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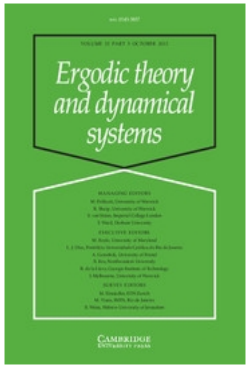

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Cyclicity in families of circle maps

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Abstract. In this paper we will study families of circle maps of the form $x \mapsto x + 2\pi r + af(x) (\text{mod } 2\pi)$ and investigate how many periodic trajectories maps from this family can have for a 'typical' function f provided the parameter a is small.

1. *Introduction*

The Hilbert–Arnold problem ([**AI85**]; see also the discussion in [**Ily02**]) asks if a generic family of planar vector fields has a uniformly bounded number of isolated limit cycles. Originally this problem was formulated just for planar vector fields (or vector fields on the two-dimensional sphere); however, the same question can be posed for vector fields on other manifolds or even for families of a maps from some manifold to itself. In this paper we consider a case of specific families of diffeomorphisms of the circle.

Consider a family of circle diffeomorphisms of this form:

$$F_{r,a} : x \mapsto x + 2\pi r + af(x) \quad (\text{mod } 2\pi),$$

where f is some periodic function, $r \in \mathbb{R}$, $a \in [-a_0, a_0]$. For a given 2π-periodic function f this family will be called the *corresponding family*.

There have been various studies of such families for different choices of the function f. If $f = \sin$, F is usually called Arnold's family.

For these families we will study periodic trajectories which originate from periodic points of the rigid rotation and investigate whether for a 'typical' family F there is a bound on the number of periodic trajectories which are born when a is small. In the original Hilbert–Arnold problem 'typical' means Baire generic. Here we will consider several other notions of typicality and we will see that the answer might depend on what notion of typicality is used.

We will see that in many cases a 'typical' family will have infinite cyclicity: this means that for any $N \in \mathbb{N}$ there exist parameter values r and a such that the map $F_{r,a}$ has more than N attracting periodic trajectories. On the other hand, families which have finite cyclicity certainly exist: if f is a trigonometric polynomial of degree d, then the number of attracting periodic trajectories of the map $F_{r,a}$ is bounded by d; see [**YAK85**].

All the questions we pose here for the specific family $F_{r,a}$ can be asked for general families of circle maps. Interestingly enough, a generic non-trivial family of C^k circle diffeomorphisms has infinite cyclicity. This follows from Herman's theorem, though to the best of my knowledge it cannot be found in the literature. It seems that such a statement holds only for families of diffeomorphisms. I conjecture that a generic family of critical circle maps (that is, maps which have points where the derivative of the map is zero) has finite cyclicity. The critical points in such families do not create significant problems for the proof of this conjecture because periodic attractors near critical points can be controlled by the negative Schwarzian derivative condition (see [**KOZ00**]). On the other hand, periodic attractors of high period born in a perturbation of a neutral fixed point are hard to analyse.

2. *Statements of results*

For a diffeomorphism g of a circle let $N(g)$ denote the number of attracting periodic trajectories of the map g and let $\rho(g)$ denote the rotation number of g. For the family $F_{r,a} : x \mapsto x + 2\pi r + af(x) \pmod{2\pi}$ of circle maps define

$$B(F, \rho_0) = \limsup_{a_1 \to 0+} \sup_{\rho(F_{r,a})=\rho_0, |a|<a_1} N(F_{r,a}).$$

So, B quantifies how many periodic attractors of a given rotation number are born when we perturb the rigid rotation.

Definition 1. The family of circle maps $F_{r,a} : x \mapsto x + 2\pi r + af(x) \pmod{2\pi}$ is said to have finite infinitesimal cyclicity if $B(F, \rho)$ is uniformly bounded for all ρ. Otherwise, F has infinite infinitesimal cyclicity.

We will study cyclicity of families F in different functional spaces and use the following notation. Let X be some space of real functions on the circle. $\mathcal{IC}_N(X) \subset X$ will denote a set of functions f in X whose corresponding families $F_{r,a} : x \mapsto x + 2\pi r + af(x) \pmod{2\pi}$ have infinitesimal cyclicity bounded by N, that is, $B(F, \rho) \le N$ for all $\rho \in \mathbb{R}$. $\mathcal{IC}_{\text{fin}}(X) = \bigcup_{N=1}^{\infty} \mathcal{IC}_N(X)$ will denote functions whose corresponding families have finite infinitesimal cyclicity, and $\mathcal{IC}_\infty(X) = X \backslash \mathcal{IC}_{\text{fin}}(X)$ functions whose corresponding families have infinite cyclicity.

2.1. *Topologically small sets.*

A standard way to define a topologically small set is to set a subset E of a topological space X to be small if E is a countable union of nowhere dense subsets of X. Such a subset E is also called *meagre*. The complement of a meagre set is called a *residual* set and if some property holds for all elements of a residual set, we say that this property is Baire generic.

Our first theorem states that infinite cyclicity is a Baire generic property.

THEOREM A. *The set $\mathcal{IC}_\infty(C^k(S))$, $k = 2, \ldots,$ as well as the set $\mathcal{IC}_\infty(C^\omega_\delta(S))$, is residual. In other words, for a Baire generic function $f \in C^k(S)$, $k = 2, \ldots, \omega$, the corresponding family F has infinite infinitesimal cyclicity.*

Here $C^\omega_\delta(S)$ denotes the space of real analytic 2π-periodic functions which can be analytically continued to the strip $\mathbb{R} \times (-\delta i, \delta i) \subset \mathbb{C}$. The norm in this space is given by

$$\|f\|_{C^{\omega}_{\delta}(S)} = \sup_{\operatorname{Im}(z)<\delta} |f(z)|.$$

2.2. *Γ^l_b prevalence.* It is well known that sets that are large in a topological sense can have zero Lebesgue measure and be invisible from a 'probabilistic' point of view. The best-known example here is the set of Liouville numbers: this set is residual but has Lebesgue measure zero.

The space of circle functions is infinite-dimensional and there is no natural definition of the Lebesgue measure in infinite-dimensional spaces. Thus, there is no natural (or unique) way to define zero Lebesgue measure sets. Several different approaches to define metric prevalent sets have been suggested; we will study a few of them and investigate their relations to the cyclicity of families of circle maps.

First we consider a definition of a metric prevalent set which employs a mixture of the topological and finite-dimensional metric prevalence.

Definition 2. Let X be some vector space. Let I^b denote a unit cube in $\mathbb{R}^b$. A subset $P \subset X$ is called Γ^l_b*-prevalent* if for a generic C^l family $g : I^b \to X$ the set $g^{-1}(P)$ has full Lebesgue measure.

Though this notion of prevalence is arguably most popular in dynamics, Γ^l_b-prevalent sets cannot be considered as a true generalization of full Lebesgue measure sets: one can construct a zero Lebesgue measure set in $\mathbb{R}^2$ which is Γ^l_1-prevalent; see [**JLT12**] for the details.

In our setting the space X is a space of circle functions, so to make use of the Γ^l_b prevalence we will consider families with values in X as in the paragraph above. On the other hand, for a map in X we will also consider the corresponding family of circle maps. Notice that we use the same word 'family' in these two different settings; however, we hope this will not cause any confusion.

THEOREM B. *For a Baire generic C^l family $f : I^b \to C^k(S)$, for every parameter $t \in I^b$ the infinitesimal cyclicity of the corresponding family $x \mapsto x + 2\pi r + af_t(x)$ is infinite. Here $b \geq 1$, $l \geq 0$, $k = 2, \ldots, \omega$†.*

Notice that the claim of this theorem is stronger than just Γ^l_b prevalence: the corresponding family has infinite infinitesimal cyclicity for *every* value of the parameter.

2.3. *Haar null sets.* Another way do describe 'small' sets in infinite-dimensional spaces was suggested in [**Chr72, HSY92**]. Let X be a complete separable normed linear space.

Definition 3. A Borel subset E of X is called a *Haar null set* if there is a Borel probability measure μ such that $\mu(x + E) = 0$ for all $x \in X$.

In [**HSY92**] such sets are called *shy*. The complement of a Haar null set will be called *Haar prevalent* to avoid confusion with prevalence in the Γ^l_b sense.

† If $k = \omega$, then we mean the space $C^{\omega}_{\delta}(S)$ for some $\delta > 0$.

Haar null sets have many properties which zero Lebesgue measure sets enjoy in finite-dimensional spaces; for example, the countable union of Haar null sets is a Haar null set as well. See [**BL00, HSY92, OY05**] for more information.

The next theorem shows that in the finite smoothness case the infinite cyclicity is Haar prevalent.

THEOREM C. *The set* $\mathcal{IC}_\infty(C^k(S))$ *of functions* f *such that the corresponding families* $F_{r,a} : x \mapsto x + 2\pi r + af(x) \pmod{2\pi}$ *have infinite infinitesimal cyclicity is Haar prevalent. Here* $k = 2, 3, \ldots$.

Surprisingly enough, in the analytic case the situation is completely the opposite: the infinite infinitesimal cyclicity is Haar null. Moreover, a Haar typical family has only one attracting cycle if its period is high enough. To make this precise we need the following definition.

Definition 4. We say that the infinitesimal cyclicity of the family $F_{r,a}$ is essentially bounded by N if there exists $Q \in \mathbb{N}$ such that for any $p, q \in \mathbb{N}, q \geq Q$ we have $B(F, p/q) \leq N$.

In other words, if the period is large enough, at most N periodic attractors of this period can appear from the rigid rotation. It is easy to show that if the infinitesimal cyclicity of a family is essentially bounded by some N, then it is finite. For a given space X of circle functions, $\mathcal{IC}^e_N(X)$ will denote the set of functions f whose corresponding families $F_{r,a} : x \mapsto x + 2\pi r + af(x) \pmod{2\pi}$ have infinitesimal cyclicity essentially bounded by N.

THEOREM D. *The set* $\mathcal{IC}^e_1(C^\omega_\delta(S))$ *is Haar prevalent.*

The notion of Haar null set can be strengthened, and next we define *cube null* sets. From the definition it will be clear that any cube null set is also Haar null.

Consider the Hilbert cube $[0, 1]^{\aleph_0}$ and a linear map $T : [0, 1]^{\aleph_0} \to X$ defined by $T((t_1, t_2, \ldots)) = x_0 + \sum_{n=1}^\infty t_n x_n$, where $x_0, x_1, \ldots \in X$ such that the vectors $x_1, x_2, \ldots$ are linearly independent, have a dense linear span in X and $\sum_{n=1}^\infty \|x_n\| < \infty$. The image of the standard product measure on $[0, 1]^{\aleph_0}$ under T is called a *cube* measure and denoted by μ_T. By definition, we call a set E *cube null* if $\mu_T(E) = 0$ for any cube measure. The complement of a cube null set we will call cube prevalent.

Similarly, one can define *Gauss* null sets where the Gauss distributions are used instead of the uniform distributions. The Gauss null sets defined in this way appear to be cube null and vice versa.

The next statement shows that the previous theorems cannot be strengthened to the case of cube prevalence.

THEOREM E.
- *The set* $\mathcal{IC}_\infty(C^k(S))$ *is* not *cube prevalent.*
- *The sets* $\mathcal{IC}^e_1(C^\omega_\delta(S))$ *and* $\mathcal{IC}_{\text{fin}}(C^\omega_\delta(S))$ *are* not *cube prevalent.*

Notice that since the sets $\mathcal{IC}_\infty(C^k(S))$ and $\mathcal{IC}_{\text{fin}}(C^\omega_\delta(S))$ are Haar prevalent they cannot be cube null either.

2.4. *σ-porous sets.* Yet another approach to define typical sets in infinite-dimensional spaces is to use a notion of σ-porous sets.

Let X be a normed linear space. $B(x, r) = \{y \in X : \|y - x\| < r\}$ will denote a ball of radius r centred at $x \in X$.

Definition 5. A subset E of X is called *porous at x* if there is $c > 0$ such that for any $\epsilon > 0$ there is $y \in X$ such that $\|y - x\| < \epsilon$ and $B(y, c\|y - x\|) \cap E = \emptyset$. A subset $E \subset X$ is called *porous* if E is porous at every $x \in E$. A subset is called *σ-porous* if it is a union of countably many porous sets.

It is clear from the definition that porous sets are nowhere dense and that the complement of a σ-porous set is residual. The Lebesgue density theorem implies that if X is finite-dimensional, then every σ-porous set has zero Lebesgue measure.

There is a direct link between complements of σ-porous sets and Γ^1_1-prevalent sets discussed in one of the previous sections.

THEOREM 2.1. [**JLT12**] *Let X be a Banach space with a separable dual and $E \subset X$ be σ-porous. Then for a Baire generic C^1 family $f : [a, b] \to X$ the set $f^{-1}(E)$ has zero Lebesgue measure.*

We will show that the C^k functions with finite cyclicity form a σ-porous set if k is finite; however, Theorem 2.1 would not imply Theorem B because the dual of the space $C^k(S)$ is not separable. Considering Sobolev spaces instead of $C^k(S)$ spaces would make application of Theorem 2.1 possible (and all theorems we prove here can be easily generalized to Sobolev spaces); however, we would still get a statement much weaker than that of Theorem B.

The next theorem tells us that in the case of smooth (non-analytic) functions infinite cyclicity prevails once again.

THEOREM F. *The set $\mathcal{IC}_{\text{fin}}(C^k(S))$, $k = 2, \dots,$ of functions f whose corresponding families $F_{r,a} : x \mapsto x + 2\pi r + af(x) \pmod{2\pi}$ have finite infinitesimal cyclicity is σ-porous.*

It is not clear if the sets $\mathcal{IC}_{\text{fin}}(C^\omega_\delta(S))$ and $\mathcal{IC}_\infty(C^\omega_\delta(S))$ are σ-porous or not.

2.5. *Summary.* For the convenience of the reader all results formulated in the previous sections are summarized in the following table.

Prevalence	$C^k(S)$	$C^\omega_\delta(S)$
Baire residual	$\mathcal{IC}_\infty$	$\mathcal{IC}_\infty$
Γ^l_b prevalent	$\mathcal{IC}_\infty$	$\mathcal{IC}_\infty$
Haar prevalent	$\mathcal{IC}_\infty$	$\mathcal{IC}^e_1$
Cube prevalent	None	None
Complement to σ-porous	$\mathcal{IC}_\infty$	?

2.6. *Torus vector fields.* Instead of perturbations of rigid rotations of the circle we can study perturbations of constant vector fields on the torus.

For two functions v_1, v_2 on the torus consider a family $V_{\alpha,a}$ of vector fields on the torus given by

$$V_{\alpha,a}(x_1, x_2) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + a \begin{pmatrix} v_1(x_1, x_2) \\ v_2(x_1, x_2) \end{pmatrix},$$

where $\alpha \in \mathbb{R}$ and $a \in (-a_0, a_0)$ are parameters. We could also consider a family

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + \begin{pmatrix} a_1 v_1(x_1, x_2) \\ a_2 v_2(x_1, x_2) \end{pmatrix},$$

which depends on two parameters a_1 and a_2 instead of only one parameter a. The definition of the infinitesimal cyclicity for such families is straightforward; one should count the number of attracting periodic limit cycles. Then the statements of Theorems A–F can be reformulated in the obvious way for such families of vector fields and the statements remain correct. In §8 we will justify why this can be done, but we will not re-prove all these theorems as their proofs are almost identical to those corresponding to the families of circle maps.

3. *Perturbations of rigid rotations*

Consider a family $F_{r,a} : x \mapsto x + 2\pi r + af(x) \pmod{2\pi}$. The nth iterate of $F_{r,a}$ is

$$F^n_{r,a}(x) = x + 2\pi rn + af(x) + af(x + 2\pi r + af(x)) + \cdots .$$

This formula shows that if $F_{r,a}$ has a periodic trajectory of period q, then there is $p \in \mathbb{Z}$ such that

$$\left| r - \frac{p}{q} \right| \leq \frac{a}{2\pi} \|f\|_{C^0(S)}.$$

If $f \in C^2(S)$, and a is small, then the above formula implies that

$$F^n_{r,a}(x) = x + 2\pi rn + a \sum_{k=0}^{n-1} f(x + kr) + O(a^2).$$

Suppose that the function f can be represented as the following Fourier series:

$$f(x) = \sum_{l=-\infty}^{+\infty} \tilde{f}_l e^{ilx}.$$

A straightforward computation shows that if $r = p/q$, where $p, q \in \mathbb{N}$, then

$$\sum_{k=0}^{q-1} f(x + kr) = q \sum_{l=-\infty}^{+\infty} \tilde{f}_{ql} e^{iqlx}.$$

The following proposition immediately follows from the discussion above.

PROPOSITION 3.1. *Let f be in $C^2(S)$. Suppose that there exists $R \in \mathbb{R}$ such that the equation*

$$\sum_{l=-\infty}^{+\infty} \tilde{f}_{ql} e^{iqlx} = R$$

has d simple roots. Then for all sufficiently small non-zero a there exists $r(a) \in [0, 2\pi)$ such that the map $F_{r(a),a}$ has at least d isolated periodic trajectories of period q.

The converse also holds at least in the analytic case.

PROPOSITION 3.2. *Let $f \in C^{\omega}_{\delta}(S)$. Suppose that for any $R \in \mathbb{R}$ the equation*

$$\sum_{l=-\infty}^{+\infty} \tilde{f}_{ql} e^{iqlx} = R$$

has at most d real roots (counted with multiplicity). Then there exists $a_0 > 0$ such that for any $r \in \mathbb{R}$ and $a \in (-a_0, a_0)\backslash\{0\}$ the map $F_{r,a}$ has at most d periodic trajectories of period q.

The proof of this proposition is a straightforward application of the argument principle. Indeed, denote $g(x) = \sum_{l=-\infty}^{+\infty} \tilde{f}_{ql} e^{iqlx}$. For every value $R_0 \in \mathbb{R}$ there exist $\beta(R_0) \in (0, \delta)$ and $A(R_0) > 0$ such that the equation $g(x) = R_0$ has at most d roots in the strip $\mathrm{Im}(x) \le \beta(R)$ and $\sup_{y\in\mathbb{R}} |g(\pm i\beta(R_0) + y) - R_0| > A(R_0)$. Moreover, there exists $r(R_0) > 0$ such that $\sup_{y\in\mathbb{R}} |g(\pm i\beta(R_0) + y) - R| > A(R_0)$ for all $R \in (R_0 - r(R_0), R_0 + r(R_0))$. By the argument principle for these values of R the equation $g(x) = R$ still has at most d roots in the strip $\mathrm{Im}(x) \le \beta(R)$. Also, if $|R| > \|g\|_{C^{\omega}_{\delta}(S)}$, then the equation $g(x) = R$ does not have roots at all. Consider a cover of the segment $(-2\|g\|_{C^{\omega}_{\delta}(S)}, 2\|g\|_{C^{\omega}_{\delta}(S)})$ by intervals of the form $(R - r(R), R + r(R))$ and take a finite subcover $(R_1 - r(R_1), R_1 + r(R_1)), \ldots, (R_L - r(R_L), R_L + r(R_L))$. Set $\beta = \min(\|g\|_{C^{\omega}_{\delta}(S)}, A(R_1), \ldots, A(R_L))$. It is easy to see that due to the argument principle for all $R \in \mathbb{R}$ the equation $g(x) + h(x) = R$ has at most d roots if $\|h\|_{C^{\omega}_{\delta}(S)} < \beta$. This proves the proposition.

4. *Sturm–Hurwitz theorem*

In order to provide a lower bound on the number of zeros of a function we employ a version of the Sturm–Hurwitz theorem. The original Sturm–Hurwitz theorem states that if all harmonic components of order less than n of a real continuous function g vanish, then the function g has at least $2n$ zeros. We need a somewhat sharper statement, where the harmonic components do not vanish but are small.

THEOREM 4.1. *Let ϕ be a real differentiable 2π-periodic function and a_n be its Fourier coefficients, $a_n = (1/2\pi) \int_0^{2\pi} \phi(x) e^{-inx}\, dx$. Moreover, assume that for some $n \in \mathbb{N}$,*

$$\sum_{k=-n+1}^{n-1} |a_k| < 2^{-2n+3} |a_n|.$$

Then on the interval $[0, 2\pi)$ the function ϕ changes its sign at least $2n$ times.

In the proof of this theorem the following statement will be used.

PROPOSITION 4.1. *Let ϕ be a* positive *real differentiable 2π-periodic function and a_n be its Fourier coefficients. Then*

$$|a_0| \geq 2|a_n|$$

for all $n \neq 0$.

The proof of this proposition can be found in [**PS76**, Part IV, N 51, p. 71], where it is proved in the case where ϕ is a trigonometric polynomial. Since the Fourier partial sums of a differentiable function converge uniformly the statement follows.

Proof of Theorem 4.1. We will prove this theorem by induction following the original proof of Hurwitz [**Hur03**].

The case $n = 1$ follows immediately from the above proposition.

Now let us make an induction step and suppose that the theorem holds for some n. Suppose that

$$\sum_{k=-n}^{n} |a_k| < 2^{-2n+1}|a_{n+1}|,$$

but ϕ changes its sign less than $2n + 2$ times. Again, due to the previous proposition the function ϕ cannot be positive (or negative) for all x and it changes its sign at least twice. Denote these points where ϕ is zero by x_1 and x_2. Then the function $\hat{\phi}(x) = \phi(x)\sin((x - x_1)/2)\sin((x - x_2)/2)$ is 2π-periodic and has less than $2n$ changes of sign.

Let us compute the Fourier coefficients of the function $\hat{\phi}$. Note that

$$\begin{aligned} &\sin\left(\frac{x - x_1}{2}\right)\sin\left(\frac{x - x_2}{2}\right) \\ &\quad = -\frac{1}{4}e^{((x_1+x_2)/2)i}e^{-ix} + \frac{1}{2}\cos\left(\frac{x_1 - x_2}{2}\right) - \frac{1}{4}e^{-((x_1+x_2)/2)i}e^{ix}. \end{aligned}$$

This implies that the Fourier coefficients of $\hat{\phi}$ are

$$\hat{a}_k = -\frac{1}{4}e^{-((x_1+x_2)/2)i}a_{k-1} + \frac{1}{2}\cos\left(\frac{x_1 - x_2}{2}\right)a_k - \frac{1}{4}e^{((x_1+x_2)/2)i}a_{k+1}.$$

Now we can estimate $|\hat{a}_k|$ in terms of the Fourier coefficients of the function ϕ:

$$\begin{aligned} |\hat{a}_n| &\geq \tfrac{1}{4}|a_{n+1}| - \tfrac{1}{2}|a_n| - \tfrac{1}{4}|a_{n-1}|, \\ |\hat{a}_k| &\leq \tfrac{1}{4}|a_{k-1}| + \tfrac{1}{2}|a_k| + \tfrac{1}{4}|a_{k+1}|. \end{aligned}$$

Combining these inequalities, we obtain

$$\begin{aligned} \sum_{k=-n+1}^{n-1} |\hat{a}_k| &\leq -\frac{3}{4}|a_{-n}| - \frac{1}{4}|a_{-n+1}| + \sum_{k=-n}^{n} |\hat{a}_k| - \frac{1}{4}|a_{n-1}| - \frac{3}{4}|a_n| \\ &< 2^{-2n+1}|a_{n+1}| - \frac{1}{2}|a_{n-1}| - \frac{3}{2}|a_n| \\ &\leq 2^{-2n+3}|\hat{a}_n|. \end{aligned}$$

We see that we can apply the induction assumption to $\hat{\phi}$, therefore $\hat{\phi}$ changes its sign at least $2n$ times and ϕ changes its sign at least $2n + 2$ times. □

5. *Baire genericity of infinite infinitesimal cyclicity*

Proof of Theorem A. We will give the proof for the $C^k(S)$ space; for the $C^\omega_\delta(S)$ spaces it works in the same way.

Let p be a trigonometric polynomial of degree N. Let $f(x) = p(x) + c\sin((N+1))$, where $d \in \mathbb{N}$, $c \in \mathbb{R}$ are constants, and let $F_{r,a}(x) = x + 2\pi r + af(x) \pmod{2\pi}$ be the corresponding family. Proposition 3.1 implies that for small values of a the map $F_{1/(N+1),a}$ has d periodic attractors since

$$F^{N+1}_{1/(N+1),a}(x) = x + ac(N+1)\sin((N+1)\,x) + O(a^2) \pmod{2\pi}.$$

From the previous theorem we know that there is a neighbourhood of f such that for any function in this neighbourhood the corresponding family will also have d periodic attracting trajectories of period $N+1$.

The trigonometric polynomials are dense in the space of C^k 2π-periodic functions, and the constant c can be taken arbitrarily small, thus we have proved that the set of functions f such that the cyclicity of the corresponding family is bounded by d is nowhere dense. The union of these sets is meagre and the complement of this union is residual. □

Proof of Theorem B. This proof uses similar ideas to those used in the proof of Theorem A. First, consider the space $C^k(S)$ for a finite k.

Let $f : I^b \to C^k(S)$ be a C^l family and let $\tilde{f}_n(t) = (1/2\pi)\int_0^{2\pi} f_t(x)e^{-inx}\,dx$ be its Fourier coefficients. The derivative of f with respect to the parameter t will be denoted by $f^{(m)}$ where $m = (m_1, \ldots, m_b)$ is a multi-index.

LEMMA 5.1. *For any $\epsilon_0 > 0$ there exists $N = N(\epsilon_0)$ such that for all $t \in I^b$, all $n \in \mathbb{Z}$, $|n| \geq N$,*

$$|\tilde{f}^{(m)}_n(t)| \leq \epsilon_0 |n|^{-k}, \tag{1}$$

for all $|m| \leq l$.

Proof. Fix $t_0 \in I^b$. The Fourier coefficients of the kth derivative of f_{t_0} with respect to x have the form $n^k\tilde{f}_n(t_0)$. From the Riemann–Lebesgue lemma we know that $|n^k\tilde{f}_n(t_0)| \to 0$ as $n \to \pm\infty$. Hence there exists $N(\epsilon_0, t_0)$ such that for all $|n| > N(\epsilon_0, t_0)$ one has $|\tilde{f}_n(t_0)| \leq \epsilon_0|n|^{-k}/2$.

Take $\delta(\epsilon_0, t_0) > 0$ so small that $\|f_t - f_{t_0}\|_{C^k(S)} < \epsilon_0/2$ if $|t - t_0| < \delta$. Then $|n|^k|\tilde{f}_n(t) - \tilde{f}_n(t_0)| < \epsilon_0/2$ for all $n \in \mathbb{Z}$. Combining this inequality with the inequality in the previous paragraph, we get that for $t \in I^b$, $|t - t_0| < \delta(\epsilon_0, t_0)$ and $|n| > N(\epsilon_0, t_0)$ inequality (1) holds for $m = 0$.

The set I^b is covered by open balls $B(t, \delta(\epsilon_0, t))$. Using compactness of I^b we can take a finite subcover $B(t_1, \delta(\epsilon_0, t_1)), \ldots, B(t_L, \delta(\epsilon_0, t_L))$ and set $N(\epsilon_0) = \max_{i \leq L} N(\epsilon_0, t_i)$. Obviously, inequality (1) holds for this choice of N if $m = 0$. For the other values of m the argument is the same. □

Now fix some positive integer d and $\epsilon > 0$. Consider the family

$$\hat{f}_t(x) = f_t(x) - \sum_{m=1}^{d+1}(\tilde{f}_{-mN}(t)e^{-imNx} + \tilde{f}_{mN}(t)e^{imNx}) + \epsilon N^{-k}(d+1)^{-k-1}\sin(N(d+1)x),$$

where N is given by the claim above for $\epsilon_0 = \epsilon(d+1)^{-1}/2$. The norm of the difference of f_t and $\hat{f}_t$ can be estimated as

$$\|\hat{f}_t - f_t\|_{C^{l,k}(S)} < 2\epsilon.$$

Arguing as before, due to Proposition 3.1 and Theorem 4.1 for arbitrary fixed $t \in I^b$ the corresponding family $\hat{F}_{t,r,a} : x \mapsto x + 2\pi r + a\hat{f}_t(x) \pmod{2\pi}$ has $d+1$ periodic attractors of period N, when $r = 1/N$ and a is small. Moreover, there exists a neighbourhood of the family $\hat{f}_t$ in $C^{l,k}(S)$ such that the corresponding family has the same property.

Thus, once again we have shown that the set of families f_t whose infinitesimal cyclicity is bounded by d for just one value of parameter t is nowhere dense. The claim of the theorem follows.

The case of analytic functions can be dealt with in exactly the same way, except that instead of inequality (1) one should use

$$|\tilde{f}_n^{(m)}(t)| \le \epsilon_0 e^{-\delta|n|}. \qquad \square$$

6. *The case of finitely differentiable functions*

Proof of Theorem F. Recall that $\mathcal{IC}_N(C^k(S)) \subset C^k(S)$ denotes a set of functions such that the cyclicity of the corresponding families is bounded by N. Let f be in $\mathcal{IC}_{N-1}(C^k(S))$, $\tilde{f}_n$ denote its Fourier coefficients and fix $\delta > 0$.

Due to the Riemann–Lebesgue lemma we know that $\lim_{n\to\pm\infty} n^k|\tilde{f}_n| = 0$. Hence, there is $d > 0$ such that $\sum_{n=1}^{N}(dn)^k|\tilde{f}_{dn}| < \delta/8$. Define a new function $\hat{f} \in C^k(S)$ by

$$\hat{f}(x) = f(x) - \sum_{n=1}^{N}(\tilde{f}_{-dn}e^{-idnx} + \tilde{f}_{dn}e^{idnx}) + \frac{\delta}{2(dN)^k}\sin(dNx).$$

It is easy to see that $\|f - \hat{f}\|_{C^k(S)} < \delta$.

Consider an arbitrary function $g \in C^k(S)$ such that $\|g - \hat{f}\|_{C^k(S)} < c\delta$, where

$$c = \frac{1}{8N^{k+1}4^N} \tag{2}$$

and let $\tilde{g}_n$ denote the Fourier coefficients of g. From the inequality above it follows that

$$|\tilde{g}_n| < c\delta d^{-k}$$

for $n = d, 2d, \ldots, (d-1)N$ and

$$|\tilde{g}_{dN}| > \left(\frac{1}{4(dN)^k} - \frac{c}{(dN)^k}\right)\delta > \frac{\delta}{8(dN)^k}.$$

Therefore,

$$\begin{aligned}\sum_{n=-N+1}^{N-1}|\tilde{g}_{dn}| &< |\tilde{g}_0| + \frac{2c(N-1)\delta}{d^k}\\ &< |\tilde{g}_0| + \frac{\delta}{4(dN)^k 4^N}\\ &< |\tilde{g}_0| + 2^{-2N+1}|\tilde{g}_{dN}|.\end{aligned}$$

Let $g_d(x)$ denote the function $\sum_{l=0}^{d-1} g(x + (2\pi/d)l)$. The Fourier series for this function is

$$g_d(x) = d \sum_{n=-\infty}^{+\infty} \tilde{g}_{dn} e^{idnx}.$$

Having the estimate on $\sum_{n=-N+1}^{N-1} |\tilde{g}_{dn}|$, we cannot apply Theorem 4.1 directly to the function g_d, as it would require the coefficient 2^{-2dN+3} in front of $|\tilde{g}_{dN}|$ and we have only 2^{-2N+1}. However, g_d is a $(2\pi/d)$-periodic function and we can apply Proposition 3.1 and Theorem 4.1 together with the estimate above to the restriction $g_d|_{[0,2\pi/d]}$ and get that the equation $g_d(x) = d\tilde{g}_0$ has at least $2N$ zeros in the interval $[0, 2\pi/d)$ and, therefore, at least $2dN$ zeros in $[0, 2\pi)$.

Thus, the family $G_{r,a} : x \mapsto x + 2\pi r + ag(x) \pmod{2\pi}$ has cyclicity at least N because for small values of a the map $G_{(1/d)-(\tilde{g}_0/2\pi)a,a}$ has at least N periodic attracting trajectories of period d.

We have proved that arbitrarily near any function $f \in C^k(S)$ we can find a function $\hat{f}$ such that the ball $B(\hat{f}, c\|f - \hat{f}\|)$, where c is given by (2), does not contain elements from the set $\mathcal{IC}_{N-1}(C^k(S))$. Hence, the set $\mathcal{IC}_{N-1}(C^k(S))$ is porous, and the union $\bigcup_{N=1}^{\infty} \mathcal{IC}_N(C^k(S))$ is σ-porous. □

Proof of Theorem C. Fix $N \in \mathbb{N}$. We will construct a measure on $C^k(S)$ and show that the set $\mathcal{IC}_{N-1}(C^k(S))$ is Haar null with respect to this measure.

Let the sequence t_m be given by

$$t_m = \frac{1}{\sqrt{m} N^{2km}},$$

for $m = 1, 2, \ldots$. Consider a sequence of independent random variables ϕ_m on $[0, 1]$ such that

$$|\{w \in [0, 1] : \phi_m(w) = 0\}| = 1 - \frac{1}{m},$$
$$|\{w \in [0, 1] : \phi_m(w) = t_m\}| = \frac{1}{m},$$

where $|\cdot|$ denotes the Lebesgue measure on $[0, 1]$.

For $w \in [0, 1]$ consider a function $\beta_w(x) = \sum_{m=1}^{\infty} \phi_m(w) N^{2km} \sin(N^{2m}x)$. Due to Markov's inequality we know that

$$\left|\left\{w \in [0, 1] : \sum_{m=1}^{\infty} \phi_m(w) N^{2km} > X\right\}\right| \le \frac{1}{X} \sum_{m=1}^{\infty} E(\phi_m) N^{2km},$$

where $X \in \mathbb{R}$ and E denotes the expectation of a random variable. The sum $\sum_{m=1}^{\infty} E(\phi_m) N^{2km} = \sum_{m=1}^{\infty} m^{-3/2}$ is convergent, therefore the sum $\sum_{m=1}^{\infty} \phi_m(w) N^{2km}$ converges for almost every $w \in [0, 1]$. This implies that the function given by the Fourier series

$$\sum_{m=1}^{\infty} \phi_m(w) N^{2km} \sin(N^{2m}x)$$

is continuous almost surely. Integrating this function k times, we see that for almost every $w \in [0, 1]$ the function

$$g_w(x) = \sum_{m=1}^{\infty} \phi_m(w) \sin(N^{2m} x)$$

is in $C^k(S)$. The push-forward of the Lebesgue measure by the map $[0, 1] \to C^k(S)$ given by $w \mapsto g_w$ defines a measure on $C^k(S)$ which we will denote by μ.

Take $f \in C^k(S)$ and let $\tilde{f}_n$ be its Fourier coefficients. Our goal is to show that $\mu(f + \mathcal{IC}_{N-1}(C^k(S))) = 0$. Let $g \in f + \mathcal{IC}_{N-1}(C^k(S))$ and $\tilde{g}_n$ be the Fourier coefficients of g.

Since $g - f \in \mathcal{IC}_{N-1}(C^k(S))$, and due to Proposition 3.1 and Theorem 4.1, we know that for all $g \in f + \mathcal{IC}_{N-1}(C^k(S))$,

$$\sum_{d=1}^{N-1} |\tilde{g}_{N^{2m-1}d} - \tilde{f}_{N^{2m-1}d}| \geq 2^{-2N+2} |\tilde{g}_{N^{2m}} - \tilde{f}_{N^{2m}}|,$$

for all $m \in \mathbb{N}$. If this inequality did not hold for some m, then the map $x \mapsto x + 2\pi N^{1-2m} - \tilde{f}_0 a + a(g(x) - f(x))$ would have N periodic attractors of period N^{2m-1} for sufficiently small values of a.

Let us observe that μ-almost surely $\tilde{g}_n = 0$ if n is not of the form $\pm N^{2m}$ for some natural m; thus the above inequality implies that for μ-almost all $g \in f + \mathcal{IC}_{N-1}(C^k(S))$,

$$\sum_{d=1}^{N} |\tilde{f}_{N^{2m-1}d}| \geq 2^{-2N+2} |\tilde{g}_{N^{2m}}|.$$

Let $c_m = 2^{2N-2} \sum_{d=1}^{N} |\tilde{f}_{N^{2m-1}d}|$. We have just shown that

$$\mu(f + \mathcal{IC}_{N-1}(C^k(S))) \leq \mu(\{g \in C^k(S) : \forall m \in \mathbb{N} |\tilde{g}_m| \leq c_m\}).$$

A direct computation shows that

$$\begin{aligned} \mu(\{g \in C^k(S) : \forall m \in \mathbb{N} \quad |\tilde{g}_m| \leq c_m\}) &= \prod_{t_m > c_m} \left(1 - \frac{1}{m}\right) \\ &\leq \exp\left(- \sum_{t_m > c_m} \frac{1}{m}\right). \end{aligned} \tag{3}$$

The function f is in $C^k(S)$, hence its kth derivative is continuous and also belongs to $L^2(S)$. The Fourier coefficients of $f^{(k)}$ are $n^k \tilde{f}_n$ and by Perceval's identity we have $\sum_{n=-\infty}^{\infty} |n|^{2k} |\tilde{f}_n|^2 < \infty$. Then

$$\begin{aligned} \sum_{m=1}^{\infty} N^{4km} |c_m|^2 &\leq 4^{2N-2} \sum_{m=1}^{\infty} N^{4km+1} \sum_{d=1}^{N} |\tilde{f}_{N^{2m-1}d}|^2 \\ &\leq 4^{2N-2} \sum_{d=1}^{N} N^{2k+1} d^{-2k} \sum_{m=1}^{\infty} N^{2(2m-1)k} d^{2k} |\tilde{f}_{N^{2m-1}d}|^2 < \infty. \end{aligned}$$

Therefore,

$$\sum_{t_m \le c_m} \frac{1}{m} = \sum_{t_m \le c_m} t_m^2 N^{4km} \le \sum_{t_m \le c_m} c_m^2 N^{4km} \le \sum_{m=1}^{\infty} c_m^2 N^{4km} < \infty.$$

This implies that the series

$$\sum_{t_m > c_m} \frac{1}{m} = \sum_{m=1}^{\infty} \frac{1}{m} - \sum_{t_m \le c_m} \frac{1}{m}$$

diverges and, because of inequality (3), we get $\mu(f + \mathcal{IC}_{N-1}(C^k(S))) = 0$.

So, we have proved that $\mathcal{IC}_{N-1}(C^k(S))$ is a Haar null set. A countable union of Haar null sets is Haar null [**HSY92**], hence $\mathcal{IC}_{\text{fin}}(C^k(S))$ is Haar null too. □

7. *The case of analytic functions*

First, we will prove the following simple lemma.

LEMMA 7.1. *Let $n \in \mathbb{N}$ and the function g be in $C^\omega_\delta(S)$ with $\|g\|_{C^\omega_\delta(S)} < (\cosh(n\delta) - 1)/2$. Then the equation*

$$r + a\cos(nx) + b\sin(nx) + g(x) = 0 \tag{4}$$

has at most $2n$ real roots in $[0, 2\pi)$ for any $r \in \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$.

Proof. The proof is elementary. By shifting x we can assume that $a = 1$ and $b = 0$. Consider two cases. If $|r| \ge (\cosh(n\delta) + 1)/2$, then (4) has no real roots as g is less than $(\cosh(n\delta) - 1)/2$ on the real line.

The image of the segment $[i\delta, i\delta + 2\pi)$ under the map $z \mapsto \cos(nz)$ is an ellipse with semi-major axis $\cosh(n\delta)$ and semi-minor axis $\sinh(n\delta)$. It is easy to check that the distance between the segment $[-(\cosh(\delta) + 1)/2, (\cosh(\delta) + 1)/2]$ and this ellipse is $(\cosh(n\delta) - 1)/2$. Due to the argument principle this implies that the function $r + \cos(nx) + g(x)$, where $r \in (-(\cosh(n\delta) + 1)/2, (\cosh(n\delta) + 1)/2)$, has exactly $2n$ zeros in the rectangle with vertices $i\delta$, $2\pi + i\delta$, $2\pi - i\delta$, $-i\delta$. The lemma follows. □

Proof of Theorem D. Let f be in $C^\omega_\delta(S)$. Because of the Cauchy theorem the Fourier coefficients of f can be written as

$$\tilde{f}_n = \frac{1}{2\pi} \int_{-i\delta}^{-i\delta + 2\pi} e^{-inx} f(x)\, dx = \frac{e^{-\delta n}}{2\pi} \int_0^{2\pi} e^{-inx} f(x - i\delta)\, dx.$$

Thus, the Fourier coefficients decay exponentially: $\|\tilde{f}_n\| \le e^{-\delta|n|} \|f\|_{C^\omega_\delta(S)}$. On the other hand, if $\delta' > \delta$, $\|b_n\| \le C e^{-\delta'|n|}$ for some $C > 0$, and $b_{-n} = \bar{b}_n$, then the function $\sum_{n=-\infty}^{\infty} b_n e^{inx}$ is in $C^\omega_\delta(S)$.

Let $D_1 \subset \mathbb{C}$ be a unit disk in the complex plain. Denote the complex Hilbert cube $D_1^{\aleph_0}$ as Q and let τ be the standard product measure on Q. Consider the map $\phi : Q \to C^{\omega}_{\delta}(S)$ defined by

$$\phi((w_1, w_2, \ldots)) = \sum_{n=1}^{\infty} e^{-3/2\delta n}(w_n e^{inx} + \bar{w}_n e^{-inx}).$$

Since the Fourier coefficients decay exponentially faster than $e^{-\delta n}$, the map is properly defined for every element of Q. We denote by μ the image of the measure τ under this map ϕ.

Let E be a set of functions in $C^{\omega}_{\delta}(S)$ such that the corresponding families have essential infinitesimal cyclicity at least 2. Now we will show that for every $f \in C^{\omega}_{\delta}(S)$ one has $\mu(f + E) = 0$.

Let g belong to the support of the measure μ, which is equivalent to $g \in \mathrm{Image}(\phi)$, and $\tilde{g}_n$ be the Fourier coefficients of g. Arguing as before and due to Proposition 3.2 and the lemma above, for arbitrarily small non-zero values of the parameter a there is $r \in \mathbb{R}$ such that the map $x \mapsto x + 2\pi r + a(g(x) - f(x))$ can have at least two periodic attracting trajectories of period N if

$$\begin{aligned}
|\tilde{g}_N - \tilde{f}_N| &\le \frac{2}{\cosh(N\delta/6) - 1}\left\| \sum_{k=2}^{\infty} (\tilde{g}_{-kN} - \tilde{f}_{-kN})e^{-ikNx} + (\tilde{g}_{kN} - \tilde{f}_{kN})e^{ikNx} \right\|_{C^{\omega}_{\delta/6}(S)} \\
&\le \frac{4}{\cosh(N\delta/6) - 1} \sum_{k=2}^{\infty} (|\tilde{g}_{kN}| + |\tilde{f}_{kN}|)e^{\delta kN/6} \\
&\le \frac{4}{\cosh(N\delta/6) - 1} \sum_{k=2}^{\infty} (e^{-4/3\delta kN} + \|f\|_{C^{\omega}_{\delta}(S)} e^{-5/6\delta kN}) \\
&\le \frac{4}{\cosh(N\delta/6) - 1} \left(\frac{e^{-8/3\delta N}}{1 - e^{-4/3\delta N}} + \|f\|_{C^{\omega}_{\delta}(S)} \frac{e^{-5/3\delta N}}{1 - e^{-5/6\delta N}} \right) \\
&\le Ce^{-5/3\delta N},
\end{aligned} \tag{5}$$

where the constant C depends on δ and f, but not on N.

We know that $\tilde{g}_N$ is uniformly distributed in the disk of radius $e^{-3/2\delta N}$. Thus, the probability that inequality (5) holds is less than $C^2 e^{-\delta N/3}$.

Consider a sequence of random numbers $p_N = p_N(g)$ such that p_N is 1 if for arbitrarily small non-zero values of the parameter a there is $r \in \mathbb{R}$ such that the map $x \mapsto x + 2\pi r + a(g(x) - f(x))$ can have at least two periodic attracting trajectories of period N and zero otherwise.

According to Markov's inequality, we have

$$\mu\left(\left\{ g : \sum_{N=1}^{\infty} p_N(g) > X \right\}\right) < \frac{1}{X} \sum_{N=1}^{\infty} E(p_N) < \frac{1}{X} \sum_{N=1}^{\infty} C^2 e^{-\delta N/3}.$$

The last series converges, therefore $\mu(\{g : \sum_{N=1}^{\infty} p_N(g) = \infty\}) = 0$. The sum $\sum_{N=1}^{\infty} p_N(g)$ diverges exactly when the family under consideration has essential infinitesimal cyclicity at least 2. Thus, $\mu(f + E) = 0$ and the theorem is proved. □

Proof of Theorem E. First, consider the case of finitely differentiable functions $C^k(S)$.

In the definition of the cube measure set $x_0 = (1 - e^{-1}\cos(x))/(1 + e^{-2} - 2e^{-1}\cos(x))$, $x_n = e^{-n^2}\sin(nx/2)$ if $2|n$ and $x_n = e^{-n^2}\cos((n-1)x/2)$ otherwise. Let μ be a cube measure on $C^k(S)$ defined by these settings. Notice that the Fourier expansion of x_0 is $2 + \sum_{m=1}^{\infty} e^{-m}\cos(mx)$.

Let $f \in \mathrm{supp}(\mu)$ and $\tilde{f}_n$ be the Fourier coefficients of f. A direct computation shows that for large values of N,

$$|\tilde{f}_N| > \frac{2}{\cosh N - 1}\left\| \sum_{m=2}^{\infty} \tilde{f}_{-Nm}e^{-iNmx} + \tilde{f}_{Nm}e^{iNmx} \right\|_{C_1^{\omega}(S)}.$$

Arguing as in the proof of Theorem D and using Proposition 3.2 and Lemma 7.1, we see that the family corresponding to f has infinitesimal cyclicity essentially bounded by one. Thus, $\mu(\mathcal{IC}_\infty(C^k(S))) = 0$.

Now let us prove that the set $\mathcal{IC}_{\mathrm{fin}}(C_\delta^\omega(S))$ is not cube prevalent.

We leave the choice of x_n as before, but now set $x_0 = \sum_{m=1}^{\infty} e^{-m^{m!}\delta}\cos(m^{m!}x)$ and let μ be the corresponding cube measure on $C_\delta^\omega(S)$. In this case one can check that for any $f \in \mathrm{supp}(\mu)$ and for N sufficiently large,

$$\frac{1}{4^{M-1}}|\tilde{f}_{NM}| \geq \sum_{m=1}^{N-1} |\tilde{f}_{M}m|,$$

where $M = N^{(N-1)!}$. Using the same reasoning as in the proof of Theorem F and using Proposition 3.1 and Theorem 2.1, we can conclude that the family corresponding to f has at least N periodic attractors for a suitable choice of the parameters. Since N can be arbitrarily large we can see that $\mu(\mathcal{IC}_{\mathrm{fin}}(C_\delta^\omega(S))) = 0$. □

8. *Perturbations of constant vector fields on the torus*

Let $\phi_{\alpha,a}^t$ be the flow of the vector field $V_{\alpha,a}$ which we assume to be at least C^2. Perturbation theory tells us that for small values of a this flow can be expressed as

$$\begin{aligned}\phi_{\alpha,a}^t : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} t \\ + a\int_0^t \begin{pmatrix} v_1(x_1 + \cos\alpha\tau, x_2 + \sin\alpha\tau) \\ v_2(x_1 + \cos\alpha\tau, x_2 + \sin\alpha\tau) \end{pmatrix} d\tau + O(a^2).\end{aligned}$$

Now assume that $\cos\alpha \neq 0$ and compute the Poincaré map to the circle $x_1 = 0$. This map is

$$\begin{aligned}x_2 \mapsto x_2 + 2\pi\tan\alpha - a\tan\alpha \int_0^{2\pi/\cos\alpha} v_1(\cos\alpha\tau, x_2 + \sin\alpha\tau)\, d\tau \\ + a\int_0^{2\pi/\cos\alpha} v_2(\cos\alpha\tau, x_2 + \sin\alpha\tau)\, d\tau + O(a^2).\end{aligned}$$

Let $\tilde{v}_{1,m,n}$, $\tilde{v}_{2,m,n}$ be the Fourier coefficients of v_1 and v_2. Define

$$\tilde{f}_n(\alpha) = \sum_{m=-\infty}^{\infty} (-\tan\alpha\,\tilde{v}_{1,m,n} + \tilde{v}_{2,m,n})\psi_{m,n}(\alpha),$$

where $\psi_{m,n}(\alpha)$ is given by

$$\psi_{m,n}(\alpha) = \begin{cases} 2\pi/\cos\alpha & \text{if } m\cos\alpha + n\sin\alpha = 0, \\ -i\,\dfrac{e^{2\pi i n \tan\alpha} - 1}{m\cos\alpha + n\sin\alpha} & \text{otherwise.} \end{cases}$$

Define the function $f(x, \alpha) = \sum_{n=-\infty}^{\infty} \tilde{f}_n(\alpha)e^{inx}$. It is easy to see that the Poincaré map computed above can be written as

$$x \mapsto x + 2\pi \tan\alpha + af(x, \alpha) + O(a^2).$$

This family looks almost like the families F we have been studying, but here the function f depends on the parameter α. However, it does not make any difference for the proof of the analog of Theorem A: if v_1 and v_2 are trigonometric polynomials, then f is also a trigonometric polynomial (with coefficients depending on the parameter α) and all arguments in the proof of Theorem A go through.

Another way to compute the Fourier coefficients $\tilde{f}_n$ is the following. Carry out the Fourier transform of v_1 and v_2 only with respect to x_2 and denote the Fourier coefficients by $\breve{v}_{1,n}$, $\breve{v}_{2,n}$, that is,

$$v_j(x_1, x_2) = \sum_{n=-\infty}^{\infty} \breve{v}_{j,n}(x_1)e^{inx_2},$$

where $j = 1, 2$. Then

$$\tilde{f}_n(\alpha) = \int_0^{2\pi/\cos\alpha} (-\tan\alpha\, \breve{v}_{1,n}(\cos(\alpha\tau)) + \breve{v}_{2,n}(\cos(\alpha\tau)))e^{in\sin(\alpha\tau)}\, d\tau.$$

In the proofs in previous sections we make several uses of the Riemann–Lebesgue lemma. The analog of this lemma also holds for $\breve{v}_{j,n}$. We will formulate this lemma when v_1 and v_2 depend on a parameter $t \in I^b$, so this statement can be applied to the proof of the analog of Theorem B directly. If v_1, v_2 do not depend on a parameter, just set $b = 0$.

LEMMA 8.1. *For any $\epsilon_0 > 0$ there exists $N = N(\epsilon_0)$ such that for all $t \in I^b$, $x_1 \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq N$,*

$$|\breve{v}_{j,n}^{(k_1,m)}(x_1, t)| < \epsilon_0|n|^{-k_2},$$

for all $m \leq l$, $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 = k$, where $\breve{v}_{j,n}^{(k_1,m)}(x_1, t)$ denotes the k_1th derivative of $\breve{v}_{j,n}(x_1, t)$ with respect to x_1 and mth derivative with respect to t.

The proof of this lemma is identical to the proof of Lemma 5.1.

Notice that this lemma implies that for any $\epsilon_0 > 0$ there is N such that for all $n \in \mathbb{Z}$, $|n| > N$,

$$|\tilde{f}_n| < \epsilon_0|n|^{-k}.$$

Using these settings, the proofs of the analogs of Theorems A, B, C and F go along the same lines as before. For example, as the perturbed family in the proof of Theorem B one should consider

$$\hat{v}_j(x_1, x_2, t) = v_j(x_1, x_2, t) - \sum_{m=1}^{d+1} (\breve{v}_{j,-mN}(x_1, t)e^{-imNx_2} + \breve{v}_{j,mN}(x_1, t)e^{imNx_2}) + \epsilon N^{-k}(d+1)^{-k}\sin(N(d+1)x_2).$$

In the case of Theorem D the measure μ is defined as the image of the measure τ under the map $\phi : Q^{\aleph_0} \to C^{\omega}_{\delta}(T^2) \times C^{\omega}_{\delta}(T^2)$ defined by

$$\phi((w_1, w_2, \ldots)) = \left(0, \sum_{n=1}^{\infty} e^{-3/2\delta n}(w_n e^{inx_2} + \bar{w}_n e^{-inx_2})\right).$$

Again, the rest of the proof can be easily adjusted.

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