtain $\hat{\rho}(y)$. Hence:

$$\hat{\rho}(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{\rho}(y) = \begin{pmatrix} 4 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We now lift $\overline{Q}$ back up to $\mathbb{Z}$. In this scenario, $\det(\overline{Q}) = 1 \in \mathbb{F}_7^*$. $\overline{Q}$ is unimodular and so when we break it up into elementary matrices and lift, the obvious lifting works. Conveniently $\overline{Q} = Q$.

All we have left to do is conjugate the original representation by Q. We define:

$$\tilde{\rho}(x) = Q\rho_\alpha(x)Q^{-1}$$
$$\tilde{\rho}(y) = Q\rho_\alpha(y)Q^{-1}$$

$$\tilde{\rho}(x) = \begin{pmatrix} 1 & 1 & 0 \\ \alpha - 3 & \alpha - 2 & 1 \\ 2\alpha + 1 & 2\alpha + 1 & 1 \end{pmatrix}, \quad \tilde{\rho}(y) = \begin{pmatrix} \alpha + 1 & \alpha & 0 \\ -(\alpha + 4) & -(\alpha + 2) & 1 \\ \alpha - 3 & \alpha - 3 & 1 \end{pmatrix}$$

We replace α with π. We saw earlier that $\pi = \alpha - 3$ and $\pi^2 = \pm 7$. This results in a representation of the form $\mathscr{T}_3(R_0, \pi)$ as we wanted:

$$\tilde{\rho}(x) = \begin{pmatrix} 1 & 1 & 0 \\ \pi & \pi + 1 & 1 \\ (2\pi + 7) & (2\pi + 7) & 1 \end{pmatrix}, \quad \tilde{\rho}(y) = \begin{pmatrix} \pi + 4 & \pi + 3 & 0 \\ -(\pi + 7) & -(\pi + 5) & 1 \\ \pi & \pi & 1 \end{pmatrix}$$

Therefore there exists an injective ring homomorphism $i : \mathscr{C}_3(R, \theta_{[3]}, 1) \to \mathscr{T}_3(R_0, \pi)$. In order to verify Rosen's Theorem we prove Stage 2 below using the argument from Proposition 3.3.8 with $q = 3$.

Proposition 7.1.5 (Stage 2)**.** *The injective ring homomorphism defined by* $i : \mathscr{C}_3(R, \theta_{[3]}, 1) \to \mathscr{T}_3(R_0, \pi)$ *is an isomorphism.*

We have established $\mathscr{C}_3(R, \theta_{[3]}, 1) \cong \mathscr{T}_3(R_0, \pi)$. We want to see what ideals each of the three rows in $\mathscr{T}_3(R_0, \pi)$ represent. In Rosen's Theorem he claims for $\mathbb{Z}[G(21)]$, where $p = 7$,$q = 3$ that $\mathscr{C}_3(R, \theta_{[3]}, 1) = R \oplus P \oplus P^2$, where $P^e = (\zeta - 1)^e\mathbb{Z}[\zeta]$ and $e = 0, 1, 2$. In Section 4.3 we show this result formally for $\mathbb{Z}[G(pq)]$. Here we shall calculate Rosen's result from first principles. Thus at this stage it is not clear why only the three ideals R, P, P^2 could exist. Nonetheless it is evident that any ideal contained in $\mathscr{C}_3(R, \theta_{[3]}, 1)$ is of the form P^e, where e is a positive integer. Using

Rosen's result we show which of the three modules R, P, P^2 each row represents, followed by an explicit calculation verifying that R, P, P^2 are the only three possible non-isomorphic distinct modules that exist in $\mathscr{C}_3(R, \theta_{[3]}, 1)$.

Take the representation of $\tilde{\rho} : \mathbb{Z}[G(21)] \to \mathscr{T}_3(R_0, \pi)$ as calculated earlier. Observe that $\mathscr{T}_3(R_0, \pi)$ decomposes as a direct sum of right ideals:

$$\mathscr{T}_3(R_0, \pi) = \mathfrak{R}(1) \oplus \mathfrak{R}(2) \oplus \mathfrak{R}(3)$$

where:

$$\mathfrak{R}(1) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, a_i \in R_0 \right\}$$

$$\mathfrak{R}(2) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \pi b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{pmatrix}, b_i \in R_0 \right\}$$

$$\mathfrak{R}(3) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \pi c_1 & \pi c_2 & c_3 \end{pmatrix}, c_i \in R_0 \right\}$$

Let:

$$T = \begin{pmatrix} 0 & 0 & 1 \\ \pi & 0 & 0 \\ 0 & \pi & 0 \end{pmatrix} \in \mathscr{T}_3(R_0, \pi)$$

We notice that $T(\mathfrak{R}(1)) \subset \mathfrak{R}(2)$, $T(\mathfrak{R}(2)) \subset \mathfrak{R}(3)$, $T(\mathfrak{R}(3)) \subset \mathfrak{R}(1)$. Furthermore $\det(T) = \pi^2 = 7^2$, and each inclusion here changes by a factor of $P = (\zeta - 1)\mathbb{Z}[\zeta]$. We also have $\det(P) = 7$, resulting in a chain of inclusions each with index 7^2:

$$\mathfrak{R}(1) \subset \mathfrak{R}(2) \subset \mathfrak{R}(3) \subset \mathfrak{R}(1) \subset \mathfrak{R}(2) \subset \mathfrak{R}(3) \ldots$$

For the present assume that R, P, P^2 repeat as a chain of inclusions. In other words we will take $R \cong P^{3e}$, $P \cong P^{3e+1}$, $P^2 \cong P^{3e+2}$, where e is a positive integer. This result is proved at the end of this subsection, and is quite cumbersome so it is assumed for now. With this assumption it is easy to see that there is another chain of inclusions each with index 7^2:

$$R \subset P \subset P^2 \subset R \subset P \subset P^2 \ldots$$

With these inclusions in place we just need to find which one of $\mathfrak{R}(1), \mathfrak{R}(2), \mathfrak{R}(3)$ the module R represents. The other two modules will then just follow the chain as above. As a left module R has underlying $\mathbb{Z}[C_7]$-module structure of the ring

$R = \mathbb{Z}[\zeta]$ where ζ is a primitive seventh root of unity. The left action is the effected by:

$$x \cdot \zeta^a = \zeta^{a+1} \quad ; \quad y \cdot \zeta^a = \zeta^{2a}$$

We want to consider right ideals. So by using the usual device to convert left modules to right modules:

$$\mathbf{v} \cdot g = g^{-1} \cdot \mathbf{v}$$

Understanding this for the module R, the action of x, y is as follows:

$$\zeta^a \cdot x = \zeta^{a-1} \quad ; \quad \zeta^a \cdot y = \zeta^{4a}$$

We take as a $\mathbb{Z}$-basis for R the set $\phi_r = \zeta^{6-r+1}(1 \leq r \leq 6)$. Then the action of x, y is given by:

$$\begin{array}{lcl}
\phi_1 \cdot x = \phi_2 & ; & \phi_1 \cdot y = \phi_4 \\
\phi_2 \cdot x = \phi_3 & ; & \phi_2 \cdot y = \phi_1 \\
\phi_3 \cdot x = \phi_4 & ; & \phi_3 \cdot y = \phi_5 \\
\phi_4 \cdot x = \phi_5 & ; & \phi_4 \cdot y = \phi_2 \\
\phi_5 \cdot x = \phi_6 & ; & \phi_5 \cdot y = \phi_6 \\
\phi_6 \cdot x = -\Sigma_{r=1}^{6} \phi_r & ; & \phi_6 \cdot y = \phi_3
\end{array}$$

Proposition 7.1.6. $R \cong \mathfrak{R}(3)$

Proof. We make a change of units to our $\tilde{\rho}$ for calculation purposes. Let $\varpi = 2\alpha + 1$. It is straightforward to check that ϖ is also a unique prime in R_0 over 7. Furthermore observe that $\pi = (\alpha + 1)(2\alpha + 1) = (\alpha + 1)\varpi$. As $(\alpha + 1)$ is a unit, we have a mapping $\tilde{\varrho}$ that is structurally the same as $\tilde{\rho}$,

$$\tilde{\varrho} : \mathbb{Z}[G(21)] \to \mathscr{T}_3(R_0, \varpi)$$

So all we need to do is change the matrices from $\tilde{\rho}(x)$ and $\tilde{\rho}(y)$ into a convenient form of $\tilde{\varrho}$. The advantage of this change is that $\varpi^2 = -7$ explicitly. Thus, using these identities we attain:

$$\tilde{\varrho}(x) = \begin{pmatrix} 1 & 1 & 0 \\ (\alpha+1)\varpi & \alpha - 2 & 1 \\ \varpi & \varpi & 1 \end{pmatrix}, \quad \tilde{\varrho}(y) = \begin{pmatrix} \alpha + 1 & \alpha & 0 \\ \alpha\varpi & -(\alpha+2) & 1 \\ (\alpha+1)\varpi & (\alpha+1)\varpi & 1 \end{pmatrix}$$

Define the elements $\Phi_r (1 \leq r \leq 6)$ in $\mathfrak{R}(3)$ as follows:

$$\Phi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varpi & \varpi & -\alpha \end{pmatrix}; \qquad \Phi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2\varpi & -\varpi & \alpha+1 \end{pmatrix}$$

$$\Phi_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\varpi & \varpi & -\alpha \end{pmatrix}; \qquad \Phi_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\varpi & 0 & \alpha+1 \end{pmatrix}$$

$$\Phi_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha\varpi & \alpha\varpi & \alpha+1 \end{pmatrix}; \quad \Phi_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\varpi^2 & (\alpha+1)\varpi & -3 \end{pmatrix}$$

It is straightforward to see that $\{\Phi_1, \ldots, \Phi_6\}$ spans a $\mathbb{Z}[G(21)]$ submodule of $\mathfrak{R}(3)$ isomorphic to R:

$$\begin{array}{lll}
\Phi_1 \cdot \tilde{\varrho}(x) = \Phi_2 & ; & \Phi_1 \cdot \tilde{\varrho}(y) = \Phi_4 \\
\Phi_2 \cdot \tilde{\varrho}(x) = \Phi_3 & ; & \Phi_2 \cdot \tilde{\varrho}(y) = \Phi_1 \\
\Phi_3 \cdot \tilde{\varrho}(x) = \Phi_4 & ; & \Phi_3 \cdot \tilde{\varrho}(y) = \Phi_5 \\
\Phi_4 \cdot \tilde{\varrho}(x) = \Phi_5 & ; & \Phi_4 \cdot \tilde{\varrho}(y) = \Phi_2 \\
\Phi_5 \cdot \tilde{\varrho}(x) = \Phi_6 & ; & \Phi_5 \cdot \tilde{\varrho}(y) = \Phi_6 \\
\Phi_6 \cdot \tilde{\varrho}(x) = -\Sigma_{r=1}^{6} \Phi_r & ; & \Phi_6 \cdot \tilde{\varrho}(y) = \Phi_3
\end{array}$$

To show that $\mathfrak{R}(3) \cong R$ it suffices to show that $span\{\Phi_1, \ldots \Phi_6\} = \mathfrak{R}(3)$. We consider the following as a $\mathbb{Z}$-basis for $\mathfrak{R}(3)$:

$$\varphi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varpi & 0 & 0 \end{pmatrix}; \quad \varphi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha\varpi & 0 & 0 \end{pmatrix}; \quad \varphi_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \varpi & 0 \end{pmatrix}$$

$$\varphi_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha\varpi & 0 \end{pmatrix}; \quad \varphi_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \varphi_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

Let $A = (A_{ij})_{1 \leq j, i \leq 6}$ be the matrix $\Phi_i = \Sigma_j \varphi_j A_{ij}$. Then:

$$A = \begin{pmatrix} 1 & -2 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -3 \\ -1 & 1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

Clearly $\det(A) = -1$. Hence $span\{\Phi_1, \ldots, \Phi_6\} = \mathfrak{R}(3)$ and so $\mathfrak{R}(3) \cong R$. $\square$

It is now possible to obtain the other two module representations.

Corollary 7.1.7. *We take* $R \cong \mathfrak{R}(3)$. *Hence:*

- $\mathfrak{R}(1) \cong P$
- $\mathfrak{R}(2) \cong P^2$

Proof. This is shown simply by using the chains of inclusion shown above. We know both of the following chains have index 7^2:

$$\begin{aligned}&\mathfrak{R}(1) \subset \mathfrak{R}(2) \subset \mathfrak{R}(3) \subset \mathfrak{R}(1) \subset \mathfrak{R}(2) \subset \mathfrak{R}(3) \dots\\ &R \subset P \subset P^2 \subset R \subset P \subset P^2 \dots\end{aligned}$$

All that remains is to match up the two chains. As $R \cong \mathfrak{R}(3)$, and $\mathfrak{R}(3)$ is contained in $\mathfrak{R}(1)$ with index 7^2, we conclude that $\mathfrak{R}(1) \cong P$ and in the same manner we obtain $\mathfrak{R}(2) \cong P^2$. □

This shows that $\mathscr{T}_3(R_0, \pi)$ can be written as $R \oplus P \oplus P^2$. All that is left to show here is that the only options for the three non-isomorphic indecomposable modules in $\mathscr{T}_3(R_0, \pi)$ are in fact R, P, P^2. In the process, we will also obtain the duality relations between R, P, P^2.

We start by looking at $\mathbb{Z}[C_7]$. Over $\mathbb{Z}[C_7]$, the dual augmentation ideal $\mathcal{I}^* = R$. Inside the ring R there consists ideals of the form $P^e = (\zeta - 1)^e R$ as established earlier. In fact, all modules of the form $P^e \cong R$ over $\mathbb{Z}[C_7]$. Using the cyclic algebra we are able look at modules over $\mathbb{Z}[G(21)]$ via the twisting relation $y\zeta = \zeta^2 y$. We consider R over $\mathbb{Z}[G(21)]$. Unlike over $\mathbb{Z}[C_7]$, it is not obvious how many distinct non-isomorphic P^e modules exist over $\mathbb{Z}[G(21)]$.

Again, we define $R = \mathbb{Z}[\zeta]/c(\zeta)$ where ζ is the image of x under $\mathbb{Z}[x] \to R$. This means that $\zeta^6 = -(1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5)$. So we can note that the generator x of C_7 in each P^e as a right module acts the same way as ζ.

$$\lambda \cdot x = \lambda \cdot \zeta$$

In fact x can be represented in the following manner $\rho : G \to GL_6(\mathbb{Z})$:

$$\rho(x) = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Extend from C_7 to $G(21)$ by allowing y to act as $y \cdot \zeta = \alpha(\zeta)$ where :

$$\alpha : C_7 \to C_7$$
$$\alpha(x) = x^2$$

So $R, P, P^2, \ldots$ all become modules over $\mathbb{Z}[G(21)]$, and each P^e can be represented by taking $\rho_e : G \to GL_6(\mathbb{Z})$. It can be seen that $\rho_e(x) = \rho(x)$ for all $e \geqslant 1$. However the $\rho_e(y)$ vary. The question of how many distinct non-isomorphic $P^e s$ in $G(21)$ can now be shown. Firstly:

Proposition 7.1.8. $R \cong_\Lambda P^6$

Proof. This is a question of units.

$$\begin{aligned}(\zeta - 1)^6 &= \zeta^6 - 6\zeta^5 + 15\zeta^4 - 20\zeta^3 + 15\zeta^2 - 6\zeta + 1 \\ &= -7(\zeta^5 - 2\zeta^4 + 3\zeta^3 - 2\zeta^2 + \zeta) \\ &= -7u\end{aligned}$$

$u = \zeta^5 - 2\zeta^4 + 3\zeta^3 - 2\zeta^2 + \zeta$. The claim is that u is a unit in R. Taking the determinants of both sides.

$$\begin{aligned}\det(RHS) &= 7^6 \det(u) \\ \det(LHS) &= \det(\zeta - 1)^6 \\ &= \det(\rho(x) - 1)^6\end{aligned}$$

From the representation $\rho(x)$ we see that $\det(\rho(x) - 1) = \pm 7$. This implies that $\det(u) = \pm 1$. Hence:

$$R \to P^6$$
$$\lambda \mapsto -7\lambda u$$

is an isomorphism. $\square$

Thus, as a priori, we only need to consider R, P, P^2, P^3, P^4, P^5. By looking at the duality of each module, we shall establish which of these modules are distinct.

Proposition 7.1.9. *The following duality relations hold:*

i) $R \cong P^*$

ii) $P^2 \cong (P^2)^*$

iii) $P^3 \cong P^*$

iv) $P^4 \cong R^*$

v) $P^5 \cong (P^2)^*$

Proof. For an isomorphism $A : R \to P^*$ the following relations need to hold:

$$\begin{aligned} & A\rho_0(g) = \rho_1^*(g)A \\ i.e. \quad & A\rho_0(g) = \rho_1(g^{-1})^T A \end{aligned}$$

This will need to be checked for the two generators x, y. Hence an invertible matrix $A \in GL_6(\mathbb{Z})$ is needed, for which the following conditions hold:

(I) $A\rho(x) = \rho(x^{-1})^T A$

(II) $A\rho_0(y) = \rho_1(y^{-1})^T A$

(III) $det(A) = \pm 1$

Throughout each of the proofs, the representation for x will be the same:

$$\rho(x) = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As a result of $\rho(x)$ being the same throughout, a general matrix A satisfying ($\mathbf{I}$) looks like:

$$A = \begin{pmatrix} a & b & c & d & e & f \\ \Sigma & a & b & c & d & e \\ f & \Sigma & a & b & c & d \\ e & f & \Sigma & a & b & c \\ d & e & f & \Sigma & a & b \\ c & d & e & f & \Sigma & a \end{pmatrix}$$

where $\Sigma = -(a+b+c+d+e+f)$. In order to obtain A, the representations for y in each P^e are needed. They are described by $\rho_e(y)$:

$$\rho_0(y) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \qquad \rho_1(y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$

$$\rho_2(y) = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 & 2 & 0 \\ 0 & 2 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix} \qquad \rho_3(y) = \begin{pmatrix} 1 & 0 & -2 & 2 & 0 & -1 \\ 3 & 0 & -3 & 2 & 1 & -1 \\ 3 & 1 & -3 & 0 & 3 & -1 \\ 1 & 3 & -3 & -1 & 3 & 0 \\ 0 & 3 & -2 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$\rho_4(y) = \begin{pmatrix} 1 & -1 & -2 & 3 & 0 & -3 \\ 4 & -1 & -5 & 5 & 1 & -4 \\ 6 & 0 & -6 & 3 & 4 & -4 \\ 4 & 3 & -6 & 0 & 6 & -3 \\ 1 & 5 & -5 & -1 & 4 & 0 \\ 0 & 3 & -2 & -1 & 1 & 2 \end{pmatrix} \quad \rho_5(y) = \begin{pmatrix} 1 & -4 & 0 & 4 & -1 & -5 \\ 5 & -5 & -5 & 9 & 0 & -9 \\ 10 & -4 & -9 & 9 & 4 & -10 \\ 10 & 0 & -10 & 4 & 9 & -9 \\ 5 & 5 & -9 & 0 & 9 & -5 \\ 1 & 5 & -5 & -1 & 4 & 0 \end{pmatrix}$$

For each P^e it can be checked that $\rho(x^7) = \rho_e(y^3) = 1$. Also $\rho_e(y)\rho(x) = \rho(x^2)\rho_e(y)$. For each duality relation it is now possible to obtain A subject to conditions (**II**) and (**III**):

i) $R \cong P^*$

$$A\rho_0(y) = \rho_1(y^{-1})^T A \qquad A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

ii) $P^2 \cong (P^2)^*$

$$A\rho_2(y) = \rho_2(y^{-1})^T A \qquad A = \begin{pmatrix} 0 & 0 & -1 & 2 & -2 & 1 \\ 0 & 0 & 0 & -1 & 2 & -2 \\ 1 & 0 & 0 & 0 & -1 & 2 \\ -2 & 1 & 0 & 0 & 0 & -1 \\ 2 & -2 & 1 & 0 & 0 & 0 \\ -1 & 2 & -2 & 1 & 0 & 0 \end{pmatrix}$$

iii) $P^3 \cong P^*$

$$A\rho_3(y) = \rho_1(y^{-1})^T A \qquad A = \begin{pmatrix} 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ -1 & 0 & 0 & 0 & 1 & -2 \\ 2 & -1 & 0 & 0 & 0 & 1 \\ -2 & 2 & -1 & 0 & 0 & 0 \end{pmatrix}$$

iv) $P^4 \cong R^*$

$$A\rho_4(y) = \rho_0(y^{-1})^T A \qquad A = \begin{pmatrix} 1 & -2 & 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ -1 & 0 & 0 & 0 & 1 & -2 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

v) $P^5 \cong (P^2)^*$

$$A\rho_5(y) = \rho_2(y^{-1})^T A \qquad A = \begin{pmatrix} 3 & -1 & 0 & 1 & -3 & 5 \\ -5 & 3 & -1 & 0 & 1 & -3 \\ 5 & -5 & 3 & -1 & 0 & 1 \\ -3 & 5 & -5 & 3 & -1 & 0 \\ 1 & -3 & 5 & -5 & 3 & -1 \\ 0 & 1 & -3 & 5 & -5 & 3 \end{pmatrix}$$

$\square$

Corollary 7.1.10. *The duality relations give rise to the following:*

- $R \cong P^3$
- $P \cong P^4$
- $P^2 \cong P^5$

Proof. These results drop out simply by manipulating the duality relation results. $\square$

Therefore it is clear that there are only three distinct non-isomorphic P^e modules contained over $\mathbb{Z}[G(21)]$, namely R, P, P^2. Also, the duality relations show that P^* is the dual of R, and that P^2 is self-dual.

The 'basic' indecomposable modules

In order to obtain a complete description of all the indecomposable modules in $\mathbb{Z}[G(21)]$, we must first describe the basic indecomposable modules in $\mathbb{Z}[G(21)]$. We described most of the basic indecomposable modules earlier. We look at the integral commutative square over $\mathbb{Z}[G(21)]$:

$$\begin{array}{ccc} \mathbb{Z}[G(21)] & \longrightarrow & \mathscr{C}_3(R, \theta_{[q]}, 1) \\ \downarrow & & \downarrow \\ \mathbb{Z}[C_3] & \longrightarrow & \mathbb{F}_7[C_3] \end{array}$$

From this square arise basic indecomposable modules over $\mathbb{Z}[C_3]$ and $\mathscr{C}_3(R, \theta_{[3]}, 1)$. There are three indecomposable modules over $\mathbb{Z}[C_3]$:

i) The trivial module: $\mathbb{Z}$

ii) The augmentation ideal, namely $\mathcal{J} = \mathrm{Ker}(\mathbb{Z}[C_3] \to \mathbb{Z})$

iii) The group ring itself $\mathbb{Z}[C_3]$.(This may be obtained by the non-split extension $0 \to \mathcal{J} \to \mathbb{Z}[C_3] \to \mathbb{Z} \to 0$.)

The non-isomorphic distinct modules over $\mathscr{C}_3(R, \theta, 1)$ were established earlier, and are:

iv) $R = P^0 = \mathbb{Z}[\zeta] = \mathbb{Z}[x]/c(x)$ where $c(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

v) $P = P^1 = (\zeta - 1)R$

vi) $P^2 = (\zeta - 1)^2 R$

All the remaining indecomposable modules can be obtained using these six indecomposable modules.

Cohomological relations

Here we state the cohomological relations between the two types of basic indecomposable modules. The proofs, and reasoning for these results can be found in Section 4.4.

- $\mathrm{Ext}^1_\Lambda(\mathbb{Z}, P^e) \cong \begin{cases} \mathbb{Z}/7 & e=1 \\ 0 & e=0,2 \end{cases}$
- $\mathrm{Ext}^1_\Lambda(\mathcal{J}, P^e) \cong \begin{cases} 0 & e=1 \\ \mathbb{Z}/7 & e=0,2 \end{cases}$
- $\mathrm{Ext}^1_\Lambda(\mathbb{Z}[C_q], P^e) \cong \mathbb{Z}/7$ for all $e \geq 0$.

The duality arguments are as follows:

- $\mathrm{Ext}^1_\Lambda(P^e, \mathbb{Z}) \cong \begin{cases} 0 & e=1 \\ \mathbb{Z}/7 & e=0,2 \end{cases}$
- $\mathrm{Ext}^1_\Lambda(P^e, \mathcal{J}) \cong \begin{cases} \mathbb{Z}/7 & e=1 \\ 0 & e=0,2 \end{cases}$
- $\mathrm{Ext}^1_\Lambda(P^e, \mathbb{Z}[C_q]) \cong \mathbb{Z}/7$ for all $e \geq 0$

A complete list of the indecomposable modules over $G(21)$

There are a total of 21 indecomposable modules over $\mathbb{Z}[G(21)]$. From Proposition 4.4.8 there exist a total of 17 distinct non-isomorphic genera of indecomposable modules. The remaining four indecomposable modules (Swan like modules) can be constructed from unit considerations of certain genera of indecomposable modules. They arise from elements of the projective class group $\widetilde{K}_0(\mathbb{Z}[G(21)]$. We proceed by reviewing the basic modules again:

I. There are three distinct modules over $\mathbb{Z}[C_3]$:

- **i)** The trivial module: $\mathbb{Z}$. $(\mathrm{rank} = 1)$
- **ii)** The augmentation ideal, namely $\mathcal{J} = \mathrm{Ker}(\mathbb{Z}[C_3] \to \mathbb{Z})$. $(\mathrm{rank} = 2)$
- **iii)** The group ring itself $\mathbb{Z}[C_3]$. $(\mathrm{rank} = 3)$

These are modules over $\mathbb{Z}[G(21)]$ via the quotient map $G \to C_3$

II. There are three distinct non-isomorphic modules over $\mathscr{C}_3(R, \theta, 1)$. They are all of $\mathrm{rank} = 6$.

iv) $R = P^0 = \mathbb{Z}[\zeta] = \mathbb{Z}[x]/c(x)$ where $c(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

v) $P = P^1 = (\zeta - 1)R$.

vi) $P^2 = (\zeta - 1)^2 R$.

These are all $\mathbb{Z}[G(21)]$ modules via the twisting relation $y\zeta = \zeta^2 y$.

The non-split extensions can now be described in the form $0 \to \mathcal{A} \to ? \to \mathcal{B} \to 0$ where $\mathcal{A}$ is a direct sum of modules of a combination of R, P, P^2, but none of these modules can be used more than once in any extension. If one does use more than one of these modules in any extension the extension will split up into smaller bits, and so is not a non-split extension. $\mathcal{B} = \mathbb{Z}, \mathcal{J}$ or $\mathbb{Z}[C_3]$, where only one of the three can be used in any extension. See proposition 4.4.9.

III. There is only one extension for when $\mathcal{B} = \mathbb{Z}$. This is a result of the cohomological properties, where when $M = R, P^2$ the extension splits.

vii) $0 \to P \to \overline{\mathbb{Z}[C_7]} \to \mathbb{Z} \to 0$

where $\overline{\mathbb{Z}[C_7]}$ is $\mathbb{Z}[C_7]$ acting by conjugation.

IV. $\mathrm{rank} = 8$

viii) $0 \to R \to V_1 \to \mathcal{J} \to 0$

ix) $0 \to P \to V_2 \to \mathcal{J} \to 0$

V. $\mathrm{rank} = 9$

x) $0 \to R \to Y_1 \to \mathbb{Z}[C_3] \to 0$

xi) $0 \to P \to Y_2 \to \mathbb{Z}[C_3] \to 0$

xii) $0 \to P^2 \to Y_3 \to \mathbb{Z}[C_3] \to 0$

VI. $\mathrm{rank} = 14$

xiii) $0 \to R \oplus P \to X \to \mathcal{J} \to 0$

VII. $\mathrm{rank} = 15$

xiv) $0 \to R \oplus P \to \mathcal{Q}(P^2) \to \mathbb{Z}[C_3] \to 0$

xv) $0 \to R \oplus P^2 \to \mathcal{Q}(P) \to \mathbb{Z}[C_3] \to 0$

xvi) $0 \to P \oplus P^2 \to \mathcal{Q}(R) \to \mathbb{Z}[C_3] \to 0$

VIII. rank $= 21$

xvii) $0 \to R \oplus P \oplus P^2 \to \mathbb{Z}[G(21)] \to \mathbb{Z}[C_3] \to 0$. This module was seen earlier, and is the non-split extension that gives the group ring.

On completion of the 17 genera of indecomposable modules, there are still four more modules that need to be considered. Looking at the extension for the module of rank 7, it can be seen there are in fact more non-isomorphic indecomposable modules of rank 7:

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, "id"] & \varprojlim(\pi,\alpha) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & P \arrow[r] & \overline{\mathbb{Z}[C_7]} \arrow[r, "\pi"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

As a result of $\mathrm{Ext}^1(\mathbb{Z}, P) \cong \mathbb{Z}/7$, there seems to be six extensions corresponding to $(\mathbb{Z}/7)^*$. However as a result of conjugation, there are in fact three indecomposable modules of this form:

$$\mathcal{Q}_0 = \overline{\mathbb{Z}[C_7]} = \varprojlim(\pi, 1) \cong \varprojlim(\pi, 6)$$

$$\mathcal{Q}_1 = \varprojlim(\pi, 2) \cong \varprojlim(\pi, 5)$$

$$\mathcal{Q}_2 = \varprojlim(\pi, 3) \cong \varprojlim(\pi, 4)$$

The remaining indecomposable modules are obtained by calculating $\widetilde{K}_0(\mathbb{Z}[G(21)])$. From the results of Cassou-Nogues([4], p. 265-266), we have the following sequence for this group:

$$0 \to D_0(\mathbb{Z}[G(21)]) \to D(\mathbb{Z}[G(21)]) \to D(\mathbb{Z}[C_3]) \to 0$$

where $D(\mathbb{Z}[C_3])$ is a well-defined subgroup of $K_0(\mathbb{Z}[C_3])$. But $\widetilde{K}_0(\mathbb{Z}[C_3]) = 0$. Next as a result of $D(\mathbb{Z}[C_7]) = 0$, $D(\mathbb{Z}[G(21)]) \cong C_3$. So from the sequence we are left with $D_0(\mathbb{Z}[G(21)]) \cong D(\mathbb{Z}[G(21)]) \cong C_3$. Thus $Cl(\mathbb{Z}[G(21)]) \cong C_3$, implying there are three distinct stable equivalence classes of projective modules. This gives three indecomposable modules of rank 21. $\widetilde{K}_0(\mathbb{Z}[G(21)]) \cong C_3$ can also be calculated directly from Theorem 4.4.13.

In fact the three modules of rank 21 can be obtained by tensoring the three modules of rank 7 by $\mathbb{Z}[C_3]$. So the three projective modules are, the group ring itself $\overline{\mathbb{Z}[C_7]} \otimes \mathbb{Z}[C_3] = \mathbb{Z}[G(21)]$, and the two remaining projective modules, $\mathcal{Q}_1 \otimes \mathbb{Z}[C_3]$ and $\mathcal{Q}_2 \otimes \mathbb{Z}[C_3]$, which are all of rank 21. Pu explains in the case of $\mathbb{Z}[G(21)]$ that any other potential indecomposable modules arising from the non-trivial elements of $\widetilde{K}_0(\mathbb{Z}[G(21)])$ are not distinct and are actually isomorphic to indecomposable modules from the trivial-element of $\widetilde{K}_0(\mathbb{Z}[G(21)])$ using unit arguments. Thus, there are 21 distinct non-isomorphic indecomposable modules for $\mathbb{Z}[G(21)]$.

7.2 Syzygy decompositions and a free resolution for $G(21)$

In this section, we construct a minimal free resolution for $G(21)$. This is achieved by obtaining all the syzygies in terms of indecomposable elements. From here it will be easier to obtain the maps for the free resolution. We commence by describing the free resolution for $G(21)$:

$$\begin{tikzcd}
 & & & & & & & \mathcal{L}^* \arrow[dr] & & & & \mathcal{L} \arrow[dr] & & & & & & & \\
0 \arrow[rr] & & \mathbb{Z} \arrow[rr, "\epsilon^*"] & & \Lambda \arrow[rr, "\partial_5"] \arrow[dr] & & \Lambda^2 \arrow[rr, "\partial_4"] \arrow[ur] & & \Lambda^2 \arrow[rr, "\partial_3"] \arrow[dr] & & \Lambda^2 \arrow[rr, "\partial_2"] \arrow[ur] & & \Lambda^2 \arrow[rr, "\partial_1"] \arrow[dr] & & \Lambda \arrow[rr, "\epsilon"] & & \mathbb{Z} \arrow[rr] & & 0 \\
 & & & & & \mathcal{I}_{\mathcal{G}}^* \arrow[ur] & & & & \mathcal{K} \arrow[ur] & & & & \mathcal{I}_{\mathcal{G}} \arrow[ur] & & & & &
\end{tikzcd}$$

By describing $\mathcal{I}_{\mathcal{G}}$ in terms of indecomposable modules, we will be able to establish the minimal representatives for the other syzygies by using certain numerical syzygy ideas, and the elimination of modules.

Theorem 7.2.1. *At the minimal level, the first syzygy $\Omega_1(\mathbb{Z})$ can be described as $\mathcal{I}_{\mathcal{G}} = P \oplus X$.*

Proof. Take the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \oplus P \oplus P^2 \arrow[r] \arrow[d, "id"] & \mathcal{I}_{\mathcal{G}} \arrow[r] \arrow[d] & \mathcal{J} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \oplus P \oplus P^2 \arrow[r] & \mathbb{Z}[G(21)] \arrow[r] \arrow[dr, "\epsilon"] & \mathbb{Z}[C_3] \arrow[r] \arrow[d, "\eta"] & 0 \\
 & & & \mathbb{Z} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

We established that $\mathrm{Ext}^1_\Lambda(\mathcal{J}, P) = 0$, so this term can split off from the extension. This means that the quotient $\mathcal{I}_{\mathcal{G}}/P$, is of dimension 14, and is contained in $\mathcal{I}_{\mathcal{G}}$. As X is the only indecomposable of rank 14, we must therefore have $\mathcal{I}_{\mathcal{G}}/P = X$. Hence, from this, one can see $\mathcal{I}_{\mathcal{G}} = X \oplus P$. $\mathcal{I}_{\mathcal{G}}$ is minimal in its stability class because rank($\mathcal{I}_{\mathcal{G}}$) $= 20 < 21 =$ rank(Λ). □

Now that $\mathcal{I}_{\mathcal{G}}$ is known, it is possible to describe minimal representatives of the other syzygies. We will start by looking at $\mathcal{I}_{\mathcal{G}}^*$ and$\mathcal{K}$.

Proposition 7.2.2.

i) $\mathcal{I}_{\mathcal{G}}^* = R \oplus X$

ii) $\mathcal{K} = P^2 \oplus X$

Proof. The free resolution for $G(21)$ is periodic, and so we know that both $\mathcal{I}_{\mathcal{G}}^*$ and $\mathcal{K}$ are also of rank = 20 $<$ rank(Λ). This in turn means that they must also be minimal representatives of their respective syzygies. From the free resolution being periodic, it is possible to go a step further and see that both $\mathcal{I}_{\mathcal{G}}^*, \mathcal{K} = \langle 6 \rangle \oplus \langle 14 \rangle$, where $\langle \mathscr{N} \rangle$ are indecomposable modules of rank $\mathscr{N}$. There is only one indecomposable of rank 14, and hence is self-dual, so in both cases we can take $\langle 14 \rangle = X$.

i) $\mathcal{I}_{\mathcal{G}}^*$ can be written in this case as $\Omega_1(\mathbb{Z})^*$. Looking at it like this we can see that $\Omega_1(\mathbb{Z})^* = P^* \oplus X^*$. As X is self-dual, and $P^* \cong R$. So $\mathcal{I}_{\mathcal{G}}^* = R \oplus X$.

ii) $\mathcal{K}$ is self-dual. This is obvious from the free resolution. This in turn implies that both the indecomposable modules are self-dual. X we know is self-dual. The only rank 6 module over $G(21)$ that is self-dual is P^2. By uniqueness, this must be the module of rank 6.

Hence all the odd syzygy representatives are easily obtained. □

The even syzygy representatives take a little more manipulation to obtain.

Proposition 7.2.3. *The even syzygies are described by:*

i) $\mathcal{L} = \overline{\mathbb{Z}[C_7]} \oplus \mathcal{Q}(P^2)$

ii) $\mathcal{L}^* = \overline{\mathbb{Z}[C_7]} \oplus \mathcal{Q}(R)$

Proof. Directly from the free resolution, both $\mathcal{L}$ and $\mathcal{L}^*$ have rank 22. They also must be minimal in their stable classes otherwise they would have to split $\mathcal{L} \cong \langle 1 \rangle \oplus \Lambda$. However the only possibility for $\langle 1 \rangle$ is $\mathbb{Z}$ which would imply that G has cohomological period 2, which would mean that G is cyclic. We know that G is not cyclic, so $\mathcal{L}$ and $\mathcal{L}^*$ are both minimal in their respective syzygies.

Knowing that $\mathcal{L}$ and $\mathcal{L}^*$ are of rank 22, it is actually possible to see that they also split up into two indecomposable modules; $\mathcal{L}$, $\mathcal{L}^* = \langle 7 \rangle \oplus \langle 15 \rangle$. This observation is made clearer by noticing that the free resolution for $G(21)$ can be 'untwisted' to form two separate monogenic infinite resolutions:

$$
\begin{tikzcd}
 & & & & & \langle 15\rangle \arrow[dr] & & & & \langle 15\rangle \arrow[dr] & & & & & \\
\ldots \arrow[r] & \Lambda \arrow[r, "\partial(0)_1"] & \Lambda \arrow[rr, "\partial(5)_1"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(4)_1"] \arrow[ur] & & \Lambda \arrow[rr, "\partial(3)_1"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(2)_1"] \arrow[ur] & & \Lambda \arrow[rr, "\partial(1)_1"] \arrow[dr] & & \Lambda \arrow[r, "\partial(0)_1"] & \Lambda \arrow[r] & \ldots \\
 & & & R \arrow[ur] & & & & P^2 \arrow[ur] & & & & P \arrow[ur] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & & & \langle 7\rangle \arrow[dr] & & & & \langle 7\rangle \arrow[dr] & & & & & \\
\ldots \arrow[r] & \Lambda \arrow[r, "\partial(0)_2"] & \Lambda \arrow[rr, "\partial(5)_2"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(4)_2"] \arrow[ur] & & \Lambda \arrow[rr, "\partial(3)_2"] \arrow[dr] & & \Lambda \arrow[rr, "\partial(2)_2"] \arrow[ur] & & \Lambda \arrow[rr, "\partial(1)_2"] \arrow[dr] & & \Lambda \arrow[r, "\partial(0)_2"] & \Lambda \arrow[r] & \ldots \\
 & & & X \arrow[ur] & & & & X \arrow[ur] & & & & X \arrow[ur] & & &
\end{tikzcd}
$$

where $\partial(m) = \begin{pmatrix} \partial(m)_1 & 0 \\ 0 & \partial(m)_2 \end{pmatrix}$. So it is now possible to obtain the minimal syzygy representatives using the two resolutions then summing the terms. The indecomposable module $\langle 7\rangle$ is interpreted as the module of rank seven acting by conjugation, $\overline{\mathbb{Z}[C_7]}$. In the second of the infinite resolutions the odd syzygies are all X, implying implies that the even syzygies here must all be the same to give the repetition for each syzygy.

All that is left is to obtain are the two $\langle 15\rangle$ indecomposable modules. We look at the following sequence within the resolution, $0 \to P^2 \to \Lambda \to \langle 15\rangle \to 0$. From this it is possible to take the quotient of the sequence to get $\langle 15\rangle = \Lambda/P^2$. This can be taken as the indecomposable module from $(0 \to R \oplus P \to \mathcal{Q}(P^2) \to \mathbb{Z}[C_3] \to 0)$. Then $\mathcal{L} = \overline{\mathbb{Z}[C_7]} \oplus \mathcal{Q}(P^2)$.

To obtain $\mathcal{L}^*$, the same as above is applied except with a different short exact sequence from the resolution. $0 \to R \to \Lambda \to \langle 15\rangle \to 0$, gives $\langle 15\rangle = \Lambda/P^2$. This can be described by $\mathcal{Q}(R)$. Hence $\mathcal{L}^* = \overline{\mathbb{Z}[C_7]} \oplus \mathcal{Q}(R)$. $\square$

Now that we have obtained a complete minimal resolution for $G(21)$ in terms of syzygies broken up into indecomposable modules, it is possible to obtain the mappings for the free resolution by combining the mappings of the two 'untwisted' resolutions, giving a diagonalised resolution. This is simpler than trying to obtain mappings directly from the full resolution, which had been the original method to obtain free resolutions for $\mathbb{Z}[D_{4n+2}]$. This is a useful result when looking at the $\mathcal{R}2)$-$\mathcal{D}(2)$ problem, and also helps simplify calculations such as obtaining cohomology groups. Below is a description of each of the indecomposable modules needed in the resolution described in terms of polynomials.

$X = [y^2 - y)$

$\overline{\mathbb{Z}[C_7]} = \Sigma_y = [1 + y + y^2)$

$$R = [1 + x - x^4 - x^5 + (1 + x^2 - x^4 - x^6)y + (1 - x)y^2)$$

$$P = [-1 + x - x^2 + x^6 + (-x^3 + x^6)y + (x - x^3 - x^5 + x^6)y^2)$$

$$P^2 = [1 - x^4 + (1 - x + x^2 - x^3)y + (1 - x)y^2)$$

$$Q(P^2) = [1 + x - y^2)$$

$$Q(R) = [1 + x^4 - y)$$

Using the above descriptions for the indecomposable modules, the following is a full free diagonalised resolution for $G(21)$.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{\partial_5} \Lambda^2 \xrightarrow{\partial_4} \Lambda^2 \xrightarrow{\partial_3} \Lambda^2 \xrightarrow{\partial_2} \Lambda^2 \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where the mappings can be described as:

a) $\epsilon = \Sigma_x(1 + y + y^2)$

b) $\partial_1 = (-1 + x - x^2 + x^6 + (-x^3 + x^6)y + (x - x^3 - x^5 + x^6)y^2, \quad y^2 - y)$

c) $\partial_2 = \begin{pmatrix} 1 + x - y^2 & 0 \\ 0 & \Sigma_y \end{pmatrix}$

d) $\partial_3 = \begin{pmatrix} 1 - x^4 + (1 - x + x^2 - x^3)y + (1 - x)y^2 & 0 \\ 0 & y^2 - y \end{pmatrix}$

e) $\partial_4 = \begin{pmatrix} 1 + x^4 - y & 0 \\ 0 & \Sigma_y \end{pmatrix}$

f) $\partial_5 = \begin{pmatrix} 1 + x - x^4 - x^5 + (1 + x^2 - x^4 - x^6)y + (1 - x)y^2 \\ y^2 - y \end{pmatrix}$

g) $\epsilon^* = \Sigma_x(1 + y + y^2)$

Finally from this free resolution, the cohomology groups for the non-abelian group of order 21 can be obtained in a straightforward manner. Below are the cohomology groups using trivial coefficients:

$$H^r(\mathbb{Z}, \mathbb{Z}) = \mathrm{Ext}^r_\Lambda(\mathbb{Z}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & r = 0 \\ 0 & r = \text{odd} \\ \mathbb{Z}/3 & r = 2, 4 \bmod 6 \\ \mathbb{Z}/21 & r = 0, 6 \bmod 6, r > 0 \end{cases}$$

7.3 Verification of the $\mathcal{R}(2)$-$\mathcal{D}(2)$ Problem for $G(21)$

In this final section we prove the two stages needed to verify a positive result for the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem as described in Chapter 6. We commence by looking at the tree structure of $\Omega_3(\mathbb{Z})$ for $\mathbb{Z}[G(21)]$.

Proposition 7.3.1. *If* $\mathcal{K}$ *is a minimal representative of* $\Omega_3(\mathbb{Z})$ *then* $\mathcal{K} \cong \mathcal{K}_1 \oplus \mathcal{K}_2$ *where* $\mathrm{rank}_{\mathbb{Z}}(\mathcal{K}_1) = 6$*, and* $\mathrm{rank}_{\mathbb{Z}}(\mathcal{K}_2) = 14$

Proof. From Proposition 5.1.3 we know that at the minimal level $\Omega_3(\mathbb{Z})$ must decompose into a maximum of two indecomposable modules.

There are no indecomposable modules of $\mathrm{rank}_{\mathbb{Z}} = 20$. So $\mathcal{K}$ must be of the form $\mathcal{K}_1 \oplus \mathcal{K}_2$. When looking at the full list of indecomposable modules in $\mathbb{Z}[G(21)]$, the only way to obtain $\mathrm{rank}_{\mathbb{Z}}(\mathcal{K}) = 20$ by using only two indecomposable modules is:

$$\mathcal{K} \cong \langle 6 \rangle \oplus \langle 14 \rangle$$

□

This establishes that there exists a unique decomposition structure at the minimal level of $\Omega_3(\mathbb{Z})$. We show that there is only one way to obtain this structure in terms of the indecomposable modules available.

Proposition 7.3.2. $\Omega_3(\mathbb{Z})$ *has no branching at the minimal level, and so* $\Omega_3(\mathbb{Z})$ *is straight.*

Proof. Let $\mathcal{K}$ be a minimal representative of $\Omega_3(\mathbb{Z})$. We know from the decomposition of syzygies that $\mathcal{K} \cong \langle 6 \rangle \oplus \langle 14 \rangle$. In fact, we can take this a step further, and take $\mathcal{K} \cong P^2 \oplus X$. From the information about the indecomposable modules, X is the unique module of rank 14, and so we only need to show that at the minimal level P^2 is the only option for the module of rank 6 in $\Omega_3(\mathbb{Z})$. Suppose that $\mathcal{K}'$ is another module, where $\mathcal{K}' \cong \langle 6 \rangle' \oplus X$ such that $\mathcal{K}'$ is stably equivalent to $\mathcal{K}$. Hence:

$$\mathcal{K} \oplus \Lambda^a \cong \mathcal{K}' \oplus \Lambda^a$$

As X is the unique indecomposable module of rank 14:

$$P^2 \oplus X \oplus \Lambda^a \cong \langle 6 \rangle' \oplus X \oplus \Lambda^a$$

Consider the following cohomological computation:

$$\mathrm{Ext}^1_\Lambda(\overline{\mathbb{Z}[C_7]}, N) \begin{cases} = 0 & \text{if } N \cong P^2 \text{ or } \langle 6 \rangle' \\ = 0 & \text{if } N \cong \Lambda \\ \neq 0 & \text{if } N \cong X \end{cases}$$

As X is a finite abelian group we can cancel it from both sides leaving:

$$P^2 \oplus \Lambda^a \cong \langle 6 \rangle' \oplus \Lambda^a$$

The three candidates for $\langle 6 \rangle'$ are R, P, P^2. Since they are all cohomologically distinct and stabilising preserves cohomological properties then the only possibility for $\langle 6 \rangle'$ is P^2 itself. As a result:

$$\mathcal{K} \cong \mathcal{K}'$$

That is, there is no branching of $\Omega_3(\mathbb{Z})$ at the minimal level as $\mathcal{K}$ is the only possible representative. Hence, by using the Swan-Jacobinski Theorem we determine that $\Omega_3(\mathbb{Z})$ is straight. □

The Swan map for $G(21)$

We have shown that there only exists a single homotopy type $\mathcal{K} \in \Omega_3(\mathbb{Z})$, so $G(21)$ is realisable by the standard presentation of the group. Finally, by realising the Swan map we verify that the module $\mathcal{K}$ is full. Thus we ensure geometric realisability for a complex involving $\mathcal{K}$ that is homotopically equivalent to a geometric 2-complex as wanted.

All the maps in the free resolution are diagonalised, so it suffices to realise the Swan map over the augmentation ideal $\mathcal{I}_\mathcal{G}$. With diagonal maps there is no complication to lift from $\mathcal{I}_\mathcal{G}$ to the minimal homotopy module $\mathcal{K}$.

The augmentation ideal decomposes as $\mathcal{I}_\mathcal{G} \cong P \oplus [y-1)$. There exists $\alpha \in \Lambda$ such that the following commutes where $\lambda_\alpha : \Lambda \to \Lambda$ is taken as $\lambda_\alpha(x) = \alpha x$. Thus:

(7.i)
$$\begin{tikzcd}
0 \arrow[r] & P \oplus [y^2-y) \arrow[r, "i"] \arrow[d, "\alpha"] & \Lambda \arrow[r] \arrow[d, "\lambda_\alpha"] & \mathbb{Z} \arrow[r] \arrow[d, "\epsilon(\alpha)"] & 0 \\
0 \arrow[r] & P \oplus [y^2-y) \arrow[r] & \varinjlim(\alpha, i) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

So we need to realise all elements in $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{I}_\mathcal{G})$. The kernel elements of $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{I}_\mathcal{G})$ are explicitly calculated using suitable α's. Earlier we showed $\widetilde{K}_0(\mathbb{Z}[G(21)]) \cong C_3$. Furthermore $(\mathbb{Z}/21)^* \cong C_2 \times C_2 \times C_3$. So the Swan map is:

$$s : C_6 \times C_2 \twoheadrightarrow C_3$$

We want to see what is contained in $\mathrm{Ker}(s)$. Straight away it is possible to see that a copy of C_2 must be $\{\pm Id\}$ using Proposition 6.3.3. We then have the following surjective map:

$$\bar{s} : (C_6 \times C_2)/\{\pm Id\} \twoheadrightarrow C_3$$

It is well-known that $C_6 \cong C_2 \times C_3$. By Proposition 6.3.4, we observe that there exists an injective homomorphism $i : C_3 \hookrightarrow \widetilde{K}_0(\mathbb{Z}[G(21)])$.

We have therefore realised $C_3 \times C_2$ in the Swan map. This leaves a maximum of two minimal algebraic complexes with the same homotopy type $\pi_2(K_{\mathcal{G}})$. Thus, in order to verify the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem all that remains is to realise the C_2 that is left. This C_2 must be an element of Ker(s), as $C_2 \not\subset \widetilde{K}_0(\mathbb{Z}[G(21)])$.

Proposition 7.3.3. $\mathrm{Ker}(s) \cong C_2 \times C_2$

Proof. To realise Ker(s), look at the commutative diagram (7.i) again:

$$\begin{tikzcd} 0 \arrow[r] & P \oplus [y^2 - y) \arrow[r, "i"] \arrow[d, "\alpha"] & \Lambda \arrow[r] \arrow[d, "\lambda_\alpha"] & \mathbb{Z} \arrow[r] \arrow[d, "\epsilon(\alpha)"] & 0 \\ 0 \arrow[r] & P \oplus [y^2 - y) \arrow[r] & \varinjlim(\alpha, i) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Take the standard presentation for $C_2 \times C_2$:

$$C_2 \times C_2 = \langle s, t \mid s^2 = t^2 = 1 \rangle$$

From $(\mathbb{Z}/21)^*$ we note that the unique option for Ker(s) in terms of $\epsilon(\alpha)$ is

$$\mathrm{Ker}(s) \cong C_2 \times C_2 = \{1, 8, 13, 20\}$$

as $8^2 = 13^2 = 20^2 = 1$ mod 21. We already know that when we take $\alpha = \pm 1$, we realise a cyclic group of order two:

$$C_2 = \{1, st\} = \{1, 20\} = \pm Id$$

As $st = 20 = 8 \times 13$ mod 21, we can associate $s = 8$, $t = 13$. Thus to show that Ker$(s) \cong C_2 \times C_2$, it suffices to realise either $\epsilon(\alpha) = 13$ or $\epsilon(\alpha) = 8$. This is sufficient as $st = 20 \in$ Ker(s) has been realised, so the only options possible are both $8, 13 \in$ Ker(s), or both $8, 13 \notin$ Ker(s).

Hence we shall only realise α (non unique) such that $\epsilon(\alpha) = 13$ (unique). α must satisfy the following conditions:

$$\alpha[\eta) = -[\eta) \qquad \text{and} \qquad \alpha[y-1) = [y-1)$$

where we take $P = [\eta) = [-1 + x - x^2 + x^6 + (-x^3 + x^6)y + (x - x^3 - x^5 + x^6)y^2)$.

So we want an α that gives:

$$\begin{tikzcd}
0 \arrow[r] & P \oplus [y^2 - y) \arrow[r, "i"] \arrow[d, "\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}"] & \Lambda \arrow[r] \arrow[d, "\lambda_\alpha"] & \mathbb{Z} \arrow[r] \arrow[d, "13"] & 0 \\
0 \arrow[r] & P \oplus [y^2 - y) \arrow[r] & \varinjlim(\alpha, i) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Let $\alpha = 1 + 2(x + x^6)\Sigma_y$ and the result follows. $\square$

The Swan map for $\mathbb{Z}[G(21)]$ is now fully realised. Therefore a positive result for the $\mathcal{R}(2)$-$\mathcal{D}(2)$ problem has been achieved for $G(21)$.

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