Rank of divisors on tropical curves

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Abstract

We investigate, using purely combinatorial methods, structural and algorithmic properties of linear equivalence classes of divisors on tropical curves. In particular, we confirm a conjecture of Baker asserting that the rank of a divisor D on a (non-metric) graph is equal to the rank of D on the corresponding metric graph, and construct an algorithm for computing the rank of a divisor on a tropical curve.

1 Introduction

Tropical geometry investigates properties of tropical varieties, objects which are commonly considered to be combinatorial counterparts of algebraic varieties. There are several survey articles on this recent branch of mathematics [14, 16, 17]. In particular, [6] concentrates on topics which are particularly close to the subject of this paper.

Tropical varieties share many important features with their algebro-geometric analogues, and allow for a variety of algebraic, combinatorial and geometric techniques to be used. We illustrate this on several important examples related to the current paper.

- In [3], a version of the Riemann-Roch theorem for graphs was proved by purely combinatorial methods. Shortly afterwards Gathmann and Kerber [7] used the result to prove Riemann-Roch theorem for tropical curves. Their contribution was a method of approximating a tropical curve by graphs.
- Mikhalkin and Zharkov [15] gave (among others) another proof of the Riemann-Roch theorem for tropical curves. Their approach used a combination of algebraic and combinatorial techniques.
- Recently, a machinery which allows one to transfer certain results from Riemann surfaces to tropical curves has been developed in [2]. For example, Baker [2] introduced a tropical version of Weierstrass points, and proved using this machinery that every tropical curve of genus more than one contains at least one such point, a fact well known in the context of algebraic curves. Note that the method necessarily has some limitations. Indeed, it is known that analogues of some theorems about Riemann surfaces do not hold in the tropical context, Pappus' Theorem being one such example ([16, §7]).

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In this paper, we contribute further towards the theory by proving new structural results on divisors on tropical curves. In particular, we confirm a conjecture of Baker [2] relating the ranks of a divisor on a graph and on a tropical curve (see Theorem 1.3), and construct an algorithm for computing the rank of a divisor on a tropical curve (see Theorem 4.1). All the proofs in the paper are purely combinatorial. In an expanded version of this article [8] we have employed these results to obtain an alternative proof of the Riemann-Roch theorem for tropical curves.

1.1 Overview and notation

Throughout the paper, a *graph* G is a finite connected multigraph that can contain loops, i.e., G is a pair consisting of a set $V(G)$ of *vertices* and a multiset $E(G)$ of *edges*, which are unordered pairs of not necessarily distinct vertices. The degree $\deg_G(v)$ of a vertex v is the number of edges incident with it (counting loops twice). The k-th *subdivision* of a graph G is the graph G^k obtained from G by replacing each edge with a path with k inner vertices.

Graphs have been considered as analogues of Riemann surfaces in several contexts, in particular, in [3, 4] in the context of linear equivalence of divisors. In this paper we further investigate the properties of linear equivalence classes of divisors. We primarily concentrate on metric graphs, but let us start the exposition by recalling the definitions and results from [3] related to (non-metric) graphs.

A *divisor* D on a graph G is an element of the free abelian group $\mathrm{Div}(G)$ on $V(G)$. We can write each element $D \in \mathrm{Div}(G)$ uniquely as

$$D = \sum_{v \in V(G)} D(v)(v)$$

with $D(v) \in \mathbb{Z}$. We say that D is *effective*, and write $D \geq 0$, if $D(v) \geq 0$ for all $v \in V(G)$. For $D \in \mathrm{Div}(G)$, we define the *degree* of D by the formula

$$\deg(D) = \sum_{v \in V(G)} D(v).$$

Analogously, we define

$$\deg^+(D) = \sum_{v \in V(G)} \max\{0, D(v)\}.$$

For a function $f : V(G) \to \mathbb{Z}$, the *divisor associated to* f is given by the formula

$$\mathrm{div}(f) = \sum_{v \in V(G)} \sum_{e = vw \in E(G)} (f(v) - f(w))\,(v).$$

Divisors associated to integer-valued functions on $V(G)$ are called *principal*. An equivalence relation $\sim$ on $\mathrm{Div}(G)$, is defined as $D \sim D'$, if and only if $D - D'$ is principal. We sometimes write $\sim_G$ instead of $\sim$ when the graph is not clearly understood from the context. For a divisor D, $|D|$ denotes the set of effective divisors equivalent to it, i.e.,

$$|D| = \{E \in \mathrm{Div}(G) \ : \ E \geq 0 \text{ and } E \sim D\}.$$

We refer to $|D|$ as the *(complete) linear system* associated to D. Sometimes, we write $|D|_G$ for $|D|$ if the underlying graph G is not clear. If $D \sim D'$, we call the divisors D and D' *equivalent* (or *linearly equivalent*).

The *rank* of a divisor D on a graph G is defined as

$$r_G(D) = \min_{\substack{E\geq 0 \\ |D-E|=\emptyset}} \deg(E) - 1 \ . \tag{1}$$

We frequently omit the subscript G in $r_G(D)$ when the graph G is clear from the context. Also note that $r(D)$ depends only on the linear equivalence class of D. In the classical case, $r(D)$ is usually referred to as the dimension of the linear system $|D|$. In our setting, however, we are not aware of any interpretation of $r(D)$ as the topological dimension of a physical space. Thus, we refer to $r(D)$ as "the rank" rather than "the dimension". See Remark 1.13 of [3] for further discussion about similarities and differences between our definition of $r(D)$ and the classical definition in the Riemann surface case.

The *canonical divisor* on G is the divisor K_G defined as

$$K_G = \sum_{v\in V(G)} (\deg(v) - 2)(v).$$

The *genus* of G is the number $g = |E(G)| - |V(G)| + 1$. In graph theory, g is called the *cyclomatic number* of G.

The following graph-theoretical analogue of the classical Riemann-Roch theorem is one of the main results of [3].

Theorem 1.1. *If D is a divisor on a loopless graph G of genus g, then*

$$r(D) - r(K_G - D) = \deg(D) + 1 - g.$$

Let us note that while the graph-theoretical results, such as Theorem 1.1, can be viewed as simply being analogous to classical results from algebraic geometry, there exist deep relations between the two contexts, e.g., a connection arising from the specialization of divisors on arithmetic surfaces is explored in [2].

Tropical geometry provides another connection between graph theory and the theory of algebraic curves. The analogue of an algebraic curve in tropical geometry is an *(abstract) tropical curve*, which following Mikhalkin [13], can be considered simply as a metric graph. A *metric graph* Γ is a graph with each edge being assigned a positive length. Each edge of a metric graph is associated with an interval of the length assigned to the edge with the end points of the interval identified with the end vertices of the edge. A special type of edges are *loops*, which are edges where the two end points coincide. The points of these intervals are referred to as *points* of Γ. The internal points of the interval are referred to as *internal* points of the edge and they form the *interior* of the edge. Subintervals of these intervals are then referred to as *segments*.

This geometric representation of Γ equips the metric graph with a topology, in particular, we can speak about open and closed sets. The distance $\mathrm{dist}_\Gamma(v, w)$ between two points v and w of Γ is measured in the metric space corresponding to the geometric representation of Γ. For an edge e of Γ and two points $x, y \in e$ we use $\mathrm{dist}_e(x, y)$ to denote the distance between x and y measured on the edge e. Note that in general, $\mathrm{dist}_\Gamma(x, y)$ can be strictly smaller than $\mathrm{dist}_e(x, y)$.

The vertices of Γ are called *branching points* and the set of branching vertices of Γ is denoted by $B(\Gamma)$. We allow branching points of degree two. As usual, we assume that the number of branching points of Γ is finite.

A *tropical curve* is a metric graph where edges incident with vertices of degree one (leaves) are allowed to have infinite length. Such edges are identified with the interval $[0, \infty]$, such that ∞ is identified with the vertex of degree one, and are called *infinite edges*. The points corresponding to ∞ are referred to as *unbounded ends*. The unbounded ends are also considered to be points of the tropical curve.

The notions of genus, divisor, degree of a divisor and canonical divisor K_Γ readily translate from graphs to metric graphs and tropical curves (with basis of the free abelian group of divisors $\mathrm{Div}(\Gamma)$ being the infinite set of all the points of Γ). In order to define linear equivalence on $\mathrm{Div}(\Gamma)$, the notion of rational function has to be adapted.

A *rational function* on a tropical curve Γ is a continuous function $f : \Gamma \to \mathbb{R} \cup \{\pm\infty\}$ which is a piecewise linear function with integral slopes on every edge. We require that the number of linear parts of a rational function on every edge is finite and the only points v with $f(v) = \pm\infty$ are unbounded ends.

The *order* $\mathrm{ord}_v(f)$ of a point v of Γ with respect to a rational function f is the sum of outgoing slopes of all the segments of Γ emanating from v. In particular, if v is not a branching point of Γ and the function f does not change its slope at v, $\mathrm{ord}_v(f) = 0$. Hence, there are only finitely many points v with $\mathrm{ord}_v(f) \neq 0$. Therefore, we can associate a divisor $\mathrm{div}(f)$ to the rational function f by setting $\mathrm{div}(f)(v) = \mathrm{ord}_v(f)$ for every point v of Γ. Observe that $\deg(\mathrm{div}(f))$ is equal to zero as each linear part of f with slope s contributes towards the sum defining $\deg(\mathrm{div}(f))$ by $+s$ and $-s$ (at its two boundary points). Note that $\mathrm{ord}_v(f)$ need not be zero for unbounded ends v.

Rational functions on tropical curves lead to a definition of principal divisors on tropical curves. In particular, we say that divisors D and D' on Γ are *equivalent* and write $D \sim D'$ if there exists a rational function f on Γ such that $D = D' + \mathrm{div}(f)$. With this notion of equivalence the linear system and the rank of a divisor on a tropical curve are defined in the same manner as for finite graphs above, in particular:

$$|D| = \{E \in \mathrm{Div}(\Gamma) \ : \ E \geq 0 \text{ and } E \sim D\},$$

$$r_\Gamma(D) = \min_{\substack{E\geq 0,\ E\in\mathrm{Div}(\Gamma)\\ |D-E|=\emptyset}} \deg(E) - 1\ .$$

We may occasionally use $|D|_\Gamma$ for $|D|$ if the underlying tropical curve Γ is not clear from the context. Gathmann and Kerber [7] and, independently, Mikhalkin and Zharkov [15] have proved the following version of the Riemann-Roch theorem for tropical curves.

Theorem 1.2. *Let D be a divisor on a tropical curve Γ of genus g. Then*

$$r(D) - r(K_\Gamma - D) = \deg(D) + 1 - g.$$

Theorem 1.2 is also proven in an expanded version of this paper [8].

We prove in Section 3 the following theorem relating the ranks of divisors on ordinary and metric graphs. Before stating the theorem we need to introduce a definition. We say that a metric graph Γ *corresponds* to the graph G if Γ is obtained from G by setting the length of each edge of G to be equal to one.

Theorem 1.3. *Let D be a divisor on a loopless graph G and let Γ be the metric graph corresponding to G. Then,*

$$r_G(D) = r_\Gamma(D).$$

The sets of effective divisors and principal divisors on Γ are both strictly larger than the respective sets for G. Hence, Theorem 1.3 is not a priori obvious.

Theorem 1.3 implies a conjecture of Baker [2] that the rank of a divisor on a loopless graph G is the same as its rank on the graph G^k, the graph where every edge of G is k times subdivided (see Corollary 3.4). Gathmann and Kerber proved in [7, Proposition 2.4] the following statement: Given a divisor D on a loopless graph G there exist infinitely many subdivisions G' of G such that $r_G(D) = r_{G'}(D)$.

Corollary 3.4 therefore strengthens quantification of [7, Proposition 2.4] by allowing all possible subdivisions as opposed to just an infinite family of subdivisions.

We finish the paper by considering algorithmic applications of the results established in Sections 2 and 3, and design an algorithm for computing the rank of divisors on tropical curves.

1.2 The Riemann-Roch criterion

In this section, we recall an abstract criterion from [3] giving necessary and sufficient conditions for the Riemann-Roch formula to hold (Theorem 1.4). By Theorem 1.2 we will be able to utilize these conditions in the context of tropical curves in Section 2. In an expanded version of this paper [8], we proceed the other way around: we show that the abstract conditions of Theorem 1.4 are met for divisors on tropical curves, thereby giving another proof of Theorem 1.2.

The setting for the results of this section is as follows. Let X be a non-empty set, and let $\mathrm{Div}(X)$ be the free abelian group on X. Elements of $\mathrm{Div}(X)$ are called *divisors* on X, divisors E with $E \geq 0$ are called *effective*. Let $\sim$ be an equivalence relation on $\mathrm{Div}(X)$ satisfying the following two properties:

(E1) If $D \sim D'$, then $\deg(D) = \deg(D')$.

(E2) If $D_1 \sim D_1'$ and $D_2 \sim D_2'$, then $D_1 + D_1' \sim D_2 + D_2'$.

As before, given $D \in \mathrm{Div}(X)$, define

$$|D| = \{E \in \mathrm{Div}(X) \ : \ E \geq 0 \text{ and } E \sim D\}$$

and

$$r(D) = \min_{\substack{E \geq 0 \\ |D-E| = \emptyset}} \deg(E) - 1 \ .$$

For a nonnegative integer g (which will correspond to the *abstract genus* of X), let us define the set of *non-special divisors*

$$\mathcal{N} = \{D \in \mathrm{Div}(X) \ : \ \deg(D) = g-1 \text{ and } |D| = \emptyset\} \ . \tag{2}$$

Some heed is needed when comparing our notion of non-special divisors to the classic notion from the theory of Riemann surfaces. Indeed, suppose that Z is a compact Riemann surface Z of genus g_Z with its canonical divisor K_Z. A divisor D on Z is called *special* whenever its rank satisfies $r_Z(K_Z - D) \geq 0$. Thus, classically, non-special divisors do not necessarily have rank of the genus decreased by one, a property which will be guaranteed by our later choice of the abstract genus g. However, when we additionally assume that $\deg_Z(D) = g_Z - 1$ then, by the Riemann-Roch Theorem for Riemann surfaces, our definition (2) is consistent with the notion of non-special divisors.

Finally, let K be an element of $\mathrm{Div}(X)$ having degree $2g-2$. The following theorem from [3] gives necessary and sufficient conditions for the Riemann-Roch formula to hold for elements of $\mathrm{Div}(X)/\sim$.

Theorem 1.4. *Define $\epsilon : \mathrm{Div}(X) \to \mathbb{Z}/2\mathbb{Z}$ by declaring that $\epsilon(D) = 0$ if $|D| \neq \emptyset$ and $\epsilon(D) = 1$ if $|D| = \emptyset$. Then the Riemann-Roch formula*

$$r(D) - r(K-D) = \deg(D) + 1 - g$$

holds for all $D \in \mathrm{Div}(X)$ if and only if the following two properties are satisfied:

(RR1) *For every $D \in \mathrm{Div}(X)$, there exists $\nu \in \mathcal{N}$ such that*

$$\epsilon(D) + \epsilon(\nu - D) = 1 \ .$$

(RR2) *For every* $D \in \mathrm{Div}(X)$ *with* $\deg(D) = g - 1$*, we have*

$$\epsilon(D) + \epsilon(K - D) = 0 \ .$$

In addition to Theorem 1.4, we will later use the following lemma from [3] that also holds in the abstract setting.

Lemma 1.5. *If (RR1) holds, then for every* $D \in \mathrm{Div}(X)$ *we have*

$$r(D) = \left(\min_{\substack{D' \sim D \\ \nu \in \mathcal{N}}} \deg^+(D' - \nu) \right) - 1 \ .$$

1.3 Reducing tropical curves to loopless metric graphs

We finish the introductory part of the paper by reducing the study of divisors on tropical curves to the corresponding situation on loopless metric graphs. Let Γ be a tropical curve, and let Γ' be the metric graph obtained from Γ by removing interiors of infinite edges and their unbounded ends. There exists a natural retraction map $\psi_\Gamma : \Gamma \to \Gamma'$ that maps deleted points of infinite edges of Γ to the ends of those edges that belong to Γ' and is an identity on the points of Γ'. This map induces a map from $\mathrm{Div}(\Gamma)$ to $\mathrm{Div}(\Gamma')$, which is denoted by ψ_Γ. The following proposition combines the results of Lemma 3.4, Remark 3.5, Lemma 3.6 and Remark 3.7 of [7].

Proposition 1.6. *Let* Γ *be a tropical curve, and let* Γ' *and* ψ_Γ *be defined as above. Let* $D \in \mathrm{Div}(\Gamma)$*, and set* $D' = \psi_\Gamma(D)$*. We have* $D \sim_\Gamma D'$, $\deg(D) = \deg(D')$*, and* $r_\Gamma(D) = r_{\Gamma'}(D')$*. In addition, it holds that* $K_{\Gamma'} = \psi_\Gamma(K_\Gamma)$*.*

It follows from Proposition 1.6 that Theorem 1.2 restricted to metric graphs implies Theorem 1.2 in full generality. It also follows that given an algorithm to compute the rank of divisors on metric graphs one can readily design an algorithm to compute rank of divisors on tropical curves. Based on these observations we concentrate our further investigations on metric graphs.

Further, we shall restrict ourselves to *loopless* metric graphs in auxiliary lemmas leading to our main results. The general case can be reduced to the loopless one by introducing a branching point of degree two on each edge. This transformation does not change the set of divisors or their properties.

1.4 Rank-determining sets

The results of this paper have been substantially extended since the first version of this manuscript was posted on the arXiv in 2007. Most importantly, Luo [11] introduced the notion of rank-determining set, and using this notion he extended Theorem 1.3. Indeed, our Theorem 1.3 asserts that the rank of a divisor on a (loopless) metric graph with edges of integral lengths can be determined on a much simpler (and finite) object, that is, on a graph. Luo's main result roughly speaking says that such a finitization is possible even for general metric graphs. To state Luo's result precisely, we need to give the definition of rank-determing sets which in turn builds on the notion of restricted rank. Let A be a non-empty set of points of a metric graph Γ. We then define the *A-restricted rank* of a divisor $D \in \mathrm{Div}(\Gamma)$ by

$$r_A(D) = \min_{\substack{E \in \mathrm{Div}(A), E \geq 0 \\ |D-E| = \emptyset}} \deg(E) - 1 \ . \qquad (3)$$

The set A is *rank-determining* if $r_\Gamma(D) = r_A(D)$ for each $D \in \mathrm{Div}(\Gamma)$.

We are now ready to state our main result concerning rank-determining sets, a result originally due to Luo [11].[1]

Theorem 1.7. *The set $B(\Gamma)$ of branching points of any loopless metric graph Γ is rank-determining.*

We give a proof of Theorem 1.7 in Section 3. See also [1] for an alternative proof of Theorem 1.7.

As we show in Section 3, Theorem 1.7 implies Theorem 1.3. While not difficult, the argument is not entirely trivial. In particular, mind that the condition $|D-E| = \emptyset$ in (1) and in (3) refer to different equivalence relations on the sets of divisors.

2 Non-special divisors, alignments, and rank-pairs

As in Subsection 1.2, we define the set of *non-special divisors* on a metric graph Γ to be

$$\mathcal{N} = \{D \in \mathrm{Div}(\Gamma) \ : \ \deg(D) = g-1 \text{ and } |D| = \emptyset\}$$

where g is the genus of Γ. The main results of this section is an alternative formula for computing the rank of a divisor on a metric graph (Corollary 2.5).

We rely on results from [15]. An alternative, self-contained approach which gives as a by-product another (purely combinatorial) proof of the Riemann-Roch theorem for tropical curves can be found in an expanded version of this paper [8].

2.1 A formula for the rank of a divisor

We present a class of non-special divisors that is of primary interest to us in our later considerations. Let P be an ordered sequence of finitely many points of Γ. We say that the set of points in P is the *support* of P and denote it by $\mathrm{supp}\, P$. The sequence P can also be viewed as a linear order $<_P$ on $\mathrm{supp}\, P$. If $B(\Gamma) \subseteq \mathrm{supp}\, P$ then P is an *alignment* of points of Γ. The set of all alignments of points of Γ is denoted by $\mathcal{P}(\Gamma)$.

We now define a divisor ν_P corresponding to an alignment P. A segment L of Γ is a *P-segment* if both ends of L belong to $\mathrm{supp}\, P$, and the interior of L is disjoint from $\mathrm{supp}\, P$. For $v \in \mathrm{supp}\, P$, let $S_P(v)$ denote the set of P-segments of Γ with one end at v and the other end preceding v in the order determined by P. Finally, let

$$\nu_P = \sum_{v \in \text{ supp } P} (|S_p(v)| - 1)(v).$$

It is easy to verify that $\deg(\nu_P) = g - 1$, where g is the genus of Γ. We start our investigation of divisors corresponding to alignments by giving two simple propositions.

Proposition 2.1. *Let P be an alignment of points of a metric graph Γ. For every point v of Γ that is not contained in* $\mathrm{supp}\, P$*, there exists an alignment P' such that* $\mathrm{supp}\, P' = \mathrm{supp}\, P \cup \{v\}$ *and $\nu_P = \nu_{P'}$.*

Proof. Such an alignment P' can be obtained by inserting the point v in the sequence P between the (distinct) boundary points of the (unique) segment containing v. $\square$

Proposition 2.2 ([15, Lemma 7.8])**.** *If P is an alignment of points of a metric graph Γ, then $\nu_P \in \mathcal{N}$.*

[1] The proof of Theorem 1.3 as given in the original version of the manuscript (version 1 on the arXiv) can actually serve as a proof of Theorem 1.7 due to Luo if phrased in the right terminology.

The next fact asserts that every divisor is either equivalent to an effective divisor, or is equivalent to a divisor dominated by ν_P for some alignment P, and not both.

Corollary 2.3 ([15, Corollary 7.9])**.** *Let Γ be a metric graph. For every $D \in \mathrm{Div}(G)$, exactly one of the following holds*

(a) $r(D) \geq 0$; or

(b) $r(\nu_P - D) \geq 0$ for some alignment P.

Corollary 2.4 ([15, Corollary 7.10])**.** *If ν is a non-special divisor on a metric graph Γ of genus g, then $\nu \sim \nu_P$ for some alignment P of a finite set of points of Γ.*

Corollary 2.3 is a consequence of Proposition 2.2. Corollary 2.4 in turn follows from Corollary 2.3 applied to non-special divisors. The arguments to derive these corollaries are rather straightforward. We refer the reader to [15, 8].

We finish this section with establishing a formula for rank of divisors on metric graphs that will be central in our later analysis of the rank.

Corollary 2.5. *If D is a divisor on a metric graph Γ, then the following formula holds:*

$$r(D) = \min_{\substack{D' \sim D \\ P \in \mathcal{P}(\Gamma)}} \deg^+(D' - \nu_P) - 1. \tag{4}$$

Proof. The Riemann-Roch formula holds for divisors on metric graphs (Theorem 1.2), and hence condition (RR1) from Theorem 1.4 is satisfied. Lemma 1.5 can be applied and we infer that

$$r(D) = \min_{\substack{D' \sim D \\ \nu \in \mathcal{N}}} \deg^+(D' - \nu) - 1 \ . \tag{5}$$

By Proposition 2.2, the minimum in (4) is taken over smaller set of parameters than the minimum in (5). Hence, it is enough to show that there exist $D'' \sim D$ and $P \in \mathcal{P}(\Gamma)$ such that $r(D) = \deg^+(D'' - \nu_P) - 1$. Let $D' \sim D$ and $\nu \in \mathcal{N}$ be chosen so that $r(D) = \deg^+(D' - \nu) - 1$.

By Corollary 2.4, we have $\nu \sim \nu_P$ for some alignment P of points of Γ. Setting $D'' = D' + (\nu_P - \nu)$ yields

$$r(D) = \deg^+(D' - \nu) - 1 = \deg^+(D'' - \nu_P) - 1,$$

as desired. $\square$

Motivated by Corollary 2.5, we say that the pair (D', P) is a *rank-pair for D* if $D' \sim D$, and $r(D) = \deg^+(D' - \nu_P) - 1$.

Note that the result analogous to Corollary 2.5 also holds for non-metric graphs, as shown in [3]. Let $\mathcal{P}(G)$ denote the set of all alignments of $V(G)$. As in the case of metric graphs, we can define the divisor ν_P corresponding to $P \in \mathcal{P}(G)$ by setting $\nu_P(v)$ to be equal to the number of edges from v to vertices in $V(G)$ preceding v, decreased by one. The next formula for the rank of a divisor on a finite graph G was established by Baker and the last author [3].

Lemma 2.6. *The following formula holds for the rank of every divisor D on a graph G:*

$$r(D) = \min_{\substack{D' \sim D \\ P \in \mathcal{P}(G)}} \deg^+(D' - \nu_P) - 1 \ . \tag{6}$$

It should be remarked that Lemma 2.6 follows in a straightforward fashion from the results above; at least for loopless graphs. Indeed, we use the Riemann-Roch Theorem 1.1 to conclude that property (RR1) in Theorem 1.4 holds. Then Lemma 2.6 is just Lemma 1.5. In [3], where these results were first obtained the arguments proceeded the other way round: the Riemann-Roch Theorem 1.1 was established as a consequence of abstract combinatorial criteria from Theorem 1.4.

3 Rank of divisors on metric graphs

In this section we show that the divisor and the alignment in Corollary 2.5 can be assumed to have a very special structure. We establish a series of lemmas strengthening our assumptions on this structure. It will then follow from our results that the rank of a divisor on a graph and on the corresponding metric graph are the same, thereby establishing Theorem 1.3.

Lemma 3.1. *Let D be a divisor on a loopless metric graph Γ. Suppose there exists an alignment P of points of Γ such that $r(D) = \deg^+(D - \nu_P) - 1$. Then there also exists an alignment P' of the points of $B(\Gamma) \cup \operatorname{supp} D$ such that $r(D) = \deg^+(D - \nu_{P'}) - 1$.*

Proof. By Proposition 2.1, we can assume that the support of P contains all the points of $B(\Gamma) \cup \operatorname{supp}\ D$. Choose among all alignments P' satisfying $r(D) = \deg^+(D - \nu_{P'}) - 1$ and $B(\Gamma) \cup \operatorname{supp} D \subseteq \operatorname{supp} P'$ an alignment such that $|\operatorname{supp} P'|$ is minimal.

If $\operatorname{supp} P' = B(\Gamma) \cup \operatorname{supp} D$, then the lemma holds. Assume that there exists a point $v_0 \in \operatorname{supp}\ P' \setminus (B(\Gamma) \cup \operatorname{supp}\ D)$. Let $v_1, v_2 \in \operatorname{supp}\ P'$ be such that the segments in Γ with ends v_0 and v_i, for $i = 1, 2$, contain no other points of $\operatorname{supp} P'$. We can assume by symmetry that $v_1 <_{P'} v_2$.

Consider now the alignment P'' obtained from P' by removing the point v_0. We shall distinguish three cases based on the mutual order of v_0, v_1 and v_2 in P', and conclude in each of the cases that $\deg^+(D - \nu_{P''}) \le \deg^+(D - \nu_{P'})$. This, together with the fact that $\operatorname{supp} P'' \subsetneq \operatorname{supp} P'$ will contradict the choice of P'.

If $v_0 <_{P'} v_1$ and $v_0 <_{P'} v_2$, then $\nu_{P'}(v_0) = -1$. Observe that $\nu_{P''}(v_1) = \nu_{P'}(v_1) - 1$, $\nu_{P''}(v_0) = 0$, and $\nu_{P''}(v) = \nu_{P'}(v)$ for $v \neq v_0, v_1$. We infer that

$$\begin{aligned} &\deg^+(D - \nu_{P'}) - \deg^+(D - \nu_{P''}) = \\ &\quad = 1 + \max\{D(v_1) - \nu_{P'}(v_1), 0\} - \max\{D(v_1) - \nu_{P'}(v_1) + 1, 0\} \ge 0 \,. \end{aligned}$$

Therefore $\deg^+(D - \nu_{P''}) \le \deg^+(D - \nu_{P'})$.

If $v_1 <_{P'} v_0 <_{P'} v_2$, then $\nu_{P'} = \nu_{P''}$ and again $\deg^+(D - \nu_{P''}) = \deg^+(D - \nu_{P'})$.

It remains to consider the case $v_1 <_{P'} v_0$ and $v_2 <_{P'} v_0$. Observe that $\nu_{P'}(v_0) = 1$, $\nu_{P''}(v_0) = 0$, $\nu_{P''}(v_2) = \nu_{P'}(v_2) + 1$, and $\nu_{P''}(v) = \nu_{P'}(v)$ for $v \neq v_0, v_2$. We conclude that

$$\begin{aligned} &\deg^+(D - \nu_{P'}) - \deg^+(D - \nu_{P''}) = \\ &\quad = \max\{D(v_2) - \nu_{P'}(v_2), 0\} - \max\{D(v_2) - \nu_{P'}(v_2) - 1, 0\} \ge 0 \,. \end{aligned}$$

Consequently, $\deg^+(D - \nu_{P''}) \le \deg^+(D - \nu_{P'})$. □

Next, we show that the divisor $D' \sim D$ that minimizes $\min_{P \in \mathcal{P}(\Gamma)} \deg^+(D' - \nu_P)$ can be assumed to be non-negative everywhere except for the points of $B(\Gamma)$.

Lemma 3.2. *Let D be a divisor on a loopless metric graph Γ. There exists a rank-pair (D', P) for D such that P is an alignment of the points of $B(\Gamma) \cup \operatorname{supp} D'$ and D' is non-negative on the interior of every edge of Γ.*

Proof. By Corollary 2.5 and Lemma 3.1, there exist a divisor D_0 equivalent to D and an alignment P_0 of the points of $B(\Gamma) \cup \operatorname{supp} D_0$ such that $r(D) = \deg^+(D_0 - \nu_{P_0}) - 1$. Among all such divisors let us consider the divisor D_0 such that the sum

$$S = \sum_{v \,\in\, \operatorname{supp}\, D_0 \setminus B(\Gamma)} \min\{0, D_0(v)\}$$

is maximal. If $S = 0$, then the divisor D_0 is non-negative on the interior of every edge of Γ, and there is nothing to prove. Hence, we assume $S < 0$ in the rest, i.e., there exists an edge e with an internal point where D_0 is negative.

Let $v_1, \ldots, v_k$ be the longest sequence of points of supp D_0 in the interior of e, such that $D_0(v_i) < 0$ for $i = 1, \ldots, k$ and the points are consecutive points, i.e., there is no point of supp D_0 on the segment between v_i and v_{i+1}, $i = 1, \ldots, k-1$. Let w_1 be the point of $B(\Gamma) \cup \text{supp } D_0$ such that the segment between v_1 and w_1 contains no point of $B(\Gamma) \cup \text{supp } D_0$ and $w_1 \neq v_2$, and let w_2 be the point of $B(\Gamma) \cup \text{supp } D_0$ such that the segment between v_k and w_2 contains no point of $B(\Gamma) \cup \text{supp } D_0$ and $v_{k-1} \neq w_2$.

We now modify the divisor D_0 and the alignment P_0. By symmetry, we can assume that $\text{dist}_e(w_1, v_1) \leq \text{dist}_e(w_2, v_k)$. Let $d_0 = \text{dist}_e(w_1, v_1)$, let L be the segment from w_1 to w_2 that contains $v_1, \ldots, v_k$ and let v'_k be the point of L at distance d_0 from w_2. Consider the rational function f equal to 0 on L between v_1 and v'_k and $f(v) = \min\{d_0, \text{dist}_e(v, v_1), \text{dist}_e(v, v'_k)\}$ elsewhere. Observe that $\text{ord}_{w_1}(f) = \text{ord}_{w_2}(f) = -1$ if $w_1 \neq w_2$, $\text{ord}_{w_1}(f) = -2$ if $w_1 = w_2$, $\text{ord}_{v_1}(f) = \text{ord}_{v'_k}(f) = 1$ if $v_1 \neq v'_k$, $\text{ord}_{v_1}(f) = 2$ if $v_1 = v'_k$, and $\text{ord}_v(f) = 0$ if $v \neq w_1, w_2, v_1, v'_k$. In addition, observe that if $w_1 = w_2$, then L is a loop in Γ, and $w_1 = w_2$ is a branching point of Γ.

Let $D'_0 = D_0 + \text{div}(f)$. We first show that the sum

$$S' = \sum_{v \,\in\, \text{supp } D'_0 \setminus B(\Gamma)} \min\{0, D'_0(v)\}$$

is strictly larger than S. The value of D'_0 is smaller than the value of D_0 only at w_1 and w_2. If w_1 is a branching point, then the change of the value of the divisor at w_1 does not affect the sum. Otherwise, the points w_1 and w_2 are distinct (as we have observed earlier), and $D_0(w_1) \geq 1$ by the choice of w_1. Hence, $D'_0(w_1) \geq 0$ and the sum is not affected by the corresponding summand. Analogous statements are true for the point w_2. We infer from $\text{ord}_{v_1}(f) > 0$ that $D'_0(v_1) > D_0(v_1)$. Since $D_0(v_1) < 0$, this change increases the sum by one. Finally, the change at v'_k either increases the sum by one (if $D_0(v'_k) < 0$) or does not affect the sum (if $D_0(v'_k) \geq 0$) at all. We conclude that $S' \geq S + 1$.

We next modify the alignment P_0 to P'_0 in such a way that $\deg^+(D_0 - \nu_{P_0}) = \deg^+(D'_0 - \nu_{P'_0})$. Without loss of generality, we assume that $v'_k \in \text{supp } P_0$ (cf. Proposition 2.1). The alignment P'_0 is obtained from P_0 as follows: all the points of supp P_0 distinct from $v_1, \ldots, v_k$ and v'_k form the initial part of the alignment in the same order as in P_0, and the points $v_1, v_2, \ldots, v_k, v'_k$ then follow (in this order).

Let $W = \{w_1, w_2, v_1, \ldots, v_k, v'_k\}$. For simplicity, let us assume that the points w_1 and w_2 are distinct, as well as the points v_1, v_k and v'_k. It is easy to verify that all our arguments translate to the setting when some of these points coincide. Since $D_0(v) = D'_0(v)$ and $\nu_{P_0}(v) = \nu_{P'_0}(v)$ for all points $v \notin W$, the following holds:

$$\begin{aligned}
&\deg^+(D_0 - \nu_{P_0}) - \deg^+(D'_0 - \nu_{P'_0}) = \\
&\quad = \sum_{v \in W} \left(\max\{0, (D_0 - \nu_{P_0})(v)\} - \max\{0, (D'_0 - \nu_{P'_0})(v)\}\right) \ .
\end{aligned}$$

By the choice of the points $v_1, \ldots, v_k$, we have $D_0(v_i) \leq -1$ and therefore $(D_0 - \nu_{P_0})(v) \leq 0$ for $v \in W \setminus \{v'_k, w_1, w_2\}$. Note also that $\nu_{P'_0}(v_i) = 0$ and $\nu_{P'_0}(v'_k) = 1$. Finally, note that $D'_0(v_i) \leq D_0(v_i) + 1 \leq 0$, unless $v_i = v'_k$, and $D'_0(v'_k) \leq 1$. As a result, we have $(D'_0 - \nu_{P'_0})(v) \leq 0$ for $v \in W \setminus \{w_1, w_2\}$. Consequently, we obtain the following:

$$\begin{aligned}
&\deg^+(D_0 - \nu_{P_0}) - \deg^+(D_0' - \nu_{P_0'}) = \\
&\quad = \max\{0, (D_0 - \nu_{P_0})(w_1)\} - \max\{0, (D_0' - \nu_{P_0'})(w_1)\} \\
&\quad\quad + \max\{0, (D_0 - \nu_{P_0})(w_2)\} - \max\{0, (D_0' - \nu_{P_0'})(w_2)\} \ .
\end{aligned}$$

Since $\mathrm{ord}_{w_1}(f) = -1$, we have $D_0'(w_1) = D_0(w_1) - 1$. On the other hand, the value $\nu_{P_0'}(w_1)$ is either equal to $\nu_{P_0}(w_1)$, or to $\nu_{P_0}(w_1) - 1$ (the latter is the case if $w_1 >_{P_0} v_1$). We conclude that $(D_0' - \nu_{P_0'})(w_1)$ is equal to either $(D_0 - \nu_{P_0})(w_1)$ or $(D_0 - \nu_{P_0})(w_1) - 1$. Hence,

$$\max\{0, (D_0 - \nu_{P_0})(w_1)\} - \max\{0, (D_0' - \nu_{P_0'})(w_1)\} \geq 0 \ .$$

An entirely analogous argument yields that

$$\max\{0, (D_0 - \nu_{P_0})(w_2)\} - \max\{0, (D_0' - \nu_{P_0'})(w_2)\} \geq 0 \ .$$

Consequently, we obtain that

$$\deg^+(D_0 - \nu_{P_0}) - \deg^+(D_0' - \nu_{P_0'}) \geq 0 \ .$$

Since $r(D) = \deg^+(D_0 - \nu_{P_0}) - 1$, and D_0' is equivalent to D, the inequality above must be the equality, and thus $r(D) = \deg^+(D_0' - \nu_{P_0'}) - 1$. By Lemma 3.1, there exists an alignment P_0'' of points of $B(\Gamma) \cup \mathrm{supp}\, D_0'$ such that $r(D) = \deg^+(D_0' - \nu_{P_0''}) - 1$. As $S' > S$, the existence of D_0' contradicts the choice of D_0. □

Next, we show that the divisor D' can be assumed to be zero outside $B(\Gamma)$, except possibly for a single point on each edge, where its value could be equal to one.

Lemma 3.3. *Let D be a divisor on a loopless metric graph Γ. There exists rank-pair (D', P) for D such that P is an alignment of the points of $B(\Gamma) \cup \mathrm{supp}\, D'$, and every edge e of Γ contains at most one point v where D' is non-zero, and if such a point v exists, then $D'(v) = 1$.*

Furthermore, the alignment P can be assumed to be such that all the non-branching points of supp P *follow the branching points in the order determined by P.*

Proof. By Lemma 3.2, there exist a divisor D_0 and an alignment P_0 of the points of $B(\Gamma) \cup \mathrm{supp}\, D_0$ such that (D_0, P) is a rank-pair for D, and D_0 is non-negative in the interior of every edge of Γ. Among all such divisors, consider the divisor D_0 such that the sum

$$S = \sum_{v \,\in\, \mathrm{supp}\ D_0 \setminus B(\Gamma)} D_0(v)$$

is minimal. If every edge e contains at most one point v where D_0 is non-zero, and $D_0(v) = 1$ at such a point v, then the lemma holds. We assume that D_0 does not have this property for a contradiction.

Choose an edge e such that the sum of the values of D_0 in the interior of e is at least two. Let w_1 and w_2 be the end points of e and $v_1, \ldots, v_k$ all the points of supp D_0 inside e ordered from w_1 to w_2. In the rest we assume that $v_1 \neq v_k$. As in the proof of the previous lemma, our arguments readily translate to the setting when some of these points are the same, but this assumption helps us to avoid technical complications during the presentation of the proof. Let us note, in order to assist the reader with the verification of the remaining cases, that if $v_1 = v_k$, then $D_0(v_1) \geq 2$.

By symmetry, we can assume that $\mathrm{dist}_e(w_1, v_1) \le \mathrm{dist}_e(w_2, v_k)$. Let $d_0 = \mathrm{dist}_e(w_1, v_1)$ and let w_2' be the point on the segment between v_k and w_2 at distance d_0 from v_k. For the sake of simplicity, we assume that $w_2 \ne w_2'$; again, our arguments readily translate to the setting when $w_2 = w_2'$. Consider the rational function f equal to 0 on the points outside the edge e and on the segment between w_2 and w_2' and $f(v) = \min\{\mathrm{dist}_e(v, w_1), \mathrm{dist}_e(v, w_2'), d_0\}$ elsewhere. Observe that $\mathrm{ord}_{w_1}(f) = \mathrm{ord}_{w_2'}(f) = 1$, $\mathrm{ord}_{v_1}(f) = \mathrm{ord}_{v_k}(f) = -1$, and $\mathrm{ord}_v(f) = 0$ if $v \ne w_1, w_2', v_1, v_k$.

Let $D_0' = D_0 + \mathrm{div}(f)$. Since $D_0'(v_1) = D_0(v_1) - 1 \ge 0$, $D_0'(v_k) = D_0(v_k) - 1 \ge 0$ and $D_0'(w_2') = 1$, the sum

$$S' = \sum_{v \in \mathrm{supp}\ D_0' \setminus B(\Gamma)} D_0'(v)$$

is equal to $S - 1$, and D_0' is non-negative in the interior of all the edges of Γ.

Next, we construct an alignment P_0' such that $r(D) = \deg^+(D_0' - \nu_{P_0'}) - 1$. First, insert w_2' into P_0 between v_k and w_2, preserving the order of v_k and w_2 (this did not change ν_{P_0}, see Proposition 2.1). The alignment P_0' is obtained from P_0 as follows: the points $v_1, \ldots, v_k$ form the initial part of P_0' in the same order as they appear in P_0, and the remaining points form the final part of P_0', again in the same order as they appear in P_0.

It is easy to verify that $\nu_{P_0}(v) = \nu_{P_0'}(v)$ for all points $v \notin \{w_1, w_2', v_1, v_k\}$. Hence,

$$\begin{aligned}
&\deg^+(D_0 - \nu_{P_0}) - \deg^+(D_0' - \nu_{P_0'}) = \\
&\quad = \max\{0, (D_0 - \nu_{P_0})(w_1)\} - \max\{0, (D_0' - \nu_{P_0'})(w_1)\} \\
&\qquad + \max\{0, (D_0 - \nu_{P_0})(w_2')\} - \max\{0, (D_0' - \nu_{P_0'})(w_2')\} \\
&\qquad + \max\{0, (D_0 - \nu_{P_0})(v_1)\} - \max\{0, (D_0' - \nu_{P_0'})(v_1)\} \\
&\qquad + \max\{0, (D_0 - \nu_{P_0})(v_k)\} - \max\{0, (D_0' - \nu_{P_0'})(v_k)\} .
\end{aligned}$$

Let us first consider the points v_1 and w_1. We distinguish two cases based on the mutual order of v_1 and w_1 in P_0.

The case we consider first is that $v_1 <_{P_0} w_1$. We have $\nu_{P_0}(v_1) = \nu_{P_0'}(v_1) \le 0$ and $\nu_{P_0}(w_1) = \nu_{P_0'}(w_1) \ge 0$. As $D_0'(v_1) = D_0(v_1) - 1 \ge 0$, we have that

$$\max\{0, (D_0 - \nu_{P_0})(v_1)\} - \max\{0, (D_0' - \nu_{P_0'})(v_1)\} = 1 .$$

As $D_0'(w_1) = D_0(w_1) + 1$, we have that

$$\max\{0, (D_0 - \nu_{P_0})(v_1)\} - \max\{0, (D_0' - \nu_{P_0'})(v_1)\} \ge -1 .$$

We conclude that

$$\begin{aligned}
&\max\{0, (D_0 - \nu_{P_0})(w_1)\} - \max\{0, (D_0' - \nu_{P_0'})(w_1)\} \\
&\quad + \max\{0, (D_0 - \nu_{P_0})(v_1)\} - \max\{0, (D_0' - \nu_{P_0'})(v_1)\} \ge 0 .
\end{aligned}$$

Let us deal with the other case when $v_1 >_{P_0} w_1$. Since $\nu_{P_0}(v_1) = \nu_{P_0'}(v_1) + 1$ and $D_0'(v_1) = D_0(v_1) - 1$, we have

$$\max\{0, (D_0 - \nu_{P_0})(v_1)\} = \max\{0, (D_0' - \nu_{P_0'})(v_1)\} .$$

Similarly, since $\nu_{P_0}(w_1) = \nu_{P_0'}(w_1) - 1$ and $D_0'(w_1) = D_0(w_1) + 1$, we have

$$\max\{0, (D_0 - \nu_{P_0})(w_1)\} = \max\{0, (D_0' - \nu_{P_0'})(w_1)\} .$$

Therefore, in this case we also obtain that

$$\max\{0,(D_0-\nu_{P_0})(w_1)\}-\max\{0,(D_0'-\nu_{P_0'})(w_1)\}$$
$$+\max\{0,(D_0-\nu_{P_0})(v_1)\}-\max\{0,(D_0'-\nu_{P_0'})(v_1)\}=0\ .$$

A symmetric argument yields that

$$\max\{0,(D_0-\nu_{P_0})(w_2')\}-\max\{0,(D_0'-\nu_{P_0'})(w_2')\}$$
$$+\max\{0,(D_0-\nu_{P_0})(v_k)\}-\max\{0,(D_0'-\nu_{P_0'})(v_k)\}=0\ .$$

Hence,

$$\deg^+(D_0-\nu_{P_0})-\deg^+(D_0'-\nu_{P_0'})=0\ .$$

Since $r(D)=\deg^+(D_0-\nu_{P_0})-1$, we have that $r(D)=\deg^+(D_0'-\nu_{P_0'})$. Since the alignment P_0' can be chosen in such a way that supp $P_0'=B(\Gamma)\cup$ supp D_0' by Lemma 3.1 and $S'<S$, the existence of D_0' and P_0' contradict the choice of D_0 and P_0.

Last, we prove the "furthermore" part of the statement. That is, we show that the alignment P can be assumed to be such that all the non-branching points of supp P follow the branching points in the order determined by P.

Consider the alignment P' obtained from P by moving a point $v\in$ supp $D'\setminus B(\Gamma)$ to the end of the alignment. We claim that $r(D)=\deg^+(D'-\nu_{P'})-1$.

By Corollary 2.5, it suffices to show that $\deg^+(D'-\nu_{P'})\le\deg^+(D'-\nu_P)$. Let w_1 and w_2 be the end points of the edge containing v. We consider in detail the case when $v<_P w_1$ and $v<_P w_2$; the other cases are analogous. As $D'(v)=1$ and $\nu_P(v)=-1$, it holds that $D'(v)-\nu_P(v)=2$ and $D'(v)-\nu_{P'}(v)=0$. Similarly, $\nu_{P'}(w_i)=\nu_P(w_i)-1$, and thus $D'(w_i)-\nu_{P'}(w_i)=D'(w_i)-\nu_P(w_i)+1$ for $i=1,2$. We conclude that

$$\begin{aligned}
&\deg^+(D'-\nu_P)-\deg^+(D'-\nu_{P'})=\\
&\quad=\max\{0,D'(w_1)-\nu_P(w_1)\}-\max\{0,D'(w_1)-\nu_{P'}(w_1)\}\\
&\qquad+\max\{0,D'(w_2)-\nu_P(w_2)\}-\max\{0,D'(w_2)-\nu_{P'}(w_2)\}\\
&\qquad+\max\{0,D'(v)-\nu_P(v)\}-\max\{0,D'(v)-\nu_{P'}(v)\}\\
&\quad\ge(D'(w_1)-\nu_P(w_1))-(D'(w_1)-\nu_{P'}(w_1))\\
&\qquad+(D'(w_2)-\nu_P(w_2))-(D'(w_2)-\nu_{P'}(w_2))+2\ge0\ .
\end{aligned}$$

The claim now follows. Hence, we can assume without loss of generality that all the points of supp $D'\setminus B(\Gamma)$ follow the points of $B(\Gamma)$ in the order determined by P, i.e., $\nu_P(v)=1$ for $v\in$ supp $D'\setminus B(\Gamma)$. □

Lemma 3.3 allows us to prove Theorem 1.7. We are grateful to an anonymous referee for suggesting the proof.

Proof of Theorem 1.7. Consider an arbitrary divisor D on a loopless metric graph Γ. By Lemma 3.3 there exists a rank-pair (D',P) for D such that P is an alignment of supp $(D')\cup B(\Gamma)$. Every edge of Γ contains at most one inner point v with non-zero value $D'(v)$, and if such a point v exists, then $D'(v)=1$. Furthermore, the points of $B(\Gamma)$ precede the other points of supp (D').

Let E be the non-negative part of $D'-\nu_P$. Note that supp $(E)\subseteq B(\Gamma)$. We have, $D'-E\le\nu_P$. Applying the rank on this inequality and using the fact that ν_P is non-special, we get $r_\Gamma(D'-E)\le r_\Gamma(\nu_P)=-1$. By the choice of D' and P, we have $r_\Gamma(D)=\deg(E)-1$. Consequently, $|D'-E|=\emptyset$, that is, E is as in (3), showing that $r_{B(\Gamma)}(D')\le\deg(E)-1$. Consequently,

$$r_\Gamma(D)\le r_{B(\Gamma)}(D)=r_{B(\Gamma)}(D')\le\deg(E)-1=\deg^+(D'-\nu_P)-1=r_\Gamma(D)\ .$$

It follows that $r_\Gamma(D)=r_{B(\Gamma)}(D)$, proving the theorem. □

With Theorem 1.7 at hand, we can now give a short proof of Theorem 1.3.

Proof of Theorem 1.3. The following claim establishes one of the two desired inequalities between the ranks.

Claim 3.3.1. *For an arbitrary divisor $F \in \mathrm{Div}(G)$ we have $r_G(F) \geq r_\Gamma(F)$.*

Proof of Claim 3.3.1. If F' is a divisor equivalent to F on the graph G then F' is equivalent to F also on the metric graph Γ. Thus the range of minimization in (4) is a superset of that in (6), and the claim follows. □

Consider an arbitrary divisor D on G. By Theorem 1.7 the set $B(\Gamma) = V(G)$ is rank-determining, i.e., we have $r_\Gamma(D) = r_{V(G)}(D)$, where $r_{V(G)}(D)$ refers to $V(G)$-restricted rank on Γ. To prove the theorem, it thus suffices to show that we range over the same sets of divisors E in the formulea (1) (for the rank on G) and (3) (for the $V(G)$-restricted rank on Γ). In other words, it needs to be shown that for a divisor $E \in \mathrm{Div}(V(G))$ we have $|D - E|_G = \emptyset$ if and only if $|D - E|_\Gamma = \emptyset$. Clearly, if $|D - E|_\Gamma = \emptyset$ then $|D - E|_G = \emptyset$. The other direction follows from Claim 3.3.1 which tells us that $r_G(D - E) = -1$ implies $r_\Gamma(D - E) = -1$. □

As a corollary of Theorem 1.3 we can prove that the rank of a divisor on a graph is preserved under subdivision. We say that a bijection φ between the points of a metric graph Γ and the points of a metric graph Γ' is a *homothety* if there exists a real number $\alpha > 0$ such that $\mathrm{dist}_\Gamma(v, w) = \alpha \cdot \mathrm{dist}_{\Gamma'}(\varphi(v), \varphi(w))$ for every two points v and w of Γ. Note that composition of a rational function with a homothety is a rational function, and thus a homothety preserves the rank of divisors.

Corollary 3.4. *Let D be a divisor on a loopless graph G and let G^k be the graph obtained from G by subdividing each edge of G exactly k times. The ranks of D in G and in G^k are the same.*

Proof. Let Γ be the metric graph corresponding to G. Observe that there exists a homothety from Γ to the metric graph Γ' corresponding to G^k. Since the rank of D in G is equal to the rank of D in Γ by Theorem 1.3 and the rank of D in G^k is equal to the rank of D in Γ' by the same theorem, the ranks of D in G and in G^k are the same. □

Finally, we show that, in addition to the conditions given in Lemma 3.3, the divisor D' can be assumed to be zero inside edges of a spanning tree of Γ. This will be used as a main auxiliary result in Section 4.1 to give an algorithm for computing the rank of divisors on metric graphs.

Lemma 3.5. *Let D be a divisor on a loopless metric graph Γ. There exists a divisor D', a spanning tree T of Γ, and an alignment $P \in \mathcal{P}(\Gamma)$ such that (D', P) is a rank-pair for D, D' is zero in the interior of every edge of T, and every edge $e \notin T$ contains at most one interior point v where $D'(v) \neq 0$, and, if such a point v exists, then $D'(v) = 1$.*

Proof. Let D' be a divisor equivalent to D, and let P be an alignment of the points of $B(\Gamma) \cup \mathrm{supp}\, D'$ as in Lemma 3.3.

Let us now color the edges of Γ with red and blue, so that the red edges contain in their interior a point v in Γ with $D'(v) = 1$ and the blue edges do not. Let $V_1, \ldots, V_k$ be the components of Γ formed by blue edges. Choose among all divisors D' equivalent to D, and alignments P, satisfying the conditions of Lemma 3.3, the divisor D' such that the number k of the components $V_1, \ldots, V_k$ is the smallest possible. If $k = 1$, there exists a spanning tree of Γ formed by the blue edges, and there is nothing to prove.

Assume now that $k \geq 2$ for the divisor D' which minimizes k. Recall that $v <_P v'$ for every $v \in B(\Gamma)$ and $v' \in \text{supp } D' \setminus B(\Gamma)$. Let us call the red edges connecting points of V_1 to points of $B(\Gamma) \setminus V_1$ *orange* edges. We can assume that the points of $V_1 \cap B(\Gamma)$ follow all the other points of $B(\Gamma)$ in the order determined by P, as every orange edge contains a point in supp $D' \setminus B(\Gamma)$. For an orange edge e incident with a branching point v_1 of V_1, let $d(e)$ be the distance between v_1 and the point of supp D' in the interior of e. Let d_0 be the minimum $d(e)$ taken over all orange edges e.

Consider the following rational function f: $f(v) = 0$ for points v on edges between two branching points of V_1,

$$f(v) = \min\{d_0, \max\{0, \text{dist}_e(v_1, v) + d_0 - d(e)\}\}$$

for points v on any orange edge e incident with any branching point v_1 of V_1, and $f(v) = d_0$ for the remaining points of Γ. Set $D'' = D' + \text{div}(f)$. Clearly, D'' is a divisor equivalent to D that is non-zero on at most one point in the interior of every edge of Γ and is equal to one at such a point. Moreover, since the blue edges remain blue and the orange edges e with $d(e) = d_0$ become blue, the number of components formed by blue edges in D'' is smaller than this number in D'.

We now find an alignment P' of the points of $B(\Gamma) \cup \text{supp } D''$ such that $\deg^+(D'' - \nu_{P'}) = \deg^+(D' - \nu_P)$. The existence of such an alignment P' would contradict our choice of D'. The alignment P' is defined as follows: The branching points of Γ are ordered as in P, and they precede all the points of supp $D'' \setminus B(\Gamma)$. The points of supp $D'' \setminus B(\Gamma)$ are ordered arbitrarily. If v is a point of $B(\Gamma) \cup \text{supp } D''$ that is not contained inside an orange edge, and that is not a branching point of V_1, then $D''(v) = D'(v)$ and $\nu_{P'}(v) = \nu_P(v)$. Hence, such points do not affect $\deg^+(D'' - \nu_{P'})$.

We now consider a branching point v of V_1. Let ℓ be the number of edges incident with v that are orange with respect to D' and blue with respect to D''. Clearly, ℓ is the number of orange edges e incident with v such that $d(e) = d_0$. By the choice of f, $D''(v) = D'(v) + \ell$. In addition, since the other branching points, incident with such edges, precede v in the order determined by P', $\nu_{P'}(v) = \nu_P(v) + \ell$. Hence, $(D'' - \nu_{P'})(v) = (D' - \nu_P)(v)$.

It remains to consider internal points of orange edges. Let v be such a point. If v is contained in the interior of an orange edge e with respect to D', then $D'(v) = 1$ and $\nu_P(v) = 1$, i.e., $(D' - \nu_P)(v) = 0$. If v is contained in the interior of an orange edge e with respect to D'', then $D''(v) = 1$ and $\nu_{P'}(v) = 1$, i.e., $(D'' - \nu_{P'})(v) = 0$. We conclude that such points do not affect $\deg^+(D'' - \nu_{P'})$ at all. Consequently, $\deg^+(D'' - \nu_{P'}) = \deg^+(D' - \nu_P)$, as desired. □

4 An algorithm for computing the rank

We now present the main algorithmic result of this paper. We describe an algorithm which, given a metric graph Γ and a divisor D on it, computes its rank. It is not a priori clear that such an algorithm has to exist[2]. If the lengths of all the edges of D and all the distances of non-zero values of D to the branching points are rational, then the problem is solvable on a Turing machine. However, this need not be the case in general. As the input can contain irrational numbers, we assume real arithmetic operations with infinite precision to be allowed in our computational model. The bound on the running time of our algorithm can easily be read from its construction; it is a simple function depending on the number of edges, number of

[2]Indeed, let us recall as a negative example in a similar setting that there exists no universal algorithm for solving Diophantine equations, a result due to Matiyasevich [12].

vertices of Γ, the ratio between the longest and the shortest edge in Γ, and the values of D. The running time is not more than exponential in any of these parameters.

There are several papers dealing with algorithmic aspects of tropical geometry, as [5, 9, 19] for a sample. Many of these papers rely on machinery of commutative algebra, while our algorithm utilizes combinatorial properties of divisors on tropical curves which were developed in previous parts of the paper.

Theorem 4.1. *There exists an algorithm that for a divisor D on a metric graph Γ computes the rank of D.*

As a tool for proving Theorem 4.1 we shall need the following auxiliary result of Gathmann and Kerber [7, Lemma 1.8].

Lemma 4.2. *Let a metric graph Γ and an integer p be given. Then there exists a computable integer U such that any rational function f on Γ with $\deg^+(\mathrm{div}(f)) \leq p$ has slope at most U at every point.*

Remark 4.3. It follows from the proof in [7] that $U = (\Delta + p)^m$ in Lemma 4.2 is a sufficient bound; here Δ and m are the maximum degree and the number of edges of Γ, respectively.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. To prove the theorem, it is enough to show that there are only finitely many divisors equivalent to a given divisor D that satisfy the conditions in the statement of Lemma 3.5.

We write ℓ_e for the length of an edge e. Without loss of generality, we can assume that Γ is loopless, and that $\operatorname{supp} D \subseteq B(\Gamma)$ (introduce new branching points incident with only two edges if needed). We can also assume that the length of each edge of Γ is at least one. Let n be the number of branching points of Γ, m the number of edges of Γ, and $M = \max_{v \in \operatorname{supp} D} |D(v)|$. We assume that $n \geq 2$ (and thus $m \geq 1$) since otherwise Γ is formed by a single point w_0 and $r(D) = \max\{D(w_0), -1\}$. Similarly, we can also assume that $M \geq 1$ since otherwise D is equal to zero at all points and thus $r(D) = 0$. Finally, we let U be given by Lemma 4.2 for

$$p = 2n^2M + 3mn \geq \frac{1}{2}(3n^2M + 3mn + m - n + 1)\ .$$

We first describe the algorithm and then verify its correctness. Fix an arbitrary vertex $w \in B(\Gamma)$. The algorithm ranges through all spanning trees T of Γ (here, T is the set of edges of the tree, i.e., $|T| = n - 1$) and all functions $F : T \to \{-U, -U+1, \ldots, U-1, U\}$.

The algorithm then constructs all rational functions f on Γ such that for every branching point $v \in B(\Gamma)$ we have

$$f(v) = \sum_{i=1}^{k} F(e_i)\ell_{e_i}\ ,$$

where $e_1, e_2, \ldots, e_k$ are the edges of T on the path from w to v, f is linear on every edge of T, and $\mathrm{ord}_v(f) \neq 0$ for at most one point v on every edge not in T (and $\mathrm{ord}_v(f) = 1$ for such a point v if it exists).

Let us observe that there is only one rational function f satisfying the above constraints. Indeed, the function f is uniquely defined on edges of T as it should be linear on such edges. Consider now an edge e between branching points v_1 and v_2 that is not contained in T. By symmetry, we can assume that $f(v_1) \leq f(v_2)$. Then e contains a point v either with $\mathrm{ord}_v(f) = 1$ or $v = v_2$ such that f is linear on e everywhere except for v. The average slope of f from v_1 to v_2 along e is

$(f(v_1) - f(v_2))/\ell_e$. Therefore, we must have a slope of $\lfloor (f(v_1) - f(v_2))/\ell_e \rfloor$ from v_1 to v and a slope of $\lfloor (f(v_1) - f(v_2))/\ell_e \rfloor + 1$ from v to v_2. This determines the position of v on e as well. We conclude that there are only finitely many rational functions f that satisfy conditions described in the previous paragraph.

The algorithm now computes the divisor $D' = D + \mathrm{div}(f)$, and then ranges through all alignments P of the points $B(\Gamma) \cup \mathrm{supp}\ D'$. For each such alignment, the value of $\deg^+(D' - \nu_P) - 1$ is computed and the minimum of all such values over all the choices of T, F (and thus f) and P is output as the rank of D. Since the number of choices of T, F and P is finite, the algorithm eventually finishes and outputs the rank of D.

We have to verify that the above algorithm is correct. By Corollary 2.5, the output value is greater than or equal to the rank of D. Hence, we have to show that the algorithm at some point of its execution considers $D' \in \mathrm{Div}(\Gamma)$ and $P \in \mathcal{P}(\Gamma)$ such that $\deg^+(D' - \nu_P) - 1 = r(D)$. Consider now the divisor D' and the alignment P as in Lemma 3.5. Since $\mathrm{supp}\ P = B(\Gamma) \cup \mathrm{supp}\ D'$, and the algorithm ranges through all alignments P of $B(\Gamma) \cup \mathrm{supp}\ D'$ for every constructed divisor D', it is enough to show that the algorithm constructs a rational function f such that $D' = D + \mathrm{div}(f)$.

Consider the step when the algorithm ranges through T as in Lemma 3.5 and let f_0 be the rational function given by the lemma. We can assume without loss of generality that $f_0(w) = 0$.

We establish that there exists a function $F : T \to \{-U, \ldots, U\}$ such that f_0 can be constructed (as described above) from F. The existence of such a function F will yield the correctness of the presented algorithm. In order to establish the existence of F, it is enough to show that absolute value of the slope of f_0 is bounded by U on every edge of T. Due to the relation between U and p it suffices to prove that

$$\deg^+(\mathrm{div}(f_0)) \le p\ . \tag{7}$$

We devote the rest of the proof to establishing (7).

It can be inferred from the definition of the rank that $r(D) \le \deg(D)$. Hence, $r(D) \le nM$. We now show that $|D'(v)| \le 2(nM + m)$ for every $v \in B(\Gamma)$. If there exists a branching point v_0 with $D'(v_0) > 2(nM + m)$, then

$$\begin{aligned} nM \ge r(D) &= \deg^+(D' - \nu_P) - 1 \\ &\ge D'(v_0) - \nu_P(v_0) - 1 > 2(nM + m) - m - 1 \ge 2nM\ , \end{aligned}$$

which is impossible. On the other hand, if there exists a branching point v_0 with $D'(v_0) \le -2(nM + m)$, then $D'(v_0) - \nu_P(v_0) \le -2(nM + m) + 1 < 0$ and thus $\deg^+(D' - \nu_P) = \deg^+(D'')$, where $D''(v_0) = 0$ and $D''(v) = (D' - \nu_P)(v)$ for $v \ne v_0$. Observe that

$$\begin{aligned} \deg(D'') &= \deg(D' - \nu_P) - (D'(v_0) - \nu_P(v_0)) \\ &\ge \deg(D') - \deg(\nu_P) + 2(nM + m) - 1 \\ &\ge -nM - (m - n) + 2(nM + m) - 1 = nM + m + n - 1 \\ &\ge r(D) + m + 1\ . \end{aligned}$$

We therefore have

$$\deg^+(D' - \nu_P) = \deg^+(D'') \ge \deg(D'') > nM + m \ge r(D) + m + 1,$$

which contradicts our choice of D' and P. We conclude that $|D'(v)| \le 2(nM + m)$, and that

$$|\mathrm{ord}_v(f_0)| \le M + |D'(v)| \le 3(nM + m)\ , \tag{8}$$

for every $v \in B(\Gamma)$.

We express

$$\deg^+(\operatorname{div}(f_0)) = \frac{1}{2}\sum_{v\in\Gamma} |\operatorname{ord}_v(f_0)| = \frac{1}{2}\left(\sum_{v\in B(\Gamma)} |\operatorname{ord}_v(f_0)| + \sum_{v\in\Gamma\setminus B(\Gamma)} |\operatorname{ord}_v(f_0)|\right) .$$

The first sum on the right-hand side has n summands, each can be bounded using (8). The second sum has at most $m-(n-1)$ non-zero summands, each of them equal to one. Plugging in these bounds we establish (7). □

Proposition 1.6 and Theorem 4.1 now imply the existence of an algorithm for computing the rank of a divisor on tropical curves.

Corollary 4.4. *There exists an algorithm that for a divisor D on a tropical curve Γ computes the rank of D.*

The algorithm which we presented is finite, i.e., it terminates for every input, however, its running time is exponential (as can be seen by plugging the bound from Remark 4.3 into Theorem 4.1) in the size of the input. It seems natural to ask whether it is possible to design a polynomial-time algorithm for computing the rank of divisors. In the case of graphs the question was posed by Hendrik Lenstra [10], and, to the best of our knowledge, is still open. Tardos [18] presented an algorithm which decides whether a divisor D on a graph has a non-negative rank. His algorithm is weakly polynomial, i.e., the running time is bounded by a polynomial in the size of the graph and $\deg^+(D)$ (note that Tardos was using a different language to state the result). It is possible to modify his algorithm in such a way that the running time becomes polynomial in the size of the graph and $\log(\deg^+(D))$, i.e., to obtain a truly polynomial-time algorithm for deciding whether a given divisor on a graph has a non-negative rank. We omit further details.

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References

[1] O. Amini: Reduced divisors and embeddings of tropical curves, to appear in Trans. Amer. Math. Soc., arXiv: 1007.5364.

[2] M. Baker: Specialization of linear systems from curves to graphs, Algebra & Number Theory 2, no. 6 (2008), 613–653.

[3] M. Baker, S. Norine: Riemann-Roch and Abel-Jacobi theory on a finite graph, Adv. Math. 215 (2007) no. 2,766–788.

[4] M. Baker, S. Norine: Harmonic morphisms and hyperelliptic graphs, International Math. Research Notices 15 (2009), 2914–2955.

[5] T. Bogart, A. N. Jensen, D. Speyer, B. Sturmfels, and R. R. Thomas: Computing tropical varieties, J. Symbolic Comput. 42 (2007) no. 1-2, 54–73.

[6] A. Gathmann: Tropical algebraic geometry, Jahresbericht der DMV 108(1) (2006), 3–32.

[7] A. Gathmann, M. Kerber: A Riemann-Roch theorem in tropical geometry, Math. Zeitschrift 259 (2008), 217–230.

[8] J. Hladký, D. Král', and S. Norine: Rank of divisors on tropical curves (expanded version), arXiv: 0709.4485.

[9] A. N. Jensen, H. Markwig, and T. Markwig: An Algorithm for Lifting Points in a Tropical Variety, Collect. Math. 59(2) (2008), 129–165.

[10] H. Lenstra, private communication.

[11] Y. Luo: Rank-determining sets of metric graphs, J. Combin. Theory Ser. A, 118(6) (2011), 1775–1793.

[12] Y. Matiyasevich: Enumerable sets are Diophantine, Dokl. Akad. Nauk 191 (1970), 279-282.

[13] G. Mikhalkin: Tropical geometry, `http://www.math.toronto.edu/mikha/book.pdf`.

[14] G. Mikhalkin: Tropical geometry and its applications, International Congress of Mathematicians, vol. II (2006), 827–852.

[15] G. Mikhalkin and I. Zharkov. Tropical curves, their Jacobians and theta functions. Valery Alexeev (ed.) et al., Curves and abelian varieties. International conference, Athens, GA, USA, March 30–April 2, 2007. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics 465, 203-230, 2008.

[16] J. Richter-Gebert, B. Sturmfels, T. Theobald: First steps in tropical geometry, Idempotent mathematics and mathematical physics (2005), 289–317.

[17] D. Speyer, B. Sturmfels: Tropical Mathematics, Mathematics Magazine 82 (2009) 163–173.

[18] G. Tardos: Polynomial bound for a chip firing game on graphs, SIAM J. Discrete Math. 1 (1988) no. 3, 397–398.

[19] T. Theobald: On the frontiers of polynomial computations in tropical geometry, J. Symbolic Comput. 41 (2006) no. 12, 1360–1375.