A new bound for the 2/3 conjecture*

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Abstract

We show that any n-vertex complete graph with edges colored with three colors contains a set of at most four vertices such that the number of the neighbors of these vertices in one of the colors is at least $2n/3$. The previous best value, proved by Erdős, Faudree, Gould, Gyárfás, Rousseau and Schelp in 1989, is 22. It is conjectured that three vertices suffice.

1 Introduction

Erdős and Hajnal [9] made the observation that for a fixed positive integer t, a positive real ϵ, and a graph G on $n > n_0$ vertices, there is a set of t vertices that have a neighborhood of size at least $(1-(1+\epsilon)(2/3)^t)n$ in either G or its complement. They further inquired whether $2/3$ may be replaced by $1/2$. This was answered in the affirmative by Erdős,

*This work was done in the framework of LEA STRUCO.

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Faudree, Gyárfás and Schelp [7], who not only proved the result but also dispensed with the $(1 + \epsilon)$ factor. They also phrased the question as a problem of vertex domination in a multicolored graph.

Given a color c in an r-coloring of the edges of the complete graph, a subset A of the vertex set *c-dominates* another subset B if, for every $y \in B \setminus A$, there exists a vertex $x \in A$ such that the edge xy is colored c. The subset A *strongly* c-dominates B if, in addition, for every $y \in B \cap A$, there exists a vertex $x \in A$ such that xy is colored c. (Thus, the two notions coincide when $A \cap B = \emptyset$.) The result of Erdős *et al.* [7] may then be stated as follows.

Theorem 1. *For any fixed positive integer t and any 2-coloring of the edges of the complete graph K_n on n vertices, there exist a color c and a subset X of size at most t such that all but at most $n/2^t$ vertices of K_n are c-dominated by X.*

In a more general form, they asked: *Given positive integers r, t, and n along with an r-coloring of the edges of the complete graph K_n on n vertices, what is the largest subset B of the vertices of K_n necessarily monochromatically dominated by some t-element subset of K_n?* However, in the same paper [7], the authors presented a 3-coloring of the edges of K_n — attributed to Kierstead — which shows that if $r \geqslant 3$, then it is not possible to monochromatically dominate all but a small fraction of the vertices with any fixed number t of vertices. This 3-coloring is defined as follows: the vertices of K_n are partitioned into three sets V_1, V_2, V_3 of equal sizes and an edge xy with $x \in V_i$ and $y \in V_j$ is colored i if $1 \leqslant i \leqslant j \leqslant 3$ and $j - i \leqslant 1$ while edges between V_1 and V_3 are colored 3. Observe that, if t is fixed, then at most $2n/3$ vertices may be monochromatically dominated.

In the other direction, it was shown in the follow-up paper of Erdős, Faudree, Gould, Gyárfás, Rousseau and Schelp [8], that if $t \geqslant 22$, then, indeed, at least $2n/3$ vertices are monochromatically dominated in any 3-coloring of the edges of K_n. The authors then ask if 22 may be replaced by a smaller number (specifically, 3). We prove here that $t \geqslant 4$ is sufficient.

Theorem 2. *For any 3-coloring of the edges of K_n, where $n \geqslant 2$, there exist a color c and a subset A of at most four vertices of K_n such that A strongly c-dominates at least $2n/3$ vertices of K_n.*

In Kierstead's coloring, the number of colors appearing on the edges incident with any given vertex is precisely 2. As we shall see later on, this property plays a central role in our arguments. In this regard, our proof seems to suggest that Kierstead's coloring is somehow extremal, giving more credence to the conjecture that three vertices would suffice to monochromatically dominate a set of size $2n/3$ in any 3-coloring of the edges of K_n.

We note that there exist 3-colorings of the edges of K_n such that no pair of vertices monochromatically dominate $2n/3 + O(1)$ vertices. This can be seen by realizing that in a random 3-coloring, the probability that an arbitrary pair of vertices monochromatically dominate more than $5n/9 + o(n)$ vertices is $o(1)$ by Chernoff's bound.

Our proof of Theorem 2 utilizes the flag algebra theory introduced by Razborov, which has recently led to numerous results in extremal graph and hypergraph theory. In the following section, we present a brief introduction to the flag algebra framework. The proof of Theorem 2 is presented in Section 3.

We end this introduction by pointing out another interesting question: what happens when one increases r, the number of colors? Constructions in the vein of that of Kierstead — for example, partitioning K_n into s parts and using $r = \binom{s}{2}$ colors — show that the size of dominated sets decreases with increasing r. While it may be difficult to determine the minimum value of t dominating a certain proportion of the vertices, it would be interesting to find out whether such constructions do, in fact, give the correct bounds.

2 Flag Algebras

Flag algebras were introduced by Razborov [23] as a tool based on the graph limit theory of Lovász and Szegedy [20] and Borgs *et al.* [5] to approach problems pertaining to extremal graph theory. This tool has been successfully applied to various topics, such as Turán-type problems [25], super-saturation questions [24], jumps in hypergraphs [2], the Caccetta-Häggkvist conjecture [17], the chromatic number of common graphs [14] and the number of pentagons in triangle-free graphs [12, 15]. This list is far from being exhaustive and results keep coming [1, 3, 4, 6, 11, 10, 13, 16, 18, 19, 21, 22].

Let us now introduce the terminology related to flag algebras needed in this paper. Since we deal with 3-colorings of the edges of complete graphs, we restrict our attention to this particular case. Let us define a *tricolored graph* to be a complete graph whose edges are colored with 3 colors. If G is a tricolored graph, then $V(G)$ is its vertex-set and $|G|$ is the number of vertices of G. Let $\mathbb{F}_\ell$ be the set of non-isomorphic tricolored graphs with ℓ vertices, where two tricolored graphs are considered to be isomorphic if they differ by a permutation of the vertices and a permutation of the edge colors. (Therefore, which specific color is used for each edge is irrelevant: what matters is whether or not pairs of edges are assigned the same color.) The elements of $\mathbb{F}_3$ are shown in Figure 1. We set $\mathbb{F} := \cup_{\ell \in \mathbf{N}} \mathbb{F}_\ell$. Given a tricolored graph σ, we define $\mathbb{F}_\ell^\sigma$ to be the set of tricolored graphs F on ℓ vertices with a fixed embedding of σ, that is, an injective mapping ν from $V(\sigma)$ to $V(F)$ such that $\mathrm{Im}(\nu)$ induces in F a subgraph that differs from σ only by a permutation of the edge colors. The elements of $\mathbb{F}_\ell^\sigma$ are usually called σ*-flags* within the flag algebras

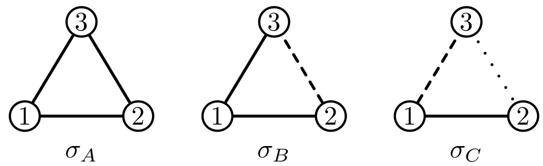

Figure 1: The elements of $\mathbb{F}_3$. The edges of color 1, 2 and 3 are represented by solid, dashed and dotted lines, respectively.

framework. We set $\mathbb{F}^\sigma := \cup_{\ell \in \mathbf{N}} \mathbb{F}^\sigma_\ell$.

The central notions are factor algebras of $\mathbb{F}$ and $\mathbb{F}^\sigma$ equipped with addition and multiplication. Let us start with the simpler case of $\mathbb{F}$. If $H \in \mathbb{F}$ and $H' \in \mathbb{F}_{|H|+1}$, then $p(H, H')$ is the probability that a randomly chosen subset of $|H|$ vertices of H' induces a subgraph isomorphic to H. For a set F, we define $\mathbf{R}F$ to be the set of all formal linear combinations of elements of F with real coefficients. Let $\mathcal{A} := \mathbf{R}\mathbb{F}$ and let $\mathcal{F}$ be $\mathcal{A}$ factorised by the subspace of $\mathbf{R}\mathbb{F}$ generated by all combinations of the form

$$H - \sum_{H' \in \mathbb{F}_{|H|+1}} p(H, H')H'.$$

Next, we define the multiplication on $\mathcal{A}$ based on the elements of $\mathbb{F}$ as follows. If H_1 and H_2 are two elements of $\mathbb{F}$ and $H \in \mathbb{F}_{|H_1|+|H_2|}$, then $p(H_1, H_2; H)$ is the probability that two randomly chosen disjoint subsets of vertices of H with sizes $|H_1|$ and $|H_2|$ induce subgraphs isomorphic to H_1 and H_2, respectively. We set

$$H_1 \cdot H_2 := \sum_{H \in \mathbb{F}_{|H_1|+|H_2|}} p(H_1, H_2; H)H.$$

The multiplication is linearly extended to $\mathbf{R}\mathbb{F}$. Standard elementary probability computations [23, Lemma 2.4] show that this multiplication in $\mathbf{R}\mathbb{F}$ gives rise to a well-defined multiplication in the factor algebra $\mathcal{A}$.

The definition of $\mathcal{A}^\sigma$ follows the same lines. Let H and H' be two tricolored graphs in $\mathbb{F}^\sigma$ with embeddings ν and ν' of σ. Informally, we consider the copy of σ in H' and we extend it into an element of $\mathbb{F}^\sigma_{|H|}$ by randomly choosing additional vertices in H'. We are interested in the probability that this random extension is isomorphic to H and the isomorphism preserves the embeddings of σ. Formally, we let $p(H, H')$ be the probability that $\nu'(V(\sigma))$ together with a randomly chosen subset of $|H| - |\sigma|$ vertices in $V(H') \setminus \nu'(V(\sigma))$ induce a subgraph that is isomorphic to H through an isomorphism f that preserves the embeddings, that is, $\nu' = f \circ \nu$. The set $\mathcal{A}^\sigma$ is composed of all formal real linear combinations of elements of $\mathbf{R}\mathbb{F}^\sigma$ factorised by the subspace of $\mathbf{R}\mathbb{F}^\sigma$ generated by all combinations of the form

$$H - \sum_{H' \in \mathbb{F}^{\sigma}_{|H|+1}} p(H, H')H'.$$

Similarly, $p(H_1, H_2; H)$ is the probability that $\nu(V(\sigma))$ together with two randomly chosen disjoint subsets of $|H_1| - |\sigma|$ and $|H_2| - |\sigma|$ vertices in $V(H) \setminus \nu(V(\sigma))$ induce subgraphs isomorphic to H_1 and H_2, respectively, with the isomorphisms preserving the embeddings of σ. The definition of the product is then analogous to that in $\mathcal{A}$.

Consider an infinite sequence $(G_i)_{i \in \mathbf{N}}$ of tricolored graphs with an increasing number of vertices. Recall that if $H \in \mathbb{F}$, then $p(H, G_i)$ is the probability that a randomly chosen subset of $|H|$ vertices of G_i induces a subgraph isomorphic to H. The sequence $(G_i)_{i \in \mathbf{N}}$ is *convergent* if $p(H, G_i)$ has a limit for every $H \in \mathbb{F}$. A standard argument (using Tychonoff's theorem [26]) yields that every infinite sequence of tricolored graphs has a convergent (infinite) subsequence.

The results presented in this and the next paragraph were established by Razborov [23]. Fix now a convergent sequence $(G_i)_{i \in \mathbf{N}}$ of tricolored graphs. We set $q(H) := \lim_{i \to \infty} p(H, G_i)$ for every $H \in \mathbb{F}$, and we linearly extend q to $\mathcal{A}$. The obtained mapping q is a homomorphism from $\mathcal{A}$ to $\mathbf{R}$. Moreover, for $\sigma \in \mathbb{F}$ and an embedding ν of σ in G_i, define $p_i^{\nu}(H) := p(H, G_i)$. Picking ν at random thus gives rise to a random distribution of mappings from $\mathcal{A}^{\sigma}$ to $\mathbf{R}$, for each $i \in \mathbf{N}$. Since $p(H, G_i)$ converges (as i tends to infinity) for every $H \in \mathbb{F}$, the sequence of these distributions must also converge. In fact, q itself fully determines the random distributions of q^{σ} for all σ. In what follows, q^{σ} will be a randomly chosen mapping from $\mathcal{A}^{\sigma}$ to $\mathbf{R}$ based on the limit distribution. Any mapping q^{σ} from support of the limit distribution is a homomorphism from $\mathcal{A}^{\sigma}$ to $\mathbf{R}$.

Let us now have a closer look at the relation between q and q^{σ}. The "averaging" operator $[\![\cdot]\!]_{\sigma} : \mathcal{A}^{\sigma} \to \mathcal{A}$ is a linear operator defined on the elements of $\mathbb{F}^{\sigma}$ by $[\![H]\!]_{\sigma} := p \cdot H'$, where H' is the (unlabeled) tricolored graph in $\mathbb{F}$ corresponding to H and p is the probability that a random injective mapping from $V(\sigma)$ to $V(H')$ is an embedding of σ in H' yielding H. The key relation between q and q^{σ} is the following:

$$\forall H \in \mathcal{A}^{\sigma}, \quad q([\![H]\!]_{\sigma}) = \int q^{\sigma}(H), \tag{1}$$

where the integration is over the probability space given by the limit random distribution of q^{σ}. We immediately conclude that if $q^{\sigma}(H) \geqslant 0$ almost surely, then $q([\![H]\!]_{\sigma}) \geqslant 0$. In particular,

$$\forall H \in \mathcal{A}^{\sigma}, \quad q\left(\left[\!\left[H^2\right]\!\right]_{\sigma}\right) \geqslant 0. \tag{2}$$

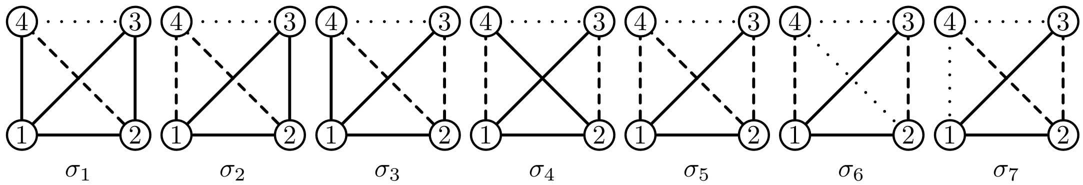

Figure 2: The elements $\sigma_1, \ldots, \sigma_7$ of $\mathbb{F}_4$. The edges of color 1, 2 and 3 are represented by solid, dashed and dotted lines, respectively.

2.1 Particular Notation Used in our Proof

Before presenting the proof of Theorem 2, we need to introduce some notation and several lemmas. Recall that σ_A, σ_B and σ_C, the elements of $\mathbb{F}_3$, are given in Figure 1. For $i \in \{A, B, C\}$ and a triple $t \in \{1,2,3\}^3$, let F^i_t be the element of $\mathbb{F}_4^{\sigma_i}$ in which the unlabeled vertex of F^i_t is joined by an edge of color t_j to the image of the j-th vertex of σ_i for $j \in \{1,2,3\}$. Two elements of $\mathcal{A}^{\sigma_B}$ and two of $\mathcal{A}^{\sigma_C}$ will be of interest in our further considerations:

$$w_B := 165F^B_{113} + 165F^B_{333} - 279F^B_{123} - 44F^B_{131} + 328F^B_{133} + 10F^B_{233} + 421F^B_{323},$$
$$w'_B := -580F^B_{113} - 580F^B_{333} + 668F^B_{123} - 264F^B_{131} + 10F^B_{133} + 725F^B_{233} + 632F^B_{323},$$
$$w_C := 100F^C_{112} + 100F^C_{312} - 100F^C_{113} - 100F^C_{133} + 162F^C_{122} + 163F^C_{221}, \quad \text{and}$$
$$w'_C := -10F^C_{112} - 10F^C_{312} + 10F^C_{113} + 10F^C_{133} - 77F^C_{122} + 89F^C_{221}.$$

We make use of seven elements $\sigma_1, \ldots, \sigma_7$ out of the 15 elements of $\mathbb{F}_4$. They are depicted in Figure 2. For $i \in \{1, \ldots, 7\}$ and a quadruple $d \in \{1,2,3\}^4$, let F^i_d be the element of $\mathbb{F}_5^{\sigma_i}$ such that the unlabeled vertex of F^i_d is joined by an edge of color d_j to the j-th vertex of σ_i for $j \in \{1,2,3,4\}$. If $i \in \{1, \ldots, 7\}$ and $c \in \{1,2,3\}$, then $F^i_{(c)}$ is the element of $\mathcal{A}^{\sigma_i}$ that is the sum of all the five-vertex σ_i-flags F^i_d such that the unlabeled vertex is joined by an edge of color c to at least one of the vertices of σ_i, i.e., at least one of the entries of d is c.

Finally, we define $H_1, \ldots, H_{142}$ to be the elements of $\mathbb{F}_5$ in the way depicted in Appendix A.

3 Proof of Theorem 2

In this section, we prove Theorem 2 by contradiction: in a series of lemmas, we shall prove some properties of a counterexample which eventually allow us to establish the nonexistence of counterexamples. Specifically, we first find a number of flag inequalities

	i=1	i=2	i=3	i=4	i=5	i=6	i=7
$c = 1$	-1/3	0	-1/3	-1/3	0	0	0
$c = 2$	1/2	0	1/6	-1/3	-1/3	-1/3	0
$c = 3$	1/2	1/2	1/2	1/2	1/2	0	0

Table 1: The values $\varepsilon_c(\sigma_i)$ for $i \in \{1, \dots, 7\}$ and $c \in \{1, 2, 3\}$.

by hand and then we combine them with appropriate coefficients to obtain a contradiction. The coefficients are found with the help of a computer.

Let G be a tricolored complete graph. For a vertex v of G, let A_v be the set of colors of the edges incident with v. Consider a sequence of graphs $(G_k)_{k\in\mathbf{N}}$, obtained from G by replacing each vertex v of G with a complete graph of order k with edges colored uniformly at random with colors in A_v; the colors of the edges between the complete graphs corresponding to the vertices v and v' of G are assigned the color of the edge vv'. This sequence of graphs converges asymptotically almost surely; let q_G be the corresponding homomorphism from $\mathcal{A}$ to $\mathbf{R}$.

Let $n \geqslant 2$. We define a *counterexample* to be a tricolored graph with n vertices such that for every color $c \in \{1, 2, 3\}$, each set W of at most four vertices strongly c-dominates less than $2n/3$ vertices of G. A counterexample readily satisfies the following property.

Observation 3. *If G is a counterexample, then every vertex is incident with edges of at least two different colors.*

In the next lemma, we establish an inequality that q_G satisfies if G is a counterexample. To do so, define the quantity $\varepsilon_c(\sigma_i)$ for $i \in \{1, \dots, 7\}$ and $c \in \{1, 2, 3\}$ to be $1/2$ if σ_i contains a single edge with color c, $-1/3$ if each vertex of σ_i is incident with an edge colored c, $1/6$ if σ_i contains at least two edges with color c and a vertex incident with edges of a single color different from c, and 0, otherwise. These values are gathered in Table 1. Let us underline that, unlike in most of the previous applications of flag algebras, we do need to deal with second-order terms (specifically, $O(1/n)$ terms) in our flag inequalities to establish Theorem 2.

Lemma 4. *Let G be a counterexample with n vertices. For every $i \in \{1, \dots, 7\}$ and $c \in \{1, 2, 3\}$, a homomorphism $q_G^{\sigma_i}$ from $\mathcal{A}^{\sigma_i}$ to $\mathbf{R}$ almost surely satisfies the inequality*

$$q_G^{\sigma_i}(F^i_{(c)}) \leqslant \frac{2}{3} + \frac{\varepsilon_c(\sigma_i)}{n}.$$

Proof. Fix $i \in \{1, \ldots, 7\}$ and $c \in \{1, 2, 3\}$. Consider the graph G_k for sufficiently large k. Let (w_1, w_2, w_3, w_4) be a randomly selected quadruple of vertices of G_k inducing a subgraph isomorphic to σ_i. Further, let W be the set of vertices strongly c-dominated by $\{w_1, \ldots, w_4\}$. We show that $|W| \leqslant \frac{2nk}{3} + \varepsilon_c(\sigma_i)k + o(k)$ with probability tending to one as k tends to infinity. This will establish the inequality stated in the lemma. Indeed, it implies that for every $\eta > 0$, there exists k_η such that if $k > k_\eta$, then $q^{\sigma_i}_{G_k}(F^i_{(c)}) \leqslant \frac{2}{3} + \frac{\varepsilon_c(\sigma_i)}{n} + \eta$ with probability at least $1 - \eta$. As $q^{\sigma_i}_{G_k}(F^i_{(c)})$ tends to $q^{\sigma_i}_{G}(F^i_{(c)})$ as k tends to infinity, we obtain the stated inequality with probability 1.

For $i \in \{1, 2, 3, 4\}$, let v_i be the vertex of G corresponding to the clique W_i of G_k containing w_i. Let V be the set of vertices of G that are strongly c-dominated by $\{v_1, \ldots, v_4\}$. Since G is a counterexample, $|V| < 2n/3$, and hence, $|V| \leqslant 2n/3 - 1/3$. If w_j and $w_{j'}$ are joined by an edge of color c and, furthermore, $v_j = v_{j'}$, then v_j is added to V as well. Since V is still strongly c-dominated by a quadruple of vertices in G (replace $v_{j'}$ by any of its c-neighbors), it follows that $|V| \leqslant 2n/3 - 1/3$.

The set W can contain the $|V|\,k$ vertices of the cliques corresponding to the vertices in V, and, potentially, it also contains some additional vertices if w_i has no c-neighbors among $w_1, \ldots, w_4$. In this case, the additional vertices in W are the c-neighbors of w_i in W_i. With high probability, there are at most $k/3 + o(k)$ such vertices if v_i is incident with edges of all three colors in G, and at most $k/2 + o(k)$ if v_i is incident with edges of only two colors in G.

If $\varepsilon_c(\sigma_i) = -1/3$, then all the vertices $w_1, \ldots, w_4$ have a c-neighbor among $w_1, \ldots, w_4$ and thus W contains only vertices of the cliques corresponding to the vertices V. We conclude that $|W| \leqslant \frac{(2n-1)k}{3} + o(k)$, as required.

If $\varepsilon_c(\sigma_i) = 0$, then all but one of the vertices $w_1, \ldots, w_4$ have a c-neighbor among $w_1, \ldots, w_4$ and the vertex w_j that has none is incident in σ_i with edges of the two colors different from c. In particular, either w_j has no c-neighbors inside W_j or v_j is incident with edges of three distinct colors in G. This implies that $|W| \leqslant \frac{(2n-1)k}{3} + o(k)$ in the former case and $|W| \leqslant \frac{2nk}{3} + o(k)$ in the latter case. So, the bound holds.

If $\varepsilon_c(\sigma_i) = 1/6$, then all but one of the vertices among $w_1, \ldots, w_4$ have a c-neighbor among $w_1, \ldots, w_4$. Let w_j be the exceptional vertex. Since w_j has at most $k/2 + o(k)$ c-neighbors in W_j, it follows that $|W| \leqslant \frac{2nk}{3} + \frac{k}{6} + o(k)$.

Finally, if $\varepsilon_c(\sigma_i) = 1/2$, then two vertices w_j and $w_{j'}$ among $w_1, \ldots, w_4$ have no c-neighbors in $\{w_1, \ldots, w_4\}$. The vertices w_j and $w_{j'}$ have at most $k/2 + o(k)$ c-neighbors each in W_j and $W_{j'}$, respectively. Moreover, since σ_i contains edges of all three colors, one of w_j and $w_{j'}$ is incident in σ_i with edges of the two colors different from c. Hence, this vertex has at most $k/3 + o(k)$ c-neighbors in W_j. We conclude that the set W contains at most $|V|\,k + 5k/6 + o(k) \leqslant \frac{2nk}{3} + \frac{k}{2} + o(k)$ vertices. □

As a consequence of (1), we have the following corollary of Lemma 4.

Lemma 5. *Let G be a counterexample with n vertices. For every $i \in \{1, \ldots, 7\}$ and $c \in \{1, 2, 3\}$ such that $\varepsilon_c(\sigma_i) \leqslant 0$, it holds that*

$$q_G(\llbracket 2\sigma_i/3 - F^i_{(c)} \rrbracket_{\sigma_i}) \geqslant 0.$$

We now prove that in a counterexample, at most two colors are used to color the edges incident with any given vertex. As we shall see, this structural property of counterexamples directly implies their nonexistence, thereby proving Theorem 2.

Lemma 6. *No counterexample contains a vertex incident with edges of all three colors.*

Proof. Let G be a counterexample and $w_3 \in \mathbf{R}\mathbb{F}_5$ be the sum of all elements of $\mathbb{F}_5$ that contain a vertex incident with at least three colors. By the definition of q_G, the graph G has a vertex incident with edges of all three colors if and only if $q_G(w_3) > 0$. Lemma 5 implies that $q_G(H)$ is non-negative for each element H of $\mathcal{A}$ corresponding to any column of Table 2 (in Appendix B). In addition, (2) ensures that $q_G(H)$ is also non-negative for each element H of $\mathcal{A}$ corresponding to any of the first four columns of Table 3 (in Appendix B). Note that these elements can be expressed as elements of $\mathbf{R}\mathbb{F}_5$. Summing these columns with coefficients

$$\frac{23457815885978657985}{1029505785512512}, \frac{15852088219609163945}{514752892756256}, \frac{3956624143678293415}{772129339134384}, \frac{74313622711306287405}{2059011571025024}, \frac{15977347300925119}{32944185136400384},$$

$$\frac{134730108347752975}{4596007971038}, \frac{196791037567187109905}{12354069426150144}, \frac{30762195734543710715}{772129339134384}, \frac{48968798259015}{514752892756256}, \frac{8880723226482731}{24708138852300288},$$

$$\frac{134730108347752975}{4596007971038}, \frac{33245823856447882025}{24708138852300288}, \frac{20816545085118359705}{4118023142050048}, \frac{39315342699665}{6177034713075072},$$

respectively, yields an element w_0 of $\mathcal{A}$ given in the very last column of Table 3. Notice that for every $H \in \mathbb{F}_5$, the coefficient of H in $-w_0$ is at least the coefficient of H in w_3. In particular, the sum $w_3 + w_0$, which belongs to $\mathbf{R}\mathbb{F}_5$, has only non-positive coefficients. We now view both w_0 and w_3 as elements of $\mathcal{A}$ and use that q_G is a homomorphism from $\mathcal{A}$ to $\mathbf{R}$. First of all, $q_G(w_3 + w_0) \leqslant 0$. So, we derive that $q_G(w_3) \leqslant -q_G(w_0)$. As noted earlier, $q_G(H) \geqslant 0$ for each element H used to define w_0. Hence, since none of the above (displayed) coefficients is negative, we deduce that $q_G(w_0) \geqslant 0$. Consequently, $q_G(w_3) \leqslant 0$, which therefore implies that $q_G(w_3) = 0$. This means that G has no vertex incident with edges of all three colors. □

We are now in a position to prove Theorem 2, whose statement is recalled below.

Theorem 2. *Let $n \geqslant 2$. Every tricolored graph with n vertices contains a subset of at most four vertices that strongly c-dominates at least $2n/3$ vertices for some color c.*

Proof. Suppose, on the contrary, that there exists a counterexample G. Recall that A_v is the set of colors that appear on the edges incident to the vertex v. Now, by Observation 3 and Lemma 6, it holds that $|A_v| = 2$ for every vertex v of G. Hence, $V(G)$ can be partitioned into three sets V_1, V_2 and V_3, where $v \in V_i$ if and only if $i \notin A_v$. Without loss of generality, assume that $|V_1| \geqslant |V_2| \geqslant |V_3|$. Pick $u \in V_1$ and $v \in V_2$. As $A_u \cap A_w = \{3\}$ for all $w \in V_2$, we observe that V_2 is 3-dominated by $\{u\}$. Similarly, V_1 is 3-dominated by $\{v\}$. Therefore, the set $\{u, v\}$ strongly 3-dominates $V_1 \cup V_2$, which has size at least $2n/3$. □

4 Concluding remarks

It is natural to ask what bound can be proven for domination with three vertices. Here, it does not seem that the trick we used in this paper helps. We can prove only that every tricolored graph with n vertices contains a subset of at most three vertices that c-dominates at least $0.66117n$ vertices for some color c.

We believe the difficulty we face is caused by the following phenomenon. The average number of vertices dominated by a triple isomorphic to σ_A or σ_B (see Figure 1 for notation) is bounded away from $2/3$ in the graphs $(G_k)_{k\in\mathbf{N}}$, which are described at the beginning of Section 3, for G being the rainbow triangle. So, if any of these two configurations is used, a tight bound cannot be proven since the inequalities analogous to that in Lemma 5 are not tight and no triple of vertices dominates more than $2/3$ of the vertices in $(G_k)_{k\in\mathbf{N}}$ to compensate this deficiency.

We see that if we aimed to prove a tight result, we can only average over rainbow triangles (which are isomorphic to σ_C). Now consider the following graph G: start from the disjoint union of a large clique of order $2m$ with all edges colored 1 and a rainbow triangle. For $i \in \{1, 2\}$, join exactly m vertices of the clique to all three vertices of the rainbow triangle by edges colored i. The obtained simple complete graph has exactly one rainbow triangle, which dominates about half of the vertices. Thus, the average proportion of vertices dominated by triples isomorphic to σ_C in the graphs $(G_k)_{k\in\mathbf{N}}$ is close to $1/2$. This phenomenon does not occur for quadruples of vertices.

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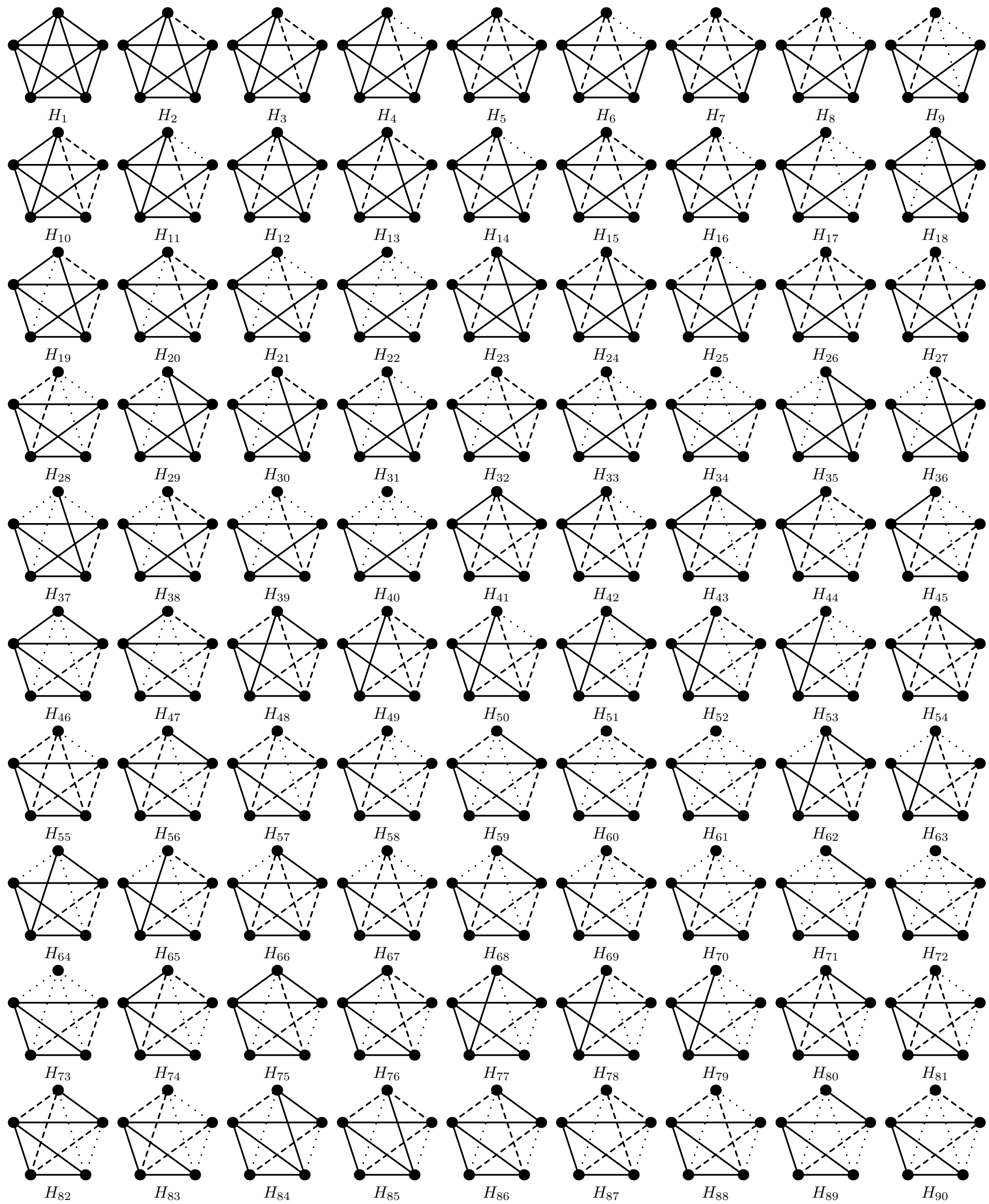
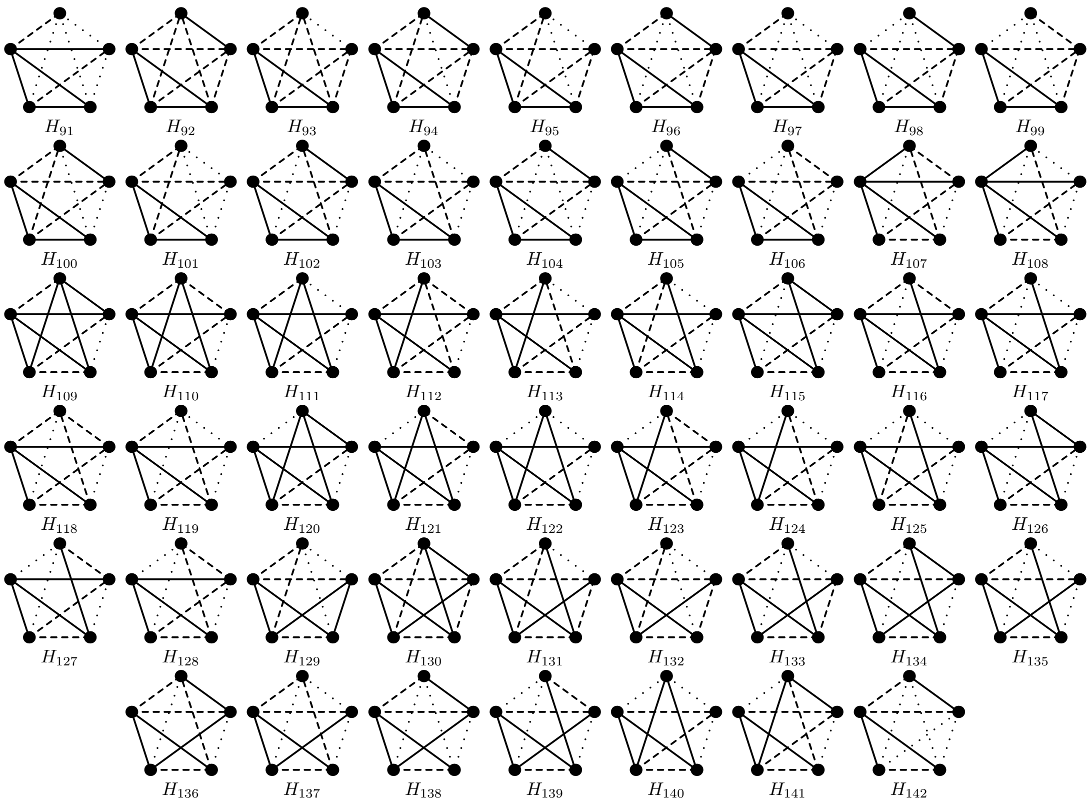
A The Elements of $\mathbb{F}_5$




H_{91} H_{92} H_{93} H_{94} H_{95} H_{96} H_{97} H_{98} H_{99}
H_{100} H_{101} H_{102} H_{103} H_{104} H_{105} H_{106} H_{107} H_{108}
H_{109} H_{110} H_{111} H_{112} H_{113} H_{114} H_{115} H_{116} H_{117}
H_{118} H_{119} H_{120} H_{121} H_{122} H_{123} H_{124} H_{125} H_{126}
H_{127} H_{128} H_{129} H_{130} H_{131} H_{132} H_{133} H_{134} H_{135}
H_{136} H_{137} H_{138} H_{139} H_{140} H_{141} H_{142}

B Vectors Used in the Proof of Lemma 6

Table 2: The first ten vectors

	$[\![2\sigma_1/3 - F^1_{(1)}]\!]_{\sigma_1}$	$[\![2\sigma_2/3 - F^2_{(1)}]\!]_{\sigma_2}$	$[\![2\sigma_2/3 - F^2_{(2)}]\!]_{\sigma_2}$	$[\![2\sigma_3/3 - F^3_{(1)}]\!]_{\sigma_3}$	$[\![2\sigma_4/3 - F^4_{(1)}]\!]_{\sigma_4}$	$[\![2\sigma_4/3 - F^4_{(2)}]\!]_{\sigma_4}$	$[\![2\sigma_5/3 - F^5_{(1)}]\!]_{\sigma_5}$	$[\![2\sigma_5/3 - F^5_{(2)}]\!]_{\sigma_5}$	$[\![2\sigma_6/3 - F^6_{(1)}]\!]_{\sigma_6}$	$[\![2\sigma_7/3 - F^7_{(2)}]\!]_{\sigma_7}$
H_1	0	0	0	0	0	0	0	0	0	0
H_2	0	0	0	0	0	0	0	0	0	0
H_3	0	0	0	0	0	0	0	0	0	0
H_4	-1/90	0	0	0	0	0	0	0	0	0
H_5	0	0	0	0	0	0	0	0	0	0
H_6	-1/90	-1/180	1/90	0	0	0	0	0	0	0
H_7	0	0	0	0	0	0	0	0	0	0
H_8	0	-1/60	-1/60	0	0	0	0	0	0	0
H_9	0	-1/45	2/45	0	0	0	0	0	0	0
H_{10}	0	0	0	0	0	0	0	0	0	0
H_{11}	0	0	0	-1/90	0	0	0	0	0	0
H_{12}	0	0	0	0	0	0	0	0	0	0
H_{13}	0	0	0	0	0	0	0	0	0	0
H_{14}	-1/90	0	0	0	-1/180	1/90	0	0	0	0
H_{15}	0	0	0	0	0	0	0	0	0	0
H_{16}	-1/180	0	0	-1/180	0	0	-1/360	1/180	0	0
H_{17}	-1/90	0	0	-1/180	0	0	0	0	1/180	0
H_{18}	0	0	0	0	0	0	0	0	0	0
H_{19}	-1/180	0	0	0	0	0	1/180	-1/360	0	0
H_{20}	-1/90	1/90	-1/180	0	0	0	0	0	0	0
H_{21}	-1/180	0	0	-1/180	0	0	0	0	0	1/120
H_{22}	0	0	0	1/180	0	0	0	0	0	0
H_{23}	0	0	0	0	0	0	0	0	0	0
H_{24}	0	0	0	0	0	0	0	0	0	0
H_{25}	0	-1/180	-1/180	0	-1/90	-1/90	0	0	0	0
H_{26}	0	0	0	0	0	0	0	0	0	0
H_{27}	0	-1/180	-1/180	0	0	0	-1/180	-1/180	0	0
H_{28}	0	-1/90	-1/90	0	0	0	0	0	1/90	0

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Table 2 – *Continued from previous page*

	$[\![2\sigma_1/3 - F^1_{(1)}]\!]_{\sigma_1}$	$[\![2\sigma_2/3 - F^2_{(1)}]\!]_{\sigma_2}$	$[\![2\sigma_2/3 - F^2_{(2)}]\!]_{\sigma_2}$	$[\![2\sigma_3/3 - F^3_{(1)}]\!]_{\sigma_3}$	$[\![2\sigma_4/3 - F^4_{(1)}]\!]_{\sigma_4}$	$[\![2\sigma_4/3 - F^4_{(2)}]\!]_{\sigma_4}$	$[\![2\sigma_5/3 - F^5_{(1)}]\!]_{\sigma_5}$	$[\![2\sigma_5/3 - F^5_{(2)}]\!]_{\sigma_5}$	$[\![2\sigma_6/3 - F^6_{(1)}]\!]_{\sigma_6}$	$[\![2\sigma_7/3 - F^7_{(2)}]\!]_{\sigma_7}$
H_{29}	-1/90	0	0	0	0	0	0	0	0	0
H_{30}	-1/180	-1/180	-1/180	0	0	0	-1/360	-1/360	0	0
H_{31}	-1/180	-1/180	1/90	0	-1/180	1/90	1/180	-1/360	0	0
H_{32}	0	-1/60	-1/60	0	0	0	0	0	0	0
H_{33}	0	-1/90	1/180	0	0	0	-1/360	1/180	0	0
H_{34}	0	-1/90	-1/90	1/90	0	0	0	0	-1/90	0
H_{35}	0	0	0	0	0	0	0	0	0	0
H_{36}	0	-1/180	1/90	0	0	0	1/90	-1/180	0	0
H_{37}	0	0	0	0	0	0	-1/180	-1/180	0	0
H_{38}	0	1/90	1/90	0	0	0	0	0	0	0
H_{39}	0	-1/180	-1/180	0	0	0	0	0	0	0
H_{40}	0	0	0	-1/90	0	0	0	0	0	0
H_{41}	0	0	0	0	0	0	0	0	0	0
H_{42}	0	0	0	-1/90	0	0	0	0	0	0
H_{43}	-1/90	0	0	0	1/90	-1/180	0	0	0	0
H_{44}	1/180	0	0	-1/180	0	0	0	0	0	0
H_{45}	-1/180	0	0	-1/90	0	0	1/180	1/180	0	0
H_{46}	-1/90	0	0	0	0	0	0	0	1/45	0
H_{47}	0	0	0	0	0	0	0	0	0	0
H_{48}	0	0	0	0	0	0	0	0	0	0
H_{49}	0	0	0	0	0	0	0	0	0	0
H_{50}	0	0	0	0	-1/180	-1/180	-1/180	-1/180	0	0
H_{51}	-1/180	0	0	0	-1/180	-1/180	-1/360	-1/360	0	0
H_{52}	-1/180	0	0	-1/180	0	0	-1/360	-1/360	0	0
H_{53}	-1/180	0	0	0	-1/180	-1/180	0	0	1/180	0
H_{54}	0	0	0	0	0	0	0	0	0	0
H_{55}	0	0	0	0	0	0	-1/180	-1/180	0	0
H_{56}	0	-1/180	-1/180	0	-1/90	-1/90	0	0	0	0
H_{57}	-1/180	-1/180	-1/180	0	0	0	-1/360	-1/360	0	0
H_{58}	0	-1/180	-1/180	0	0	0	1/360	-1/180	1/180	0
H_{59}	0	-1/180	1/90	0	-1/90	-1/90	0	0	1/180	0
H_{60}	0	-1/180	1/90	-1/180	0	0	-1/180	-1/180	0	0

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Table 2 – *Continued from previous page*

	$[\![2\sigma_1/3 - F^1_{(1)}]\!]_{\sigma_1}$	$[\![2\sigma_2/3 - F^2_{(1)}]\!]_{\sigma_2}$	$[\![2\sigma_2/3 - F^2_{(2)}]\!]_{\sigma_2}$	$[\![2\sigma_3/3 - F^3_{(1)}]\!]_{\sigma_3}$	$[\![2\sigma_4/3 - F^4_{(1)}]\!]_{\sigma_4}$	$[\![2\sigma_4/3 - F^4_{(2)}]\!]_{\sigma_4}$	$[\![2\sigma_5/3 - F^5_{(1)}]\!]_{\sigma_5}$	$[\![2\sigma_5/3 - F^5_{(2)}]\!]_{\sigma_5}$	$[\![2\sigma_6/3 - F^6_{(1)}]\!]_{\sigma_6}$	$[\![2\sigma_7/3 - F^7_{(2)}]\!]_{\sigma_7}$
H_{61}	0	-1/180	-1/180	1/90	0	0	0	0	-1/45	0
H_{62}	-1/180	0	0	0	0	0	-1/360	-1/360	0	0
H_{63}	-1/180	0	0	0	0	0	1/360	1/360	0	0
H_{64}	0	0	0	0	0	0	1/90	-1/180	0	0
H_{65}	0	0	0	1/90	0	0	-1/360	-1/360	0	0
H_{66}	0	-1/180	-1/180	0	0	0	-1/180	-1/180	0	0
H_{67}	0	-1/180	-1/180	0	0	0	0	0	0	0
H_{68}	0	-1/180	1/90	0	1/90	-1/180	1/360	-1/180	0	0
H_{69}	1/90	-1/90	1/180	0	0	0	0	0	0	0
H_{70}	0	-1/180	-1/180	1/90	0	0	-1/360	1/180	0	-1/120
H_{71}	0	0	0	0	0	0	-1/180	-1/180	1/180	0
H_{72}	0	0	0	1/90	0	0	0	0	0	-1/60
H_{73}	0	0	0	-1/60	0	0	0	0	0	0
H_{74}	-1/90	1/90	1/90	-1/180	0	0	0	0	0	0
H_{75}	-1/45	0	0	0	0	0	0	0	0	0
H_{76}	-1/90	0	0	-1/90	1/90	1/90	0	0	0	0
H_{77}	-1/90	0	0	0	0	0	1/90	-1/180	0	0
H_{78}	-1/90	0	0	0	0	0	0	0	0	0
H_{79}	-1/90	1/45	-1/90	0	-1/180	1/90	0	0	0	0
H_{80}	-1/180	0	0	0	-1/180	-1/180	-1/360	-1/360	0	0
H_{81}	1/90	0	0	0	0	0	1/360	-1/180	0	0
H_{82}	-1/180	-1/180	-1/180	0	0	0	1/180	-1/360	1/180	0
H_{83}	1/90	1/60	-1/60	0	0	0	-1/360	1/180	0	0
H_{84}	-1/90	0	0	0	-1/90	1/45	0	0	0	0
H_{85}	-1/90	0	0	0	0	0	-1/180	1/360	-1/90	0
H_{86}	-1/180	-1/180	-1/180	0	-1/180	-1/180	-1/360	-1/360	0	0
H_{87}	1/90	-1/180	-1/180	-1/180	-1/180	1/90	-1/360	-1/360	0	0
H_{88}	1/90	-1/180	-1/180	0	1/90	-1/180	0	0	-1/90	-1/120
H_{89}	-1/180	-1/180	-1/180	0	1/180	1/180	-1/360	-1/360	0	0
H_{90}	1/90	-1/180	-1/180	0	0	0	-1/180	1/90	0	-1/120
H_{91}	1/45	-1/90	-1/90	0	0	0	0	0	-1/90	0
H_{92}	0	0	0	0	0	0	0	0	0	0

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Table 2 – *Continued from previous page*

	$[\![2\sigma_1/3-F^1_{(1)}]\!]_{\sigma_1}$	$[\![2\sigma_2/3-F^2_{(1)}]\!]_{\sigma_2}$	$[\![2\sigma_2/3-F^2_{(2)}]\!]_{\sigma_2}$	$[\![2\sigma_3/3-F^3_{(1)}]\!]_{\sigma_3}$	$[\![2\sigma_4/3-F^4_{(1)}]\!]_{\sigma_4}$	$[\![2\sigma_4/3-F^4_{(2)}]\!]_{\sigma_4}$	$[\![2\sigma_5/3-F^5_{(1)}]\!]_{\sigma_5}$	$[\![2\sigma_5/3-F^5_{(2)}]\!]_{\sigma_5}$	$[\![2\sigma_6/3-F^6_{(1)}]\!]_{\sigma_6}$	$[\![2\sigma_7/3-F^7_{(2)}]\!]_{\sigma_7}$
H_{93}	0	0	0	0	0	0	0	0	0	0
H_{94}	0	-1/180	-1/180	0	-1/90	-1/90	0	0	0	0
H_{95}	0	1/90	-1/180	0	0	0	1/90	-1/180	0	0
H_{96}	0	-1/180	1/90	0	-1/90	-1/90	0	0	1/180	0
H_{97}	0	1/90	-1/180	1/90	0	0	-1/180	-1/180	0	0
H_{98}	0	0	0	0	0	0	0	0	-1/30	0
H_{99}	0	0	0	-1/60	0	0	0	0	0	0
H_{100}	0	-1/90	-1/90	0	0	0	0	0	1/90	0
H_{101}	0	2/45	-1/45	0	0	0	0	0	0	0
H_{102}	0	-1/90	-1/90	0	-1/90	-1/90	0	0	0	0
H_{103}	0	1/45	-1/90	0	1/90	-1/180	-1/180	-1/180	0	0
H_{104}	0	-1/90	-1/90	0	1/45	-1/90	0	0	-1/90	0
H_{105}	0	-1/90	-1/90	0	0	0	0	0	-1/90	0
H_{106}	0	1/45	-1/90	-1/180	-1/90	-1/90	0	0	0	0
H_{107}	0	0	0	-1/45	0	0	0	0	0	0
H_{108}	0	0	0	-1/45	0	0	0	0	0	0
H_{109}	0	0	0	-1/90	0	0	0	0	0	0
H_{110}	0	0	0	-1/180	0	0	-1/180	-1/180	0	0
H_{111}	0	0	0	-1/180	-1/180	-1/180	1/180	-1/360	0	0
H_{112}	0	0	0	-1/90	0	0	0	0	0	0
H_{113}	0	0	0	-1/180	-1/180	-1/180	1/180	-1/360	0	0
H_{114}	0	0	0	-1/90	0	0	1/360	-1/180	0	0
H_{115}	0	0	0	-1/180	-1/90	-1/90	0	0	-1/90	0
H_{116}	0	0	0	-1/180	-1/90	1/180	-1/180	-1/180	0	0
H_{117}	0	0	0	-1/180	-1/180	-1/180	-1/180	1/360	-1/90	0
H_{118}	0	0	0	1/180	0	0	-1/180	-1/180	0	0
H_{119}	0	0	0	1/90	0	0	-1/360	-1/360	-1/90	-1/120
H_{120}	0	0	0	-1/90	0	0	0	0	0	0
H_{121}	0	0	0	-1/180	-1/180	1/90	-1/180	1/360	0	0
H_{122}	0	0	0	-1/180	0	0	-1/360	-1/360	0	0
H_{123}	0	0	0	-1/180	0	0	-1/180	-1/180	-1/90	0
H_{124}	0	0	0	-1/180	0	0	-1/90	1/180	0	0

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Table 2 – *Continued from previous page*

	$\llbracket 2\sigma_1/3 - F^1_{(1)} \rrbracket_{\sigma_1}$	$\llbracket 2\sigma_2/3 - F^2_{(1)} \rrbracket_{\sigma_2}$	$\llbracket 2\sigma_2/3 - F^2_{(2)} \rrbracket_{\sigma_2}$	$\llbracket 2\sigma_3/3 - F^3_{(1)} \rrbracket_{\sigma_3}$	$\llbracket 2\sigma_4/3 - F^4_{(1)} \rrbracket_{\sigma_4}$	$\llbracket 2\sigma_4/3 - F^4_{(2)} \rrbracket_{\sigma_4}$	$\llbracket 2\sigma_5/3 - F^5_{(1)} \rrbracket_{\sigma_5}$	$\llbracket 2\sigma_5/3 - F^5_{(2)} \rrbracket_{\sigma_5}$	$\llbracket 2\sigma_6/3 - F^6_{(1)} \rrbracket_{\sigma_6}$	$\llbracket 2\sigma_7/3 - F^7_{(2)} \rrbracket_{\sigma_7}$
H_{125}	0	0	0	1/180	0	0	-1/360	-1/360	0	-1/120
H_{126}	0	0	0	-1/180	0	0	-1/180	-1/180	1/180	0
H_{127}	0	0	0	-1/180	1/180	-1/90	-1/360	-1/360	0	-1/120
H_{128}	0	0	0	1/90	0	0	0	0	-1/90	-1/60
H_{129}	0	0	0	0	0	0	-1/180	-1/180	0	0
H_{130}	0	0	0	0	0	0	0	0	0	0
H_{131}	0	0	0	0	-1/180	-1/180	-1/180	-1/180	0	0
H_{132}	0	0	0	0	0	0	1/180	-1/90	0	0
H_{133}	0	0	0	0	-1/180	-1/180	-1/360	-1/360	1/180	0
H_{134}	0	0	0	0	-1/60	0	-1/180	-1/180	0	0
H_{135}	0	0	0	0	0	-1/60	0	0	0	-1/60
H_{136}	0	0	0	0	-1/90	-1/90	-1/180	-1/180	-1/90	0
H_{137}	0	0	0	0	1/90	-1/180	-1/120	0	0	-1/120
H_{138}	0	0	0	0	-1/180	-1/180	-1/180	1/360	-1/45	0
H_{139}	0	0	0	0	-1/180	-1/180	-1/360	-1/360	-1/90	-1/120
H_{140}	0	0	0	0	0	0	-1/180	1/90	0	-1/60
H_{141}	0	0	0	0	0	0	-1/90	-1/90	0	0
H_{142}	0	0	0	0	0	0	0	0	-2/45	0

Table 3: The last six vectors

	$[\![w_B \cdot w_B]\!]_{\sigma_B}$	$[\![w'_B \cdot w'_B]\!]_{\sigma_B}$	$[\![w_C \cdot w_C]\!]_{\sigma_C}$	$[\![w'_C \cdot w'_C]\!]_{\sigma_C}$	w_3	w_0
H_1	0	0	0	0	0	0
H_2	0	0	0	0	0	0
H_3	0	0	0	0	0	0
H_4	0	0	0	0	1	$\frac{-1563854392398577199}{6177034713075072}$
H_5	0	0	0	0	0	0
H_6	29161/60	101524/15	0	0	1	-1
H_7	0	0	0	0	0	0
H_8	0	0	2000	20	0	0
H_9	0	0	-4000/3	-40/3	0	0
H_{10}	0	0	0	0	0	0
H_{11}	1815/2	33640/3	0	0	1	$\frac{-10173977739002723}{55152095652456}$
H_{12}	0	0	0	0	0	0
H_{13}	0	0	0	0	0	0
H_{14}	-242	5104	0	0	1	$\frac{-734882450141728337}{2316388017403152}$
H_{15}	0	0	0	0	0	0
H_{16}	-9922/15	-422/3	0	0	1	$\frac{-5722046702587908817}{37062208278450432}$
H_{17}	57013/60	-65634/5	0	0	1	$\frac{-57717650068438077139}{148248833113801728}$
H_{18}	0	0	0	0	0	0
H_{19}	0	0	0	0	1	$\frac{-703462682135213465}{3369291661677312}$
H_{20}	29161/60	101524/15	0	0	1	-1
H_{21}	781/2	-37700/3	0	0	1	$\frac{-5891700664190917297}{148248833113801728}$
H_{22}	-1804	-580/3	0	0	1	$\frac{-326148889774443071}{18531104139225216}$
H_{23}	0	0	0	0	0	0
H_{24}	0	0	0	0	0	0
H_{25}	0	0	19723/20	2047/20	1	$\frac{-15461491234942018543}{5929953324552069120}$
H_{26}	0	0	0	0	0	0
H_{27}	0	0	540	77/3	1	$\frac{-88140807390257339}{289548502175394}$
H_{28}	10/3	105125/6	0	0	1	$\frac{-35834405989042100849}{74124416556900864}$
H_{29}	0	0	0	0	1	$\frac{-1563854392398577199}{6177034713075072}$
H_{30}	0	0	270	77/6	1	$\frac{-5456161234717178191}{12354069426150144}$
H_{31}	0	0	0	0	1	$\frac{-4427934211353668633}{37062208278450432}$
H_{32}	0	0	2000	20	0	0
H_{33}	0	0	-1810/3	-97/6	1	$\frac{-1570031427111652271}{6177034713075072}$
H_{34}	-328/3	725/3	2000/3	20/3	1	$\frac{-327323049775204219}{6738583323354624}$
H_{35}	0	0	0	0	0	0

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Table 3 – *Continued from previous page*

	$[\![w_B \cdot w_B]\!]_{\sigma_B}$	$[\![w'_B \cdot w'_B]\!]_{\sigma_B}$	$[\![w_C \cdot w_C]\!]_{\sigma_C}$	$[\![w'_C \cdot w'_C]\!]_{\sigma_C}$	w_3	w_0
H_{36}	0	0	0	0	1	$\frac{-2040849950139277}{1323650295658944}$
H_{37}	0	0	2187/5	5929/60	1	$\frac{-324486989357699}{150414806324880}$
H_{38}	0	0	-4000/3	-40/3	0	0
H_{39}	177241/60	99856/15	0	0	1	$\frac{-14389006173379021}{6177034713075072}$
H_{40}	53792/15	10/3	0	0	1	-1
H_{41}	0	0	0	0	0	0
H_{42}	53792/15	10/3	0	0	1	-1
H_{43}	-242	5104	0	0	1	$\frac{-244418926250621773 1}{32944185136400384}$
H_{44}	-19723/60	8018	0	0	1	$\frac{-13098050481821622 5}{5294601182635776}$
H_{45}	19251/20	-46426/5	0	0	1	$\frac{-7679414109492555 31}{4118023142050048}$
H_{46}	0	0	0	0	1	$\frac{-521981733736779125 3}{37062208278450432}$
H_{47}	-4743/20	-27388/5	0	0	1	$\frac{-3547091272571898 91}{6177034713075072}$
H_{48}	0	0	0	0	0	0
H_{49}	0	0	0	0	0	0
H_{50}	0	0	0	0	1	$\frac{-1025220090062617489 33}{296497666227603456}$
H_{51}	0	0	4401/10	-6853/60	1	$\frac{-1038167017674141157 97}{592995332455206912}$
H_{52}	1331/4	24476/3	0	0	1	$\frac{-179483076061126408 7}{5294601182635776}$
H_{53}	-7157/30	72838/15	0	0	1	$\frac{-552264157000706688 35}{296497666227603456}$
H_{54}	0	0	0	0	0	0
H_{55}	0	0	2187/5	5929/60	1	$\frac{-324486989357699}{150414806324880}$
H_{56}	0	0	1630/3	-89/3	1	$\frac{-1487068889448541 03}{561548610279552}$
H_{57}	0	0	270	77/6	1	$\frac{-545616123471717819 1}{12354069426150144}$
H_{58}	0	0	0	0	1	$\frac{-748267710550291959 07}{148248833113801728}$
H_{59}	0	0	0	0	1	-1
H_{60}	0	0	-540	-77/3	1	$\frac{-1273469138371545 13}{240663690119808}$
H_{61}	-93	48430/3	0	0	1	$\frac{-1046673264968855 5}{5294601182635776}$
H_{62}	0	0	0	0	1	$\frac{-932073916095866548 1}{37062208278450432}$
H_{63}	0	0	0	0	1	$\frac{-6238719343279771 3}{37062208278450432}$
H_{64}	0	0	0	0	1	$\frac{-2176090233065440 37}{1323650295658944}$
H_{65}	-34522/15	316/3	0	0	1	-1
H_{66}	0	0	540	77/3	1	$\frac{-881408073902573 39}{289548502175394}$
H_{67}	177241/60	99856/15	0	0	1	$\frac{-14389006173379021}{6177034713075072}$
H_{68}	0	0	-815/3	89/6	1	$\frac{-4262141302871120 5}{37062208278450432}$
H_{69}	4631/4	-18328/3	-1000/3	-10/3	1	$\frac{-1330374174754201}{1029505785512512}$
H_{70}	-39153/20	105544/15	-270	-77/6	1	$\frac{-583464589838574219 5}{16472092568200192}$
H_{71}	0	0	0	0	1	$\frac{-365223320589775545 9}{16472092568200192}$

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Table 3 – *Continued from previous page*

	$[\![w_B \cdot w_B]\!]_{\sigma_B}$	$[\![w'_B \cdot w'_B]\!]_{\sigma_B}$	$[\![w_C \cdot w_C]\!]_{\sigma_C}$	$[\![w'_C \cdot w'_C]\!]_{\sigma_C}$	w_3	w_0
H_{72}	-39153/10	211088/15	0	0	1	$\frac{-40194399986166687563}{741244165569008864}$
H_{73}	17391/4	114896/15	0	0	1	$\frac{-6737646243561340l}{1323650295658944}$
H_{74}	-3069/2	-38744/3	0	0	1	-1
H_{75}	-968	20416	0	0	1	$\frac{-206704879201250857}{441216765219648}$
H_{76}	-13706/15	-92396/15	0	0	1	$\frac{-87225018889329233387}{16472092568200192}$
H_{77}	0	0	0	0	1	$\frac{-70346268213521346 5}{1684645830838656}$
H_{78}	4631/2	-36656/3	0	0	1	$\frac{-205076529391967946 7}{18531104139225216}$
H_{79}	0	0	0	0	1	-1
H_{80}	0	0	0	0	1	$\frac{-34340368851241376879}{988325554092011 52}$
H_{81}	-4631/15	-13904/5	0	0	1	-1
H_{82}	0	0	0	0	1	$\frac{-10725188546965769537}{21178404730543104}$
H_{83}	0	0	-2810/3	-39/2	1	-1
H_{84}	0	0	0	0	1	$\frac{-7417316739041385395}{18531104139225216}$
H_{85}	121/6	-30595/3	0	0	1	$\frac{-21505715322664188433}{741244165569008864}$
H_{86}	0	0	815/3	-89/6	1	$\frac{-1005157507446338 5}{18384031884152}$
H_{87}	1331/4	24476/3	270	77/6	1	$\frac{-2065655974432544177}{9265552069612608}$
H_{88}	-8657/30	-194687/15	-815/3	89/6	1	$\frac{-10513889487465286471}{21178404730543104}$
H_{89}	0	0	270	77/6	1	$\frac{-102492676367157469795}{296497666227603456}$
H_{90}	4631/4	-18328/3	-270	-77/6	1	$\frac{-3470421686575164043}{148248833113801728}$
H_{91}	121/3	-61190/3	2000/3	20/3	1	$\frac{-1253905713964428 5}{8236046284100096}$
H_{92}	0	0	0	0	0	0
H_{93}	0	0	0	0	0	0
H_{94}	0	0	19723/20	2047/20	1	$\frac{-15461491234942018543}{5929953324552069120}$
H_{95}	0	0	0	0	1	$\frac{-2040849950139277}{1323650295658944}$
H_{96}	0	0	-1630/3	89/3	1	$\frac{-37632094249561791565}{148248833113801728}$
H_{97}	0	0	-540	-77/3	1	$\frac{-1472297168476995 67}{9265552069612608}$
H_{98}	0	0	0	0	1	$\frac{-4163309017023671941}{24708138852300288}$
H_{99}	77841/20	111556/5	0	0	1	-1
H_{100}	10/3	105125/6	0	0	1	$\frac{-35834405989042100849}{741244165569008864}$
H_{101}	0	0	-4000/3	-40/3	0	0
H_{102}	0	0	1630/3	-89/3	1	$\frac{-10943236189159518679}{18531104139225216}$
H_{103}	0	0	-1630/3	89/3	1	$\frac{-244304794290394685}{329441851364003 84}$
H_{104}	0	0	-1630/3	89/3	1	$\frac{-6580270239524616359}{105892023652715 52}$
H_{105}	0	0	0	0	1	$\frac{-17483526616286112727}{24708138852300288}$
H_{106}	0	0	-1630/3	89/3	1	$\frac{-2389728277891266261}{8236046284100096}$
H_{107}	107584/15	20/3	0	0	1	-2

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Table 3 – *Continued from previous page*

	$[\![w_B \cdot w_B]\!]_{\sigma_B}$	$[\![w'_B \cdot w'_B]\!]_{\sigma_B}$	$[\![w_C \cdot w_C]\!]_{\sigma_C}$	$[\![w'_C \cdot w'_C]\!]_{\sigma_C}$	w_3	w_0
H_{108}	30504/5	1336/3	0	0	1	$\frac{-66935245670393753}{661825147829472}$
H_{109}	1815/2	33640/3	0	0	1	$\frac{-10173977739002723}{55152095652456}$
H_{110}	0	0	0	0	1	$\frac{-866621514187196297}{2059011571025024}$
H_{111}	4631/4	-18328/3	0	0	1	$\frac{-7493427555720786047}{26954333293418496}$
H_{112}	-1804	-580/3	0	0	1	$\frac{-4771933910371470719}{9265552069612608}$
H_{113}	-34522/15	316/3	0	0	1	$\frac{-56090180193586615063}{98832555409201152}$
H_{114}	-3069/4	-19372/3	0	0	1	$\frac{-6146435219180552237}{9265552069612608}$
H_{115}	55	-42050/3	26569/60	7921/60	1	$\frac{-476307820942045182061}{1976651108184023040}$
H_{116}	0	0	0	0	1	$\frac{-19450524422641811549}{32944185136400384}$
H_{117}	-93/2	24215/3	0	0	1	$\frac{-8220306420511019599}{42356809461086208}$
H_{118}	-1804	-580/3	2187/5	5929/60	1	$\frac{-3629584303315557939}{92655520696126080}$
H_{119}	-70439/30	8177	0	0	1	$\frac{-15367554150816847711}{49416277704600576}$
H_{120}	1815/2	33640/3	0	0	1	$\frac{-10173977739002723}{55152095652456}$
H_{121}	0	0	0	0	1	$\frac{-1003343753899617143}{6177034713075072}$
H_{122}	4631/4	-18328/3	0	0	1	$\frac{-2082303347082636833}{9265552069612608}$
H_{123}	-328/3	725/3	0	0	1	$\frac{-12006211264574547431}{24708138852300288}$
H_{124}	0	0	0	0	1	$\frac{-137657019589129 19}{2059011571025024}$
H_{125}	-27249/10	8684/15	0	0	1	$\frac{-75624575885139732659}{148248833113801728}$
H_{126}	55	-42050/3	0	0	1	$\frac{-70683455524198969843}{148248833113801728}$
H_{127}	0	0	0	0	1	$\frac{-11080743495118222157}{21178404730543104}$
H_{128}	170681/60	103481/15	0	0	1	-1
H_{129}	0	0	0	0	1	$\frac{-11572939959407334 71}{4632776034806304}$
H_{130}	0	0	0	0	0	0
H_{131}	0	0	4401/5	-6853/30	1	-1
H_{132}	0	0	0	0	1	$\frac{-4264279061141416 89}{1029505785512512}$
H_{133}	421/6	22910/3	0	0	1	$\frac{-12354978221722842 83}{8984777764472832}$
H_{134}	0	0	4401/5	-6853/30	1	$\frac{-5617524380783071181}{32944185136400384}$
H_{135}	0	0	0	0	1	$\frac{-26428837039774952311}{42356809461086208}$
H_{136}	0	0	0	0	1	$\frac{-73815219170205621743}{148248833113801728}$
H_{137}	0	0	0	0	1	$\frac{-51576322752518046641}{296497666227603456}$
H_{138}	0	0	0	0	1	$\frac{-37389454791911250173}{296497666227603456}$
H_{139}	421/6	22910/3	0	0	1	$\frac{-17214181054154995319}{32944185136400384}$
H_{140}	0	0	0	0	1	$\frac{-1542818265173952315}{8236046284100096}$
H_{141}	0	0	0	0	1	$\frac{-1157293995940733471}{2316388017403152}$
H_{142}	20/3	105125/3	0	0	1	-1